APPLIED MATHEMATICS

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1. Computer Languages, Mathematical Algorithms, Analytical and Numerical Mathematics

Mathematics and mathematical applications used toseperate as analytical math and numerical maths. In modern applied mathematical applications, it is harder to seperate math as such. In complex mathematical universe applied to simulate nature, even analytically derived equations should be calulated and evaluated numerically. Therefore in this text we will approach to math problems in a mixed approach of solution technics. Problems that an analytical formulations can be derived, is usually simplified to solve by computers compare to completly numerical approaches. In order to communucate with computers computer languages or package programs utilising computer lanuages used. As a startin points we would like to give some basic informations about some of the compute languages used in scientific calculations. Note that what is given here is just an introduction to such languages, for more information you should refer to textbooks written specifically for the language teaachings. This book is about applied mathematics. So we will try to give concepts with science and engineering examples. Authour's own field of expertise is thermal sciences, so you might find more examples from this field, but ı'll try to expand range of applications to other fields as well.

1.1 COMPUTER LANGUAGES

There are several languges existed in computer environment. In this work most of the codes of numerical algorithms will be given in java language. The basic approach for the compiler and computer languge selection in this text is to be free for the use in learning environment. As additional to java language other languages are also defined in here. We will be given some details in here where you can find each computer language and how to compile and run them.

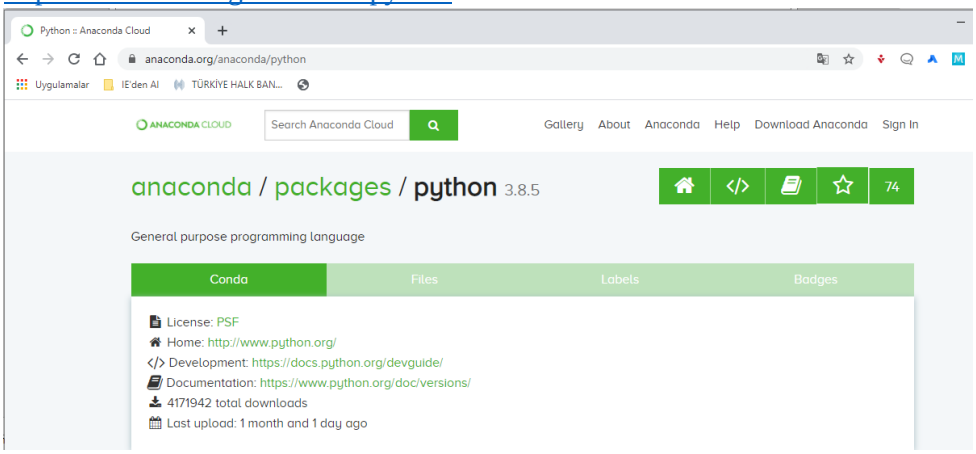
Computer language	İnternet adress to upload language source	compile	run
java	https://www.java.com	**javac** name.java	**java** name
octave	https://www.gnu.org/software/octave/	Write as functions in editor and save	Call functions in run window
C++	Octave above is written in c++ so you can find it under mingw64/bin subdirectory as c++ or g++ or gcc Alternatively https://mingww64.org/doku.php/download	g++ name.cpp –o name.exe	name
Fortran	Octave above has under mingw64/bin subdirectory as gfortran Alternatively https://mingww64.org/doku.php/download	gfrortran name.f90 –o name.exe	name
Python(Anaconda)	https://www.anaconda.com/	-	pythonw name.py
Julia	https://julialang.org/		Julia name.jl

In the year 1849 Ada Lowlace, considered as first computer programmer, described a code for Charles Babbage's Analytical Engine to calculate Bernoulli numbers. First relatively modern electrical computers developed in 1940's and assembly lanuage programs used in these

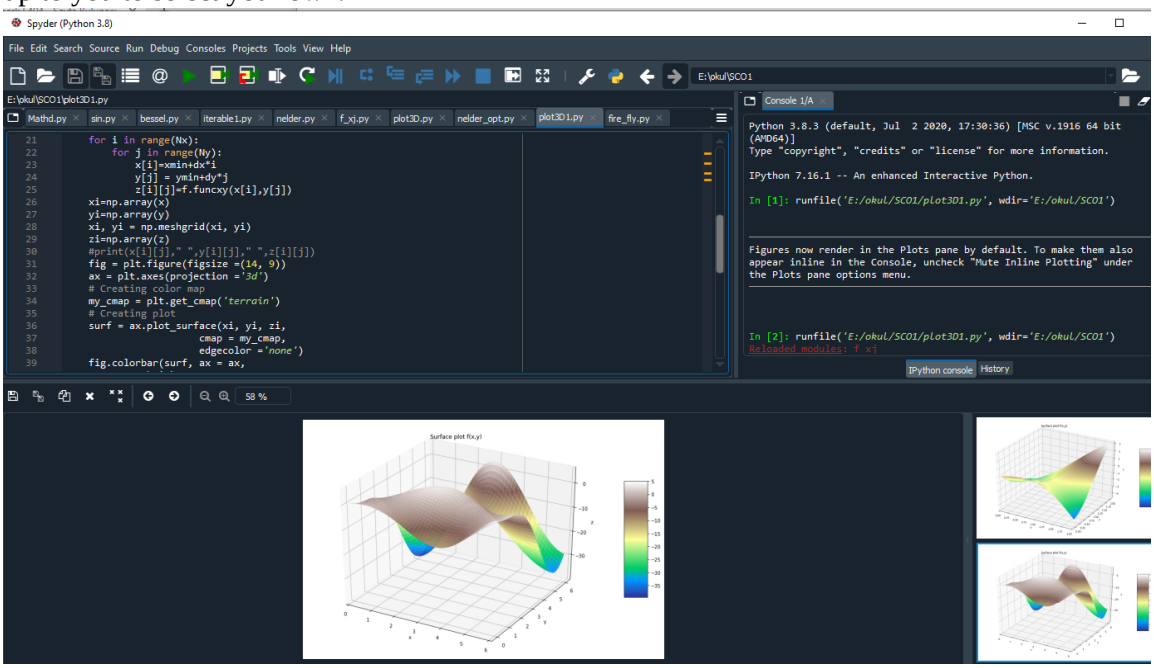
machines. Due to very high afford requirement to write assembly codes, some languages that compile to machine code is developed. The most important of this first generation languages are fortran which developed by IBM group, LISP and COBOL. Other computer languages of 1960's that should be noted are APL (Advanced programming language), Simula, BASIC. A second generation language such as Pascal, C, Smalltalk, Prolog and SQL(sequential querry language) is developed in late 60's. In 1980's comptuter languages started to enter into class age and class based languges such as Ada and C++ is came into the picture. Matlab is also came in this era as structural language. In 1990's languages such as Java, R, PHP is introduced into the picture. Some of the languages developed in 2000's are C#, Go, Julia, Swift. As You see there are so many languages can be utilised and use for scientific calculation purposes. In this section we would like to look some of them in a litle bit more details, Fortran for as the first serious scientific language code that still is in use, C group of languages (C, C++ and Java) , Python and Julia as a recent candidates of scientific calculation and Mathlab-Octave as another package type scientific language.

1.2 Python language basics

We are recommending anaconda python version. You can obtain this from https://anaconda.org/anaconda/python



Adress. Furthermore package include several editors to work with. We are using spyder editor, but it is up to you to select your own.



Python is an interpreted language. An **interpreted language** is a type of language for which most of its implementations execute instructions directly, without previously compiling to machine language (binary) code instructions. The Python interpreter executes the program directly, translating each statement into macine language code. Elimination of compiling process makes running of the program simpler, but execution of the program is much slower compare to compiled programming languages such as C. Authors personel preference for complex numerical modelling and processes languages like C, C++ or java, of which C and C++ are compiled languages and java has two step procedure. In the first step it is compiled to a virtual code close to machine language, and then interpret this code into machine language.

1.2.1 BASIC PROCESSES AND VARIABLES IN PYTHON LANGUAGE

In Python language when defining the variables, method and classes variable types is not specified. When the data is loaded to a value into the variable language determine the variable type. This has both positive and negatve effects. Very beginner programmers does not have to learn variable type

immadetly, but yet because the language using them, ıt is necassary to learn them and using them properly. Passaging of variables into methods(functions) and classes are a little different compare to languages with enforced variable type specifications.

You can run the python statements directly in console environment

```
In [4]: 2*sin(pi/3.0)
Out[4]: 1.7320508075688772

In [5]: x=2.3*sin(pi/4.0)

In [6]: y=x*2+3.2*x+5

In [7]: 2.3*3.4-5.5
Out[7]: 2.3199999999999994
```

But it is only practical when the codes are simple. For the complex cases programs should be written and running (through enterpreter). Programs should be saved as **name.py**. Simple examples explain variables and input andoutput will be given here.

Welcome message (String)

```
print("welcome to python programming language" )
```

```
runfile('E:/okul/SCO1/if_x_test.py', wdir='E:/okul/SCO1')
welcome to python programming language
```

Pyton supports integer (int) and floating numbers(float). Declaration of the type is not necassary in python, when you need an integer numbers declare it as x=15 and when you need a floating numbers declare it as x=15.0.
Integer numbers are sored in memory of computer a s a series of binary bits. They are called bytes. Different number of bits can be located in a byte. In python bytes of an integer can be changed acording to actual numbers stored

Table writing integer 15 as an integer variable in computer memory as binary bits

+/-	2^6	2^5	2^4	2^3	2^2	2^1	2^0
+	**64**	**32**	**16**	**8**	**4**	**2**	**1**
0	**0**	**0**	**0**	**1**	**1**	**1**	**1**

```
import sys
x=75
y=sys.getsizeof(x)
print(y)
28
w=2**75
y1=sys.getsizeof(w)
print(y1)
36
```

Sys.getsizeof(x) returns the size of integer in bits that stored in computer memory

writing $22.625x10^{-17}$ in a floating numbers as binary bits

+/-	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	+\-	2^4	2^3	2^2	2^1	2^0
+	**16**	**8**	**4**	**2**	**1**	**0.5**	**0.25**	**0.125**	**0.0625**	-	**16**	**8**	**4**	**2**	**1**

0	**1**	**0**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**

Python supports four different numerical types −

- int (signed integers)
- long (long integers, they can also be represented in octal and hexadecimal)
- float (floating point real values)
- complex (complex numbers)

If a string variable is consider, you use double or single quote signs
x=”Turhan” or x=’Turhan’

```
x=15
y=15.0;
z=int(15);
w='15'
q=int(w)
r=float(w)
s=float(x)
print("x=",x,"y=",y,"z=",z,"w=",w,"q=",q,"r=",r,"s=",s)
```

```
runfile('E:/okul/SCO1/number.py', wdir='E:/okul/SCO1')
x= 15 y= 15.0 z= 15 w= 15 q= 15 r= 15.0 s= 15.0
```

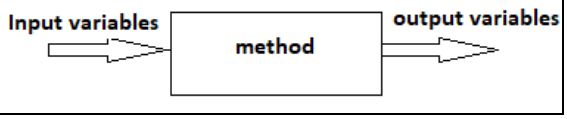
Arithmetic processes of integer + add ; - substract * multiply; / divide; % remaining in divide
Integer variable reading, calculating & printing

```
x=int(input("x="))
y=2*x+5
print("x =",x,"y = ",y)
```

```
runfile('E:/okul/SCO1/int1.py', wdir='E:/okul/SCO1')

x=5
x = 5 y =  15
```

Methods or functions are program pieces that has input variables and output variables. Structure is as follows:

```
def method_name(method input variable,method input variable,..):
    program statements
    …..
    return  method output variable
```



Integer variable reading, calculating by using method(function) & printing

```
def f(x):
    y=2*x+5
    return y;

x=int(input("x="))
print("x =",x,"y = ",f(x))
```

```
runfile('E:/okul/SCO1/int2.py', wdir='E:/okul/SCO1')

x=5
x = 5 y =  15
```

Integer variable reading, calculating by using method(function) & printing

```
def divide(x,y):
    z=x/y
    return int(z);
def remain(x,y):
    z=x%y
    return z;

x=int(input("x="))
y=int(input("y="))
print("x =",x,"y = ",y,"x/y = ",divide(x,y),"x%y = ",remain(x,y))
```

```
runfile('E:/okul/SCO1/int3.py', wdir='E:/okul/SCO1')

x=7
```

```
y=3
x = 7 y =  3 x/y =  2 x%y =  1
```

```
number1=int(input("number1 = "))
number2=int(input("number2 = "))
sum = number1+number2;
substract = number1-number2
multiply = number1*number2
divide=int(number1/number2)
remain=number1%number2
s=str(number1)+" +  "+str(number2)+" = "+str(sum)+"\n"
s+=str(number1)+" -  "+str(number2)+" = "+str(substract)+"\n"
s+=str(number1)+" x  "+str(number2)+" = "+str(multiply)+"\n"
s+=str(number1)+" /  "+str(number2)+" = "+str(divide)+"\n"
s+=str(number1)+" %  "+str(number2)+" = "+str(remain)+"\n"
print(s)
```

```
runfile('E:/okul/SCO1/string2.py', wdir='E:/okul/SCO1')

number1 = 7

number2 = 3
7 +  3 = 10
7 -  3 = 4
7 x  3 = 21
7 /  3 = 2
7 %  3 = 1
```

Note when adding strings together in tor float will be converted dto string by using str(variable) method

Float variable(real numbers) reading, calculating and printing

```
x=float(input("x="))
y=x*x-2.3*x+5.7
print("x=",x,"y=",y)
```

```
x=1.235
x= 1.235 y= 4.384725
```

Float variable(real numbers) reading, calculating by using method(function) and printing

```
def f(x):
    y=x*x-2.3*x+5.7
    return y;

x=float(input("x="))
print("x=",x,"y=",f(x))
```

```
runfile('E:/okul/SCO1/float2.py', wdir='E:/okul/SCO1')

x=1.235
x= 1.235 y= 4.384725
```

Float variable(real numers) reading, calculating by using method(function) and printing

```
def divide(x,y):
    z=x/y
    return z

x=float(input("x="))
y=float(input("y="))
print("x =",x,"y = ",y,"x/y = ",divide(x,y))
```

```
runfile('E:/okul/SCO1/float3.py', wdir='E:/okul/SCO1')

x=7

y=3
x = 7.0 y =  3.0 x/y =  2.3333333333333335
```

```
number1=float(input("number1 = "))
number2=float(input("number2 = "))
sum = number1+number2;
substract = number1-number2
multiply = number1*number2
divide=number1/number2
s=str(number1)+" +  "+str(number2)+" = "+str(sum)+"\n"
```

```
s+=str(number1)+" -  "+str(number2)+" = "+str(substract)+"\n"
s+=str(number1)+" x  "+str(number2)+" = "+str(multiply)+"\n"
s+=str(number1)+" /  "+str(number2)+" = "+str(divide)+"\n"
print(s)
```

```
runfile('E:/okul/SCO1/float4.py', wdir='E:/okul/SCO1')

number1 = 7.86

number2 = 2.31
7.86 +  2.31 = 10.17
7.86 -  2.31 = 5.550000000000001
7.86 x  2.31 = 18.1566
7.86 /  2.31 = 3.4025974025974026
```

Reading and writing string

```
name=input("what is your name =")
print("welcome to pyhon programming ",name)
```

```
runfile('E:/okul/SCO1/prog2.py', wdir='E:/okul/SCO1')

what is your name =turhan
welcome to pyhon programming  turhan
```

Reading and writing string by using method name()

```
def name():
    n=input("what is your name =")
    return n

print("welcome to pyhon programming ",name())
```

```
what is your name =turhan
welcome to pyhon programming  turhan
```

Reading and writing string by using method hello()

```
def hello():
    n=input("what is your name =")
    s="welcome to python programming "+n
    return s

print(hello())
```

```
runfile('E:/okul/SCO1/name3.py', wdir='E:/okul/SCO1')

what is your name =turhan
welcome to python programming turhan
```

Boolean variables:

Boolean variable tales values True/False. It takes only one bit binary to define a boolean number.
Boolean operators:
AND OR NOT

Boolean Logic table

First variable	**Second variable**	**AND**	**OR**
True	True	True	True
True	False	False	True
False	True	False	True
False	False	False	False
First variable		**NOT**	
True		False	
False		True	

```
x=True
y=False
z=x and y
w=x or y
```

```
u=not x
print("x=",x,"y=",y,"z=",z,"w=",w,"u=",u)
```

```
runfile('E:/okul/SCO1/bool1.py', wdir='E:/okul/SCO1')
x= True y= False z= False w= True
```

```
def x_and_y(x,y):
    return x and y
def x_or_y(x,y):
    return x or y
def not_x(x):
    return not x
x=True
y=True
print("x=",x,"y=",y,"x and y=",x_and_y(x,y),"x or y=",x_or_y(x,y),"not x ",not_x(x))
x=True
y=False
print("x=",x,"y=",y,"x and y=",x_and_y(x,y),"x or y=",x_or_y(x,y),"not x ",not_x(x))
x=False
y=True
print("x=",x,"y=",y,"x and y=",x_and_y(x,y),"x or y=",x_or_y(x,y),"not x ",not_x(x))
x=False
y=False
print("x=",x,"y=",y,"x and y=",x_and_y(x,y),"x or y=",x_or_y(x,y),"not x ",not_x(x))
```

```
runfile('E:/okul/SCO1/bool2.py', wdir='E:/okul/SCO1')
x= True y= True x and y= True x or y= True not x  False
x= True y= False x and y= False x or y= True not x  False
x= False y= True x and y= False x or y= True not x  True
x= False y= False x and y= False x or y= False not x  True
```

Comparison of two numbers:

sign	meaning
>	**greater**
>=	**Grater and equal**
<	**smaller**
<=	**Smaller and equal**
==	**equal**
!=	**Not equal**

```
def greater(x,y):
    return x>y
def smaller(x,y):
    return x<y
def greater_and_equal(x,y):
    return x>=y
def smaller_and_equal(x,y):
    return x<=y
def equal(x,y):
    return x==y
def not_equal(x,y):
    return x!=y
x=int(input("x="))
y=int(input("x="))
print("x=",x,"y=",y,"x>y=",greater(x,y),"x<y=",smaller(x,y),"x>=y",greater_and_equal(x,y),"x<=y",smaller_and_equal(x,y),"equal",equa
l(x,y),"not equal ",not_equal(x,y))
```

```
runfile('E:/okul/SCO1/bool3.py', wdir='E:/okul/SCO1')

x=2

x=3
x= 2 y= 3 x>y= False x<y= True x>=y False x<=y True equal False not equal  True
```

```
runfile('E:/okul/SCO1/bool3.py', wdir='E:/okul/SCO1')

x=3
x=3
x= 3 y= 3 x>y= False x<y= False x>=y True x<=y True equal True not equal  False
```

Character variable type and Unicode characters

Character (char) variable type define actual characters in computers. In order to see characters in python **ISO Unicode** character standards are used. Unicode character code consist of 4 hexagonal (base 16) number. In case you are not familiar with orthogonal number system, a table of equiavalancy is given

Tablo Heksagonal(sixteen base), and 10 base and binary number systems equvalency

Heksagonal	10 base	2 base (binary)
0	**0**	**0000**
1	**1**	**0001**
2	**2**	**0010**
3	**3**	**0011**
4	**4**	**0100**
5	**5**	**0101**
6	**6**	**0110**
7	**7**	**0111**
8	**8**	**1000**
9	**9**	**1001**
A	**10**	**1010**
B	**11**	**1011**
C	**12**	**1100**
D	**13**	**1101**
E	**14**	**1110**
F	**15**	**1111**

```
x = int(input("Input an integer: "))
print("x ten base =",x,"x 2 base ",bin(x),"x octogonal base",oct(x),"x hexagonal base=",hex(x))
```

```
Input an integer: 15
x ten base = 15 x 2 base  0b1111 x octogonal base 0o17 x hexagonal base= 0xf
```

```
Input an integer: 12
x ten base = 12 x 2 base  0b1100 x octogonal base 0o14 x hexagonal base= 0xc
```

in ISO unicode '\u0041' defines character 'A'. or '\u03E1'. Or code '\u03E1' defines 'a'. When the first two number of unicode equals to 0, it defines ASCII characte codes. You can find very detailed list of characters in ISO unicode by looking the internet site **http:\\unicode.org**

In the small code sample below character A and 'α' is loaded to computer as char variables in python

```
A1='\u0041'
A2='A'
alpha1='\u03B1'
print(A1,A2,alpha1)
```

```
runfile('E:/okul/SCO1/char1.py', wdir='E:/okul/SCO1')
A A α
```

Table Some special control characters

Chracter sysmbol	**action**	**Unicode symbol**
\b	**Back one character**	**\u0008**
\t	**tab**	**\u0009**
\n	**ctrl-return**	**\u000A**
\"	**“ sign**	**\u0022**
\'	**‘ sign**	**\u0027**
****	**\ sign**	**\u005C**

Some additional unicode characters are given in tables below. By using control characters several actions in print can be carried out such as back space, carriage-return, tab etc. When character variables are used, they can merge in strings, In fact strings are character arrays. When something with a character that does not available in your editor you can add that character into your String variable as char variable character.

```
b1='ü'
b2='ğ'
b3='ç'
b4='a'
b5='b'
b6='c'
s=""+b1+b2+b3+b4+b5+b6
print(s)
```

```
runfile('E:/okul/SCO1/char2.py', wdir='E:/okul/SCO1')
üğçabc
```

```
b1='\u03B1'
b2='\u03B2'
b3='\u03B3'
b4='\u03B4'
b5='\u03B5'
b6='\u03B6'
s=""+b1+b2+b3+b4+b5+b6
print(s)
```

```
runfile('E:/okul/SCO1/char3.py', wdir='E:/okul/SCO1')
αβγδεζ
```

```
b1='\u0394'
b2='\u00B2'
s=b1+"T = "+b1+"x"+b2+"+"+b1+"y"+b2
print(s)
```

```
runfile('E:/okul/SCO1/char1.py', wdir='E:/okul/SCO1')
ΔT = Δx²+Δy²
```

Table Some unicode character codes

	000	001	002	003	004	005	006	007
0	NUL 0000	DLE 0010	SP 0020	0 0030	@ 0040	P 0050	` 0060	p 0070
1	SOH 0001	DC1 0011	! 0021	1 0031	A 0041	Q 0051	a 0061	q 0071
2	STX 0002	DC2 0012	" 0022	2 0032	B 0042	R 0052	b 0062	r 0072
3	ETX 0003	DC3 0013	# 0023	3 0033	C 0043	S 0053	c 0063	s 0073
4	EOT 0004	DC4 0014	$ 0024	4 0034	D 0044	T 0054	d 0064	t 0074
5	ENQ 0005	NAK 0015	% 0025	5 0035	E 0045	U 0055	e 0065	u 0075
6	ACK 0006	SYN 0016	& 0026	6 0036	F 0046	V 0056	f 0066	v 0076
7	BEL 0007	ETB 0017	' 0027	7 0037	G 0047	W 0057	g 0067	w 0077
8	BS 0008	CAN 0018	(0028	8 0038	H 0048	X 0058	h 0068	x 0078
9	HT 0009	EM 0019	) 0029	9 0039	I 0049	Y 0059	i 0069	y 0079
A	LF 000A	SUB 001A	* 002A	: 003A	J 004A	Z 005A	j 006A	z 007A
B	VT 000B	ESC 001B	+ 002B	; 003B	K 004B	[005B	k 006B	{ 007B
C	FF 000C	FS 001C	, 002C	< 003C	L 004C	\ 005C	l 006C	| 007C
D	CR 000D	GS 001D	- 002D	= 003D	M 004D	] 005D	m 006D	} 007D
E	SO 000E	RS 001E	. 002E	> 003E	N 004E	^ 005E	n 006E	~ 007E
F	SI 000F	US 001F	/ 002F	? 003F	O 004F	_ 005F	o 006F	DEL 007F

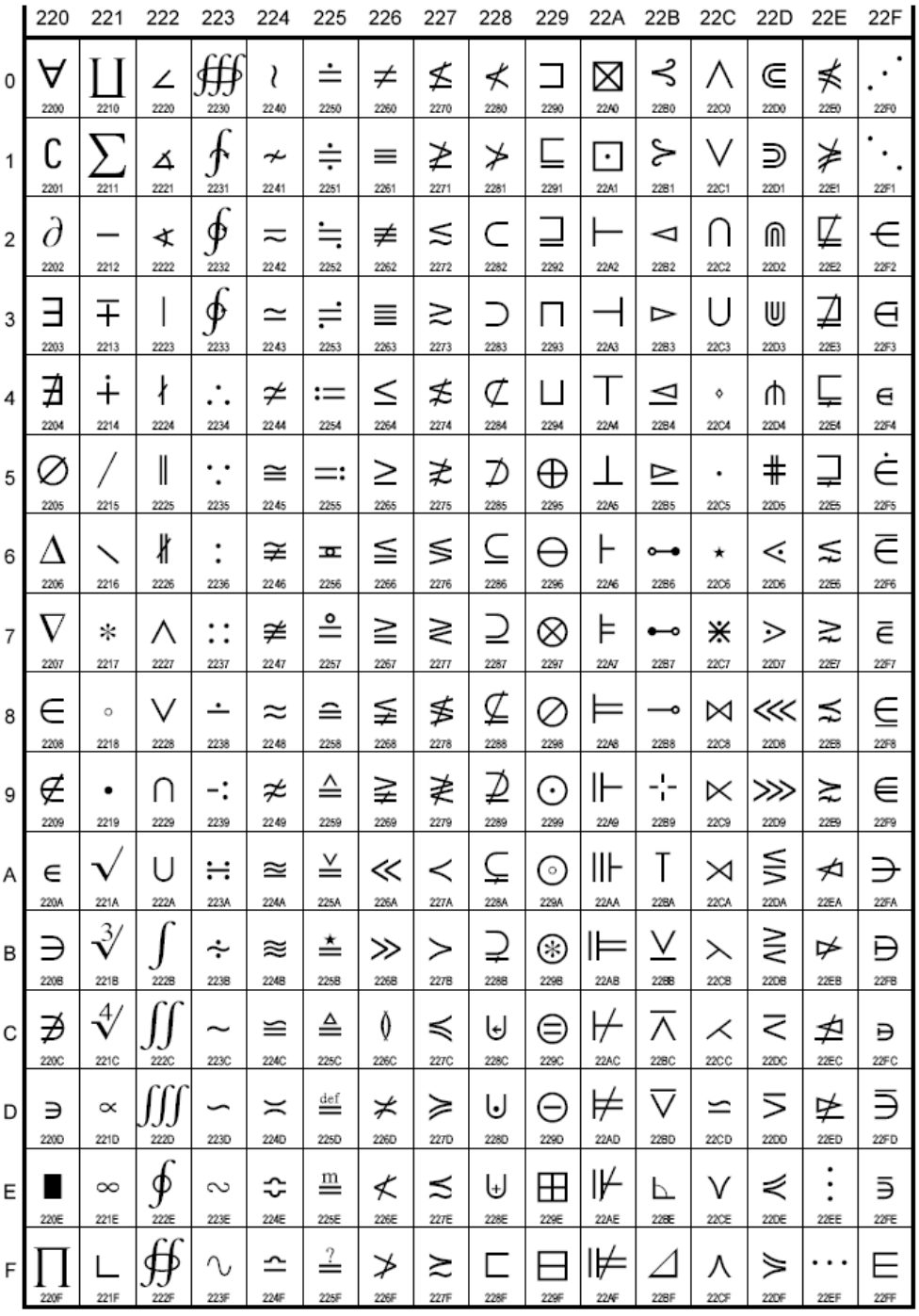
	008	009	00A	00B	00C	00D	00E	00F
0	XXX 0080	DCS 0090	NB SP 00A0	° 00B0	À 00C0	Ð 00D0	à 00E0	ð 00F0
1	XXX 0081	PU1 0091	¡ 00A1	± 00B1	Á 00C1	Ñ 00D1	á 00E1	ñ 00F1
2	BPH 0082	PU2 0092	¢ 00A2	2 00B2	Â 00C2	Ò 00D2	â 00E2	ò 00F2
3	NBH 0083	STS 0093	£ 00A3	3 00B3	Ã 00C3	Ó 00D3	ã 00E3	ó 00F3
4	IND 0084	CCH 0094	¤ 00A4	´ 00B4	Ä 00C4	Ô 00D4	ä 00E4	ô 00F4
5	NEL 0085	MW 0095	¥ 00A5	µ 00B5	Å 00C5	Õ 00D5	å 00E5	õ 00F5
6	SSA 0086	SPA 0096	¦ 00A6	¶ 00B6	Æ 00C6	Ö 00D6	æ 00E6	ö 00F6
7	ESA 0087	EPA 0097	§ 00A7	· 00B7	Ç 00C7	× 00D7	ç 00E7	÷ 00F7
8	HTS 0088	SOS 0098	¨ 00A8	¸ 00B8	È 00C8	Ø 00D8	è 00E8	ø 00F8
9	HTJ 0089	XXX 0099	© 00A9	1 00B9	É 00C9	Ù 00D9	é 00E9	ù 00F9
A	VTS 008A	SCI 009A	ª 00AA	º 00BA	Ê 00CA	Ú 00DA	ê 00EA	ú 00FA
B	PLD 008B	CSI 009B	« 00AB	» 00BB	Ë 00CB	Û 00DB	ë 00EB	û 00FB
C	PLU 008C	ST 009C	¬ 00AC	¼ 00BC	Ì 00CC	Ü 00DC	ì 00EC	ü 00FC
D	RI 008D	OSC 009D	SHY 00AD	½ 00BD	Í 00CD	Ý 00DD	í 00ED	ý 00FD
E	SS2 008E	PM 009E	® 00AE	¾ 00BE	Î 00CE	Þ 00DE	î 00EE	þ 00FE
F	SS3 008F	APC 009F	¯ 00AF	¿ 00BF	Ï 00CF	ß 00DF	ï 00EF	ÿ 00FF

0180 **Latin Extended-B** **024**

	018	019	01A	01B	01C	01D	01E	01F	020	021	022	023	024
0	ƀ 0180	Ɛ 0190	Ơ 01A0	ư 01B0	ǀ 01C0	ǐ 01D0	Ā 01E0	ǰ 01F0	À 0200	Ȑ 0210	ɳ 0220	Ō 0230	ɀ 0240
1	Ɓ 0181	Ƒ 0191	ơ 01A1	Ʊ 01B1	ǁ 01C1	Ǒ 01D1	ā 01E1	DZ 01F1	à 0201	ȑ 0211	ɖ 0221	ō 0231	ɂ 0241
2	Ƃ 0182	ƒ 0192	Ƣ 01A2	Ʋ 01B2	ǂ 01C2	ǒ 01D2	Ē 01E2	Dz 01F2	Â 0202	Ȓ 0212	ȣ 0222	Ȳ 0232	ɔ 0242
3	ƃ 0183	Ɠ 0193	ƣ 01A3	Ƴ 01B3	! 01C3	Ǔ 01D3	ē 01E3	dz 01F3	â 0203	î 0213	ȣ 0223	ȳ 0233	Ƀ 0243
4	Ƅ 0184	Ɣ 0194	Ƥ 01A4	ƴ 01B4	DŽ 01C4	ǔ 01D4	Ġ 01E4	Ǵ 01F4	È 0204	Ù 0214	Ȝ 0224	ɻ 0234	Ʉ 0244
5	ƅ 0185	ƕ 0195	ƥ 01A5	Ƶ 01B5	Dž 01C5	Ǖ 01D5	g 01E5	ǵ 01F5	è 0205	ù 0215	ȝ 0225	ɲ 0235	Ʌ 0245
6	Ɔ 0186	ƚ 0196	Ʀ 01A6	ƶ 01B6	dž 01C6	ǖ 01D6	Ǧ 01E6	Ƕ 01F6	Ê 0206	Û 0216	Ȧ 0226	ȶ 0236	Ɇ 0246
7	Ƈ 0187	ƛ 0197	Ƨ 01A7	Ʒ 01B7	LJ 01C7	Ǘ 01D7	ǧ 01E7	ƿ 01F7	ê 0207	û 0217	ȧ 0227	ɟ 0237	ɇ 0247
8	ƈ 0188	Ƙ 0198	ƨ 01A8	Ǯ 01B8	Lj 01C8	ǘ 01D8	Ǩ 01E8	Ǹ 01F8	Ì 0208	Ș 0218	Ȩ 0228	ɐ 0238	ɉ 0248
9	Ɖ 0189	ƙ 0199	Σ 01A9	ǯ 01B9	lj 01C9	Ǚ 01D9	ǩ 01E9	ǹ 01F9	ì 0209	ș 0219	ȩ 0229	ɑ 0239	ɉ 0249
A	Ɗ 018A	ƚ 019A	ƪ 01AA	ƺ 01BA	NJ 01CA	ǚ 01DA	Ǫ 01EA	Ǻ 01FA	Î 020A	Ț 021A	Ɵ 022A	Ⱥ 023A	ɐ 024A
B	Ƌ 018B	ƛ 019B	ƫ 01AB	Ƹ 01BB	Nj 01CB	Ǜ 01DB	ǫ 01EB	ǻ 01FB	î 020B	ț 021B	ȫ 022B	Ȼ 023B	ɑ 024B
C	ƌ 018C	Ɯ 019C	Ƭ 01AC	Ƽ 01BC	nj 01CC	ǜ 01DC	Ǭ 01EC	Æ 01FC	Ò 020C	Ȝ 021C	Ȭ 022C	ȼ 023C	Ʀ 024C
D	ƍ 018D	Ɲ 019D	ƭ 01AD	ƽ 01BD	Ǎ 01CD	ǝ 01DD	ǭ 01ED	æ 01FD	ò 020D	ȝ 021D	ȭ 022D	Ł 023D	ɾ 024D
E	Ǝ 018E	ƞ 019E	Ʈ 01AE	ƾ 01BE	ǎ 01CE	Ǟ 01DE	Ǯ 01EE	Ǽ 01FE	Ô 020E	Ȟ 021E	Ȯ 022E	Ⱦ 023E	Ɏ 024E
F	Ə 018F	Ɵ 019F	Ư 01AF	ƿ 01BF	Ǐ 01CF	ǟ 01DF	ǯ 01EF	ǽ 01FF	ô 020F	ȟ 021F	ȯ 022F	ɀ 023F	ɏ 024F

0370 **Greek and Coptic** **03FF**

	037	038	039	03A	03B	03C	03D	03E	03F
0	Ͱ 0370		ΐ 0390	Π 03A0	ΰ 03B0	π 03C0	ϐ 03D0	Ϣ 03E0	ϰ 03F0
1	ͱ 0371		Α 0391	Ρ 03A1	α 03B1	ρ 03C1	ϑ 03D1	ϣ 03E1	ϱ 03F1
2	Ͳ 0372		Β 0392		β 03B2	ς 03C2	ϒ 03D2	Ϥ 03E2	ϲ 03F2
3	ͳ 0373		Γ 0393	Σ 03A3	γ 03B3	σ 03C3	ϓ 03D3	ϥ 03E3	ϳ 03F3
4	ʹ 0374	΄ 0384	Δ 0394	Τ 03A4	δ 03B4	τ 03C4	ϔ 03D4	Ϧ 03E4	ϴ 03F4
5	͵ 0375	΅ 0385	Ε 0395	Υ 03A5	ε 03B5	υ 03C5	ϕ 03D5	ϧ 03E5	ϵ 03F5
6	Ͷ 0376	Ά 0386	Ζ 0396	Φ 03A6	ζ 03B6	φ 03C6	ϖ 03D6	Ϩ 03E6	϶ 03F6
7	ͷ 0377	· 0387	Η 0397	Χ 03A7	η 03B7	χ 03C7	ϗ 03D7	ϩ 03E7	Ϸ 03F7
8		Έ 0388	Θ 0398	Ψ 03A8	θ 03B8	ψ 03C8	Ϙ 03D8	Ϫ 03E8	ϸ 03F8
9		Ή 0389	Ι 0399	Ω 03A9	ι 03B9	ω 03C9	ϙ 03D9	ϫ 03E9	Ϲ 03F9
A	ͺ 037A	Ί 038A	Κ 039A	Ϊ 03AA	κ 03BA	ϊ 03CA	Ϛ 03DA	Ϭ 03EA	Ϻ 03FA
B	ͻ 037B		Λ 039B	Ϋ 03AB	λ 03BB	ϋ 03CB	ϛ 03DB	ϭ 03EB	ϻ 03FB
C	ͼ 037C	Ό 038C	Μ 039C	ά 03AC	μ 03BC	ό 03CC	Ϝ 03DC	Ϯ 03EC	ϼ 03FC
D	ͽ 037D		Ν 039D	έ 03AD	ν 03BD	ύ 03CD	ϝ 03DD	ϯ 03ED	Ͻ 03FD
E	; 037E	Ύ 038E	Ξ 039E	ή 03AE	ξ 03BE	ώ 03CE	Ϟ 03DE	ϰ 03EE	Ͼ 03FE
F		Ώ 038F	Ο 039F	ί 03AF	ο 03BF	ϗ 03CF	ϟ 03DF	ϱ 03EF	Ͽ 03FF



Binary , octagonal and hexagonal integers

Integer numbers can be expressed either in 10 digit form we are familiar with or binary form which is what computer understand or octagonal form and hexagonal form which basically derived from binary bit combinations (octagonal=8 digit=2^3, hexagonal=16 digit=2^4) If you want to output a 10 base integer into binary, octagonal or hexagonal form use bin,oct and hex method

```
x=256
print("x=",x,"binary ",bin(x),"octagonal ",oct(x),"hexagonal ",hex(x))
```

```
runfile('E:/okul/SCO1/greek_char.py', wdir='E:/okul/SCO1')
x= 256 binary  0b100000000 octagonal  0o400 hexagonal  0x100
```

We can enter a binary integer by using 0b in front, for example 0b001101, 0o for octogonal number, for example 0o134, 0x for hexagonal number for example 0x12af

```
x=134
x1=0b001101
x2=0o134
x3=0x12af
print("x=",x,"binary ",bin(x),"octagonal ",oct(x),"hexagonal ",hex(x))
print("x1=",x1,"binary ",bin(x1),"octagonal ",oct(x1),"hexagonal ",hex(x1))
print("x12",x2,"binary ",bin(x2),"octagonal ",oct(x2),"hexagonal ",hex(x2))
print("x3=",x3,"binary ",bin(x3),"octagonal ",oct(x3),"hexagonal ",hex(x3))
```

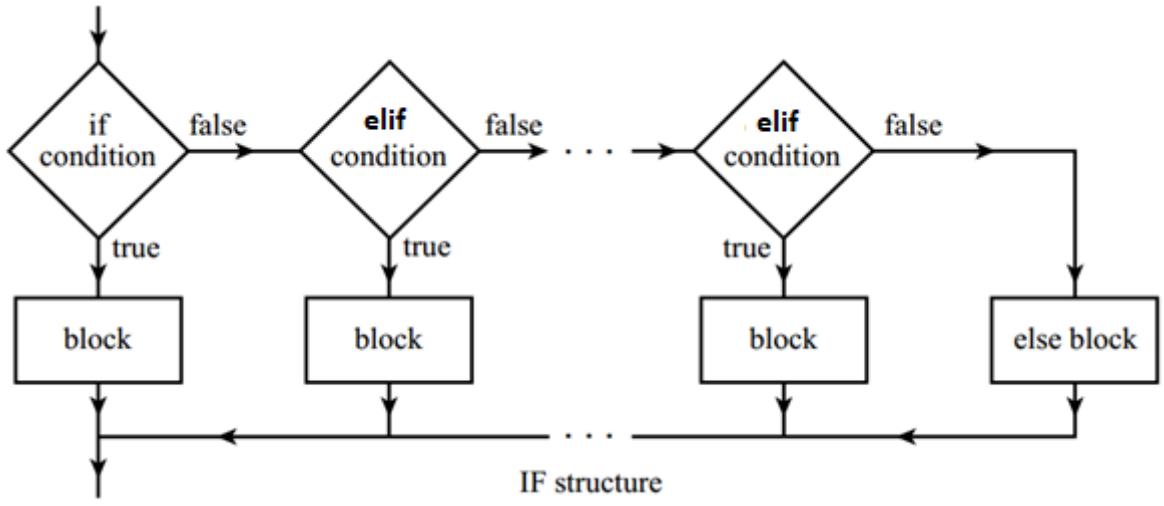
```
runfile('E:/okul/SCO1/greek_char.py', wdir='E:/okul/SCO1')
x= 134 binary  0b10000110 octagonal  0o206 hexagonal  0x86
x1= 13 binary  0b1101 octagonal  0o15 hexagonal  0xd
x12 92 binary  0b1011100 octagonal  0o134 hexagonal  0x5c
```

1.2.3 IF STATEMENT IN PYTHON LANGUAGE

When a decision should be made by a computer program, if control structure used. If structure has the following form :

an example of boolean variable

if boolean variable 1:
    executes statements here if boolean variable 1 is true
elif boolean variable 2:
  executes statements here if boolean variable 2 is true
elif (boolean variable 3)
   executes statements here if boolean variable 3 is true
……. (more else if statements are possible)
Else:
  executes statements here if none of the above boolean variables are true



IF structure

In our first example program name and grade of a students read and process the grade according to number grade.

```
name=input("Enter student name=")
grade=float(input("enter student\'s grade="))
s1="Grade of "+name+" is "
if grade>=90: s1=s1+"A"
elif grade>=75: s1=s1+"B"
elif grade>=60: s1=s1+"C"
elif grade>=50: s1=s1+"D"
elif grade>=40: s1=s1+"E"
else:veli
78  s1=s1+"F"
print(s1);
```

```
runfile('E:/okul/SCO1/if5.py', wdir='E:/okul/SCO1')

Enter student name=ali

enter student's grade=16
Grade of ali is F

runfile('E:/okul/SCO1/if5.py', wdir='E:/okul/SCO1')

Enter student name=veli

enter student's grade=78
Grade of veli is B
```

```
x=float(input("x="))
if x<2: y=x*x-2
elif x>=2 & x<4: y=x+5.5
else:   y=3
print("x=",x,"y=",y)
```

```
runfile('E:/okul/SCO1/if1.py', wdir='E:/okul/SCO1')

x=1.5
x= 1.5 y= 0.25
```

```
wheather=input("how is weather to day?")
x1=wheather=="rainy"
x2=wheather=="sunny"
x3=wheather=="cloudy"
if x1: action="stay at home"
elif x2: action="go to beach"
elif x3: action="study"
else: action="do nothing"
print(action)
```

```
runfile('E:/okul/SCO1/wheather.py', wdir='E:/okul/SCO1')

how is weather to day?sunny
go to beach
```

```
def compare(x,y):
    if x<y: s=str(x)+" is smaller than "+str(y)
    elif x>y: s=str(x)+" is bigger than "+str(y)
    elif x==y: s=str(x)+" is equal to "+str(y)
    return s

x=float(input("x="))
y=float(input("x="))
print(compare(x,y))
```

```
runfile('E:/okul/SCO1/bool3.py', wdir='E:/okul/SCO1')

x=7.35

x=2.23
7.35 is bigger than 2.23
```

```
def guess_color(s):
    b1=s=="red"
    if b1: s1="Correct guess color you entered is red"
    else:  s1="Color you entered is npt red, it is "+s
    return s1

x=input("guess the color =")
print(guess_color(x))
```

```
runfile('E:/okul/SCO1/bool4.py', wdir='E:/okul/SCO1')

guess the color =red
Correct guess color you entered is red
```

Nested if

```
x=float(input("x = "))
y=float(input("y = "))
if x>5:
    if y>5: s="x and y are greater than 5"
    elif y==5: s="x is greater that 5 and y is equal to 5"
    else: s="x is greater that 5 and y is smaller than 5"
else:
    if y>5: s="x is smaller or equal to 5 and  y is greater than 5"
    elif y==5: s="x is smaller or equal to 5 and y is equal to 5"
    else: s="x is smaller or equal to 5 and y is smaller than 5"
print(s)
```

```
runfile('E:/okul/SCO1/bool5.py', wdir='E:/okul/SCO1')

x = 1.23

y = 2.35
x is smaller or equal to 5 and y is smaller than 5
```

Composite if

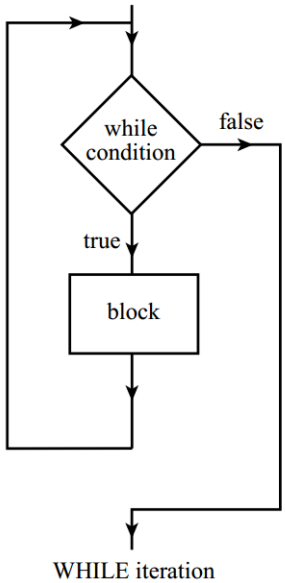
```
x=float(input("x = "))
y=float(input("y = "))
if x>5 and y>5: s="x and y are greater than 5"
elif x>5 and y==5: s="x is greater that 5 and y is equal to 5"
elif x>5 and y<5: s="x is greater that 5 and y is smaller than 5"
elif x<=5 and y>5: s="x is smaller or equal to 5 and  y is greater than 5"
elif x<=5 and y==5: s="x is smaller or equal to 5 and y is equal to 5"
elif x<=5 and y>5: s="x is smaller or equal to 5 and y is smaller than 5"
print(s)
```

```
runfile('E:/okul/SCO1/bool6.py', wdir='E:/okul/SCO1')

x = 3
y = 2
x is smaller or equal to 5 and y is smaller than 5
```

1.2.4 FOR AND WHILE STATEMENTS IN PYTHON LANGUAGE

In the if structure of the previous section, it is seen that if the given boolean value is true, computer carries out instruction inside of the if section. While structure is similar to if structure, with one difference: In the if statement inside of the statement structure instructions will be carried out once. In while statements, it will go inside of while statement as repeatedly, as long as the boolean is true.

while boolean variable:
processes while boolean variable is true




```
def sum(n):
    total=0;
```

```
    i=1
    while i<(n+1):
        total=total+i;
        i+=1
    return total;

n=int(input("n = "));
r=sum(n);
print("total=",r);
```

```
n = 100
total= 5050
```

For loop structure is also similar to while structure. Except initial condition and incrimental conditions are also given in the structure

for initial variable **in** a vector
  body

A set of numbers (or any type of varibles) is called lists. Lists are given within [] and seperated with ,
A set of numbers(or any type of variables) is called tupples . Tupples are given within () and seperated with ,
Basic difference of list and tuple is: elements of list can be changed. Elements of tuple can not be changed define only ones and then used as is. In math, List and tuple is represent a **vector**.

```
list = [ 'abcd', 786 , 2.23, 'john', 70.2 ]

print(list)          # Prints complete list
print(list[0])       # Prints first element of the list
print(list[1:3])     # Prints elements starting from 2nd till 3rd
print(list[2:])      # Prints elements starting from 3rd element
```

```
runfile('E:/okul/SCO1/list.py', wdir='E:/okul/SCO1')
['abcd', 786, 2.23, 'john', 70.2]
abcd
[786, 2.23]
[2.23, 'john', 70.2]
```

```
tuple = ( 'abcd', 786 , 2.23, 'john', 70.2 )

print(tuple)          # Prints complete list
print(tuple[0])       # Prints first element of the list
print(tuple[1:3])     # Prints elements starting from 2nd till 3rd
print(tuple[2:])      # Prints elements starting from 3rd element
```

```
runfile('E:/okul/SCO1/list.py', wdir='E:/okul/SCO1')
('abcd', 786, 2.23, 'john', 70.2)
abcd
(786, 2.23)
(2.23, 'john', 70.2)
```

There are buit in functions to be used with the lists

Method	Description
append()	Adds an element at the end of the list
clear()	Removes all the elements from the list
copy()	Returns a copy of the list
count()	Returns the number of elements with the specified value
extend()	Add the elements of a list (or any iterable), to the end of the current list
index()	Returns the index of the first element with the specified value
insert()	Adds an element at the specified position

pop()	Removes the element at the specified position
remove()	Removes the first item with the specified value
reverse()	Reverses the order of the list
sort()	Sorts the list

```
list = [ 'abcd', 786 , 2.23, 'john', 70.2 ]

print(list)          # Prints complete list
print(list[0])       # Prints first element of the list
print(list[1:3])     # Prints elements starting from 2nd till 3rd
print(list[2:])      # Prints elements starting from 3rd element
list.append("xxx")
print(list)
list1=[1,2,3]
list.extend(list1)
print(list)
list.pop(5)
print(list)
```

```
runfile('E:/okul/SCO1/list_test1.py', wdir='E:/okul/SCO1')
['abcd', 786, 2.23, 'john', 70.2]
abcd
[786, 2.23]
[2.23, 'john', 70.2]
['abcd', 786, 2.23, 'john', 70.2, 'xxx']
['abcd', 786, 2.23, 'john', 70.2, 'xxx', 1, 2, 3]
['abcd', 786, 2.23, 'john', 70.2, 1, 2, 3]
```

vector can be defined y using range method or given directly

```
y=[1,2,3,5,7,9,11]
i=0
for x in y:
    print("i=",i,"x=",x)
    i=i+1
```

```
runfile('E:/okul/SCO1/for3.py', wdir='E:/okul/SCO1')
i= 0 x= 1
i= 1 x= 2
i= 2 x= 3
i= 3 x= 5
i= 4 x= 7
i= 5 x= 9
i= 6 x= 11
```

```
y=(1,2,3,5,7,9,11)
i=0
for x in y:
    print("i=",i,"x=",x)
    i=i+1
```

```
runfile('E:/okul/SCO1/list.py', wdir='E:/okul/SCO1')
i= 0 x= 1
i= 1 x= 2
i= 2 x= 3
i= 3 x= 5
i= 4 x= 7
i= 5 x= 9
i= 6 x= 11
```

```
y=["What's"," Up"," Docs"]
i=0
for x in y:
    print("i=",i,"x=",x)
    i=i+1
```

```
runfile('E:/okul/SCO1/for4.py', wdir='E:/okul/SCO1')
i= 0 x= What's
i= 1 x=  Up
i= 2 x=  Docs
```

Range method can defined vectors

range(3) defines [0,1,2]
range(1,4) defines [1,2,3]
range(0,2,10) defines [0,2,4,6,8]
range(5,-1,0) defines [5,4,3,2,1]

```
i=0
for x in range(5):
    print("i=",i,"x=",x)
    i=i+1
```

```
runfile('E:/okul/SCO1/for5.py', wdir='E:/okul/SCO1')
i= 0 x= 0
i= 1 x= 1
i= 2 x= 2
i= 3 x= 3
i= 4 x= 4
```

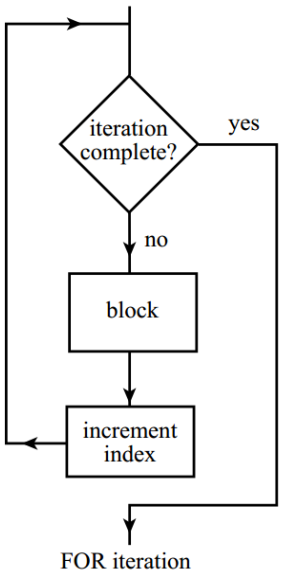
```
i=0
y=range(5)
for x in y:
    print("i=",i,"x=",x)
    i=i+1
```

```
runfile('E:/okul/SCO1/for5.py', wdir='E:/okul/SCO1')
i= 0 x= 0
i= 1 x= 1
i= 2 x= 2
i= 3 x= 3
i= 4 x= 4
```

A matrix(double array) example

```
m=3
n=5
x= [[0 for j in range(n)] for i in range(m)]
s=""
s1=""
s2=""
for i in range(m):
    for j in range(n):
        s="("+str(i)+str(j)+") "
        s1=str(i)+str(j)
        s2=s2+s
        x[i][j]=int(s1)
    s2=s2+"\n"
print("x = ")
print(x)
print("s2 = ")
print(s2)
print(x)
```

```
x =
[[0, 1, 2, 3, 4], [10, 11, 12, 13, 14], [20, 21, 22, 23, 24]]
s2 =
(00) (01) (02) (03) (04)
(10) (11) (12) (13) (14)
(20) (21) (22) (23) (24)

[[0, 1, 2, 3, 4], [10, 11, 12, 13, 14], [20, 21, 22, 23, 24]]
```




```
 from math import *

def sum(n):
    total=0;
    for i in range(1,(n+1)):
        total=total+i;
    return total;

n=int(input("n = "));
r=sum(n);
print("total=",r);
```

```
from math import *

def average(x):
    n=len(x);
    total=0;
    i=1;
    for i in range(0,n):
        total=total+x[i];
    return total/n;

x=[1.1179,2.21,3.31,4.41,5.51,6.61];
r=average(x);
print ("average=",r);
```

```
a=["ali","veli","49","elli"]
for i in a:
    print("i =",i)
```

```
runfile('E:/okul/SCO1/sum.py', wdir='E:/okul/SCO1')
i = ali
i = veli
i = 49
i = elli
```

```
a=[1.0,3.5,7.5,9.1,10.5]
sum=0
for i in a:
    sum+=i
    print("i =",i,"sum=",sum)
```

```
runfile('E:/okul/SCO1/sum.py', wdir='E:/okul/SCO1')
i = 1.0 sum= 1.0
i = 3.5 sum= 4.5
```

```
i = 7.5 sum= 12.0
i = 9.1 sum= 21.1
i = 10.5 sum= 31.6
```

List of greek characters

```
c1=ord('\u03B1')
c2=ord('\u03D2')
s=""
for i in range(c1,c2):
    s=s+" "+chr(i)
print(s)
```

```
runfile('E:/okul/SCO1/greek_char.py', wdir='E:/okul/SCO1')
 α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ ς σ τ υ φ χ ψ ω ϊ ϋ ό ύ ώ Ϗ ϐ ϑ
```

ASCII characters

```
c1=ord('\u0021')
c2=ord('\u007E')
s=""
for i in range(c1,c2):
    s=s+" i: "+str(i)+" "+chr(i)
print(s)
```

```
runfile('E:/okul/SCO1/greek_char.py', wdir='E:/okul/SCO1')
 i: 33 ! i: 34 " i: 35 # i: 36 $ i: 37 % i: 38 & i: 39 ' i: 40 ( i: 41 ) i: 42 * i: 43 + i: 44 , i: 45 - i: 46 . i: 47 / i: 48 0 i: 49 1 i: 50 2 i: 51 3 i: 52 4 i: 53 5 i: 54 6 i: 55 7 i: 56 8 i: 57 9 i: 58 : i: 59 ; i: 60 < i: 61 = i: 62 > i: 63 ? i: 64 @ i: 65 A i: 66 B i: 67 C i: 68 D i: 69 E i: 70 F i: 71 G i: 72 H i: 73 I i: 74 J i: 75 K i: 76 L i: 77 M i: 78 N i: 79 O i: 80 P i: 81 Q i: 82 R i: 83 S i: 84 T i: 85 U i: 86 V i: 87 W i: 88 X i: 89 Y i: 90 Z i: 91 [ i: 92 \ i: 93 ] i: 94 ^ i: 95 _ i: 96 ` i: 97 a i: 98 b i: 99 c i: 100 d i: 101 e i: 102 f i: 103 g i: 104 h i: 105 i i: 106 j i: 107 k i: 108 l i: 109 m i: 110 n i: 111 o i: 112 p i: 113 q i: 114 r i: 115 s i: 116 t i: 117 u i: 118 v i: 119 w i: 120 x i: 121 y i: 122 z i: 123 { i: 124 | i: 125 }
```

1.2.5 CLASS CONCEPT IN PYTHON LANGUAGE

```
class Person:
  def __init__(self, name, age,ssno):
    self.name = name
    self.age = age
    self.ssno=ssno

  def print(self):
      print("name = ",self.name,"age=",self.age,"social security number = ",self.ssno)

p1 = Person("John", 36,20321)
p2=Person("Daphne",18,65655)
p1.print()
p2.print()
```

```
runfile('E:/okul/SCO1/Person.py', wdir='E:/okul/SCO1')
name =  John age= 36 social security number =  20321
name =  Daphne age= 18 social security number =  65655
```

```
class Box:
  def __init__(self, length,width,height,color):
    self.length = length
    self.width = width
    self.height=height
    self.color=color
  def volume(self):
      return self.length*self.width*self.height
  def area(self):
      return 2.0*(self.length*self.width+self.length*self.height+self.width*self.height)

  def print(self):
      print("length = ",self.length,"width=",self.width,"height = ",self.height,"color = ",self.color,"volume =
",self.volume(),"area=",self.area())

p1 = Box(1.0,1.2,0.4,"blue")
p2=Box(1.2,1.2,1.0,"yellow")
p1.print()
```

```
p2.print()
```

```
runfile('E:/okul/SCO1/Box.py', wdir='E:/okul/SCO1')
length =  1.0 width= 1.2 height =  0.4 color =  blue volume =  0.48 area= 4.16
length =  1.2 width= 1.2 height =  1.0 color =  yellow volume =  1.44 area= 7.68
```

```
class redBox:
  def __init__(self, length,width,height):
    self.p = Box(length,width,height,"red")

  def print(self):
      print("length = ",self.p.length,"width=",self.p.width,"height = ",self.p.height,"color = ",self.p.color,"volume =
",self.p.volume(),"area=",self.p.area())

p1 = redBox(1.0,1.2,0.4)
p2=redBox(1.2,1.2,1.0)
p1.print()
p2.print()
```

```
runfile('E:/okul/SCO1/redBox.py', wdir='E:/okul/SCO1')
length =  1.0 width= 1.2 height =  0.4 color =  red volume =  0.48 area= 4.16
length =  1.2 width= 1.2 height =  1.0 color =  red volume =  1.44 area= 7.68
```

```
from math import *
class robot:
  def __init__(self, name):
    self.name = name
    self.R=0.0
    self.theta=0.0
  def turn_north(self):
      self.theta=pi/2.0
  def turn_south(self):
      self.theta=3.0*pi/2.0
  def turn_west(self):
      self.theta=pi
  def turn_east(self):
      self.theta=0.0
  def turn(self,angle):
      self.theta=angle
  def forward(self):
      self.R=self.R+1.0
  def forward(self,Ri):
      self.R=self.R+Ri
  def backward(self):
      self.R=self.R-1.0
  def backward(self,Ri):
      self.R=self.R-Ri
  def print(self):
      print("name = ",self.name,"R=",self.R,"theta=",self.theta)

p1 = robot("R2D2")
p1.turn_north()
p1.forward(2.5)
p1.print()
```

```
runfile('E:/okul/SCO1/robot.py', wdir='E:/okul/SCO1')
name =  R2D2 R= 2.5 theta= 1.5707963267948966
```

Class inheritance:

```
class yellowBox(Box):
  def __init__(self, length,width,height):
    self.length=length
    self.width=width
    self.height=height
    self.color="yellow"

p1 = yellowBox(1.0,1.2,0.4)
p2 = yellowBox(1.2,1.2,1.0)
p1.print()
p2.print()
```

```
runfile('E:/okul/SCO1/yellowBox.py', wdir='E:/okul/SCO1')
length =  1.0 width= 1.2 height =  0.4 color =  yellow volume =  0.48 area= 4.16
length =  1.2 width= 1.2 height =  1.0 color =  yellow volume =  1.44 area= 7.68
```

1.2.6 ABSTRACT CLASS IN PYTHON LANGUAGE

An abstract class or interface is an empty class for reference purposes, although they can have additional methods. Most of the modern third generation languages such as java, C++, python has abstract class and/or interface buid into the language. Lambda variables a short-cut definition to define a new class derived from an interface. This concepts are very useful for us in numerical methods, because function definitions in our numerical codes become independent of actual function. As an input function an object of abstract class or interface can be defined, and when we will use the numerial analysis codes (suc as root finding, integration etc.) actual function can be defined as derived form abstract/class and/or interface. Abstract class/interface examples will be given here for simple function f(x). Please note that inside of these classes some additional codes for derivatives, root finding methods and integrals might exist. In order to see details of these codes please refer to related chapters in this book.

```
# Python program showing
# abstract base class work

from abc import ABC, abstractmethod

class Polygon(ABC):

    @abstractmethod
    def noofsides(self):
        pass

class Triangle(Polygon):

    # overriding abstract method
    def noofsides(self):
        print("I have 3 sides")

class Pentagon(Polygon):

    # overriding abstract method
    def noofsides(self):
        print("I have 5 sides")

class Hexagon(Polygon):

    # overriding abstract method
    def noofsides(self):
        print("I have 6 sides")

R = Triangle()
R.noofsides()

R = Pentagon()
R.noofsides()

K = Hexagon()
K.noofsides()
```

```
runfile('E:/okul/SCO1/Polygon.py', wdir='E:/okul/SCO1')
I have 3 sides
I have 5 sides
I have 6 sides
```

Abstract Class if_x (for mathematical function f(x))

```
from abc import ABC, abstractmethod
from math import *

class if_x(ABC):

    @abstractmethod
    def func(self,x):
        pass;

    def dfunc(self,x,n):
        dx=0.001;
        if n==0: df=self.func(x);
```

```
        elif n==1: df=(3.0*self.func(x-4.0*dx)-32.0*self.func(x-3.0*dx)+168.0*self.func(x-2.0*dx)-672.0*self.func(x-
dx)+672.0*self.func(x+dx)-168.0*self.func(x+2.0*dx)+32.0*self.func(x+3.0*dx)-3.0*self.func(x+4.0*dx))/840.0/dx;
        elif n==2: df=(-14350.0*self.func(x)-9.0*self.func(x-4.0*dx)+128.0*self.func(x-3.0*dx)-1008.0*self.func(x-
2.0*dx)+8064.0*self.func(x-dx)+8064.0*self.func(x+dx)-1008.0*self.func(x+2.0*dx)+128.0*self.func(x+3.0*dx)-
9.0*self.func(x+4.0*dx))/5040.0/dx/dx;
        elif n==3: df=(-7.0*self.func(x-4.0*dx)+72.0*self.func(x-3.0*dx)-338.0*self.func(x-2.0*dx)+488.0*self.func(x-dx)-
488.0*self.func(x+dx)+338.0*self.func(x+2.0*dx)-72.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx;
        elif n==4: df=(2730.0*self.func(x)+7.0*self.func(x-4.0*dx)-96.0*self.func(x-3.0*dx)+676.0*self.func(x-2*dx)-1952.0*self.func(x-
dx)-1952.0*self.func(x+dx)+676.0*self.func(x+2.0*dx)-96.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx/dx;
        elif n==5: df=(self.func(x-4.0*dx)-9.0*self.func(x-3.0*dx)+26.0*self.func(x-2.0*dx)-29.0*self.func(x-dx)+29.0*self.func(x+dx)-
26.0*self.func(x+2.0*dx)+9.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/6.0/dx/dx/dx/dx/dx;
        elif n==6: df=(-150.0*self.func(x)-self.func(x-4.0*dx)+12.0*self.func(x-3.0*dx)-52.0*self.func(x-2.0*dx)+116.0*self.func(x-
dx)+116.0*self.func(x+dx)-52.0*self.func(x+2.0*dx)+12.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/4.0/dx/dx/dx/dx/dx/dx;
        elif n==7: df=(-self.func(x-4.0*dx)+6.0*self.func(x-3.0*dx)-14.0*self.func(x-2.0*dx)+14.0*self.func(x-dx)-
14.0*self.func(x+dx)+14.0*self.func(x+2.0*dx)-6.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/2.0/dx/dx/dx/dx/dx/dx/dx;
        elif n==8: df=(70.0*self.func(x)+self.func(x-4.0*dx)-8.0*self.func(x-3.0*dx)+28.0*self.func(x-2.0*dx)-56.0*self.func(x-dx)-
56.0*self.func(x+dx)+28.0*self.func(x+2.0*dx)-8.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/dx/dx/dx/dx/dx/dx/dx/dx;
        else: df=0;
        return df;

    def gauss_legendre_coefficients(self,x1,x2,n):
        #calculates legendre gauss-coefficients as coefficients of the integral
        #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

    def integral(self,x1,x2,n):
        #integral func(x)dx
        #integral of a function by using gauss-legendre quadrature
        #between x1 and x2
        a=self.gauss_legendre_coefficients(x1,x2,n)
        z=0
        for i in range(n):
            z=z+a[1][i]*self.func(a[0][i])
        return z
```

```
from math import *
from if_x import *;

class f1(if_x):
  def func(self,x):
      y=x*x-2;
      return y;

x=[1.2,2.3,3.4,4.5,5.6,6.0,5.7,5.1,4.6]
n=len(x)
f = f1()
for i in range(n):
    print("i = ",i," x = ",x[i],"y=f(x) = ",f.func(x[i]))
```

```
runfile('E:/okul/SCO1/if_x_test.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
i =  0  x =  1.2 y=f(x) =  -0.56
```

```
i =  1  x =  2.3 y=f(x) =  3.2899999999999999
i =  2  x =  3.4 y=f(x) =  9.5599999999999999
i =  3  x =  4.5 y=f(x) =  18.25
i =  4  x =  5.6 y=f(x) =  29.359999999999996
i =  5  x =  6.0 y=f(x) =  34.0
i =  6  x =  5.7 y=f(x) =  30.490000000000002
i =  7  x =  5.1 y=f(x) =  24.009999999999998
i =  8  x =  4.6 y=f(x) =  19.159999999999997
```

Lambda version of class f1

```
from math import *
from if_x import *;

x=[1.2,2.3,3.4,4.5,5.6,6.0,5.7,5.1,4.6]
n=len(x)
class f1(if_x):func=lambda self,x: x*x
f = f1()
for i in range(n):
    print("i = ",i," x = ",x[i],"y=f(x) = ",f.func(x[i]))
```

```
runfile('E:/okul/SCO1/if_x_test.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
i =  0  x =  1.2 y=f(x) =  1.44
i =  1  x =  2.3 y=f(x) =  5.2899999999999999
i =  2  x =  3.4 y=f(x) =  11.559999999999999
i =  3  x =  4.5 y=f(x) =  20.25
i =  4  x =  5.6 y=f(x) =  31.359999999999996
i =  5  x =  6.0 y=f(x) =  36.0
i =  6  x =  5.7 y=f(x) =  32.49
i =  7  x =  5.1 y=f(x) =  26.009999999999998
i =  8  x =  4.6 y=f(x) =  21.159999999999997
```

The same function can be defined with **lambda variable method(function**) as:

```
x=[1.2,2.3,3.4,4.5,5.6,6.0,5.7,5.1,4.6]
n=len(x)
f = lambda x: x*x-2;
for i in range(n):
    print("i = ",i," x = ",x[i],"y=f(x) = ",f(x[i]))
```

```
runfile('E:/okul/SCO1/if_x_test1.py', wdir='E:/okul/SCO1')
i =  0  x =  1.2 y=f(x) =  -0.56
i =  1  x =  2.3 y=f(x) =  3.2899999999999999
i =  2  x =  3.4 y=f(x) =  9.5599999999999999
i =  3  x =  4.5 y=f(x) =  18.25
i =  4  x =  5.6 y=f(x) =  29.359999999999996
i =  5  x =  6.0 y=f(x) =  34.0
i =  6  x =  5.7 y=f(x) =  30.490000000000002
i =  7  x =  5.1 y=f(x) =  24.009999999999998
i =  8  x =  4.6 y=f(x) =  19.159999999999997
```

Abstract class f_xj (for multi independent function f(x0,x1,x2..))

```
from abc import ABC, abstractmethod
from math import *

class f_xj(ABC):


  # multifunction multi independent variable
  # vector of dependent variables are returned
  # example f[0]=x[0]+sin(x[1])
  #         f[1]=x[0]*x[1]-x[1]
  # func(x) returns the value of f[0] and f[1]
  # as a two dimensional vector

  @abstractmethod
  def func(self,x):
    pass;

  def funcxy(self,x1,y1):
    n=2;
    x=[0 for i in range(n)];
    x[0]=x1;
    x[1]=y1;
    return self.func(x);
```

```
def dfunc_(self,x,x_ref):
    #derivative of the function with respect to x_ref
    h0=0.015675863;
    m=7;
    n=len(x);
    x1=[0 for i in range(n)];
    x2=[0 for i in range(n)];
    for i in range(n):
        x1[i]=x[i];
        x2[i]=x[i];
    #derivative of a simple function
    T=[[0 for i2 in range(m)] for j2 in range(m)];
    h=[0 for i in range(m)];
    for i in range(m):
        h[i]=0;
        for j in range(m):
            T[i][j]=0;
    h[0]=h0;
    r=0.5;
    for i in range(m):
        h[i]=h0*pow(r,i);
    for i in range(m):
        x1[x_ref]+=h[i];
        x2[x_ref]-=h[i];
        f1=self.func(x1);
        f2=self.func(x2);
        T[i][0]=( f1 - f2)/(2.0*h[i]);
        x1[x_ref]=x[x_ref];
        x2[x_ref]=x[x_ref];
    for k in range(1,m):
         for i in range(0,m-k):
             T[i][k]=(h[i]*h[i]*T[i+1][k-1] - h[i+k]*h[i+k]*T[i][k-1])/(h[i]*h[i]
             - h[i+k]*h[i+k]);
    xx=T[0][n-1];
    return xx;

def d2func_(self,x,x_ref):
    #second derivative of the function with respect to x_ref
    h0=0.015675863;
    m=7;
    n=len(x);
    x1=[0 for i in range(n)];
    x2=[0 for i in range(n)];
    for i in range(n):
        x1[i]=x[i];
        x2[i]=x[i];
    #derivative of a simple function
    T=[[0 for i2 in range(m)] for j2 in range(m)];
    h=[0 for i in range(m)];
    for i in range(m):
        h[i]=0;
        for j in range(m):
            T[i][j]=0;
    h[0]=h0;
    r=0.5;
    for i in range(m):
        h[i]=h0*pow(r,i);
    for i in range(m):
        x1[x_ref[1]]+=h[i];
        x2[x_ref[1]]-=h[i];
        f1=self.dfunc_(x1,x_ref[0]);
        f2=self.dfunc_(x2,x_ref[0]);
        T[i][0]=( f1 - f2)/(2.0*h[i]);
        x1[x_ref[1]]=x[x_ref[1]];
        x2[x_ref[1]]=x[x_ref[1]];
    for k in range(1,m):
         for i in range(0,m-k):
             T[i][k]=(h[i]*h[i]*T[i+1][k-1] - h[i+k]*h[i+k]*T[i][k-1])/(h[i]*h[i]
             - h[i+k]*h[i+k]);
    xx=T[0][n-1];
    return xx;

def dfunc(self,x):
    n=len(x);
    c=[0 for i in range(n)];
    for i in range(n):
```

```
            c[i]=self.dfunc_(x,i);
        return c;

    def d2func(self,x):
        n=len(x);
        x_ref=[0 for i2 in range(2)];
        c=[[0 for i2 in range(n)] for j2 in range(n)];
        for i in range(n):
            for j in range(n):
                x_ref[0]=i;
                x_ref[1]=j;
                c[i][j]=self.d2func_(x,x_ref);
        return c;
```

Separate class version

```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 24 16:46:48 2018

@author: Mustafa Turhan Çoban
"""
from math import *
from f_xj import *

class f2(f_xj):
  def func(self,x):
      y=sin(x[0])*x[0]+sin(x[1])*x[1];
      return y;

f=f2();
x=[1,1];
df=f.dfunc(x);
print("df",df);
d2f=f.d2func(x);
print("d2f",d2f);
```

```
runfile('E:/okul/SCO1/f_xj.py', wdir='E:/okul/SCO1')

runfile('E:/okul/SCO1/f_xjtest.py', wdir='E:/okul/SCO1')
df [1.3817732900787727, 1.3817732900787727]
d2f [[0.23913362632367666, -3.682833747686372e-13], [-3.682833747686372e-13, 0.23913362632367666]]
```

Lambda version:

```
from math import *
from f_xj import *

class f1(f_xj):func=lambda self,x: sin(x[0])*x[0]+sin(x[1])*x[1]
f=f1()
x=[1,1]
df=f.dfunc(x)
print("df",df)
d2f=f.d2func(x)
```

```
runfile('E:/okul/SCO1/f_xjtest1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj
df [1.3817732900787727, 1.3817732900787727]
d2f [[0.23913362632367666, -3.682833747686372e-13], [-3.682833747686372e-13, 0.23913362632367666]]
```

Abstract class fi_xi (for multi independent function-vector function)

f_1(x0,x1,x2..)

f_2(x0,x1,x2..)

…

```
from abc import ABC, abstractmethod
from math import *

class fi_xi(ABC):

  # multifunction multi independent variable
  # vector of dependent variables are returned
  # example f[0]=x[0]+sin(x[1])
  #         f[1]=x[0]*x[1]-x[1]
  # func(x) returns the value of f[0] and f[1]
  # as a two dimensional vector

  @abstractmethod
```

```
  def func(self,x):
     pass;

  def xp(self,x,j,m):
     k=len(x);
     h=1.0e-6
     xx=[0 for i in range(k)];
     for i in range(k):
        if i==m:
           xx[i]=x[i]+j*h;
        else: xx[i]=x[i];
     return xx;

  def dfunc(self,x,n):
     h=1.0e-6;
     k=len(x);
     df=[[0 for i in range(k)] for j in range(k)];
     for i in range(k):
        for j in range(k):
           df[i][j]=self.dfun(x,n,i,j);
     return df;

  def dfun(self,x,n,j,m):
     k=len(x);
     ff=[[0 for i2 in range(k)] for j2 in range(9)];
     xp1=[[0 for i2 in range(k)] for j2 in range(9)];
     h=1.0e-6;
     hh=1/h;
     x1=x[m];
     for jj in range(-4,5):
        j1=jj+4;
        xp1[j1]=self.xp(x,jj,m);
        ff[j1]=self.func(xp1[j1]);
     if n==0: df=ff[4][j];
     elif n==1:
        df=(3.0*ff[0][j]-32.0*ff[1][j]+168.0*ff[2][j]-672.0*ff[3][j]+672.0*ff[5][j]-168.0*ff[6][j]+32.0*ff[7][j]-3.0*ff[8][j])/840.0*hh;
     elif n==2:
        df=(-14350.0*ff[4][j]-9.0*ff[0][j]+128*ff[1][j]-1008*ff[2][j]+8064*ff[3][j]+8064.0*ff[5][j]-1008.0*ff[6][j]+128.0*ff[7][j]-
9.0*ff[8][j])/5040.0*hh*hh;
     elif n==3:
        df=(-7.0*ff[0][j]+72.0*ff[1][j]-338.0*ff[2][j]+488.0*ff[3][j]-488.0*ff[5][j]+338.0*ff[6][j]-
72.0*ff[7][j]+7.0*ff[8][j])/240.0*hh*hh*hh;
     elif n==4:
        df=(2730.0*ff[4][j]+7.0*ff[0][j]-96.0*ff[1][j]+676.0*ff[2][j]-1952.0*ff[3][j]-1952.0*ff[5][j]+676.0*ff[6][j]-
96.0*ff[7][j]+7.0*ff[8][j])/240.0*hh*hh*hh*hh;
     elif n==5:
        df=(ff[0][j]-9.0*ff[1][j]+26.0*ff[2][j]-29.0*ff[3][j]+29.0*ff[5][j]-26.0*ff[6][j]+9.0*ff[7][j]-ff[8][j])/6.0*hh*hh*hh*hh*hh;
     elif n==6:
        df=(-150.0*ff[4][j]-ff[0][j]+12.0*ff[1][j]-52.0*ff[2][j]+116.0*ff[3][j]+116.0*ff[5][j]-52.0*ff[6][j]+12.0*ff[7][j]-
ff[8][j])/4.0*hh*hh*hh*hh*hh*hh;
     elif n==7:
        df=(-ff[0][j]+6.0*ff[1][j]-14.0*ff[2][j]+14.0*ff[3][j]-14.0*ff[5][j]+14.0*ff[6][j]-6.0*ff[7][j]+ff[8][j])/2.0*hh*hh*hh*hh*hh*hh*hh;
     elif n==8:
        df=(70.0*ff[4][j]+ff[0][j]-8.0*ff[1][j]+28.0*ff[2][j]-56.0*ff[3][j]-56.0*ff[5][j]+28.0*ff[6][j]-
8.0*ff[7][j]+ff[8][j])*hh*hh*hh*hh*hh*hh*hh*hh;
     else: df=0;
     return df
```

```
from math import *
from fi_xi import *

class f2(fi_xi):
  def func(self,x):
     y=[0 for i2 in range(2)]
     y[0]=sin(x[0])*x[0]+sin(x[1])*x[1]
     y[1]=x[0]*x[0]-x[1]
     return y;

f=f2();
x=[1.0,1.0]
ff=f.func(x)
print("f=\n",ff)
df=f.dfunc(x,1)
print("df=\n",df)
```

```
runfile('E:/okul/SCO1/fi_xjtest.py', wdir='E:/okul/SCO1')
```

```
Reloaded modules: fi_xi
f=
 [1.682941969615793, 0.0]
df=
 [[1.3817732906328843, 1.3817732906328843], [2.000000000016142, -0.999999999954873]]
```

Abstract class fij_xj (for multi independent function-matrix function)
f_{11}(x0,x1,x2..) f_{12}(x0,x1,x2..) …
f_{21}(x0,x1,x2..) f_{22}(x0,x1,x2..) …
…..
f_{n1}(x0,x1,x2..) f_{n2}(x0,x1,x2..) …

```
from abc import ABC, abstractmethod
from math import *

class fij_xi(ABC):

  # multifunction multi independent variable
  # for n independent variable n*n matrix of
  # dependent variables are returned
  # example f[0][0]=x[0]+Math.sin(x[1])   f[0][1]=x[0]-x[1]
  #         f[1][0]=x[0]*x[0]-x[1]        f[1][1]=Math.exp(x[0]+x[1]*x[1]
  # func(x) returns the value of
  # multifunction multi independent variable
  # for n independent variable n*n matrix of
  # dependent variables are returned
  # example f[0][0]=x[0]+Math.sin(x[1])   f[0][1]=x[0]-x[1]
  #         f[1][0]=x[0]*x[0]-x[1]        f[1][1]=Math.exp(x[0]+x[1]*x[1]
  # func(x) returns the value of
  #         f[0][0], f[0][1]
  #         f[1][0], f[1][1]
  # as a two dimensional matrix

  @abstractmethod
  def func(self,x):
     pass;
```

```
from math import *
from fij_xi import *

class f2(fij_xi):
  def func(self,x):
      y=[[0 for j in range(2)] for i in range(2)]
      y[0][0]=2.0*x[0]+x[1];
      y[0][1]=x[0];
      y[1][0]=3.0*x[1]*x[1];
      y[1][1]=1.0+6.0*x[0]*x[1];
      return y;

f=f2();
x=[1.0,1.0]
ff=f.func(x)
print("f=\n",ff)
```

```
runfile('E:/okul/SCO1/fij_xitest.py', wdir='E:/okul/SCO1')
Reloaded modules: fij_xi
f=
 [[3.0, 1.0], [3.0, 7.0]]
```

Iterable in Python:
An iteratable is a Python object that can be used as a sequence. You can go to the next item of the sequence using the next() method.
You can loop over an iterable, but you cannot access individual elements directly.
It’s a container object: it can only return one of its element at the time.

```
d = { "one": 1, "two": 2, "three": 3, "four": 4, "five": 5 }
iterable = d.keys()
for item in iterable:
   print(item)
```

```
runfile('E:/okul/SCO1/iterable1.py', wdir='E:/okul/SCO1')
one
two
three
four
```

five

1.2..7 MATH LIBRARY IN PYTHON LANGUAGE

Several mathematical functions(methods) are defined in math library of python. Detail information can be obtained from https://docs.python.org/3/library/math.html internet site. A list of methods and constants are given below as a table

`ceil(x)`	Return the ceiling of *x*, the smallest integer greater than or equal to *x*. If *x* is not a float,
`copysign(x, y)`	Return a float with the magnitude (absolute value) of *x* but the sign of *y*.
`comb(n, k)`	Return the number of ways to choose *k* items from *n* items without repetition
`fabs(x)`	Return the absolute value of *x*.
`factorial(x)`	Return *x* factorial as an integer.
`floor(x)`	Return the floor of *x*, the largest integer less than or equal to *x*.
`fmod(x, y)`	Return fmod(x, y)
`frexp(x)`	Return the mantissa and exponent of *x* as the pair (m, e). *m* is a float and *e* is an integer
`fsum(iterable)`	Return an accurate floating point sum of values in the iterable.
`gcd(a, b)`	Return the greatest common divisor of the integers *a* and *b*. If either *a* or *b* is nonzero
`modf(x)`	Return the fractional and integer parts of *x*.
`prod(iterable, *, start=1)`	Calculate the product of all the elements in the input *iterable*. The default *start* value for the product is 1.
`remainder(x, y)`	Return the IEEE 754-style remainder of *x* with respect to *y*.
`trunc(x)`	Return the Real value *x* truncated to an integer
`exp(x)`	Return *e* raised to the power *x*, where *e* = 2.718281… is the base of natural logarithms.
`expm1(x)`	Return *e* raised to the power *x*, minus 1.
`log10(x)`	Return the base-10 logarithm of *x*.
`log(x[, base])`	With one argument, return the natural logarithm of *x* (to base *e*).
`pow(x, y)`	Return x raised to the power y.
`sqrt(x)`	Return the square root of *x*.
`acos(x)`	Return the arc cosine of *x*, in radians.
`asin(x)`	Return the arc sine of *x*, in radians.
`atan(x)`	Return the arc tangent of *x*, in radians.
`atan2(y, x)`	Return atan(y / x), in radians. The result is between -pi and pi. T
`cos(x)`	Return the cosine of *x* radians.
`dist(p, q)`	Return the Euclidean distance between two points *p* and *q*,
`hypot(*coordinates)`	Return the Euclidean norm, sqrt(sum(x**2 for x in coordinates)).
`sin(x)`	Return the sine of *x* radians.
`tan(x)`¶	Return the tangent of *x* radians.
`degrees(x)`	Convert angle *x* from radians to degrees.
`radians(x)`	Convert angle *x* from degrees to radians.
`acosh(x)`	Return the inverse hyperbolic cosine of *x*.
`asinh(x)`	Return the inverse hyperbolic sine of *x*
`atanh(x)`	Return the inverse hyperbolic tangent of *x*.
`cosh(x)`	Return the hyperbolic cosine of *x*.
`sinh(x)`	Return the hyperbolic sine of *x*
`tanh(x)`	Return the hyperbolic tangent of *x*.
`erf(x)`	Return the error function at *x*.
`phi(x)`	Cumulative distribution function for standard normal distribution
`erfc(x)`	Return the complementary error function at *x*.
`lgamma(x)`	Return the natural logarithm of the absolute value of the Gamma function at *x*.
`CONSTANTS`	
pi	The mathematical constant π = 3.141592…, to available precision.
e	The mathematical constant *e* = 2.718281…, to available precision.
tau	The mathematical constant τ = 6.283185…, to available precision. Tau is a circle constant equal to 2π,
inf	A floating-point positive infinity.
nan	A floating-point “not a number” (NaN) value.

```
x = [0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1]
y=fsum(x)
print("y=",y)
```

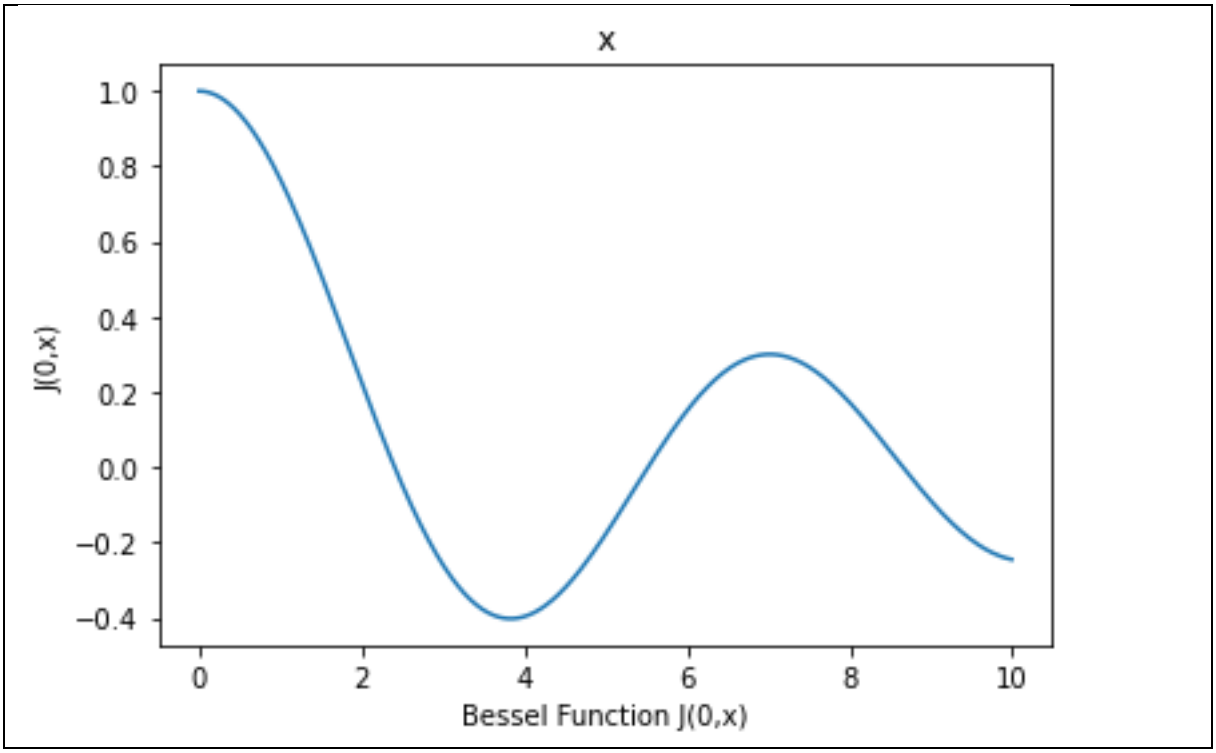
```
runfile('E:/okul/SCO1/iterable1.py', wdir='E:/okul/SCO1')
y= 1.0
```

1.2.8 PLOTTING IN PYTHON LANGUAGE

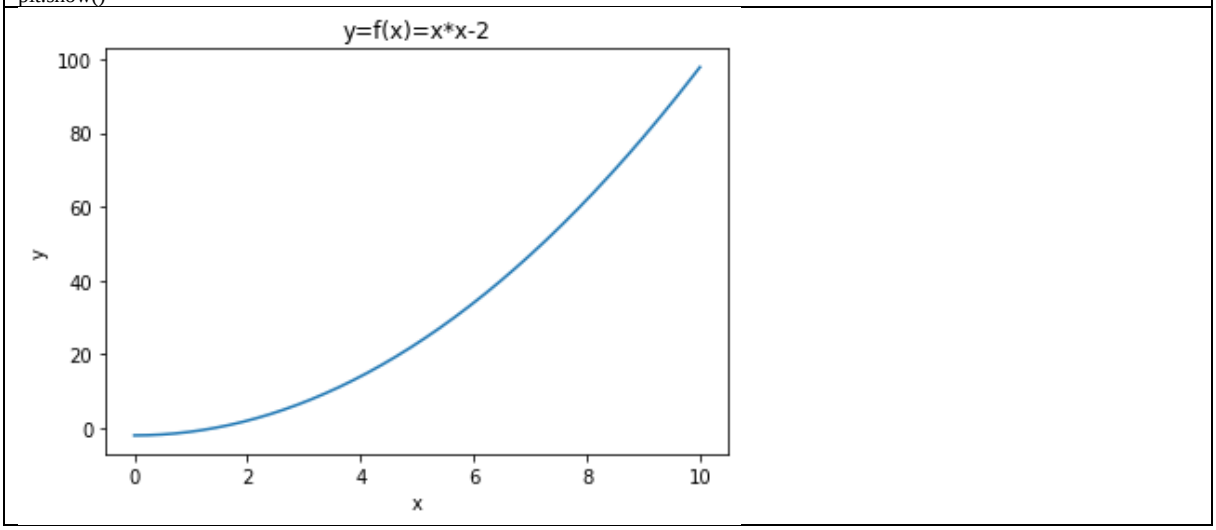
Python has the ability to create graphs by using the matplotlib library. It has numerous packages and functions which generate a wide variety of graphs and plots. It is also very simple to use. It along with

numpy and other python built-in functions achieves the goal. In this article we will see some of the different kinds of graphs it can generate.
Matplotlib library documentation:
https://matplotlib.org/3.3.2/contents.html#

```
from math import *
from matplotlib import pyplot as plt
import numpy as np

def J(v,x):
    JJ=1e-6
    a=1.0
    fact=1
    b=0.0
    if x <=20:
        x5=1
        lna=0.0;
        x1=v*log(0.5*x)
        lna1=log(0.25*x*x)
        for k in range(0,100):
            x2=lna
            x3=lgamma(k+1)
            x4=lgamma(v+k+1)
            b=x1+x2-x3-x4
            b=x5*exp(b)
            lna=lna+lna1
            x5=-1*x5
            JJ=JJ+b
    elif (v==0 and x==0):
        JJ=1
    else:
        JJ=sqrt(2.0/PI/x)*cos(x-0.5*v*PI-0.25*PI)
    return JJ


n=1000
dx=1.0/100.0
y=[0 for z in range(n)];
x=[0 for z in range(n)];
for i in range(n):
    x[i]=0.00001+dx*i
    y[i]=J(0.0,x[i])
    print(x[i]," ",y[i])

plt.plot(x,y)
plt.xlabel("Bessel Function J(0,x)")
plt.ylabel("J(0,x)")
plt.title('x')
plt.show()
```



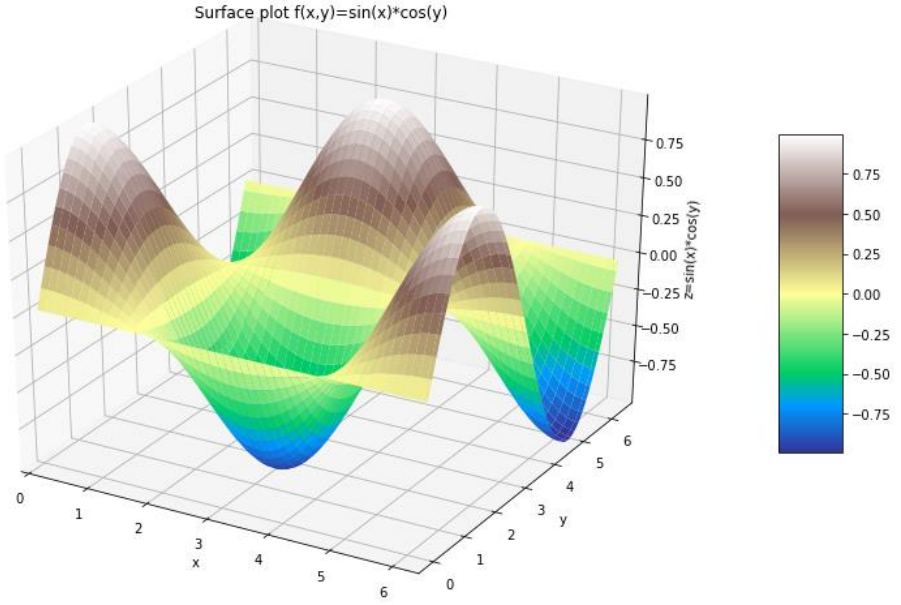

```
from math import *
from matplotlib import pyplot as plt
import numpy as np

f = lambda x: x*x-2;
n=1000
dx=1.0/100.0
y=[0 for z in range(n)];
x=[0 for z in range(n)];
for i in range(n):
    x[i]=0.00001+dx*i
    y[i]=f(x[i])
    print(x[i]," ",y[i])

plt.plot(x,y)
plt.xlabel("x")
plt.ylabel("y")
plt.title('y=f(x)=x*x-2')
plt.show()
```



3D Plot example is given below. The function is f(x)=sin(x)*cos(y)

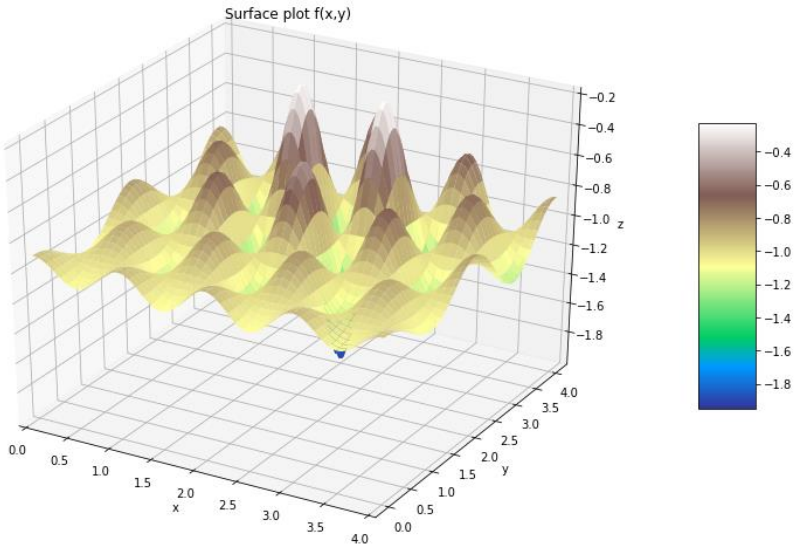
```
# Import libraries
```

```
from mpl_toolkits import mplot3d
import numpy as np
import matplotlib.pyplot as plt
from f_xj import *;

#Function to plot
class f2(f_xj):
  def func(self,x):
     y=sin(x[0])*cos(x[1])
     return y;

# Creating dataset
def plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLAbel,XLAbel,YLable,ZLabel):
    #f=f2()
    x=[0.0 for j in range(Nx)]
    y=[0.0 for j in range(Ny)]
    z=[[0.0 for j in range(Ny)] for i in range(Nx)]
    dx=(xmax-xmin)/(Nx-1)
    dy=(ymax-ymin)/(Ny-1)
    for i in range(Nx):
        for j in range(Ny):
            x[i]=xmin+dx*i
            y[j] = ymin+dy*j
            z[i][j]=f.funcxy(x[i],y[j])
    xi=np.array(x)
    yi=np.array(y)
    xi, yi = np.meshgrid(xi, yi)
    zi=np.array(z)
    #print(x[i][j]," ",y[i][j]," ",z[i][j])
    fig = plt.figure(figsize =(14, 9))
    ax = plt.axes(projection ='3d')
    # Creating color map
    my_cmap = plt.get_cmap('terrain')
    # Creating plot
    surf = ax.plot_surface(xi, yi, zi,
                           cmap = my_cmap,
                           edgecolor ='none')
    fig.colorbar(surf, ax = ax,
                 shrink = 0.5,
                 aspect = 5)

    ax.set_title(PLabel)
    ax.set_xlabel(XLabel)
    ax.set_xlim(xmin, xmax)
    ax.set_ylabel(YLabel)
    ax.set_xlim(ymin, ymax)
    ax.set_zlabel(ZLabel)
    ax.set_zlim(np.min(zi), np.max(zi))
    plt.show()

f=f2()
xmin=0.0
xmax=6.283185307
Nx=100
ymin=0.0
ymax=6.283185307
Ny=100
PLabel="Surface plot f(x,y)=sin(x)*cos(y)"
XLabel="x"
YLabel="y"
ZLabel="z=sin(x)*cos(y)"
plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```



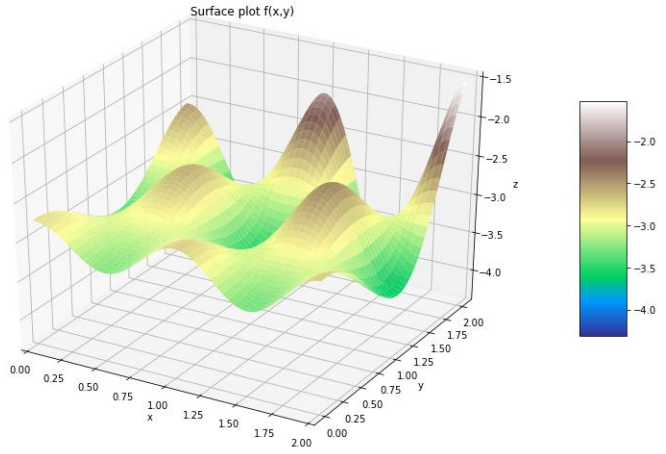
$$z(x,y) = (1 + cos(\pi(x-3) * \frac{cos(2\pi(y-2)}{1+(x-3)^2+(y-2)^2}$$

```
#Function to plot
class f2(f_xj):
  def func(self,x):
     z=-(1.0+cos(pi*(x[0]-3.0))*cos(2.0*pi*(x[1]-2.0))/(1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0)))
     return z;

f=f2()
xmin=0.0
xmax=4
Nx=100
ymin=0.0
ymax=4
Ny=100
PLabel="Surface plot f(x,y)"
XLabel="x"
YLabel="y"
ZLabel="z"
plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```



```
from math import *
from f_xj import *
import  plot3D1 as P3D

class f1(f_xj):func=lambda self,x:-3.0*(1.0+cos(pi*(x[0]-3.0))*cos(2.0*pi*(x[1]-
2.0))/(1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0)))
f=f1()
a=[-1.0,-1.0]
b=[1.0,1.0]
f=f1()
xmin=0.0
xmax=2.0
Nx=100
ymin=0.0
ymax=2.0
Ny=100
PLabel="Surface plot f(x,y)"
XLabel="x"
YLabel="y"
ZLabel="z"
P3D.plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```

```
runfile('E:/okul/SCO1/f_xjtest1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, plot3D1
```

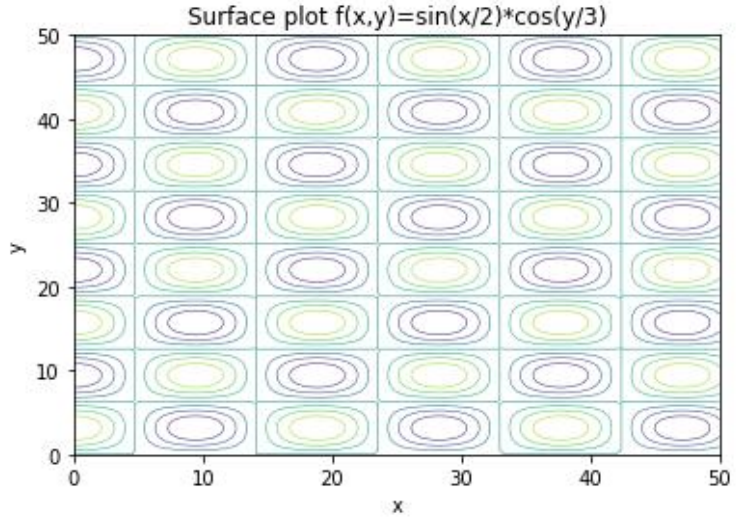


Plotting contour_plot

$$z(x,y) = sin\left(\frac{x}{2}\right) cos\left(\frac{y}{3}\right)$$

```
from mpl_toolkits import mplot3d
import numpy as np
import matplotlib.pyplot as plt
from f_xj import *

#Function to plot
class f2(f_xj):
  def func(self,x):
      y=sin(x[0]/2.0)*cos(x[1]/3.0)
      return y;

# Creating dataset
def contour_plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLable,ZLabel):
    #f=f2()
    x=[0.0 for j in range(Nx)]
    y=[0.0 for j in range(Ny)]
    z=[[0.0 for j in range(Ny)] for i in range(Nx)]
    dx=(xmax-xmin)/(Nx-1)
    dy=(ymax-ymin)/(Ny-1)
    for i in range(Nx):
        for j in range(Ny):
            x[i]=xmin+dx*i
            y[j] = ymin+dy*j
```

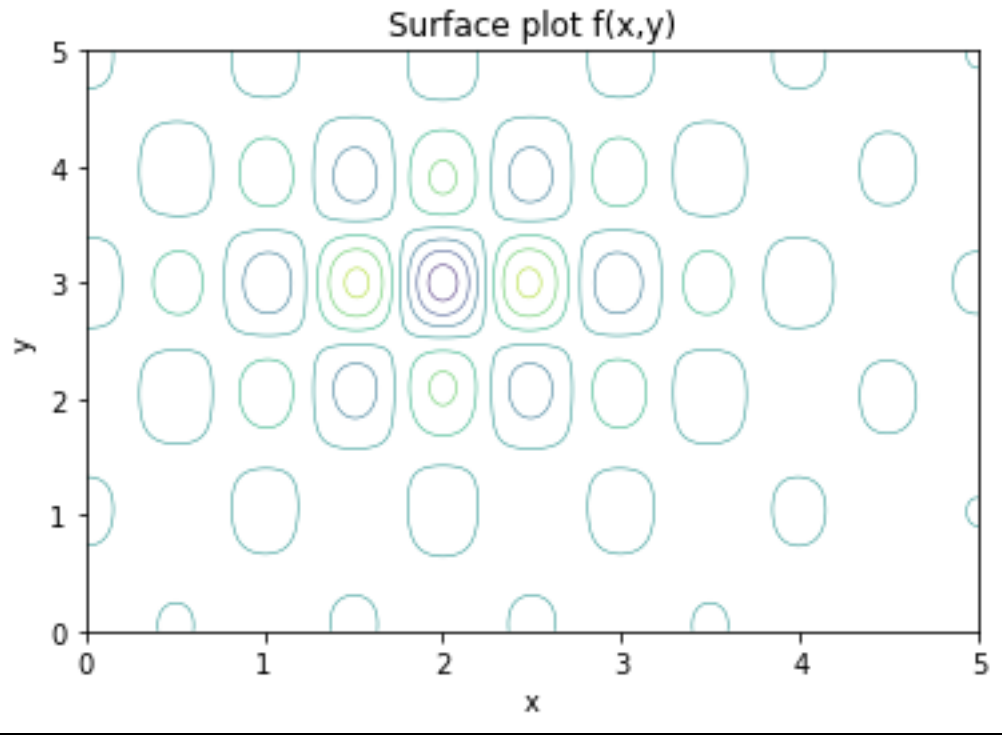
```
            z[i][j]=f.funcxy(x[i],y[j])
    xi=np.array(x)
    yi=np.array(y)
    xi, yi = np.meshgrid(xi, yi)
    zi=np.array(z)
    fig, ax = plt.subplots(1, 1)
    # plots contour lines
    ax.contour(xi, yi, zi,linewidths=0.5)
    ax.set_title(PLabel)
    ax.set_xlabel(XLabel)
    ax.set_xlim(xmin, xmax)
    ax.set_ylabel(YLabel)
    ax.set_xlim(ymin, ymax)
    plt.show()

f=f2()
xmin=0.0
xmax=50.0
Nx=100
ymin=0.0
ymax=50.0
Ny=100
PLabel="Surface plot f(x,y)=sin(x/2)*cos(y/3)"
XLabel="x"
YLabel="y"
ZLabel="z=sin(x)*cos(y)"
contour_plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```



$$z(x,y) = (1 + cos(\pi(x-3) * \frac{cos(2\pi(y-2)}{1+(x-3)^2+(y-2)^2}$$

```
 from mpl_toolkits import mplot3d
import numpy as np
import matplotlib.pyplot as plt
from f_xj import *

# Creating dataset
def contour_plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLable,ZLabel):
    #f=f2()
    x=[0.0 for j in range(Nx)]
    y=[0.0 for j in range(Ny)]
    z=[[0.0 for j in range(Ny)] for i in range(Nx)]
    dx=(xmax-xmin)/(Nx-1)
    dy=(ymax-ymin)/(Ny-1)
    for i in range(Nx):
        for j in range(Ny):
            x[i]=xmin+dx*i
            y[j] = ymin+dy*j
            z[i][j]=f.funcxy(x[i],y[j])
    xi=np.array(x)
    yi=np.array(y)
    xi, yi = np.meshgrid(xi, yi)
    zi=np.array(z)
    fig, ax = plt.subplots(1,1)
```

```
    # plots contour lines
    ax.contour(xi, yi, zi,linewidths=0.5)
    ax.set_title(PLabel)
    ax.set_xlabel(XLabel)
    ax.set_xlim(xmin, xmax)
    ax.set_ylabel(YLabel)
    ax.set_xlim(ymin, ymax)
    plt.show()

class f1(f_xj):func=lambda self,x:-3.0*(1.0+cos(pi*(x[0]-3.0))*cos(2.0*pi*(x[1]-2.0))/(1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-
2.0)*(x[1]-2.0)))
f=f1()
xmin=0.0
xmax=5.0
Nx=500
ymin=0.0
ymax=5.0
Ny=500
PLabel="Surface plot f(x,y)"
XLabel="x"
YLabel="y"
ZLabel="z=sin(x)*cos(y)"
contour_plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```



1.2.9 READING AND WRITING DATA FROM A SEQUENTIAL FILE IN PYTHON LANGUAGE

In order to read/write data into a sequaential file open statement will be used. This statement has the following form:
File_object=open("filename","mode")
Where "filename" is the name of input/output file and mode will take the values of

- '***r***' – Read mode which is used when the file is only being read
- '***w***' – Write mode which is used to edit and write new information to the file (any existing files with the same name will be erased when this mode is activated)
- '***a***' – Append mode, which is used to add new data to the end of the file; that is new information is automatically amended to the end
- '***r+***' – Special read and write mode, which is used to handle both actions when working with a file

```
path = 'e:/okul/SCO1/b.txt'
file1 = open(path,'r')
days = file1.read()
print(days)
```

```
runfile('E:/okul/SCO1/read1.py', wdir='E:/okul/SCO1')

1.234 3.24 5.67 7.78
```

```
path = 'e:/okul/SCO1/d.txt'
file1 = open(path,'w')
s="once upon a time, in a country far far away"
file1.write(s)
file1.close()
file1 = open(path,'r')
s1 = file1.read()
print(s1)
```

```
runfile('E:/okul/SCO1/write1.py', wdir='E:/okul/SCO1')
once upon a time, in a country far far away
```

Reading file b.txt (first two varibles as x,y in each line)
File b.txt

```
1.234 3.24 5.67 7.78
1.345 5.25 5.67 7.78
3.748 6.38 5.67 7.78
2.175 5.24 5.67 7.78
1.222 3.45 5.67 7.78
```

```
path = 'b.txt'
file1 = open(path,'r')
file1.seek(0,2)  #jumps to the end
endLocation=file1.tell()
file1.seek(0) #jumps to the start
nn=0
while True:
   line=file1.readline()
   nn+=1
   if  file1.tell()==endLocation:
      break
x=[0.0 for i in range(nn)]
y=[0.0 for i in range(nn)]
file1.seek(0) #jumps to the start
for i in range(nn):
   line=file1.readline()
   ee=line.split()
   x[i]=float(ee[0])
   y[i]=float(ee[1])
   print(x[i]," ",y[i])
file1.close()
```

```
runfile('E:/okul/SCO1/read2.py', wdir='E:/okul/SCO1')
1.234   3.24
```

```
1.345   5.25
3.748   6.38
2.175   5.24
1.222   3.45
```

```
def read_array2D(d):
    path = d
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    m=0
    while True:
        line=file1.readline()
        ee=line.split()
        nn+=1
        if  file1.tell()==endLocation:
            break
    m=len(ee)
    print("m=",m)
    a=[[0.0 for i in range(nn)] for j in range(m)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        for j in range(m):
            a[j][i]=float(ee[j])
    file1.close()
    return a

a=read_array2D('b.txt')
print(a)
```

```
runfile('E:/okul/SCO1/read3.py', wdir='E:/okul/SCO1')
m= 4
[[1.234, 1.345, 3.748, 2.175, 1.222], [3.24, 5.25, 6.38, 5.24, 3.45], [5.67, 5.67, 5.67, 5.67, 5.67], [7.78, 7.78, 7.78, 7.78, 7.78]]
```

1.2.10 TAYLOR SERIES

Taylor equation can be considered the roots of many numerical analysis methods. It will give us serial. **Taylor series** is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point. It is named after the English mathematician Brook Taylor. If the series is centered at zero, the series is also called a **Maclaurin series**, named after the Scottish mathematician Colin Maclaurin. It is common practice to use a finite number of terms of the series to approximate a function. Taylor series around a point a can be written as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f"(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Or if it is written in summation notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Or if a is replaces with x^k and x is replaced with x^{k+1} equation becomes

$$f(x^{k+1}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x^k)}{n!}(x^{k+1}-x^k)^n$$

+..

Computers could not calculate anything more complex than addition, the rest usually accomplish through Taylor series approximations. For example function $f(x)=e^x$ can be approximated as

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

Some of the formulation used in Numerical analysis is direct derivation from the Taylor series. For example for the Newton root finding equation, above linear terms of the equation is ignorede and function value is replaced with 0

$f(x^{k+1}) = 0 = f(x^k) + f'(x^k)(x^{k+1} - x^k)$ Will yield to

$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$

Taylor series can also be used to estimate error. The series usually used up to a certain term and error will be proportinal to one upper term in the series. For example If exponetial function is taken as

$f(x) = e^x = 1 + \mathrm{x} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

The error term will be proportional to the next term of the series

$$Error \cong \frac{x^5}{120}$$

SERİES SOLUTIONS OF FUNCTIONS

Digital computers can only calculate addition process, therefore most of the mathematical expressions are calculated by using series. Series solutions of a few functions will be listed in here.

1. $\ln(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}$ $[0 < x]$
2. $\exp(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
3. $\exp(-\mathrm{x}) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}$
4. $\exp(-\mathrm{x}^2) = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$
5. $\exp(\mathrm{x})(1+x) = \sum_{k=1}^{\infty} \frac{x^k(k+1)}{k!}$
6. $a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$
7. $\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x}$ [a>1 and x<0 or 0<a<1 x>0]
8. $\sin(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
9. $\sin h(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$
10. $\cos(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
11. $\cos\ \mathrm{h}(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$
12. $\tan(\mathrm{x}) = \frac{\sin(x)}{\cos(x)}$
13. $\tanh(\mathrm{x}) = \frac{\sinh(x)}{\cos\ \mathrm{h}(x)}$
14. $\arcsin(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)}$ $x^2 \leq 1$
15. $\mathrm{arcsinh}(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)}$ $x^2 \leq 1$
16. $\arctan(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)}$ $x^2 \leq 1$
17. $\mathrm{arctanh}(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)}$ $x^2 \leq 1$
18. $\mathrm{erf}(\mathrm{x}) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$
19. $\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{2}{8k+6}\right)$ Bailey, Borwein Plouffe(BBP) formula[90]
20. $J_n(z) = \left(\frac{1}{2z}\right) \sum_{k=1}^{\infty} (-1)^k \frac{\left(\frac{1}{2z}\right)^{2k}}{k!\,\Gamma(n+1)}$ Bessel function
21. $\Gamma(z) = \int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$ where

    $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant

22. $Y_n\,(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}$ Bessel function

23. $\gamma = \lim_{n\to\infty}\left(-log(n) + \sum_{k=1}^{n}\frac{1}{k}\right)$ Euler-Mascheroni constant

$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

24. $K = \frac{3}{64}\sum_{k=0}^{\infty}\frac{(-1)^k}{64^k}\left(\frac{32}{(12k+1)^2} - \frac{32}{(12k+2)^2} - \frac{32}{(12k+3)^2} - \frac{8}{(12k+5)^2} - \frac{16}{(12k+6)^2} - \frac{4}{(12k+7)^2} - \frac{4}{(12k+9)^2} - \frac{2}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right)$ Catalan's constant (K=0.915965594177)

25. $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4(396^{4k})}$ Srinavasa Ramanujan π calculation series

26. $\frac{1}{\pi} = \frac{12}{640320^{3/2}}\sum_{k=0}^{\infty}\frac{(6k)!(13591409+545140134k)}{(3k)!(k!)^3(-640320)^{3k}}$ Chudnovsky formula π calculation series

27. $\pi = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$ Bailey-Borvein-Plouffe formula π calculation series

Exponential function $\exp(x) = \sum_{k=1}^{\infty}\frac{x^k}{k!}$

```
from math import *
def exp(x):
    total=0;
    factorial=1;
    power=1;
    for i in range(1,100):
      total=total+power/factorial;
      power=power*x;
      factorial=factorial*i;
    return total;
r=exp(1.0);
print("r=",r);
```

```
runfile('E:/okul/SCO1/exp.py', wdir='E:/okul/SCO1')
r= 2.7182818284590455
```

Natural logarithm $\ln(x) = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}$ $[0 < x]$

```
from math import *

def ln(x):
   total=0;
   y=(x-1.0)/(x+1.0);
   power=y;
   for k in range(1,100):
      total=total+power/(2.0*k-1.0);
      power=power*y*y;
   return 2*total;
e=r=2.7182818284590455;
r=ln(e);
print("r=",r);
```

```
runfile('E:/okul/SCO1/func2.py', wdir='E:/okul/SCO1')
r= 0.9999999999999999
```

ln(x)

k	x=	2,7182818285
	y=(x-1)/(x+1)	0,4621171573
	power	total
0	0,4621171573	0
1	0,0986861666	0,9242343145
2	0,0210746546	0,9900250922
3	0,0045005403	0,9984549541
4	0,0009611006	0,9997408227
5	0,0002052452	0,9999544006
6	4,38306E-005	0,9999917179
7	9,36012E-006	0,9999984611
8	1,99887E-006	0,9999997091
9	4,26864E-007	0,9999999443
10	9,11578E-008	0,9999999892
11	1,94670E-008	0,9999999979
12	4,15721E-009	0,9999999996
13	8,87782E-010	0,9999999999
14	1,89588E-010	1
15	4,04869E-011	1
16	8,64608E-012	1
17	1,84639E-012	1
18	3,94301E-013	1
19	8,42038E-014	1
20	1,79819E-014	1

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

```
import math;

def sin(x):
   total=0;
   factorial=1;
   power=x;
   plusminus=1;
   i=0;
   for k in range(1,80):
      total=total+power/factorial*plusminus;
      power=power*x*x;
      i=2*k+1;
      factorial=factorial*(i-1)*i;
      plusminus*=-1;
   return total;

pi=math.pi;
x=pi/3.0;
print ("sin="+str(sin(x))+"math.sin = "+str(math.sin(x)));
```

```
runfile('E:/okul/SCO1/sin.py', wdir='E:/okul/SCO1')
sin=0.8660254037844385math.sin = 0.8660254037844386
```

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \leq 1$$

```
import math;

def asin(x):
   total=0;
   factorial1=1;
   factorial2=1;
   power=x;
   power2=1;
   i=0;
   for k in range(1,80):
      total=total+power*factorial1/(power2*factorial2*factorial2)/(2.0*(k-1)+1.0);
      power*=x*x;
      power2*=4;
      i=2*k;
      factorial1=factorial1*(i-1)*i;
```

```
        factorial2=factorial2*k;
    return total;

pi=math.pi;
x1=pi/3.0;
x2=math.sin(x1);
print ("asin="+str(asin(x2))+" math.asin = "+str(math.asin(x2)));
```

```
runfile('E:/okul/SCO1/asin.py', wdir='E:/okul/SCO1')
asin=1.0471975511964204 math.asin = 1.0471975511965976
```

$$a^x = e^{xln(a)}$$

```
from math import *

def ln(x):
    total=0
    y=(x-1.0)/(x+1.0)
    power=y
    for k in range(1,100):
        total=total+power/(2.0*k-1.0)
        power=power*y*y
    return 2*total;
def exp(x):
      total=0;
      factorial=1;
      power=1;
      for i in range(1,100):
        total=total+power/factorial;
        power=power*x;
        factorial=factorial*i;
      return total;
def pow(a,x):
    return exp(x*log(a));

r=pow(2.0,3.0);
print("r=",r);
```

```
runfile('E:/okul/SCO1/pow.py', wdir='E:/okul/SCO1')
r= 7.999999999999999
```

1. $J_n(z) = \left(\frac{1}{2z}\right)\sum_{k=1}^{\infty}(-1)^k \frac{\left(\frac{1}{2z}\right)^{2k}}{k!\,\Gamma(n+1)}$ Bessel function

Note : Γ(x) gamma function existed in python math library as math.gamma(x) and natural logarithm of gamma function as math.lgamma(x)

```
from math import *

def J(v,x):
    JJ=0
    a=1.0
    fact=1
    b=0.0
    if x <=20:
        x5=1
        lna=0.0;
        x1=v*log(0.5*x)
        lna1=log(0.25*x*x)
        for k in range(0,100):
            x2=lna
            x3=lgamma(k+1)
            x4=lgamma(v+k+1)
            b=x1+x2-x3-x4
            b=x5*exp(b)
            lna=lna+lna1
            x5=-1*x5
            JJ=JJ+b
    elif (v==0 and x==0):
        JJ=1
```

```
    else:
        JJ=sqrt(2.0/PI/x)*cos(x-0.5*v*PI-0.25*PI)
    return JJ

x=0.1
y=J(0.0,x)
print("J(0.0,",x,") = ",y)
```

```
runfile('E:/okul/SCO1/bessel.py', wdir='E:/okul/SCO1')
J(0.0, 0.1 ) =  0.9975015620660401
```

$$\text{erf(x)} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

```
import  math

def erf(x):
   A=2.0/math.sqrt(math.pi)
   total=0.0
   plusminus=1
   power=x
   factorial=1.0
   for n in range(1,200):
        total=total+power*plusminus/factorial/(2.0*(n-1)+1.0)
        power=power*x*x
        factorial=factorial*n
        plusminus*=plusminus*(-1);
   return A*total

x=0.1
y1=erf(x)
y=math.erf(x)
print("x=",x,"y1=",y,"y=",y)
```

```
runfile('E:/okul/SCO1/erf1.py', wdir='E:/okul/SCO1')
x= 0.1 y1= 0.1124629160182849 y= 0.1124629160182849
```

Some of the algorithms might not be as clear as a directly given function. For example let us look at the **Russian peasant multiplication problem:**

Take two integer for example 21 and 37 then double the first one(left hand side) and divide by two the second one(right side) ignore remaining in division process and stop when the right hand side reached to 1 and then cross each line where the right side number is even add up remaining left hand side numbers for example. If the given integer numbers to be multiplied is 21 and 17, the process will follow the path given below:

```
21 37
~~42 18~~ cross out (18 is even)
84 9
~~168 4~~ cross out(4 is even)
~~336 2~~ cross out(2 is even)
672 1
then 21+84+672=777
21*37=777
```

Now let us convert this to an algorithmic structure:

```
 from math import *

def isOdd(x):
  return bool(x %2 != 0 );

def multiply(x1,x2):
  y2=x2;
  y1=x1;
  b2=0;
  total=0;
  while y2>=1:
    b2=isOdd(y2);
    if y2 %2 != 0 :
```

```
    total=total+y1;
   print("total=",total,"y1=",y1,"y2=",y2);
   y1=y1*2;
   y2=(int)(y2/2);
 return total;

x1=21;
x2=37;
x=multiply(x1,x2);
print("Russian multiplication :\nx = ",x)
```

```
runfile('E:/okul/SCO1/russian.py', wdir='E:/okul/SCO1')
total= 21 y1= 21 y2= 37
total= 21 y1= 42 y2= 18
total= 105 y1= 84 y2= 9
total= 105 y1= 168 y2= 4
total= 105 y1= 336 y2= 2
total= 777 y1= 672 y2= 1
Russian multiplication :
x =  777
```

Euler-Mascheroni constant $\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$

$$\gamma = \lim_{n\to\infty}\left(-log(n) + \sum_{k=1}^{n}\frac{1}{k}\right)$$

```
#Euler-Mascheroni constant
from math import *
from f_x import *

def euler_contant():
   n=100000
   total=0.0
   for k in range(1,n):
      total+=1.0/k
   return -log(n)+total

print(euler_contant())
```

```
runfile('D:/okul/SCO1/euler_contant.py', wdir='D:/okul/SCO1')
Reloaded modules: f_x
0.5772106648931068
```

$\pi = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$ Bailey-Borvein-Plouffe formula π calculation series

```
#              Bailey-Borvein-Plouffe formula π calculation series
from math import *
from f_x import *

def pi():
   n=1000
   total=4.0-2.0/4.0-1.0/5.0-1.0/6.0
   pow=1.0;
   for k in range(1,n):
      pow*=16
      k8=8.0*k
      total+=1.0/pow*(4.0/(k8+1.0)-2.0/(k8+4.0)-1.0/(k8+5.0)-1.0/(k8+6.0))
   return total

print("pi=",pi())
```

```
runfile('D:/okul/SCO1/pi_Plouffe.py', wdir='D:/okul/SCO1')
Reloaded modules: f_x
pi= 3.141592653589793
```

Bailey-Borvein-Plouffe formula π

k	pow	total
0	1	3.133333333333333000000000E+00
1	16	3.141422466422470000000000E+00

2	256	3.14158739034658000000000E+00
3	4096	3.14159245756744000000000E+00
4	65536	3.14159264546034000000000E+00
5	1048576	3.14159265322809000000000E+00
6	16777216	3.14159265357288000000000E+00
7	2.68E+08	3.14159265358897000000000E+00
8	4.29E+09	3.14159265358975000000000E+00
9	6.87E+10	3.14159265358979000000000E+00
10	1.1E+12	3.14159265358979000000000E+00

Further mathematical functions are defined as class Mathd. Some of the methods defined here can be used in numerical math.

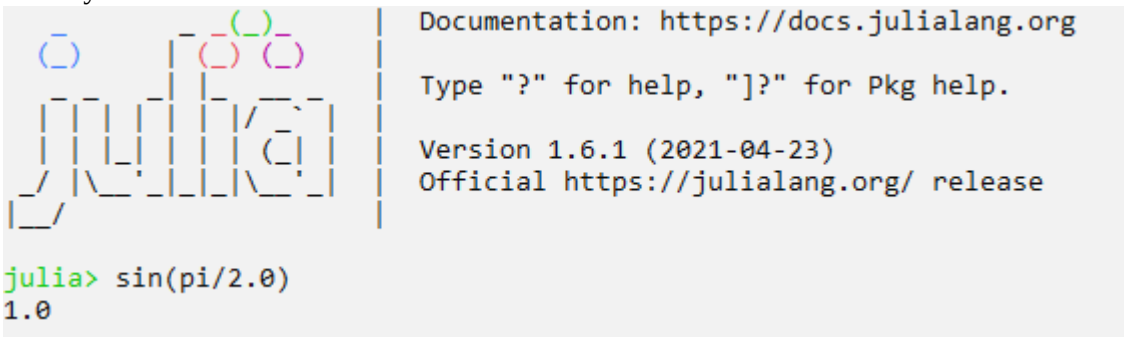
1.3 Julia language basics

1.3.1 LOADING JULIA ENVIRONMENT INTO YOUR COMPUTER

The main source for julia programming language is https://julialang.org



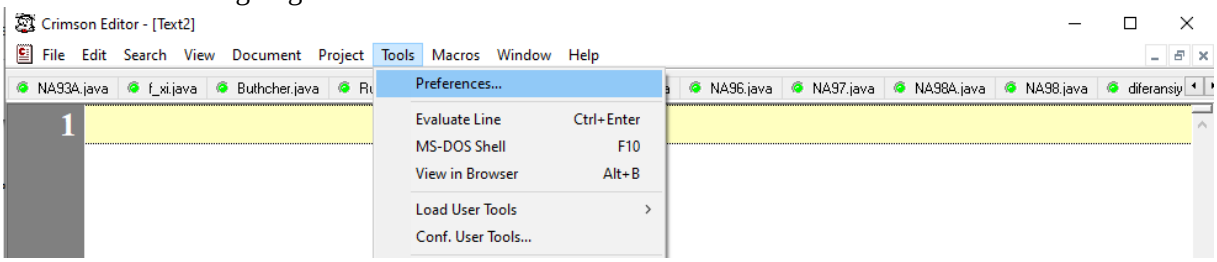
It is a free Access programming language with MIT licence.
After loading up julia programming language, you will find bin command under julia_home/bin directory. When you run this julia command, console environment will be opened. If you wish, you can run your command in console.

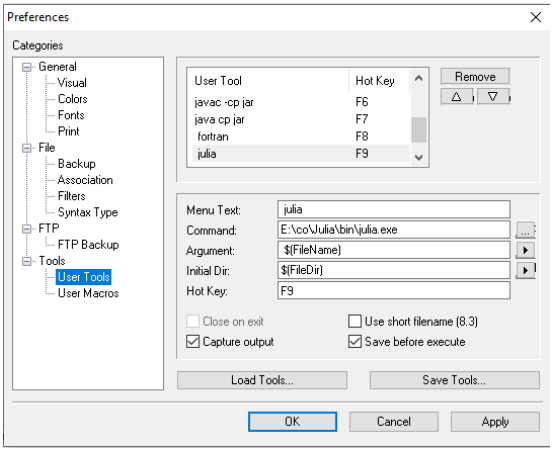


Or alternatively, write your program in a editor environment and then run by using julia command. As

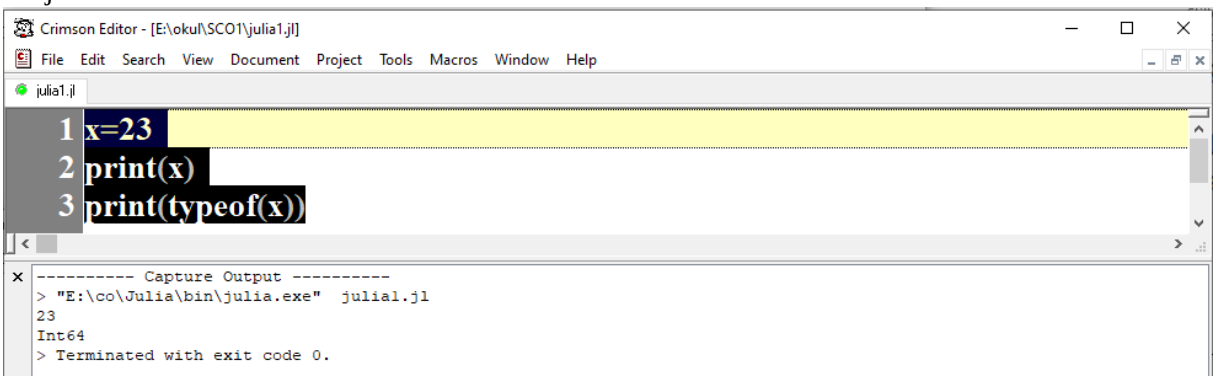
julia>julia script.jl arg1 arg2...

One of my favorite editor is crimson editor(http://www.crimsoneditor.com/) to run the programs. It is a simple editor with build in console environment command runner so you can run your julia commands without going into console.

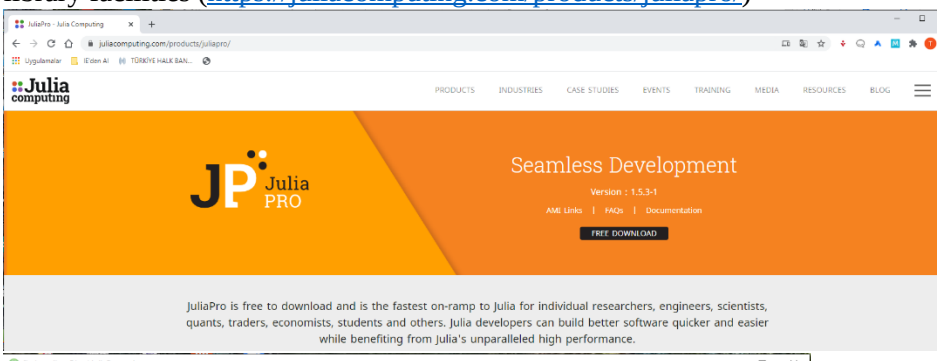


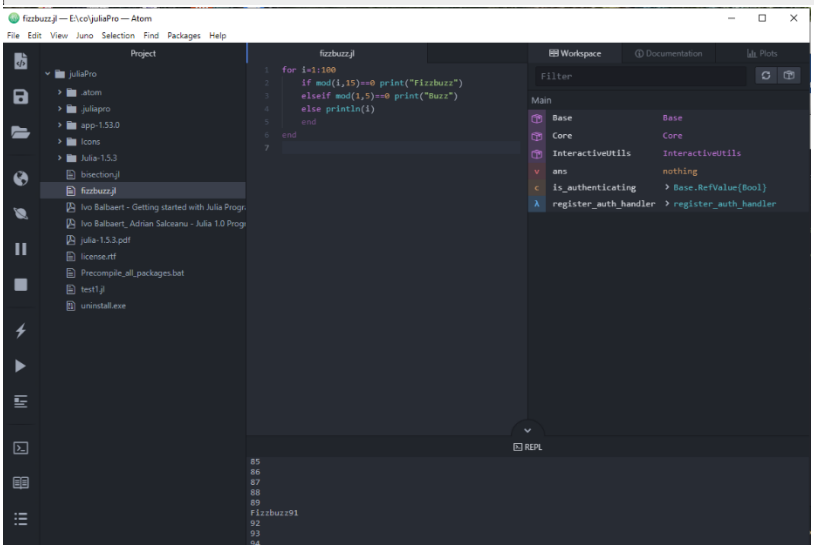


After setting up write your program (name.jl) into the editor and run it by using the selected hot key for julia command



Another good running and editing environment is JuliaPro. It is an environment with editors and more library facilities (https://juliacomputing.com/products/juliapro/)





1.3.2 BASIC PROCESSES AND VARIABLES IN JULIA LANGUAGE

In Julia language when defining the variables, method and classes variable types is not specified. When the data is loaded to a value into the variable language determine the variable type. Julia can use variable base definitions for numbers and strings.

Integer types:

Type	Java/C equivalent	Signed?	Number of Bits	Smallest value	Largest value
Int8	byte	x	8	-2^7=-128	2^7-1=127
UInt8			8	0	$2^8=256$
Int16	short	x	16	$-2^{15}=-32768$	$2^{15}-1$=-32767
UInt16			16	0	2^{16} =65536
Int32	int	x	32	-2^{31} $=-2147483648$	$2^{31}-1$ $=2147483647$
UInt32			32	0	2^{32} =4294967296
Int64	long	x	64	-2^{63} $=-9.223372e18$	$2^{63}-1$ $=9.223372e18$
UInt64			64	0	1.844674e19
Int128		x	128	-2^{127} $=1.701411e38$	$2^{127}-1$ $=1.701411e38$
UInt128			128	0	3.402823669e38
Bool			8	false(0)	true(1)

Floating-point (real) types

Type	Java equivalent	Signed?	Number of Bits	Smallest value	Largest value
Float16		x	16	-2^7=-128	2^7-1=127
Float32	float		32	0	$2^8=256$
Float64	double	x	64	$-2^{15}=-32768$	$2^{15}-1$=-32767

Variable type can be investigated by using typeof method

```
 julia> sin(pi/2.0)
1.0
julia> x=277
277
julia> typeof(x)
Int64
julia> y=1.2454353e38
1.2454353e38
julia> typeof(y)
Float64
```

Additionally, full support for Complex and Rational Numbers is built on top of these primitive numeric types. Binary and octoganal types can also be used

```
julia> x=0b01001
0x09
julia> y=0o14
0x0c
```

Types can be converted by using their variable types

```
julia> x=-1.2
-1.2
julia> y=Float32(x)
-1.2f0
julia> z=0b0010101
0x15
julia> w=Float64(z)
21.0
```

Floating point numbers have two zeros, positive zero and negative zero. Values are equal to each other, but bit values are different

```
 julia> x=0.0
0.0
julia> y=-0.0
-0.0
julia> z=(x==y)
true
julia> bitstring(x)
"0000000000000000000000000000000000000000000000000000000000000000"
julia> bitstring(y)
"1000000000000000000000000000000000000000000000000000000000000000"
```

Infinities and Notanumber type is also
Positive infinite Infinite numbers: Inf16 Inf32 Inf
Negative infinite numbers: -Inf16 -Inf32 -Inf

Not a number: NaN16 NaN32 NaN

An epsilon value is also defined

```
 julia> eps1=eps(Float32)
1.1920929f-7
julia> eps2=eps(Float64)
2.220446049250313e-16
```

The following arithmetic operators are defined in Julia

Expression	Name	Description
+x	Unary plus	The identitiy operation
-x	Unary minus	Map values to their additive inverses
x+y	Binary plus	Perform addition
x-y	Binary minus	Perform substraction
x*y	multiplication	Perform multiplication
x/y	divide	Performs division
x ÷y	integer divide	Division truncated to an integer

x\y	İnverse divide	Equaivalent to y/x
x^y	power	Raises x to the yth power
x%y	remainder	

negation of Bool type

!x	negation	Change true to false false to true

Bitwise operators

The following bitwise operators are supported on all primitive integer types:

Expression	Name
~x	bitwise not
x&y	bitwise and
x|y	bitwise or
x ⊻y	bitwise xor(exclusive or)
x>>>y	Logical shift right
x>>y	Arithmetic shift right
x<<y	Logical/arithmetic shift left

```
julia> 123 | 234
251
julia> xor(123,234)
145
julia> 123 & 234
106
julia>
```

Numeric comparison operators:

==	Equality
!=, ≠	Not equal
<	Less than
<=, ≤	Less than or eqaul to
>	Greater than
>= , ≥	Greater than or equal to

```
julia> x=1
1
julia> y=2
2
julia> z=x<=2
true
julia>
```

Some additional functions for comparisons are available

Function	Test if
isequal(x,y)	x and y are identical
isfinite(x)	x is a finite number
isinf(x)	x is an infinite number
isnan(x)	x is a nan

Rounding functions

Function	Description	Return type
round(x)	round x to the nearest integer	typeof(x)
round(T, x)	round x to the nearest integer	T
floor(x)	round x towards -Inf	typeof(x)
floor(T,x)	round x towards -Inf	T
ceil(x)	round x towards +Inf	typeof(x)
ceil(T,x)	round x towards +Inf	T
trunc(x)	round x towards zero	typeof(x)
trunc(T,x)	round x towards zero	T

Division functions

Function	Description
div(x,y)	Truncated division, quotinent rounded towards zero
fld(x,y)	floored division, quotinent rounded towards -Inf
cld(x,y)	ceeling division, quotinent rounded towards Inf
rem(x,y)	remainder
mod(x,y)	modulus
mod1(x,y)	modulus with offset 1
Mod2p(x)	modulus with respect to 2π
divrem(x,y)	returns (div(x,y),rem(x,y)
Fldmod(x,y)	returns (div(x,y),mod(x,y)
gcd(x,y,…	greatest common divisor pf xy,..
Lcm(x,y,…)	least positive common multiple of x,y,…

Sign and absolute value functions

Function	Description
abs(x)	a positive value with the magnitude of x
abs2(x)	the squared magnitude of x
sign(x)	indicates the sign of x, returning -1, 0, or +1
signbit(x)	indicates whether the sign bit is on (true) or off (false)
copysign(x,y)	a value with the magnitude of x and the sign of y
flipsign(xy)	a value with the magnitude of x and the sign of x*y

Powers logs and roots

Function	Description
sqrt(x), √x	Square root of x
cbrt(x), ∛x	Cube root of x
hypot(x,y)	Hypotenuse of right-angled triangle $\sqrt{x^2 + y^2}$
exp(x)	natural exponent of x e^x
expm1(x)	$e^x - 1$
lxdexp(x,n)	$x2^n$ where n integer
log(x)	natural logarithm of x
log(b,x)	base b logarithm
log2(x)	base 2 logarithm
log10(x)	base 10 logarithm
log1p(x)	log(1+x)
exponent(x)	Binary exponent of x
Significant(x)	binary significand (a.k.a. mantissa) of a floating-point number x

```
julia> x=1
1
julia> y=exp(x)
2.718281828459045
julia>
```

Trigonometric and hyperbolic functions
sin cos tan cot sec csc sinh cosh tanh coth sech csch asin acos atan acot asec acsc asinh acosh atanh acoth asech acsch sinc cosc
Trigonometric functions in degree
sind cosd tand cotd secd cscd asind acosd atand acotd asecd acscd

```
 julia> x=1
1
julia> y=exp(x)
2.718281828459045
julia> z=sin(pi/2.0)
1.0
julia> w=cos(pi/2.0)
6.123233995736766e-17
```

Complex Numbers:

```
julia> x=1+2.5im
1.0 + 2.5im
julia> y=1.25+2im
1.25 + 2.0im
julia> z=x*y
-3.75 + 5.125im
julia> w=sin(x)
5.160143667579705 + 3.268939432079549im
julia>
```

Rational numbers

```
julia> x=6//9
2//3
julia> y=-4//8
-1//2
julia> z=x+y
1//6
julia>
```

Special functions
Special functions are a package. In Julia in order to add a package, Package should be defined first. Pkg comes with a REPL. Enter the Pkg REPL **by pressing]** from the Julia REPL. To get back to the Julia REPL, press backspace or ^C.

```
 (@v1.6) pkg> add SpecialFunctions
    Updating registry at `C:\Users\meral\.julia\registries\General`
    Updating registry at `C:\Users\meral\.julia\registries\JuliaComputingRegistry`
   Installed Libiconv_jll ──────────── v1.16.0+8
   Installed OpenSpecFun_jll ────────── v0.5.5+0
```

```
   Installed Qt_jll ──────────────────── v5.15.2+3
   Installed LogExpFunctions ──────── v0.2.5
   Installed DocStringExtensions ─ v0.8.5
   Installed SpecialFunctions ─────── v1.5.1
   Installed ChainRulesCore ──────────── v0.10.11
  Downloaded artifact: Qt
  Downloaded artifact: Libiconv
  Downloaded artifact: OpenSpecFun
    Updating `C:\Users\meral\.julia\environments\v1.6\Project.toml`
  [276daf66] + SpecialFunctions v1.5.1
    Updating `C:\Users\meral\.julia\environments\v1.6\Manifest.toml`
  [d360d2e6] + ChainRulesCore v0.10.11
  [ffbed154] + DocStringExtensions v0.8.5
  [2ab3a3ac] + LogExpFunctions v0.2.5
  [276daf66] + SpecialFunctions v1.5.1
  [94ce4f54] ↑ Libiconv_jll v1.16.0+7 ⇒ v1.16.0+8
  [efe28fd5] + OpenSpecFun_jll v0.5.5+0
  [ede63266] ↑ Qt_jll v5.15.2+1 ⇒ v5.15.2+3
  [3f19e933] + p7zip_jll
Precompiling project...
  117 dependencies successfully precompiled in 150 seconds

(@v1.6) pkg> ^C

julia>
```

To remove packages, use rm:

```
(v1.1) pkg> rm SpecialFunctions
```

The same addition can be done in directly julia REPLY as:

```
julia>import Pkg
julia>Pkg.add("SpecialFunctions")
```

So far, we have referred only to registered packages. Pkg also supports working with unregistered packages. To add an unregistered package, specify a URL:

```
(v1.1) pkg> add https://github.com/JuliaLang/Example.jl
```

Gamma Function

Function	**Description**
gamma(z)	gamma function \Gamma(z)Γ(z)
loggamma(x)	accurate log(gamma(x)) for large x
logabsgamma(x)	accurate log(abs(gamma(x))) for large x
logfactorial(x)	accurate log(factorial(x)) for large x; same as loggamma(x+1) for x > 1, zero otherwise
digamma(x)	digamma function (i.e. the derivative of loggamma at x)
invdigamma(x)	invdigamma function (i.e. inverse of digamma function at x using fixed-point iteration algorithm)
trigamma(x)	trigamma function (i.e the logarithmic second derivative of gamma at x)
polygamma(m,x)	polygamma function (i.e the (m+1)-th derivative of the loggamma function at x)

gamma(a,z)	upper incomplete gamma function \Gamma(a,z)Γ(a,z)
loggamma(a,z)	accurate log(gamma(a,x)) for large arguments
gamma_inc(a,x,IND)	incomplete gamma function ratio P(a,x) and Q(a,x) (i.e evaluates P(a,x) and Q(a,x)for accuracy specified by IND and returns tuple (p,q))
beta_inc(a,b,x,y)	incomplete beta function ratio Ix(a,b) and Iy(a,b) (i.e evaluates Ix(a,b) and Iy(a,b) and returns tuple (p,q))
gamma_inc_inv(a,p,q)	inverse of incomplete gamma function ratio P(a,x) and Q(a,x) (i.e evaluates x given P(a,x)=p and Q(a,x)=q
beta(x,y)	beta function at x,y
logbeta(x,y)	accurate log(beta(x,y)) for large x or y
logabsbeta(x,y)	accurate log(abs(beta(x,y))) for large x or y
logabsbinomial(x,y)	accurate log(abs(binomial(n,k))) for large n and k near n/2

Exponential and Trigonometric Integrals

Function	**Description**
expint(ν, z)	exponential integral \operatorname{E}_\nu(z)Eν(z)
expinti(x)	exponential integral \operatorname{Ei}(x)Ei(x)
expintx(x)	scaled exponential integral e^z \operatorname{E}_\nu(z)ezEν(z)
sinint(x)	sine integral \operatorname{Si}(x)Si(x)
cosint(x)	cosine integral \operatorname{Ci}(x)Ci(x)

Error Functions, Dawson's and Fresnel Integrals

Function	**Description**
erf(x)	error function at xx
erf(x,y)	erf(*y*)−erf(*x*)
erfc(x)	complementary error function, i.e. the accurate version of 1-
erfcinv(x)	inverse function to erfc()
erfcx(x)	scaled complementary error function, i.e. accurate e^{x^2}
logerfc(x)	log of the complementary error function, i.e.
logerfcx(x)	log of the scaled complementary error function, i.e.
erfi(x)	imaginary error function defined as -i
erfinv(x)	inverse function to erf()
dawson(x)	scaled imaginary error function, a.k.a. Dawson function, i.e.

Airy and Related Functions

Function	**Description**
airyai(z)	Airy Ai function at z
airyaiprime(z)	derivative of the Airy Ai function at z
airybi(z)	Airy Bi function at z
airybiprime(z)	derivative of the Airy Bi function at z
airyaix(z), airyaiprimex(z), airybix(z), airybiprimex(z)	scaled Airy Ai function and kth derivatives at z

Bessel Functions

Function	**Description**
besselj(nu,z)	Bessel function of the first kind of order nu at z

besselj0(z)	besselj(0,z)
besselj1(z)	besselj(1,z)
besseljx(nu,z)	scaled Bessel function of the first kind of order nu at z
sphericalbesselj(nu,z)	Spherical Bessel function of the first kind of order nu at z
bessely(nu,z)	Bessel function of the second kind of order nu at z
bessely0(z)	bessely(0,z)
bessely1(z)	bessely(1,z)
besselyx(nu,z)	scaled Bessel function of the second kind of order nu at z
sphericalbessely(nu,z)	Spherical Bessel function of the second kind of order nu at z
besselh(nu,k,z)	Bessel function of the third kind (a.k.a. Hankel function) of
hankelh1(nu,z)	besselh(nu, 1, z)
hankelh1x(nu,z)	scaled besselh(nu, 1, z)
hankelh2(nu,z)	besselh(nu, 2, z)
hankelh2x(nu,z)	scaled besselh(nu, 2, z)
besseli(nu,z)	modified Bessel function of the first kind of order nu at z
besselix(nu,z)	scaled modified Bessel function of the first kind of order nu at z
besselk(nu,z)	modified Bessel function of the second kind of order nu at z
besselkx(nu,z)	scaled modified Bessel function of the second kind of order nu at z
jinc(x)	scaled Bessel function of the first kind divided by x. A.k.a. sombrero

Elliptic Integrals

Function	Description
ellipk(m)	complete elliptic integral of 1st kind K(m)K(m)
ellipe(m)	complete elliptic integral of 2nd kind E(m)E(m)

Zeta and Related Functions

Function	Description
eta(x)	Dirichlet eta function at x
zeta(x)	Riemann zeta function at x

```
using SpecialFunctions
x=0:5.0
y1=besselj0.(x)
print("x = $x y = $y1")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" bessel2.jl
x = 0.0:1.0:5.0 y = [1.0, 0.7651976865579666, 0.22389077914123567, -0.2600519549019335, -0.3971498098638474, -
0.1775967713143383]
> Terminated with exit code 0.
```

Strings and character

Strings are finite sequences of characters. Of course, the real trouble comes when one asks what a character is. Chracters is letters, punctuations, numbers etc. In starting of computer times a standart called ASCII (American Standart code for Information Interchange) is developed which basicaly includes control characters, letters and numbers of american alphabet.

ASCII Control code chart

Binary	Oct	Dec	Hex	Abbreviation			[c]	[d]	Name (1967)
				1963	1965	###			
000 0000	0	0	0	NULL	NUL		^@	\0	Null
000 0001	1	1	1	SOM	SOH		^A		Start of Heading
000 0010	2	2	2	EOA	STX		^B		Start of Text
000 0011	3	3	3	EOM	ETX		^C		End of Text
000 0100	4	4	4	EOT			^D		End of Transmission
000 0101	5	5	5	WRU	ENQ		^E		Enquiry
000 0110	6	6	6	RU	ACK		^F		Acknowledgement
000 0111	7	7	7	BELL	BEL		^G	\a	Bell
000 1000	10	8	8	FE0	BS		^H	\b	Backspace[e][f]

000 1001	11	9	9	HT/SK	HT	^I	\t	Horizontal Tab[g]
000 1010	12	10	0A	LF		^J	\n	Line Feed
000 1011	13	11	0B	VTAB	VT	^K	\v	Vertical Tab
000 1100	14	12	0C	FF		^L	\f	Form Feed
000 1101	15	13	0D	CR		^M	\r	Carriage Return[h]
000 1110	16	14	0E	SO		^N		Shift Out
000 1111	17	15	0F	SI		^O		Shift In
10 000	20	16	10	DC0	DLE	^P		Data Link Escape
10 001	21	17	11	DC1		^Q		Device Control
10 010	22	18	12	DC2		^R		Device Control 2
10 011	23	19	13	DC3		^S		Device Control
10 100	24	20	14	DC4		^T		Device Control 4
10 101	25	21	15	ERR	NAK	^U		Negative
10 110	26	22	16	SYNC	SYN	^V		Synchronous Idle
10 111	27	23	17	LEM	ETB	^W		End of Transmission Block
11 000	30	24	18	S0	CAN	^X		Cancel
11 001	31	25	19	S1	EM	^Y		End of Medium
11 010	32	26	1A	S2	SS SUB	^Z		Substitute
11 011	33	27	1B	S3	ESC	^[	\e[i]	Escape[j]
11 100	34	28	1C	S4	FS	^\		File Separator
11 101	35	29	1D	S5	GS	^]		Group Separator
11 110	36	30	1E	S6	RS	^^[k]		Record Separator
11 111	37	31	1F	S7	US	^_		Unit Separator
1 111 111	177	127	7F	DEL		^?		Delete[l][f]

ASCII Character chart

100 000	40	32	20	space		
100 001	41	33	21	!		
100 010	42	34	22	"		
100 011	43	35	23	#		
100 100	44	36	24	$		
100 101	45	37	25	%		
100 110	46	38	26	&		
100 111	47	39	27	'		
101 000	50	40	28	(		
101 001	51	41	29	)		
101 010	52	42	2A	*		
101 011	53	43	2B	+		
101 100	54	44	2C	,		
101 101	55	45	2D	-		
101 110	56	46	2E	.		
101 111	57	47	2F	/		
110 000	60	48	30	0		
110 001	61	49	31	1		
110 010	62	50	32	2		
110 011	63	51	33	3		
110 100	64	52	34	4		
110 101	65	53	35	5		
110 110	66	54	36	6		
110 111	67	55	37	7		
111 000	70	56	38	8		
111 001	71	57	39	9		
111 010	72	58	3A	:		
111 011	73	59	3B	;		
111 100	74	60	3C	<		
111 101	75	61	3D	=		
111 110	76	62	3E	>		
111 111	77	63	3F	?		
1 000 000	100	64	40	@	`	@
1 000 001	101	65	41	A		

1 000 010	102	66	42	B		
1 000 011	103	67	43	C		
1 000 100	104	68	44	D		
1 000 101	105	69	45	E		
1 000 110	106	70	46	F		
1 000 111	107	71	47	G		
1 001 000	110	72	48	H		
1 001 001	111	73	49	I		
1 001 010	112	74	4A	J		
1 001 011	113	75	4B	K		
1 001 100	114	76	4C	L		
1 001 101	115	77	4D	M		
1 001 110	116	78	4E	N		
1 001 111	117	79	4F	O		
1 010 000	120	80	50	P		
1 010 001	121	81	51	Q		
1 010 010	122	82	52	R		
1 010 011	123	83	53	S		
1 010 100	124	84	54	T		
1 010 101	125	85	55	U		
1 010 110	126	86	56	V		
1 010 111	127	87	57	W		
1 011 000	130	88	58	X		
1 011 001	131	89	59	Y		
1 011 010	132	90	5A	Z		
1 011 011	133	91	5B	[		
1 011 100	134	92	5C	\	~	\
1 011 101	135	93	5D	]		
1 011 110	136	94	5E	↑	^	
1 011 111	137	95	5F	←	_	
1 100 000	140	96	60		@	`
1 100 001	141	97	61		a	
1 100 010	142	98	62		b	
1 100 011	143	99	63		c	
1 100 100	144	100	64		d	
1 100 101	145	101	65		e	
1 100 110	146	102	66		f	
1 100 111	147	103	67		g	
1 101 000	150	104	68		h	
1 101 001	151	105	69		i	
1 101 010	152	106	6A		j	
1 101 011	153	107	6B		k	
1 101 100	154	108	6C		l	
1 101 101	155	109	6D		m	
1 101 110	156	110	6E		n	
1 101 111	157	111	6F		o	
1 110 000	160	112	70		p	
1 110 001	161	113	71		q	
1 110 010	162	114	72		r	
1 110 011	163	115	73		s	
1 110 100	164	116	74		t	
1 110 101	165	117	75		u	
1 110 110	166	118	76		v	
1 110 111	167	119	77		w	
1 111 000	170	120	78		x	
1 111 001	171	121	79		y	
1 111 010	172	122	7A		z	
1 111 011	173	123	7B		{	
1 111 100	174	124	7C	ACK	¬	|
1 111 101	175	125	7D		}	
1 111 110	176	126	7E	ESC	¦	~

ASCII code does not include the characters of the other languages. A more extended coding for characters developed by Unicode organisation (Unicode.org)
Characters either ASCII or Unicode, can be defined by single quote sign. Characters can be combined to create String variables

```
julia> c1='ü'
'ü': Unicode U+00FC (category Ll: Letter, lowercase)
julia> c2='\u00c7'
'Ç': Unicode U+00C7 (category Lu: Letter, uppercase)
julia> c3='x'
'x': ASCII/Unicode U+0078 (category Ll: Letter, lowercase)
julia> c4='y'
'y': ASCII/Unicode U+0079 (category Ll: Letter, lowercase)

julia> s=string(c1,c2,c3,c4)
"üÇxy"
julia> s1="ü \u00c7 x y"
"ü Ç x y"
julia>
```

```
julia> b1='ü'
'ü': Unicode U+00FC (category Ll: Letter, lowercase)

julia> b2='ğ'
'ğ': Unicode U+011F (category Ll: Letter, lowercase)

julia> b3='ç'
'ç': Unicode U+00E7 (category Ll: Letter, lowercase)

julia> b4='a'
'a': ASCII/Unicode U+0061 (category Ll: Letter, lowercase)

julia> b5='b'
'b': ASCII/Unicode U+0062 (category Ll: Letter, lowercase)

julia> b6='c'
'c': ASCII/Unicode U+0063 (category Ll: Letter, lowercase)

julia> d="$b1$b2$b3$b4$b5$b6"
"üğçabc"
```

```
julia> b1='\u03B1'
'α': Unicode U+03B1 (category Ll: Letter, lowercase)

julia> b2='\u03B2'
'β': Unicode U+03B2 (category Ll: Letter, lowercase)

julia> b3='\u03B3'
'γ': Unicode U+03B3 (category Ll: Letter, lowercase)
```

```
julia> b4='\u03B4'
'δ': Unicode U+03B4 (category Ll: Letter, lowercase)

julia> b5='\u03B5'
'ε': Unicode U+03B5 (category Ll: Letter, lowercase)

julia> b6='\u03B6'
'ζ': Unicode U+03B6 (category Ll: Letter, lowercase)

julia> d="$b1$b2$b3$b4$b5$b6"
"αβγδεζ"
```

Welcome message (String)

```
julia> print("Welcome to Julia programming language")
Welcome to Julia programming language
julia>
```

```
julia> x="Welcome to Julia Programming language"

"Welcome to Julia Programming language"

julia> print(x)

Welcome to Julia Programming language

julia>
```

In order to combine different string together (concatenation) * is used

```
julia> x="Hello"*" world"*"\n"
"Hello world\n"

julia> print(x)
```

Constructing strings using concatenation can become a bit cumbersome, however. To reduce the need for these verbose calls to string or repeated multiplications, Julia allows interpolation into string literals using $,

```
julia> s1="Hello"
"Hello"
julia> s2="World"
"World"
julia> x="$s1,$s2\n"
"Hello,World\n"
```

The shortest complete expression after the $ is taken as the expression whose value is to be interpolated into the string. Thus, you can interpolate any expression into a string using parentheses:

```
julia> x=1
1
julia> y=2
2
julia> s="x+y=$(x+y)"
"x+y=3"

julia> c1='ü'
'ü': Unicode U+00FC (category Ll: Letter, lowercase)

julia> c2='\u00c7'
'Ç': Unicode U+00C7 (category Lu: Letter, uppercase)

julia> x="Hello,$c1,$c2"
"Hello,ü,Ç"
```

Longer than one line of strings can be defined by using tripple quote “””

```
julia> s1="""
      Hello Friend,
      how are you today?
```

```
        I am quite fine thanks
        """
  "Hello Friend,\nhow are you today?\nI am quite fine thanks\n"

Strings can be compare by using the same comparison signs given in the table above.
  julia> "1+2=3"=="1+2=$(1+2)"
  true

  julia> "Turhan">"turhan"
  false
```

You can search for the index of a particular character using the findfirst and findlast functions:

```
  julia> x="Ali veli 49 elli"
  "Ali veli 49 elli"
  julia> findfirst(isequal('e'),x)
  6
  julia> findlast(isequal('e'),x)
  13
```

You can use the occursin function to check if a substring is found within a string:

```
  julia> x="Ali veli 49 elli"
  "Ali veli 49 elli"

  julia> occursin("veli",x)
  true
```

Repeat is a usefull string function

```
  julia> repeat("Turhan ",3)
```

Some other useful String functions:

• firstindex(str) gives the minimal (byte) index that can be used to index into str (always 1 for strings, not necessarily true for other containers).
• lastindex(str) gives the maximal (byte) index that can be used to index into str.
• length(str) the number of characters in str.
• length(str, i, j) the number of valid character indices in str from i to j.
• ncodeunits(str) number of code units in a string.
• codeunit(str, i) gives the code unit value in the string str at index i.
• thisind(str, i) given an arbitrary index into a string find the first index of the character into which the index points.
• nextind(str, i, n=1) find the start of the nth character starting after index i. • prevind(str, i, n=1) find the start of the nth character starting before index i.

FUNCTIONS

In Julia, a function is an object that maps a tuple of argument values to a return value.

```
function f(x,y)
  x+y
  end

x=1.2
y=2.3
z=f(x,y)
print("z=",z)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func1.jl
z=3.5
> Terminated with exit code 0.
```

```
function f(x,y)
      return x*y
end

x=1.2
```

```
y=2.3
z=f(x,y)
s="x=$(x) y=$(y) z=$z"
print(s)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func1.jl
x=1.2 y=2.3 z=2.76
> Terminated with exit code 0.
```

```
 function f(x,y)
      return x.+y
end

x=0.0:0.1:2.2
y=2.013245
z=f(x,y)
s="x=$(x) y=$(y) z=$z"
print(s)
```

```
---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" func1.jl
x=0.0:0.1:2.2 y=2.013245 z=2.013245:0.1:4.213245
> Terminated with exit code 0.
```

```
julia> s(x,y)=x+y
s (generic function with 1 method)
julia> s(2,3.2)
5.2
```

```
function hypot(x,y)
x = abs(x)
y = abs(y)
if x > y
r = y/x
return x*sqrt(1+r*r)
end
if y == 0
return zero(x)
end
r = x/y
return y*sqrt(1+r*r)
end

x=1.2
y=2.3
z=hypot(x,y)
s="x=$(x) y=$(y) z=$z"
print(s)
```

```
---------- Capture Output ----------

> "E:\co\Julia\bin\julia.exe"  func2.jl

x=1.2 y=2.3 z=2.5942243542145693

> Terminated with exit code 0.
```

For functions that do not need to return a value (functions used only for some side effects), the Julia convention is to return the value nothing:

```
function printx(x)
println("x = $x")
return nothing
end

x=1.23
printx(x)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func3.jl
x = 1.23
```

```
> Terminated with exit code 0.
```

Function output type can be defined

```
function g(x, y)::Int8
return x * y
end;

z=g(1, 2)
w=typeof(z)
print("z = $z w = $w")
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func5.jl
z = 2 w = Int8
> Terminated with exit code 0.
```

1.3.3 TUPPLES IN JULIA LANGUAGE

Julia has a built-in data structure called a tupple that is closely related to function arguments and return values. A tuple is a fixed-length container that can hold any values, but cannot be modified (it is immutable). Tuples are constructed with commas and parentheses, and can be accessed via indexing: Tupples are in a way similar to array variables in languages like java or c++. But unlike these languages tupples can hold different types of data in its structures.

```
julia> x=(1.2,"hello",6*7)
(1.2, "hello", 42)

julia> x[2]
"hello"

julia> x[1]
1.2
```

It should also be note that topple references started from 1 (not 0)

The components of tuples can optionally be named, in which case a named tuple is constructed:

```
julia> x=(a=1.2,b="hello",c=6*7)
(a = 1.2, b = "hello", c = 42)

julia> x.a
1.2

julia> x[1]
1.2

julia> x.b
"hello"

julia> x[2]
"hello"

julia>
```

In Julia, one returns a tuple of values to simulate returning multiple values. However, tuples can be created and destructured without needing parentheses, thereby providing an illusion that multiple values are being returned, rather than a single tuple value. For example, the following function returns a pair of values:

```
function g(x, y)
return x * y,x+y
end;

z=g(1, 2)
print("z = $z")
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func6.jl
z = (2, 3)
> Terminated with exit code 0.
```

It is often possible to provide sensible default values for function arguments. This can save users from having to pass every argument on every call. For example, the function Date(y, [m, d]) from Dates module constructs a Date type for a given year y, month m and day d. However, m and d arguments are optional and their default value is 1. This behavior can be expressed concisely as:

```
function Date1(y::Int64, m::Int64=1, d::Int64=1)
err = validargs(Date, y, m, d)
err === nothing || throw(err)
return Date(UTD(totaldays(y, m, d)))
end

using Dates
d1=Date(2020,12,15)
println("d1 $d1")
d2=Date(2020,12)
println("d2 $d2")
d3=Date(2020)
println("d3 $d3")
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func7.jl
d1 2020-12-15
d2 2020-12-01
d3 2020-01-01

> Terminated with exit code 0.
```

Compound expressions:
If more than one expression should be defined and execute as a block. This can be done two ways:
The first method is begin end block

```
julia> z=begin
       x=1
       y=2
       z=3
       x+y*z
       end
7

julia>
```

The second one is chain synthax

```
julia> w=(x=1;y=2;z=3;x+y*z)
7

julia>
```

IF STRUCTURE IN JULIA LANGUAGE

Conditional evaluation (if block):

```
function greater(x,y)
b=true
```

```
if x<y
   b=false
elseif x==y
   b=false
return b
end
end

x=1
y=2
c=greater(x,y)
println("c =$c")
```

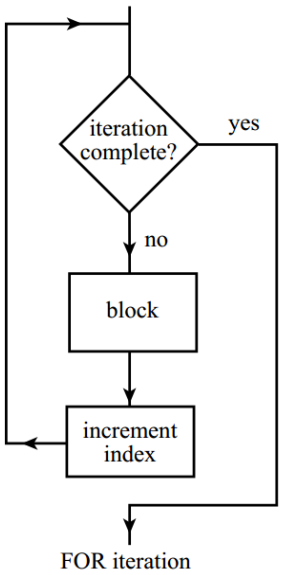
```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func8.jl
c =false

> Terminated with exit code 0.
```

```
function test(x,y)
a="greater than"
if x<y
   a="smaller than"
elseif x==y
   a="equal to"
end
b="$x is $a $y"
return b
end

x=1
y=2
c=test(x,y)
println(c)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func9.jl
1 is smaller than 2

> Terminated with exit code 0.
```

1.3.4 FOR STATEMENT IN JULIA LANGUAGE

For statement and loop structure



Sum of numbers 1 to n

```
function sum(n)
s = 0 # new local
for i = 1:n
s = s + i # assign existing local
end
return s # same local
end

x=100
a=sum(x)
print("s = $a")
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func10.jl
s = 5050
> Terminated with exit code 0.
```

Numbers can be added up by sum function

```
x=[1.1179,2.21,3.31,4.41,5.51,6.61]
a=sum(x)
print("a = $a")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" mean.jl
a = 23.1679
> Terminated with exit code 0.
```

Average of number set

```
function average(x)
s = 0 # new local
n=length(x)
for i  in x
s = s + i
end
s=s/n
return s # same local
```

```
end

x=[1.1179,2.21,3.31,4.41,5.51,6.61]
a=average(x)
print("a = $a")
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func10.jl
a = 3.8613166666666667
> Terminated with exit code 0.
```

The same thing can be achived by using mean function of Statistics library

```
using Statistics
x=[1.1179,2.21,3.31,4.41,5.51,6.61]
a=mean(x)
print("a = $a")
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" mean.jl
a = 3.8613166666666667
> Terminated with exit code 0.
```

Printing string values:

```
a=["ali","veli","49","elli"]
for i in a
   println("i =",i)
end
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func11.jl
i =ali
i =veli
i =49
i =elli

> Terminated with exit code 0.
```

List of greek characters

```
julia> c1='\u03B1'
'α': Unicode U+03B1 (category Ll: Letter, lowercase)

julia> c2='\u03D2'
'ϒ': Unicode U+03D2 (category Lu: Letter, uppercase)

julia> for i=c1:c2
       print(" ",i)
       end
 α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ ς σ τ υ φ χ ψ ω ϊ ϋ ό ύ ώ Ϗ ϐ ϑ ϒ
```

List of ASCII characters

```
c1='\u0021'
c2='\u007E'
for i =c1:c2
   print("  ",i)
end
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func12.jl
```

```
 ! " # $ % & ' ( ) * + , - . / 0 1 2 3 4 5 6 7 8 9 : ; < = > ? @ A B C D E F G H I J K
L M N O P Q R S T U V W X Y Z [ \ ] ^ _ ` a b c d e f g h i j k l m n o p q r s t u v
w x y z { | } ~
> Terminated with exit code 0.
```

1.3.6 WHILE STATEMENT IN JULIA LANGUAGE

While loop statement basically carry out the same duty as for statement

Listing of ASCII characters

```
c1='\u0021'
c2='\u007E'
i=c1
while i<c2
    print("  ",i)
   global i+=1
end
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func12.jl
 ! " # $ % & ' ( ) * + , - . / 0 1 2 3 4 5 6 7 8 9 : ; < = > ? @ A B C D
E F G H I J K L M N O P Q R S T U V W X Y Z [ \ ] ^ _ ` a b c d e f
g h i j k l m n o p q r s t u v w x y z { | }
> Terminated with exit code 0.
```

Listing of greek characters in unicode

```
julia> c1='\u03B1'
'α': Unicode U+03B1 (category Ll: Letter, lowercase)

julia> c2='\u03D2'
'ϒ': Unicode U+03D2 (category Lu: Letter, uppercase)

julia> i=c1
'α': Unicode U+03B1 (category Ll: Letter, lowercase)

julia> while i<=c2
      print(" ",i)
      global i+=1
      end
 α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ ς σ τ υ φ χ ψ ω ϊ ϋ ό ύ ώ Ϗ ϐ ϑ ϒ
julia>
```

Finding average value

```
function average(x)
s = 0 # new local
n=length(x)
i=1
s=0
while i<=n
s = s + x[i]
i=i+1
end
s=s/n
return s # same local
end

x=[1.1179,2.21,3.31,4.41,5.51,6.61]
a=average(x)
print("a = $a")
```

```
---------- Capture Output ----------
```

```
> "E:\co\Julia\bin\julia.exe"  func12.jl
a = 3.8613166666666667
> Terminated with exit code 0.
```

Anonymous function (Lambda variables??)

Array process:

```
fun = (x,y,z)->2x+y-z
x=[1,2,3]
y=[2,3,4]
z=[1,3,5]
w=fun(x, y, z)
print(w)
```

Tupple process

```
fun = (x,y,z)->2x+y-z
x=(1,2,3)
y=(2,3,4)
z=(1,3,5)
w=map(fun,x, y, z)
print(w)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func14.jl
(3, 4, 5)
> Terminated with exit code 0.
```

Read data from screen and convert it to Float64: (a line of data seperated with ,)

```
print("enter x = ")
x=readline()
y=split(x,",")
z=map(x->(v = tryparse(Float64,x)),y)
print(z)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func15.jl
enter x1,2,3,4
[1.0, 2.0, 3.0, 4.0]
> Terminated with exit code 0.
```

Read data from screen and convert it to Float64: (a line of data seperated with space)

```
print("enter x = ")
x=readline()
y=split(x," ")
z=map(x->(v = tryparse(Float64,x)),y)
print(z)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func26.jl
enter x = 1 2 3 4 5
[1.0, 2.0, 3.0, 4.0, 5.0]
> Terminated with exit code 0.
```

Read a single data from screen and convert it to Float64

```
print("enter x = ")
x=readline()
z=tryparse(Float64,x)
print(z)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func26.jl
enter x = 1.253453
1.253453
> Terminated with exit code 0.
```

Anonymous function (Lambda variables??)

```
function readdouble(s)
print(s)
```

```
x=readline()
y=split(x,",")
z=map(x->(v = tryparse(Float64,x)),y)
return z
end

fun = (x,y,z)->2x+y-z
x=readdouble("x=")
y=readdouble("y=")
z=readdouble("z=")
w=map(fun,x, y, z)
print(w)
```

```
---------- Capture Output ----------
> "E:\co\Julia\bin\julia.exe"  func17.jl
x=1,2,3
y=4,5,6
z=7,8,9
[-1.0, 1.0, 3.0]
> Terminated with exit code 0.
```

x=1.4142135623719696 y=-3.183231456205249e-12
x=1.4142135623719696 y=-3.183231456205249e-12

Arrays and matrices in Julia

Arrays and array dimensions can be defined by using several ways. The most important point should be mentioned here is that **array index in Julia start from 1**. In languages like C, C++, java, python array index start from 0.

```
B=Array{Float64}(undef,2)
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), B)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array1.jl
2-element Vector{Float64}:
 0.0
 0.0
> Terminated with exit code 0.
```

```
B=Array{Float64}(undef,4)
n=size(B)[1]
println("n = $n")
for i=1:n
  y=B[i]
  println("B[$i] = $y")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2.jl
n = 4
B[1] = 0.0
B[2] = 0.0
B[3] = 0.0
B[4] = 0.0

> Terminated with exit code 0.
```

```
B=zeros(Float64,4)
n=size(B)[1]
println("n = $n")
for i=1:n
  y=B[i]
  println("B[$i] = $y")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array3.jl
n = 4
B[1] = 0.0
B[2] = 0.0
B[3] = 0.0
```

```
B[4] = 0.0

> Terminated with exit code 0.
```

```
B=ones(Float64,4)
n=size(B)[1]
println("n = $n")
for i=1:n
  y=B[i]
  println("B[$i] = $y")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array3.jl
n = 4
B[1] = 1.0
B[2] = 1.0
B[3] = 1.0
B[4] = 1.0

> Terminated with exit code 0.
```

```
B=[1.1,2.2,3.3,4.4]
n=size(B)[1]
println("n = $n")
for i=1:n
  y=B[i]
  println("B[$i] = $y")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array4.jl
n = 4
B[1] = 1.1
B[2] = 2.2
B[3] = 3.3
B[4] = 4.4

> Terminated with exit code 0.
```

If it is two dimensional array, it is given as:

```
A = Array{Float64,2}(undef, 2, 3)
show(IOContext(stdout, :limit=>false), MIME"text/plain"(),A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_1.jl
2X3 Matrix{Float64}:
 0.0  0.0  0.0
 0.0  0.0  0.0
> Terminated with exit code 0.
```

```
julia> include("array2D_1.jl")
2×3 Matrix{Float64}:
 1.18628e-314  1.18628e-314  1.18628e-314
 1.18628e-314  1.18628e-314  1.0e-323
julia>
```

```
A = Array{Float64,2}(undef, 2, 3)
n=size(A)[1]
m=size(A)[2]
println("n = $n m = $m")
for i=1:n
  for j=1:m
  y=A[i,j]
  print("A[$i,$j] = $y ")
  end
  println(" ")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_2.jl
n = 2 m = 3
A[1,1] = 0.0 A[1,2] = 0.0 A[1,3] = 0.0
A[2,1] = 0.0 A[2,2] = 0.0 A[2,3] = 0.0
> Terminated with exit code 0.
```

```
A = Array{Float64,2}(undef, 2, 3)
print(A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_3.jl
[0.0 0.0 0.0; 0.0 0.0 0.0]
> Terminated with exit code 0.
```

```
A = zeros(Float64,2, 3)
print(A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_4.jl
[0.0 0.0 0.0; 0.0 0.0 0.0]
> Terminated with exit code 0.
```

```
A = ones(Float64,2, 3)
print(A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_4.jl
[1.0 1.0 1.0; 1.0 1.0 1.0]
> Terminated with exit code 0.
```

```
A = fill(5.34,(2,3))
show(IOContext(stdout, :limit=>false), MIME"text/plain"(),A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_5.jl
2X3 Matrix{Float64}:
 5.34  5.34  5.34
 5.34  5.34  5.34
> Terminated with exit code 0.
```

```
A = Matrix{Float64}(undef, 2, 3)
show(IOContext(stdout, :limit=>false), MIME"text/plain"(),A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" array2D_6.jl
2X3 Matrix{Float64}:
 0.0  0.0  0.0
 0.0  0.0  0.0
> Terminated with exit code 0.
```

If, it is desired that array indices start from some other values rather than 1, class OffsetArrays can be used. If a program originally written in C,C++,java or python array indices started from 1 so sometimes it is easier to use it that way instead of translating all the index variables form 0 to 1. Package OffsetArrays is used for this purpose.

```
import Pkg
Pkg.add("OffsetArrays")
```

```
 using OffsetArrays

n=size(A)[1]
m=size(A)[2]
for i=0:n-1
  for j=0:m-1
  y=10*i+j
  A[i,j]=y
  end
end
print("A= $A")
show(IOContext(stdout, :limit=>false), MIME"text/plain"(),A)
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" zeroarray_2.jl
A= [0.0 1.0 2.0 3.0; 10.0 11.0 12.0 13.0; 20.0 21.0 22.0 23.0]
3X4 OffsetArray(::
Matrix{Float64}, 0:2, 0:3) with eltype Float64 with indices 0:2X0:3:
  0.0   1.0   2.0   3.0
 10.0  11.0  12.0  13.0
 20.0  21.0  22.0  23.0
> Terminated with exit code 0.
```

```
function print_OA(A)
 n=size(A)[1]
 m=size(A)[2]
 for i=0:n-1
  for j=0:m-1
    y=A[i,j]
    print("($i,$j)$y ")
  end
  println(" ")
end
end

using OffsetArrays
A=OffsetArray{Float64}(undef,0:2,0:3)
n=size(A)[1]
m=size(A)[2]
for i=0:n-1
  for j=0:m-1
  y=10*i+j
  A[i,j]=y
  end
end
print_OA(A)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" zeroarray_2.jl
(0,0)0.0 (0,1)1.0 (0,2)2.0 (0,3)3.0
(1,0)10.0 (1,1)11.0 (1,2)12.0 (1,3)13.0
(2,0)20.0 (2,1)21.0 (2,2)22.0 (2,3)23.0

> Terminated with exit code 0.
```

1.3.7 PLOTTING DATA IN JULIA LANGUAGE

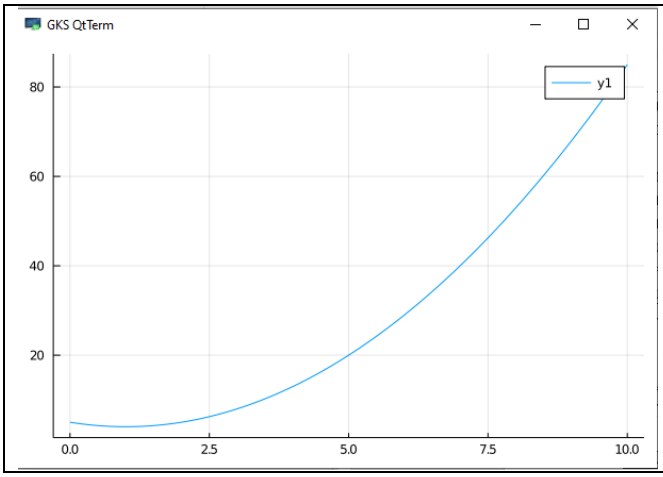
A rich plotting environment is existed in Julia. There are several packages to utilised.The basic Plot Packages are:
Plots
PyPlot
GR
PlotlyJS

In order to use a plot package it should be loaded up. For example, in order to add Plots package to julia use:

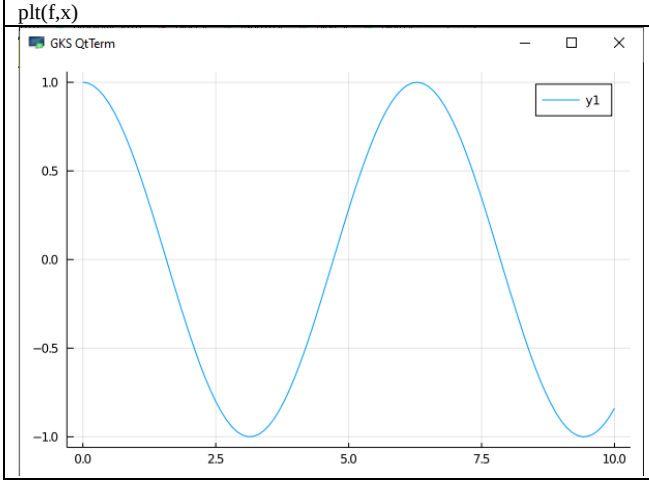
```
using Pkg
Pkg.add(“Plots”)
```

Julia> include("D:\\okul\\sco1\\Plot1.jl")

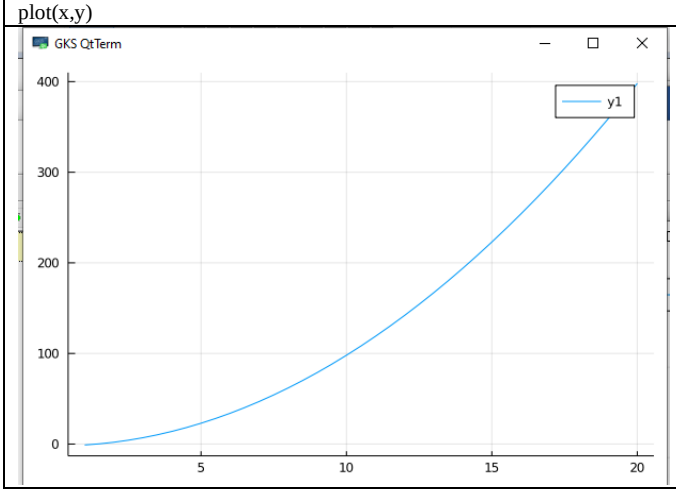
```
function plt(f,x0,x1,dx)
x=x0:dx:x1
y=map(f,x)
plot(x,y)
end

using Plots
f1(x)=x*x-2.0*x+5.0
plt(f1,0.0,10.0,0.1)
```



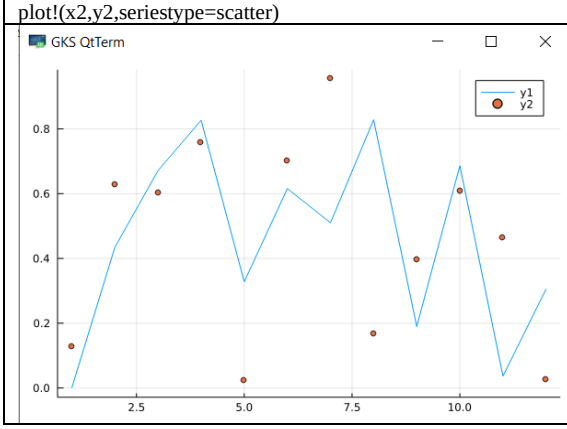

```
function plt(f,x)
y=map(f,x)
plot(x,y)
end

using Plots
f(x)=cos(x)
x=0:0.1:10
plt(f,x)
```



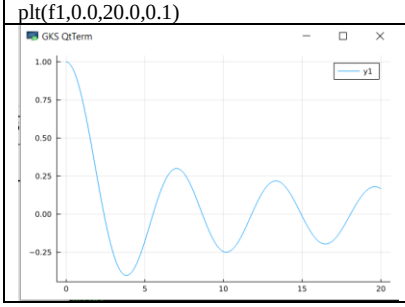

```
using Plots
f(x)=x*x-2.0
x = 1:0.5:20.0
y =map(f,x)
plot(x,y)
```



```
using Plots
x1 = collect(1:12)
y1 = rand(12)
x2 = collect(1:12)
y2= rand(12)
plot(x1,y1)
plot!(x2,y2,seriestype=scatter)
```

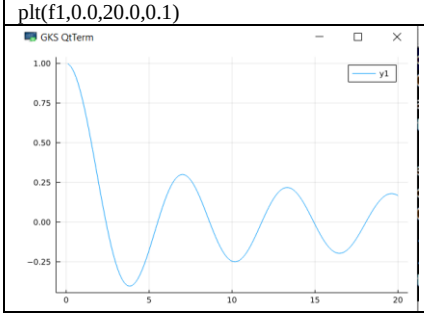



```
function plt(f,x0,x1,dx)
x=x0:dx:x1
y=map(f,x)
plot(x,y)
end

using Plots
using SpecialFunctions
f1(x)=besselj0(x)
plt(f1,0.0,20.0,0.1)
```

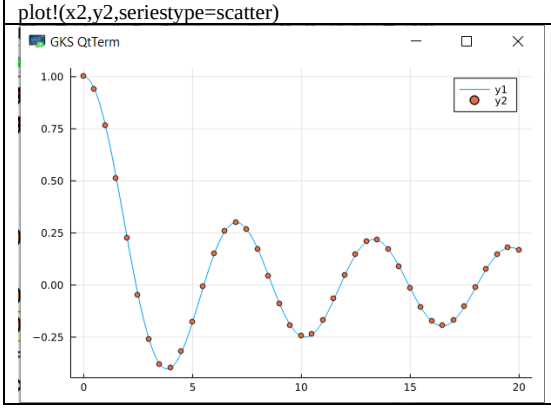



```
function plt(f,x0,x1,dx)
x=x0:dx:x1
y=map(f,x)
plot(x,y)
end
# Bessel function J
function J(v,x)
JJ=0
a=1.0
b=0.0
fact=1
if  x<=20
            x1=0.0
            x2=0.0
            x3=0.0
            x4=0.0
            x5=1
            lna=0.0
            x1=v*log(0.5*x);
            lna1=log(0.25*x*x);
            for k=1:100
                x2=lna
                x3=loggamma(k)
                x4=loggamma(v+k)
```

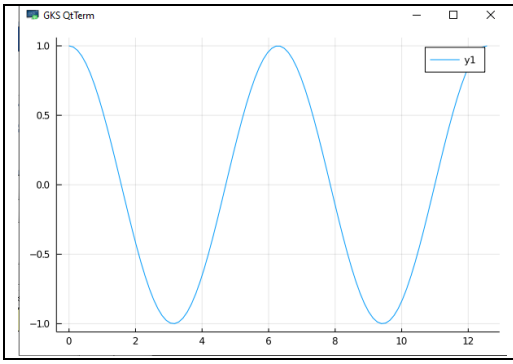
```
                    b=x1+x2-x3-x4;
                    b=x5*exp(b);
                    lna+=lna1;
                    x5*=-1;
                    JJ+=b;
            end
elseif v==0 && x==0
  JJ=1.0
else #x>30
  JJ=sqrt(2.0/pi/x)*cos(x-0.5*v*pi-0.25*pi)
end
return JJ;
end # function

using Plots
using SpecialFunctions
f1(x)=J(0,x)
plt(f1,0.0,20.0,0.1)
```



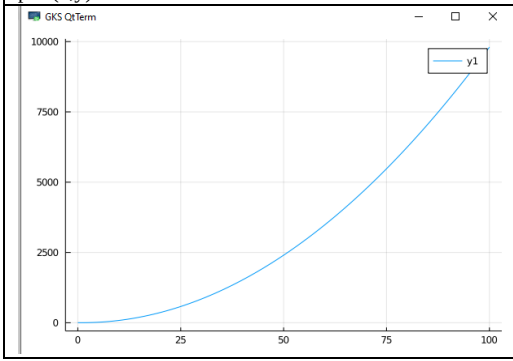

```
using Plots
using SpecialFunctions

x0=0.0
dx=0.1
x1=20.0
x=x0:dx:x1
dx1=0.5
x2=x0:dx1:x1
f(x)=besselj0(x)
y=map(f,x)
y2=map(f,x2)
plot(x,y)
plot!(x2,y2,seriestype=scatter)
```



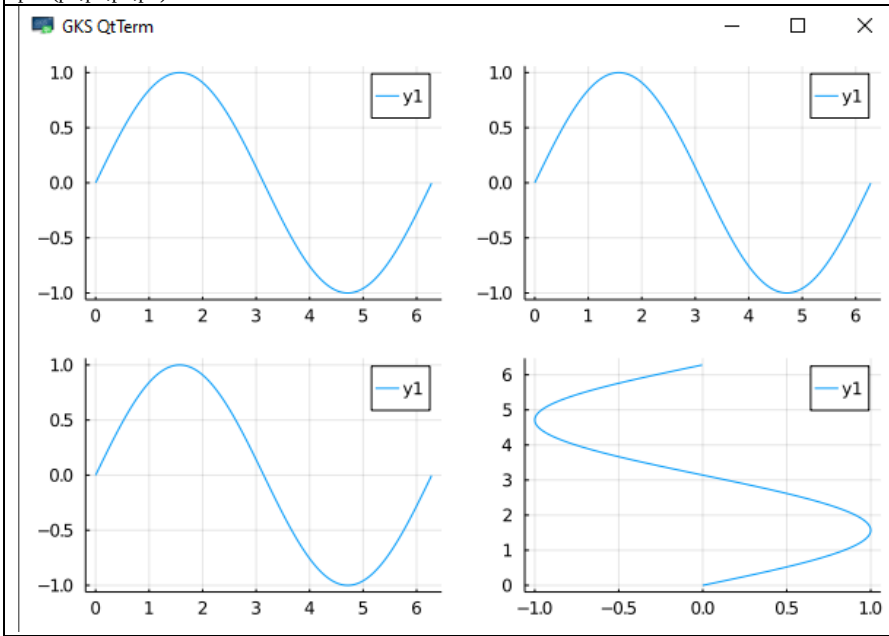

```
using Plots
m= 100
f(x)=cos(x)
x = fill(0.0, (m))
y = fill(0.0, (m))
dx=4.0*pi/(m-1)
for i=1:m
      x[i] = dx*(i-1)
      y[i] = f(x[i])
end
plot(x,y)
```



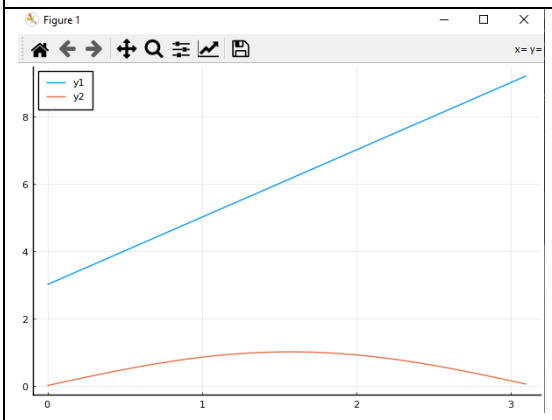

```
using Plots
f(x)=x*x-2.0*x+3.0
x = 0:0.1:100
y=map(f,x)
plot(x,y)
```



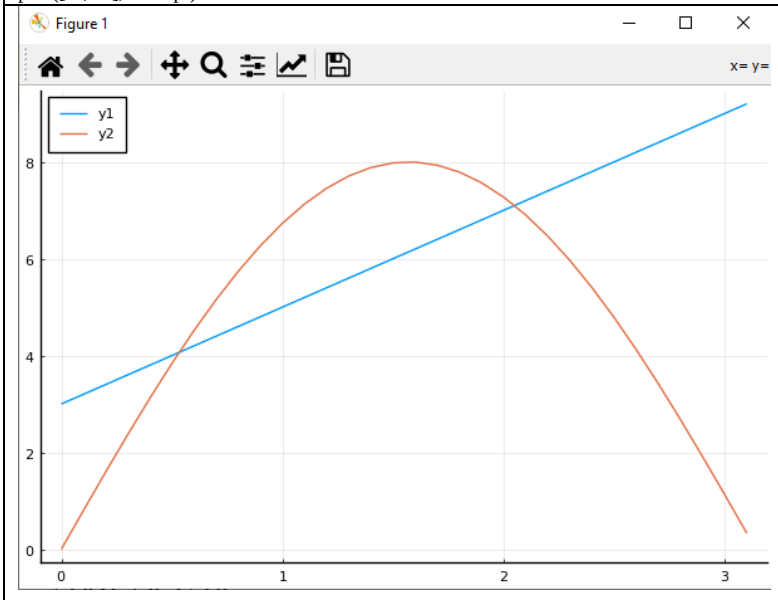

```
using Plots
p1 = plot(sin, 0:0.01:2*pi)
p2 = plot(0:0.01:2*pi, sin)
x = Array(0:0.01:2*pi);
y = sin.(x); # elementwise sin(x)
p3 = plot(x,y)
p4 = plot(y,x)
plot(p1,p2,p3,p4)
```



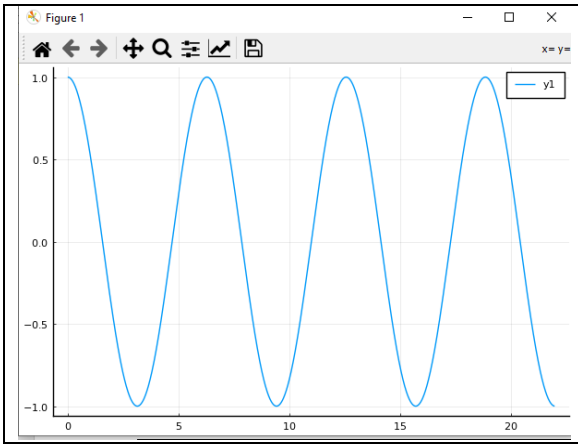
```
using Plots
f1=(x)->2.0*x+3.0
f2=(x)->sin(x)
plot([f1, f2],0:0.1:pi)
```



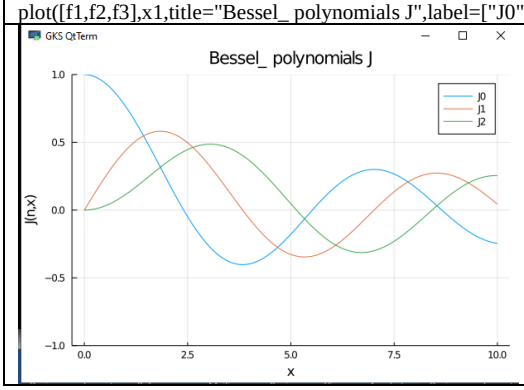

```
using Plots
f3(x)=2.0*x+3.0
f4(x)=8.0*sin(x)
plot([f3, f4],0:0.1:pi)
```



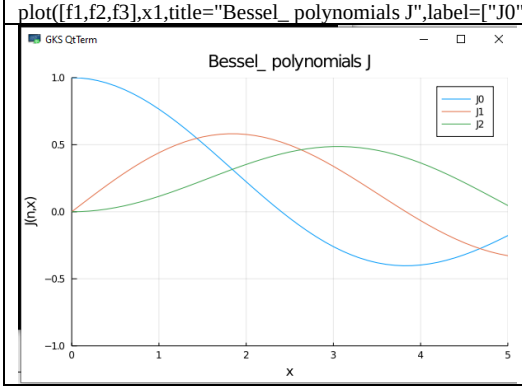

```
using Plots
x=0:pi/200:7*pi
y=cos.(x)
plot(x,y)
```




```
using SpecialFunctions
using Plots
x1=0.0:0.1:10.0
f1(x)=besselj(0,x)
f2(x)=besselj(1,x)
f3(x)=besselj(2,x)
plot([f1,f2,f3],x1,title="Bessel_ polynomials J",label=["J0" "J1" "J2"],ylims = (-1.0,1.0),xaxis = ("x"),yaxis = ("J(n,x)"))
```

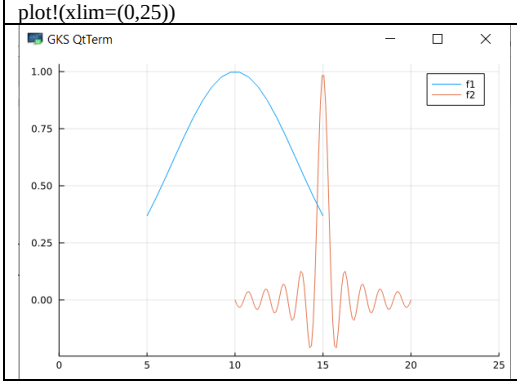



```
using SpecialFunctions
using Plots
x1=0.0:0.1:10.0
f1(x)=besselj(0,x)
f2(x)=besselj(1,x)
f3(x)=besselj(2,x)
plot([f1,f2,f3],x1,title="Bessel_ polynomials J",label=["J0" "J1" "J2"],ylims = (-1.0,1.0),xlims = (0.0,5.0),xaxis = ("x"),yaxis = ("J(n,x)"))
```




```
using Plots
x1 = range(5,15,length=20)
x2 = range(10,20,length=100)
f1(x)=exp(-(x-10)^2/5^2)
f2(x)=sin(2pi*(x-15))/(2pi*(x-15))
y1 = f1.(x1)
y2 = f2.(x2)
plot(x1,y1,label="f1")
plot!(x2,y2,label="f2")
```

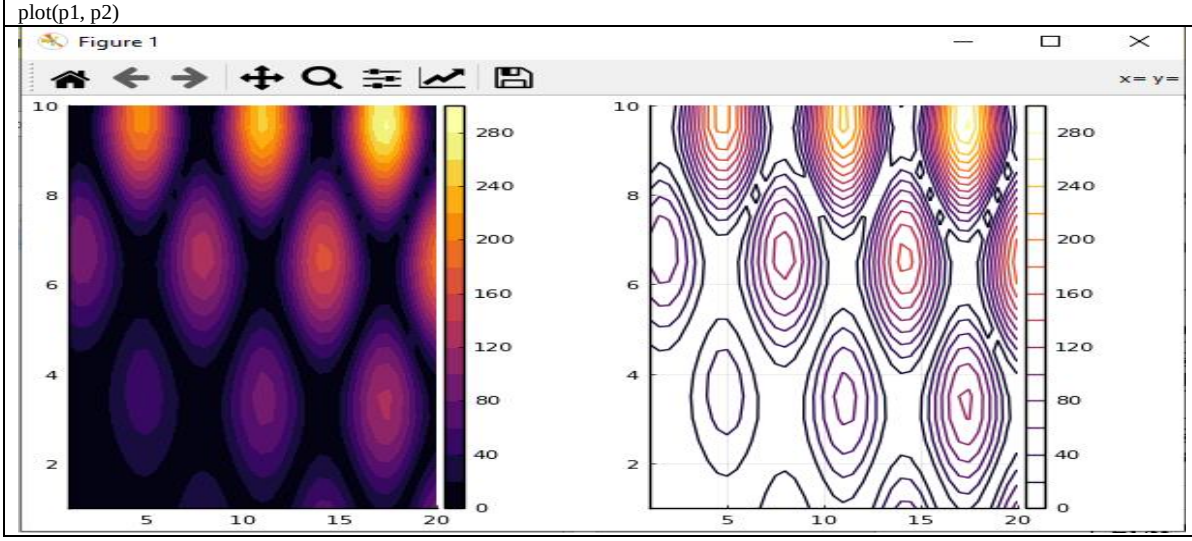
```
plot!(xlim=(0,25))
```



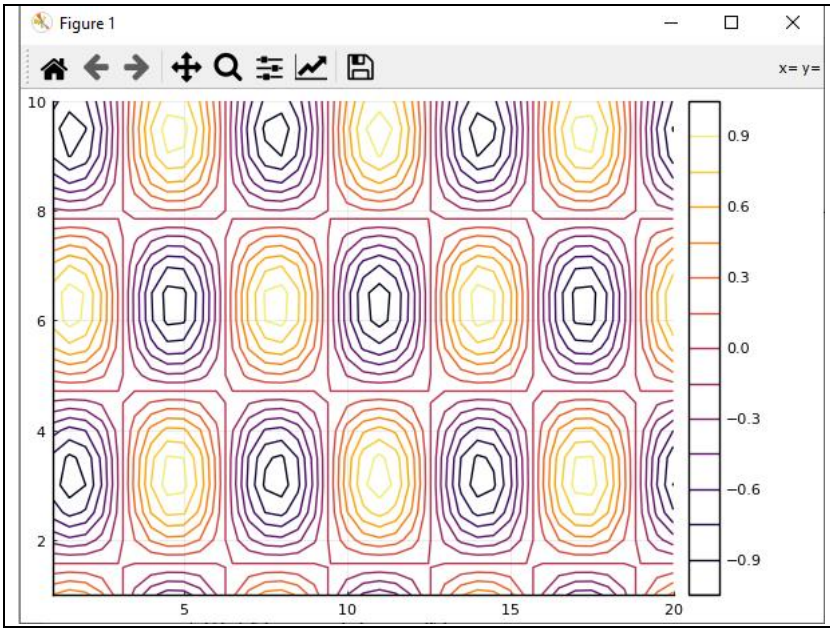
Contour Plot:

```
using Pkg
Pkg.add("PyPlot")
```

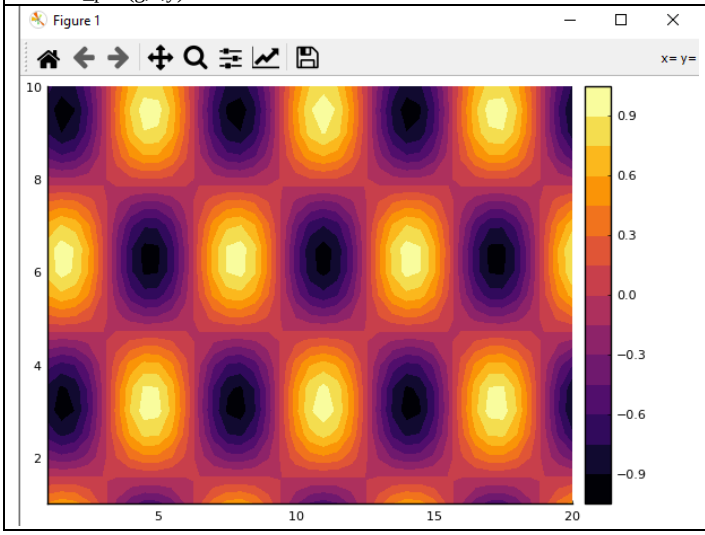
```
using Plots
Plots.pyplot() # back to pyplot
x = 1:0.5:20
y = 1:0.5:10
g(x, y) = begin
      (3x + y ^ 2) * abs(sin(x) + cos(y))
   end
X = repeat(reshape(x, 1, :), length(y), 1)
Y = repeat(y, 1, length(x))
Z = map(g, X, Y)
p1 = contour(x, y, g, fill=true)
p2 = contour(x, y, Z)
plot(p1, p2)
```




```
function contour_plt(f,x,y)
p1 = contour(x, y, f)
plot(p1)
end


using Plots
Plots.pyplot() # back to pyplot
x = 1:0.5:20
y = 1:0.5:10
g(x, y) = sin.(x)*cos.(y)
contour_plt(g,x,y)
```



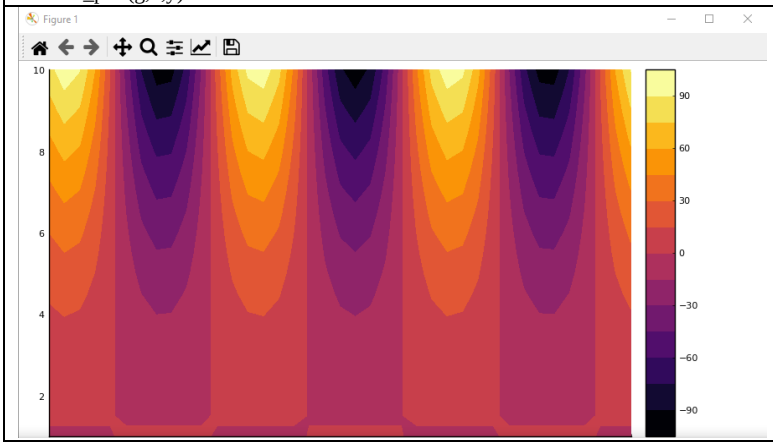

```
function contour_plt1(f,x,y)
p1 = contour(x, y, g,fill=true)
plot(p1)
end

using Plots
Plots.pyplot() # back to pyplot
x = 1:0.5:20
y = 1:0.5:10
g(x, y) = sin.(x)*cos.(y)
contour_plt1(g,x,y)
```

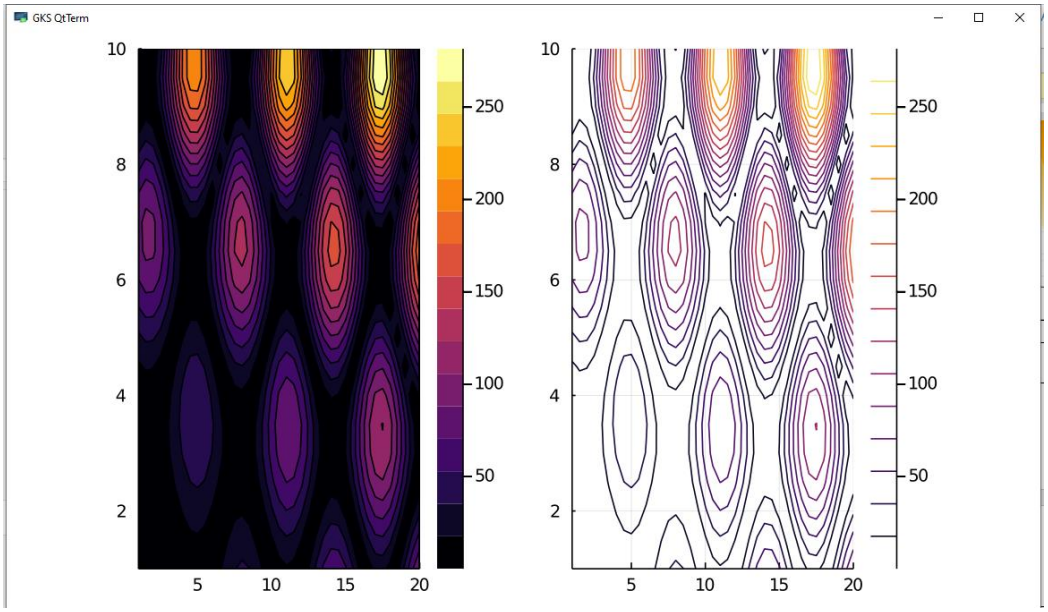


contourplot

```
function contour_plt1(f,x,y)

X = repeat(reshape(x, 1, :), length(y), 1)
Y = repeat(y, 1, length(x))
Z = map(f, X, Y)
p1 = contour(x, y, g,fill=true)
plot(p1)
end


using Plots
```

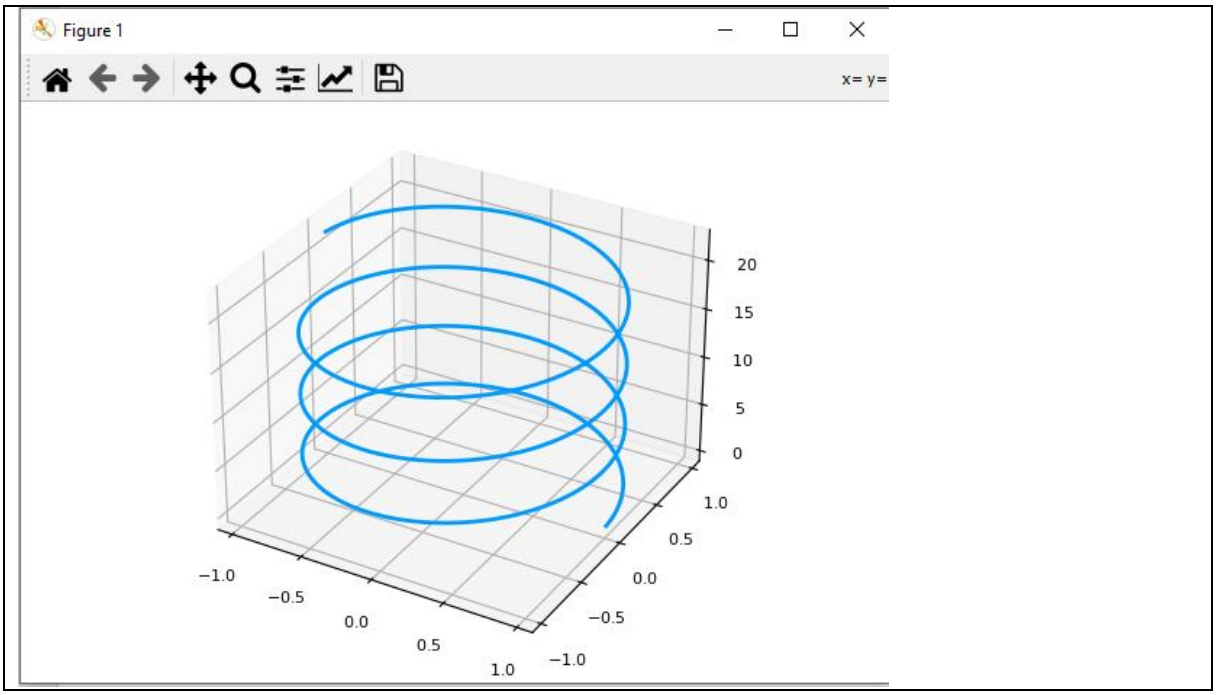
```
Plots.pyplot() # back to pyplot
x = 1:0.5:20
y = 1:0.5:10
f(x)=x*x-2.0/x
g(x, y) = sin.(x)*f.(y)
contour_plt1(g,x,y)
```





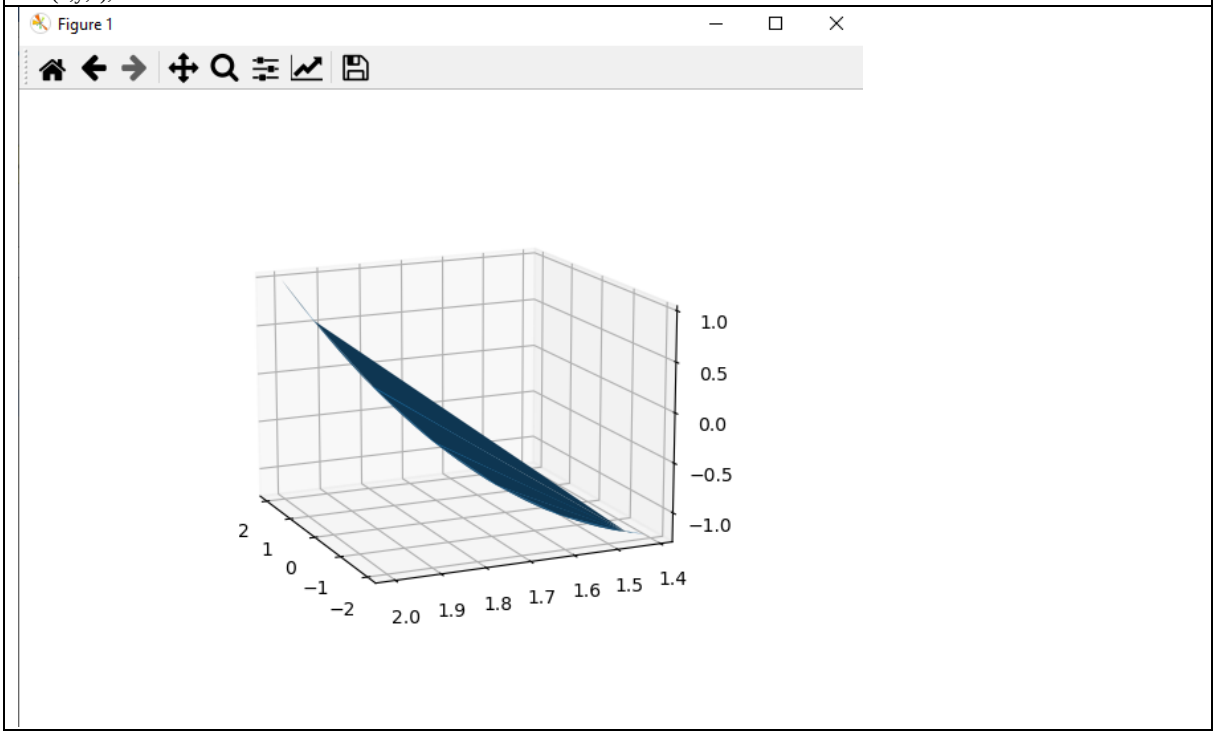
3D plot (3D line plot)

```
using Plots
t=0:pi/200:7*pi
x=cos.(t)
y=sin.(t)
plot3d(x,y,t,lw=2,leg=false)
```



Pyplot surf (3D surface plot)

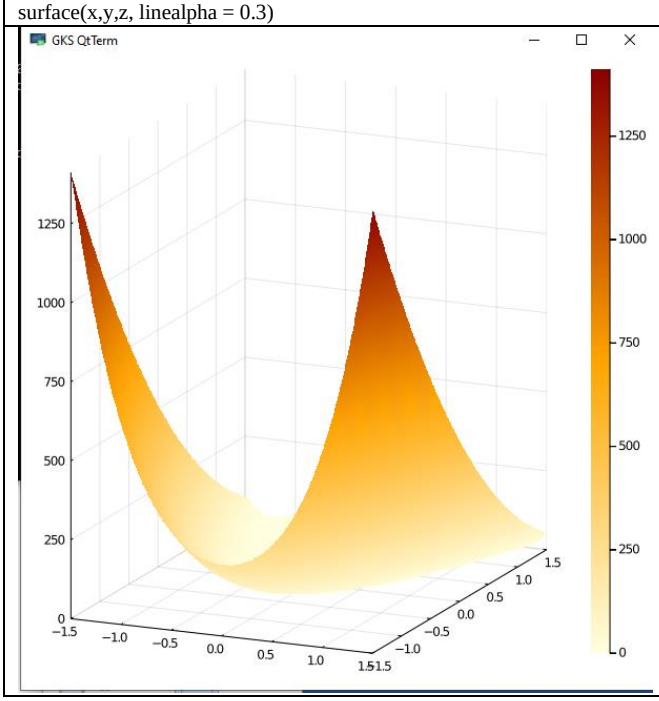
```
using PyPlot;
x = collect(Float16, range(-2,length=100,stop=2));
y = collect(Float16, range(sqrt(2),length=100, stop=2));
z = (x.*y).-y.-x.+1;
surf(x,y,z);
```



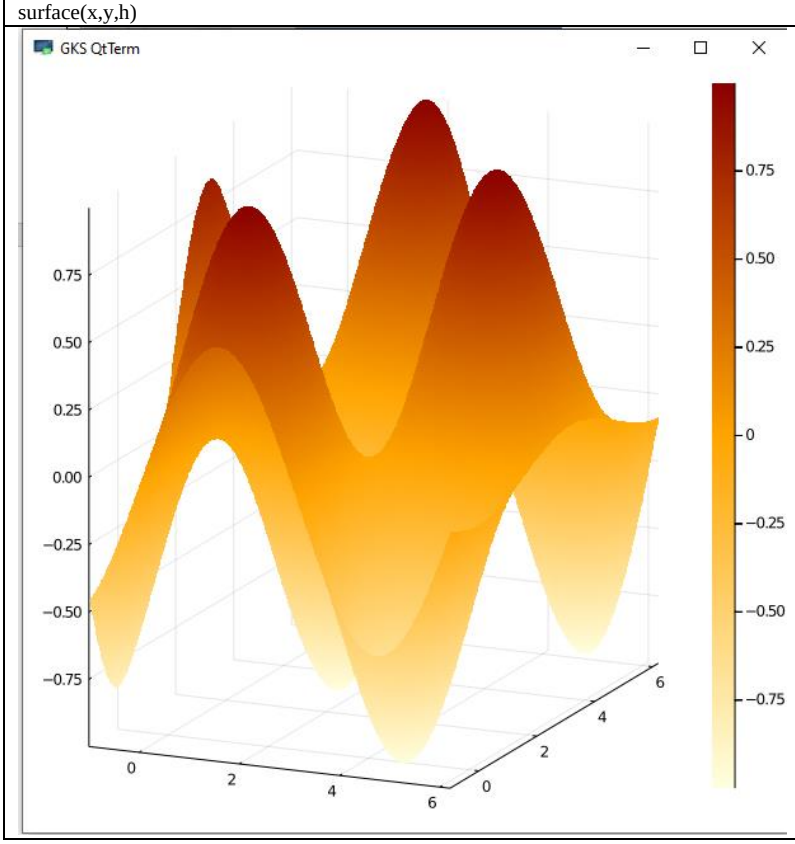
Plots surface (3D surface plot)

```
using Plots

function rosenbrock(x::Vector)
  return (1.0 - x[1])^2 + 100.0 * (x[2] - x[1]^2)^2
end
```

```
default(size=(600,600), fc=:heat)
x, y = -1.5:0.1:1.5, -1.5:0.1:1.5
z = Surface((x,y)->rosenbrock([x,y]), x, y)
surface(x,y,z, linealpha = 0.3)
```

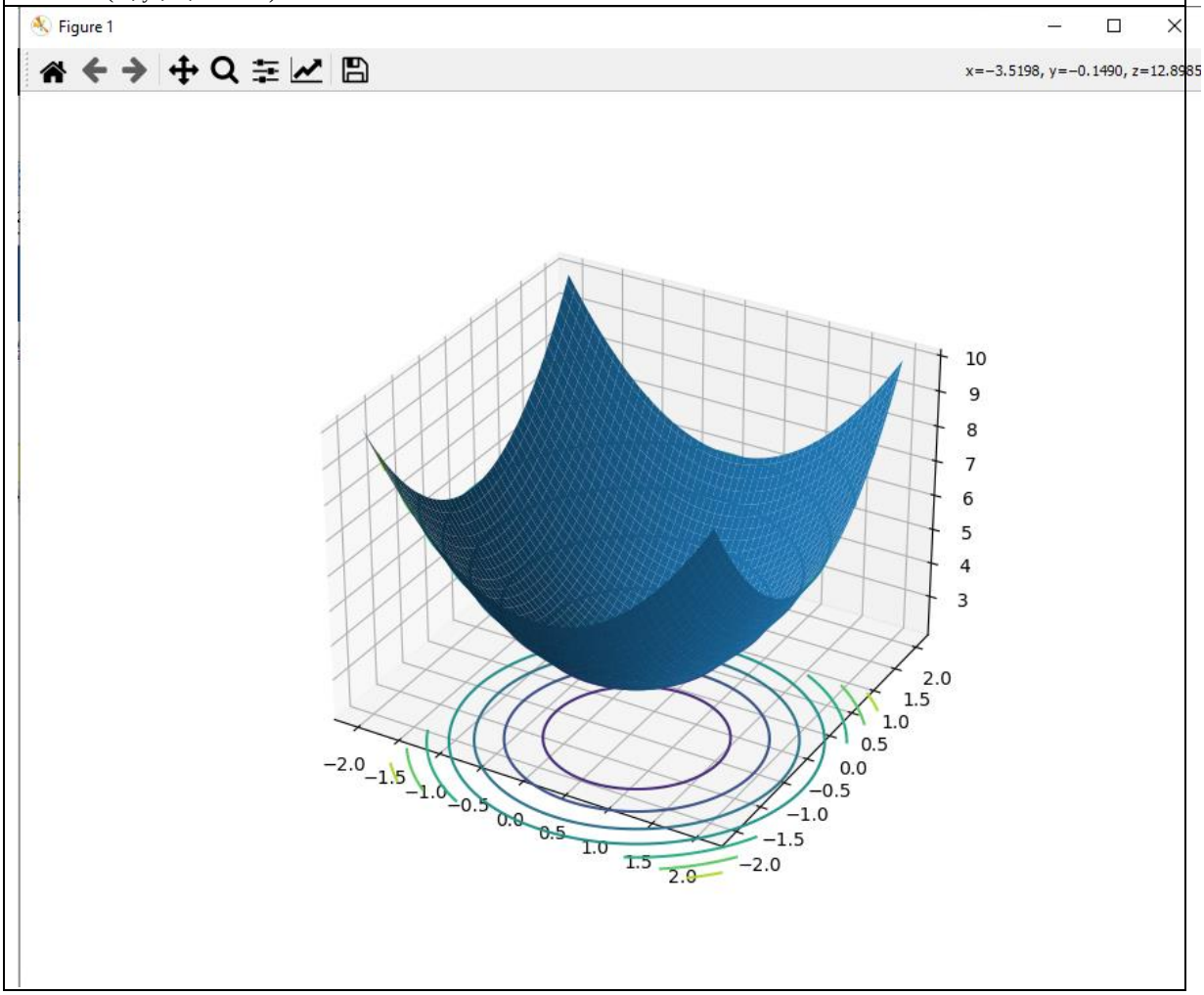



```
using Plots
x=-1:0.1:2.0*pi
y=-1:0.1:2.0*pi
h(x,y)=sin.(x)*cos.(y);

surface(x,y,h)
```

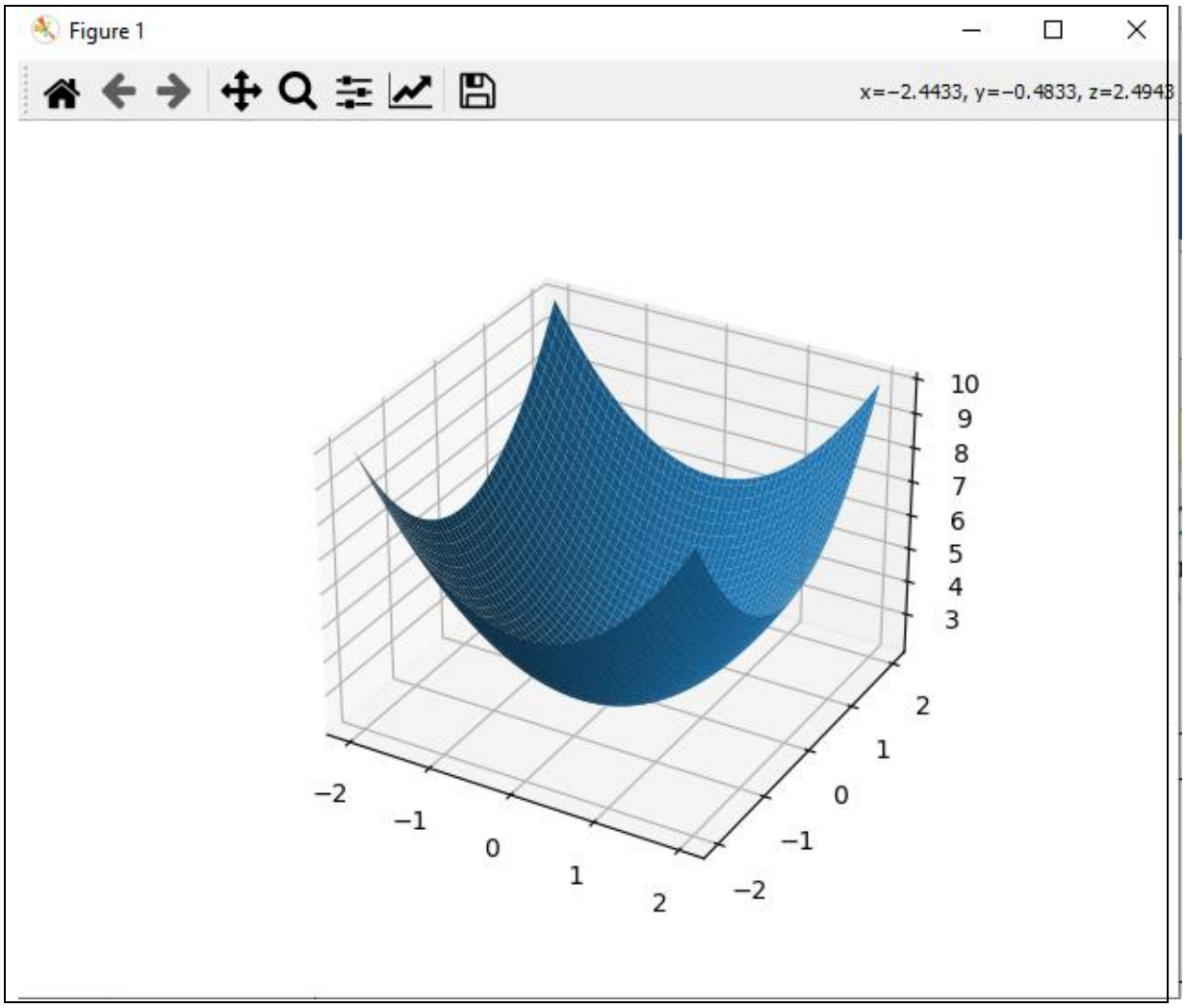



```
using PyPlot
```

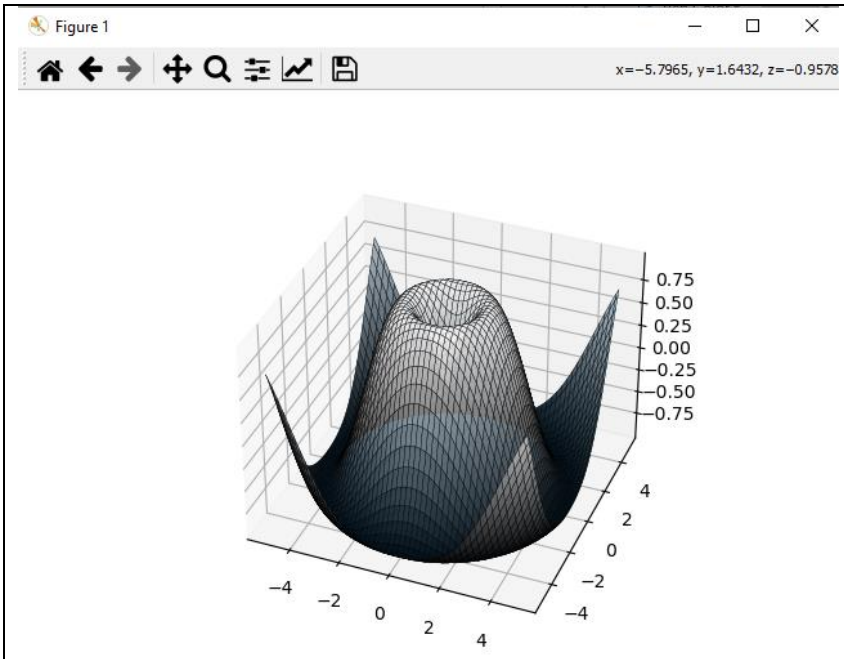
```
xs = ys = range(-2, stop=2, length=100)
f(x,y) = 2 + x^2 + y^2
zs = [f(x,y) for y in ys, x in xs]
PyPlot.plot_surface(xs, ys, zs)
PyPlot.contour3D(xs, ys, zs)
ax = PyPlot.gca()
ax.contour(xs, ys, zs, offset=0)
```



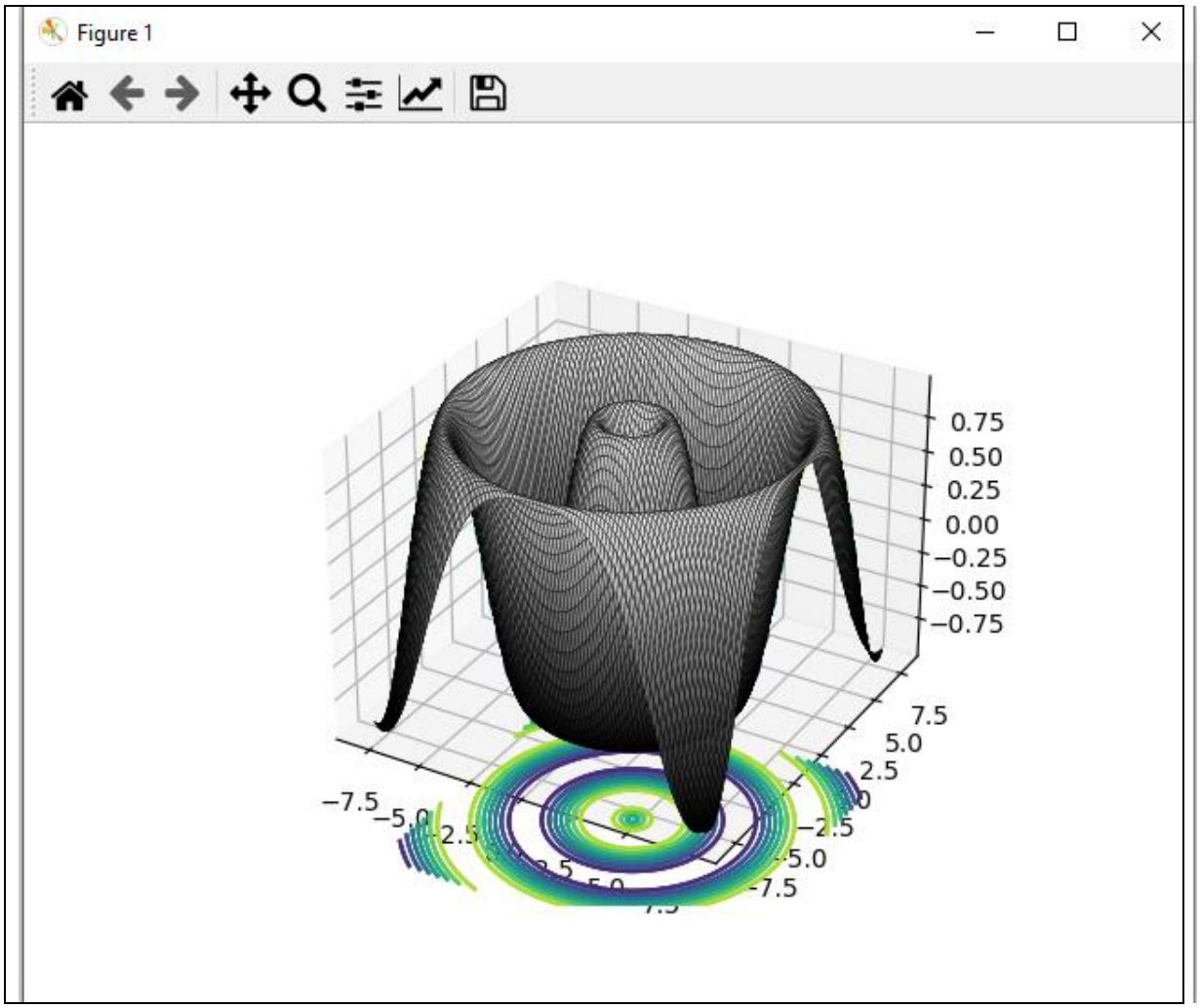

```
using PyPlot
xs = ys = range(-2, stop=2, length=100)
f(x,y) = 2 + x^2 + y^2
zs = [f(x,y) for y in ys, x in xs]
PyPlot.plot_surface(xs, ys, zs)
```




```
using PyPlot
xs = ys = range(-5, stop=5, length=100)
f(x,y) = sin(sqrt(x^2+y^2))
zs = [f(x,y) for y in ys, x in xs]
PyPlot.plot_surface(xs, ys, zs, rstride=2,edgecolors="k", cstride=2,
   cmap=ColorMap("gray"), alpha=0.8, linewidth=0.25)
```

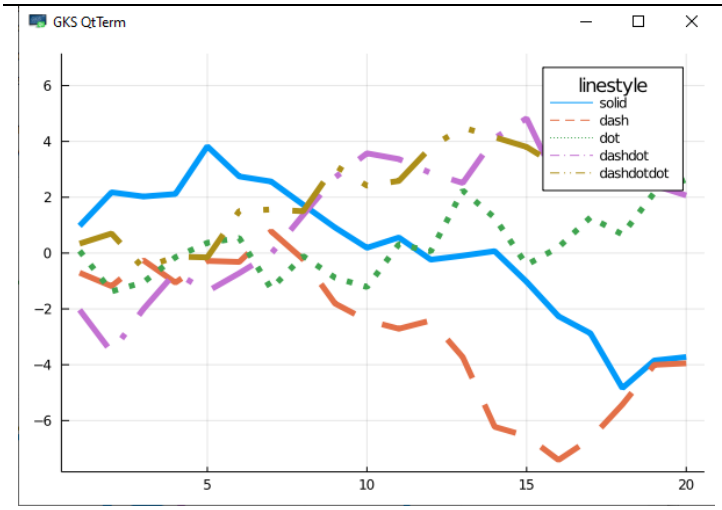



```
using PyPlot
xs = ys = range(-8, stop=8, length=50)
f(x,y) = sin(sqrt(x^2+y^2))
zs = [f(x,y) for y in ys, x in xs]
PyPlot.plot_surface(xs, ys, zs, rstride=2,edgecolors="k", cstride=2,
   cmap=ColorMap("gray"), alpha=0.8, linewidth=0.25)
 PyPlot.contour3D(xs, ys, zs)
ax = PyPlot.gca()
ax.contour(xs, ys, zs, offset=-2)
```

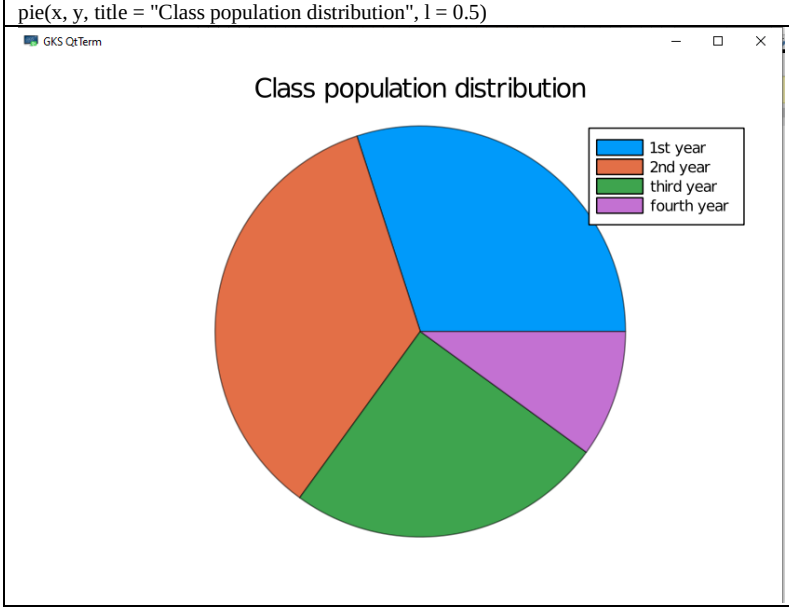


Multiline plot with GR

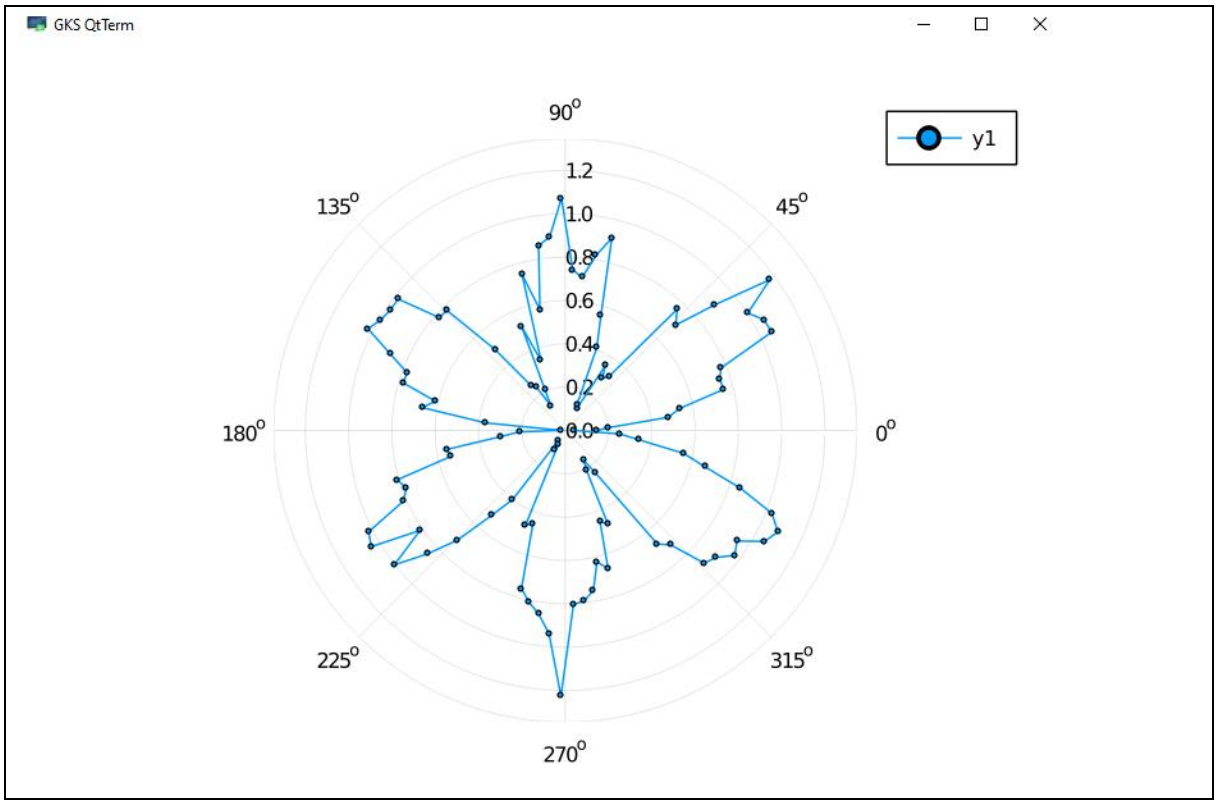
```
using Plots
gr()
styles = filter((s->begin
            s in Plots.supported_styles()
        end), [:solid, :dash, :dot, :dashdot, :dashdotdot])
styles = reshape(styles, 1, length(styles))
n = length(styles)
y = cumsum(randn(20, n), dims = 1)
plot(y, line = (5, styles), label = map(string, styles), legendtitle = "linestyle")
```




```
using Plots
gr()
x = ["1st year", "2nd year", "third year","fourth year"]
y = [0.3, 0.35, 0.25,0.1]
pie(x, y, title = "Class population distribution", l = 0.5)
```




```
using Plots
gr()
tetha = range(0, stop = 2.0*pi, length = 100)
r = abs.(0.1 * randn(100) + sin.(3*tetha))
plot(tetha, r, proj = :polar, m = 2)
```



1.3.8 READING AND WRITING DATA FROM A SEQUENTIAL FILE IN JULIA LANGUAGE

In order to read/write data into a sequaential file open statement will be used. This statement has the following form:

open(filename)

Where “filename” is the name of input/output file

```
open("test.txt") do f

  # line_number
  line = 0

  # read till end of file
  while ! eof(f)
    # read a new / next line for every iteration
    s = readline(f)
    line += 1
    println("$line . $s")
  end

end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" file1.jl
1 . once upon a time in a country far far a way
2 . there was a little princess

> Terminated with exit code 0.
```

```
julia> include("file1.jl")
1 . once upon a time in a country far far a way
2 . there was a little princess

julia>
```

Reading file b.txt (first two varibles as x,y in each line)

File b.txt

1.234 3.24 5.67 7.78
1.345 5.25 5.67 7.78
3.748 6.38 5.67 7.78
2.175 5.24 5.67 7.78
1.222 3.45 5.67 7.78

```
 function convert_to_float(column)
   new_column = map(x -> (
         x = ismissing(x) ? "" : x;  # no method matching tryparse(::Type{Float64}, ::Missing)
         x = tryparse(Float64, x);   # returns: Float64 or nothing
         isnothing(x) ? missing : x; # options: missing, or "", or 0.0, or nothing
         ), column)  # input
   # returns Array{Float64,1} OR Array{Union{Missing, Float64},1}
   return new_column
end

function readDouble(name)
  name="b.txt"
  open(name) do f
  # read till end of file
  m=0
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    global m1=m
    global n1=n
  end  #while
  close(f)
  end  #open
  global x=Matrix{Float64}(undef,n1,m1)
  open(name) do f
  # read till end of file
  m=0
  global z=Array{Float64}(undef,n1,0)
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    x1=convert_to_float(s)
    z=hcat(z,x1)
    global x=z
  end  #while
  close(f)
  end  #open
  #transpose matrix
  global y=x'
  return y
  end

function readOffsetDouble(name)
  y=readDouble(name)
  n=size(y)[1]
  m=size(y)[2]
  global x=OffsetArray(y, 0:n-1,0:m-1)
  return x
  end

using OffsetArrays
name="b.txt"
y=readDouble(name)
n=size(y)[1]
m=size(y)[2]
print("n = $n m = $m")
print("array indis starting from 1")
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), y)
print("array indis starting from 0")
x=readOffsetDouble(name)
```

```
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), x)
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" readouble.jl
n = 5 m = 4array indis starting from 1
5X4 adjoint(::Matrix{Float64}) with eltype Float64:
 1.234  3.24  5.67  1.0
 1.345  5.25  5.67  2.0
 3.748  6.38  5.67  3.0
 2.175  5.24  5.67  4.0
 1.222  3.45  5.67  5.0array indis starting from 0
5X4 OffsetArray(adjoint(::Matrix{Float64}), 0:4, 0:3) with eltype Float64 with indices 0:4X0:3:
 1.234  3.24  5.67  1.0
 1.345  5.25  5.67  2.0
 3.748  6.38  5.67  3.0
 2.175  5.24  5.67  4.0
 1.222  3.45  5.67  5.0
> Terminated with exit code 0.
```

Writing into a text file:

```
io = open("myfile.txt", "w")
write(io, "Hello world!")
close(io)
```

Text f,le myfile.txt contains:

```
Hello world!
```

1.2.1 TAYLOR SERIES

Taylor equation can be considered the roots of many numerical analysis methods. It will give us serial. **Taylor series** is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point. It is named after the English mathematician Brook Taylor. If the series is centered at zero, the series is also called a **Maclaurin series**, named after the Scottish mathematician Colin Maclaurin. It is common practice to use a finite number of terms of the series to approximate a function. Taylor series around a point a can be written as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f"(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Or if it is written in summation notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Or if a is replaces with x^k and x is replaced with x^{k+1} equation becomes

$$f(x^{k+1}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x^k)}{n!}(x^{k+1}-x^k)^n$$

+..

Computers could not calculate anything more complex than addition, the rest usually accomplish through Taylor series approximations. For example function $f(x)=e^x$ can be approximated as

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

Some of the formulation used in Numerical analysis is direct derivation from the Taylor series. For example for the Newton root finding equation, above linear terms of the equation is ignorede and function value is replaced with 0

$f(x^{k+1}) = 0 = f(x^k) + f'(x^k)(x^{k+1}-x^k)$ Will yield to

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$$

Taylor series can also be used to estimate error. The series usually used up to a certain term and error will be proportinal to one upper term in the series. For example If exponetial function is taken as

$f(x) = e^x = 1 + \mathrm{x} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

The error term will be proportional to the next term of the series

$$Error \cong \frac{x^5}{120}$$

2.2 SERİES SOLUTIONS OF FUNCTIONS

Digital computers can only calculate addition process, therefore most of the mathematical expressions are calculated by using series. Series solutions of a few functions will be listed in here.

28. $\ln(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1} \quad [0 < x]$
29. $\exp(\mathrm{x}) = \sum_{k=1}^{\infty} \frac{x^k}{k!}$
30. $\exp(-\mathrm{x}) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}$
31. $\exp(-\mathrm{x}^2) = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$
32. $\exp(\mathrm{x})(1+x) = \sum_{k=1}^{\infty} \frac{x^k(k+1)}{k!}$
33. $a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$
34. $\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x}$ [a>1 and x<0 or 0<a<1 x>0]
35. $\sin(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
36. $\sin h(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$
37. $\cos(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
38. $\cos\ \mathrm{h}(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$
39. $\tan(\mathrm{x}) = \frac{\sin(x)}{\cos(x)}$
40. $\tanh(\mathrm{x}) = \frac{\sinh(x)}{\cos\ \mathrm{h}(x)}$
41. $\arcsin(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \quad x^2 \leq 1$
42. $\mathrm{arcsinh}(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \quad x^2 \leq 1$
43. $\arctan(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)} \quad x^2 \leq 1$
44. $\mathrm{arctanh}(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)} \quad x^2 \leq 1$
45. $\mathrm{erf}(\mathrm{x}) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$
46. $\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{2}{8k+6}\right)$ Bailey, Borwein Plouffe(BBP) formula[90]
47. $J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\,\Gamma(n+k+1)}$ Bessel function
48. $\Gamma(z) = \int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}$ where
    $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant
49. $Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}$ Bessel function

Exponential function $\exp(x) = \sum_{k=1}^{\infty} \frac{x^k}{k!}$

```
function exp(x)
    total=0;
    factorial=1;
    power=1;
    for i =1:50
      total=total+power/factorial;
      power=power*x
      factorial=factorial*i
      end
    return total
    end
x=1.0
r=exp(x);
print("exp($x) = $r");
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" exp.jl
exp(1.0) = 2.7182818284590455
> Terminated with exit code 0.
```

Gamma function:

```
function gamma1(z)
  g=0.57721566490153286
  m=1.0
  for n=1:1000
    z1=z/n
    m=m/(1.0+z1)*exp(z1)
  end
  return exp(-g*z)/z*m
  end

using SpecialFunctions
z=0.1
g1=gamma1(z)
g2=gamma(z)
print("gamma1 = $g1 gamma = $g2")
```

```
---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" gamma1.jl
gamma1 = 9.513460156608664 gamma = 9.51350769866873
> Terminated with exit code 0.
```

Natural logarithm $\ln(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1} \quad [0 < x]$

```
function ln(x)
   total=0;
   y=(x-1.0)/(x+1.0);
   power=y;
   for k =1:100
      total=total+power/(2.0*k-1.0);
      power=power*y*y
      end
   return 2.0*total
   end
e=exp(1.0);
r=ln(e);
print("ln($e) = $r");
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" ln.jl
ln(2.718281828459045) = 0.9999999999999997
> Terminated with exit code 0.
```

ln(x)

k	x=	2,7182818285
	y=(x-1)/(x+1)	0,4621171573
	power	total
0	0,4621171573	0
1	0,0986861666	0,9242343145
2	0,0210746546	0,9900250922
3	0,0045005403	0,9984549541
4	0,0009611006	0,9997408227
5	0,0002052452	0,9999544006
6	4,38306E-005	0,9999917179
7	9,36012E-006	0,9999984611
8	1,99887E-006	0,9999997091
9	4,26864E-007	0,9999999443
10	9,11578E-008	0,9999999892
11	1,94670E-008	0,9999999979
12	4,15721E-009	0,9999999996
13	8,87782E-010	0,9999999999
14	1,89588E-010	1
15	4,04869E-011	1
16	8,64608E-012	1
17	1,84639E-012	1
18	3,94301E-013	1
19	8,42038E-014	1
20	1,79819E-014	1

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

```
function sin1(x)
   total=0.0
   factorial=1.0
   power=x;
   plusminus=1
   i=0;
   for k=1:80
      total=total+power/factorial*plusminus;
      power=power*x*x;
      i=2*k+1;
      factorial=factorial*(i-1)*i;
      plusminus*=-1;
      end
   return total
   end

x=pi/3.0;
y=sin(x)
y1=sin1(x)
print("sin=$y1 math library sin = $y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" sin1.jl
sin=0.8660254037844385 math library sin = 0.8660254037844386
> Terminated with exit code 0.
```

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \le 1$$

```
function asin1(x)
   total=0.0
   factorial1=1.0
   factorial2=1.0
   power=x;
   power2=1.0;
   i=0;
   for k=1:80
      total=total+power*factorial1/(power2*factorial2*factorial2)/(2.0*(k-1)+1.0);
      power*=x*x;
      power2*=4;
      i=2*k;
      factorial1=factorial1*(i-1)*i
      factorial2=factorial2*k
```

```
        end
    return total
    end

x1=pi/3.0
x2=sin(x1);
y2=asin1(x2)
y3=asin(x2)
print("asin =$y2 math.asin = $y3")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" asin.jl
asin =1.047197551196467 math.asin = 1.0471975511965976
> Terminated with exit code 0.
```

$$a^x = e^{xln(a)}$$

```
function exp1(x)
      total=0.0;
      factorial=1.0;
      power=1.0;
      for i =1:100
        total=total+power/factorial;
        power=power*x
        factorial=factorial*i
        end
      return total
      end

function ln1(x)
    total=0.0;
    y=(x-1.0)/(x+1.0);
    power=y;
    for k =1:100
        total=total+power/(2.0*k-1.0);
        power=power*y*y
        end
    return 2.0*total
    end

function pow1(x,a)
  y=x*ln1(a)
  return exp1(x*ln1(a))
  end

x=2.0
a=3.0
r=pow1(x,a)
print("pow($x,$a) = $r")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" pow.jl
pow(2.0,3.0) = 8.999999999999986
> Terminated with exit code 0.
```

$$\text{erf(x)} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

```
function erf1(x)
   A=2.0/sqrt(pi)
   total=0.0
   plusminus=1
   power=x
   factorial=1.0
```

```
  for n=1:200
      total=total+power*plusminus/factorial/(2.0*(n-1)+1.0)
      power=power*x*x
      factorial=factorial*n
      plusminus*=plusminus*(-1)
      end
  return A*total
  end

using SpecialFunctions
x=0.1
y1=erf1(x)
y=erf(x)
print("x=$x,y1=$y y=$y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" erf.jl
x=0.1,y1=0.1124629160182849 y=0.1124629160182849
> Terminated with exit code 0.
```

Some of the algorithms might not be as clear as a directly given function. For example let us look at the **Russian peasant multiplication problem:**

Take two integer for example 21 and 37 then double the first one(left hand side) and divide by two the second one(right side) ignore remaining in division process and stop when the right hand side reached to 1 and then cross each line where the right side number is even add up remaining left hand side numbers for example. If the given integer numbers to be multiplied is 21 and 17, the process will follow the path given below:

21 37
~~42 18~~ cross out (18 is even)
84 9
~~168 4~~ cross out(4 is even)
~~336 2~~ cross out(2 is even)
672 1
then 21+84+672=777
21*37=**777**

Now let us convert this to an algorithmic structure:

```
function  isOdd(x)
  return bool(x %2 != 0 )
  end

function multiply(x1,x2)
  y2=x2;
  y1=x1;
  b2=0;
  total=0;
  while y2>=1
    b2=(y2 %2 != 0 )
    if y2 %2 != 0
      total=total+y1
    end
    println("total = $total, y1= $y1,y2= $y2")
    y1=y1*2;
    y2=Int64(floor(y2/2))
    end
  return total
  end

x1=21
x2=37
x=multiply(x1,x2)
print("russian multiplication $x1*$x2 = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" russian.jl
total = 21, y1= 21,y2= 37
total = 21, y1= 42,y2= 18
total = 105, y1= 84,y2= 9
total = 105, y1= 168,y2= 4
total = 105, y1= 336,y2= 2
```

```
total = 777, y1= 672,y2= 1
russian multiplication 21*37 = 777
> Terminated with exit code 0.
```

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\,\Gamma(n+k+1)}$$ Bessel function

$$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1} e^{-t} dt = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$ gamma function

$\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant

```
using SpecialFunctions
function gammax(z)
  g=0.57721566490153286
  mul=1.0
  zn=1.0
  for n=1.0:1000
    zn=z/n
    mul*=exp(zn)/(1.0+zn)
  end
  mul*=exp(-g*z)/z
  return mul
end

function J(n,x)
  minus=1
  y=x/2.0
  y2=y*y
  yy=y^n
  total=0.0
  fact=1.0
  for k=1:200
    total+=minus*yy/fact/gamma(n+k)
    fact*=k
    yy*=y2
    minus*=-1
  end
  return total
end

n=0.0
z=0.1
gamma1=gamma(z)
gamma2=gammax(z)
print("gamma1=$gamma1 gamma2=$gamma2")
J1=J(n,z)
J2=besselj(n,z)
print("J1=$J1 J2=$J2")
```

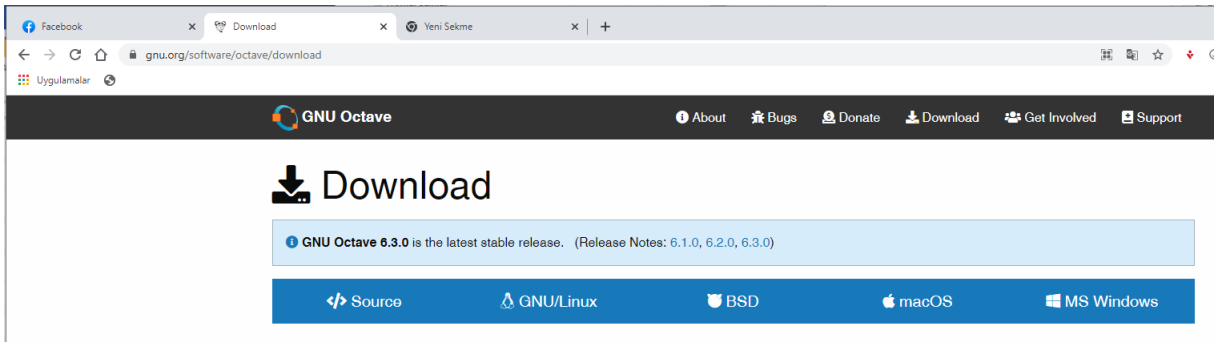
```
---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" bessel.jl
gamma1=9.51350769866873 gamma2=9.513460156608746
J1=0.9975015620660401 J2=0.99750156206604
> Terminated with exit code 0.
```

1.4 Octave-matlab language basics

Octave and Mathlab are quite smilar systems. The specification listed in here is in Octave environment small differences may exist between excel and Matlab, such as comment statement % and #, user should check and correct such differences. Yo can obtain octave from:

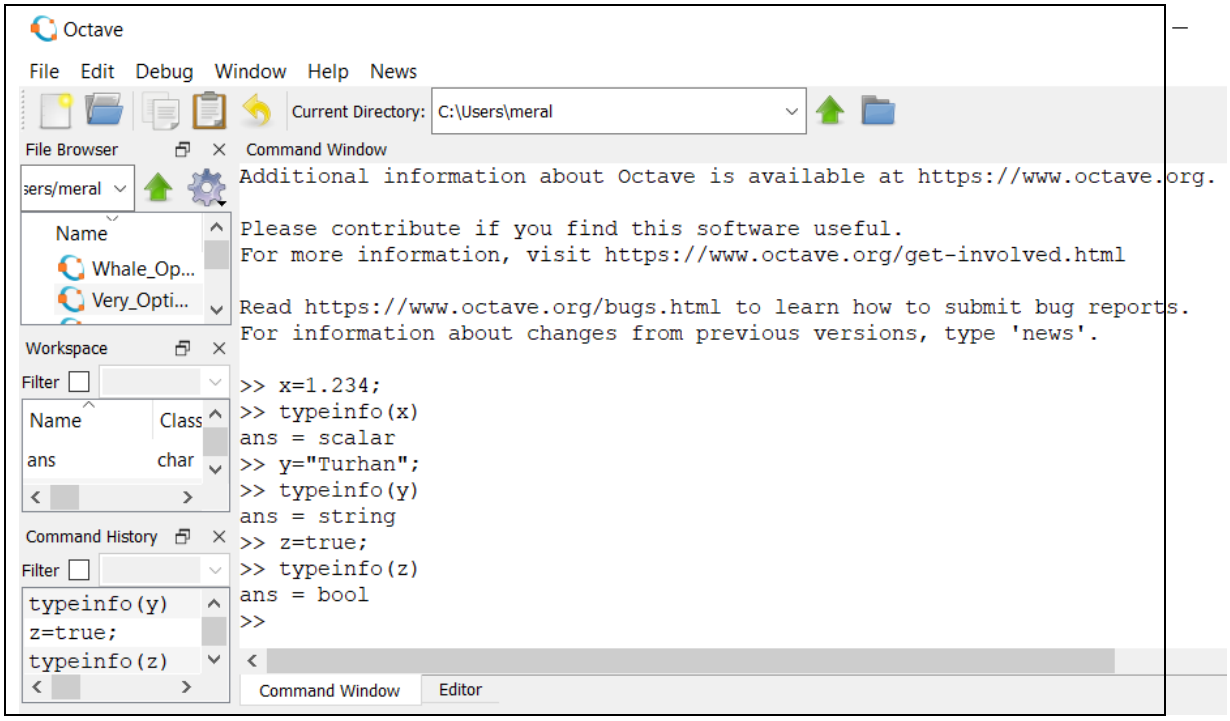
https://www.gnu.org/software/octave/download

adress



Matlab is a commercial package, you shoul purchase it in order to use it.

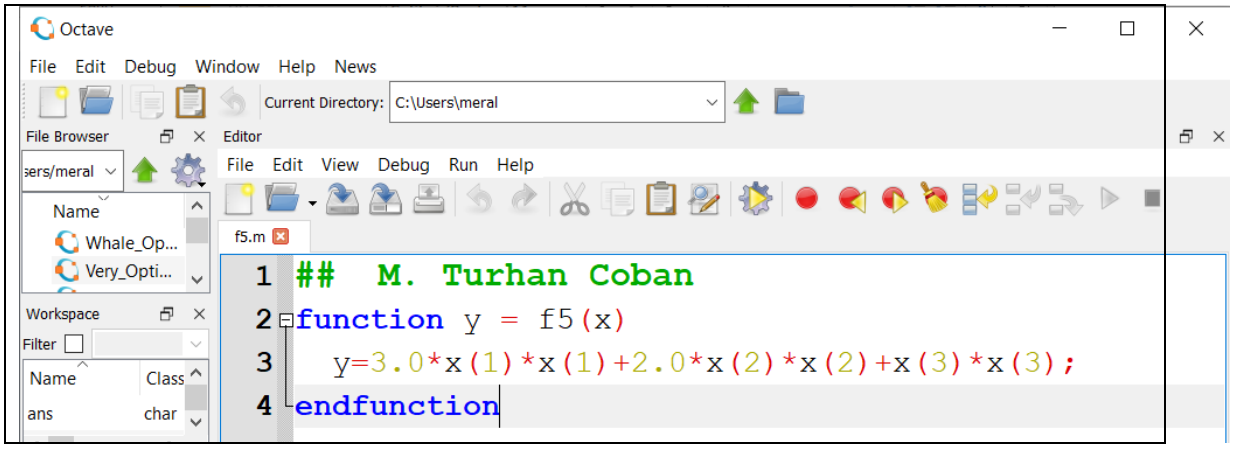
In matlab, like python and Julia, variable types is not have to be defined. Variables are assigned according to actual value is given for the variable. Variable type can be investigated and asked. Matlab-Octave environment consist of a command window and and editor window. You can write comment for calculations and run them to see the results in comment window. Or you can write your functions or commands in editor and then run them in command window.

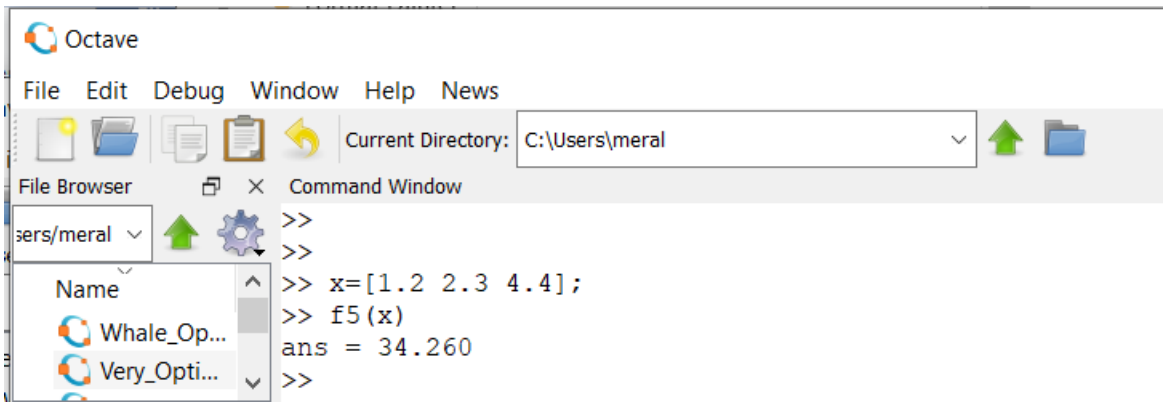



```
>> x=1.23456
```

```
x = 1.2346
>> whos x
Variables visible from the current scope:

variables in scope: top scope

  Attr Name        Size                     Bytes  Class
  ==== ====        ====                     =====  =====
       x           1x1                          8  double

Total is 1 element using 8 bytes
```

```
>> x=[1.2 2.3 4.4];
>> whos x
Variables visible from the current scope:

variables in scope: top scope

  Attr Name        Size                     Bytes  Class
  ==== ====        ====                     =====  =====
       x           1x3                         24  double

Total is 3 elements using 24 bytes
```





Function can also be defined as inline functions

```
>> y=@(x)x*x-2.0;
>> x=1.234;
>> y(x)
ans = -0.4772
>>
```

```
>> y=@(x)3.0*x(1)*x(1)+2.0*x(2)*x(2)+x(3)*x(3);
>> x=[1.1 2.34 4.123];
>> y(x)
ans = 31.580
>>
```

İn a function more than one variable can be returned

```
function [r,theta] = polarcoord (x,y)
  r=sqrt(x^2+y^2);
  theta=atan2(y,x);
endfunction
```

```
>> [r,theta]=polarcoord(3,4)
r = 5
theta = 0.9273
>>
```

Complex numbers and complex processes can also be defined directly in octave-matlab environment

```
>> x=1+1i
x =  1 + 1i
>> y=2+3i
y =  2 + 3i
>> z=x/y
z =  0.384615 - 0.076923i
>>
```

Logicals and if statement

Relational operators compare the elements in two arrays and return logical true or false values to indicate where the relation holds.

==	equality
>=	Greater than or equal to
<=	Less then or equal to
<	Less then
~= (matlab) != octave	Not equal
isequal	Array equality
isequaln	Array equality determine NaN values as equal

```
function retval = boolean1 (input1, input2)
  if input1==input2
    retval="variables are equal";
  elseif input1>=input2
    retval="variables greater then or  equal to";
  elseif input1>input2
    retval="variables greater then";
  elseif input1<=input2
    retval="variables smaller then or  equal to";
  elseif input1<input2
    retval="variables smaller then";
  else
    retval="variables are not equal";
  endif
endfunction
```

```
>> x=2
x = 2
>> y=3
y = 3
>> boolean1(x,y)
ans = variables smaller then or  equal to
```

For and while statements

For and while statemets supply continious process (loops) with some logical statements to end it

```
function total = for1 (x)
  n=length(x);
  total=0.0;
  for i=1:n
    total=total+x(i);
  endfor
endfunction
```

```
>> x=[1.2 2.3 3.3 4.12 5.25 6.2 7.7];
>> y=for1(x)
y = 30.070
>>
```

```
function total = while1 (x)
  n=length(x);
  total=0.0;
  i=1;
  while i<=n
    total=total+x(i);
    i++;
  endwhile
endfunction
```

```
>> x=[1.2 2.3 3.3 4.12 5.25 6.2 7.7];
>> y=while1(x)
y = 30.070
>>
```

Definition of a function

Function is basic definition in mathematics. Mathlab & Octave language is function based programming language . Functions are written an editor and directly called in octave command window.
f(n) $= sum1(n) = \sum_{k=1}^{n} k = 1 + 2 + .. + n$ can be approximated in computer environment as

```
function total = sum1 (n)
  total=0;
  for k=1:n
     total=total+k;
   endfor
end
```

```
>> sum1(100)
ans =  5050
>>
```

In this structure sum1 is the name of function, total is the output variable, for is a repeatableloop changing the value of k from 1 to n,and in each step adding k to total. Endfor statement is ending for loop. In the window below (command window) function is called. The same calculation can also be achieved in mathlab(octave) command window as:

```
>> n=100;
>> k=1:n;
>> total=sum(k)
total =  5050
>>
```

In here **sum** is also a function. They are called built in functions of Matlab.

f(n) $= average(n) = \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n}(1 + 2 + .. + n)$ can be approximated in computer environment as

average(x) function series solution, octave(matlab) Version

```
function total = average(n)
  total=0;
            for k=1:n
   total=total+k;
    endfor
    total=total/n;
end
```

```
>> average(100)
ans =  50.500
>>
```

The same can be achived by using buit in function mean

```
>> n=100;
>> k=1:n;
>> average=mean(k)
average =  50.500
>>
```

The function f(x) = exp(x) $= \sum_{k=1}^{\infty} \frac{x^k}{k!}$ Or in open form:

f(x) $= e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$ can be approximated in computer environment as

```
function total=exp1(x)
    total=0;
    factorial=1;
    power=1;
    for i=1:100
      total=total+power/factorial;
      power=power*x;
      factorial=factorial*i;
    endfor
endfunction
```

```
>> exp1(1)
ans =  2.71828182845905
>>
```

exp(x) function series solution, OpenOffice Calc (excel) version

	x=	1	
n	x^n	n!	total
0	1	1	1
1	1	1	2
2	1	2	2,5
3	1	6	2,66666667
4	1	24	2,70833333
5	1	120	2,71666667
6	1	720	2,71805556
7	1	5040	2,71825397
8	1	40320	2,71827877
9	1	362880	2,71828153
10	1	3628800	2,7182818
11	1	39916800	2,71828183
12	1	479001600	2,71828183
13	1	6227020800	2,71828183
14	1	8,718E+010	2,71828183

The mathlab built in function for exponent (exp) is existed

```
>> exp(1)
ans =  2.7183
>>
```

function f(x)=ln(x) can be approximated as $\mathbf{ln}(\boldsymbol{x}) = \mathbf{2}\sum_{\boldsymbol{k=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{2k-1}} \left(\frac{\boldsymbol{x-1}}{\boldsymbol{x+1}}\right)^{\mathbf{2k-1}}$ $[\mathbf{0} < \boldsymbol{x}]$

```
function total=ln(x)
    total=0;
    y=(x-1.0)/(x+1.0);
    power=y;
    for k=1:100
      total=total+power/(2.0*k-1.0);
      power=power*y*y;
    endfor
    total=total*2;
endfunction
```

```
>> ln(e)
ans =  1.000000000000000
>>
```

Built in function log is existed in Matlab(Octave)

```
>> format long
```

```
>> x=exp(1)
x =  2.718281828459045
>> y=log(x)
y =  1
```

Some of the functions are defined as combination of more than one function (or series solution). For example power function is defined as:

$$f(x) = a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$$

$f(x) = a^x$ function series solution,Octave (Matlab) version

```
function total=pow(a,x)
    total=exp1(x*ln(a));
endfunction
```

```
>> pow(2,3)
ans =  8.00000000000000
>>
```

Matlab(Octave) can directly process power function as

```
>> 2^3
ans =  8
>>
```

Some of the series solutions might not be as clear as a directly given function. For example let us look at the

Russian peasant multiplication problem:

Take two integer for example 21 and 37 then double the first one(left hand side) and divide by two the second one(right side) ignore remaining in division process and stop when the right hand side reached to 1 and then cross each line where the right side number is even add up remaining left hand side numbers for example. If the given integer numbers to be multiplied is 21 and 17, the process will follow the path given below:

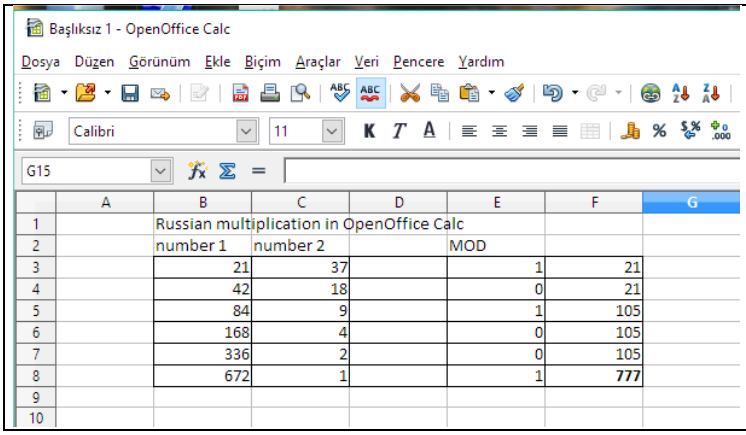
Russian peasant multiplication, Octave(Matlab) version

```
function total=russian(x1,x2)
    y2=x2;
    y1=x1;
    total=0;
    while y2>=1
     if mod(y2,2)!=0
     total=total+y1;
     endif
     y1=y1*2;y2=fix(y2/2);
    endwhile
endfunction
```

```
>> russian(21,37)
ans =  777
>>
```

Sorry guys there are no built in funct5ion for this application

Russian multiplication, OpenOffice calc (excel) version



	A	B	C	D	E	F	G
1		Russian multiplication in OpenOffice Calc					
2		number 1	number 2		MOD		
3		21	37		1	21	
4		42	18		0	21	
5		84	9		1	105	
6		168	4		0	105	
7		336	2		0	105	
8		672	1		1	**777**	
9							
10							

Sin(x) function

Octave (Mathlab) sin(x) function

```
function total=sin1(x)
% sin function
   total=0;
   factorial=1;
   power=x;
   plusminus=1;
   for k=1:100
      total=total+power/factorial*plusminus;
      power=power*x*x;
      i=2*k+1;
      factorial=factorial*(i-1)*i;
      plusminus=-1*plusminus;
   end
end
```

```
>> format long
>> x=pi/3
x =  1.047197551196598
>> sin1(x)
ans =    8.660254037844385e-01
```

Let us use the built in sin function now

```
>> x=pi/3
x =  1.047197551196598
>> sin(x)
ans =    8.660254037844386e-01
```

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \le 1$$

```
function total=asin1(x)
% asin function
   total=0;
   factorial1=1;
   factorial2=1;
   power=x;
   power2=1;
   i=0;
   for k=1:80
      total=total+power*factorial1/(power2*factorial2*factorial2)/(2*(k-1)+1);
      power=power*x*x;
      power2=power2*4;
      i=2*k;
      factorial1=factorial1*(i-1)*i;
      factorial2=factorial2*k;
   end
end
```

```
>> x=pi/3
```

```
x =  1.047197551196598
>> y=sin1(x)
y =    8.660254037844385e-01
>> z=asin1(y)
z =  1.047197551196467
>>
```

Built-in function for arcsin is asin

```
>> x=pi/3
x =  1.047197551196598
>> y=sin(x)
y =    8.660254037844386e-01
>> z=asin(y)
z =  1.047197551196598
>>
```

Error function:

$$\text{erf(x)} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

```
        ## Author: TURHAN <TURHAN@DESKTOP-E5V4CKD>
        ## Created: 2021-05-27
        function total = erf1 (x)
          A=2.0/sqrt(pi);
          total=0;
          plusminus=1;
          power=x;
          factorial=1;
          for n=1:200
            total=total+power*plusminus/factorial/(2.0*(n-1)+1.0);
            power=power*x*x;
            factorial=factorial*n;
            plusminus*=plusminus*(-1);
          end
          total=total*A;
        endfunction
>> x=0.1
x =    1.000000000000000e-01
>> y=erf1(x)
y =    1.124606592495027e-01
>>
```

Built-in function erf(x) is existed

```
>> x=0.1
x =    1.000000000000000e-01
>> y=erf(x)
y =    1.124629160182849e-01
```

$J_n(z) = \left(\frac{1}{2z}\right) \sum_{k=1}^{\infty} (-1)^k \frac{\left(\frac{1}{2z}\right)^{2k}}{k!\,\Gamma(n++1)}$ Bessel function

$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1} e^{-t} dt = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{t/n}$

where $\gamma = 0.5772156649015328 6\ldots$ is the Euler-Mascheroni constant

in this case, gamma(x) built in function will be utilised to write the bessel function J

```
 ## Author: TURHAN <TURHAN@DESKTOP-E5V4CKD>
## Created: 2021-05-27
function JJ = bessel_J(v,x)
   JJ=0;
   a=1.0;
   fact=1;
   b=0.0;
   if x <=20
      x5=1;
      lna=0.0;
      x1=v*log(0.5*x);
      lna1=log(0.25*x*x);
```

```
      for k=0:100
         x2=lna;
         x3=log(gamma(k+1));
         x4=log(gamma(v+k+1));
         b=x1+x2-x3-x4;
         b=x5*exp(b);
         lna=lna+lna1;
         x5=-1*x5;
         JJ=JJ+b;
      endfor
   elseif v==0 && x==0
      JJ=1;
   else
      JJ=sqrt(2.0/pi/x)*cos(x-0.5*v*pi-0.25*pi);
   endif
endfunction
```

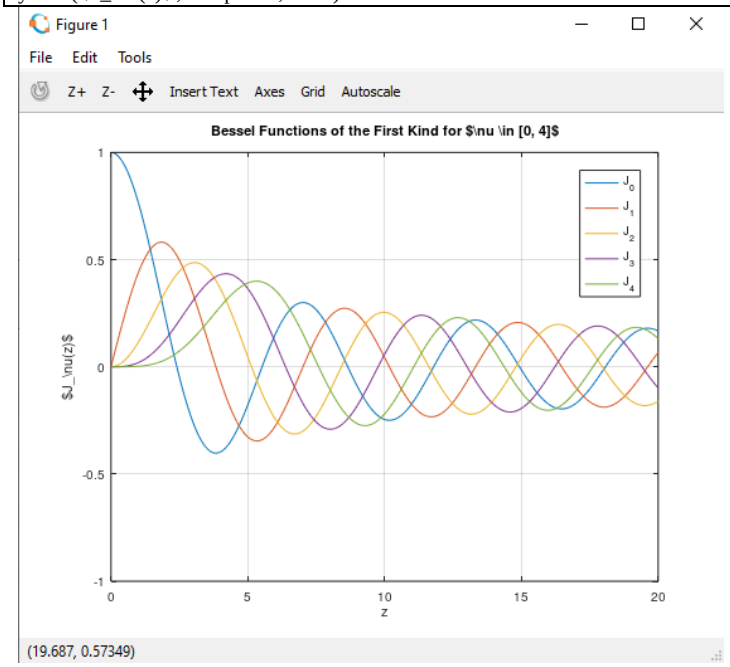
```
>> x=0.1
x =    1.000000000000000e-01
>> y=bessel_J(0,x)
y =    9.975015620660401e-01
```

Built-in function for J bessel function is besselj(v,x)

```
>> x=0.1
x =    1.000000000000000e-01
>> y=besselj(0,x)
y =    9.975015620660401e-01
```

Plot of bessel functions by using bessel_J(v,x)

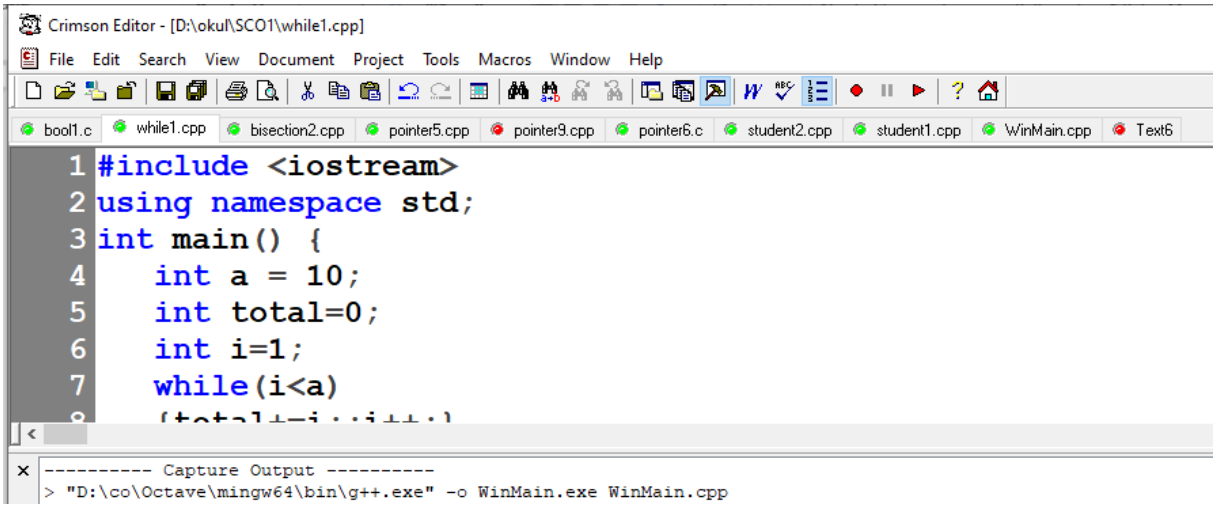
```
 z = 0:0.1:20;
J1 = zeros(5,201);
for i = 0:4
  for j = 1:201
    J1(i+1,j) = bessel_J(i,z(j));
  endfor
endfor
plot(z,J1)
grid on
legend('J_0','J_1','J_2','J_3','J_4','Location','Best')
title('Bessel Functions of the First Kind for $\nu \in [0, 4]$','interpreter','latex')
xlabel('z','interpreter','latex')
ylabel('$J_\nu(z)$','interpreter','latex')
```



1.5 C and C++ language basics

In order to operate with a computer language, a compiler is needed. A wide varity of compilers available for C languages. Out of this mingw is one of the best free access C and C++ compiler. If you already loaded octave you will find mingw C compiler inside the package. If not, you can load it up from https://www.gnu.org/software/octave/download

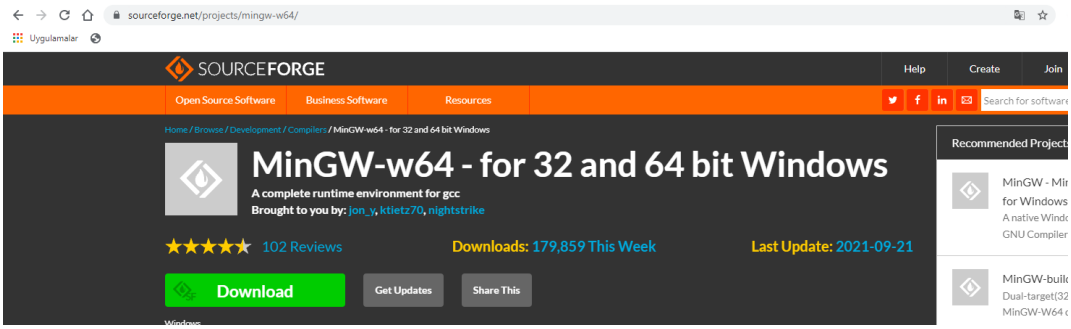
Adress. In utilisation of editor, you also have wide choices. We will be using in example here crimson editor. It is a simple multilanguage editor.



MingGW can directly be obtained from its own internet site

sourceforge.net/projects/mingw-w64

It should be note that C++ compilers are computer dependent. If you have other system like unix or apple operation systems, you should use related compilers.



A first program in C language

```
#include <stdio.h>
using namespace std;
  int main()
  {
    printf("Hello World!\n");
    return 0;
  }
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" hello

D:\okul\SCO1>hello
```

```
Hello World!

> Terminated with exit code 0.
```

A first program in C++ language

```
#include <iostream>
using namespace std;
  int main()
  {
    cout << "Hello World!\n";
    return 0;
  }
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" hello

D:\okul\SCO1>hello
Hello World!

> Terminated with exit code 0.
```

As a basic difference printf in <stdio.h> library is changed with cin in iostream library. Stream is a class type of concept which opens a channel to computer preferies such as screen, printer etc.

```
#include <iostream>
using namespace std;
#include <string.h>
int main()
{
string word;
cin >> word;
cout << "word read is: " << word << '\n';
cout << "ok: no more words to read: bye!\n";
return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" a

D:\okul\SCO1>a
Turhan
word read is: Turhan
ok: no more words to read: bye!

> Terminated with exit code 0.
```

Primitive Built-in Types

C++ offers the programmer a rich assortment of built-in as well as user defined data types. Following table lists down seven basic C++ data types –

Type	Keyword
Boolean	bool
Character	char
Integer	int
Floating point	float
Double floating point	double
Valueless	void
Wide character	wchar_t

Several of the basic types can be modified using one or more of these type modifiers −

- signed

- unsigned
- short
- long

The following table shows the variable type, how much memory it takes to store the value in memory, and what is maximum and minimum value which can be stored in such type of variables.

Type	Typical Bit Width	Typical Range
char	1byte	-127 to 127 or 0 to 255
unsigned char	1byte	0 to 255
signed char	1byte	-127 to 127
int	4bytes	-2147483648 to 2147483647
unsigned int	4bytes	0 to 4294967295
signed int	4bytes	-2147483648 to 2147483647
short int	2bytes	-32768 to 32767
unsigned short int	2bytes	0 to 65,535
signed short int	2bytes	-32768 to 32767
long int	8bytes	-2,147,483,648 to 2,147,483,647
signed long int	8bytes	same as long int
unsigned long int	8bytes	0 to 4,294,967,295
long long int	8bytes	-(2^63) to (2^63)-1
unsigned long long int	8bytes	0 to 18,446,744,073,709,551,615
float	4bytes	
double	8bytes	
long double	12bytes	
wchar_t	2 or 4 bytes	1 wide character

```
#include <iostream>
using namespace std;

int main() {
   cout << "Size of char : " << sizeof(char) <<"\n";
   cout << "Size of int : " << sizeof(int) <<"\n";
   cout << "Size of short int : " << sizeof(short int) <<"\n";
   cout << "Size of long int : " << sizeof(long int) <<"\n";
   cout << "Size of float : " << sizeof(float) <<"\n";
   cout << "Size of double : " << sizeof(double) <<"\n";
   cout << "Size of wchar_t : " << sizeof(wchar_t) <<"\n";

   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" sizeof1

D:\okul\SCO1>sizeof1
Size of char : 1
Size of int : 4
Size of short int : 2
Size of long int : 4
Size of float : 4
Size of double : 8
Size of wchar_t : 2

> Terminated with exit code 0.
```

In addition string type is defined in C++ as an array of char type. It is not defined in C. In c you should use array of character for this purpose.

```
#include <iostream>
using namespace std;
int main()
{
string name="Turhan";
int b=5;
int c=b/3;
int d=b%3;
cout <<name<< " b = " << b <<" c = "<<c<<" d = "<<d<< '\n';
return 0;
```

```
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" program2

D:\okul\SCO1>program2
Turhan b = 5 c = 1 d = 2

> Terminated with exit code 0.
```

```
#include <iostream>
using namespace std;
int main()
{
string name="Turhan";
double b=5.43543;
double c=b/3.25;
cout <<name<< " b = " << b <<" c = " <<c << '\n';
return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" program1

D:\okul\SCO1>program1
Turhan b = 5.43543 c = 1.67244

> Terminated with exit code 0.
```

String variables does not existed in C language. Instead of string, character (char) array can be used.

```
#include <stdio.h>
  int main()
  {
    char s[] ="Hello World!\n";
    printf(s);
    return 0;
  }
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" hello2

D:\okul\SCO1>hello2
Hello World!

> Terminated with exit code 0.
```

Following example reads a string of 20 character long (by using char array)

```
#include <stdio.h>
int main()
{
    char s1[20];
    printf("what is your name : ");
    scanf("%s", s1); // input is a string
    printf("reading string : *%s* \n", s1);
    return 0;
}
```

```
D:\okul\SCO1>scanf1
what is your name : Turhan
reading string : *Turhan*
```

```
D:\okul\SCO1>
```

Another C program to read String data

```
#include <stdio.h>

#define BUFFER_SIZE 1024

int main()
{
   char a[BUFFER_SIZE];
   printf("Enter the string: ");
   fgets(a, BUFFER_SIZE, stdin);
   printf("\nEntered string is: %s", a);
}
```

```
icrosoft Windows [Version 10.0.19042.1237]
(c) Microsoft Corporation. Tüm hakları saklıdır.

D:\okul\turhan_coban\sirketler\huseyin\Lumatron_H_Nazli>fgets1
Enter the string: Turhan

Entered string is: Turhan
```

Same program in C++

```
#include <iostream>
using namespace std;

int main()
{
   string s1;
   cout<<"what is your name : ";
   cin>>s1; // input is a string
   cout<<"reading string : *"<<s1<<"*\n";
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" cin1

D:\okul\SCO1>cin1
what is your name : Turhan
reading string : *Turhan*

> Terminated with exit code 0.
```

Enumerated Types

An enumerated type declares an optional type name and a set of zero or more identifiers that can be used as values of the type. Each enumerator is a constant whose type is the enumeration.

Creating an enumeration requires the use of the keyword **enum**. The general form of an enumeration type is −

enum enum-name { list of names } var-list;

Here, the enum-name is the enumeration's type name. The list of names is comma separated.

For example, the following code defines an enumeration of colors called colors and the variable c of type color. Finally, c is assigned the value "blue".

enum color { red, green, blue } c;

c = blue;

By default, the value of the first name is 0, the second name has the value 1, and the third has the value 2, and so on.

```
#include <iostream>
using namespace std;

int main() {
  enum color { red, green, blue } c;
  c = blue;
  cout<<"color = "<<c;

  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" enum1

D:\okul\SCO1>enum1
color = 2
> Terminated with exit code 0.
```

Boolean variable is not directly defined in c, but can be accessed as integer 0 and 1

Following table shows all the logical operators supported by C language. Assume variable **A** holds 1 and variable **B** holds 0, then –

Operator	Description	Example
&&	Logical AND operator. If both operands are non-zero, then the condition becomes true	A=false; B=true (A && B) is false
||	Logical OR operator. If one operands are non-zero, then the condition becomes true	A=false; B=true (A || B) is true
!	Logical not operator. It is used to reverse the logical operand	A=false !A is true

Logical statements used with if statement

```
#include <stdio.h>

main() {

  int a = 0;
  int b = 1;
  printf("a = %i b = %i\n",a,b );
  if ( a || b ) {printf("%i || %i Condition is true\n",a,b );}
  else if ( a && b ) {printf("%i && %i Condition is false\n",a,b );}
  /* lets change the value of  a and b */
  a = 1;
  b = 1;
  printf("a = %i b = %i\n",a,b );
  if ( a && b )  {printf("%i && %i Condition is true\n",a,b );}
  if ( a || b )  {printf("%i || %i Condition is true\n",a,b );}
  if ( !(a && b) ) {printf("!(%i && %i) Condition is true\n",a,b );}
  a = 0;
  b = 0;
  printf("a = %i b = %i\n",a,b );
  if ( a && b )  {printf("%i && %i Condition is true\n",a,b );}
```

```
   if ( a || b )  {printf("%i || %i Condition is true\n",a,b );}
   if ( !(a && b) ) {printf("!(%i && %i) Condition is true\n",a,b );}
}
```

Comparison of numbers can be achived by using

>	greater
<	smaller
==	equals
>=	Greater and equal
<=	smaller and equal
!=	Not equal

C version:

```
#include <stdio.h>

main() {
   int a = 21;
   int b = 35;
   printf("a = %i b = %i\n",a,b );
   if ( a ==  b ) {printf("%i == %i \n",a,b );}
   else if ( a > b ) {printf("%i > %i \n",a,b );}
   else if ( a < b ) {printf("%i < %i \n",a,b );}
   else if ( a != b ) {printf("%i != %i \n",a,b );}
}
```

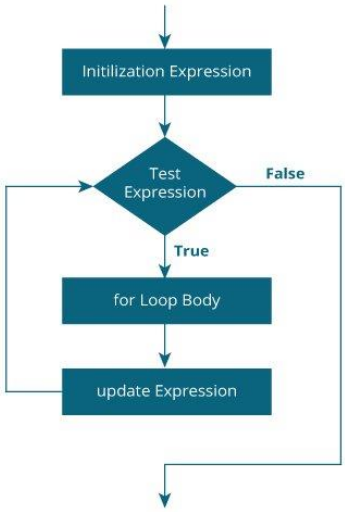
```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" boolean2

D:\okul\SCO1>boolean2
a = 21 b = 35
21 < 35

> Terminated with exit code 0.
```

C++ version will look like:

```
#include <iostream>
using namespace std;
int main() {
   int a = 21;
   int b = 21;
   cout<<" a = "<<a<<" b = "<<b<<"\n";
   if ( a ==  b ) cout<<a<<"=="<<b<<"\n";
   else if ( a > b ) cout<<a<<">"<<b<<"\n";
   else if ( a < b ) cout<<a<<"<"<<b<<"\n";
   else if ( a != b ) cout<<a<<"!="<<b<<"\n";
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" boolean3

D:\okul\SCO1>boolean3
 a = 21 b = 21
21==21

> Terminated with exit code 0.
```

For statement: In order to calculate repeated processes for or while statements will be used

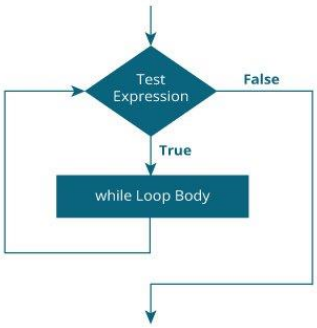


Working of for loop

```
#include <iostream>
using namespace std;
int main() {
   int a = 10;
   int total=0;
   for(int i=1;i<a;i++)
   {total+=i;}
   cout<<" total = "<<total<<"\n";

   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" for1

D:\okul\SCO1>for1
 total = 45

> Terminated with exit code 0.
```

While loop:



While is similar to for, but only boolean condition existed in loop structure and this condition repeated the loop body as long as the boolean value is true

```
#include <iostream>
using namespace std;
int main() {
   int a = 10;
   int total=0;
   int i=1;
   while(i<a)
```

```
    {total+=i;;i++;}
    cout<<" total = "<<total<<"\n";

    return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" while1

D:\okul\SCO1>while1
 total = 45

> Terminated with exit code 0.
```

SERİES SOLUTIONS OF FUNCTIONS

Digital computers can only calculate addition process, therefore most of the mathematical expressions are calculated by using series. Series solutions of a few functions will be listed in here.

$$\ln(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1} \qquad [0 < x]$$

$$\exp(x) = \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$\exp(-x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}$$

$$\exp(-x^2) = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

$$\exp(x)(1+x) = \sum_{k=1}^{\infty} \frac{x^k(k+1)}{k!}$$

$$a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$$

$\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x}$ [a>1 and x<0 or 0<a<1 x>0]

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\sin h(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\cos\ h(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cos\ h(x)}$$

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \le 1$$

$$\text{arcsinh(x)} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \le 1$$

$$\text{arctan(x)} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)} \qquad x^2 \le 1$$

$$\text{arctanh(x)} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)} \qquad x^2 \le 1$$

$$\text{erf(x)} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{2}{8k+6}\right)$ Bailey, Borwein Plouffe(BBP) formula[90]

$J_n(z) = \left(\frac{1}{2z}\right)\sum_{k=1}^{\infty}(-1)^k \frac{\left(\frac{1}{2z}\right)^{2k}}{k!\,\Gamma(n++1)}$ Bessel function

$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{t/n}$ where $\gamma = 0.5772156649015328 6\ldots$ is the Euler-Mascheroni constant

$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}$ Bessel function

Now we would like to convert some of these mathematical series formulations into computer programming codes in C++computer language. As a first series let us look at a simple form as: $\text{f(n)} = sum(n) = \sum_{k=1}^{n} k = 1 + 2 + \ldots + n$ can be approximated in computer environment as

sum(n) function series solution, C++ Version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
using namespace std;

double sum(int n)
  {double total=0;
   for(int k=1;k<=n;k++)
   {total+=k;}
   return total;
  }
int main()
{printf("sum=%20.15f\n",sum(100));
 cout<<sum(100);
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" sum1

D:\okul\SCO1>sum1
sum=5050.000000000000000
5050
> Terminated with exit code 0.
```

average(x) function series solution, C++ Version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

double average(int n)
  {double total=0;
   for(int k=1;k<=n;k++)
   {total+=k;}
   return total/n;
  }
int main()
{printf("average =%20.15f\n",average(100));
```

```
 cout<<setprecision(20);
 cout<<average(100);
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" average1

D:\okul\SCO1>average1
average =  50.500000000000000
50.5
> Terminated with exit code 0.
```

average(x_k) function series solution, C++ Version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

double average(double x[],int n)
  {double total=0;
   for(int k=0;k<n;k++)
   {total+=x[k];}
   return total/n;
  }
int main()
{   double x[]={1.11,2.21,3.31,4.41,5.51,6.61};
    int n=sizeof x/sizeof(double);
    printf("average =%20.15f\n",average(x,n));
    cout<<setprecision(20);
    cout<<average(x,n);
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" average2
D:\okul\SCO1>average2
average =   3.860000000000000
3.8599999999999999
> Terminated with exit code 0.
```

exp(x) function series solution, C++ Version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
using namespace std;
   double exp(double x)
   {double total=0;
    double factorial=1;
    double power=1;
    for(int i=1;i<=200;i++)
    {total=total+power/factorial;
     power=power*x;
     factorial=factorial*i;
    }
    return total;}
   int main()
   { double r=exp(1.0);
     printf("r=%20.15f\n",r);
     cout<<"r="<<r;
     return 0;
   }
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" func

D:\okul\SCO1>func
r=   2.718281828459046
r=2.71828
> Terminated with exit code 0.
```

function f(x)=ln(x) can be approximated as $\mathbf{ln}(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1} \quad [0 < x]$

ln(x) function series solution,C++ Version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
using namespace std;
double ln(double x)
   {double total=0;
    double y=(x-1.0)/(x+1.0);
    double power=y;
    for(int k=1;k<=200;k++)
    {  total=total+power/(2.0*k-1.0);
       power=power*y*y;
    }
    return 2.0*total;
   }
   int main()
   {   double r=2.7182818284590455;
                cout<<ln(r);}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" func2
D:\okul\SCO1>func2
1
> Terminated with exit code 0.
```

Some of the functions are defined as combination of more than one function (or series solution). For example power function is defined as:

$$f(x) = a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$$

$f(x) = a^x$ function series solution, C++ Version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
using namespace std;
   double exp(double x)
   {double total=0;
    double factorial=1;
    double power=1;
    for(int i=1;i<=200;i++)
    {total=total+power/factorial;
     power=power*x;
     factorial=factorial*i;
    }
    return total;
   }
   double ln(double x)
   {double total=0;
    double y=(x-1.0)/(x+1.0);
    double power=y;
    for(int k=1;k<=200;k++)
    {  total=total+power/(2.0*k-1.0);
       power=power*y*y;
    }
    return 2.0*total;
   }
   double pow(double a,double x)
   {return exp(x*ln(a));}
   int main()
   { double r=pow(2.0,3.0);
     printf("r=%20.15f\n",r);
     cout<<"r="<<r;
     return 0;
}
```

```
---------- Capture Output ----------
```

```
> "D:\okul\SCO1\run.bat" func3
D:\okul\SCO1>func3
r=   7.999999999999997
r=8
> Terminated with exit code 0.
```

Russian peasant multiplication problem:

Take two integer for example 21 and 37 then double the first one(left hand side) and divide by two the second one(right side) ignore remaining in division process and stop when the right hand side reached to 1 and then cross each line where the right side number is even add up remaining left hand side numbers for example. If the given integer numbers to be multiplied is 21 and 17, the process will follow the path given below:

```
21 37
~~42 18~~ cross out (18 is even)
84 9
~~168 4~~ cross out(4 is even)
~~336 2~~ cross out(2 is even)
672 1
then 21+84+672=777
21*37=777
```

Now let us convert this to an algorithmic structure:

Russian peasant multiplication, C++ version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
using namespace std;

int multiply(int x1,int x2)
{ int y2=x2;
  int y1=x1;
  int total=0;
  while(y2>=1)
  {  if(y2%2 != 0) total+=y1;
                cout<<"y1 = "<<y1<<" y2 = "<<y2<<endl;
    y1*=2;y2/=2;
  }
  return total;
}

int main()
{ cout << "multiplication = "<<multiply(21,37);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" russian

D:\okul\SCO1>russian
y1 = 21 y2 = 37
y1 = 42 y2 = 18
y1 = 84 y2 = 9
y1 = 168 y2 = 4
y1 = 336 y2 = 2
y1 = 672 y2 = 1
multiplication = 777
> Terminated with exit code 0.
```

C++ sin(x) function

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

   double sin1(double x)
```

```
   {double total=0;
    double factorial=1;
    double power=x;
    int plusminus=1;
    int i=0;
    for(int k=1;k<=200;k++)
    {total=total+power/factorial*plusminus;
     power=power*x*x;
     i=2*k+1;
     factorial=factorial*(i-1)*i;
     plusminus*=-1;
    }
    return total;
   }
   int main()
   {   double PI=atan(1.0)*4.0;
                double x=PI/3.0;
                cout<<setprecision(12)<<"sin = "<<sin1(x)<<" Math.sin = "<<sin(x);
   }
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" sin

E:\okul\SCO1>sin
sin = 0.866025403784 Math.sin = 0.866025403784
> Terminated with exit code 0.
```

C++ version arcsin(x) function

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;
    double asin1(double x)
   {double total=0;
    double factorial1=1;
    double factorial2=1;
    double power=x;
    double power2=1;
    int i=0;
    for(int k=1;k<=80;k++)
    {total=total+power*factorial1/(power2*factorial2*factorial2)/(2.0*(k-1)+1.0);
     power*=x*x;
     power2*=4;
     i=2*k;
     factorial1=factorial1*(i-1)*i;
     factorial2=factorial2*k;
    }
    return total;
   }
   int main()
   {   double PI=atan(1.0)*4.0;
                double x1=PI/3.0;
                double x2=sin(x1);
                cout<<setprecision(20)<<"asin = "<<asin1(x2)<<" math.asin = "<<asin(x2);
   }
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" asin

E:\okul\SCO1>asin
asin = 1.0471975511964670691 math.asin = 1.0471975511965976313
> Terminated with exit code 0.
```

Pointers in C and C++

If you have a variable var in your program &var will give you its adress in the memory

```
int main()
{
  int var = 5;
  printf("var: %d\n", var);
```

```
  // Notice the use of & before var
  printf("address of var: %p", &var);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer1

D:\okul\SCO1>pointer1
var: 5
address of var: 000000000065FE1C
> Terminated with exit code 0.
```

Pointers are special variables that are used to store adresses rather than values

```
#include <stdio.h>
int main()
{
  int *var;
  int c;
  c=5;
  var=&c;
  printf("var: %d\n", c);
  // Notice the use of & before var
  printf("address of var: %p", var);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer2

D:\okul\SCO1>pointer2
var: 5
address of var: 000000000065FE14
> Terminated with exit code 0.
```

Pointers can be used for array processing

```
#include <stdio.h>
int main()
{
  int *var;
  int c[]={10,11,12,13,14,15,16,17};
  var=&c[0];  //adress of array
  int n=sizeof(c)/sizeof(int);
  for(int i=0;i<n;i++)
  {   printf("i =: %d x = %d\n",i,**(var+i)**);}
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer3

D:\okul\SCO1>pointer3
i =: 0 x = 10
i =: 1 x = 11
i =: 2 x = 12
i =: 3 x = 13
i =: 4 x = 14
i =: 5 x = 15
i =: 6 x = 16
i =: 7 x = 17

> Terminated with exit code 0.
```

```
#include <stdio.h>
int main()
{
  int *var;
  int c[]={10,11,12,13,14,15,16,17};
  var=&c[0];  //adress of array
  int n=sizeof(c)/sizeof(int);
  for(int i=0;i<n;i++)
  {
                printf("i =: %d x = %d\n",i,*var);
                var++;
  }
  return 0;
}
```

```
#include <stdio.h>
int main()
{
  double *var;
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  var=&c[0];  //adress of array
  int n=sizeof(c)/sizeof(double);
  for(int i=0;i<n;i++)
  {
                printf("i =: %d x = %f\n",i,*var);
                var++;
  }
  return 0;
}
```

```
#include <stdio.h>
int main()
{
  double *var;
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  var=&c[0];  //adress of array
  int n=sizeof(c)/sizeof(double);
  for(int i=0;i<n;i++)
  {
                printf("i =: %d x = %f\n",i,*var);
                var++;
  }
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer4

D:\okul\SCO1>pointer4
i =: 0 x = 10.120000
i =: 1 x = 11.150000
i =: 2 x = 12.450000
i =: 3 x = 13.250000
i =: 4 x = 14.160000
i =: 5 x = 15.220000
i =: 6 x = 16.210000
i =: 7 x = 17.350000

> Terminated with exit code 0.
```

C++ version of the previous example:

```
#include <iostream>
using namespace std;
int main()
{
  double *var;
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  var=&c[0];  //adress of array
```

```
  int n=sizeof(c)/sizeof(double);
  for(int i=0;i<n;i++)
  {   cout<<"i ="<<i<<" c["<<i<<"] = "<<(*var)<<"\n";
      var++;
  }
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer5

D:\okul\SCO1>pointer5
i =0 c[0] = 10.12
i =1 c[1] = 11.15
i =2 c[2] = 12.45
i =3 c[3] = 13.25
i =4 c[4] = 14.16
i =5 c[5] = 15.22
i =6 c[6] = 16.21
i =7 c[7] = 17.35

> Terminated with exit code 0.
```

We can use pointer concept to pass arrays to functions

```
 #include <stdio.h>
   void pointer6( double *var,int n)
   {    for(int i=0;i<n;i++)
        { printf("i =: %d x = %f\n",i,*var);
                  var++;
        }
   }

int main()
{
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  double *var=&c[0];  //adress of array
  int n=sizeof(c)/sizeof(double);
  pointer6(var,n);
  return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer6

D:\okul\SCO1>pointer6
i =: 0 x = 10.120000
i =: 1 x = 11.150000
i =: 2 x = 12.450000
i =: 3 x = 13.250000
i =: 4 x = 14.160000
i =: 5 x = 15.220000
i =: 6 x = 16.210000
i =: 7 x = 17.350000

> Terminated with exit code 0.
```

C++ version:

```
 #include <iostream>
using namespace std;
   void pointer6( double *var,int n)
   {    for(int i=0;i<n;i++)
        { cout<<"i ="<<i<<" c["<<i<<"] = "<<(*var)<<"\n";
                  var++;
        }
   }
```

```
int main()
{
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  double *var=&c[0];  //adress of array
  int n=sizeof(c)/sizeof(double);
  pointer6(var,n);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer7

D:\okul\SCO1>pointer7
i =0 c[0] = 10.12
i =1 c[1] = 11.15
i =2 c[2] = 12.45
i =3 c[3] = 13.25
i =4 c[4] = 14.16
i =5 c[5] = 15.22
i =6 c[6] = 16.21
i =7 c[7] = 17.35

> Terminated with exit code 0.
```

you can also pass array variable names directly to function

C version

```
#include <stdio.h>
   void pointer6( double c[],int n)
   {
                    for(int i=0;i<n;i++)
        { printf("i =: %d x = %f\n",i,c[i]);
        }
   }

int main()
{
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  int n=sizeof(c)/sizeof(double);
  pointer6(c,n);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer9

D:\okul\SCO1>pointer9
i =: 0 x = 10.120000
i =: 1 x = 11.150000
i =: 2 x = 12.450000
i =: 3 x = 13.250000
i =: 4 x = 14.160000
i =: 5 x = 15.220000
i =: 6 x = 16.210000
i =: 7 x = 17.350000

> Terminated with exit code 0.
```

C++ version

```
#include <iostream>
using namespace std;
   void pointer6( double c[],int n)
   {     for(int i=0;i<n;i++)
        { cout<<"i ="<<i<<" c["<<i<<"] = "<<c[i]<<"\n";}
   }
```

```
int main()
{
  double c[]={10.12,11.15,12.45,13.25,14.16,15.22,16.21,17.35};
  int n=sizeof(c)/sizeof(double);
  pointer6(c,n);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" pointer8

D:\okul\SCO1>pointer8
i =0 c[0] = 10.12
i =1 c[1] = 11.15
i =2 c[2] = 12.45
i =3 c[3] = 13.25
i =4 c[4] = 14.16
i =5 c[5] = 15.22
i =6 c[6] = 16.21
i =7 c[7] = 17.35

> Terminated with exit code 0.
```

Class concept in C++

Class structure is not defined in C. The basic purpose of C++ is to include class structure into C language.

```
 #include <bits/stdc++.h>
using namespace std;
class student
{
   // Access specifier
   public:
   // Data Members
   string name;
   int number;
   //Constructor function
   student(string namei,int numberi)
   { name=namei;
     number=numberi;
   }
   // Member Functions()
   void print()
   { cout << "Name : " << name<<"\nNumber : "<<number;}
};

int main() {
   // Declare an object of class student
   student st1=student("Turhan Coban",12345);
   // accessing member function
   st1.print();
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" student1

D:\okul\SCO1>student1
Name : Turhan Coban
Number : 12345
> Terminated with exit code 0.
```

A class can be given without constructor class. If ıts access specifier is defined as public, variable can be input in accessing point

```
#include <iostream>
#include <math.h>
using namespace std;
class student
{
   // Access specifier
   public:
   // Data Members
   string name;
   int number;
   //Constructor function

   // Member Functions()
   void print()
   { cout << "Name : " << name<<"\nNumber : "<<number;}
};

int main() {
   // Declare an object of class student
   student st1;
   st1.name="Turhan Coban";
   st1.number=12345;
   // accessing member function
   st1.print();
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" student2

D:\okul\SCO1>student2
Name : Turhan Coban
Number : 12345
> Terminated with exit code 0.
```

A class can be used in another class or class function as an input variable. Classes are basically defined user defined variable types

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f1
{ public: double func(double x)
  { double ff=x*x-2.0;
              return ff;
  }
  public: double dfunc(double x)
  { double ff=2.0*x;
              return ff;
  }
};

double newton(f1 &f,double x)
{
int nmax=100;
double tolerance=1.0e-8;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=f.dfunc(x);
  x-=fx/dfx;
  cout <<"i="<<i<<"x="<<x<<"fx="<<fx<<"dfx="<<dfx<<"\n";
  if(fabs(fx)<tolerance) { return x;}
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;
}
```

```
int main()
{  double xa,r;
   cout << " first guess for the root value : ";
   cin >>xa;
   f1 f=f1();
   cout<<setprecision(20);
   r=newton(f,xa);
   cout<<"root value : ="<<r;
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" newton

D:\okul\SCO1>newton
 first guess for the root value : 1
i=0x=1.5fx=-1dfx=2
i=1x=1.416666666666667407fx=0.25dfx=3
i=2x=1.414215862745098866fx=0.006944444444444646418174dfx=2.8333333333333334814
i=3x=1.414213562374689869 8fx=6.0073048828712671821e-006dfx=2.8284313725490197733
i=4x=1.414213562373095145 5fx=4.5106141044470859924e-012dfx=2.8284271247493797397
root value : =1.4142135623730951455
> Terminated with exit code 0.
```

An example class:

```
#include <iostream>
using namespace std;

class Box {
   public:
      Box(double  len,double hei,double wid): length(len), height(hei), width(wid)
      {cout<<"object created length="<<length<<"height = "<<height<<"width = "<<width;
      }
      Box()
      {length=1.0;
       width=1.0;
       height=1.0;
      }
      double getVolume(void) {
         return length * height * height;
      }
      void setLength( double len ) {
         length = len;
      }
      void setwidth( double bre ) {
         width = bre;
      }
      void setHeight( double hei ) {
         height = hei;
      }

      // Overload + operator to add two Box objects.
         Box operator+(const Box& b) {
         Box box;
         box.length = this->length + b.length;
         box.width =  this->width + b.width;
         box.height = this->height + b.height;
         return box;
      }

   private:
      double length;      // Length of a box
      double width;     // width of a box
      double height;      // Height of a box
};

// Main function for the program
int main() {
   Box Box1=Box(6.0,7.0,5.0);               // Declare Box1 of type Box
   Box Box2=Box(1.0,2.0,3.0);
```

```
   Box Box3;
   // volume of box 1
   volume = c;
   cout << "Volume of Box1 : " <<Box1.getVolume()<<endl;
   cout << "Volume of Box2 : " <<Box1.getVolume()<<endl;
   cout << "Volume of Box3 : " << volume <<endl; <<endl;


   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" Box1

D:\okul\SCO1>Box1
Volume of Box1 : 210
Volume of Box2 : 1560
Volume of Box3 : 5400

> Terminated with exit code 0.
```

An example class a simple **complex number** calculation class. Note that +,-,* and / operations can be defined in a C++ class.

```
 //Define complex numbers
#include <bits/stdc++.h>
#include <math.h>
using namespace std;
class cn
{
   // Access specifier
   public:
   // Data Members
   double real;
   double imag;
   //Constructor function
   cn(double r,double im)
   { real=r;
     imag=im;
   }
   cn()
   {}
   cn(const cn &c1)
   {real=c1.real;
    imag=c1.imag;
   }
   // Member Functions()
   void set(double r,double im)
   { real=r;
     imag=im;
   }
   void operator+=(const cn& c1)
   {real+=c1.real;
    imag+=c1.imag;
   }
   cn operator+(const cn& c1)
   {  cn c3;
      c3.real=real+c1.real;
      c3.imag+=real+c1.imag;
      return c3;
   }
   void operator-=(const cn& c1)
   {real-=c1.real;
    imag-=c1.imag;
   }
   cn operator-(const cn& c1)
   {  cn c3;
      c3.real=real-c1.real;
      c3.imag+=real-c1.imag;
      return c3;
   }
   void operator*=(const cn& c1)
   { double r1=real*c1.real-imag*c1.imag;
     double r2=real*c1.imag+imag*c1.real;
```

```
      real=r1;
      imag=r2;
    }
    cn operator*(const cn& c1)
    {  cn c3;
       double r1=real*c1.real-imag*c1.imag;
       double r2=real*c1.imag+imag*c1.real;
       c3.real=r1;
       c3.imag=r2;
       return c3;
    }
    void operator/=(const cn& c1)
    { double r1=real*c1.real+imag*c1.imag;
      double r2=-real*c1.imag+imag*c1.real;
      double r3=c1.real*c1.real+c1.imag*c1.imag;
      real=r1/r3;
      imag=r2/r3;
    }
    cn operator/(const cn& c1)
    {  cn c3;
       double r1=real*c1.real+imag*c1.imag;
       double r2=-real*c1.imag+imag*c1.real;
       double r3=c1.real*c1.real+c1.imag*c1.imag;
       c3.real=r1/r3;
       c3.imag=r2/r3;
       return c3;
    }
    //Mathematical definitions====================
    double R()
    {// Return radius of polar coordinate equivalent of c number
      return sqrt(real*real+imag*imag);}
    double theta()
    {// Return angle of polar coordinate equivalent of c number in radian
     return atan2(imag,real);}
    double dtheta()
    {// Return angle of polar coordinate equivalent of c number in degree
      return atan2(imag,real)*180.0/M_PI;}
    //Power of a complex number

    cn pow(double n)
    { //power of a complex number
       double Rad,th;
       double R1=R();
       Rad=exp(n*log(R1));
       th=n*theta();
       cn result=cn((Rad*cos(th)),(Rad*sin(th)));
       return result;
    }


     //=====================================
    void print()
    { cout <<real<<" + i*" << imag;}
    void print(string s1)
    { cout <<s1<<" = "<<real<<" + i*" << imag;}
    void println()
    { cout <<real<<" + i*" << imag<<endl;}
    void println(string s1)
    { cout <<s1<<" = "<<real<<" + i*" << imag<<endl;}
};

int main() {
    // Declare an object of class student
    cn c1=cn(1.0,2.0);
    // accessing member function
    //cout<<"\nc1 = ";
    c1.println("c1");
    cn c2;
    c2.set(2.0,3.0);
    c2.println("c2");
    cn c3(c1);
    c3.println("c3");
    c3+=c2;
    c3.println("c3");
    cn c4=c3+c1;
    c4.println("c4");
```

```
    cn c5=c3*c1;
    c5.println("c5");
    cn c6=c3/c1;
    c6.println("c6");
    c6.pow(2);
    cn c7=cn(2.0,0.0);
    c7.println("c7");
    cn c8=c7.pow(2.0);
    c8.println("c8");
    return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" cn

D:\okul\SCO1>cn
c1 = 1 + i*2
c2 = 2 + i*3
c3 = 1 + i*2
c3 = 3 + i*5
c4 = 4 + i*5
c5 = -7 + i*11
c6 = 2.6 + i*-0.2
c7 = 2 + i*0
c8 = 4 + i*0

> Terminated with exit code 0.
```

Note that built-in complex class is existed in C++ (in latest versions)

```
#include <iostream>
#include <complex>
using namespace std;

// Main function for the program
int main() {
 complex<double> x1(3.0,2.0);
 complex<double> x2(1.0,1.0);
 complex<double> x3=x1+x2;
 cout<<"x3 = "<<x3;
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" complex3

D:\okul\SCO1>complex3
x3 = (4,3)
> Terminated with exit code 0.
```

As another axample following is a **rational number** class, it is relatively similar definition like complex class

```
//Define rational numbers
#include <bits/stdc++.h>
using namespace std;
class rn
{
   // Access specifier
   public:
   // Data Members
   int   numerator;
   int   denominator;
   //Constructor function
   rn(int nr,int dnr)
   {   numerator=nr;
       denominator=dnr;
       simplify();
```

```
    }
    rn(int c,int nr,int dnr)
    {   numerator=c*dnr+nr;
        denominator=dnr;
        simplify();
    }
    rn()
    {}
    rn(const rn &c1)
    {numerator=c1.numerator;
     denominator=c1.denominator;
    }
int leastcommondivisor()
{
// least common divisor of two iteger numbers
int n=numerator;
int m=denominator;
if(n==0)
   return m;
if(m==0)
   return n;
while(m != n)
{
  if(n>m)
     n=n-m;
  else
     m=m-n;
}
return n;
}

void simplify()
{
//bayağı kesiri sadeleştirir
int isaret=1;
if(numerator<0)
{
 isaret=-isaret;
 numerator=-numerator;
}
if(denominator<0)
{
 isaret=-isaret;
 denominator=-denominator;
}
if(denominator==0)
   { cout<<"rational number divide by zero error";}
int ebob=leastcommondivisor();
ebob=abs(ebob);
numerator=isaret*numerator/ebob;
denominator=denominator/ebob;
}
    // Member Functions()
    void set(int nr,int dnr)
    {   numerator=nr;
        denominator=dnr;
        simplify();
    }
double toDouble()
{
//rational sayinin gercek sayi esidi
return ((double)numerator/(double)denominator);
}

rn convert_to_rational(double d)
{
rn b;
b.set((int)d*10000,10000);
b.simplify();
return b;
}
    // Mathematical operations + +=  - -= * *= / /=
    void operator+=(const rn& c1)
    {  numerator = numerator*c1.denominator + c1.numerator*denominator;
       denominator = denominator*c1.denominator;
       simplify();
```

```
}
rn operator+(const rn& c1)
{  rn c3;
   c3.numerator = numerator*c1.denominator + c1.numerator*denominator;
   c3.denominator = denominator*c1.denominator;
   simplify();
   return c3;
}

void operator-=(const rn& c1)
{  numerator = numerator*c1.denominator - c1.numerator*denominator;
   denominator = denominator*c1.denominator;
   simplify();
}
rn operator-(const rn& c1)
{  rn c3;
   c3.numerator = numerator*c1.denominator - c1.numerator*denominator;
   c3.denominator = denominator*c1.denominator;
   simplify();
   return c3;
}

void operator*=(const rn& c1)
{  numerator = numerator*c1.numerator;
   denominator = denominator*c1.denominator;
   simplify();
}

rn operator*(const rn& c1)
{  rn c3;
   c3.numerator = numerator*c1.numerator;
   c3.denominator = denominator*c1.denominator;
   simplify();
   return c3;
}

void operator/=(const rn& c1)
{  numerator = numerator*c1.denominator;
   denominator = denominator*c1.numerator;
   simplify();
}

rn operator/(const rn& c1)
{  rn c3;
   c3.numerator = numerator*c1.denominator;
   c3.denominator = denominator*c1.numerator;
   simplify();
   return c3;
}

//Boolean comperison operations
bool operator!=(const rn& c1)
{ return !(*this == c1);}
bool operator==(const rn& c1)
{ return (*this == c1);}
bool operator>(const rn& c1)
{ bool result;
  if ((this->numerator / this->denominator) > (c1.numerator / c1.denominator))
  {result = true;}
  else
  {result = false;}
  return result;
}
 bool operator>=(const rn& c1)
{ bool result;
  if ((this->numerator / this->denominator) >= (c1.numerator / c1.denominator))
  {result = true;}
  else
  {result = false;}
  return result;
}
bool operator<(const rn& c1)
{ bool result;
  if ((this->numerator / this->denominator) < (c1.numerator / c1.denominator))
  {result = true;}
  else
```

```
      {result = false;}
     return result;
    }
    bool operator<=(const rn& c1)
    { bool result;
     if ((this->numerator / this->denominator) < (c1.numerator / c1.denominator))
     {result = true;}
     else
     {result = false;}
     return result;
    }
    void print()
    { cout <<numerator<<"/" << denominator;}
    void print(string s1)
    { cout <<s1<<" = "<<numerator<<"/" << denominator;}
    void println()
    { cout <<numerator<<"/" << denominator<<endl;}
    void println(string s1)
    { cout <<s1<<" = "<<numerator<<"/" << denominator<<endl;}
};



int main() {
    // Declare an object of class student
    rn r1=rn(1,1,3);
    r1.println("r1");
    rn r2=rn(1,5);
    r2.println("r2");
    rn r3=r1+r2;
    r3.println("r3");
    return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" rn

D:\okul\SCO1>rn
r1 = 4/3
r2 = 1/5
r3 = 23/15

> Terminated with exit code 0.
```

Abstract classes

A class with some defined an at least one undefined function can be defined. The basic reason for this applications, to create a general class with actual definitions will be created later by defining another child class. As an example we will define a class called f_x in this class we will defined an empty function called func to define general function, and another function called dfunc to define numerical derivative of that function. Later another class f1 will be defined wich include func function with the filled function definition. We will be given a function called bisection. In the entry of function we will be defining an entry object of class f_x, but in real usage we will be using class f1 to carry out function calculations inside bisection method.

```
 #include <iomanip>
using namespace std;
class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
```

```
}
};

class f1:public f_x
{ public:double func(double x)
  {return x*x-2; };
};

double bisection(f_x &f,double xl,double xu)
{
//bisection root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr;
int maxit=100;
int iter=0;
es=0.00000001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
{
   xr=(xl+xu)/2.0;
   iter++;
   if(xr!=0)
      { ea=fabs((xu-xl)/(xu+xl))*100.0;}
   fxl= f.func(xl);
   fxr= f.func(xr);
   test= fxl*fxr;
   if(test==0.0)      ea=0;
   else if(test<0.0)  xu=xr;
   else               {xl=xr;}
}
if(iter>=maxit) cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return xr;
}
int main()
{  double a,b,r;
   f1 f=f1();
   cout << " INPUT LOWER LIMIT OF SEARCH AREA a : ";
   cin >>a;
   cout << " INPUT UPPER LIMIT OF SEARCH AREA b : ";
   cin >>b;
   r=bisection(f,a,b);
   cout<<setprecision(20);
   cout<<"r="<<r;
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" bisection4

D:\okul\SCO1>bisection4
 INPUT LOWER LIMIT OF SEARCH AREA a : 1
 INPUT UPPER LIMIT OF SEARCH AREA b : 4
r=1.4142135623551439494
> Terminated with exit code 0.
```

Now we will apply a small change to the previous example. New class will be defined inside of the main class

```
#include <stdio.h>
#include <iostream>
#include <math.h>
  #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;
class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
```

```
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};
double bisection(f_x &f,double xl,double xu)
{
//bisection root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr;
int maxit=100;
int iter=0;
es=0.00000001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
{
   xr=(xl+xu)/2.0;
   iter++;
   if(xr!=0)
      { ea=fabs((xu-xl)/(xu+xl))*100.0;}
   fxl= f.func(xl);
   fxr= f.func(xr);
   test= fxl*fxr;
   if(test==0.0)      ea=0;
   else if(test<0.0)  xu=xr;
   else               {xl=xr;}
}
if(iter>=maxit) cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return xr;
}
int main()
{  double a,b,r;
   class f1:public f_x{public:double func(double x){ return x*x-2; };};
   f1 f;
   cout << " INPUT LOWER LIMIT OF SEARCH AREA a : ";
   cin >>a;
   cout << " INPUT UPPER LIMIT OF SEARCH AREA b : ";
   cin >>b;
   r=bisection(f,a,b);
   cout<<setprecision(20);
   cout<<"r="<<r;
   return 0;
}
```

Results will be same as above application.
Similar application is given for newton-raphson root finding method of numerical math. In this example in order to find roots derivative function defined in class f_x is also utilised

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x)
  { double ff=x*x-2.0;
              return ff;
  }
};
```

```
double newton(f_x &f,double x)
{
int nmax=100;
double tolerance=1.0e-8;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=f.dfunc(x);
  x-=fx/dfx;
  cout <<"i="<<i<<"x="<<x<<"fx="<<fx<<"dfx="<<dfx<<"\n";
  if(fabs(fx)<tolerance) { return x;}
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;
}

int main()
{  double xa,r;
   cout << " first guess for the root value : ";
   cin >>xa;
   class f1:public f_x{public:double func(double x){ return x*x-2; };};
   f1 f;
   cout<<setprecision(20);
   r=newton(f,xa);
   cout<<"root value : ="<<r;
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" newton

D:\okul\SCO1>newton
 first guess for the root value : 1
i=0x=1.5000000000003854694fx=-1dfx=1.9999999999984581223
i=1x=1.4166666666667311336fx=0.25000000000011564083dfx=3.0000000000023181457
i=2x=1.4142156862745109969fx=0.0069444444446271624827dfx=2.8333333333347132665
i=3x=1.4142135623746898698fx=6.007304885979891651e-006dfx=2.8284313725477510104
i=4x=1.4142135623730951455fx=4.5106141044470859924e-012dfx=2.8284271247495369472
root value : =1.4142135623730951455
> Terminated with exit code 0.
```

Struct in C

Class is not existed in C language but Struct which contain variables can be defined

```
#include <stdio.h>
#include <string.h>

struct Books {
   char  title[100];
   char  author[100];
   char  subject[100];
};

void print( struct Books b )
{
   printf( "Title : %s\n", b.title);
   printf( "Author : %s\n", b.author);
   printf( "subject : %s\n\n", b.subject);
}
int main( ) {
   struct Books b1;
   strcpy( b1.title, "Applied Mathematics");
   strcpy( b1.author, "M. Turhan Coban");
   strcpy( b1.subject, "Mathematics");
   print(b1);
   struct Books b2;
   strcpy( b2.title, "Slaughterhouse");
```

```
   strcpy( b2.author, "Kurt Vonneguti");
   strcpy( b2.subject, "Science Fiction");
   print(b2);
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" Books1

D:\okul\SCO1>Books1
Title : Applied Mathematics
Author : M. Turhan Coban
subject : Mathematics

Title : Slaughterhouse
Author : Kurt Vonneguti
subject : Science Fiction


> Terminated with exit code 0.
```

Templates and generic functions in C++

A template is a simple and yet very powerful tool in C++. The simple idea is to pass data type as a parameter so that we don't need to write the same code for different data types. For example, a software company may need sort() for different data types. Rather than writing and maintaining the multiple codes, we can write one sort() and pass data type as a parameter.

```
#include <iostream>
using namespace std;

// One function works for all data types.  This would work
// even for user defined types if operator '>' is overloaded
template <typename T>
T Max(T x, T y)
{  T m=x;
   if(x>y) m=x;
   else m=y;
   return m;
}

int main()
{
  cout << "Maximum int : "<<Max<int>(3, 7) << endl;  // Call Max for int
  cout << "Maximum char : "<<Max<char>('g', 'e') << endl;   // call Max for char
  cout << "Maximum string : "<<Max<string>("Ali", "veli") << endl;   // call Max for char
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" Max_T

D:\okul\SCO1>Max_T
Maximum int : 7
Maximum char : g
Maximum string : veli

> Terminated with exit code 0.
```

```
#include <iostream>
using namespace std;

// A template function to implement bubble sort.
// We can use this for any data type that supports
// comparison operator < and swap works for it.
```

```
template <class T>
void Sort(T a[], int n) {
    for (int i = 0; i < n - 1; i++)
        for (int j = n - 1; i < j; j--)
            if (a[j] < a[j - 1])
              swap(a[j], a[j - 1]);
}
template <class T>
void print(T a[], int n)
{   // calls template function
    Sort<T>(a, n);
    cout << " Sorted array : ";
    for (int i = 0; i < n; i++)
        cout << a[i] << " ";
    cout << endl;
}

int main() {
    int a[5] = {10, 50, 30, 40, 20};
    int n = sizeof(a) / sizeof(a[0]);
    print(a,n);
    double b[5] = {10.4534, 50.94874, 30.08497, 40.123, 20.78847};
    n = sizeof(b) / sizeof(b[0]);
    print(b,n);
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" sort_T

D:\okul\SCO1>sort_T
 Sorted array : 10 20 30 40 50
 Sorted array : 10.4534 20.7885 30.085 40.123 50.9487

> Terminated with exit code 0.
```

Containers in C++

Containers are dynamic arrays that array sizes can be change during running the program. Containers usually defined with template class structure so that any variable type can be used with them

```
// C++ program to illustrate the iterators in vector
#include <iostream>
#include <vector>
using namespace std;

template <class T>
void print(vector<T> g1)
{   int i=0;
    cout << "Output of begin and end: ";
    for (auto i = g1.begin(); i != g1.end(); ++i)
        cout << *i << " ";
}

int main()
{
    vector<int> g1;
    for (int i = 1; i <= 20; i++)
        g1.push_back(i);
    print(g1);
    return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" vector_T

D:\okul\SCO1>vector_T
Output of begin and end: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
```

```
> Terminated with exit code 0.
```

A second example of container vector:

```
 // C++ program to illustrate the iterators in vector
#include <iostream>
#include <vector>
#include <bits/stdc++.h>
using namespace std;

template <class T>
void print(vector<T> g1)
{   int i=0;
    for (auto i = g1.begin(); i != g1.end(); ++i)
        cout << *i << " ";
    cout<<endl;
}

int main()
{   double a[]={1.2,2.3,3.3,4.1,7.8};
    vector<double> g1;
    int n=sizeof(a)/sizeof(a[0]);
    for (int i = 0; i < n; i++)
        g1.push_back(a[i]);
    cout<<"load double a into vector:";
    print(g1);
    double b[]={11.2,42.3,13.31,4.1345,17.8325};
    int n1=sizeof(b)/sizeof(b[0]);
    for (int i = 0; i <n1; i++)
        g1.push_back(b[i]);
    cout<<"add double b into vector";
    print(g1);
    cout<<"sort the  vector";
    sort(g1.begin(),g1.end());
     print(g1);
    return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" vector_T1

D:\okul\SCO1>vector_T1
load double a into vector:1.2 2.3 3.3 4.1 7.8
add double b into vector1.2 2.3 3.3 4.1 7.8 11.2 42.3 13.31 4.1345 17.8325
sort the  vector1.2 2.3 3.3 4.1 4.1345 7.8 11.2 13.31 17.8325 42.3

> Terminated with exit code 0.
```

Vector is just one of the generic container class. There are several other containers class, each with a different data set structures such as ques, lists and stacks.

Sequence containers

Sequence containers implement data structures which can be accessed sequentially.

array	static contiguous array
vector	dynamic contiguous array
deque	double-ended queue
forward_list	singly-linked list
list	doubly-linked list

Associative containers

Associative containers implement sorted data structures that can be quickly searched (*O(log n)* complexity).

set	collection of unique keys, sorted by keys

map	collection of key-value pairs, sorted by keys, keys are unique
multiset	collection of keys, sorted by keys
multimap	collection of key-value pairs, sorted by keys

Unordered associative containers

Unordered associative containers implement unsorted (hashed) data structures that can be quickly searched (*O(1)* amortized, *O(n)* worst-case complexity).

unordered_set	collection of unique keys, hashed by keys
unordered_map	collection of key-value pairs, hashed by keys, keys are unique
unordered_multiset	collection of keys, hashed by keys
unordered_multimap	collection of key-value pairs, hashed by keys

Container adaptors

Container adaptors provide a different interface for sequential containers.

stack	adapts a container to provide stack (LIFO data structure)
queue	adapts a container to provide queue (FIFO data structure)
Priority_queue	adapts a container to provide priority queue

Set container examples.

```
// C++ program to illustrate the iterators in set
#include <iostream>
#include <set>
#include <bits/stdc++.h>
using namespace std;

template <class T>
void print(set<T> g1)
{   int i=0;
    for (auto i = g1.begin(); i != g1.end(); ++i)
        cout << *i << " ";
    cout<<endl;
}

int main()
{
    set<string> g1;
    string a[]={"first","second","third","first","fifth","sixth","third"};
    int n=sizeof(a)/sizeof(a[0]);
    for (int i = 0; i < n; i++)
        g1.insert(a[i]);
    cout<<"load string a into set:\n";
    print(g1);
    return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" set_T

D:\okul\SCO1>set_T
load string a into set:
fifth first second sixth third

> Terminated with exit code 0.
```

Note that repeated “first” and “third” is listed only once. It is also sorted according to letter order. The same program with double variables

```
 // C++ program to illustrate the iterators in set
#include <iostream>
#include <set>
#include <bits/stdc++.h>
using namespace std;
```

```
template <class T>
void print(set<T> g1)
{   int i=0;
    for (auto i = g1.begin(); i != g1.end(); ++i)
        cout << *i << " ";
    cout<<endl;
}

int main()
{
    set<double> g1;
    double a[]={4.4,4.3,2.2,3.3,4.4,1.1,2.2,5.6,7.1,0.8};
    int n=sizeof(a)/sizeof(a[0]);
    for (int i = 0; i < n; i++)
        g1.insert(a[i]);
    cout<<"load double a into set:\n";
    print(g1);
    double b[]={7.1,4.4,2.35,2.2,3.75,1.12,3.27};
    int n1=sizeof(b)/sizeof(b[0]);
    for (int i = 0; i < n1;i++)
        g1.insert(b[i]);
    cout<<"load double b into set:\n";
    print(g1);
    return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" set_T1

D:\okul\SCO1>set_T1
load double a into set:
0.8 1.1 2.2 3.3 4.3 4.4 5.6 7.1
load double b into set:
0.8 1.1 1.12 2.2 2.35 3.27 3.3 3.75 4.3 4.4 5.6 7.1

> Terminated with exit code 0.
```

Map defines a pair of variables and one of the pair element as act reference while the second is as data

```
#include <map>
using namespace std;

int main (){
  map<char,int> first;

  //initializing
  first['a']=11;
  first['b']=22;
  first['c']=35;
  first['d']=15;
  first['e']=55;
   map<char, int>::iterator it;
   for(it=first.begin(); it!=first.end(); ++it){
      cout << it->first << " => " << it->second << '\n';
   }

  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" map1

D:\okul\SCO1>map1
a => 11
b => 22
c => 35
d => 15
e => 55

> Terminated with exit code 0.
```

Embedded for statement

For statement can be used in a short notation form. It is called emmedded form. As an old programmer, I prefer the old ways of using it, but it is up to you which way you prefer to use it.

```
#include <iostream>
#include <vector>
using namespace std;

int main()
{   //classical for
    vector<int> v {1,2,5,7,9,11,19};
    for(int i = 0; i < v.size(); ++i)
    {cout << v[i]<<" ";}
    cout<<"\n";
    // embedded for
    for(auto& num : v)
    {cout << num<<" ";}
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" embeddedfor

D:\okul\SCO1>embeddedfor
1 2 5 7 9 11 19
1 2 5 7 9 11 19
> Terminated with exit code 0.
```

If we use double array instead of vector class:

```
#include <iostream>
using namespace std;

int main()
{   //classical for
    double v[]= {1,2,5,7,9,11,19};
    int n=sizeof(v)/sizeof(v[0]);
    for(int i = 0; i < n; ++i)
    {cout << v[i]<<" ";}
    cout<<"\n";
    // embedded for
    for(auto& num : v)
    {cout << num<<" ";}
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" embeddedfor1

D:\okul\SCO1>embeddedfor1
1 2 5 7 9 11 19
1 2 5 7 9 11 19
> Terminated with exit code 0.
```

Reading and writing from files

So far, we have been using the **iostream** standard library, which provides **cin** and **cout** methods for reading from standard input and writing to standard output respectively.
This tutorial will teach you how to read and write from a file. This requires another standard C++ library called **fstream**, which defines three new data types –

Sr.No	Data Type & Description
1	**ofstream**

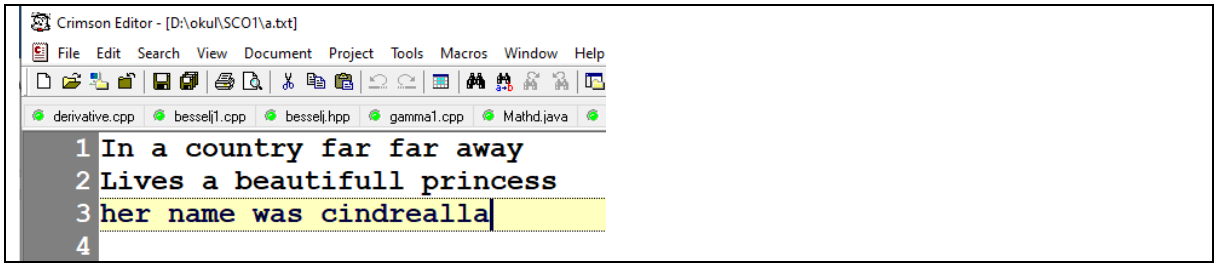
	This data type represents the output file stream and is used to create files and to write information to files.
2	**ifstream** This data type represents the input file stream and is used to read information from files.
3	**fstream** This data type represents the file stream generally, and has the capabilities of both ofstream and ifstream which means it can create files, write information to files, and read information from files.

To perform file processing in C++, header files <iostream> and <fstream> must be included in your C++ source file.

```
#include <fstream>
#include <iostream>
using namespace std;

int main () {
  string data[100];
  string data1;
  // open a file in write mode.
  ofstream fout;
  fout.open("a.txt");
  data[0]="In a country far far away";
  data[1]="Lives a beautifull princess";
  data[2]="her name was cindrealla";
  cout << "Writing to the file" << endl;
  for(int i=0;i<3;i++)
  {fout<<data[i]<<endl;}
  fout.close();

  // open a file in read mode.
  ifstream fin;
  fin.open("a.txt");
  while(getline(fin,data1))
  { cout << data1 << endl; }
  fin.close();

  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" read1

D:\okul\SCO1>read1
Writing to the file
In a country far far away
Lives a beautifull princess
her name was cindrealla

> Terminated with exit code 0.
```

a.txt



Reading and writing data in c is a bit different. There are no stream functions in c . fopen function is used to acces a file

Opening Files in C

You can use the **fopen( )** function to create a new file or to open an existing file. This call will initialize an object of the type **FILE**, which contains all the information necessary to control the stream. The prototype of this function call is as follows −

FILE *fopen(const char * filename, const char * mode);

Here, **filename** is a string literal, which you will use to name your file, and access **mode** can have one of the following values –

Sr.No.	Mode & Description
1	**r**
2	**w**
3	**a**
4	**r+**
5	**w+**
6	**a+**

If you are going to handle binary files, then you will use following access modes instead of the above mentioned ones −

"rb", "wb", "ab", "rb+", "r+b", "wb+", "w+b", "ab+", "a+b"

Closing a File

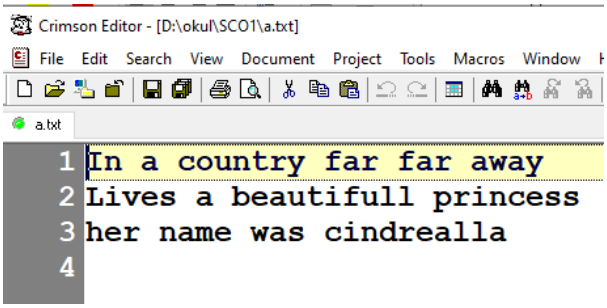
To close a file, use the fclose( ) function. The prototype of this function is −

int fclose(FILE *fp);

```
#include <stdio.h>

int main () {
   char data1[255];
   char c;
   // open a file in write mode.
   FILE *fp;
   fp=fopen("a.txt","w+");
   char data[]="In a country far far away\nLives a beautifull princess\nher name was cindrealla\n";
   fprintf(fp,data);
   fclose(fp);
   // open a file in read mode.
   char buff[255];
   fp=fopen("a.txt","r");
   int i=0;
   while((c = getc(fp)) != EOF)
   { data1[i]=c;i++; }
   printf(data1);
   fclose(fp);
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" read2

D:\okul\SCO1>read2
In a country far far away
Lives a beautifull princess
her name was cindrealla

> Terminated with exit code 0.
```

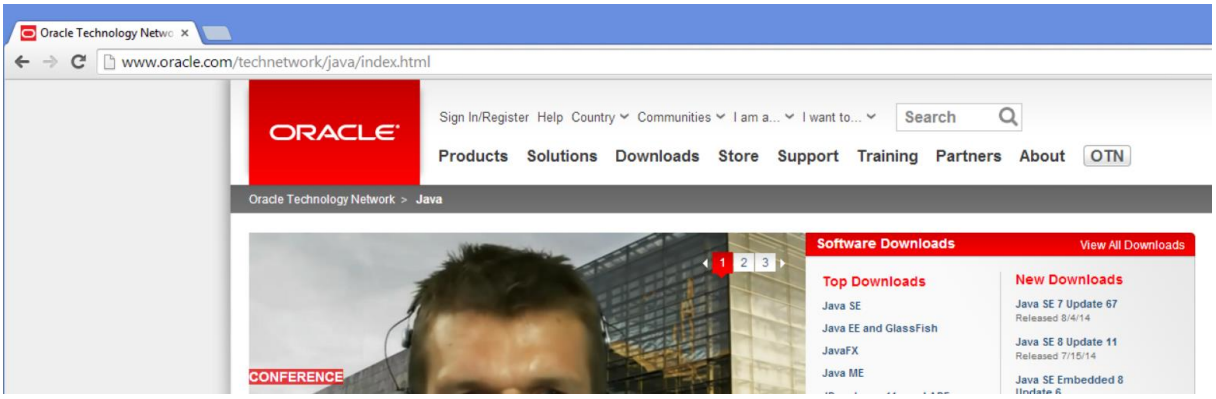
a.txt


Crimson Editor - [D:\okul\SCO1\a.txt]
File Edit Search View Document Project Tools Macros Window
a.txt
1 In a country far far away
2 Lives a beautifull princess
3 her name was cindrealla
4

1.6 JAVA LANGUAGE BASICS

1.6.4 PREAPARATION TO ESTABLISH JAVA PROGRAMMING ENVIRONMENT IN YOUR COMPUTER

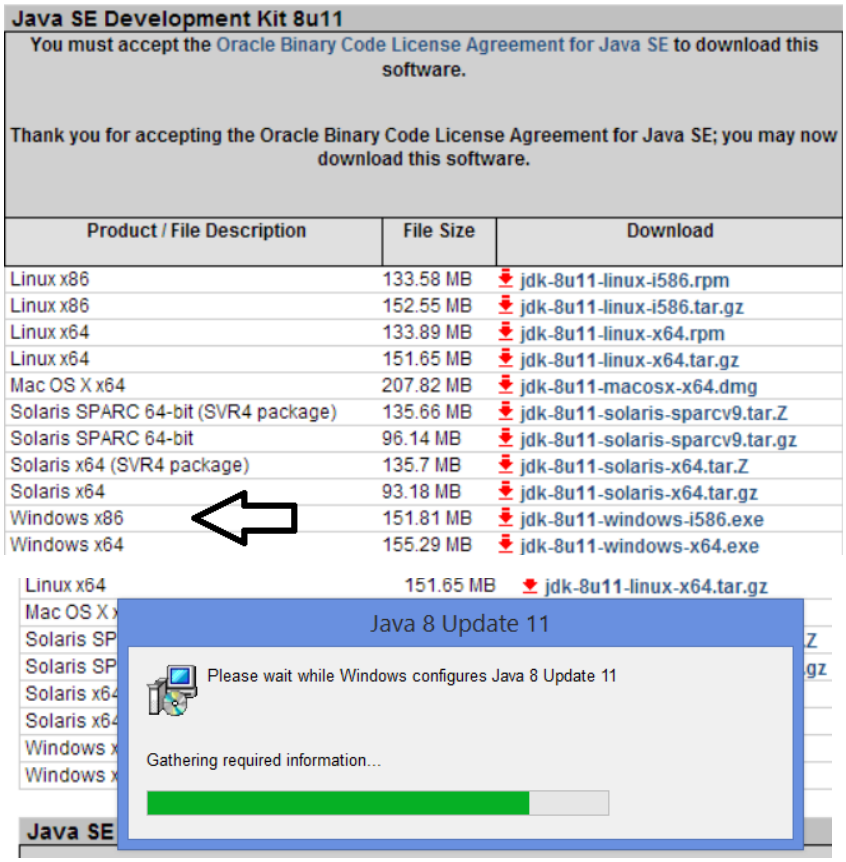
In order to start computer programming with java language you should have java development kid (JDK) in your computer. JDK package is available from sun computer company and you can get it from internet adres http://www.oracle.com/technetwork/java/index.html



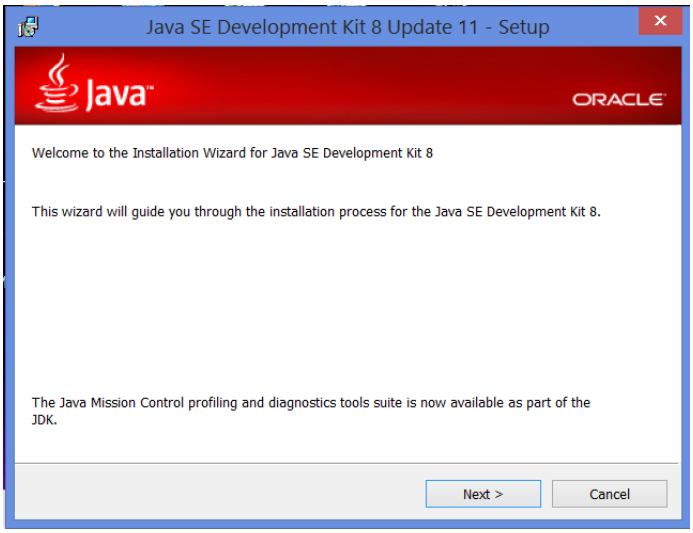
Select Java SE, then select JDK8 Update 11 or most recent JDK package available to load, push Download button



Select Licence agreement and then Select your platform (Windows, Linux etc.) . A note for 64 bit users. Java has huge library and 32 bit version has better completed libraries specially for the ones that would like to connect to external devices, so I am using 32 bit version of JDK

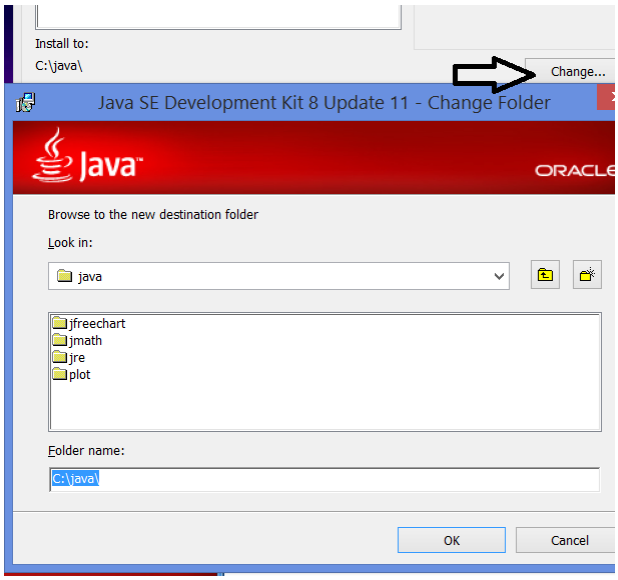


After saving the program we can start it by double click

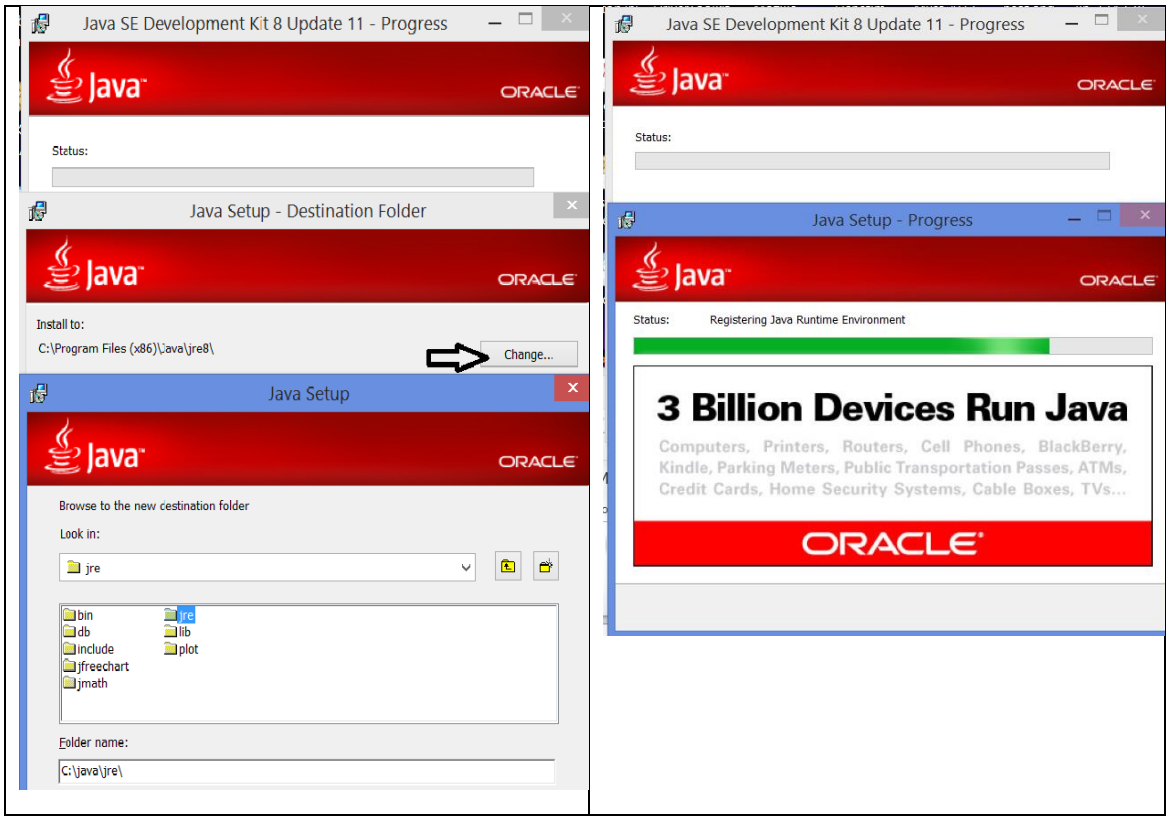


Specify the directory you would like to install java Develpment Kit (JDK), I do recommend a simple directory to easy Access , such as d:\java, Then install it



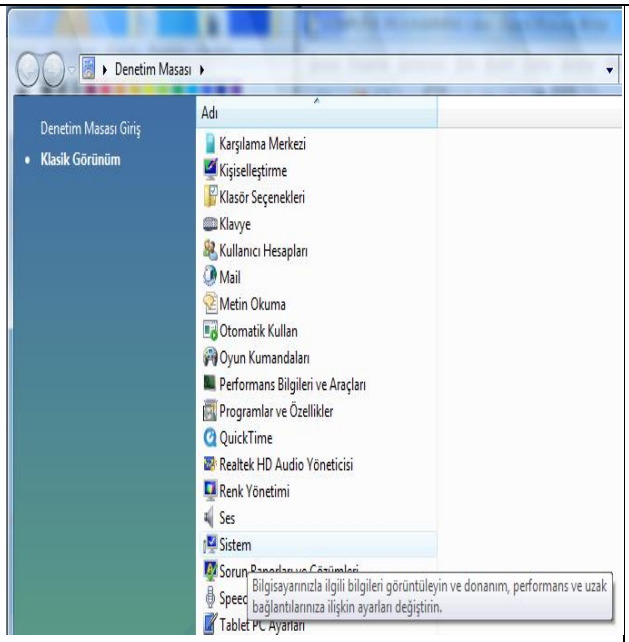


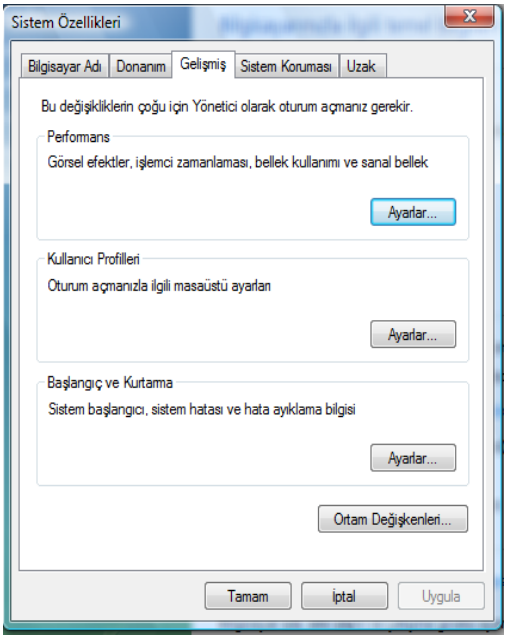
Select the folder you would like to put JDK set. I do recommend d:\java as a simple option. Then program will ask the folder for jre (Java Runtime environment) we can select jre folder under the java folder we already selected (d:\java\jre)



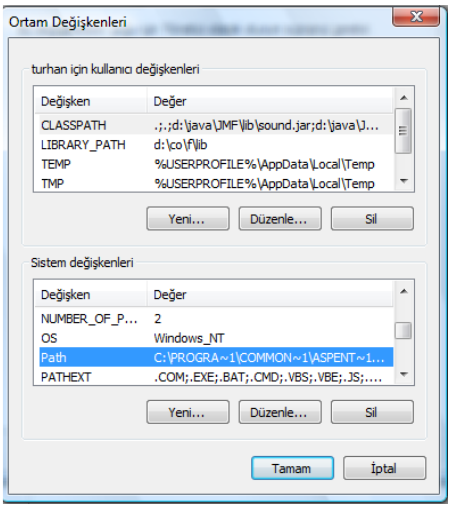


Now java is installed. The next is to add java directories to path statement. When the computer could not able to find a command it will check the directories given in path statement and if it finds it there it will use it. In order to to do this, from control panel select the system folder

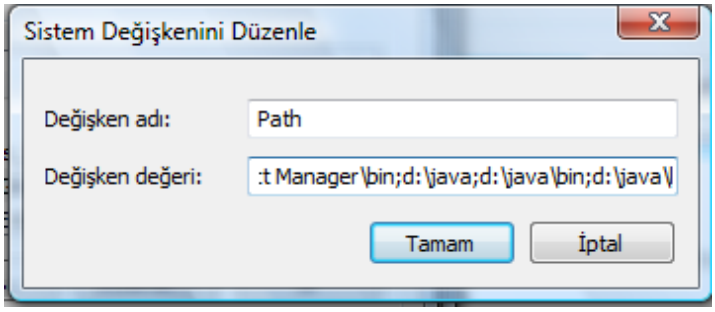




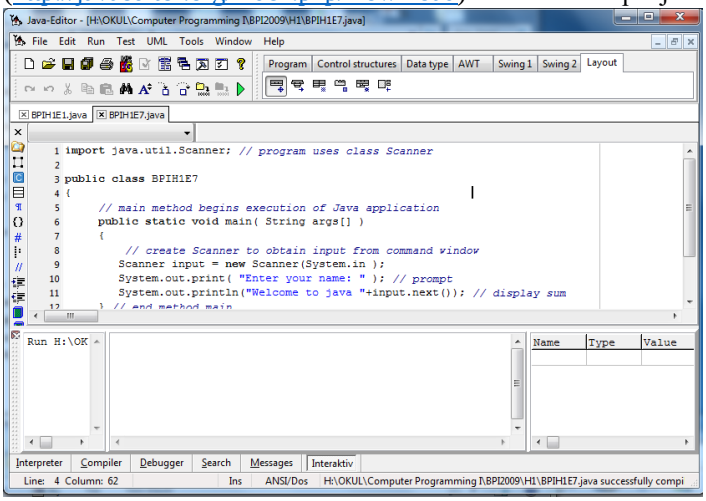
Select environmental variables and find path



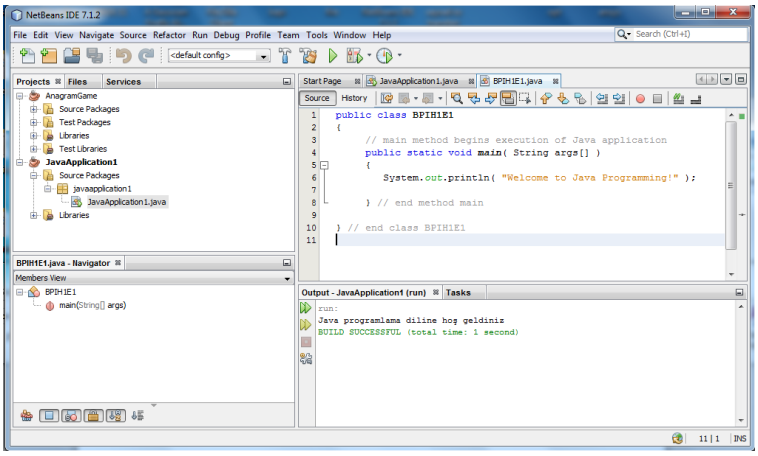
select Path statement and add bin and lib directories path in out case d:\java\bin and d:\java\lib.



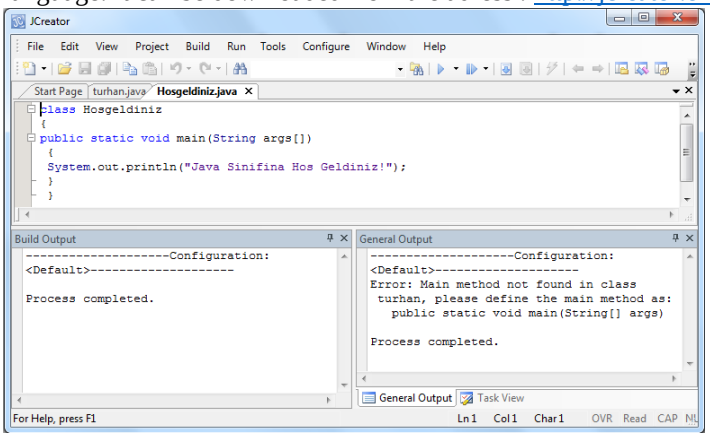
Next, I recommend you to use an editor to edit and compile java programs. There are several freeware good editors are available to use togetger wiyh java. Let us look at some of these editors. Everyone java Editor(http://sourceforge.net/projects/eje/) is a simple java editör. Java-editor (http://javaeditor.org/index.php/Download) is another simple java editör created to use directly with java.



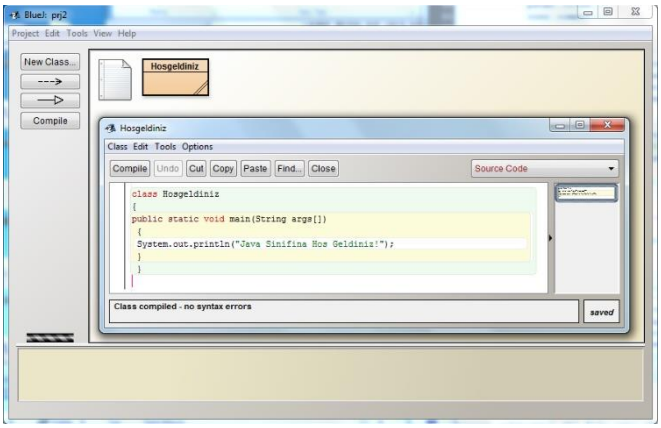
Netbeans is an advanced java editör (http://netbeans.org/downloads/index.html)



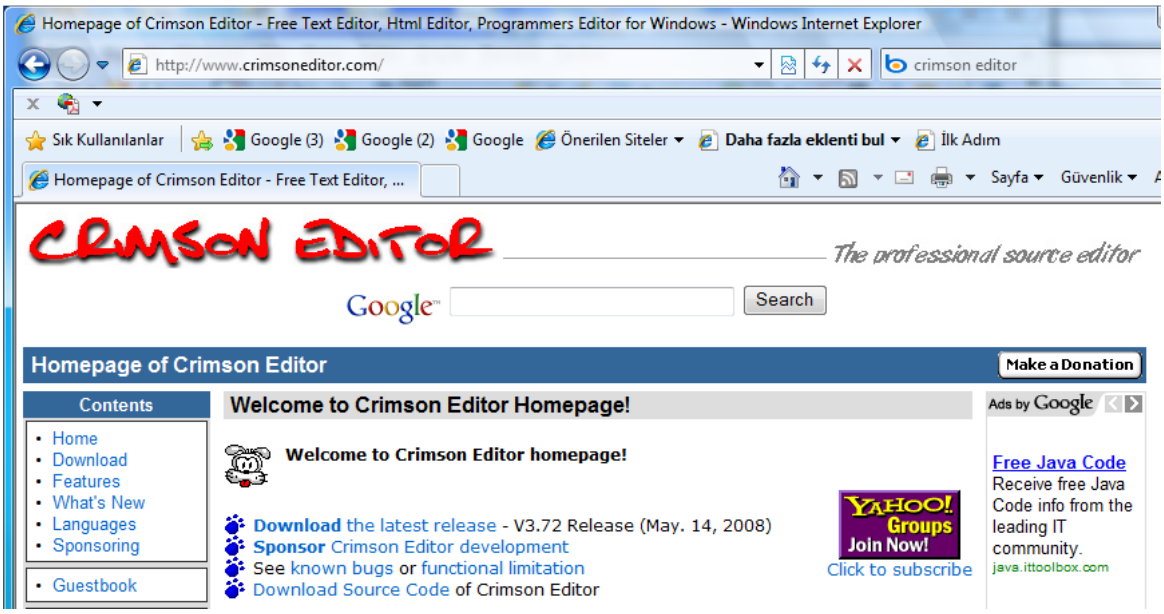
Another simple nice freware java editör is JCreator. This editör is also designed to be used only with java language. It can be downloaded from the adress : http://jcreator.en.softonic.com/



A recent editor that easy to use with java is BlueJThis editör can be downloaded from http://www.bluej.org/ adress

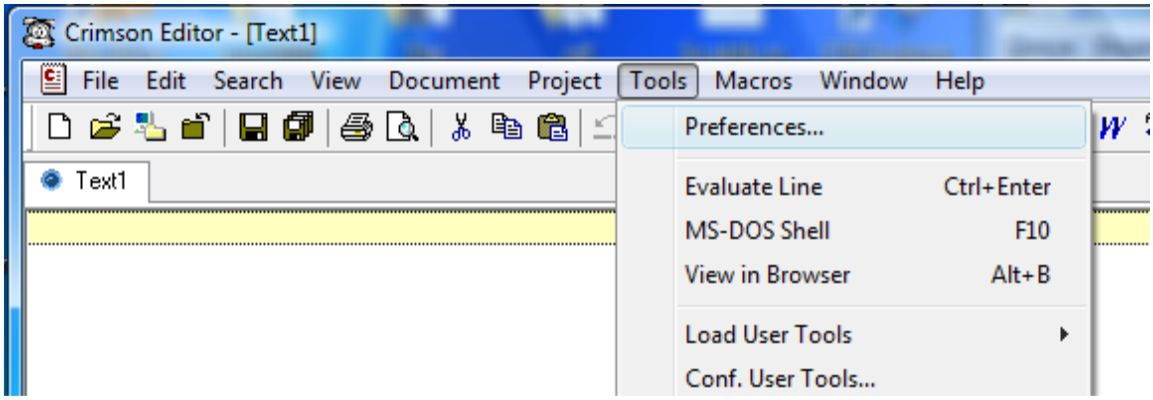


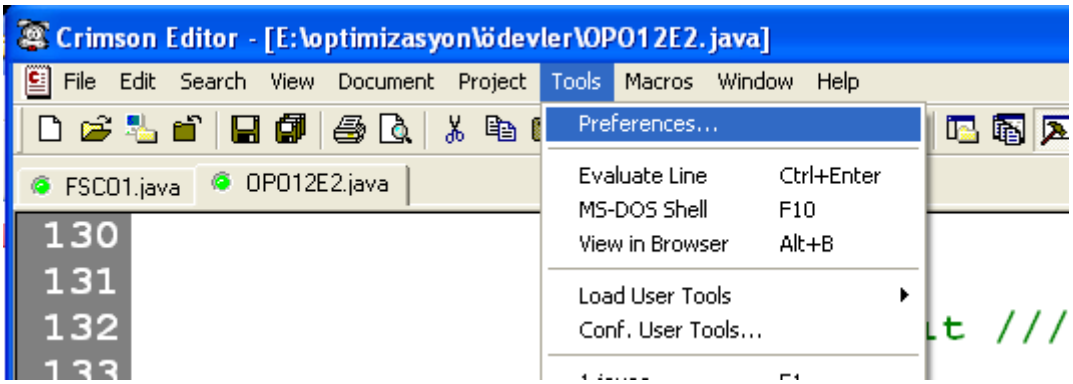
I do recommend a simpler editor called crimson editor for starting programmers. This is also a free program, you can obtain a copy from internet adress http://www.crimsoneditor.com/ .This is a multilanguage editor, which java compiling can be established as well as other programming languages.



The latest version of Crimson editor when ı looked at was Version 3.72 the file it loads named as : cedt-241-setup.exe. By double click you can load it up to your computer.

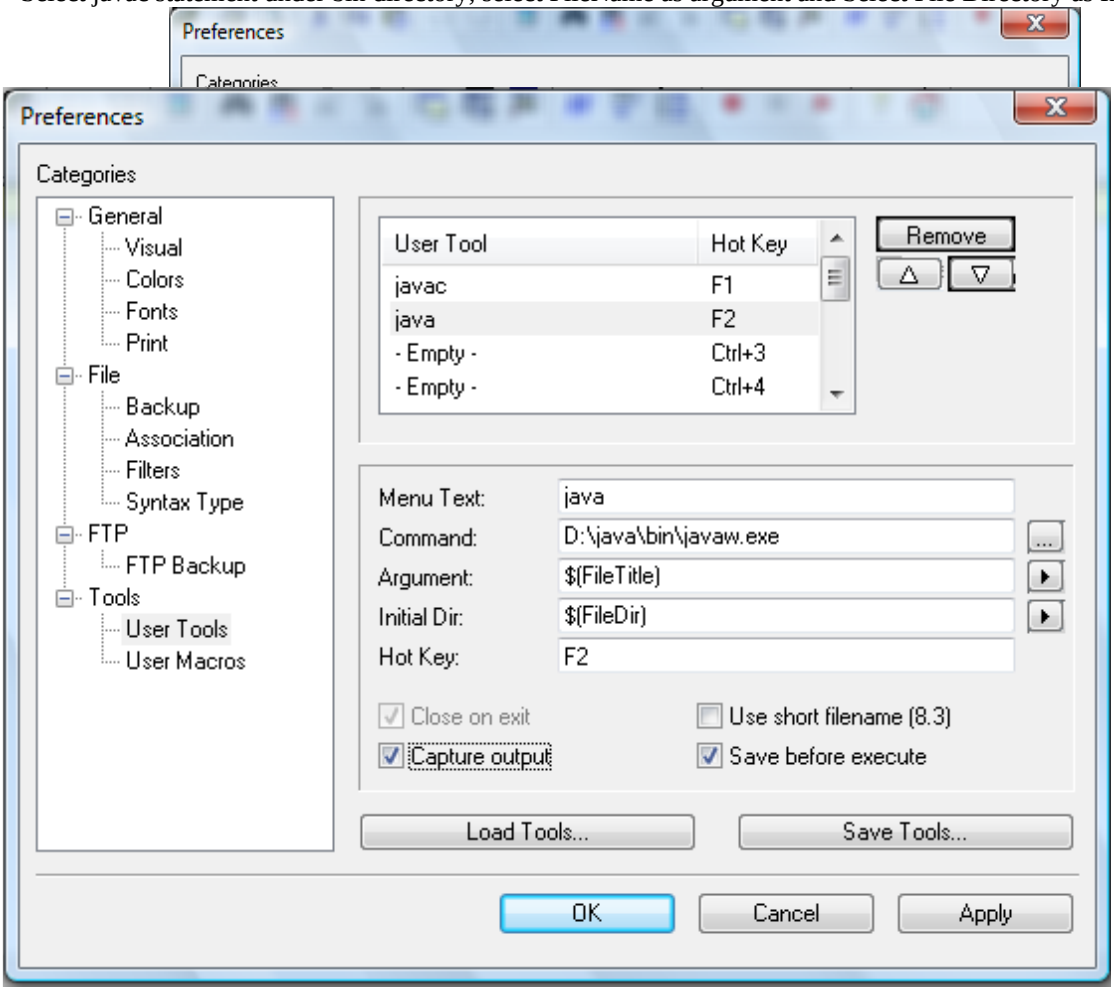
In order to compile and run java programs in crimson editor select Tools, preferences, User tools. Cedit programında java dosyanızı çalıştırmak istiyorsanız: Tools, preferences, User Tools düğmelerine basarak javac ve java komutlarını editöre aşağıdaki pencerelerde görüldüğü gibi girebiliriz. (burada java d:\java dosya sistemine yüklenmiştir.)



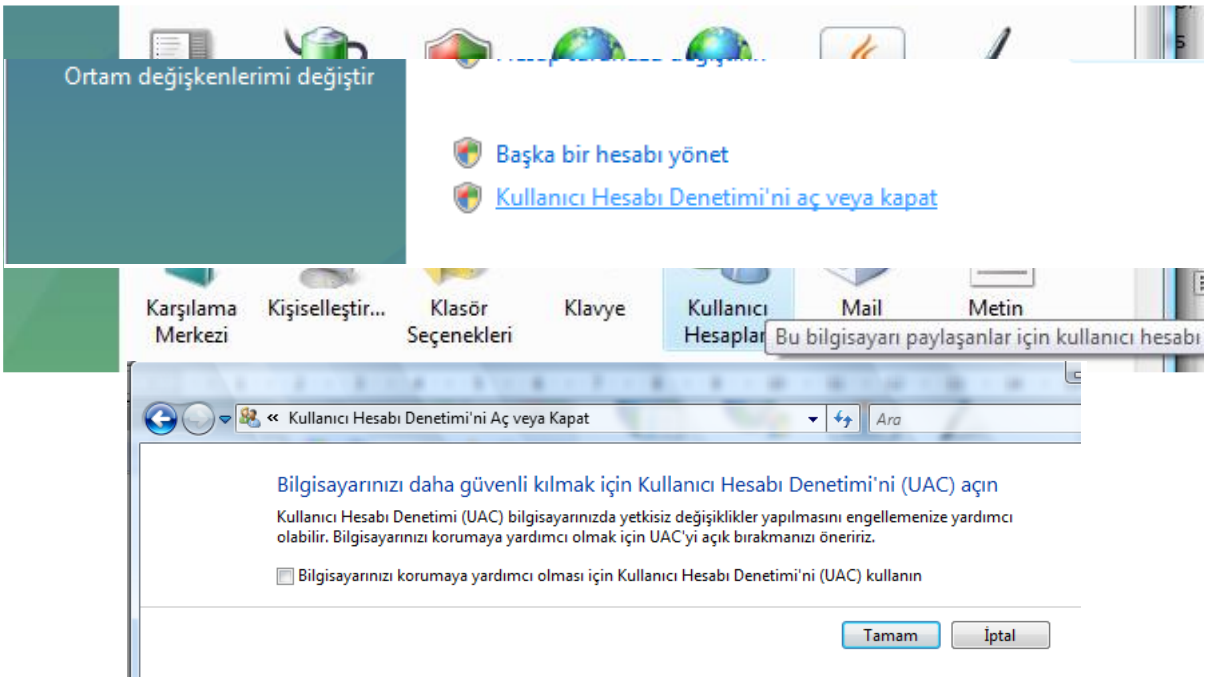


then enter java compile statement javac

Select javac statement under bin directory, select FileName as argument and Select File Directory as Initial



directory. Select F1 as hot key, select Capture output button. In order to enter java runtime enterpretation statement java enter java as Menu Text, As command enter javaw or java command under java compilers bin directory. Select FileTitle as Argument, select FileDir as initial directory, and select F2 as hot key. After doing this you can able to work with java programs, compile it by pressing F1 and run them by pressing F2. One note for MS Vista users, In order to protect you from nasty programs vista does not allow you to make changes in your editor settings, iIf you want this settings remains in its place, go to control panel and select user accounts, then open and closed the user account button



and turn off the user acount security control button.

In addition , I do use important public access java packages such as visad (3 D graphic package) . If you like to use it, obtain a copy of visad.jar from the adress http://www.java2s.com/Code/Java/Advanced-Graphics/GraphLibraryVisad.htm, the load up java 3D package and as well from the adress http://java3d.java.net/binary-builds.html, and load java3D and copy visad.jar into ../jre/lib/ext directory.

Excell spread sheet read and write package xls.jar from the adress http://jexcelapi.sourceforge.net/ and again copy jxl.jar into the ../jre/lib/ext directory

If you like to use SQLite database with java you can get necassary connection file sqlitejdbc-v056.jar from http://www.zentus.com/sqlitejdbc/ and again load it into the ../jre/lib/ext directory

1.6.5 INTRODUCTION TO JAVA PROGRAMING LANGUAGE

Now we can start to write our first program. You can use your crimson editor or any other environment to write your program. We will assume that crimson editor is used. Our first example is a welcome message. It is traditional in programming language teaching to give a welcome message as the first application

Program BPIH1E1.java programı

```
public class BPIH1E1
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       System.out.println( "Welcome to Java Programming!" );
    } // end method main

} // end class BPIH1E1
```

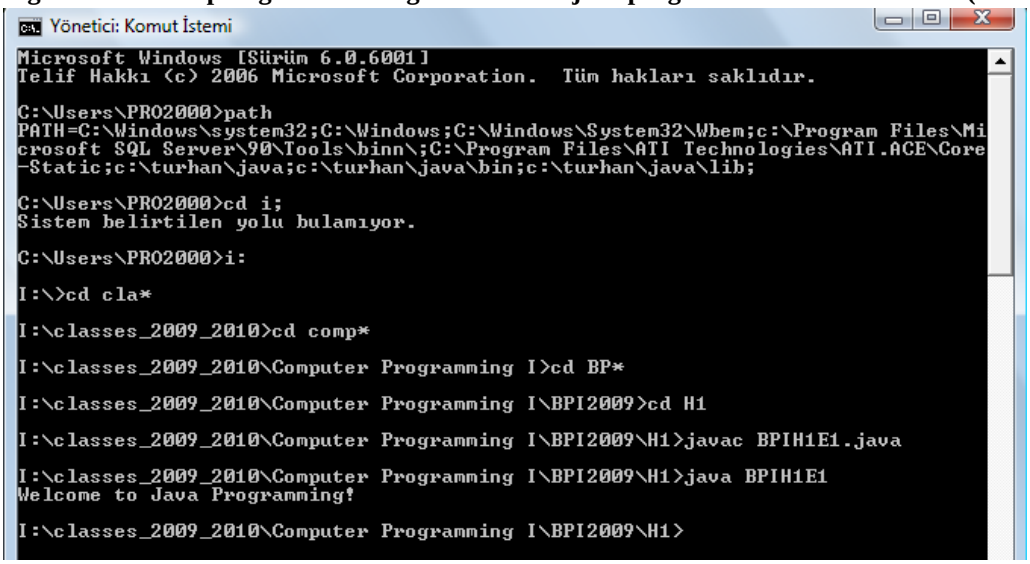
Let us assume that we write our program in any editor and save it. All Java programs has an external envelope called class. The name of the class should be same as the name of the program file. The class name fort his first example is given as BPIH1E1, therefore we will save the file as BPIH1E1.java

In this stage, we will save it in the same directory as we are working. In order to compile the program**, javac** statement is used. It can be carry out in console environment (dos for window system). After compilation, program create a class file, in this example a file called **BPIH1E1.class**. This file contains a machine language-like code that is easily convertible to machine language code for the operation systems and processor of actual computer program is running. This conversion process is done through an interpreter specific to the computer system program is running. In order to operate the interpretor (convertion the class bycode to the machine language code **java** command is used. To run the above program in console environment will require then rhe following commands;

>javac BPIH1E1.java
>java BPIH1E1

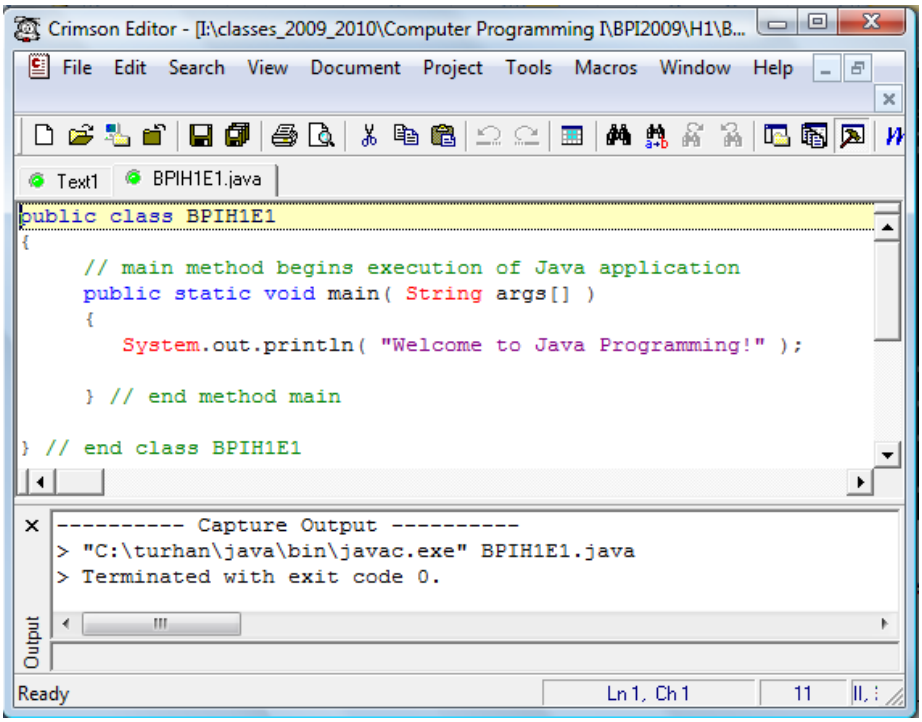
In figure 1.2.1 compiling and running of **BPIH1E1.java** program by using javac and java commands are shown

Figure 1.2.1 compiling and running of BPIH1E1.java program in Windows console (DOS) environment



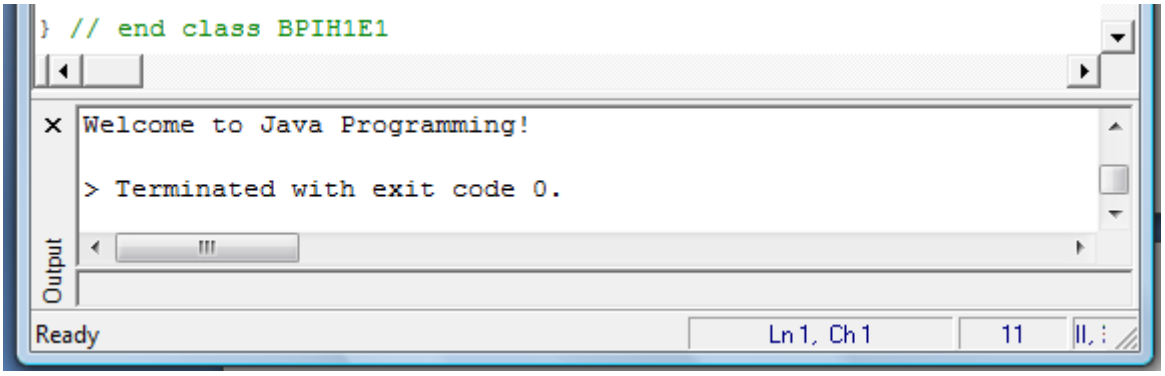
If we would like to do the same by using crimpson editor, all we have to do is to use pre-programmed F1 and F2 buttons . When F1 button pushed (javac compiler command)

Figure compiling of BPIH1E1.java program in Windows console (DOS) environment by using preprogrammed F1 button in Crimson Editor



Program compiles and **BPIH1E1.class** is created. When F2 button pushed, **BPIH1E1.class** file interpreted the machine language and program runs in the computer. Because of this dual system java codes can run free of the operating systems or computer, the samer code (java class code) will run in any computer.

Figure interpreting (running) of BPIH1E1.class bycode in Windows console (DOS) environment by using preprogrammed F2 button in Crimson Editor



Now let us look at the details of our first program, The first line

public class BPIH1E1 is given us basic class name, term public specify that the code of the program will be public (will not be hidden). Classes started with big open paranthesis, {, and ends with big close paranthesis, }. This is also valid for other java statements as well. Big paranthesis are the same as paragraph start and finish signs.

The line **// main method begins execution of Java application** is a command statement, computer does not do anything related to this line, it is for us to understand the program code.

The line **public static void main(String args[])** start a subprogram part called a method. The name of our method is **main,** the word **void** tells us that this method is not given us any output from this program part.
System.out.println("Welcome to Java Programming!");
print to console(DOS) screen **Welcome to Java Programming!** The sign **"** indicates that it is a writing (String) type of constant. The sign **;** indicates the end end of the programming command **System.out.println** is another method defined in java library **java.io.** With closed big paranthesis main method closed, and class is closed with another big paranthesis.

Second example program , **BPIH1E2.java,** will do the same as the first program, but in this one two output commands are used.

System.out.print("Welcome to ");
System.out.println("Java Programming!");

The difference between System.out.print and System.out.prinln is that the second one includes a carriage return, while the first one stops in the same line, therefore the output of this program is exactly same as the previous one.

Program BPIH1E2.java programı

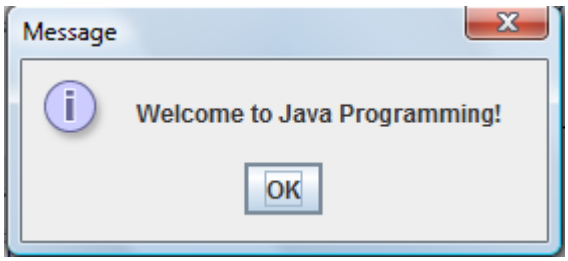
```
import javax.swing.*;
public class BPIH1E2
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
    System.out.print("Welcome to ");
    System.out.println("Java Programming!" );
    } // end method main

} // end class BPIH1E2
```

In the third code, in order to take an output, the command
JOptionPane.showMessageDialog(null,"Welcome to Java Programming!");
Is used. This program uses library method **JOptionPane.showMessageDialog.** This method is defined in java library **javax.swing.** The first line of the program indicates that we are connected to the codes in this library * indicates that all codes under this library is called. The output of this methos is given as a graphic output instead of console text output.

Program BPIH1E3.java programı

```
import javax.swing.*;// program uses class JOptionPane in swing library
public class BPIH1E3
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
    JOptionPane.showMessageDialog(null,"Welcome to Java Programming!" );
    } // end method main

} // end class BPIH1E3
```

Figure interpreting (running) of BPIH1E3.class bycode in graphic environment



A similar program to BPIH1E1.java is also given as **BPIH1E4.java.** In this program, instead of

System.out.println("Welcome to Java Programming!");
System.out.printf("%s\n%s\n","Welcome to", "Java Programming!"); is used. The result is exactly the same. This output structure is taken exactly from C programming language. Java is very similar to c and C++ as command structure. In the latest version of java also some of the same method names are alsıo introduced. Some control terms are used in this statement. Some of the very common control statements in java as follows
\n ctrl-return (next line)

\t ctrl-tab (next tab space)

\r enter key

\\ write \ sign

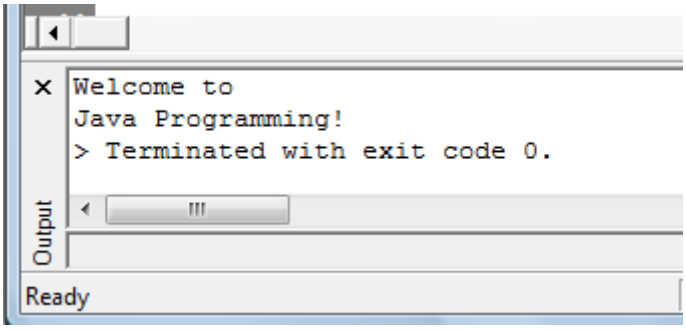
\" write " sign

There are also some printf format structures in printf statement %s means write a String (array of characters) constant or variable.

Program BPIH1E4.java programı

```
public class BPIH1E4
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
    System.out.printf( "%s\n%s\n","Welcome to", "Java Programming!" );

    } // end method main
} // end class BPIH1E4
```

The program output is



The carriage return (/n) in the middle of the formatting part of printf force the output to the next line

INTEGER AND REAL VARIABLES AND ARITHMETIC CALCULATIONS

Our next program **BPIH1E5.java** reads two integer numbers from console screen, and then adds this numbers, and output the result to the console screen. In order to read data from console environment, class **java.util.Scanner** is used. A channel to the console opens by using method Scanner. The name of the input channel (Scanner type variable) input. Scanner is basically another class structure (program) defined in java. Actual reading is done throgh a method **input.nextInt();** of class Scanner (located in the library java.util.Scanner)

Program BPIH1E5.java programı

```
import java.util.Scanner; // program uses class Scanner

public class BPIH1E5
```

```
{
   // main method begins execution of Java application
   public static void main( String args[] )
   {
      // create Scanner to obtain input from command window
      Scanner input = new Scanner( System.in );
      int number1; // first number to add
      int number2; // second number to add
      int sum; // sum of number1 and number2
      System.out.print( "Enter first int: " ); // prompt
      number1 = input.nextInt(); // read first number from user
      System.out.print( "Enter second integer: " ); // prompt
      number2 = input.nextInt(); // read second number from user
      sum = number1 + number2; // add numbers
      System.out.printf( "Sum is %d\n", sum ); // display sum
   } // end method main

} // end class BPIH1E5
```

Integer variables can be called as byte(8 bit), short(16 bit), int(32 bit) and long(64 bit) according to space they occupy in computer memory. Bits are binary structures that only contains 0 or 1. When bits coming together as byte groups of different sizes they represents integer numbers. For example maximum number can be written to 8 bit byte type of integer variable is 11111111 which represents 256. Usually one of the bits used as +/- sign in a number therefore byte carries numbers between -128 to 127 (total 256 including 0) . ın order to define an integer variable, first variable type defined and a variable name follows. An actual value can be assigned by using an = (equal) sign.

int number1;
number1=32;

or this two statement can be given only as one statement.

int number1=32;

In our program integer value is read from the console as

int number1;

number1 = input.nextInt();

Numbers represents with bits in computer memory. In table 1.3.1 different type of integer variables and their number representation limits are given.

Figure The bit sizes and 10 based limits of the integer type variables.

Integer type	**Byte size**	**Number represantation limits**
Byte	8 bit	From -128 to 127
Short	16 bit	From -32768 to 32767
int	32 bit	From -2157483648 to 2147483647
long	64 bit	From -9223372036854775808 to 9223372036854775808

In Table 1.3.2 representation of number 15 in byte type integer variable in computer bit structure.

Table writing 15 in a 8 bit integer variable (byte)

+/-	2^6	2^5	2^4	2^3	2^2	2^1	2^0

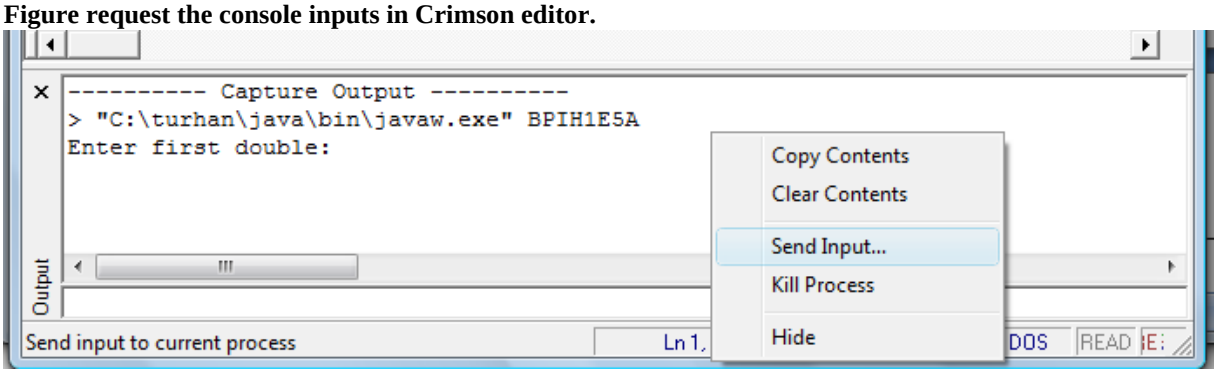
+	64	32	16	8	4	2	1
0	0	0	0	1	1	1	1

The following arithmetic proseses are defined in java for integer variables:
+ addition - substraction * multiplication / division and % module (remaining number) in integer division. For example an addition process can be defined as

sum = number1 + number2;

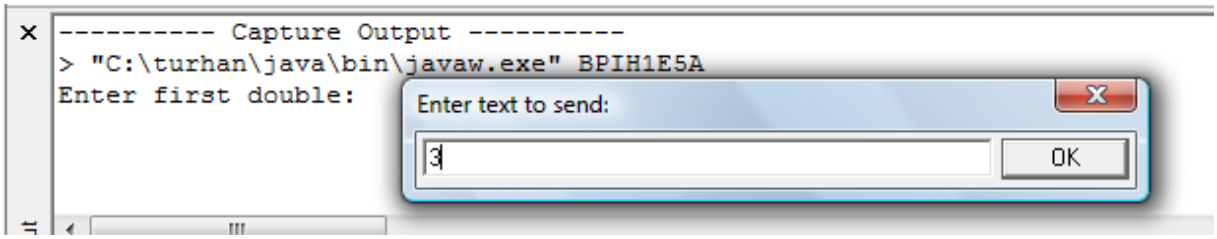
To get output of the example program in console environment by using Crimson editor, first we should send inputs through pushing send Input buton

Figure request the console inputs in Crimson editor.



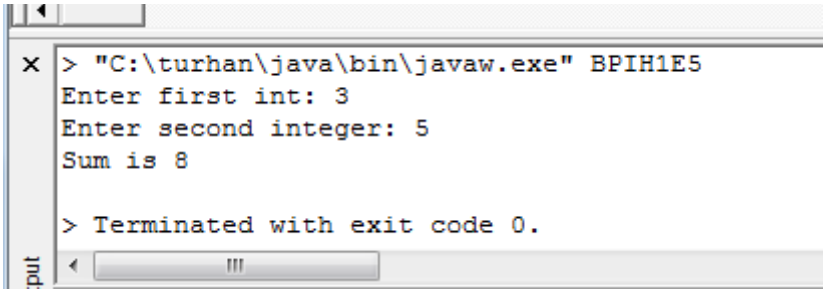
Then you can enter the number through the opening boxes

Figure sending the console inputs in Crimson editor.



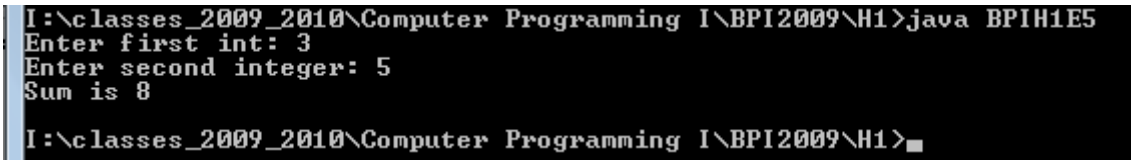
Then you can enter the second number the same way and result will apear in the capture output window as

Figure Crimson editor output of BPIH1E5.java



If we would like to do the same thing directly in dos windos we can enter the numbers directly

Figure Console(DOS) output of BPIH1E5.java



Instead of integer arithmetic, if real number arithmetic is required, float(32 bit) or double(64 bit) real variables can be used. In order to write in computer memory of binary bit structure a real number, sum digits are assumed

to carry negative powers of 2. For example to write **22.625e-17** into computer memory, following bit/byte structure can be assumed.

Table writing $22.625x10^{-17}$ in a 16 bit real variable

+/-	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	+\-	2^4	2^3	2^2	2^1	2^0
+	16	8	4	2	1	0.5	0.25	0.125	0.0625	-	16	8	4	2	1
0	1	0	1	1	0	1	0	1	0	1	1	0	0	0	1

Number representaion limits of real variables are given in Table 1.3.4.

Table Real variable types, bit sizes and number representation limits

Real variable type	Bit size	Number represantation limits
float	32	From -3.40292347e+38 to 3.40292347e+38
double	64	From -1.7976931348623157e+308 to 1.7976931348623157e+308 e

Now, a program similar to **BPIH1E5A.java and BPIH1E5B.java** with double variable can be investigated.

Program BPIH1E5A.java program

```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;

public class BPIH1E5A
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
      Scanner input = new Scanner( System.in );
      Locale us=new Locale("US");
      input.useLocale(us);
      double number1; // first number to add
      double number2; // second number to add
      double sum; // sum of number1 and number2
      System.out.print( "Enter first double: " ); // prompt
      number1 = input.nextDouble(); // read first number from user
      System.out.print( "Enter second integer: " ); // prompt
      number2 = input.nextDouble(); // read second number from user
      sum = number1 + number2; // add numbers
      System.out.printf(us,"Sum is %10.5e\n", sum ); // display sum
    } // end method main

} // end class BPIH1E5A
```

Program BPIH1E5B.java program

```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;

public class BPIH1E5B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
```

```
        Scanner input = new Scanner( System.in );
        Locale tr=new Locale("TR");
        input.useLocale(tr);
        double number1; // first number to add
        double number2; // second number to add
        double sum; // sum of number1 and number2
        System.out.print( "Enter first double: " ); // prompt
        number1 = input.nextDouble(); // read first number from user
        System.out.print( "Enter second integer: " ); // prompt
        number2 = input.nextDouble(); // read second number from user
        sum = number1 + number2; // add numbers
        System.out.printf(tr,"Sum is %10.5e\n", sum ); // display sum
    } // end method main

} // end class BPIH1E5B
```

The basic difference of this programs compare to the previous integer version is the
Locale us=new Locale("US");
Statement, with this statement and the with the following
input.useLocale(us);
statement you can define the country program is running and that specification of that countries number divisor. For example US use period (.) as number divisor, but in Turkey comma(,) is used as number divisor.

In program **BPIH1E5C.java** formatting with **String s=String.format(us,"Sum is %10.5f\n", sum);** term is used to get the formatted output. Please not that %d for integer format, %10.5e and %10.5f for double format is used.

Program BPIH1E5C.java program

```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;

public class BPIH1E5C
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
        Scanner input = new Scanner( System.in );
        Locale us=new Locale("US");
        input.useLocale(us);
        double number1; // first number to add
        double number2; // second number to add
        double sum; // sum of number1 and number2
        System.out.print( "Enter first double: " ); // prompt
        number1 = input.nextDouble(); // read first number from user
        System.out.print( "Enter second integer: " ); // prompt
        number2 = input.nextDouble(); // read second number from user
        sum = number1 + number2; // add numbers
        String s=String.format(us,"Sum is %10.5f\n", sum );
        System.out.println(s); // display sum
    } // end method main

} // end class BPIH1E5C
```

In the next program graphic type String (character array variable) defined a file name and data is read by using Scanner channel from a data file

String filename=JOptionPane.showInputDialog("Enter file name (a.txt): ");

ProgramBPIH1E5D.java program

```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;  // international format library Locale
import javax.swing.*;     //Java swing graphic library
import java.io.*;         //Java input library for File class

public class BPIH1E5D
{
    // main method begins execution of Java application
    public static void main( String args[] ) throws FileNotFoundException
    {
       // create Scanner to obtain input from a file
      String filename=JOptionPane.showInputDialog("Enter file name (a.txt): ");
      Scanner input = new Scanner(new File(filename));
      Locale us=new Locale("US");
      input.useLocale(us);
      double number1; // first number to add
      double number2; // second number to add
      double sum; // sum of number1 and number2
      number1 = input.nextDouble(); // read first number from user
      number2 = input.nextDouble(); // read second number from user
      sum = number1 + number2; // add numbers
      String s=String.format(us,"Sum is %10.5f\n", sum );
      System.out.println(s); // display sum
    } // end method main
} // end class BPIH1E5D
```

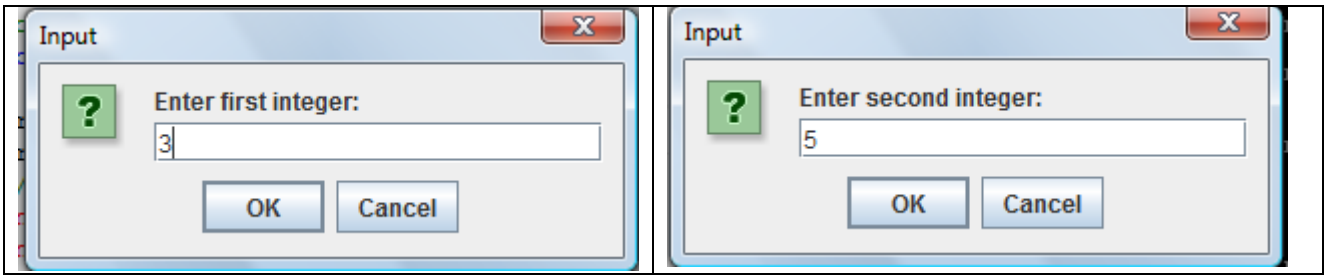
In the last program of first week, two integer numbers are read through the graphic window method through statement

number1=Integer.parseInt(JOptionPane.showInputDialog("Enter first integer: "));

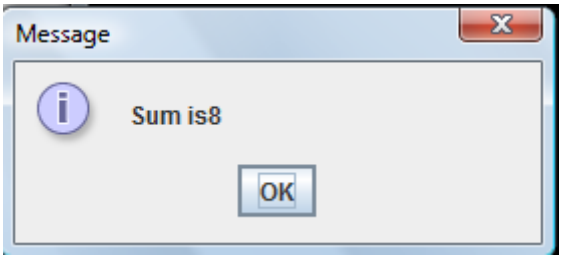
Program BPIH1E6.java program

```
import javax.swing.*;// program uses class JOptionPane in swing library

public class BPIH1E6
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      int number1; // first number to add
      int number2; // second number to add
      int sum; // sum of number1 and number2
      number1=Integer.parseInt(JOptionPane.showInputDialog("Enter first integer: "));
      number2=Integer.parseInt(JOptionPane.showInputDialog("Enter second integer: "));
      sum = number1 + number2; // add numbers
      JOptionPane.showMessageDialog(null,"Sum is"+sum ); // display sum
    } // end method main

} // end class BPIH1E4
```

The input Windows of this program will look like this :



The result will be



Not that instead of integer if we would like to read double value, the statement will be
double number1=Double.parseDouble(JOptionPane.showInputDialog("Enter first reqal number: "));

In the next program we will input a name (String variable) by using Scanner class. In order to read the variable **input.next()** statement is used.

Program BPIH1E7.java program

```
import java.util.Scanner; // program uses class Scanner

public class BPIH1E7
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
      Scanner input = new Scanner(System.in );
      System.out.print( "Enter your name: " ); // prompt
      System.out.println("Welcome to java "+input.next()); // display sum
    } // end method main

} // end class BPIH1E4
```

In the last program String type input variable will be read by using
String isim=JOptionPane.showInputDialog("Enter first integer: ");
statement

```
import javax.swing.*;// program uses class JOptionPane in swing library

public class BPIH1E8
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      String isim=JOptionPane.showInputDialog("Enter first integer: ");
      JOptionPane.showMessageDialog(null,"Welcome to java "+isim ); // display sum
    } // end method main

} // end class BPIH1E4
```

INTRODUCTION TO JAVA LANGUAGE

STRİNG VARIABLES

In this section we will continue to learn integer and doubles. We will also look at a new kind of variable called String type String type variables can be assign as:

String s=”Ali”;
or
String s=new String(“Ali”);
Strings can be added up by using + sign; for example
String s=”Ali”;
s=s+” Veli” will give us string value
“Ali Veli”

In java arithmetic proceses can be written in shorthand form as well (A notation from c language)
for example instead of **int c=c+1; int c++** can be written and Instead of int **c=c+5; c+=5;** can be written, If we give long and shorthanded form of this statements :

Long notation	**Short notation**
c=c+1;	c++;
c=c-1;	c--;
c=c+a;	c+=a;
c=c-a;	c-=a;
c=c/a;	c/=a;
c=c*a;	c*=a;
c=c%a;	c%=a;

The same short notation can be used in String type of variables as well. Instead of
String s=”Ali”;
the same statement can be written in shorthand form
String s=”Ali”;
s+=” Veli”;
The next example program will be summation of two integer values. In this program, despite reading the data from screen, they are assigned directly in the program. “\n” String format is utilised to advance lines. Note that integer or double values can be added to the String value as well. The results will be String. For example
int number=2;
String s=”number =”+number;
will give the String value:
“number = 2”
Remaining strings can directly written by an output method.

Program BPIH2E1.java program

```
import java.io.*; //calls java input output class io
class BPIH2E1
 {   // main method begins execution of Java application
     public static void main(String args[])
     {
        int number1,number2;
        int sum,substract,multiply,divide,remain;
        number1=2;
        number2=3;
        sum = number1+number2;
        substract = number1-number2;
        multiply = number1*number2;
        divide=number1/number2;
        remain=number1%number2;
        String s=number1+" +  "+number2+" = "+sum+"\n";
        s+=number1+" -  "+number2+" = "+substract+"\n";
```

```
        s+=number1+" x  "+number2+" = "+multiply+"\n";
        s+=number1+" /  "+number2+" = "+divide+"\n";
        s+=number1+" %  "+number2+" = "+remain+"\n";
        System.out.println(s);
     } // end method main
 }
```

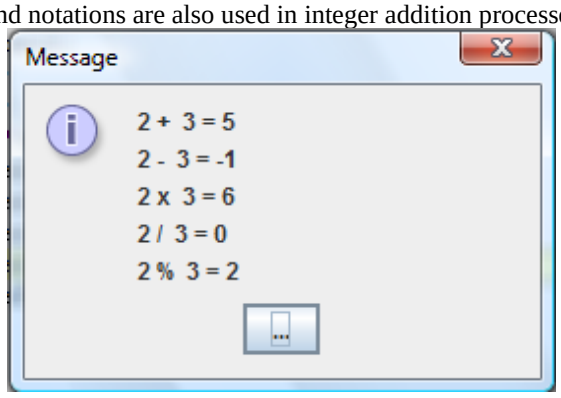
```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" BPIH2E1
2 +  3 = 5
2 -  3 = -1
2 x  3 = 6
2 /  3 = 0
2 %  3 = 2

> Terminated with exit code 0.
```

The second example is the same as the first example except output is send to a graphic window

Program BPIH2E1A.java program

```
import javax.swing.*;// program uses class JOptionPane in swing library

class BPIH2E1A
 {    // main method begins execution of Java application
     public static void main(String args[])
     {
        int number1,number2;
        int sum,substract,multiply,divide,remain;
        number1=2;
        number2=3;
        sum = number1+number2;
        substract = number1-number2;
        multiply = number1*number2;
        divide=number1/number2;
        remain=number1%number2;
        String s=number1+" +  "+number2+" = "+sum+"\n";
        s+=number1+" -  "+number2+" = "+substract+"\n";
        s+=number1+" x  "+number2+" = "+multiply+"\n";
        s+=number1+" /  "+number2+" = "+divide+"\n";
        s+=number1+" %  "+number2+" = "+remain+"\n";
        JOptionPane.showMessageDialog(null,s);
      }// end method main
 }
```

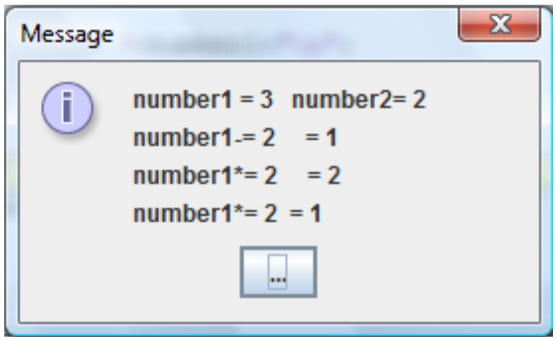
In the third example, shorthand notations are also used in integer addition processes



Program BPIH2E2.java program

```
import javax.swing.JOptionPane;

public class BPIH2E2
{  public static void main(String[] args)
```

```
        {
        int number1=1;
        int number2=2;
        number1+=number2;
        String s="number1 = "+number1+"   number2= "+number2+"\n";
        number1-=number2;
        s+="number1-= "+ number2+"     = "+number1+"\n";
        number1*=number2;
        s+="number1*= "+ number2+"     = "+number1+"\n";
        number1/=number2;
        s+="number1*= "+ number2+"  = "+number1+"\n";
        JOptionPane.showMessageDialog(null,s);
        }
}
```



The fourth example is similar the third one except, for printout printf console output is used.

Program BPIH2E2B.java program

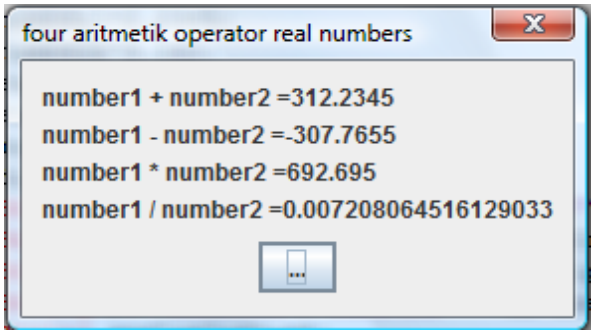
```
public class BPIH2E2B
{  public static void main(String[] args)
        {
        int number1=1;
        int number2=2;
        number1+=number2;
        String s="number1+= : "+ number1+"\n";
        number1-=number2;
        s+="number1-= : "+ number1+"\n";
        number1*=number2;
        s+="number1*= : "+ number1+"\n";
        number1/=number2;
        s+="number1/= : "+ number1;
        //\n newline position the screen curser
        //   at the beginning of the new line
        //\t Horizontal tab, Move the screen cursor
        //   to the next tab stop
        //\r Carriage return. Position the screen cursor
        //   at the beginning of the current line
        //%s String format
        //%d int format
        System.out.printf("%s",s);
        }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" BPIH2E2B
number1+= : 3
number1-= : 1
number1*= : 2
number1/= : 1
> Terminated with exit code 0.
```

A similar program is given by using a double type real numbers

Program 2.1.5 BPIH2E3.java program

```
import javax.swing.*;
class BPIH2E3
 {
    public static void main(String args[])
    {
       double number1,number2;
       double sum,substract,multiply,divide;
       number1=2.2345;
       number2=3.1e2;
       sum = number1+number2;
       substract = number1-number2;
       multiply = number1*number2;
       divide=number1/number2;
       String s1="number1 + number2 ="+sum+"\n";
       String s2="number1 - number2 ="+substract+"\n";
       String s3="number1 * number2 ="+multiply+"\n";
       String s4="number1 / number2 ="+divide+"\n";
       String s=s1+s2+s3+s4;
       JOptionPane.showMessageDialog(null,s,"four aritmetik operator real numbers",JOptionPane.PLAIN_MESSAGE);
     }
 }
```



In this last program, A heading for graphic window is also given as : "four aritmetik operator real numbers"

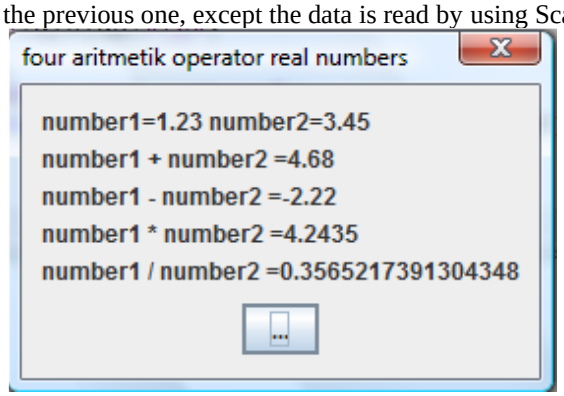
In the next example two real numbers read from graphic screen as real numbers and loaded to different String statements and at the end String statements are added into a new String.

Program BPIH2E3A.java program

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;
class BPIH2E3A
 {
    public static void main(String args[])
    {
       double number1,number2;
       double sum,substract,multiply,divide;
       number1=Double.parseDouble( JOptionPane.showInputDialog("Enter first number: "));
       number2=Double.parseDouble( JOptionPane.showInputDialog("Enter second number: "));
       sum = number1+number2;
       substract = number1-number2;
       multiply = number1*number2;
       divide=number1/number2;
       String s0="number1="+number1+" number2="+number2+"\n";
       String s1="number1 + number2 ="+sum+"\n";
       String s2="number1 - number2 ="+substract+"\n";
       String s3="number1 * number2 ="+multiply+"\n";
       String s4="number1 / number2 ="+divide+"\n";
       String s=s0+s1+s2+s3+s4;
       String sg="four aritmetik operator real numbers";
       JOptionPane.showMessageDialog(null,s,sg,JOptionPane.PLAIN_MESSAGE);
```

```
    }
}
```

The last program is similar to the previous one, except the data is read by using Scanner class.



Program BPIH2E3B.java program

```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;

class BPIH2E3B
 {
    public static void main(String args[])
    {
       double number1,number2;
       double sum,substract,multiply,divide;
      // create Scanner to obtain input from command window
       Scanner input = new Scanner( System.in );
       Locale us=new Locale("US");
       input.useLocale(us);
       System.out.print("enter the first number : ");
       number1=input.nextDouble();
       System.out.print("enter the second number : ");
       number2=input.nextDouble();
       sum = number1+number2;
       substract = number1-number2;
       multiply = number1*number2;
       divide=number1/number2;
       String s1="number1 + number2 ="+sum+"\n";
       String s2="number1 - number2 ="+substract+"\n";
       String s3="number1 * number2 ="+multiply+"\n";
       String s4="number1 / number2 ="+divide+"\n";
       String s=s1+s2+s3+s4;
       System.out.println(s);
    }
 }
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" BPIH2E3B
enter the first number : 1.234
enter the second number : 2.345
number1 + number2 =3.579
number1 - number2 =-1.1110000000000002
number1 * number2 =2.89373
number1 / number2 =0.5262260127931769
```

CHARACTER VARIABLES

Character (char) variable type define actual characters in computers. In order to see characters in java **ISO Unicode** character standards are used. Unicode character code consist of 4 hexagonal (base 16) number. In case you are not familiar with orthogonal number system, a table of equiavalancy is given

Tablo Heksagonal(sixteen base), and 10 base and binary number systems equvalency

Heksagonal	10 base	2 base (binary)
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

in ISO unicode '\u0041' defines character 'A'. or '\u03E1'. Or code '\u03E1' defines 'a'. When the first two number of unicode equals to 0, it defines ASCII characte codes. You can find very detailed list of characters in ISO unicode by looking the internet site **http:\\unicode.org**

In the small code sample below character A and 'α' is loaded to computer as char variables in Java

```
char A1,A2;
char alpha1,alpha2;
A1='\u0041';
A2='A';
alpha1='\u03B1';
alpha2='a';
```

It should be noted that A1 and alpha1 can be seen in any environment, But A2 and alpha2 can be seen only if the editor program is written recognise the correct unicode characters. Some of the control characters are listed in Table 2.2.2

Table Some special control characters

Chracter sysmbol	**action**	**Unicode symbol**
\b	**Back one character**	**\u0008**
\t	**tab**	**\u0009**
\n	**ctrl-return**	**\u000A**
\"	**“ sign**	**\u0022**
\'	**‘ sign**	**\u0027**

\\	\ **sign**	**\u005C**

Some additional unicode characters are given in table 2.2.3. By using control characters several actions in print can be carried out such as back space, carriage-return, tab etc. When character variables are used, they can merge in strings, In fact strings are character arrays. When something with a character that does not available in your editor you can add that character into your String variable as char variable character. Note that some characters might not be seen properly in console environment, but can always be seen correctly in graphic environment. Because of this, **JoptionPane.showMessageDialog** method will be used for output in the example programs. The first example, characterA.java prints some characters (ügçabc) by using Unicode characters and char variable. The second program characterB.java prints a set of greek letters, The third program characterC.java prints Turkish character set. The fourth and fifth programs prints mathematical equations.

Program characterA.java program

```
import java.io.*;
import javax.swing.*;

class characterA
 {
    public static void main(String args[])
    {
       char b1,b2,b3,b4,b5,b6;
       b1='ü';
       b2='ğ';
       b3='ç';
       b4='a';
       b5='b';
       b6='c';
       String s=""+b1+b2+b3+b4+b5+b6;
        JOptionPane.showMessageDialog(null,s," character
procesing",JOptionPane.PLAIN_MESSAGE);
     }
 }
```

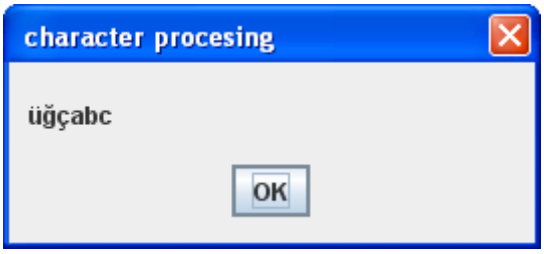


Table Some unicode character tables

	000	001	002	003	004	005	006	007
0	NUL 0000	DLE 0010	SP 0020	0 0030	@ 0040	P 0050	` 0060	p 0070
1	SOH 0001	DC1 0011	! 0021	1 0031	A 0041	Q 0051	a 0061	q 0071
2	STX 0002	DC2 0012	" 0022	2 0032	B 0042	R 0052	b 0062	r 0072
3	ETX 0003	DC3 0013	# 0023	3 0033	C 0043	S 0053	c 0063	s 0073
4	EOT 0004	DC4 0014	$ 0024	4 0034	D 0044	T 0054	d 0064	t 0074
5	ENQ 0005	NAK 0015	% 0025	5 0035	E 0045	U 0055	e 0065	u 0075
6	ACK 0006	SYN 0016	& 0026	6 0036	F 0046	V 0056	f 0066	v 0076
7	BEL 0007	ETB 0017	' 0027	7 0037	G 0047	W 0057	g 0067	w 0077
8	BS 0008	CAN 0018	(0028	8 0038	H 0048	X 0058	h 0068	x 0078
9	HT 0009	EM 0019	) 0029	9 0039	I 0049	Y 0059	i 0069	y 0079
A	LF 000A	SUB 001A	* 002A	: 003A	J 004A	Z 005A	j 006A	z 007A
B	VT 000B	ESC 001B	+ 002B	; 003B	K 004B	[005B	k 006B	{ 007B
C	FF 000C	FS 001C	, 002C	< 003C	L 004C	\ 005C	l 006C	| 007C
D	CR 000D	GS 001D	- 002D	= 003D	M 004D	] 005D	m 006D	} 007D
E	SO 000E	RS 001E	. 002E	> 003E	N 004E	^ 005E	n 006E	~ 007E
F	SI 000F	US 001F	/ 002F	? 003F	O 004F	_ 005F	o 006F	DEL 007F

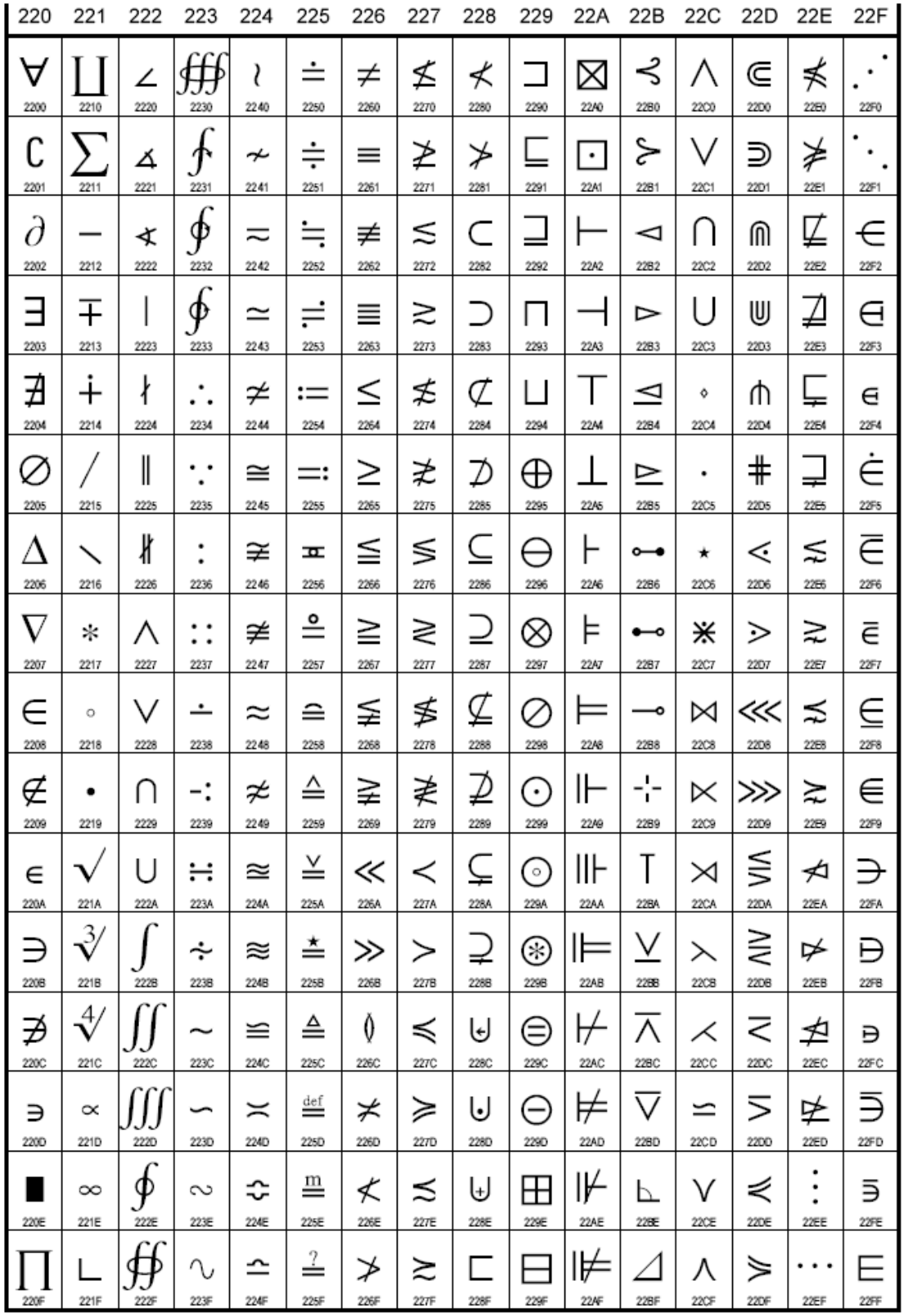
	008	009	00A	00B	00C	00D	00E	00F
0	XXX 0080	DCS 0090	NB SP 00A0	° 00B0	À 00C0	Ð 00D0	à 00E0	ð 00F0
1	XXX 0081	PU1 0091	¡ 00A1	± 00B1	Á 00C1	Ñ 00D1	á 00E1	ñ 00F1
2	BPH 0082	PU2 0092	¢ 00A2	2 00B2	Â 00C2	Ò 00D2	â 00E2	ò 00F2
3	NBH 0083	STS 0093	£ 00A3	3 00B3	Ã 00C3	Ó 00D3	ã 00E3	ó 00F3
4	IND 0084	CCH 0094	¤ 00A4	´ 00B4	Ä 00C4	Ô 00D4	ä 00E4	ô 00F4
5	NEL 0085	MW 0095	¥ 00A5	µ 00B5	Å 00C5	Õ 00D5	å 00E5	õ 00F5
6	SSA 0086	SPA 0096	¦ 00A6	¶ 00B6	Æ 00C6	Ö 00D6	æ 00E6	ö 00F6
7	ESA 0087	EPA 0097	§ 00A7	· 00B7	Ç 00C7	× 00D7	ç 00E7	÷ 00F7
8	HTS 0088	SOS 0098	¨ 00A8	¸ 00B8	È 00C8	Ø 00D8	è 00E8	ø 00F8
9	HTJ 0089	XXX 0099	© 00A9	1 00B9	É 00C9	Ù 00D9	é 00E9	ù 00F9
A	VTS 008A	SCI 009A	ª 00AA	º 00BA	Ê 00CA	Ú 00DA	ê 00EA	ú 00FA
B	PLD 008B	CSI 009B	« 00AB	» 00BB	Ë 00CB	Û 00DB	ë 00EB	û 00FB
C	PLU 008C	ST 009C	¬ 00AC	¼ 00BC	Ì 00CC	Ü 00DC	ì 00EC	ü 00FC
D	RI 008D	OSC 009D	SHY 00AD	½ 00BD	Í 00CD	Ý 00DD	í 00ED	ý 00FD
E	SS2 008E	PM 009E	® 00AE	¾ 00BE	Î 00CE	Þ 00DE	î 00EE	þ 00FE
F	SS3 008F	APC 009F	¯ 00AF	¿ 00BF	Ï 00CF	ß 00DF	ï 00EF	ÿ 00FF

0180 **Latin Extended-B** **024**

	018	019	01A	01B	01C	01D	01E	01F	020	021	022	023	024
0	ƀ 0180	Ɛ 0190	Ơ 01A0	ư 01B0	ǀ 01C0	ǐ 01D0	Ā 01E0	ǰ 01F0	À 0200	Ȑ 0210	ɳ 0220	Ō 0230	ɀ 0240
1	Ɓ 0181	Ƒ 0191	ơ 01A1	Ʊ 01B1	ǁ 01C1	Ǒ 01D1	ā 01E1	DZ 01F1	à 0201	ȑ 0211	ɖ 0221	ō 0231	ɂ 0241
2	Ƃ 0182	ƒ 0192	Ƣ 01A2	Ʋ 01B2	ǂ 01C2	ǒ 01D2	Ē 01E2	Dz 01F2	Â 0202	Ȓ 0212	ȣ 0222	Ȳ 0232	ɔ 0242
3	ƃ 0183	Ɠ 0193	ƣ 01A3	Ƴ 01B3	! 01C3	Ǔ 01D3	ē 01E3	dz 01F3	â 0203	î 0213	ȣ 0223	ȳ 0233	Ƀ 0243
4	Ƅ 0184	Ɣ 0194	Ƥ 01A4	ƴ 01B4	DŽ 01C4	ǔ 01D4	Ġ 01E4	Ǵ 01F4	È 0204	Ù 0214	Ȝ 0224	ɻ 0234	Ʉ 0244
5	ƅ 0185	ƕ 0195	ƥ 01A5	Ƶ 01B5	Dž 01C5	Ǖ 01D5	g 01E5	ǵ 01F5	è 0205	ù 0215	ȝ 0225	ɲ 0235	Ʌ 0245
6	Ɔ 0186	ƚ 0196	Ʀ 01A6	ƶ 01B6	dž 01C6	ǖ 01D6	Ǧ 01E6	Ƕ 01F6	Ê 0206	Û 0216	Ȧ 0226	ȶ 0236	Ɇ 0246
7	Ƈ 0187	ƛ 0197	Ƨ 01A7	Ʒ 01B7	LJ 01C7	Ǘ 01D7	ǧ 01E7	ƿ 01F7	ê 0207	û 0217	ȧ 0227	ɟ 0237	ɇ 0247
8	ƈ 0188	Ƙ 0198	ƨ 01A8	Ǯ 01B8	Lj 01C8	ǘ 01D8	Ǩ 01E8	Ǹ 01F8	Ì 0208	Ș 0218	Ȩ 0228	ɐ 0238	ɉ 0248
9	Ɖ 0189	ƙ 0199	Σ 01A9	ǯ 01B9	lj 01C9	Ǚ 01D9	ǩ 01E9	ǹ 01F9	ì 0209	ș 0219	ȩ 0229	ɑ 0239	ɉ 0249
A	Ɗ 018A	ƚ 019A	ƪ 01AA	ƺ 01BA	NJ 01CA	ǚ 01DA	Ǫ 01EA	Ǻ 01FA	Î 020A	Ț 021A	Ɵ 022A	Ⱥ 023A	ɐ 024A
B	Ƌ 018B	ƛ 019B	ƫ 01AB	Ƹ 01BB	Nj 01CB	Ǜ 01DB	ǫ 01EB	ǻ 01FB	î 020B	ț 021B	ȫ 022B	Ȼ 023B	ɑ 024B
C	ƌ 018C	Ɯ 019C	Ƭ 01AC	Ƽ 01BC	nj 01CC	ǜ 01DC	Ǭ 01EC	Æ 01FC	Ò 020C	Ȝ 021C	Ȭ 022C	ȼ 023C	Ʀ 024C
D	ƍ 018D	Ɲ 019D	ƭ 01AD	ƽ 01BD	Ǎ 01CD	ǝ 01DD	ǭ 01ED	æ 01FD	ò 020D	ȝ 021D	ȭ 022D	Ł 023D	ɾ 024D
E	Ǝ 018E	ƞ 019E	Ʈ 01AE	ƾ 01BE	ǎ 01CE	Ǟ 01DE	Ǯ 01EE	Ǽ 01FE	Ô 020E	Ȟ 021E	Ȯ 022E	Ⱦ 023E	Ɏ 024E
F	Ə 018F	Ɵ 019F	Ư 01AF	ƿ 01BF	Ǐ 01CF	ǟ 01DF	ǯ 01EF	ǽ 01FF	ô 020F	ȟ 021F	ȯ 022F	ɀ 023F	ɏ 024F

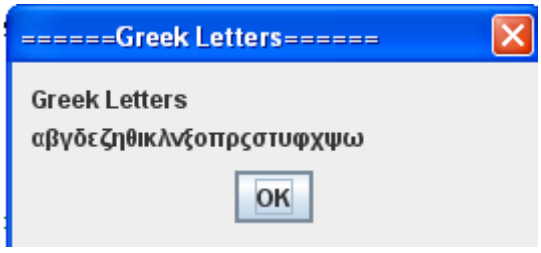
0370 **Greek and Coptic** **03FF**

	037	038	039	03A	03B	03C	03D	03E	03F
0	Ͱ 0370		ΐ 0390	Π 03A0	ΰ 03B0	π 03C0	ϐ 03D0	Ϣ 03E0	ϰ 03F0
1	ͱ 0371		Α 0391	Ρ 03A1	α 03B1	ρ 03C1	ϑ 03D1	ϣ 03E1	ϱ 03F1
2	Ͳ 0372		Β 0392		β 03B2	ς 03C2	ϒ 03D2	Ϥ 03E2	ϲ 03F2
3	ͳ 0373		Γ 0393	Σ 03A3	γ 03B3	σ 03C3	ϓ 03D3	ϥ 03E3	ϳ 03F3
4	ʹ 0374	΄ 0384	Δ 0394	Τ 03A4	δ 03B4	τ 03C4	ϔ 03D4	Ϧ 03E4	ϴ 03F4
5	͵ 0375	΅ 0385	Ε 0395	Υ 03A5	ε 03B5	υ 03C5	ϕ 03D5	ϧ 03E5	ϵ 03F5
6	Ͷ 0376	Ά 0386	Ζ 0396	Φ 03A6	ζ 03B6	φ 03C6	ϖ 03D6	Ϩ 03E6	϶ 03F6
7	ͷ 0377	· 0387	Η 0397	Χ 03A7	η 03B7	χ 03C7	ϗ 03D7	ϩ 03E7	Ϸ 03F7
8		Έ 0388	Θ 0398	Ψ 03A8	θ 03B8	ψ 03C8	Ϙ 03D8	Ϫ 03E8	ϸ 03F8
9		Ή 0389	Ι 0399	Ω 03A9	ι 03B9	ω 03C9	ϙ 03D9	ϫ 03E9	Ϲ 03F9
A	ͺ 037A	Ί 038A	Κ 039A	Ϊ 03AA	κ 03BA	ϊ 03CA	Ϛ 03DA	Ϭ 03EA	Ϻ 03FA
B	ͻ 037B		Λ 039B	Ϋ 03AB	λ 03BB	ϋ 03CB	ϛ 03DB	ϭ 03EB	ϻ 03FB
C	ͼ 037C	Ό 038C	Μ 039C	ά 03AC	μ 03BC	ό 03CC	Ϝ 03DC	Ϯ 03EC	ϼ 03FC
D	ͽ 037D		Ν 039D	έ 03AD	ν 03BD	ύ 03CD	ϝ 03DD	ϯ 03ED	Ͻ 03FD
E	; 037E	Ύ 038E	Ξ 039E	ή 03AE	ξ 03BE	ώ 03CE	Ϟ 03DE	ϰ 03EE	Ͼ 03FE
F		Ώ 038F	Ο 039F	ί 03AF	ο 03BF	ϗ 03CF	ϟ 03DF	ϱ 03EF	Ͽ 03FF


220 221 222 223 224 225 226 227 228 229 22A 22B 22C 22D 22E 22F
0 ∀ 2200 ∐ 2210 ∠ 2220 ∭ 2230 ≀ 2240 ≑ 2250 ≠ 2260 ≰ 2270 ⊀ 2280 ⊐ 2290 ⊠ 22A0 ⋩ 22B0 ⋀ 22C0 ⋐ 22D0 ⪇ 22E0 ⋰ 22F0
1 ∁ 2201 ∑ 2211 ∡ 2221 ∫ 2231 ≁ 2241 ≓ 2251 ≡ 2261 ≱ 2271 ⊁ 2281 ⊑ 2291 ⊡ 22A1 ⋪ 22B1 ⋁ 22C1 ⋑ 22D1 ⪈ 22E1 ⋱ 22F1
2 ∂ 2202 − 2212 ∢ 2222 ∮ 2232 ≂ 2242 ≒ 2252 ≢ 2262 ≲ 2272 ⊂ 2282 ⊒ 2292 ⊢ 22A2 ⋫ 22B2 ⋂ 22C2 ⋒ 22D2 ⪉ 22E2 ⋲ 22F2
3 ∃ 2203 ∓ 2213 ∣ 2223 ∯ 2233 ≃ 2243 ≓ 2253 ≣ 2263 ≳ 2273 ⊃ 2283 ⊓ 2293 ⊣ 22A3 ⋬ 22B3 ⋃ 22C3 ⋓ 22D3 ⪊ 22E3 ⋳ 22F3
4 ∄ 2204 ∔ 2214 ∤ 2224 ∴ 2234 ≄ 2244 ≔ 2254 ≤ 2264 ≴ 2274 ⊄ 2284 ⊔ 2294 ⊤ 22A4 ⋭ 22B4 ⋄ 22C4 ⋔ 22D4 ⪋ 22E4 ⋴ 22F4
5 ∅ 2205 ∕ 2215 ∥ 2225 ∵ 2235 ≅ 2245 ≕ 2255 ≥ 2265 ≵ 2275 ⊅ 2285 ⊕ 2295 ⊥ 22A5 ⋮ 22B5 ⋅ 22C5 ⋕ 22D5 ⪌ 22E5 ⋵ 22F5
6 ∆ 2206 ∖ 2216 ∦ 2226 ∶ 2236 ≆ 2246 ≖ 2256 ≦ 2266 ≶ 2276 ⊆ 2286 ⊖ 2296 ⊦ 22A6 ⋯ 22B6 ⋆ 22C6 ⋖ 22D6 ⪍ 22E6 ⋶ 22F6
7 ∇ 2207 ∗ 2217 ∧ 2227 ∷ 2237 ≇ 2247 ≗ 2257 ≧ 2267 ≷ 2277 ⊇ 2287 ⊗ 2297 ⊧ 22A7 ⋰ 22B7 ⋇ 22C7 ⋗ 22D7 ⪎ 22E7 ⋷ 22F7
8 ∈ 2208 ∘ 2218 ∨ 2228 ∸ 2238 ≈ 2248 ≘ 2258 ≨ 2268 ≸ 2278 ⊈ 2288 ⊘ 2298 ⊨ 22A8 ⋱ 22B8 ⋈ 22C8 ⋘ 22D8 ⪏ 22E8 ⋸ 22F8
9 ∉ 2209 • 2219 ∩ 2229 ∹ 2239 ≉ 2249 ≙ 2259 ≩ 2269 ≹ 2279 ⊉ 2289 ⊙ 2299 ⊩ 22A9 ⋲ 22B9 ⋉ 22C9 ⋙ 22D9 ⪐ 22E9 ⋹ 22F9
A ∊ 220A √ 221A ∪ 222A ∺ 223A ≊ 224A ≚ 225A ≪ 226A ≺ 227A ⊊ 228A ⊙ 229A ⊪ 22AA ⋊ 22BA ⋉ 22CA ⋛ 22DA ⪑ 22EA ⋺ 22FA
B ∋ 220B ∛ 221B ∫ 222B ∻ 223B ≋ 224B ≛ 225B ≫ 226B ≻ 227B ⊋ 228B ⊛ 229B ⊫ 22AB ⋋ 22BB ⋊ 22CB ⋚ 22DB ⪒ 22EB ⋻ 22FB
C ∌ 220C ∜ 221C ∬ 222C ∼ 223C ≌ 224C ≜ 225C ⧫ 226C ≼ 227C ⊌ 228C ⊜ 229C ⊬ 22AC ⋌ 22BC ⋉ 22CC ⋜ 22DC ⪓ 22EC ⋼ 22FC
D ∍ 220D ∝ 221D ∭ 222D ∽ 223D ≍ 224D ≝ 225D ≮ 226D ≽ 227D ⊍ 228D ⊝ 229D ⊭ 22AD ⋍ 22BD ⋍ 22CD ⋝ 22DD ⪔ 22ED ⋽ 22FD
E ∎ 220E ∞ 221E ∮ 222E ∾ 223E ≎ 224E ≞ 225E ≮ 226E ≺ 227E ⊎ 228E ⊞ 229E ⊯ 22AE ⋏ 22BE ⋎ 22CE ⋞ 22DE ⪕ 22EE ⋾ 22FE
F ∏ 220F ∟ 221F ∯ 222F ∿ 223F ≏ 224F ≟ 225F ≯ 226F ≿ 227F ⊏ 228F ⊟ 229F ⊮ 22AF ⋏ 22BF ∧ 22CF ⋟ 22DF ⋯ 22EF ⋿ 22FF

Program characterB.java program

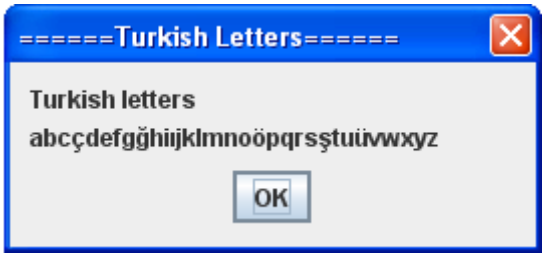
```
import java.io.*;
import javax.swing.*;
class characterB
 {
     public static void main(String args[])
     {   //Greek letters
         char b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11,b12,b13,b14;
         char b15,b16,b17,b18,b19,b20,b21,b22,b23,b24;
         String s="Greek Letters \n";
         b1='\u03B1';  // alfa
         b2='\u03B2';  // beta
         b3='\u03B3';  // gamma
         b4='\u03B4';  // delta
         b5='\u03B5';  // epsilon
         b6='\u03B6';  // fi
         b7='\u03B7';  // eta
         b8='\u03B8';  // teta
         b9='\u03B9';  // ypsilon
         b10='\u03BA'; // kappa
         b11='\u03BB'; // lambda
         b12='\u03BC'; // mü
         b12='\u03BD'; // nü
         b13='\u03BE'; // ksi
         b14='\u03BF'; // o
         b15='\u03C0'; // pi
         b16='\u03C1'; // ro
         b17='\u03C2'; //
         b18='\u03C3'; //
         b19='\u03C4'; //
         b20='\u03C5'; //
         b21='\u03C6'; //
         b22='\u03C7'; //
         b23='\u03C8'; //
         b24='\u03C9'; //

s+=""+b1+b2+b3+b4+b5+b6+b7+b8+b9+b10+b11+b12+b13+b14+b15+b16+b17+b18+b19+b20+b21+b22+b23+b24;
          JOptionPane.showMessageDialog(null,s,"======Greek Letters======",JOptionPane.PLAIN_MESSAGE);
      }
 }
```



Program characterC.java program

```
import java.io.*;
import javax.swing.*;
class charaterC
```

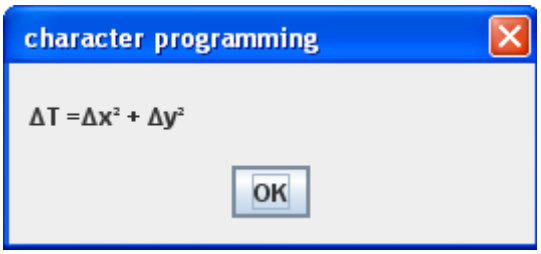
```
{    public static void main(String args[])
     {
        char b1,b2,b3,b3a,b4,b5,b6,b7,b7a,b8,b9,b9a,b10,b11,b12,b13,b14,b15,b15a,b16,b17,b18,b19,b20;
        char b21,b22,b23,b24,b25,b25a,b26,b27,b27a,b28,b29,b30,b31,b32,b33;
        String s="Turkish letters\n";
        b1='\u0061';   //a
        b2='\u0062';   //b
        b3='\u0063';   //c
        b3a='\u00E7';  //ç
        b4='\u0064';   //d
        b5='\u0065';   //e
        b6='\u0066';   //f
        b7='\u0067';   //g
        b7a='\u011F';  //ğ
        b8='\u0068';   //h
        b9a='\u0131';  //ı
        b9='\u0069';   //i
        b10='\u006A';  //j
        b11='\u006B';  //k
        b12='\u006C';  //l
        b13='\u006D';  //m
        b14='\u006E';  //n
        b15='\u006F';  //o
        b15a='\u00F6'; //ö
        b22='\u0070';  //p
        b23='\u0071';  //q
        b24='\u0072';  //r
        b25='\u0073';  //s
        b25a='\u015F'; //ş
        b26='\u0074';  //t
        b27='\u0075';  //u
        b27a='\u00FC'; //ü
        b28='\u0076';  //v
        b29='\u0077';  //w
        b30='\u0078';  //x
        b31='\u0079';  //y
        b33='\u007A';  //z
        s+=""+b1+b2+b3+b3a+b4+b5+b6+b7+b7a+b8+b9+b9a+b10+b11+b12+b13+b14+b15+b15a+b22+b23+b24+b25+b25a+b2
        JOptionPane.showMessageDialog(null,s,"======Turkish Letters======",JOptionPane.PLAIN_MESSAGE);
     }
}
```



Program characterD.java program

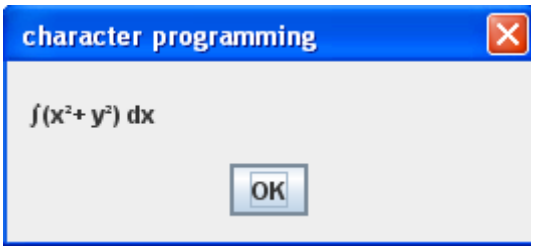
```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;
```

```
class characterD
 {
    public static void main(String args[])
    {
       char b1,b2;
       b1='\u0394';   //big delta
       b2='\u00B2';   //Square sign
       String s=""+b1+"T ="+b1+"x"+b2+" + "+b1+"y"+b2;
       JOptionPane.showMessageDialog(null,s," character programming",JOptionPane.PLAIN_MESSAGE);
     }
 }
```



Program characterE.java program

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;
class characterE
 {
    public static void main(String args[])
    {
       char b1,b2;
       b1='\u222B';   //integral
       b2='\u00B2';
       char b4='\u221E';
       String s=""+b1+"(x"+b2+"+ y"+b2+") dx";
       JOptionPane.showMessageDialog(null,s," character programming",JOptionPane.PLAIN_MESSAGE);
     }
 }
```



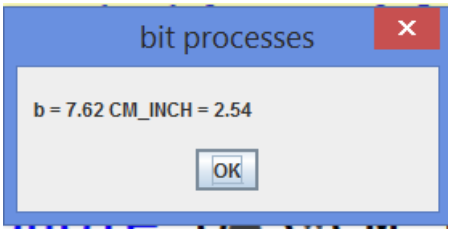
THE TERM FINAL AND CONSTANTS IN JAVA

In Java language, in order to use constants instead of variables the term **final** comes in front of variable term. A constant defined with final can not be changed. Can be defined only ones in the program. For example

final double pi=3.14159;
defines pi as a constant.

Program finalA.java program : an example of usage term final

```
 import java.io.*;
import javax.swing.*;

class finalA
 {    public static void main(String args[])
     {
       final double CM_INCH = 2.54;
        double b=3*CM_INCH;
        //CM_INCH=4;
        String s="b = "+b+" CM_INCH = "+CM_INCH;
        JOptionPane.showMessageDialog(null,s,"bit processes",JOptionPane.PLAIN_MESSAGE);
    }
 }
```



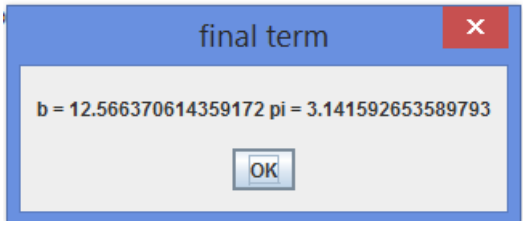

```
import java.io.*;
import javax.swing.*;

class finalA
 {    public static void main(String args[])
     {
       final double CM_INCH = 2.54;
        double b=3*CM_INCH;
        CM_INCH=4;
        String s="b = "+b+" CM_INCH = "+CM_INCH;
        JOptionPane.showMessageDialog(null,s,"bit processes",JOptionPane.PLAIN_MESSAGE);
    }
 }
```

```
---------- Capture Output ----------
> "C:\java\bin\javac.exe" finalA.java
finalA.java:9: error: cannot assign a value to final variable CM_INCH
        CM_INCH=4;
        ^
1 error

> Terminated with exit code 1.
```

Program finalC.java program : an example of usage term final

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class finalC
```

```
{    public static void main(String args[])
    {
                final double pi = 3.141592653589793;
       double b=4*pi;
       //pi=4;
       String s="b = "+b+" pi = "+pi;
       JOptionPane.showMessageDialog(null,s,"final term",JOptionPane.PLAIN_MESSAGE);
   }
}
```




```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class finalC
 {    public static void main(String args[])
     {
                final double pi = 3.141592653589793;
        double b=4*pi;
        pi=4;
        String s="b = "+b+" pi = "+pi;
        JOptionPane.showMessageDialog(null,s,"final term",JOptionPane.PLAIN_MESSAGE);
    }
 }
```

```
---------- Capture Output ----------
> "C:\java\bin\javac.exe" finalC.java
finalC.java:9: error: cannot assign a value to final variable pi
        pi=4;
        ^
1 error

> Terminated with exit code 1.
```

BOOLEAN VARIABLES

Boolean type variable is for logical processes. Key word for the boolean type variable is **boolean**. Boolean variable is only one bit long and taken value **true** or **false**

boolean x=true;
boolean y=false;

The available operators for the boolean variables are:

Table 3.1.1 Boolean operators that operate directly in boolean variables

Boolean operator	Meaning
&&	and
||	or
!	not

Usage of this variables are given in

Program booleanex1.java : an example of boolean variable

```
import java.awt.Graphics;

public class booleanex1
{
            public static void main(String arg[])
            {
             boolean b1=true;
             boolean b2=false;
             boolean b3=b1&&b2;
             boolean b4=b1||b2;
             boolean b5=!b1;
             String s="b1="+b1+"\n";
             s+="b2="+b2+"\n";
             s+="b3="+b3+"\n";
             s+="b4="+b4+"\n";
             s+="b5="+b5+"\n";
             System.out.println(s);
            }
}
```

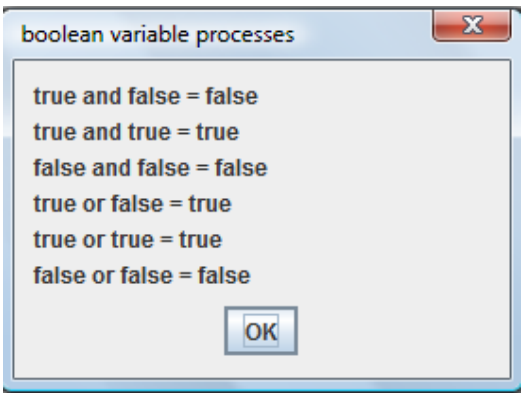
```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" booleanex1
b1=true
b2=false
b3=false
b4=true
b5=false
> Terminated with exit code 0.
```

The second example creates a complete boolean process table of all possible outputs of **boolean and(&&)** and **boolean or(||)** statements

Program booleanA.java : an example of boolean variable, creates boolean table

```
class booleanA
 {
     public static void main(String args[])
     {
        boolean b1,b2,b3,b4;
        boolean x1,x2,x3,x4,x5,x6;
```

```
        b1=true;
        b2=false;
        b3=true;
        b4=false;
        x1 = b1&&b2;
        x2 = b1&&b3;
        x3 = b2&&b4;
        x4 = b1||b2;
        x5 = b1||b3;
        x6 = b2||b4;
        String s1=b1+" and "+b2+" = "+x1+"\n";
        String s2=b1+" and "+b3+" = "+x2+"\n";
        String s3=b2+" and "+b4+" = "+x3+"\n";
        String s4=b1+" or "+b2+" = "+x4+"\n";
        String s5=b1+" or "+b3+" = "+x5+"\n";
        String s6=b2+" or "+b4+" = "+x6;
        String s=s1+s2+s3+s4+s5+s6;
        JOptionPane.showMessageDialog(null,s," boolean variable processes",JOptionPane.PLAIN_MESSAGE);
    }
}
```



Boolean variables can interact with real and integer numbers through a second set of operators.

Table Boolean operators that operates on integer and real variables variables

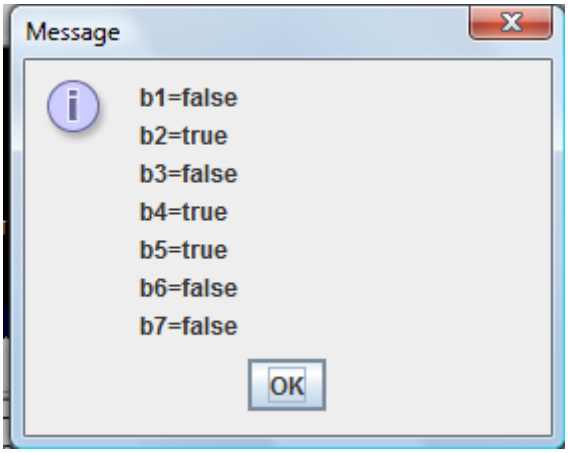
Boolean operator	Meaning
>	Grater than
<	Smaller than
==	Equal to
>=	Greater than and equal to
<=	Smaller than and equal to
!=	Not equal to

This operators operates with integer and real variables but result will be in boolean type, for example
int x=3;
int y=5;
boolean b=x>y;
will give the boolean value of false. In the second example, an example of number boolean operators. Note that due to getting result in boolean, after first process the result can be reprocessed by using operators in Table 3.1.1. Boolean Values can be added up like other type of variables to Strings to take the printput of results.

Program booleanex2.java : an example of boolean variable

```
import javax.swing.JOptionPane;
```

```
public class booleanex2
{
            public static void main(String arg[])
            {
             int x=3;
             int y=4;
             boolean b1=x>y;
             boolean b2=x<y;
             boolean b3=(x==y);
             boolean b4=(x!=y);
             boolean b5=!(x==y);
             boolean b6=(x==y) && (x!=y);
             boolean b7=b3 && b4;
             String s="b1="+b1+"\n";
             s+="b2="+b2+"\n";
             s+="b3="+b3+"\n";
             s+="b4="+b4+"\n";
             s+="b5="+b5+"\n";
             s+="b6="+b6+"\n";
             s+="b7="+b7+"\n";
             JOptionPane.showMessageDialog(null,s);
            }
}
```

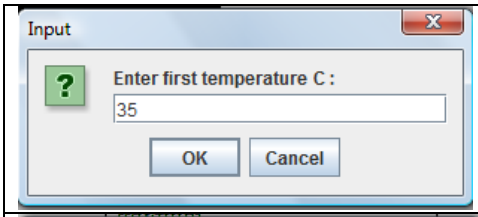


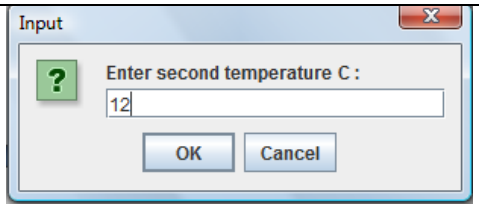
A second example will be comparison of a temperature difference and write the results booleans.

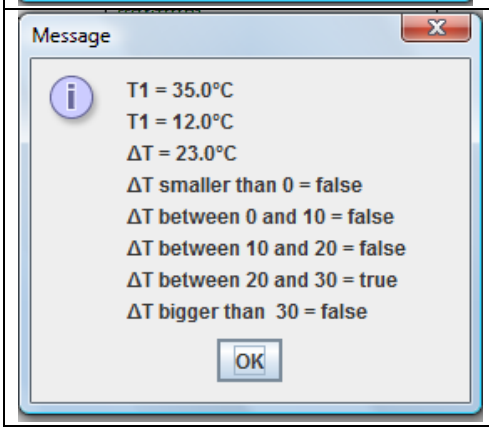
Program booleanex3.java : an example of boolean variable

```
import javax.swing.JOptionPane;
import java.util.Locale;

public class booleanex3
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {  Locale us=new Locale("US");
      char d='\u0394';
      char o='\u00B0';
      double T1;
      double T2; //
      double dT; // sum of number1 and number2
      String a1="Enter first temperature "+'\u000B'+"C : ";
      T1 = Double.parseDouble(JOptionPane.showInputDialog(a1));
      String a2="Enter second temperature "+'\u000B'+"C : ";
```

```
        T2 = Double.parseDouble(JOptionPane.showInputDialog(a2));
        dT = T1 - T2; // difference of temperatures
        boolean b0=(dT<0);
        boolean b1=(dT>=0) && (dT<10);
        boolean b2=(dT>=10) && (dT<20);
        boolean b3=(dT>=20) && (dT<30);
        boolean b4=(dT>=30);
        String s1="T1 = %4.1f"+o+"C \n";
        s1+="T1 = %4.1f"+o+"C \n";
        s1+=d+"T = %4.1f"+o+"C \n";
        s1+= d+"T smaller than 0 = "+b0+"\n";
        s1+= d+"T between 0 and 10 = "+b1+"\n";
        s1+= d+"T between 10 and 20 = "+b2+"\n";
        s1+= d+"T between 20 and 30 = "+b3+"\n";
        s1+= d+"T bigger than  30 = "+b4+"\n";
        String s4=String.format(us,s1,T1,T2,dT );
        JOptionPane.showMessageDialog(null,s4); // display sum
      } // end method main

} // end class booleanex3
```





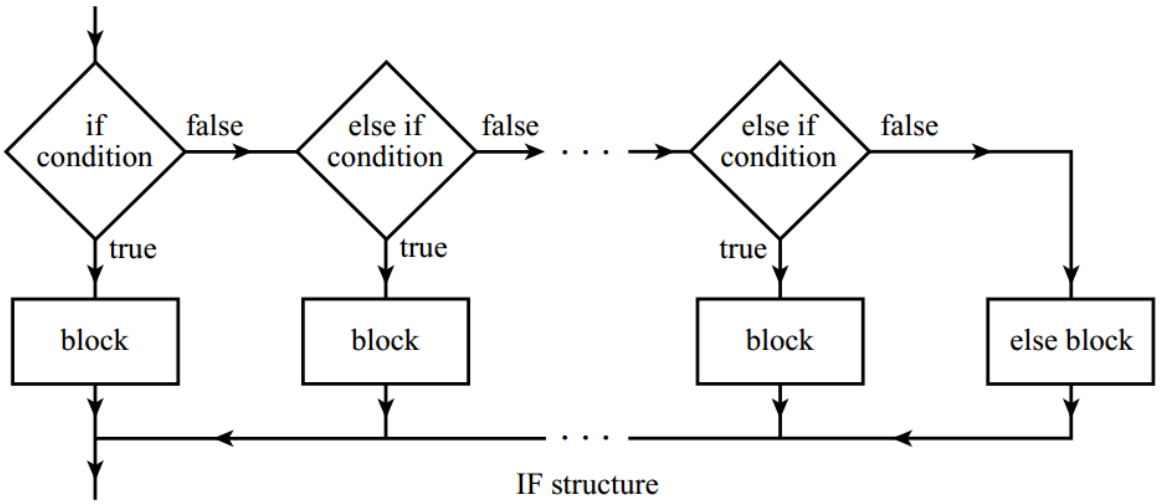


IF CONTROL STRUCTURE

When a decision should be made by a computer program, if control structure used. If structure has the following form :

Program booleanex3.java : an example of boolean variable

```
if(boolean variable 1)
{ executes statements here if boolean variable 1 is true}
else if (boolean variable 2)
{ executes statements here if boolean variable 2 is true}
else if (boolean variable 3)
{ executes statements here if boolean variable 3 is true}
……. (more else if statements are possible)
else
{ executes statements here if none of the above boolean variables are true}
```



IF structure

In our first example program name and grade of a students read and process the grade according to number grade.

```
import javax.swing.JOptionPane;
 class if1
{
    public static void main (String args[])
    {
    double grade;
    String name;
    name=JOptionPane.showInputDialog("Enter Student\'s name : ");
    grade=Double.parseDouble(JOptionPane.showInputDialog("Enter Student\'s grade(0-100) : "
    String s1="Grade of "+name+" is  ";
      if( grade >= 90)    { s1+="A";}
      else if(grade >=75) { s1+="B";}
      else if(grade >=60) { s1+="C";}
      else if(grade >=50) { s1+="D";}
      else if(grade >=40) { s1+="E";}
      else                { s1+="F";}
      s1+="("+grade+")";
    JOptionPane.showMessageDialog(null,s1,"Studen\'s grade : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

Two double numbers compared in the second example

Program if2.java : an example of boolean variable

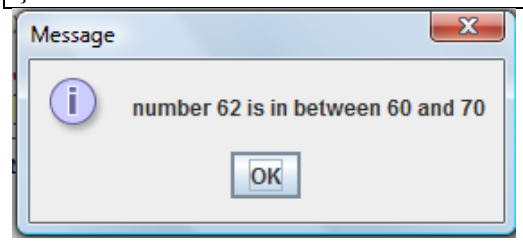
```
public class if2
{
public static void main (String args[])
{
 double x=1.245;
 double y=3.235;
 String s1="";
 if(x<y)
 { s1=x+" is smaller than "+y;}
 else if(x>y)
 { s1=x+" is bigger than "+y;}
  else if(x==y)
 { s1=x+" is equal to "+y;}
  System.out.println(s1);
 }
```

```
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" if2
1.245 is smaller than 3.235

> Terminated with exit code 0.
```

Program if3.java : an example of boolean variable

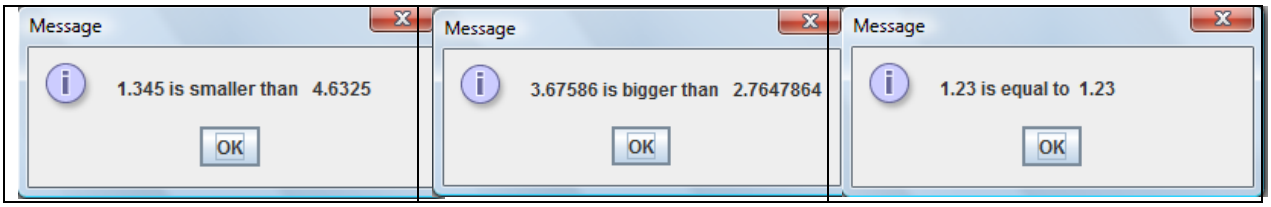
```
import java.awt.Graphics;
import javax.swing.JOptionPane;
import javax.swing.JApplet;

public class if3
{
public static void main (String args[])
{
 int i=62;
 int i1=60;
 int i2=70;
 String s1="";
 boolean b1=(i>=i1) && (i<i2);
 if(b1)
 { s1="number "+i+" is in between "+i1+" and "+i2;}
 else
  { s1="number "+i+" is not in between "+i1+" and "+i2;}
  JOptionPane.showMessageDialog(null,s1);
}
}
```



In the next example two double variables are compared

Program if_double.java : an example of double variable comparison

```
import java.io.*;
import javax.swing.JOptionPane;

class if_double
{
public static void main(String args[])
 {
 double x;
 double y;
 x=Double.parseDouble(JOptionPane.showInputDialog(" x = "));
 y=Double.parseDouble(JOptionPane.showInputDialog(" y = "));
 if(x<y)
 { JOptionPane.showMessageDialog(null,x+" is smaller than   "+y);}
 else if(x>y)
 { JOptionPane.showMessageDialog(null,x+" is bigger than   "+y);}
  else if(x==y)
 { JOptionPane.showMessageDialog(null,x+" is equal to  "+y);}
}
}
```

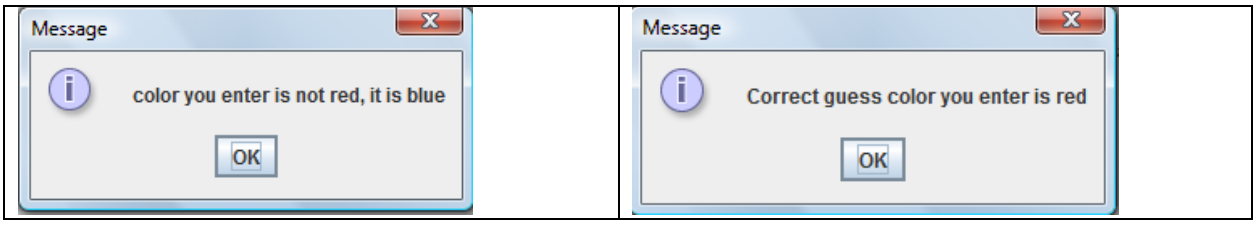


In the next example String variables compared. In order to check if two String variables are equal or not String comparision method equals can be used.

String s1=”John”;
String s2=”Jane”;
Boolean b1=s1.equals(s2);

Program if_String.java : an example of String variable comparison

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.JOptionPane;

class if_String
{
public static void main(String args[])
 {
 //Guess the color
 String s1;
 String s=JOptionPane.showInputDialog(" Enter a color = ");
 boolean b1=s.equals("red");
 if(b1) {s1="  Correct guess color you enter is red";}
 else   {s1="  color you enter is not red, it is "+s;}
 JOptionPane.showMessageDialog(null,s1);
}
}
```



When a complex decision making is required it can be achived as nested if structures (and if structure inside another if structure) or composite if structure(more than one boolean is combined together by joining and(&&), or(||) type boolean processors). In the next example problem 3.2.6 a nested if structure example is given. In the second example problem 3.2.6 a composite if structure is carried out exactly the same program output.

Program nested if.java : an example of nested if structure

```
import java.io.*;
import javax.swing.*;

class nestedif
{
public static void main(String args[])
{
double x = 7.0;
double y = 3.0;
if(x > 5)
```

```
{
   if(y > 5) { System.out.println("x and y are greater than 5.");}
   else if (y==5) { System.out.println("x is greater than 5  and y is equal to 5");}
   else { System.out.println("x is greater than 5 and y is smaller than 5");}
}
else
{
  if(y > 5) { System.out.println("x is smaller or equal to five  y is greater than 5.");}
  else if(y==5) { System.out.println("x is smaller or equal to five  y is equal to 5.");}
  else { System.out.println("x is smaller or equal to five  y is smaller than 5."); }
}
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" nestedif
x is greater than 5 and y is smaller than 5

> Terminated with exit code 0.
```

Program composite if.java : an example of composite if structure

```
import java.io.*;
import javax.swing.*;

class compositeif
{
public static void main(String args[])
{
double x = 7.0;
double y = 3.0;
if(x>5 && y>5) { System.out.println("x and y are greater than 5.");}
else if(x>5 && y==5) { System.out.println("x and y are greater than 5.");}
else if(x>5 && y<5) { System.out.println("x is greater than 5 and y is smaller than 5");}
else if(x<=5 && y>5) { System.out.println("x is smaller or equal to five  y is greater than 5.");}
else if(x<=5 && y==5) { System.out.println("x is greater than 5  and y is equal to 5");}
else if(x <= 5 && y<5) { System.out.println("x is smaller or equal to five  y is smaller than 5."); }
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" compositeif
x is greater than 5 and y is smaller than 5

> Terminated with exit code 0.
```

As you observed from the example, any complex logical processes can be achived by using if structural blocks. If structure is one of the widely use computer structure.

BIT PROCESSING IN JAVA

From time to time it could be necassary to proceed directly with the bits in computer. In java a set of processors are given to manupulate directly the bits. In order to understand the bir processes we should understand the basic way of data representation. A byte type integer data representation in bit structure in computer was given in Table 1.3.2

Table 3.3.1 writing 15 in a 8 bit integer variable (byte)

+/-	2^6	2^5	2^4	2^3	2^2	2^1	2^0
+	**64**	**32**	**16**	**8**	**4**	**2**	**1**

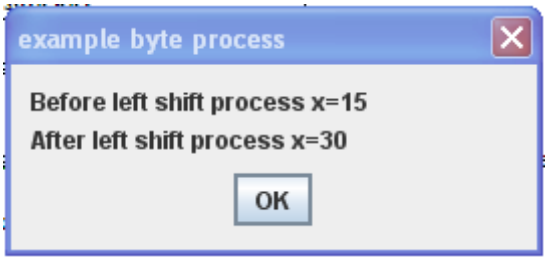
0	0	0	0	1	1	1	1

Table 3.3.2 Bit processors and their meaning

Bit processor	Meaning
&	AND
|	OR
^	XOR
~	NOT
>>	Shift one bit to the right
<<	Shift one bit to the left

Program bit.java : an example of bit proses << (shift one bit to the left)

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class bit
 {   public static void main(String args[])
     {
              int x=15;
              String s="Before left shift process x="+x+"\n";
        x=x<<1;
        s+="After left shift process x="+x;
        JOptionPane.showMessageDialog(null,s,"example byte
process",JOptionPane.PLAIN_MESSAGE);
     }
 }
```



What happened to the number. It is beter to explain it with the bit table again (table in byte type is given to avoid 32 bit table of int type integer variable)

Table shifting the bits of Table 1.3.2 one to the left a 8 bit integer variable (byte)

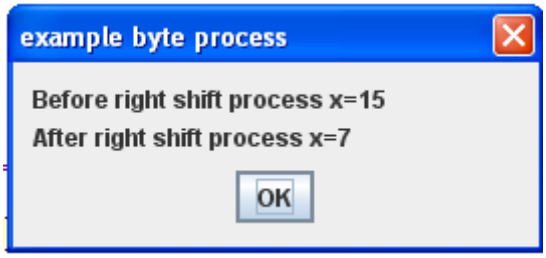
	+/-	2^6	2^5	2^4	2^3	2^2	2^1	2^0
	+	64	32	16	8	4	2	1
Before (15)	0	0	0	0	1	1	1	1
After(30)	0	0	0	1	1	1	1	0

As you see from the table, number become 30. If left shift apply to the same number(15), number becomes 7.

Program bit1.java : an example of bit proses >> (shift one bit to the right)

```
import javax.swing.*;
class bit1
 {   public static void main(String args[])
     {          int x=15;
        String s="Before right shift process x="+x+"\n";
```

```
        x=x>>1;
        s+="After right shift process x="+x;
        JOptionPane.showMessageDialog(null,s,"example byte process",JOptionPane.PLAIN_MESSAGE);
    }
}
```



If this process shown in the bit table

Table shifting the bits of Table 1.3.2 one to the right a 8 bit integer variable (byte)

	+/-	2^6	2^5	2^4	2^3	2^2	2^1	2^0
	+	64	32	16	8	4	2	1
Before (15)	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**
After(7)	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**

In reality int has 32 bit, therefore
int x=15;
will assign bit value as:
00000000000000000000000000001111
And after left shift (<<) operation it will be:
00000000000000000000000000011110

Several bit processes are shown in the next 3 example problems bitA.java, bitB.java and bitD.java

Program bitA.java : an example of bit proseses

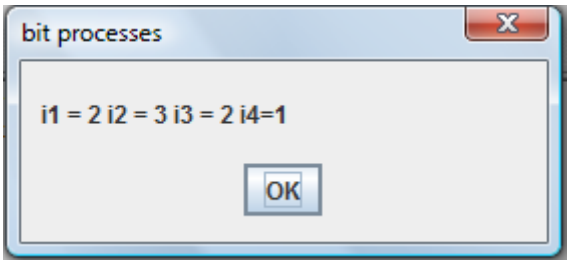
```
class bitA
 {   public static void main(String args[])
     {
        //or | bit işlemi
                int i1=1;
        int i2=2;
        int i3=i1|i2;
        int i4=~i3;
        String s="i1 = "+i1+" i2 = "+i2+" i3 = "+i3+"i4="+i4;
        System.out.println(s);
    }
 }
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" bitA
i1 = 1 i2 = 2 i3 = 3i4=-4

> Terminated with exit code 0.
```

Program bitB.java : an example of bit proseses

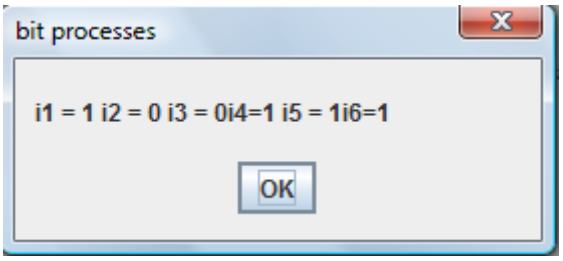
```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class bitB
 {   public static void main(String args[])
```

```
    {
       //xor  bit işlemi
       int i1=2;
       int i2=3;
       int i3=i1&i2;
       int i4=i3>>1;
       String s="i1 = "+i1+" i2 = "+i2+" i3 = "+i3+" i4="+i4;
       JOptionPane.showMessageDialog(null,s,"bit processes",JOptionPane.PLAIN_MESSAGE);
    }
}
```



Program bitD.java : an example of bit proseses

```
import java.io.*;
import javax.swing.*;

class bitD
 {    public static void main(String args[])
      {
         //or | bit process
         int i1=1;
         int i2=0;
         int i3=i1&i2;
         int i4=i1|i2;
         int i5=i1^i2;
         int i6=~i1;
         String s="i1 = "+i1+" i2 = "+i2+" i3 = "+i3+"i4="+i4+" i5 = "+i5+"i6="+i4;
         JOptionPane.showMessageDialog(null,s,"bit processes",JOptionPane.PLAIN_MESSAGE);
    }
 }
```

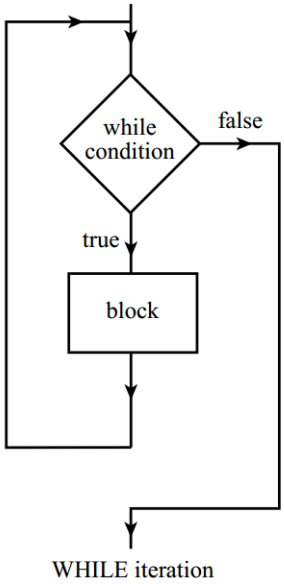


INTRODUCTION TO JAVA LANGUAGE : LOOP STRUCTURES

WHILE LOOP STRUCTURE

In the if structure of the previous section, it is seen that if the given boolean value is true, computer carries out instruction inside of the paranthesis. In the while structure it is similar to if structure, with one difference. In the if statement inside of the paranthesis instructions will be carried out once. In while statements, it will go into the paranthesis as repeatedly, as long as the boolean is true.

while(boolean variable)
{ processes while boolean variable is true}



For example while loop in the following structure will continue infinitely
while(true)
{ System.out.println("This loop will be infinite");}

In order to stop the loop the value of the boolean should change somewhere in the loop
int number=0;
while(number <= 100)
{ number += 1; }
This loop will turn 100 times and when number is bigger than 100, it will stop. The same loop could be written as
int number=0;
boolean b= number <= 100;
while(b)
{ number += 1;
b= number <= 100;
}
In our first example numbers from 1 to number1 is added and sum is calculated.
Program BPIH4E1.java : an example of while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E1
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
      Scanner input = new Scanner( System.in );
      int number1; // first number to add
      int sum=0;  // sum of numbers
      System.out.print( "Enter the number: " ); // prompt
      number1 = input.nextInt(); // read first number from user
      int i=1;
      while(i<=number1)
      { sum += i; // add numbers
        i++;
```

```
        }
        System.out.printf( "Sum is %d\n", sum ); // display sum
    } // end method main

} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E1
Enter the number: 4
Sum is 10
> Terminated with exit code 0.
```

In the second example numbers from number 1 to number 2 is added and sum is calculated.

Program BPIH4E2.java : an example of while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E2
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       Scanner input = new Scanner( System.in );
       int number1; // first number to add
       int number2;  //last number to add
       int sum=0; // sum of number1 and number2
        System.out.print( "Enter the first number: " ); // prompt
       number1 = input.nextInt(); // read first number from user
        System.out.print( "Enter the second number: " ); // prompt
       number2=input.nextInt();
       int i=number1;
       while(i<=number2)
       { sum += i; // add numbers
         i++;
       }
       System.out.printf( "Sum from %d to %d = %d \n", number1,number2,sum ); // display sum
    } // end method main
} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E2
Enter the first number: 1
Enter the second number: 100
Sum from 1 to 100 = 5050

> Terminated with exit code 0.
```

In the third example we will process Strings instead of just numbers by using while statement

Program BPIH4E3.java : an example of while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E3
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
       Scanner input = new Scanner( System.in );
       int number1; // first number to add
       System.out.print( "Enter the number: " ); // prompt
       number1 = input.nextInt(); // read first number from user
```

```
        int i=0;
        String star="*";
        String space=" ";
        String sum=star;
        while(i<=number1)
        {  sum = space+sum; // add Strings
           System.out.println(sum ); // display sum
           i++;
        }

    } // end method main

} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E3
Enter the number: 4
 *
  *
   *
    *
     *
> Terminated with exit code 0.
```

In the next example number and stars are written in on th same line

Program BPIH4E4.java : an example of while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E4
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
        // create Scanner to obtain input from command window
        Scanner input = new Scanner( System.in );
        int number1; // first number to add
        System.out.print( "Enter the number: " ); // prompt
        number1 = input.nextInt(); // read first number from user
        int i=0;
        String star="*";
        String space=" ";
        String sum=star;
        while(i<=number1)
        {  sum = star+i+space; // add Strings
           System.out.print(sum); // display sum
           i++;
        }
    } // end method main

} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E4
Enter the number: 5
*0 *1 *2 *3 *4 *5
> Terminated with exit code 0.
```

In the next example a primitice type of plot (line plot) is drawn by usıng while loop and * sign

Program BPIH4E5.java : an example of while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E5
{
   // main method begins execution of Java application
   public static void main( String args[] )
   {
     // create Scanner to obtain input from command window
     Scanner input = new Scanner( System.in );
     int x1; // first x to plot
     int x2; // last x to plot
     int y=0;
     System.out.print( "Enter the first number x1=" ); // prompt
     x1=input.nextInt(); // read first number from user
     System.out.print( "Enter the first number x1=" ); // prompt
     x2=input.nextInt(); // read second number from user
     String star="*";
     String space=" ";
     String sum;
     String s="";
     int x=x1;
     while(x<=x2)
     {   y=x*x+2*x+1;  // function to be plotted
        int i=0;
        sum=star;
        while(i<y)
        { sum = space+sum;
          i++;
        }
        s+=sum+"\n";

        x++;
     }
     System.out.println(s);
   } // end method main
} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E5
Enter the first number x1=-4
Enter the first number x1=4
       *
    *
  *
 *
*
 *
   *
      *
           *
                *


> Terminated with exit code 0.
```

When a complex numbers such as π=3.14159… , series solution is used. A series



can be used. Let us look at how a series like this can be put into the prgram by using a while loop.

Program BPIH4E6.java : an example of while loop

```
public class BPIH4E6
```

```
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      //pi = 4 - 4/3 + 4/5 -4/7+4/9-4/11+...+
      String s;
      double pi=0;
      int artieksi=-1;
      int i=1;
      while(i < 100000)
      {
      artieksi*=-1;
      pi+=4/(2.0*i-1.0)*artieksi;
      i++;
      }
      s="pi = "+pi;
      System.out.println(s);
    } // end method main
} // end class BPIH4E1
```

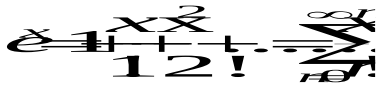
In the next example factorial will be calculated. Factorial is defined as :
n!=n*(n-1)*(n-2)….4.3.2.1 4!= 4*3*2*1 3!=3*2*1

Program BPIH4E7.java : an example of while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E7
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      // create Scanner to obtain input from command window
      Scanner input = new Scanner( System.in );
              int number; // number to get
      int y=0;
      System.out.print( "Enter the number n=" ); // prompt
      number=input.nextInt(); // read first number from user
      long factorial=1;
      String s="";
      int i=1;
      while(i<=number)
      {  factorial*=i;
        i++;
      }
      s=""+number+"! = "+factorial;
      System.out.println(s);
    } // end method main
} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E7
Enter the number n=4
4! = 24
> Terminated with exit code 0.
```

In the last example a more complex series will be calculated



note that e=2.7182818…

Program BPIH4E8.java : an example of while loop

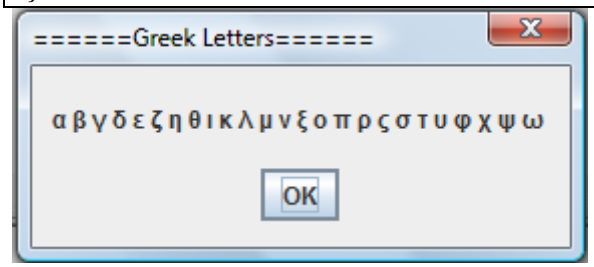
```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;

public class BPIH4E8
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      // number e = 2.718...
      // exp(x) = 1 + x /1! + x2 / 2! + x3 / 3! + x4 / 4! + ..
      String s;
      // create Scanner to obtain input from command window
      Scanner input = new Scanner( System.in );
      Locale us=new Locale("US");
      input.useLocale(us);
      System.out.print( "Enter x : " ); // prompt
      double x = input.nextDouble(); // read first number from user
      double power=1;
      double exponent=1;
      double factorial=1;
      double number=1;
      while(number<=200)
      {
      factorial*=number;
      power*=x;
      exponent+=power/factorial;
      number++;
      }
      s="exp("+x+") = "+exponent;
      System.out.println(s);
    } // end method main
} // end class BPIH4E1
```

It is possible to use also char variable in while loop counter

Program BPIH4E12.java : an example of while loop

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class BPIH4E12
 {   public static void main(String args[])
     {
        String s="";
        char b='\u03B1';
        while(b<='\u03C9')
```

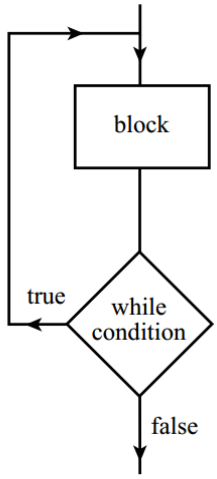
```
        {s+=b+" ";
        b++;}
        JOptionPane.showMessageDialog(null,s,"======Greek
Letters======",JOptionPane.PLAIN_MESSAGE);
    }
 }
```



DO-WHILE LOOP STRUCTURE

Do-while structure is very similar to while structure. Except in do-while statements in the paranthesis will be carried out at least ones, in while statement if the boolean statement is false, it will not go inside of the loop at all.

do
{ processes while boolean variable is true}
while(boolean variable);



In order to stop the loop the value of the boolean should change somewhere in the loop
For example in the first while loop the value of number will be 11 (it will not enter to the loop)

```
int number=11;
while(number <= 10)
{ number += 1; }
```

In the next do-while statement, the value of number will be 12. It will go into the loop once.

```
int number=11;
do
{ number += 1; }
while(number <= 10);
```

In our first example numbers from 1 to number1 is added and sum is calculated.
We will list the few codes, similar to while statements code here, but the all codes similar to while codes are available in exercise folder.

Program BPIH4E1A.java : an example of do-while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E1A
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
       Scanner input = new Scanner( System.in );
       int number1; // first number to add

       int sum=0; // sum of number1 and number2
       System.out.print( "Enter the number: " ); // prompt
       number1 = input.nextInt(); // read first number from user
       int i=1;
       do
       { sum += i; // add numbers
         i++;
       }  while(i<=number1);
       System.out.printf( "Sum is %d\n", sum ); // display sum
    } // end method main

} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E1A
Enter the number: 100
Sum is 5050

> Terminated with exit code 0.
```

Program BPIH4E4A.java : an example of do-while loop

```
import java.util.Scanner; // program uses class Scanner

public class BPIH4E4A
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
       // create Scanner to obtain input from command window
       Scanner input = new Scanner( System.in );
       int number1; // first number to add
       System.out.print( "Enter the number: " ); // prompt
       number1 = input.nextInt(); // read first number from user
       int i=0;
       String star="*";
       String space="  ";
       String sum=star;
       String s="";
       do
       {  s += star+i+space; // add Strings
          i++;
       }  while(i<=number1);
       System.out.printf("%s",s);
    } // end method main

} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E4A
```

```
Enter the number: 20
*0 *1 *2 *3 *4 *5 *6 *7 *8 *9 *10 *11 *12 *13 *14 *15 *16 *17 *18 *19 *20
> Terminated with exit code 0.
```

In this example an equation y=f(x)= 5.67*x*x+ 2.25*x-3.25 is calculated in and result is printed out.

Program BPIH4E9A.java : an example of do-while loop

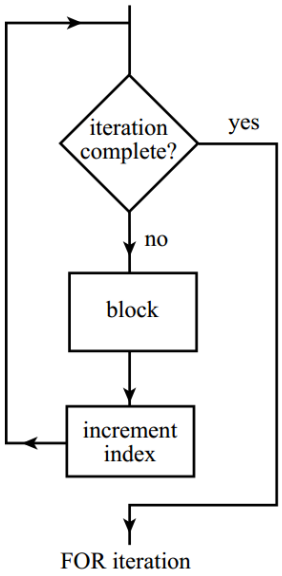
```
import java.util.Scanner; // program uses class Scanner
import java.util.Locale;

public class BPIH4E9A
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      Scanner input = new Scanner( System.in );
      Locale us=new Locale("US");
      input.useLocale(us);
      System.out.print( "Enter first double x1: " ); // prompt
      double x1 = input.nextDouble(); // read first number from user
      System.out.print( "Enter last double x2: " ); // prompt
      double x2 = input.nextDouble(); // read first number from user
      System.out.print( "Enter increment step dx: " ); // prompt
      double dx = input.nextDouble(); // read first number from user
      double x=x1;
      String s="";
      double y=0;
      do
      { y=5.67*x*x+ 2.25*x-3.25; // add numbers
        x+=dx;
        s+="x = "+x+" y = "+y+"\n";
      }  while(x<=x2);
      System.out.println(s); // display x and y
    } // end method main
} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E9A
Enter first double x1: 0
Enter last double x2: 10
Enter increment step dx: 0.2
x = 0.2 y = -3.25
x = 0.4 y = -2.5732
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E9A
Enter first double x1: 0
Enter last double x2: 2
Enter increment step dx: 0.2
x = 0.2 y = -3.25
x = 0.4 y = -2.5732
x = 0.6000000000000001 y = -1.4427999999999999
x = 0.8 y = 0.14120000000000088
x = 1.0 y = 2.1788000000000007
x = 1.2 y = 4.67
x = 1.4 y = 7.614799999999999
x = 1.5999999999999999 y = 11.0132
x = 1.7999999999999998 y = 14.865199999999998
x = 1.9999999999999998 y = 19.170799999999996
x = 2.1999999999999997 y = 23.929999999999993

> Terminated with exit code 0.
```

FOR LOOP STRUCTURE

For loop structure is also similar to while structure. Except initial condition and incrimental conditions are also given in the structure

for(initial statements, boolean condition, incremental statements)
{ processes while boolean variable is true}



Some example statements:

for(int number=0,int sum=0;number <= 10;number++)
{sum+=number; }

for(double number=0,double sum=0;number <= 10.0;number+=0.1)
{sum+=number; }

For loops are usually used as a counter (similar tos um or product statements in math), but for and while can always be replaced each other. To show this fact the example problems given in this subchapter will be the same as while examples except graphic screen input and output is used in here.
In our first example numbers from 1 to number1 is added and sum is calculated.

Program BPIH4E1B.java : an example of for loop

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E1B
{
   // main method begins execution of Java application
   public static void main( String args[] )
   {
      int number1; // first number to add
      int sum=0; // sum of number1 and number2
      number1=Integer.parseInt(JOptionPane.showInputDialog("Enter first integer: ")); // read number
      for(int i=1;i<=number1;i++)
      { sum += i; // add numbers
      }
      System.out.printf( "Sum is %d\n", sum ); // display sum
   } // end method main

} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E1B
Sum is 5050
> Terminated with exit code 0.
```

In the second example numbers from number 1 to number 2 is added and sum is calculated.

Program BPIH4E2B.java : an example of for loop

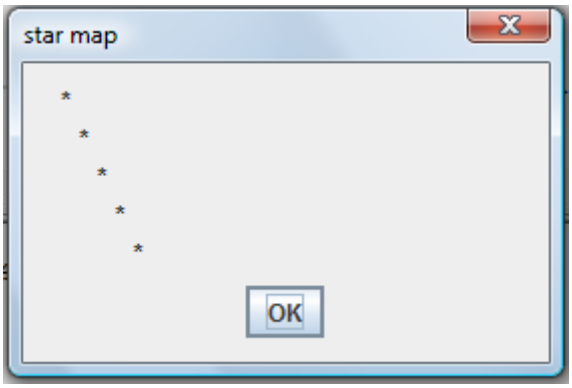
```
import javax.swing.*; // program uses class Scanner

public class BPIH4E2B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      int number1; // first number to add
      int number2;  //last number to add
      int sum=0; // sum of number1 and number2
      number1=Integer.parseInt(JOptionPane.showInputDialog("Enter first integer: ")); // read number
      number2=Integer.parseInt(JOptionPane.showInputDialog("Enter last integer: ")); // read number
      for(int i=number1;i<=number2;i++)
      { sum += i; // add numbers
      }
      System.out.printf( "Sum from %d to %d = %d \n", number1,number2,sum ); // display sum
    } // end method main
} // end class BPIH4E1
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" BPIH4E2B
Sum from 1 to 100 = 5050
> Terminated with exit code 0.
```

In the third example we will process Strings instead of just numbers by using while statement

Program BPIH4E3B.java : an example of for loop

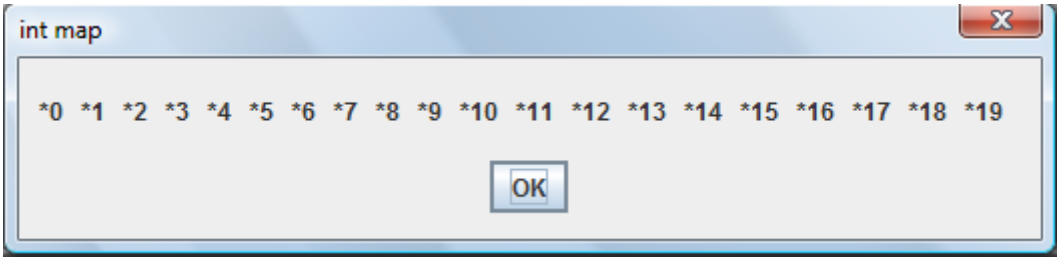
```
import javax.swing.*; // program uses class Scanner

public class BPIH4E3B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      int number1; // first number to add
      number1=Integer.parseInt(JOptionPane.showInputDialog("Enter number of stars: ")); // read number
      String star="*";
      String space="   ";
      String sum=star;
      String s="";
      for(int i=0;i<number1;i++)
      {  sum = space+sum; // add Strings
        s+=sum+"\n";
      }
      JOptionPane.showMessageDialog(null,s,"star map",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E1
```



In the next example number and stars are written in on th same line

Program BPIH4E4B.java : an example of for loop

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E4B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      int number1; // first number to add
      number1=Integer.parseInt(JOptionPane.showInputDialog("Enter number of starts: ")); // read number
      String star="*";
      String space="   ";
      String s="";
      for(int i=0;i<number1;i++)
      {  s += star+i+space; }
      JOptionPane.showMessageDialog(null,s,"int map",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E1
```

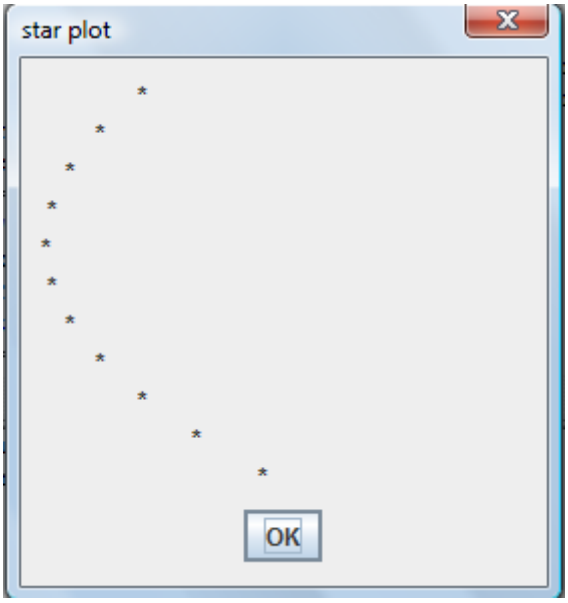


In the next example a primitive type of plot (line plot) is drawn by usıng while loop and * sign

Program BPIH4E5B.java : an example of for loop

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E5B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      int x1; // first x to plot
      int x2; // last x to plot
      int y=0;
      x1=Integer.parseInt(JOptionPane.showInputDialog("Enter first number: ")); // read number
      x2=Integer.parseInt(JOptionPane.showInputDialog("Enter last number: "));  // read number
      String star="*";
      String space=" ";
```

```
        String sum=star;
        String s="";
        for(int x=x1;x<=x2;x++)
        {   y=x*x+2*x+1;  //function to be plotted
           for(int i=0;i<y;i++)
           { sum = space+sum;}
           s+=sum+"\n";
           sum=star;
        }
        JOptionPane.showMessageDialog(null,s,"star plot",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E5B
```



When a complex numbers such as π=3.141592653589793, series solution is used. A series

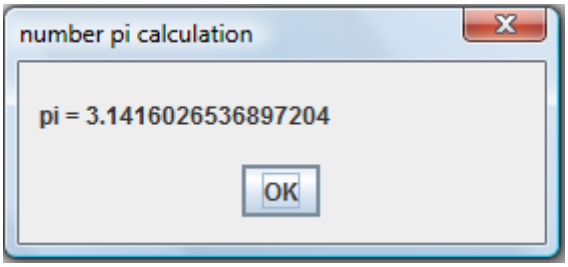


can be used. Let us look at how a series like this can be put into the program by using a while loop.

Program BPIH4E6.java : an example of for loop

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E6B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      //pi = 4 - 4/3 + 4/5 -4/7+4/9-4/11+...+
      String s;
      int i;
      double pi=0;
      int artieksi=-1;
      for(i=1;i < 100000;i++)
      {
      artieksi*=-1;
      pi+=4/(2.0*i-1.0)*artieksi;
```

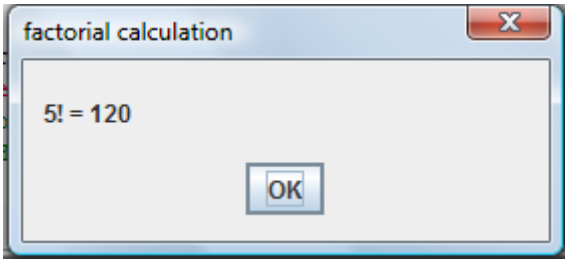
```
        }
        s="pi = "+pi;
        JOptionPane.showMessageDialog(null,s,"number pi
calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E6B
```



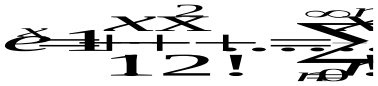
In the next example factorial will be calculated. Factorial is defined as :
n!=n*(n-1)*(n-2)….4.3.2.1 4!= 4*3*2*1 3!=3*2*1

Program BPIH4E7B.java : an example of for loop

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E7B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      // factorial
      String s;
      long factorial=1;
      int number=Integer.parseInt(JOptionPane.showInputDialog("Enter a number: ")); // read number
      for(int i=1;i <=number;i++)
      {
      factorial*=i;
      }
      s=""+number+"! = "+factorial;
      JOptionPane.showMessageDialog(null,s,"factorial calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E7B
```



In the next example a more complex series will be calculated

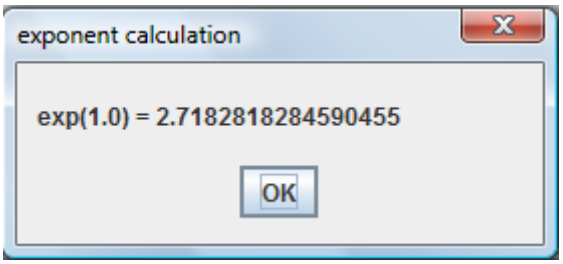


note that e=2.7182818…

Program BPIH4E8B.java : an example of for loop

```
import javax.swing.*; // program poweres class Scanner
```

```
public class BPIH4E8B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      // number e = 2.718...
      // exp(x) = 1 + x /1! + x2 / 2! + x3 / 3! + x4 / 4! + ..
      String s;
      double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: ")); // read number
      double power=1;
      double exponent=1;
      double factorial=1;
      for(double sayi=1;sayi<=200;sayi++)
      {
      factorial*=sayi;
      power*=x;
      exponent+=power/factorial;
      }
      s="exp("+x+") = "+exponent;
      JOptionPane.showMessageDialog(null,s,"exponent calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E8B
```



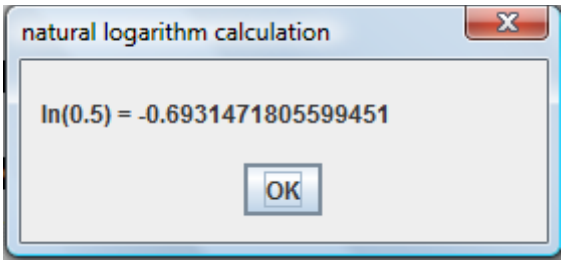
Another series example will be



Program BPIH4E13B.java : an example of for loop

```
import javax.swing.*; // program poweres class Scanner

public class BPIH4E13B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      // number e = 2.718...
      // ln(x) = (x-1) - (x-1)^2/2+(x-1)^3/3+(x-1)^4/4..
      // for x<2;
      String s;
      double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: ")); // read number
      double power=1;
      double ln=0;
      double xm1=x-1;;
      int plusminus=-1;
      for(double n=1;n<=10000000;n++)
      {
              plusminus*=-1;
              power*=xm1;
      ln+=plusminus*power /n;
      }
      s="ln("+x+") = "+ln;
      JOptionPane.showMessageDialog(null,s,"natural logarithm calculation",JOptionPane.PLAIN_MESSAGE);
```

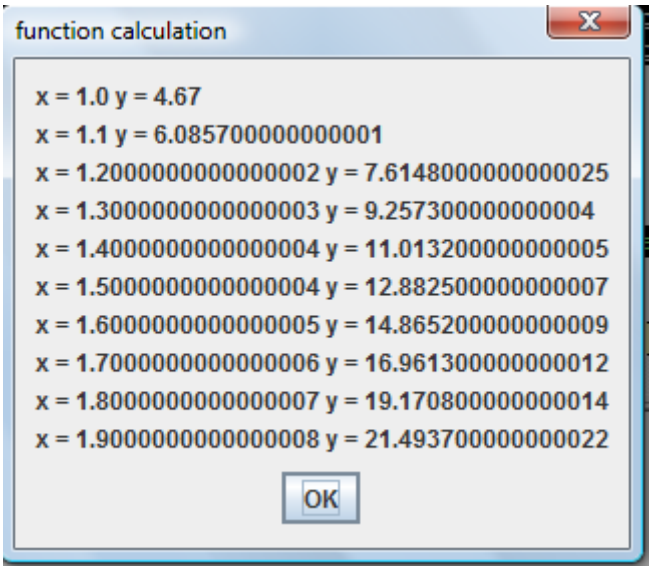
```
    } // end method main
} // end class BPIH4E13B
```



In the last example two numbers and a number increment read from the program and a function calculated results are listed.

Program BPIH4E9B.java : an example of for loop

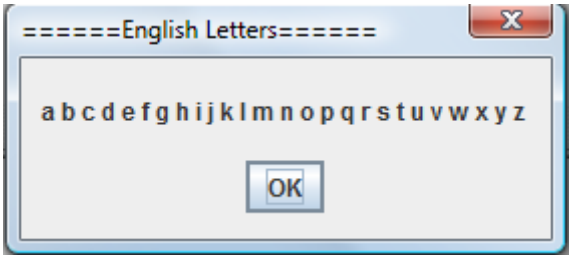
```
import javax.swing.*; // program uses class Scanner

public class BPIH4E9B
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      double x1; // first number to add
      double x2;  //last number to add
      double dx;
      double y;
      x1=Double.parseDouble(JOptionPane.showInputDialog("Enter first double: ")); // read number
      x2=Double.parseDouble(JOptionPane.showInputDialog("Enter last double: ")); // read number
      dx=Double.parseDouble(JOptionPane.showInputDialog("Enter increment step: ")); // read number
      String s="";
      for( double x=x1;x<=x2; x+=dx)
      { y=5.67*x*x+ 2.25*x-3.25;// calculate y
        s+="x = "+x+" y = "+y+"\n";
      }
      JOptionPane.showMessageDialog(null,s,"function calculation",JOptionPane.PLAIN_MESSAGE); //display x and y
    } // end method main
} // end class BPIH4E9B
```



By using for loop structure, also char type varibles can be counted

Program BPIH4E12B.java : an example of for loop

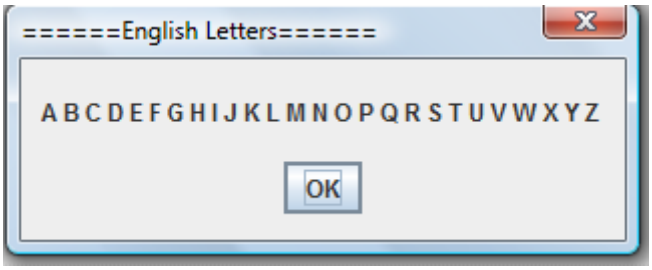
```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class BPIH4E12B
 {    public static void main(String args[])
      {
         String s="";
         for(char b='\u0061';b<='\u007A';b++)
         {s+=b+" ";}
         JOptionPane.showMessageDialog(null,s,"======English Letters======",JOptionPane.PLAIN_MESSAGE);
     }
 }
```



The same program can be given by using char names instead of their Unicode characters

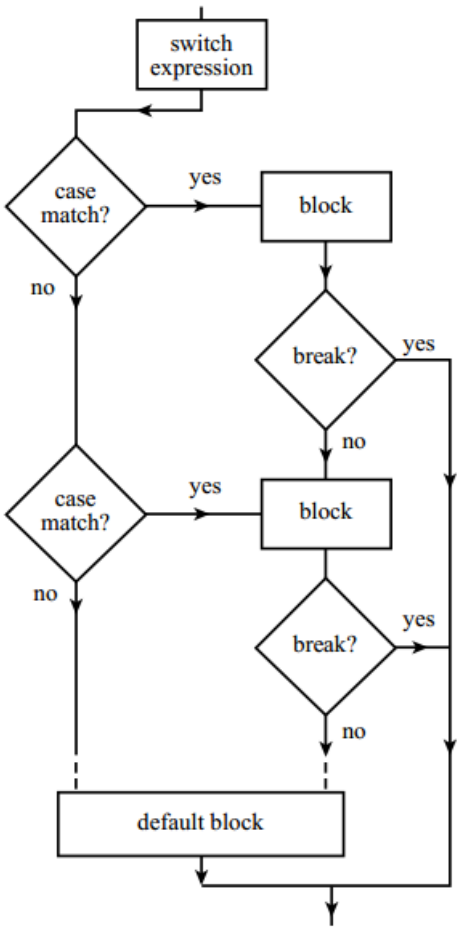
Program BPIH4E13B.java : an example of for loop

```
import java.io.*; //java girdi cikti sinifini cagir
import javax.swing.*;

class BPIH4E13B
 {    public static void main(String args[])
      {
         String s="";
         for(char b='A';b<='Z';b++)
         {s+=b+" ";}
         JOptionPane.showMessageDialog(null,s,"======English Letters======",JOptionPane.PLAIN_MESSAGE);
     }
 }
```



SWITCH-CASE STRUCTURE

Switch-case structure is a structure closer to if structure than the above loop structures. This statement is not widely use among programmers because the same thing can be achived by using if blocks. The structure has the following form :



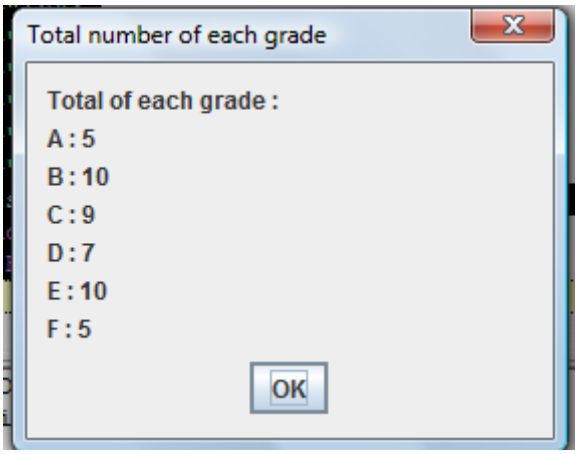

```
switch( variable name)
{
case the value of the variable :
if the variable has the value above carry out statements here
break; // end of structure
case the value of the variable:
if the variable has the value above carry out statements here
break; // end of structure
.............
default :
If non of the variable above is reached, carry out this statement
break; // end of structure
}
```

A sample program of grade counter of students are given

Program BPIH4E10C.java : an example of switch-case statement

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E10C
{
// main method begins execution of Java application
public static void main( String args[] )
{
String s1="";
char grade;
boolean b=true;
int Anumber=0,Bnumber=0,Cnumber=0,Dnumber=0,Enumber=0,Fnumber=0;
```

```
while(b)
{
s1=JOptionPane.showInputDialog("Enter Student\'s grade: "); // read grade
grade=s1.charAt(0);
switch(grade)
{
case 'A': case 'a':
Anumber++;
break;
case 'B': case 'b':
Bnumber++;
break;
case 'C': case 'c':
Cnumber++;
break;
case 'D': case 'd':
Dnumber++;
break;
case 'E': case 'e':
Enumber++;
break;
case 'F': case 'f':
Fnumber++;
break;
default:
b=false;
break;
} // end of switch-case
}//end of while
String s="Total of each grade : \n";
s+="A : "+Anumber+"\n";
s+="B : "+Bnumber+"\n";
s+="C : "+Cnumber+"\n";
s+="D : "+Dnumber+"\n";
s+="E : "+Bnumber+"\n";
s+="F : "+Fnumber+"\n";
JOptionPane.showMessageDialog(null,s,"Total number of each grade",JOptionPane.PLAIN_MESSAGE);
//display x and y
    } // end method main
} // end class BPIH4E10C
```



If the same program is written by using if blocks instead of switch-case block:

Program BPIH4E10C.java : switch-case statement replaced with if statement

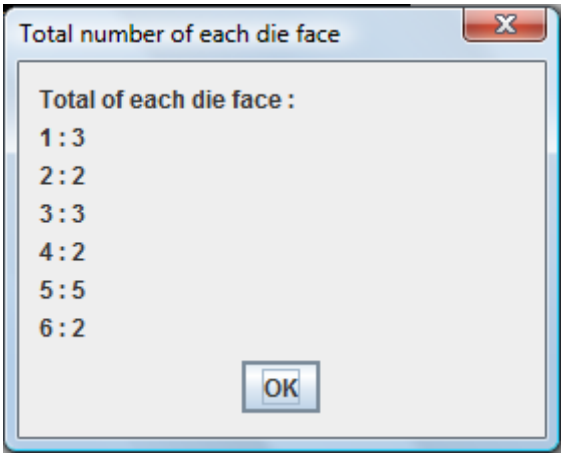
```
import javax.swing.*; // program uses class Scanner

public class BPIH4E10E
{
// main method begins execution of Java application
public static void main( String args[] )
{
String s1="";
char grade;
boolean b=true;
int Anumber=0,Bnumber=0,Cnumber=0,Dnumber=0,Enumber=0,Fnumber=0;
while(b)
{
s1=JOptionPane.showInputDialog("Enter Student\'s grade: "); // read grade
grade=s1.charAt(0);
if(grade=='A'||grade=='a')
{Anumber++;}
else if(grade=='B'||grade=='b')
{Bnumber++;}
else if(grade=='C'||grade=='c')
{Cnumber++;}
else if(grade=='D'||grade=='d')
{Dnumber++;}
else if(grade=='E'||grade=='e')
{Enumber++;}
else if(grade=='F'||grade=='f')
{Fnumber++;}
else
{b=false;}
}//end of while
String s="Total of each grade : \n";
s+="A : "+Anumber+"\n";
s+="B : "+Bnumber+"\n";
s+="C : "+Cnumber+"\n";
s+="D : "+Dnumber+"\n";
s+="E : "+Bnumber+"\n";
s+="F : "+Fnumber+"\n";
JOptionPane.showMessageDialog(null,s,"Total number of each grade",JOptionPane.PLAIN_MESSAGE); //display x
and y
    } // end method main
} // end class BPIH4E10E
```

The result is the same as the previous program. Now switch case sample will be given for a die throwing example. A die has 6 faces and we will be counting how many times each face is entered.

Program BPIH4E11C.java : an example of switch-case statement

```
import javax.swing.*; // program uses class Scanner

public class BPIH4E11C
{
// main method begins execution of Java application
public static void main( String args[] )
{
//die throw counting
int die=1;
boolean b=true;
int one=0,two=0,three=0,four=0,five=0,six=0;
```

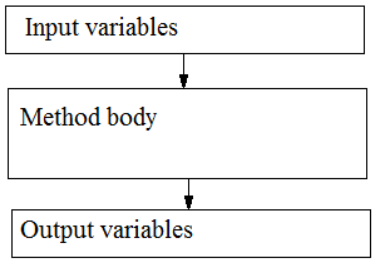
```
while(b)
{
die=Integer.parseInt(JOptionPane.showInputDialog("Enter number in die :")); // read grade
switch(die)
{
case 1:
one++;
break;
case 2:
two++;
break;
case 3:
three++;
break;
case 4:
four++;
break;
case 5:
five++;
break;
case 6:
six++;
break;
default:
b=false;
break;
} // end of switch-case
}//end of while
String s="Total of each die face : \n";
s+="1 : "+one+"\n";
s+="2 : "+two+"\n";
s+="3 : "+three+"\n";
s+="4 : "+four+"\n";
s+="5 : "+five+"\n";
s+="6 : "+six+"\n";
JOptionPane.showMessageDialog(null,s,"Total number of each die face",JOptionPane.PLAIN_MESSAGE);
//display x and y
    } // end method main
} // end class BPIH4E11C
```



METHODS AND INTRODUCTION TO CLASSES

METHODS

Methods and **variables** are the main components of java programs. The main componenets of java programs are classes. Methods are program components located in a class. Methods also have input and output variables which makes them to communucate with other program parts.



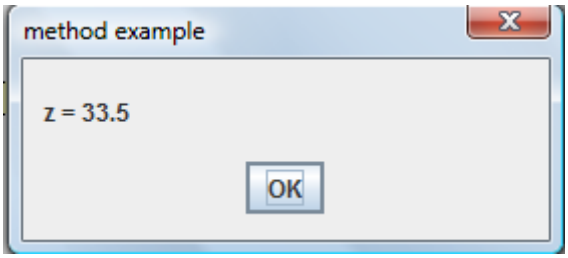
In general class has the following structure:
(**public**) (**static**) name_of_the_method(method input variables list)
{
Processes to be carried out in method
return Method output variable
}
If an example given:

```
public static double z(int x,double y)
{
double w = 3.5;
double f ;
f = w*x*x+y;
return f;
}
```

In this method, x and y is integer type input variables, z method name and f is local variable used inside the method, and because f is used with return, it is also the output variable. The word public declare that the method can be accessed by any program. There is also word private and protected for limited acsess. The word static declares that method can be used as a complete programming piece. Methods can also exist as a part of a complete class, which some of the variable decleration is given in some other part of the class structure. In this case it is not a whole structure by itself, but a subpart of class stucture. When that is the case the word static is not used.

Program BPIH5E1**: an example of static method call**

```
import javax.swing.*;

    public class BPIH5E1
    {
    public static double z(int x,double y)
    {
    double w = 3.5;
    double f ;
    f = w*x*x+y;
    return f;
    }

   public static void main( String args[] )
   {
            int i=3;
            double q=2.0;
               String s="z = "+z(i,q);
     JOptionPane.showMessageDialog(null,s,"method example",JOptionPane.PLAIN_MESSAGE);
   } // end method main
} // end class BPIH5E1
```



It should be note that in our example program two methods are declared one is **z** the other is **main**. Once a method is declared it can be called from the other programs as well (if it is declared as public)

Program BPIH5E2**: an example of static method call**

```
  import javax.swing.*; // program uses class Scanner

    public class BPIH5E2
    {

   public static void main( String args[] )
   {
      int i=3;
      double q=2.0;
     String s="z = "+BPIH5E1.z(i,q);
     JOptionPane.showMessageDialog(null,s,"method example",JOptionPane.PLAIN_MESSAGE);
   } // end method main
} // end class BPIH4E2
```

The output of second example is the same as the first one. It is also used the method from the first example by declaring the name as :

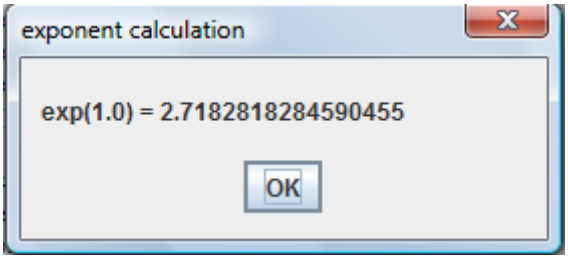
BPIH5E1.z(i,q);

BPIH5E1. in the declaration tells us that method is defined in class **BPIH5E1**. Now that we learn how to write a method, programs previously written as a whole in the main program such as BPIH4E8B can be written now by using a new method.

Program BPIH5E3**: an example of static method call to calculate function exp**

```
import javax.swing. JOptionPane; // program poweres class Scanner

public class BPIH5E3
{
    // main method begins execution of Java application
    public static double exp(double x)
    {
      // number e = 2.718...
      // exp(x) = 1 + x /1! + x2 / 2! + x3 / 3! + x4 / 4! + ..
      double power=1;
      double exponent=1;
      double factorial=1;
      for(double sayi=1;sayi<=200;sayi++)
      {
      factorial*=sayi;
      power*=x;
      exponent+=power/factorial;
      }
     return exponent;
     }// end method exp
```

```
    public static void main( String args[] )
    {
      String s;
      double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: "));
     // read number
      s="exp("+x+") = "+exp(x);
      JOptionPane.showMessageDialog(null,s,"exponent
calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E1
```



Observe the differences between program BPIH4E8B and BPIH5E3. Once a method such as exp written, it can be used in anywhere. In example BPIH5E4, exp is called from program BPIH5E3

Program BPIH5E4.java **: an example of static method call to calculate function exp**

```
import javax.swing.*;

public class BPIH5E4
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      String s;
      double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: "));
      // read number
      s="exp("+x+") = "+BPIH5E3.exp(x);
      JOptionPane.showMessageDialog(null,s,"exponent calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E4
```

It can also be noted in this point that **JOptionPane.showMessageDialog** is a method called **showMessageDialog** defined in class **JOptionPane.** Because this class definition is not located in our directory directly, it should also be defined the location of the class. It is given with import statemet as: **import javax.swing.JOptionPane;**

There are other important libraries that we can utilises methods written in. For example, class Math is one of the important libraries for mathematical processes

Table 2.1 some of the static methods defined in class Math

Method	**Definition**	**Example**
abs(x)	Absolute value of double variable x	Math.abs(9.2) = 9.2 Math.abs(-9.2) = 9.2
ceil(x)	Converts double variable x to one upper integer value	Math.ceil(9.2) = 10 Math.ceil(-9.8) = -9

floor(x)	Converts double variable x to one lower integer value	Math.floor(9.2) = 9 Math.floor(-9.8) = -10
cos(x)	Cosine of double value x(in radian)	Math.cos(1.0)=0.54030230568
sin(x)	sine of double value x(in radian)	Math.sin(1.0)=0.8414709840709
tan(x)	Tangent of double value x(in radian)	Math.tan(1.0)=1.557040724655
acos(x)	Inverse cosine (in radian) of double value x	Math.acos(0.54030230568)=1.0
asin(x)	Inverse sine (in radian) of double value x	Math.asin(0.8414709840709)=1.0
atan(x) atan2(x,y)	Inverse Tangent (in radian) of double value x Inverse Tangent (in radian) of double value x and y as in cartesian coordinate system	4.0*Math.atan(1.0)=3.14159265359 4.0*Math.atan2(1.0,1.0)=3.14159265359
exp(x)	Powers of exponent number e(2.7182818284590452354)	Math.exp(1.0) =2.718281828459 Math.exp(2.0) =7.389056098931
log(x)	Natural logarithm for power e	Math.log(2.718281828459)=1.0
pow(x,y)	Y power of x	Math.pow(3,2)=9.0
max(x,y)	Maximum of x and y	Math.max(2.3,12.7)=12.7
min(x,y)	Minimum of x and y	Math.min(2.3,12.7)=2.3
random()	Random number between 0 and y	Math.random() = any double number between 0 and 1 in random

By using Math library, **Program 5.1.4** BPIH5E4.java can be re-written as :

Program BPIH5E5.java **: an example of static method call to calculate function exp**

```
import javax.swing.JOptionPane; ; // program poweres class Scanner

public class BPIH5E5
{
    // main method begins execution of Java application
    public static void main( String args[] )
    {
      String s;
      double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: ")); // read number
      s="exp("+x+") = "+Math.exp(x);
      JOptionPane.showMessageDialog(null,s,"exponent calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E5
```

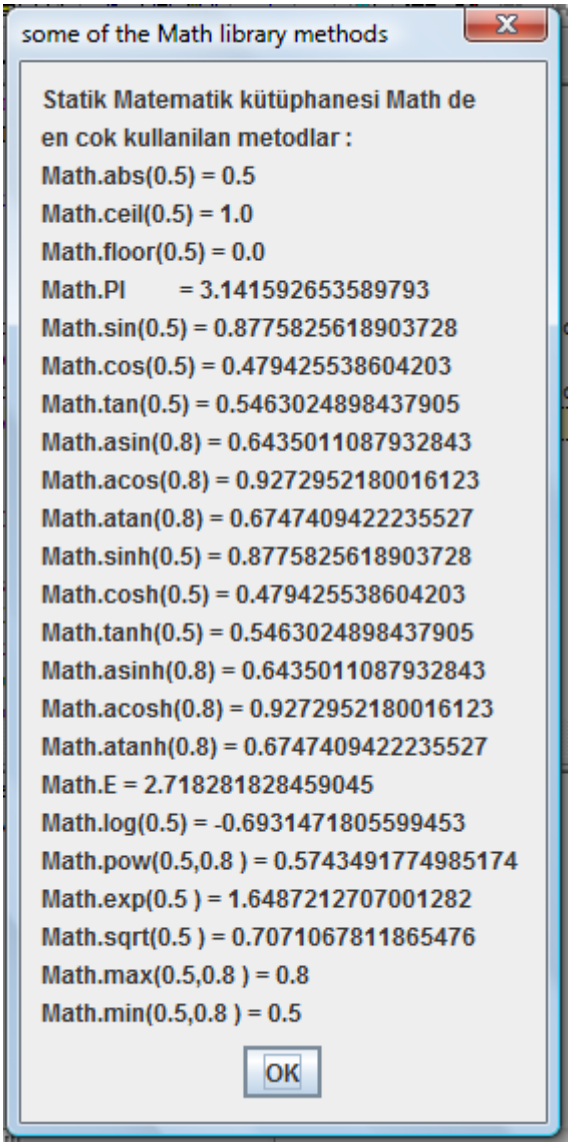
In the next example some of the Math library methods will be used.

Program BPIH5E6.java **: an example of static method call to calculate function exp**

```
import javax.swing.JOptionPane;
public class BPIH5E6
{
    public static void main (String args[])
    {
    double x,y;
    String s="";
    x=Double.parseDouble(JOptionPane.showInputDialog
```

```
    ("enter a real number : "));
    y=Double.parseDouble(JOptionPane.showInputDialog
    ("enter a second real number giriniz : "));

    s +="Statik Matematik kütüphanesi Math de \n";
    s+="en cok kullanilan metodlar :\n" ;
    // Math.abs(3.2)=3.2   Math.abs(-3.2)=3.2
    s+="Math.abs("+x+") = "+Math.abs(x)+"\n";
    // Math.ceil(9.2)=10   Math.ceil(-9.8)=-9
    s+="Math.ceil("+x+") = "+Math.ceil(x)+"\n";
    // Math.floor(9.2)=9   Math.floor(-9.8)=-10
    s+="Math.floor("+x+") = "+Math.floor(x)+"\n";
    s += "Math.PI         = "+Math.PI+"\n";
    //trigonometric function
    s += "Math.sin("+x+") = "+Math.cos(x)+"\n";
    s += "Math.cos("+x+") = "+Math.sin(x)+"\n";
    s += "Math.tan("+x+") = "+Math.tan(x)+"\n";
    s += "Math.asin("+y+") = "+Math.acos(y)+"\n";
    s += "Math.acos("+y+") = "+Math.asin(y)+"\n";
    s += "Math.atan("+y+") = "+Math.atan(y)+"\n";
    //hyperbolic functions
    s += "Math.sinh("+x+") = "+Math.cos(x)+"\n";
    s += "Math.cosh("+x+") = "+Math.sin(x)+"\n";
    s += "Math.tanh("+x+") = "+Math.tan(x)+"\n";
    s += "Math.asinh("+y+") = "+Math.acos(y)+"\n";
    s += "Math.acosh("+y+") = "+Math.asin(y)+"\n";
    s += "Math.atanh("+y+") = "+Math.atan(y)+"\n";
    // Math.E       number e = 2.718281828...
    s += "Math.E = "+Math.E+"\n";
    s += "Math.log("+x+") = "+Math.log(x)+"\n";
    s += "Math.pow("+x+","+y+" ) = "+Math.pow(x,y)+"\n";
    s += "Math.exp("+x+" ) = "+Math.exp(x)+"\n";
    s += "Math.sqrt("+x+" ) = "+Math.sqrt(x)+"\n";
    // Math.max(x,y)   Math.min(x,y)
    s += "Math.max("+x+","+y+" ) = "+Math.max(x,y)+"\n";
    s += "Math.min("+x+","+y+" ) = "+Math.min(x,y)+"\n";
    JOptionPane.showMessageDialog(null,s,
    "some of the Math library methods",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

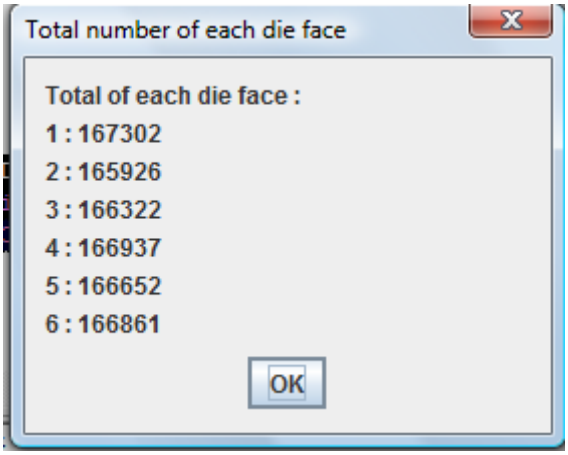
In order to show you the usage of random number, **Program 4.4.3 BPIH4E11C.java is** re-written with a throw_die method. This method will give a random number between 1 to 6 (to simulate die faces) and count how many times each faces comes. We tried 1million die throw in the program. Compare the code with **BPIH4E11C.java.**



Program BPIH5E7.java **: an example of static method die throwing and counting**

```
import javax.swing.*; // program uses class Scanner

public class BPIH5E7
{
//die method
public static int throw_die()
{return (int)(Math.random()*6+1);}

// main method
public static void main( String args[] )
{
//die throw counting
int die=1;
boolean b=true;
int one=0,two=0,three=0,four=0,five=0,six=0;
int counter=0;
while(counter<1000000)
{
die=throw_die();
if(die==1) {one++;}
else if (die==2) {two++;}
else if (die==3) {three++;}
```

```
else if (die==4) {four++;}
else if (die==5) {five++;}
else if (die==6) {six++;}
counter++;
}//end of while
String s="Total of each die face : \n";
s+="1 : "+one+"\n";
s+="2 : "+two+"\n";
s+="3 : "+three+"\n";
s+="4 : "+four+"\n";
s+="5 : "+five+"\n";
s+="6 : "+six+"\n";
JOptionPane.showMessageDialog(null,s,"Total number of each die face",JOptionPane.PLAIN_MESSAGE); //display x and y
    } // end method main
} // end class BPIH4E11C
```



As seen from the results numbers are similar to each other due to statistical possibility of each face of the the is 1/6th. Theoretically each face should come 166667 times.

In Homework 3 of the last week natural logarithm calculation was given with the Formula :



$$y=\frac{x-1}{x+1}$$

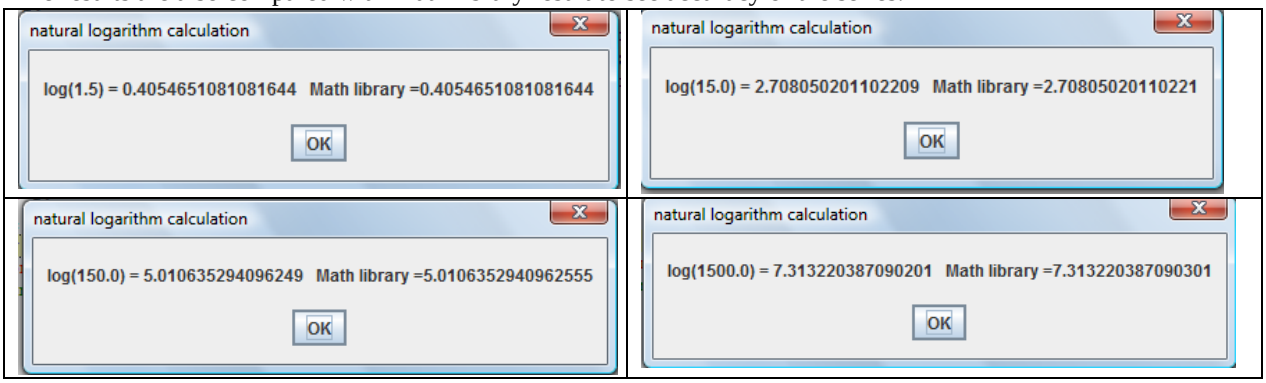
If this Formula written as a static method:

Program BPIH5E13.java **: an example of static method natural logarithm series calculation**

```
import javax.swing.*; // program poweres class Scanner

public class BPIH5E13
{
   public static double log(double x)
   {
     // ln(x) = 1+y^2/3+y^4/5+y^6/7+...   y=(x-1)/(x+1)
      double power=1;
      double ln=1;
      double y=(x-1)/(x+1);
      double n=1;
      do
      { power*=y*y;
      ln+=power /(2.0*n+1.0);
      n++;
```

```
      } while(n<=100000);
      ln*=2*y;
      return ln;
    }
   // main method begins execution of Java application
    public static void main( String args[] )
    {
      String s;
      double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: ")); // read number
      s="log("+x+") = "+log(x)+"   "+"Math library ="+Math.log(x);
      System.out.println(s);
      JOptionPane.showMessageDialog(null,s,"natural logarithm calculation",JOptionPane.PLAIN_MESSAGE);
    } // end method main
} // end class BPIH4E1
```

The results are also compared with Math library result to see accuracy of the series.



In order to improve the accuracy further, big digit number can be converted small digit number before the calculation is carried out. For example

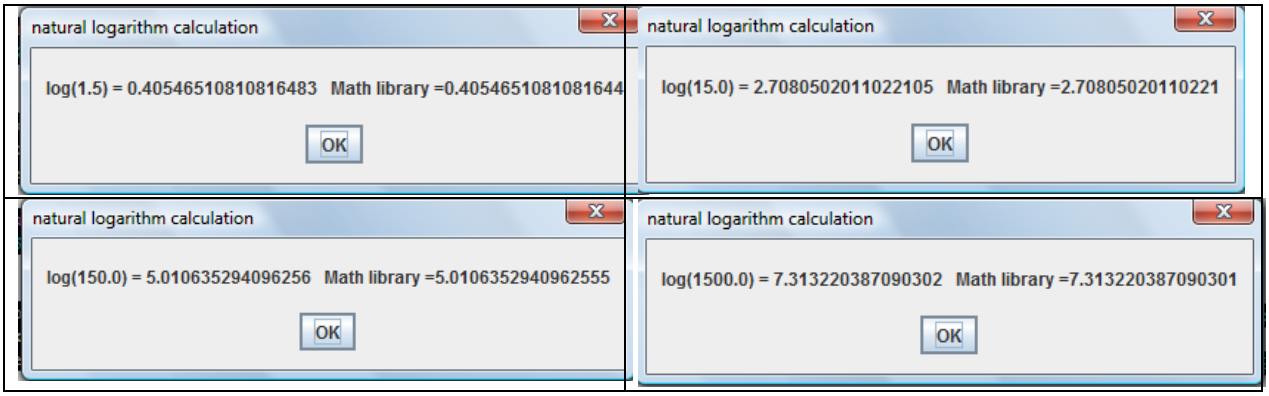
$\ln(123.456)=\ln(1.23456*10^2)= \ln(1.23456)+\ln(10^2)= \ln(1.23456)+2*\ln(10)=$
$\ln(1.23456)+2*2.302585092994046$
$\ln(1234.56)= \ln(1.23456)+3*2.302585092994046$
$\ln(12345.6)= \ln(1.23456)+4*2.302585092994046$

We will write a second version to utilise this property

Program BPIH5E14.java **: an example of static method natural logarithm series calculation**
With digit correction

```
import javax.swing.*; // program poweres class Scanner

public class BPIH5E14
{
   public static double log(double x)
   {
            // ln(x) = 1+y^2/3+y^4/5+y^6/7+...   y=(x-1)/(x+1)
            //
            double ln10=2.302585092994046;
     int m=0;
     int j=1;
     double z=x;
     j=(int)z;
     while(j>0)
     {
            if(j>0) {x=z;m++;}
            z=x/10;
            j=(int)z;
            }
            x=z;
```

```
      double power=1;
      double ln=1;
      double y=(x-1)/(x+1);
      double n=1;
      do
      {
              power*=y*y;
      ln+=power /(2.0*n+1.0);
      n++;
      } while(n<=100000);
      ln*=2*y;
      ln+=m*ln10;
              return ln;
   }
   // main method begins execution of Java application
   public static void main( String args[] )
   {
     String s;
     double x=Double.parseDouble(JOptionPane.showInputDialog("Enter a real number: ")); // read number
     s="log("+x+") = "+log(x)+"   "+"Math library ="+Math.log(x);
     JOptionPane.showMessageDialog(null,s,"natural logarithm calculation",JOptionPane.PLAIN_MESSAGE);
   } // end method main
} // end class BPIH4E1
```



The next example will plot the integer data by using a given character

Program BPIH5E18.java **: an example of static method line plotting of an integer function**

```
import java.io.*;     //java girdi cikti sinifini cagir
import javax.swing.*;

class BPIH5E18
 {   public static String emptychar(int emptyspace,char x1)
     {String s="";
        for(int i=0;i<emptyspace;i++)
        {s+=" ";}
        s+=x1+"\n";
        return s;
     }

     public static int y(int x)
     {return x*x;}

     public static void  plot(int x1,int x2,char c1)
     { String s="";
       for(int x=x1;x<=x2;x++)
       {s+=emptychar(y(x),c1);}
       print(s);
```

```
    }

            public static void print(String s)
    {
              JOptionPane.showMessageDialog(null,s,"     ",JOptionPane.PLAIN_MESSAGE);
            }

            public static void main(String args[])
    {
       plot(-10,10,'*');
    }
 }
```

6. CLASS STRUCTURE

6.1 GENERAL DEFINITION OF CLASS

Class concept and structure are used in all our programs so far, but we only barely mentioned its actual use and potential. The most important thing about classes, is that classes behaves as a new type of variable**.** The variables declared in classes are called objects. Classes are the general definitions of objects. Classes are the general definitions of objects, and objects are the variables defined out of objects. For example when we say pen, it is a class. When we mentione the red pen in Ali's pocket it becomes an object.

The general definition of the class is :

```
class class_name
{
//class variable list
variable type_1  variable_name_1;
variable type_2  variable_name_2;
.........
//class method list

variable_type mrthod_name(lists of method input variables)
{
//calculation and processes done by the method
return output_variable;
}
.............
}
```

Methods (or variables) does not have to be existed in the class definition, if they exist, then they are variables or methods belongs to that class. As the first class example, let us give a box clas that does not contain any methods in it. To define a box geometrically, we should define its three dimension, length, width and height

Program 6.1.1 box.java : a simple class application

```
class box
{
double length;
double width;
double height;
}
```

By this defination, a new type of variable, box, defined. In order to class this in a program,

box boxnumber1= new box();

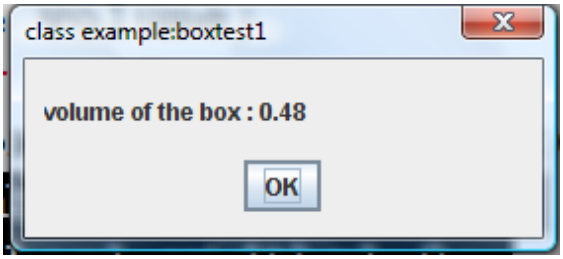
definition is used.

boxnumber1, is a variable(object) defined in class box type. This variable contains three subvariable length, width and height. We can assign values to sub variables in the program as

boxnumber1.length=1.0;
boxnumber1.width=1.2;
boxnumber1.height=0.4;

Program 6.1.2 boxtest1.java : a test program for class box : a simple class application

```
import javax.swing.JOptionPane;

class boxtest1
{
  public static void main(String args[])
  {
  double volume;
  String s="";
  box boxnumber1= new box();
  boxnumber1.length=1.0;
  boxnumber1.width=1.2;
  boxnumber1.height=0.4;
  volume=boxnumber1.length*boxnumber1.width*boxnumber1.height;
  s=s+"volume of the box : "+volume+"\n";
  JOptionPane.showMessageDialog(null,s,
   "class example:boxtest1",JOptionPane.PLAIN_MESSAGE);
   System.exit(0);}
}
```



The adress structure for a class variables will be as following :

Term	**Computer main object adress**	**Object class variables adresses**
Box1 **boxnumber1**;	null	
boxnumber1=new Box1();	Box1 **boxnumber1** adresss (for example: 23436A3F)	length width height (an adress for each variable, for example 13F23A3B,…)

In the second example, a method is also given in the box class

Program 6.1.3 box1.java : a simple class application

```
class box1
{
double length;
double width;
double height;

box1(double l,double w, double h)
{length=l;
width=w;
```

```
height=h;
}
}
```

This method box1 has the same name as the class(box1). This method is called the **constructor** method. By using this method, values to the class variables directly assigned to class variables.

box1 boxnumber2= new box1(1.0,1.2,.4);

Program 6.1.4 boxtest2.java : a test program for class box1 : a simple class application

```
import javax.swing.JOptionPane;
class boxtest2
{
  public static void main(String args[])
  {
  double volume;
  String s="";
  box1 boxnumber2= new box1(1.0,1.2,0.4);
  volume=boxnumber2.length*boxnumber2.width*boxnumber2.height;
  s=s+"volume of the box : "+volume+"\n";
  JOptionPane.showMessageDialog(null,s,
    "class example:boxtest1",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);}
}
```

In the next example volume calculation of the box2 is taken inside the box class as another method. Note that this method use **class variables** and defined as a **dynamic method** (word static is not existed)

Program 6.1.5 box2.java : a simple class application

```
class box2
{
double length;
double width;
double height;

box2(double l,double w, double h)
{length=l;
width=w;
height=h;
}
double volume()
{return length*width*height;}
}
```

When box2 is called, method volume can be used to calculate volumeof the box

Program 6.1.6 boxtest3.java : a test program for class box2 : a simple class application

```
import javax.swing.JOptionPane;
class boxtest3
{
  public static void main(String args[])
  {
  double volume;
  String s="";
  box2 boxnumber3= new box2(1.0,1.2,0.4);
  volume=boxnumber3.volume();
  s=s+"volume of the box : "+volume+"\n";
  JOptionPane.showMessageDialog(null,s,
    "class example:boxtest3",JOptionPane.PLAIN_MESSAGE);
```

```
    System.exit(0);}
}
```

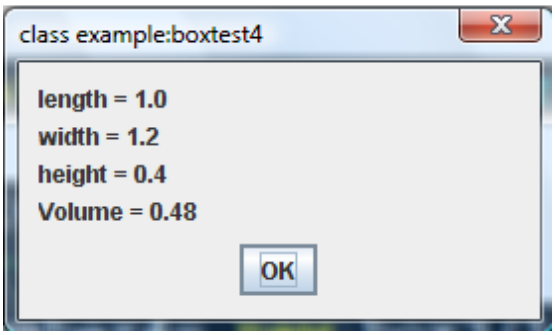
Also output process can be defined as another method inside the class

Program 6.1.7 box3.java : a simple class application

```
class box3
{
double length;
double width;
double height;

box3(double l,double w, double h)
{length=l;
width=w;
height=h;
}

double volume()
{return length*width*height;}

public String toString()
{ String s="";
  s+="length = "+length+"\n";
  s+="width = "+width+"\n";
  s+="height = "+height+"\n";
  s+="Volume = "+volume()+"\n";
  return s;
}

}
```

Program 6.1.8 boxtest4.java : a test program for class box3 : a simple class application

```
import javax.swing.JOptionPane;

class boxtest4
{
   public static void main(String args[])
   {
   double volume;
   String s="";
   box3 boxnumber4= new box3(1.0,1.2,0.4);
   JOptionPane.showMessageDialog(null,boxnumber4.toString(),
     "class example:boxtest4",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);}
}
```



The most important thing about classes that they defined a new kind of variable and this variables, with all their methods and class variables can be used like simple variables (double,int, etc). In the next example constructor method of class box4 will be defined with box4 as an input variable. In this way a copy of a given box4 class variable can be obtained.

box4 firstbox= new box4(1.0,1.2,0.4);
box4 secondbox= new box4(firstbox);

Program 6.1.9 box4.java : a simple class application

```
class box4
{
double length;
double width;
double height;
box4(double l,double w, double h)
{length=l;
width=w;
height=h;
}
box4(box4 bi)
{length=bi.length;
width=bi.width;
height=bi.height;
}

double volume()
{return length*width*height;}

public String toString()
{ String s="";
  s+="length = "+length+"\n";
  s+="width = "+width+"\n";
  s+="height = "+height+"\n";
  s+="Volume = "+volume()+"\n";
  return s;
}

}
```

Program 6.1.10 boxtest5.java : a test program for class bo43 : a simple class application

```
import javax.swing.JOptionPane;

class boxtest5
{
   public static void main(String args[])
   {
   double volume;
```

```
   box4 firstbox= new box4(1.0,1.2,0.4);
   box4 secondbox= new box4(firstbox);
   String s="First box : \n"+firstbox.toString()+"\n";
   s+="Second box : \n"+secondbox.toString()+"\n";
   JOptionPane.showMessageDialog(null,s,
     "class example:boxtest4",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);}
}
```

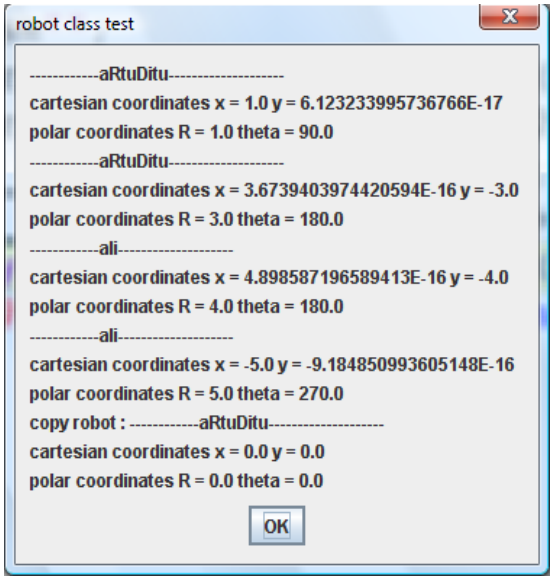
In the next example a robot class is defined. The coordinate of robot is given in the constructor method original coordinates and by using several methods turn to required direction and move in that direction.Fourdifferent overloaded contructor methods are given in this class. The fourth one is defined robot class as input variable

Program 6.1.11 robot.java : a simple class application

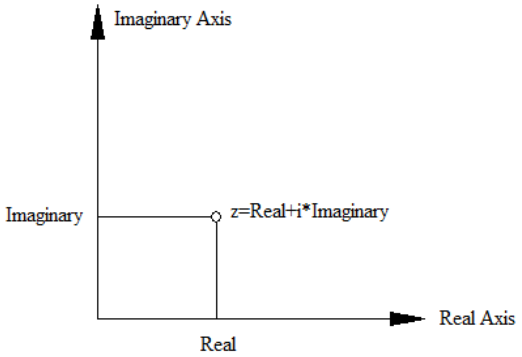
```
public class robot
{
  public String name;
  public double R;
  public double theta;
  public robot(String is) {R=0;theta=0;name=is;}
  public robot(String is,double Ri) {name=is;R=Ri;theta=0;}
  public robot(String is,double Ri,double theta_angle) {name=is;R=Ri;theta=theta_angle*Math.PI/180.0;}
  public robot(robot r1) {name=r1.name;R=r1.R;theta=r1.theta;}
  public void  turn_north() {theta=Math.PI/2.0;}
  public void  turn_south() {theta=3.0*Math.PI/2.0;}
  public void  turn_west() {theta=Math.PI;}
  public void  turn_east() {theta=Math.PI;}
  public void  turn(double angle) {theta+=angle*Math.PI/180.0;}
  public void  forward()  {R+=1;}
  public void  forward(double Ri)  {R+=Ri;}
  public void  backward()  {R-=1;}
  public void  backward(double Ri)  {R-=Ri;}
  public String toString()
  { String s="----------"+name+"----------\n";
    s+="cartesian coordinates x = "+R*Math.sin(theta)+" y = "+R*Math.cos(theta)+" \n";
    s+="polar coordinates R = "+R+" theta = "+theta*180/Math.PI+" \n";return s; }
}
```

Program 6.1.12 robottest.java : a test program for class robot : a simple class application

```
import javax.swing.*;public class robottest
{  public static void main(String args[])
   {robot R2D2=new robot("aRtuDitu");
    robot z2=new robot("ali",1.0,90.0);
    robot z3=new robot(R2D2);
    String s="";
    R2D2.turn_north();R2D2.forward();s+=R2D2.toString();
    R2D2.turn_west();R2D2.forward(2.0);s+=R2D2.toString();
    z2.turn_east();z2.forward(3.0);s+=z2.toString();
    z2.turn_south();z2.forward();s+=z2.toString();
    s+="copy robot : "+z3.toString();
    JOptionPane.showMessageDialog(null,s, "robot class test",JOptionPane.PLAIN_MESSAGE);
   }
}
```



In the next example, a complex variable is defined as a class. Complex numbers are expressed with two subnumber a real and an imaginary number. As [redacted], where $i = \sqrt{-1}$.



The first class is complex1. In this class only two variable, real and imaginary is defined.

Program 6.1.13 complex1.java : a simple class application

```
public class complex1
{ public double real;
  public double imaginary;
}
```

Program 6.1.14 complex1test.java : a test program for class complex1 : a simple class application

```
import javax.swing.*;
public class complex1test
{  public static void main(String args[])
   {complex1 z1=new complex1();
    z1.real=2.0;
    z1.imaginary=3.25;
    String s="complex number = "+z1.real+" + i*"+z1.imaginary+" \n";
    complex1 z2=new complex1();
    z2.real=1.1;
```

```
    z2.imaginary=2.0;
    s+="complex number = "+z2.real+" + i*"+z2.imaginary+" \n";
    complex1 z3=new complex1();
    z3.real=z1.real+z2.real;
    z3.imaginary=z1.imaginary+z2.imaginary;
    s+="total of complex number = "+z3.real+" + i*"+z3.imaginary+" \n";
   JOptionPane.showMessageDialog(null,s, "complex1 sınıfı testi",JOptionPane.PLAIN_MESSAGE);
    }
}
```

When class complex1 is used most of the process are still carried out in the main method. The purpose of using classes is to cary out most of the calculations in the class structure. In the next example constructors and methods for adding complex variables are defined in the class complex3.

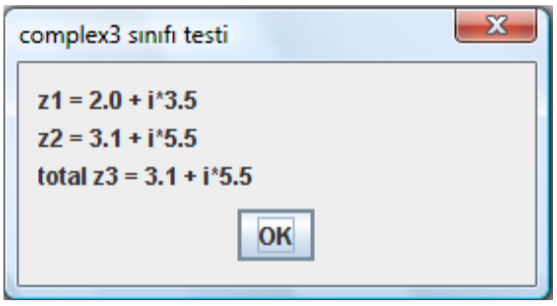
Program 6.1.15 complex3.java : a simple class application

```
public class complex3
{ public double real;
  public double imaginary;
  public complex3(double g,double s)
  {real=g;
   imaginary=s;
  }
  public complex3(complex3 s1)
  {real=s1.real;
   imaginary=s1.imaginary;
  }
  public void add(complex3 s1)
  {real+=s1.real;
   imaginary+=s1.imaginary;
  }
  public  static complex3 add(complex3 s1,complex3 s2)
  {complex3 z3=new complex3((s1.real+s2.real),(s1.imaginary+s2.imaginary));
   return z3;
  }
   public String toString()
  { String s=+real+" + i*"+imaginary+" \n";
    return s;
  }
}
```

Two different test programs are given. In the first one, class complex3test, three complex class object z1,z2 and z3 crated. Z2 is copied into z3 and z1 is added into the z3, terefore we are ended up adding z2 and z3.

Program 6.1.16 complex3test.java : a simple class application

```
import javax.swing.*;
public class complex3test
{  public static void main(String args[])
   { complex3 z1=new complex3(2.0,3.5);
     complex3 z2=new complex3(1.1,2.0);
     complex3 z3=new complex3(z2);
     z3.add(z1);
     String s="z1 = "+z1.toString()+"z2 = "+z3.toString()+"total z3 = "+z3.toString();
   JOptionPane.showMessageDialog(null,s, "complex3 sınıfı testi",JOptionPane.PLAIN_MESSAGE);
   }
}
```



In the second class, complex3test1, two complex class object z1,z2 are crated. While a third onject z3 is created, the results of addition of z1 and z2 are direcl assigned as vale, by usingsattic add method.

Program 6.1.17 complex3test1.java : a simple class application

```
import javax.swing.*;
public class complex3test1
{  public static void main(String args[])
   { complex3 z1=new complex3(2.0,3.5);
     complex3 z2=new complex3(1.1,2.0);
     complex3 z3=complex3.add(z1,z2);
     String s="z1 = "+z1.toString()+"z2 = "+z3.toString()+"total z3 = "+z3.toString();
    JOptionPane.showMessageDialog(null,s, "complex3 sınıfı testi",JOptionPane.PLAIN_MESSAGE);
   }
}
```

The last example will be a company class. This class keep account balances when uy and sell products.

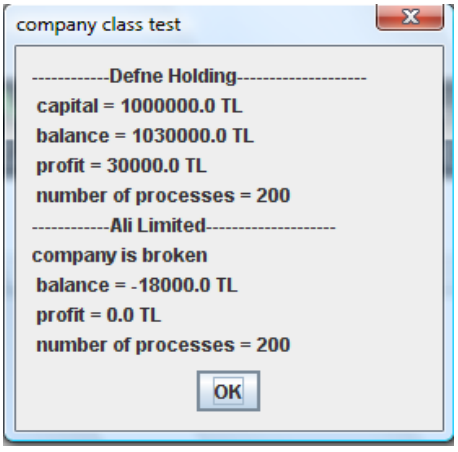
Program 6.1.18 company.java : a simple class application

```
public class company
{ public String name;
  public double capital,balance,profit;
  public int process;
  //this method establish the company

  //constructor method
  public company(String is,double Ri)
  {name=is;
  capital=Ri;
  balance=Ri;
  profit=0;
  process=0;
  }

  public void  buy(double x)
  {balance-=x;
   process++;
   if(balance<=0)
     {System.out.println("company is broken "+process);}
  }

  public void  sell(double x)
  {balance+=x;
   profit=balance-capital;
   process++;
  }

  public String acount()
```

```
  { String s="------------"+name+"--------------------\n";
  if(balance<=0)
    s+="company is broken \n";
  else
    s+=" capital = "+capital+" TL\n";
    s+=" balance = "+balance+" TL\n";
    s+=" profit = "+profit+" TL\n";
    s+=" number of processes = "+process+" \n";
  return s;
  }
}
```

Program 6.1.19 companytest.java : a simple class application test program for class company

```
import javax.swing.*;

public class companytest
{  public static void main(String args[])
   {
   company T=new company("Defne Holding",1.0e6);
   for(int i=0;i<100;i++) {T.buy(100);T.sell(400);}
   String s=T.acount();

   company A=new company("Ali Limited",1000.0);
   for(int i=0;i<100;i++) {A.buy(100);A.buy(90);}
   s+=A.acount();

   JOptionPane.showMessageDialog(null,s, "company class test",JOptionPane.PLAIN_MESSAGE);
   }}
```



6.2 USAGE OF TERM THIS

If the name of class variable and the name of a local variable in a method are the same, to diferentiate both, we can use term this to distinguish the class variables. For example in class box1 constructor method was given as:

box1(double l,double w, double h)
{length=l;
width=w;

height=h;
}

If instead of l,w and h variable names length, width and height is used they might confuse with class variables of the same name therefore, this method can be written as:

box1(double length ,double width, double height)
{this.length=length;
this.width=width;
this.height=height;
}

6.3 HIDING OF CLASS VARIABLES FROM OUTSIDE WORLD

Sometimes free access to the class variable outside of the class might not be desirable. For example if the class variable is carried out a bank başance it is desired to be reached to this variable only by the owner of the account if he/she known the access pasword. This kind of application should be done through methods (you can ask password and if password is correct, you can give the further information) . If class box5 is defined with private class variables

Program 6.3.1 box5.java : a simple class application with private class variables

```
class box5
{
private double length;
private double width;
private double height;

box5(double l,double w, double h)
{length=l;
width=w;
height=h;
}
// This methods are written to access private length,
// width and height variables
public double read_length() {return length;}
public double read_width() {return width;}
public double read_height() {return height;}
//This methods are written to change private length,
// width and height variables
public void write_length(double l) {length=l;}
public void write_width(double w) {width=w;}
public void write_height(double h) {height=h;}

box5(box5 bi)
{length=bi.length;
width=bi.width;
height=bi.height;
}

double volume()
{return length*width*height;}

public String toString()
{ String s="";
  s+="length = "+length+"\n";
  s+="width = "+width+"\n";
  s+="height = "+height+"\n";
  s+="Volume = "+volume()+"\n";
  return s;
```

```
}

}
```

We will try to access the class variables through the following test program :

Program 6.3.2 box5test.java : a simple class application test program for class boxtest5

```
import javax.swing.JOptionPane;

class boxtest6
{
  public static void main(String args[])
  {
  double volume;
  box5 firstbox= new box5(1.0,1.2,0.4);
  String s="First box : \n"+firstbox.toString()+"\n";
  firstbox.length=3.0;
  firstbox.width=2.0;
  firstbox.height=1.0;
  s+="value changed first box : \n"+firstbox.toString()+"\n";
  JOptionPane.showMessageDialog(null,s,
    "class example:boxtest4",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);}
}
```

The result of compiling the program is :

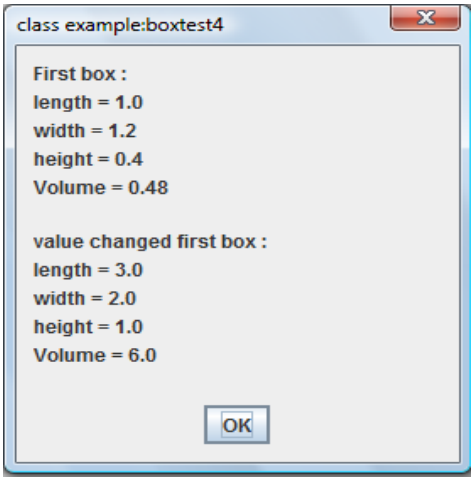
```
---------- Capture Output ----------
> "C:\turhan\java\bin\javac.exe" boxtest6.java
boxtest6.java:10: length has private access in box5
  firstbox.length=3.0;
        ^
boxtest6.java:11: width has private access in box5
  firstbox.width=2.0;
        ^
boxtest6.java:12: height has private access in box5
  firstbox.height=1.0;
        ^
3 errors
> Terminated with exit code 1.
```

The correct acccess tot the private variables will be as follows:

Program 6.3.4 box5testA.java : a simple class application test program for class company

```
import javax.swing.JOptionPane;

class boxtest6A
{
  public static void main(String args[])
  {
  double volume;
  box5 firstbox= new box5(1.0,1.2,0.4);
  String s="First box : \n"+firstbox.toString()+"\n";
  firstbox.write_length(3.0);
```

```
   firstbox.write_width(2.0);
   firstbox.write_height(1.0);
   s+="value changed first box : \n"+firstbox.toString()+"\n";
   JOptionPane.showMessageDialog(null,s,
     "class example:boxtest4",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);}
}
```



Another example is given as a bank account . In this class varaible accountno is private and can be reached only if a correct password is given. Class also has a counter to aalow only a given number of wrong password entry

Program 6.3.5 bankaccount.java

```
import javax.swing.*;
public class bankaccount
{
private String accountno;
private int code;
private int number;

public bankaccount(String an,int c)
{
accountno=an;
code=c;
number=0;
}
public String output()
{ while(number<3)
  {          number++;
  int c1=Integer.parseInt(JOptionPane.showInputDialog("enter the code = "));
  {if(c1==code) return accountno;
  }
  }
  return "wrong code is given";
}
}
```

Program 6.3.6 bankaccounttest1.java

```
public class bankaccounttest1
{
public static void main(String arg[])
```

```
{
 bankaccount isim=new bankaccount("123A4B5",12345);
 System.out.println(isim.output());
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" bankaccounttest1
123A4B5
> Terminated with exit code 0.
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" bankaccounttest1
wrong password is given 3 times system is locked
> Terminated with exit code 0.
```

WEEK 6 CLASS EXERSIZES (Wil be shown at the and of Lab class)

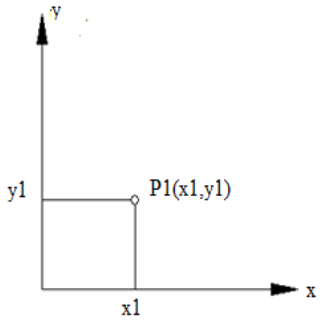
EX1) Multiplication of two complex number z1=Re1+i*Im1 and z2= z1=Re2+i*Im2 is defined as: z1*z2=(Re1*Re2-Im1*Im2) + i*(Re1*Im2+Re2*Im1) add two methods to complex3 class

public void multiply(complex3 s1)

public static complex3 multiply(complex3 s1,complex3 s2)

and test the methods through test programs

EX2) A point in two dimensional cartesian coordinate system is shown as x and y. Write a class to represent the point with output method to show the coordinates of the point



EX3) Add private String variable color of the box to box5 class and add up methods to Access the color and to change the color to to the class

public String read_color() {…….}

public void write_color(String col) {…….}

WEEK 6 HOMEWORK EXERSIZES (Will bring to next class -Tuesday)

HW1) A Force vector is defined with force components as F=Fx* i+Fy*j where i and j are unit vectors in x and y directions. Write a class to define a Force vector. In your definition also add a method to add two vectors. Note : rule of adding two vectors as F1=Fx1*i+Fy1*j and F2=Fx2*i+Fy2*j is F=F1+F2=(Fx1+Fx2)*i+(Fy1+Fy2)*j

HW2) Add the point method you developed in class exercise Ex2, a method to calculate the distance between to points. For the points P1(x1,y1) and P2(x2,y2) the distance between two points are :

 write a test program to test it

HW3) we would like to give robot we defined in robot class more direction flexibility so we will add
public void turn_north_east()
public void turn_north_west()
public void turn_south_east()
public void turn_south_west()
methods to class. Write the methods and test it with a test program

HW4) In the company class, we would like to give company possibility to borrow Money. Borrowed Money should be add to balance statement, when borrowed Money paid, so we will also define

public void borrowmoney(double x)
public void payborrowedmoney(double x)

After each 10 buying or selling 10% interest should be added to borrowed Money class variable (borrowed Money variable should be multiply with 1.1), but this amount will not be added to balance statement. Not that variable process counts total number of processes and when process%10 is zero, 10 buying or selling is completed regardless of the actual value of process variable
if((process%10)==0) {…….} is the statement to write

CLASS STRUCTURE: COMPOSITION AND INHERITANCE

COMPOSITION AND PROTECTED, PUBLIC AND PRIVATE

It is already mentioned that class structures are defined a new objects, similar but more complex compered to variable types. Since they are similar to variable, they can used as class variables in the definition of the other classes. Our first example consist of three classes point, circle and cylinder. As center of circle, point class is used and as base of cylinder circle class is used, so there is a chain relation between point, circle and cylinder classes.

Program point.java : a simple class with protected class variables

```
public class point
{
   protected double x, y;

   public point(double a, double b)
   {enterpoint(a,b);}

   public point(point A)
   {enterpoint(A.readX(),A.readY());}

   public void enterpoint(double a, double b)
   { x=a;y=b;}

   public double readX()
   {return x;}

   public double readY()
   {return y;}

   public String toString()
   {return "["+x+","+y+"]";}

   public String name(){return "point";}

}
```

Program circle.java : a simple class with protected class variables, use class point as class variable

```
public class circle
{
   protected double radius;
   protected point p;

   public circle()
   {   p=new point(0.0,0.0);
       enterradius(0.0);}

   public circle(double r, double a, double b)
   {  p=new point(a,b);
      enterradius(r);
   }

   public circle(double r, point n)
   {  p=new point(n.readX(),n.readY());
      enterradius(r);
```

```
    }

    public circle(circle c)
    {  p=new point(c.readpoint().readX(),c.readpoint().readY());
       enterradius(c.readradius());
    }

    public void enterradius(double r)
    {   if(r >= 0.0)
            radius=r;
        else
            radius=0.0;
    }

    public point readpoint()
    {return p;}

    public double readradius()
    {return radius;}

    public double area()
    {return Math.PI*radius*radius;}

    public String toString()
    { return "Center = "+"["+p.x+","+p.y+"]"+"; radius="+radius;}

    public String name(){return "circle";}
}
```

Program cylinder.java : a simple class with protected class variables, use class circlr as class variable

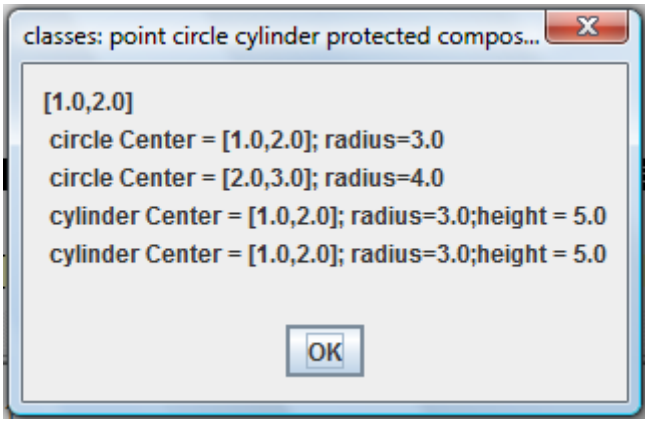
```
public class cylinder
 {
 protected double height;
 protected circle c;

 public cylinder()
 {c=new circle(0.0,0.0,0.0);
  enterheight(0);}

 public cylinder( double h,circle d)
 {c=new circle(d);
  height=h;
 }

 public cylinder( double h, double r, double a, double b)
 {c=new circle(r,a,b);
  height=h;
 }

 public void enterheight (double h )
 { if(h>=0) height=h;
   else height=0;
 }

 public double readheight(){return height;}

 public double area()
 {return 2*c.area()+2*Math.PI*c.radius*height;}

 public double volume(){return c.area()*height;}
```

```
 public String toString()
 {return c.toString()+";height = "+height;}

 public String name(){return "cylinder";}
}
```

Program test program for classes point, circlr and cylinder with protected class variables

```
import javax.swing.*;

class compositiontest1
{
  public static void main(String args[])
  {
  point n=new point(1.0,2.0);
  circle d1=new circle(3.0,n);
  circle d2=new circle(4.0,2.0,3.0);
  cylinder s1=new cylinder(5.0,d1);
  cylinder s2=new cylinder(5.0,3.0,1.0,2.0);
  String s=n.toString()+"\n "+
  d1.name()+" "+d1.toString()+"\n "+
  d2.name()+" "+d2.toString()+"\n "+
  s1.name()+" "+s1.toString()+"\n "+
  s2.name()+" "+s2.toString()+"\n ";
  JOptionPane.showMessageDialog(null,s,
    "classes: point circle cylinder protected composition test",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);}
}
```

As it is seen from the example , this classes are used inside another and has similar methods inside as well, They can be called as a family of classes composed from each other. Variables of these classes are also defined as **protected.** The word protected is more public than private but protected variables and methods are stil hard to Access, we usually use methods to get Access to protected variables. Output of the example class is :



Similar methods can be written with public variables as follows:

Program point.java : a simple class with public class variables

```
public class point
{
   public double x, y;

   public point(double a, double b)
   {x=a;y=b;}
```

```
   public point(point A)
   {x=A.x;
    y=A.y;
   }

   public String toString()
   {return "["+x+","+y+"]";}

   public String name(){return "point";}

}
```

Program circle.java : a simple class with public class variables, use class point as class variable

```
public class circle
{
   public double radius;
   public point p;

   public circle()
   {   p=new point(0.0,0.0);
       radius=0.0;}

   public circle(double r, double a, double b)
   {  p=new point(a,b);
      radius=r;
   }

   public circle(double r, point n)
   {  p=new point(n.x,n.y);
      radius=r;
   }

   public circle(circle c)
   {  p=new point(c.p.x,c.p.y);
      radius=c.radius;
   }

   public double area()
   {return Math.PI*radius*radius;}

   public String toString()
   { return "Center = "+"["+p.x+","+p.y+"]"+"; radius="+radius;}

   public String name(){return "circle";}
}
```

Program cylinder.java : a simple class with public class variables, use class circle as class variable

```
public class cylinder
 {
 public double height;
 public circle c;

 public cylinder()
 {c=new circle(0.0,0.0,0.0);
  height=0.0;}

 public cylinder( double h,circle d)
 {c=new circle(d);
  height=h;
 }
```

```
 public cylinder( double h, double r, double a, double b)
 {c=new circle(r,a,b);
  height=h;
 }

 public double area()
 {return 2*c.area()+2*Math.PI*c.radius*height;}

 public double volume(){return c.area()*height;}

 public String toString()
 {return c.toString()+";height = "+height;}

 public String name(){return "cylinder";}
}
```

Results are the same as the previous program, so why we need protected or private, In the next example a class, bank_account, is given. This class can only be written as private due to security concern of electronic money theft from the acount.

Program class bank_account with private class variables

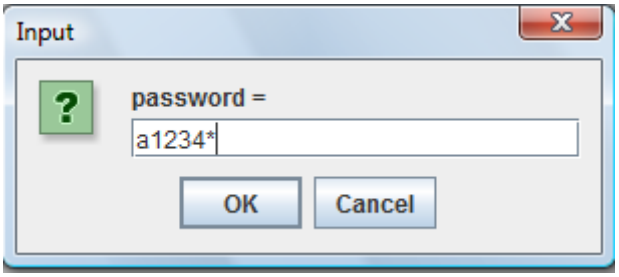
```
import javax.swing.*;

public class bank_account
{
 private String name;
 private int account_number;
 private String password;
 private int password_trial;
 private static int maximum_password_trial=3;
 private double account_balance;

public bank_account(String iname,int i_account_number,String i_password,double pgir)
 {name=iname;
  account_number=i_account_number;
  password=i_password;
  account_balance=pgir;
  password_trial=0;
 }

 public String read_account()
 { String s1="null";
   String password_read;
   int account_number_read=Integer.parseInt(JOptionPane.showInputDialog("account number = "));
   while(password_trial<maximum_password_trial)
   {
   password_read=JOptionPane.showInputDialog("password = ");
   if(password_read.equals(password) ){password_trial=0;return write();}
   else{password_trial++;JOptionPane.showMessageDialog(null,"sorry wrong password, please try again","Wrong
password",JOptionPane.PLAIN_MESSAGE);}
   }
   return s1;
 }

 public String change_password()
 {String s1="change password";
  String password_read;
  while(password_trial<maximum_password_trial)
  {
  password_read=JOptionPane.showInputDialog("password = ");
```

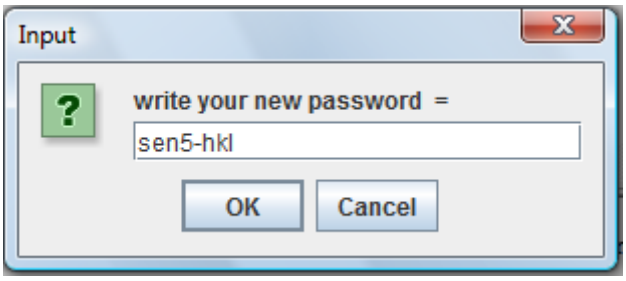
```
 if(password_read.equals(password) )
 {
            String newpassword1=JOptionPane.showInputDialog("write your new password  = ");
            String newpassword2=JOptionPane.showInputDialog("re-write your new password  = ");
            if(newpassword1.equals(newpassword2))
            {password=newpassword2;s1="Password is changed sucsessfully";break;}
            else
            {JOptionPane.showMessageDialog(null,"new password1 and new password2 is not the same","wrong new
password",JOptionPane.PLAIN_MESSAGE);s1="Password change failuare";}
    password_trial=0;
 }
 else
 {JOptionPane.showMessageDialog(null,"sorry, wrong password please give it again","Wrong
password",JOptionPane.PLAIN_MESSAGE);
 password_trial++;s1="Password change failure";
 }
}
return s1;
}

 private String write()
 {String s="Account name = "+name+"\n";
  s+="Account number = "+account_number+"\n";
  s+="Account balance = "+account_balance+" T.L.\n";
  return s;
 }

}
```

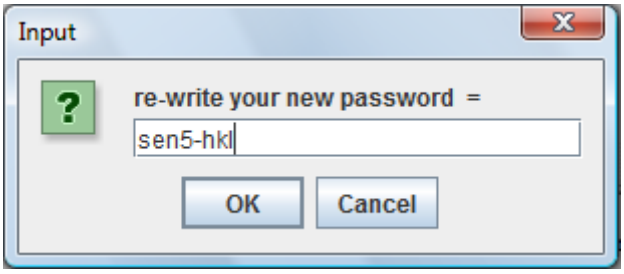
Program A test program for class bank_account with private class variables

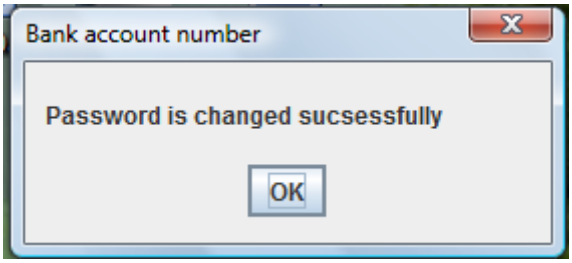
```
import javax.swing.*;

public class bank_account_test
{
public static void main(String arg[])
{
bank_account h1=new bank_account("ali veli",12345,"a1234*",100000.0);
JOptionPane.showMessageDialog(null,h1.change_password(),"Bank account number",JOptionPane.PLAIN_MESSAGE);
JOptionPane.showMessageDialog(null,h1.read_account(),"Bank account information",JOptionPane.PLAIN_MESSAGE);
}
}
```

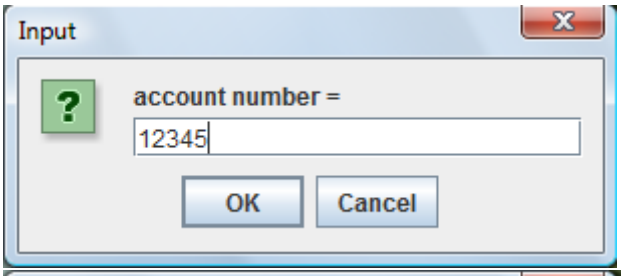
Class bank_account allows the password changes and graphical output of bank acount information. It has a sophisticated control. If wrong password is given some countable time, program stops.

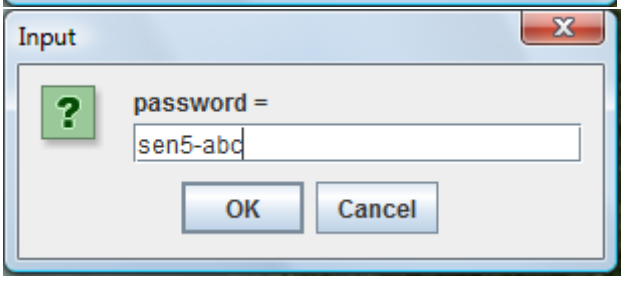


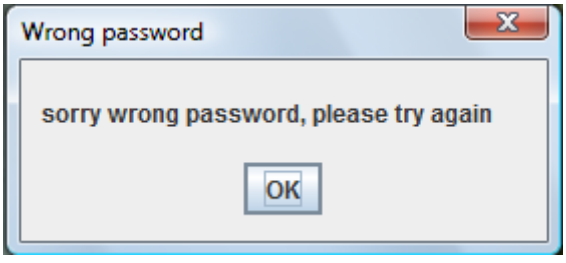

Input
write your new password =
sen5-hkl
OK
Cancel

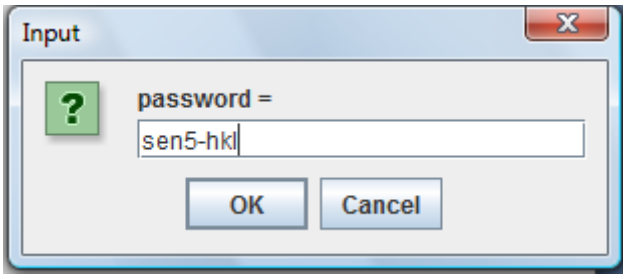

Input
re-write your new password =
sen5-hkl
OK
Cancel

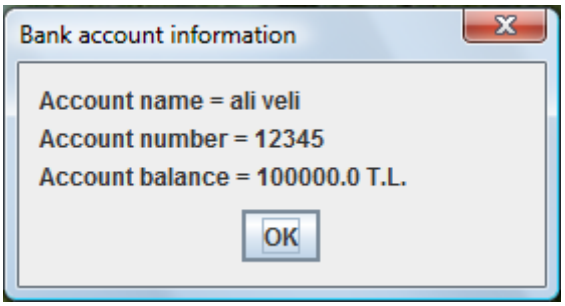

Bank account number
Password is changed sucsessfully
OK


Input
account number =
12345
OK
Cancel


Input
password =
sen5-abc
OK
Cancel


Wrong password
sorry wrong password, please try again
OK





As it is seen from the example program secrecy of the code sometime is required. The case of protected and private is used for this kind of cases.

Now another group of classes are introduced as composition example, this case is presented only with public variables

Program vehicle.java : a simple class with public class variables

```
import javax.swing.*;

public class vehicle
{
//valid vehicle types bus,truck,minitruck,minivan,auto
public String vehicletype;
public String brandName;
public double power; //KW olarak
public String color;
public double weight;          //yüksüz net ağırlık
//kurucu metodlar

public vehicle()
{vehicletype="not defined";
brandName="not defined";
power=0;
color="not defined";
weight=0;
}

public vehicle(String i_vehicletype,String i_brandName,double i_motor,String i_color,double i_weight)
{//vehicle type "bus","truck","mini truck","mini bus","auto","tır"
//if not define as "not defined"
if(i_vehicletype.equals("truck") ||
   i_vehicletype.equals("minitruck") ||
   i_vehicletype.equals("bus")||
   i_vehicletype.equals("minibus")||
   i_vehicletype.equals("auto") ||
   i_vehicletype.equals("motorcycle") ||
   i_vehicletype.equals("bicycle") ||
   i_vehicletype.equals("off road vehicle")
   )
```

```
{vehicletype=i_vehicletype;}
else
{vehicletype="not defined";}
brandName=i_brandName;
power=i_motor;
color=i_color;
weight=i_weight;
}

public vehicle(vehicle A)
{//gerekli tanımları yap
vehicletype=A.vehicletype;
brandName=A.brandName;
power=A.power;
color=A.color;
weight=A.weight;
}

public void define_vehicle(String i_vehicletype,String i_brandName,double i_motor,String i_color,double i_weight)
{//vehicle type "bus","truck","mini truck","mini bus","auto","tır"
//if not define as "not defined"
if(i_vehicletype.equals("truck") ||
   i_vehicletype.equals("minitruck") ||
   i_vehicletype.equals("bus")||
   i_vehicletype.equals("minibus")||
   i_vehicletype.equals("auto") ||
   i_vehicletype.equals("motorcycle") ||
   i_vehicletype.equals("bicycle") ||
   i_vehicletype.equals("off road vehicle"))
   {vehicletype=i_vehicletype;}
else
{vehicletype="not defined";}
brandName=i_brandName;
power=i_motor;
color=i_color;
weight=i_weight;
}

public String toString()
{String s1="";
s1+="Vehicle Type : "+vehicletype+"\n";
s1+="Brand Name : "+brandName+"\n";
s1+="Motor Power : "+power+" KW\n";
s1+="Color : "+color+"\n";
s1+="Weight : "+weight+" kg\n";
return s1;//gerekli tanımları yap
}

public void outputwindow()
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,toString(),s1, JOptionPane.PLAIN_MESSAGE);
}

}
```

Program truck.java : a simple class with public class variables, use class vehicle as class variable

```
public class truck
{
vehicle v;
```

```
public truck(String i_marka,double i_motor,String i_renk,double i_agirlik)
{v=new vehicle("truck",i_marka,i_motor,i_renk,i_agirlik);}

public truck()
{ v=new vehicle();
  v.vehicletype="truck";
}
public truck(truck A)
{v=new vehicle(A.v);
 v.vehicletype="truck";
}

}
```

Program bus.java : a simple class with public class variables, use class vehicle as class variable

```
public class bus extends vehicle
{
vehicle v;

public bus(String i_marka,double i_motor,String i_renk,double i_agirlik)
{v=new vehicle("bus",i_marka,i_motor,i_renk,i_agirlik);}

public bus()
{ v=new vehicle();
  v.vehicletype="bus";
}
public bus(bus A)
{ v=new vehicle(A.v);
 v.vehicletype="bus";

}

}
```

Program minibus.java : a simple class with public class variables, use class vehicle as class variable

```
public class minibus
{
vehicle v;

public minibus(String i_marka,double i_motor,String i_renk,double i_agirlik)
{v=new vehicle("minibus",i_marka,i_motor,i_renk,i_agirlik);}

public minibus()
{ v=new vehicle();
  v.vehicletype="minibus";
}
public minibus(minibus A)
{v=new vehicle(A.v);
 v.vehicletype="minibus";
}

}
```

Program A test program for classes vehicle,bus,truck and minibus

```
import javax.swing.*;
class vehicletesti1
{

public static void outputwindow(String s)
{
String s1="Vehicle class output window";
```

```
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{
vehicle truck=new vehicle("truck","MAN",200.0,"red",15000);
vehicle bus=new vehicle("bus","Mersedes-Chrysler",300.0,"white",10000);
vehicle a=new vehicle();
String s=truck.toString()+"\n"+bus.toString()+"\n"+a.toString();
outputwindow(s);
}

}
```

Program A test program for classes vehicle,bus,truck and minibus

```
import javax.swing.*;
class vehicletest2
{

public static void outputwindow(String s)
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{

truck k1=new truck("MAN",200.0,"red",15000);
bus b1 =new bus("Mersedes-Chrysler",300.0,"white",10000);
minibus mb1=new minibus("BMC",70,"blue",1070);
String s=k1.toString()+"\n"+b1.toString()+"\n"+mb1.toString();
outputwindow(s);
}

}
```

Program A test program for classes vehicle,bus,truck and minibus

```
import javax.swing.*;
class vehicletest3
{

public static void outputwindow(String s)
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{

truck t1=new truck("MAN",200.0,"red",15000);
bus b1 =new bus("Mersedes-Chrysler",300.0,"white",10000);
minibus mb1=new minibus("BMC",70,"blue",1070);
vehicle v1=t1;
vehicle v2=b1;
vehicle v3=mb1;
String s=v1.toString()+"\n"+v2.toString()+"\n"+v3.toString();
outputwindow(s);
}
```

```
}
```

INHERITANCE

Instead of deriving a class by composition from another class through variable definition, new class can be **extended** from the previous class. To do this the word extend is used. If a class is extended from another class, it is similar to copying the whole codes (variables and methods) from the first class to the extended class. In order to check the basic mechanism, point-circle-cyclinder clas trio is re-defined by using inheritance.

Program point.java : a simple class with protected class variables

```
public class point
{   protected double x, y;

    public point(double a, double b)
    {enterpoint(a,b);}

    public point(point A)
    {enterpoint(A.readX(),A.readY());}

    public void enterpoint(double a, double b)
    { x=a;y=b;}

    public double readX()
    {return x;}

    public double readY()
    {return y;}

    public String toString()
    {return "["+x+","+y+"]";}

    public String name(){return "point";}
}
```

Program circle.java : a simple class with protected class variables, extended from class point

```
public class circle extends point
{
    protected double radius;
    public circle()
    {   super(0.0,0.0);
        enterradius(0.0);}

    public circle(double r, double a, double b)
    {  super(a,b);
       enterradius(r);
    }

    public circle(double r, point n)
    {  super(n.readX(),n.readY());
       enterradius(r);
    }

    public circle(circle c)
    {  super(c.x,c.y);
       enterradius(c.radius);
    }

    public void enterradius(double r)
```

```
    {   if(r >= 0.0)
            radius=r;
        else
            radius=0.0;
    }

    public double readradius()
    {return radius;}

    public double area()
    {return Math.PI*radius*radius;}

    public String toString()
    { return "Center = "+"["+x+","+y+"]"+"; radius="+radius;}

    public String name(){return "circle";}
}
```

Program cylinder.java : a simple class with protected class variables, extended from class circle

```
public class cylinder extends circle
 {
   protected double height;

 public cylinder()
 { super(0.0,0.0,0.0);
   height=0.0;
 }

 public cylinder( double h,circle d)
 { super(d);
   height=h;
 }

 public cylinder( double h, double r, double a, double b)
 { super(r,a,b);
   height=h;
 }

 public void enterheight (double h )
 { if(h>=0) height=h;
   else height=0;
 }

 public double readheight(){return height;}

 public double area()
 {return 2*super.area()+2*Math.PI*radius*height;}

 public double volume(){return super.area()*height;}

 public String toString()
 {return super.toString()+";height = "+height;}

 public String name(){return "cylinder";}
}
```

Program A test program for classes point,circle and cylinder (inheritance example)

```
import javax.swing.*;

class inheritancetest1
{
```

```
   public static void main(String args[])
   {
   point n=new point(1.0,2.0);
   circle d1=new circle(3.0,n);
   circle d2=new circle(4.0,2.0,3.0);
   cylinder s1=new cylinder(5.0,d1);
   cylinder s2=new cylinder(5.0,3.0,1.0,2.0);
   String s=n.toString()+"\n "+
   d1.name()+" "+d1.toString()+"\n "+
   d2.name()+" "+d2.toString()+"\n "+
   s1.name()+" "+s1.toString()+"\n "+
   s2.name()+" "+s2.toString()+"\n ";
   JOptionPane.showMessageDialog(null,s,
     "classes: point circle cylinder protected inheritance test",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);}
}
```

One thing is to note at in an extended class, if a contructor method of parent class is to be used, it can be used by using word **super.**

Another inheritance test program group is vehicle-truck-bus-minibus group

Program vehicle.java : a simple class with public class variables

```
import javax.swing.*;

public class vehicle
{
//valid vehicle types bus,truck,minitruck,minivan,auto
public String vehicletype;
public String brandName;
public double power; //KW olarak
public String color;
public double weight;           //yüksüz net ağırlık
//kurucu metodlar

public vehicle()
{vehicletype="not defined";
brandName="not defined";
power=0;
color="not defined";
weight=0;
}

public vehicle(String i_vehicletype,String i_brandName,double i_motor,String i_color,double i_weight)
{//vehicle type "bus","truck","mini truck","mini bus","auto","tır"
//if not define as "not defined"
if(i_vehicletype.equals("truck") ||
   i_vehicletype.equals("minitruck") ||
   i_vehicletype.equals("bus")||
   i_vehicletype.equals("minibus")||
   i_vehicletype.equals("auto") ||
   i_vehicletype.equals("motorcycle") ||
   i_vehicletype.equals("bicycle") ||
   i_vehicletype.equals("off road vehicle")
   )
{vehicletype=i_vehicletype;}
else
{vehicletype="not defined";}
brandName=i_brandName;
power=i_motor;
color=i_color;
```

```
weight=i_weight;
}

public vehicle(vehicle A)
{//gerekli tanımları yap
vehicletype=A.vehicletype;
brandName=A.brandName;
power=A.power;
color=A.color;
weight=A.weight;
}

public void define_vehicle(String i_vehicletype,String i_brandName,double i_motor,String i_color,double i_weight)
{//vehicle type "bus","truck","mini truck","mini bus","auto","tır"
//if not define as "not defined"
if(i_vehicletype.equals("truck") ||
   i_vehicletype.equals("minitruck") ||
   i_vehicletype.equals("bus")||
   i_vehicletype.equals("minibus")||
   i_vehicletype.equals("auto") ||
   i_vehicletype.equals("motorcycle") ||
   i_vehicletype.equals("bicycle") ||
   i_vehicletype.equals("off road vehicle"))
   {vehicletype=i_vehicletype;}
else
{vehicletype="not defined";}
brandName=i_brandName;
power=i_motor;
color=i_color;
weight=i_weight;
}

public String toString()
{String s1="";
s1+="Vehicle Type : "+vehicletype+"\n";
s1+="Brand Name : "+brandName+"\n";
s1+="Motor Power : "+power+" KW\n";
s1+="Color : "+color+"\n";
s1+="Weight : "+weight+" kg\n";
return s1;//gerekli tanımları yap
}

public void outputwindow()
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,toString(),s1, JOptionPane.PLAIN_MESSAGE);
}

}
```

Program bus.java : a simple class with public class variables, extended from class vehicle

```
public class bus extends vehicle
{
public bus(String i_marka,double i_motor,String i_renk,double i_agirlik)
{super("bus",i_marka,i_motor,i_renk,i_agirlik);}
public bus()
{ super();
  vehicletype="bus";
}
public bus(bus A)
{super(A);
```

```
 vehicletype="bus";
}

}
```

Program truck.java : a simple class with public class variables, extended from class vehicle

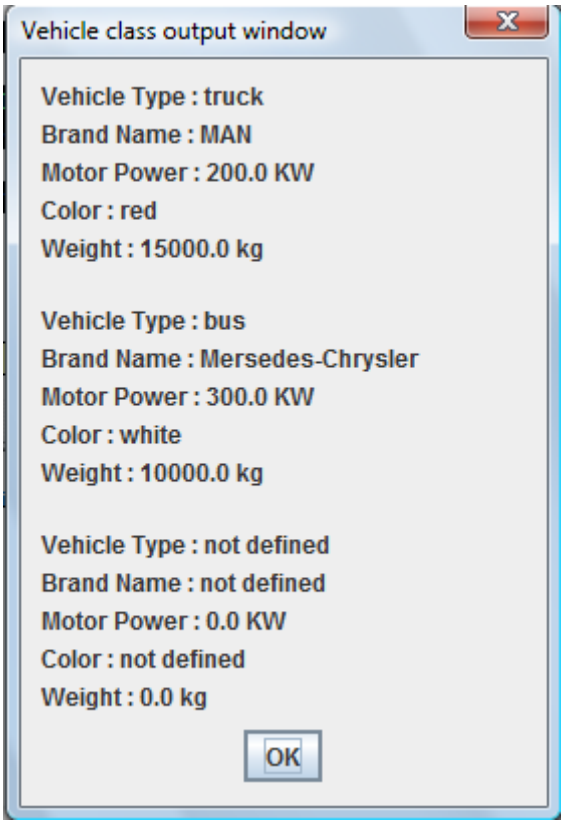
```
public class truck extends vehicle
{
public truck(String i_marka,double i_motor,String i_renk,double i_agirlik)
{super("truck",i_marka,i_motor,i_renk,i_agirlik);}
public truck()
{ super();
  vehicletype="truck";
}
public truck(truck A)
{super(A);
 vehicletype="truck";
}

}
```

Program minibus.java : a simple class with public class variables, extended from class vehicle

```
public class minibus extends vehicle
{
public minibus(String i_marka,double i_motor,String i_renk,double i_agirlik)
{super("minibus",i_marka,i_motor,i_renk,i_agirlik);}
public minibus()
{ super();
  vehicletype="minibus";
}
public minibus(minibus A)
{super(A);
 vehicletype="minibus";
}

}
```

Program A test program for classes vehicle,bus,truck and minibus

```
import javax.swing.*;
class vehicletesti1
{

public static void outputwindow(String s)
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{
vehicle truck=new vehicle("truck","MAN",200.0,"red",15000);
vehicle bus=new vehicle("bus","Mersedes-Chrysler",300.0,"white",10000);
vehicle a=new vehicle();
String s=truck.toString()+"\n"+bus.toString()+"\n"+a.toString();
outputwindow(s);
}

}
```



In the next example problem **indirect referencing** is shown. If classes derived from the same parent class, they can be referenced, indirectly, through the parent (root) class. In c and c++ languages indirect referencing applied by using reference concept. What we are establishing in here is the same thing, but references can not be adressed directly due to security reasıns. Instead they can be reached by indirect class referencing mechanism.

Program A test program for classes vehicle,bus,truck and minibus

```
import javax.swing.*;
class vehicletest3
{

public static void outputwindow(String s)
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{

truck t1=new truck("MAN",200.0,"red",15000);
bus b1 =new bus("Mersedes-Chrysler",300.0,"white",10000);
minibus mb1=new minibus("BMC",70,"blue",1070);
vehicle v1=t1;
vehicle v2=b1;
vehicle v3=mb1;
String s=v1.toString()+"\n"+v2.toString()+"\n"+v3.toString();
outputwindow(s);
}

}
```

Output of this program is the same as the previous output.

As a third example of inherited class chain book-fiction-poetry-encyclopedia group is defined

Program book.java : a simple class with public class variables

```
public class book
{
String name;
String author;
String clasification;
String language;

public book(String kname,String kauthor,String kclasification,String klanguage)
{name=kname;
 author=kauthor;
 clasification=kclasification;
 language=klanguage;
}

public book(book k)
{name=k.name;
 author=k.author;
 clasification=k.clasification;
 language=k.language;
}

public String toString()
{
String s="name = "+name+"\n";
s+="author = "+author+"\n";
s+="book clasification = "+clasification+"\n";
s+="language = "+language+"\n";
return s;
}

}
```

Program fiction.java : a simple class with public class variables, extended from class book

```
public class fiction extends book
{
public fiction(String kisim,String kyazar,String klanguage)
{ super(kisim,kyazar,"fiction",klanguage);}

}
```

Program poetry.java : a simple class with public class variables, extended from class book

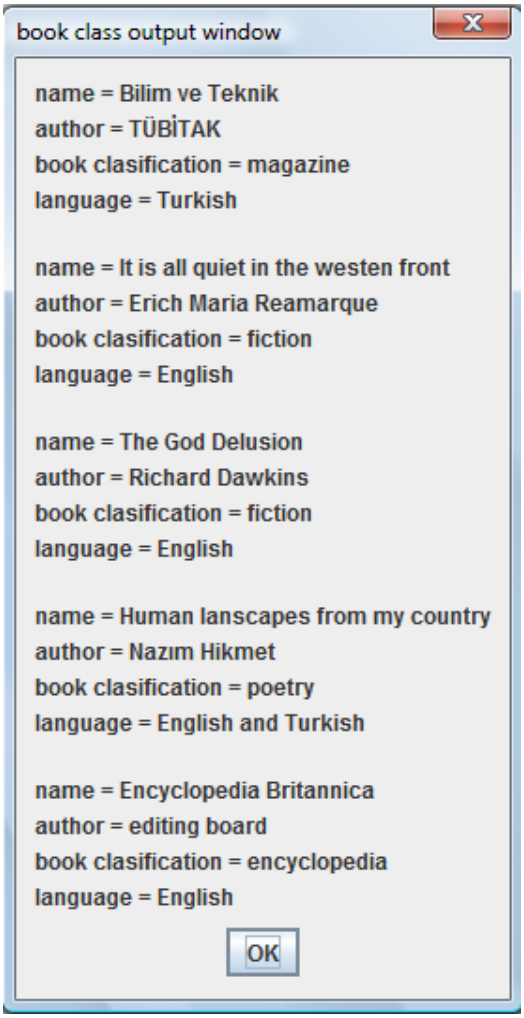
```
public class poetry extends book
{
public poetry(String kname,String kauthor,String klanguage)
{ super(kname,kauthor,"poetry",klanguage);}
}
```

Program encyclopedia.java : a simple class with public class variables, extended from class book

```
public class encyclopedia extends book
{
public encyclopedia(String kname,String klanguage)
{ super(kname,"editing board","encyclopedia",klanguage);}
}
```

Program A test program for classes book,fiction,poetry,encyclopedia

```
import javax.swing.*;
class booktest1
{
public static void outputwindow(String s)
{
String s1="book class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}
public static void main(String args[])
{
book magazine=new book("Bilim ve Teknik","TÜBİTAK","magazine","Turkish");
fiction r1=new fiction("It is all quiet in the westen front ","Erich Maria Reamarque","English");
book r2=new book("The God Delusion","Richard Dawkins","fiction","English");
poetry  s1=new poetry("Human lanscapes from my country","Nazım Hikmet","English and Turkish");
encyclopedia a1=new encyclopedia("Encyclopedia Britannica","English");
String s=magazine.toString()+"\n"+r1.toString()+"\n"+r2.toString()+"\n"+s1.toString()+"\n"+a1.toString();
outputwindow(s);
}
```



ARRAYS

GENERAL DEFINITION OF ONE DIMENSIONAL ARRAY VARIABLES

Arrays defines more than one variable or object with the same group name. Each individual variable can be reached with the assigned index. In General Array type variables are defined as :

Variable_type variable_name[];

Or
Variable_type[] variable_name;
An example definition :

int days_of_the_month[]; or
int[] days_of_the_month;

This definition defines that variable has more than one sub-variable in it, but it does not define how many sub-variables it containes. In order to define array size
variable_name=new Variable_type[variable_size];
structure used. example definition :
days_of_the_month=new int[12];
After array variable is defined, any subvariable can be reached by using its index number. For example:
days_of_the_month[0]=31;
days_of_the_month[1]=28;
Note that index variable is always starts from 0
If array concept is given in a program form :
Program The first array example array1a.java

```
import java.io.*;
class array1a
{
public static void main(String args[] )
{
int days_of_the_month[];
days_of_the_month=new int[12];
days_of_the_month[0]=31;
days_of_the_month[1]=28;
days_of_the_month[2]=31;
days_of_the_month[3]=30;
days_of_the_month[4]=31;
days_of_the_month[5]=30;
days_of_the_month[6]=31;
days_of_the_month[7]=31;
days_of_the_month[8]=30;
days_of_the_month[9]=31;
days_of_the_month[10]=30;
days_of_the_month[11]=31;
System.out.println("Month April is "+days_of_the_month[3]+" days long");
}}
```

The output will be:

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" array1a
Month April is 30 days long

> Terminated with exit code 0.
```

In the first creation of arrays arrays assignments can also be made by using big paranthesis,{}.
Variable_type[] variable_name={value_0,value_1,..,value_n};

Not that in this type of definition the size of the variable is not defined seperately, program counts the number of variables inside of the paranthesis and assign the size as well as actual valu for each sub-variable. An example can be given as:

int days_of_the_month[ ]={31,28,31,30,31,30,31,31,30,31,30,31};

If array concept is given in a program form :

Program The second array example array1b.java

```
import java.io.*;
class array1b
{
public static void main(String args[] )
{
int days_of_the_month[]={31,28,31,30,31,30,31,31,30,31,30,31};
System.out.println("Month April is "+days_of_the_month[3]+" days long");
}
}
```

The result will be same as the previous program. Since array type of variables are reached by using indices, they can be used easily by using loop structures. The following program finds avarage of given set of numbers by using for loop structures an arrays

Program math1.java an example of acessing array indices through for loop structure

```
class math1
{public static void main(String args[])
{int number[]={10,5,7,9,11,13,14,18};
double total=0;
int i;
for(i=0;i<number.length;i++)
{total+=number[i];}
System.out.println("Average = "+total/number.length);
}}
```

The output will be :

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" math1
Average = 10.875
> Terminated with exit code 0.
```

Of course the same result can be achived by using while structure

Program math2.java an example of acessing array indices through while loop structure

```
class math2
{public static void main(String args[])
{int number[]={10,5,7,9,11,13,14,18};
double total=0;
int i=0;
while(i<number.length)
{total+=number[i];i++;}
System.out.println("Average = "+total/number.length);
}}
```

The result will be the same as the previous program. In another example the values in two arrays are added up together. In this example formatted output is used to get a nice output for arrays. Some of the format varible types are given in table 8.1.1

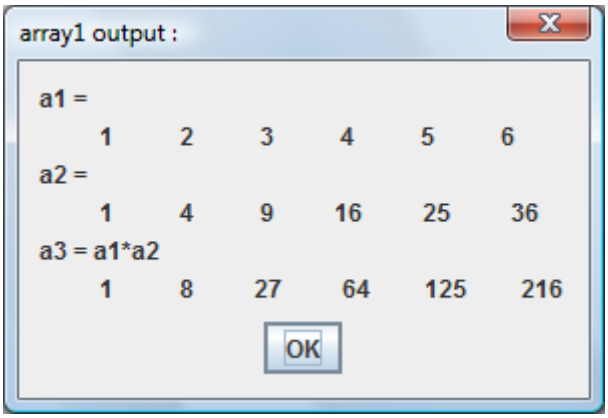
Table Format structures

Conversion	Argument Category	Description
'b', 'B'	general	If the argument *arg* is null, then the result is "false". If *arg* is a boolean or Boolean, then the result is the string returned by String.valueOf(). Otherwise, the result is "true".
'h', 'H'	general	If the argument *arg* is null, then the result is "null". Otherwise, the result is obtained by invoking Integer.toHexString(arg.hashCode()).
's', 'S'	general	If the argument *arg* is null, then the result is "null". If *arg* implements Formattable, then arg.formatTo is invoked. Otherwise, the result is obtained by invoking arg.toString().
'c', 'C'	character	The result is a Unicode character
'd'	integer	The result is formatted as a decimal integer
'o'	integer	The result is formatted as an **octal** (base 8) integer
'x', 'X'	integer	The result is formatted as a **hexadecimal** (base 16) integer
'e', 'E'	floating point	The result is formatted as a decimal number in computerized scientific notation
'f'	floating point	The result is formatted as a decimal number
'g', 'G'	floating point	The result is formatted using computerized scientific notation or decimal format, depending on the precision and the value after rounding.
'a', 'A'	floating point	The result is formatted as a hexadecimal floating-point number with a significand and an exponent
't', 'T'	date/time	Prefix for date and time conversion characters. See Date/Time Conversions.
'%'	percent	The result is a literal '%' ('\u0025')
'n'	line separator	The result is the platform-specific line separator

Program array1.java an example of acessing array indices through for loop structure

```
import javax.swing.*;
import java.util.Locale;

 public class array1
 {
 public static void main(String[] args)
 {
    int a1[]={1,2,3,4,5,6};
    int a2[]={1,4,9,16,25,36};
    int a3[]=new int[a2.length];
    String s="array1 output : ";
    String s1="a1 =\n"+output(a1);
    s1+="a2 = \n"+output(a2);
    for(int i=0;i<a1.length;i++) {a3[i]=a1[i]*a2[i];}
    s1+="a3 = a1*a2 \n"+output(a3);
    JOptionPane.showMessageDialog(null,s1,s,JOptionPane.PLAIN_MESSAGE);
}
 public static String output(int a[])
 {
 Locale tr=new Locale("TR");
 int x=25;
 String s1="";
  for(int i=0;i<a.length;i++)
   { s1+=String.format(tr," %10d ",a[i]);}
  s1+="\n";
  return s1;
 }

 }
```

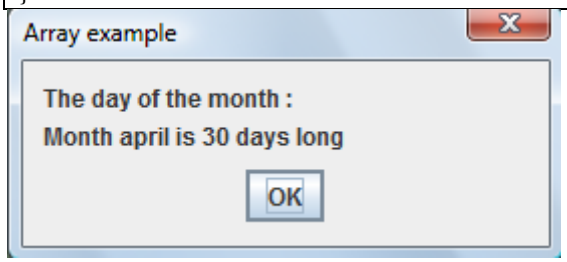
The result will be :



In the next example String type of array variable is introduced.

Program H8Ex2B.java an example of acessing array indices through for loop structure

```
import java.io.*;
import javax.swing.*;
class H8Ex2B {
public static int month(String s)
{

String month[]={"january","february","march","april","may","june","july",
          "august","september","october","november","december"};
int i;
int j=-1;
for(i=0;i<12;i++)
{if(s.equals(month[i])) {j=i;break;}}
return j;
}


public static void main(String args[] )
{ int the_day_of_the_month[]={31,28,31,30,31,30,31,31,30,31,30,31};
String s="The day of the month : \n";
String s2="april";
int i=month(s2);
s+="Month "+s2+" s "+ayin_gunleri[i]+" days long \n";
String s1="Array example";
JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```



ENHANCED FOR STATEMENT

In latest java versions a simpler for statement is introduced to use with for arrays. As an old programmer I have not used it much but you have right to know the new structure. In the normal for structure, the for loop was given as :

Program ForTest.java an example of acessing array indices through for loop structure

```
public class ForTest
 {
 public static void main( String args[] )
 {
 int array[] = { 87, 68, 94, 100, 83, 78, 85, 91, 76, 87 };
 int total = 0;
   for ( int i=0;i<array.length;i++)
     {total += array[i];}
   System.out.println( "Total = "+total );
 } // end main
 }
```

In the enhanced for loop the same can be achived as :

Program ForTest.java an example of acessing array indices through enhanced for loop structure

```
public class ForTest1
 {
 public static void main( String args[] )
 {
 int array[] = { 87, 68, 94, 100, 83, 78, 85, 91, 76, 87 };
 int total = 0;
     for ( int i:array)
     {total += number;}
   System.out.println( "total = "+total );
 } // end of main }
```

As it is seen for structure in the second program is given as:

```
    for ( int number:array)
    {total += number;}
```

In the original for structure this procedure was

```
  for ( int i=0;i<array.length;i++)
    {total += array[i];}
```

Note that in the second structure number is not counter but represent array[i] value in the previous program. Using this new form certainly has advantages and disadvantages. Original form is best known and understood by everyone. Understanding by everybody is an important issue in languages computer as well as human languages. Therefore mostly original form is used in this book, but when it is used with the object array the new form has advantages in referencing.

ARRAYS OF OBJECTS

Array structures can be used for objects as well as simple variables.

Program box.java

```
class box
{
private double length;
private double width;
private double height;

box(double l,double w, double h)
{length=l;
width=w;
```

```
height=h;
}
// This methods are written to access private length,
// width and height variables
public double read_length() {return length;}
public double read_width() {return width;}
public double read_height() {return height;}
//This methods are written to change private length,
// width and height variables
public void write_length(double l) {length=l;}
public void write_width(double w) {width=w;}
public void write_height(double h) {height=h;}

box(box bi)
{length=bi.length;
width=bi.width;
height=bi.height;
}

double volume()
{return length*width*height;}

public String toString()
{ String s="";
  s+="length = "+length+"\n";
  s+="width = "+width+"\n";
  s+="height = "+height+"\n";
  s+="Volume = "+volume()+"\n\n";
  return s;
}

}
```

Program An example program using box.java in array form

```
import java.io.*;
import javax.swing.*;
class H8Ex3
{
public static void main(String args[] )
{   box x[]=new box[4];
    x[0]=new box(1.1,2.2,3.3);
    x[1]=new box(1.23,2.11,3.12);
    x[2]=new box(2.5,2.11,3.0);
    x[3]=new box(2.5,2.11,3.1);
  String s1="Array example ";
  String s="";
  for(int i=0;i<x.length;i++)
    {s+="Box number = "+i+"\n"+x[i].toString();}
  JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

If this program written with enhanced for statement:

Program An example program using box.java in array form written with enhanced for statement

```
import java.io.*;
```

```
import javax.swing.*;
class H8Ex3A
{
public static void main(String args[] )
{   box x[]=new box[4];
    x[0]=new box(1.1,2.2,3.3);
    x[1]=new box(1.23,2.11,3.12);
    x[2]=new box(2.5,2.11,3.0);
    x[3]=new box(2.5,2.11,3.1);
  String s1="Array example ";
  String s="";
  for(box number:x)
    {s+=number.toString();}
  JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

The advantages of using an enhanced for with arrays of objects are clear in here, it is a shorter form. But as it is seen it is not necassary. The basic difference of the object type arays is that we should initilize them by using constructor classes. Each individual array element should be constructed individually. In our second example a vector class will be defined and vectors will be added directly in the output example. It should also be note that dimension variables can be used when methods area called

Program vector.java class

```
import java.util.Locale;
public class vector
 {
 public double V[];
 //kurucu metodlar
 public vector(double Vi[])
 { V=new double[Vi.length];
   input_vector(Vi);
 }

 public vector(vector Vi)
 { V=new double[Vi.V.length];
   input_vector(Vi.V);
 }

 public vector(int n)
 { V=new double[n];
   for(int i=0;i<n;i++)
              { V[i]=0.0;}
 }

 public void input_vector(double Vi[])
 {   for(int i=0;i<V.length;i++)
               {V[i]=Vi[i];}
 }

 public static vector add(vector V1,vector V2)
 {
 //
 int n=V1.V.length;
 vector V3=new vector(n);
  for(int i=0;i<n;i++)
               {V3.V[i]=V1.V[i]+V2.V[i];}
```

```
 return V3;
 }

 public static vector add(double[] V1,double[] V2)
 {
 //
 int n=V1.length;
 vector M=new vector(n);
  for(int i=0;i<n;i++)
                {M.V[i]=V1[i]+V2[i];}
 return M;
 }

 public void add(vector V1)
 {
 //
 int n=V.length;
  for(int i=0;i<n;i++)
                {V[i]+=V1.V[i];}
 }

 public void add(double [] B)
 {
 //
 int n=V.length;
  for(int i=0;i<n;i++)
                {V[i]+=B[i];}
 }

 public static String toString(vector M)
 { int n=M.V.length;
   String s="";
  for(int i=0;i<n;i++)
                {s+=M.V[i]+" ";}
 s+="\n";
 return s;
 }

 public String toString()
 {  Locale tr=new Locale("TR");
              int n=V.length;
    String s1="";
    for(int i=0;i<n;i++)
    { s1+=String.format(tr," %10.5f ",V[i]);}
   s1+="\n";
 return s1;
 } //

 }//
```

Program An example using vector class

```
import javax.swing.*;
class H8Ex7 {
public static void main(String args[] )
{ double A[]={1.0,2.0,3.0,4.0};
  double B[]={1.0,1.0,1.0,1.0};
  vector M1=new vector(A);
  vector M2=new vector(B);
  M1.add(M2);
  String s="summation of two vector : \n"+M1.toString();
  String s1="class vector";
```

```
  JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
} }
```

As a third example, book class defined in the previous chapter will be used in array structure

Program library class that use book class as an array object variable

```
public class library
{
String name;
int number_of_books;
book list[];

public library(String kname,book klist[])
{name=kname;
 number_of_books=klist.length;
 list=new book[number_of_books];
 for(int book_number=0;book_number<number_of_books;book_number++)
 {list[book_number]=klist[book_number];}
}


 public String toString()
 {            String s="Library name : "+name+"\n";
    s+="Total number of books : "+number_of_books+"\n\n";
    for(int book_number=0;book_number<number_of_books;book_number++)
    {s+=list[book_number].toString()+"\n";}
    return s;
 }
}
```

Program An example program using book and library class that use book class as an array object variable

```
import javax.swing.*;
class booktest1
{
public static void outputwindow(String s)
{
String s1="library class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{

book b[]=new book[5];
b[0]=new book("Bilim ve Teknik","TÜBİTAK","magazine","Turkish");
b[1]=new fiction("It is all quiet in the westen front ","Erich Maria Reamarque","English");
b[2]=new book("The God Delusion","Richard Dawkins","fiction","English");
b[3]=new poetry("Human lanscapes from my country","Nazım Hikmet","English and Turkish");
b[4]=new encyclopedia("Encyclopedia Britannica","English");
library l=new library("Library of congress",b);
String s=l.toString();
outputwindow(s);
}
```

It should be noted that, in the last examples fiction poetry and encyclopedia classes are used as well in the book definition, because, tese classes are extended from class book.

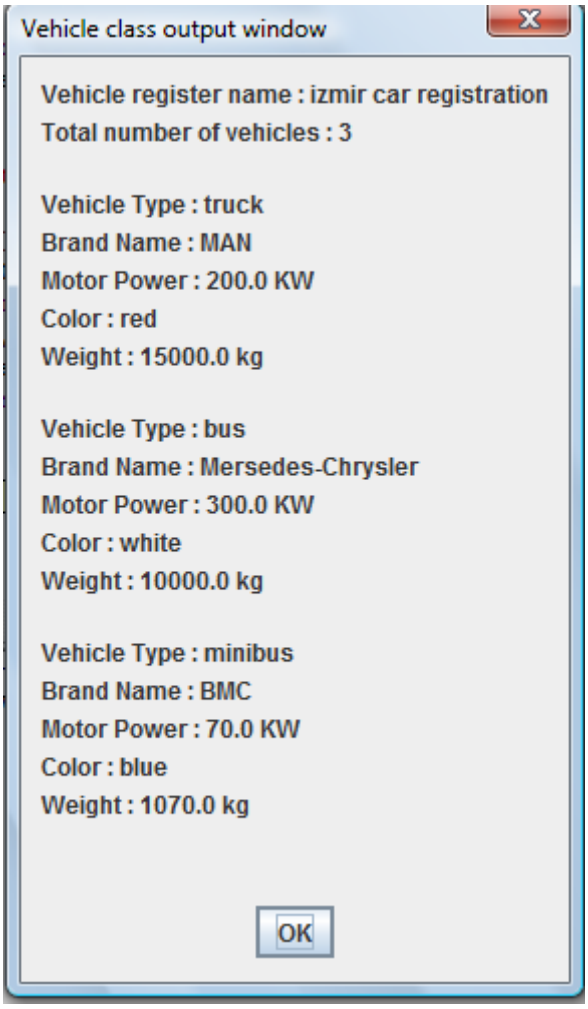
A similar example will be given for vehicle class of the previous chapter

Program vehicle_register class that use vehicle class as an array object variable

```
public class vehicle_register
{
String name;
int number_of_vehicles;
vehicle list[];

public vehicle_register(String kname,vehicle klist[])
{name=kname;
 number_of_vehicles=klist.length;
 list=new vehicle[number_of_vehicles];
 for(int vehicle_number=0;vehicle_number<number_of_vehicles;vehicle_number++)
 {list[vehicle_number]=klist[vehicle_number];}
}

 public String toString()
 {          String s="Vehicle register name : "+name+"\n";
    s+="Total number of vehicles : "+number_of_vehicles+"\n\n";
    for(int vehicle_number=0;vehicle_number<number_of_vehicles;vehicle_number++)
    {s+=list[vehicle_number].toString()+"\n";}
    return s;
 }

}
```

Program An example program utilises vehicle class and vehicle_registration class

```
import javax.swing.*;
class vehicletest1
{

public static void outputwindow(String s)
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}

public static void main(String args[])
{
vehicle v[]=new vehicle[3];
v[0]=new truck("MAN",200.0,"red",15000);
v[1]=new bus("Mersedes-Chrysler",300.0,"white",10000);
v[2]=new minibus("BMC",70,"blue",1070);
vehicle_register vr=new vehicle_register("izmir car registration",v);
String s=vr.toString()+"\n";
outputwindow(s);
}

}
```



Program Another example program Student and course classes are given, this two classes specify students in the class and avarage grade from the class for each student and letter grades of the students according to class average and standard deviation

```
public class Student
{
private String firstName;
private String lastName;
private String StudentNumber;
private String school;
private double grade[];
private String lettergrade;
private double weight[];
private double totalweight;
public Student( String first, String last, String sn,String school_name )
{
firstName = first;
lastName = last;
StudentNumber = sn;
if(school_name.equals("mechanical") || school_name.equals("chemical") ||
school_name.equals("electrical") || school_name.equals("industrial") ||
school_name.equals("civil") || school_name.equals("mining"))
{school=school_name;}
else
{school="not defined";}
```

```
lettergrade=" ";
}

public void setFirstName( String first )
{
firstName = first;
}
public String getFirstName()
{
return firstName;
}

public void setLastName( String last )
{
lastName = last;
}

public String getLastName()
{
return lastName;
}

public void setStudentNumber( String ssn )
{
StudentNumber = ssn;
}
public void setLetterGrade( String ssn )
{lettergrade = ssn; }

public String getStudentNumber()
{
return StudentNumber;
}
public String getSchool()
{ return school; }

public void setGrade(double gradi[])
{int n=gradi.length;
grade=new double[n];
for(int i=0;i<n;i++)
{grade[i]=gradi[i];}
}
public void setWeight(double weighti[])
{int n=weighti.length;
weight=new double[n];
totalweight=0;
for(int i=0;i<n;i++)
{weight[i]=weighti[i];totalweight+=weight[i];}
}
public double[] getGrade()
{ return grade;
}
public double[] getWeight()
{ return weight;
}
public double getAverage()
```

```
{ double avg=0;
  int n=weight.length;
  for(int i=0;i<n;i++)
  {avg+=weight[i]*grade[i];}
  avg/=totalweight;
  return avg;
}
public String toString()
{ String s=getFirstName()+" "+getLastName()+" "+getStudentNumber()+" "+getSchool()+"
"+getAverage()+"Letter grade = "+lettergrade+"\n";
 return s;
}
}
```

Program class course

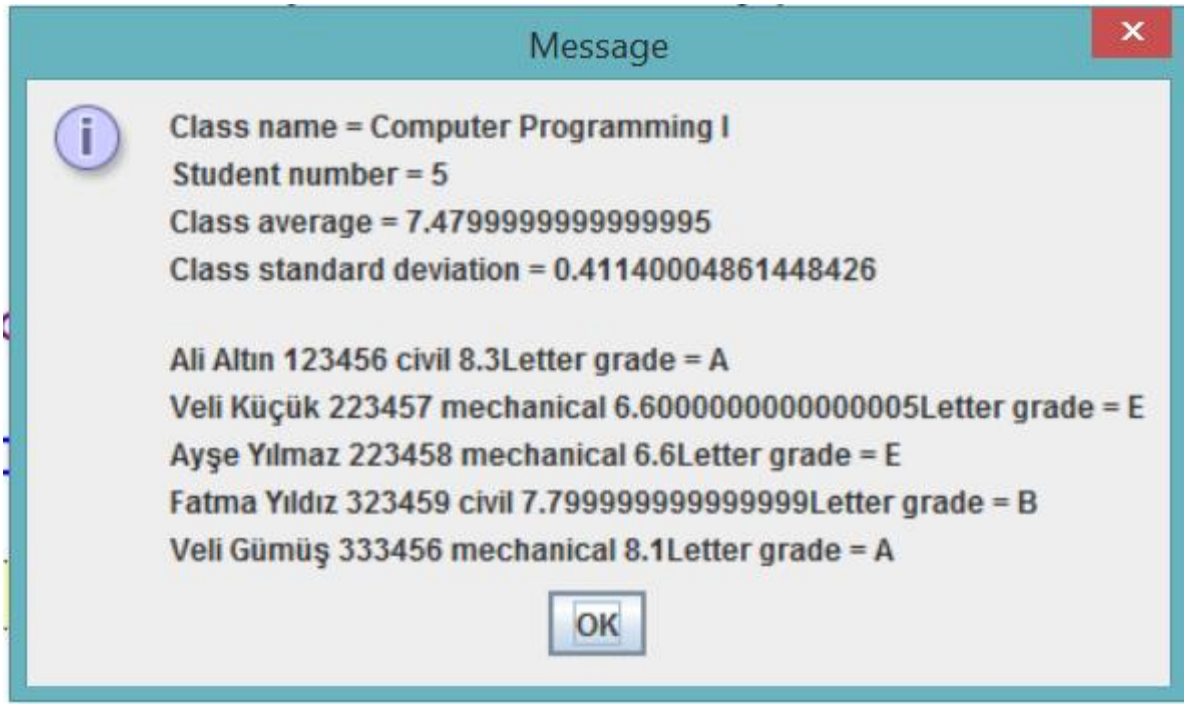
```
public class course
{
public String name;
public Student[] student;
private double weight[];
private double totalweight;

public course(String namei,Student[] studenti,double weighti[])
{
name=namei;
int n=studenti.length;
student=studenti;
setWeight(weighti);
for(int i=0;i<n;i++)
{student[i].setWeight(weighti);}
}

public void setWeight(double weighti[])
{int n=weighti.length;
weight=new double[n];
totalweight=0;
for(int i=0;i<n;i++)
{weight[i]=weighti[i];totalweight+=weight[i];}
}
public void setGrade(double grade[][])
{int n=weight.length;
 int m=student.length;
 for(int i=0;i<m;i++)
 {student[i].setGrade(grade[i]);}
}


public double getAverage()
{ double avg=0;
  int n=student.length;
  for(int i=0;i<n;i++)
  {
   avg+=student[i].getAverage();
  }
  avg/=n;
  return avg;
}
```

```
public double getStandardDeviation()
{ double avg=getAverage();
  double studentavg=0;
  int n=student.length;
  double total=0;
  for(int i=0;i<n;i++)
  { studentavg=student[i].getAverage();
   total+=(studentavg-avg)*(studentavg-avg);
  }
  total=Math.sqrt(total)/(n-1);
  return total;
}

public void setLettergrades()
{  double avg=getAverage();
   double std=getStandardDeviation();
   double studentavg=0;
   int n=student.length;
   for(int i=0;i<n;i++)
   { studentavg=student[i].getAverage();
     if(studentavg>(avg+3.0*std/2.0)) student[i].setLetterGrade("A");
     else if(studentavg<=(avg+3.0*std/2.0) && studentavg>=(avg+std/2.0))
student[i].setLetterGrade("B");
     else if(studentavg<=(avg+std/2.0) && studentavg>=(avg-std/2.0))
student[i].setLetterGrade("C");
     else if(studentavg<=(avg-std/2.0) && studentavg>=(avg-3.0*std/2.0))
student[i].setLetterGrade("D");
     else if(studentavg<=(avg-3.0*std/2.0) && studentavg>=(avg-5.0*std/2.0))
student[i].setLetterGrade("E");
     else if(studentavg<=(avg-5.0*std/2.0)) student[i].setLetterGrade("F");
   }
}

public String toString()
{int n=student.length;
 setLettergrades();
 String s="Class name = "+name+"\n";
 s+="Student number = "+n+"\n";
 s+="Class average = "+getAverage()+"\n";
 s+="Class standard deviation = "+getStandardDeviation()+"\n\n";
 for(int i=0;i<n;i++)
 {s+=student[i].toString();}
return s;
}
}
```

Program class coursetest

```
import javax.swing.*;
public class coursetest
{
public static void main(String arg[])
{
Student s1[]=new Student[5];
s1[0]=new Student("Ali","Altın","123456","civil");
double a1[]={10.0,7.0,8.0};
```

```
s1[0].setGrade(a1);

s1[1]=new Student("Veli","Küçük","223457","mechanical");
double a2[]={7.0,7.0,6.0};
s1[1].setGrade(a2);

s1[2]=new Student("Ayşe","Yılmaz","223458","mechanical");
double a3[]={9.0,5.0,6.0};
s1[2].setGrade(a3);

s1[3]=new Student("Fatma","Yıldız","323459","civil");
double a4[]={9.0,5.0,9.0};
s1[3].setGrade(a4);

s1[4]=new Student("Veli","Gümüş","333456","mechanical");
double a5[]={10.0,5.0,9.0};
s1[4].setGrade(a5);

double weight[]={0.3,0.3,0.4};
course c1=new course("Computer Programming I",s1,weight);
String s=""+c1;
JOptionPane.showMessageDialog(null,s);
}
}
```



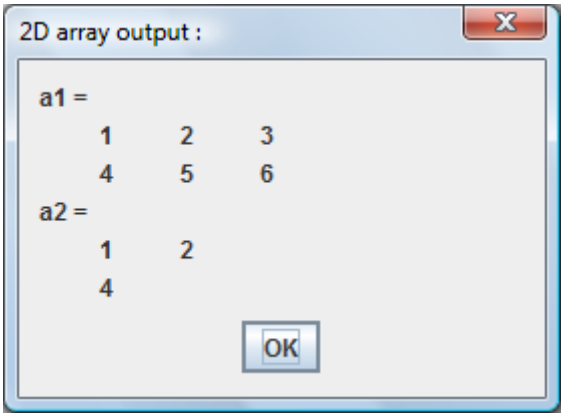
MULTI-DIMENSIONAL ARRAY VARIABLES OF OBJECTS

Arrays can be given in more than one dimensions. When arrays are more than one dimension, they can be considered as arrays of arrays.

Program array2D.java 2 dimensional arry example

```
import javax.swing.*;
import java.util.Locale;
```

```
public class array2D
{
public static void main(String[] args)
{
    int a1[][]={{1,2,3},{4,5,6}};
    int a2[][]={{1,2},{4}};
    String s="2D array output : ";
    String s1="a1 =\n"+output2D(a1);
    s1+="a2 = \n"+output2D(a2);
    JOptionPane.showMessageDialog(null,s1,s,JOptionPane.PLAIN_MESSAGE);
}
public static String output2D(int a[][])
{
Locale tr=new Locale("TR");
int x=25;
String s1="";
  for(int i=0;i<a.length;i++)
  {  for(int j=0;j<a[i].length;j++)
   { s1+=String.format(tr," %10d ",a[i][j]);}
  s1+="\n";
  }
  return s1;
}}
```



The same concept can be applied to class objects as well

Program matrix.java class with two dimensional double variables

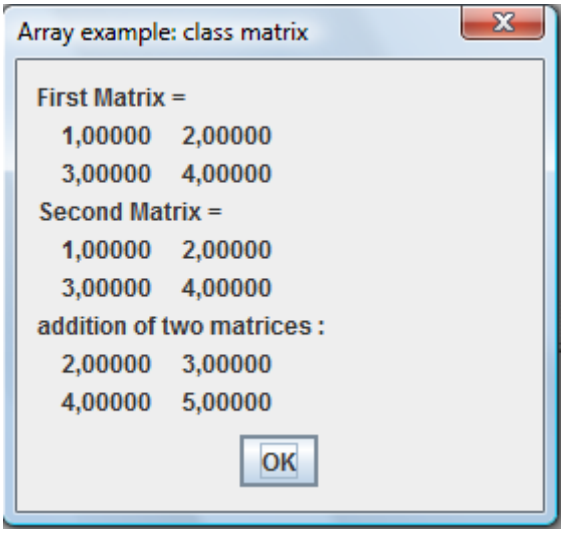
```
import java.util.Locale;
public class matrix
 {

 public double A[][];
 //constructor methods
 public matrix(double Ai[][])
 { A=new double[Ai.length][Ai[0].length];
   input_matrix(Ai);
 }

 public matrix(matrix M)
 { A=new double[M.A.length][M.A[0].length];
   input_matrix(M.A);
 }

 public matrix(int n,int m)
 { A=new double[n][m];
```

```
  for(int i=0;i<n;i++)
            { for(int j=0;j<m;j++)
              {A[i][j]=0.0;}
   }
}

public void input_matrix(double Ai[][])
{   for(int i=0;i<A.length;i++)
            { for(int j=0;j<A[0].length;j++)
              {A[i][j]=Ai[i][j];}
   }
}

public static matrix add(matrix M1,matrix M2)
{
//
int n=M1.A.length;
int m=M1.A[0].length;
matrix M3=new matrix(n,m);
 for(int i=0;i<n;i++)
            { for(int j=0;j<m;j++)
              {M3.A[i][j]=M1.A[i][j]+M2.A[i][j];}
   }
return M3;
}

public static matrix add(double[][] A1,double[][] A2)
{
//
int n=A1.length;
int m=A1[0].length;
matrix M=new matrix(n,m);
 for(int i=0;i<n;i++)
            { for(int j=0;j<m;j++)
              {M.A[i][j]=A1[i][j]+A2[i][j];}
   }
return M;
}

public void add(matrix M1)
{
//
int n=A.length;
int m=A[0].length;
 for(int i=0;i<n;i++)
            { for(int j=0;j<m;j++)
              {A[i][j]+=M1.A[i][j];}
   }
}

public void add(double [][] B)
{
//
int n=A.length;
int m=A[0].length;
 for(int i=0;i<n;i++)
            { for(int j=0;j<m;j++)
              {A[i][j]+=B[i][j];}
   }
}

public static String toString(matrix M)
```

```
 { int n=M.A.length;
   int m=M.A[0].length;
   String s="";
  for(int i=0;i<n;i++)
              { for(int j=0;j<m;j++)
                {s+=M.A[i][j]+" ";}
              s+="\n";
              }
return s;
}

public String toString()
{  Locale tr=new Locale("TR");
            int n=A.length;
  int m=A[0].length;
  String s1="";
  for(int i=0;i<n;i++)
  {  for(int j=0;j<m;j++)
   { s1+=String.format(tr," %10.5f ",A[i][j]);}
  s1+="\n";
  }
return s1;
} //end of method toString

}//matrix sınıfının sonu
```

Program An example program using class matrix to add two matrices

```
import javax.swing.*;
class H8Ex4 {
public static void main(String args[] )
{ double A[][]={{1.0,2.0},{3.0,4.0}};
double B[][]={{1.0,1.0},{1.0,1.0}};
matrix M1=new matrix(A);
String s="First Matrix = \n"+M1.toString();
matrix M2=new matrix(B);
s+="Second Matrix = \n"+M1.toString();
M1.add(M2);
s+="addition of two matrices : \n"+M1.toString();
String s1="Array example: class matrix";
JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
} }
```



In a different for of the program, matrix is also used as an array structure. In this version static add method is used to get output.

Program An example program using class matrix to add two matrices

```
import javax.swing.*;
class H8Ex5 {
public static void main(String args[] )
{ double A[][]={{1.0,2.0},{3.0,4.0}};
  double B[][]={{1.0,1.0},{1.0,1.0}};
  matrix M[]=new matrix[3];
  M[0]=new matrix(A);
  M[1]=new matrix(B);
  M[2]=matrix.add(M[0],M[1]);
  String s="First Matrix = \n"+M[0].toString();
  s+="Second Matrix = \n"+M[1].toString();
  s+="addition of two matrices : \n"+M[2].toString();
  String s1="Array example: class matrix";
  JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
} }
```

The result will be same as the previous example.

CHANGING ARRAY DIMENSIONS

Dimensions of the arrays can not be changed due to the fact that memory is assigned in the definition of the array. But by using indirect methods it can be changed, namely by creating a different array with a different dimension and reloaded array values into this new array.

```
double a[]={3.0,5.0,7.0}
double b[]=new double[4];
for(int i=0;i<a.length;i++)
   {b[i]=a[i];}
a=b;
```

In this small program part dimension of the array a is changed from 3 to 4. Now this basic process is carried out for a double array through a class, doubleArray

Program: dimension changing example doubleArray

```
public class doubleArray
{
  //sınıf değişkenleri
  public double a[];
  public int length;

  public doubleArray(double x[] )
  {
  length=x.length;
  a=new double[length];
     for(int i=0;i<length;i++)
       a[i]=x[i];
  }

  public doubleArray(int n)
  {
  a=new double[n];
  length=a.length;
  }
```

```
 public void addDimension(int n)
 {
 if(n>0)
   {
   int z=a.length+n;
   double[] b=new double[z];
   for(int i=0;i<a.length;i++)
     {b[i]=a[i];}
   a=b;
   length=a.length;
   }
 }

 public void addDimension()
 {
 addDimension(1);
 }

 public void removeDimension(int n)
 {
 int z=a.length-n;
  if(z>1)
  {
  double[] b=new double[a.length-n];
  for(int i=0;i<z;i++)
     b[i]=a[i];
  a=b;
  }
  else
  {
  double[] b=new double[1];
  b[0]=a[0];
  a=b;
  }
  length=a.length;
 }

 public void removeDimension()
 {
removeDimension(1);
 }

 public double getValue(int i)
 {
 return a[i];
 }

 public double[] getValue()
 {
 return a;
 }

 public void setValue(double x,int i)
 {
 a[i]=x;
 }
```

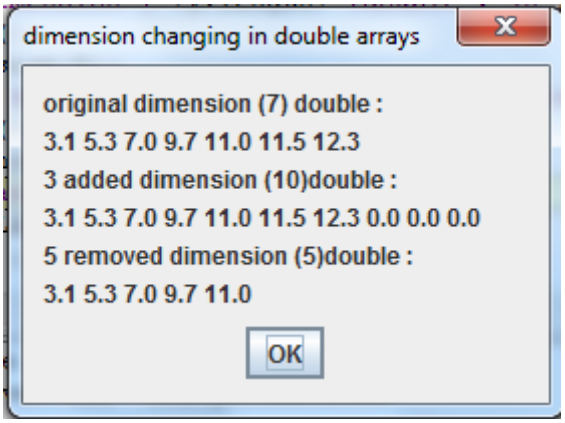
```
  public void setValue(double[] x )
  {
  length=x.length;
  a=new double[length];
  for(int i=0;i<length;i++)
     {
     a[i]=x[i];
     }
  }

  public String toString(int i)
  {
    return ""+a[i];
  }

  public String toString()
  {
  String s="";
  for(int i=0;i<length;i++)
    {
    s+=a[i]+" ";
    }
    s+="\n";
    return s;
  }
}
```

DoubleBoyut sınıfında boyutu değiştirilebilen değişken (nesne) tanımladık. Bu değişkeni boyutDegistir sınıfında test edelim :

Program: dimension changing example : changeDimension

```
import javax.swing.JOptionPane;

public class changeDimension
{
  public static void main(String args[])
  {
  double x[]={3.1,5.3,7.0,9.7,11.0,11.5,12.3};
  doubleArray y=new doubleArray(x);
  String s="original dimension ("+y.length+") double : \n";
  s+=y.toString();
  y.addDimension(3);
  s+="3 added dimension ("+y.length+")double : \n";
  s+=y.toString();
  y.removeDimension(5);
  s+="5 removed dimension ("+y.length+")double : \n";
  s+=y.toString();
  JOptionPane.showMessageDialog(null,s,
  "dimension changing in double arrays",
  JOptionPane.PLAIN_MESSAGE);
  System.exit(0);
  }
}
```

Program output:



EXAMPLE : Create a double array and calculate sum

```
public class array1
{
public static void main(String arg[])
{
double x[]=new double[3];
x[0]=1.2;
x[1]=2.3;
x[2]=3.1;
double total=0;
for(int i=0;i<x.length;i++)
{total+=x[i];
System.out.println("x["+i+"] = "+x[i]);
}
System.out.println("total="+total);
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array1
x[0] = 1.2
x[1] = 2.3
x[2] = 3.1
total=6.6
> Terminated with exit code 0.
```

EXAMPLE : Create a double array and calculate sum

```
public class array2
{
public static void main(String arg[])
{
double x[]={1.2,2.3,3.3};
double total=0;
for(int i=0;i<x.length;i++)
{total+=x[i];
System.out.println("x["+i+"] = "+x[i]);
}
System.out.println("total="+total);
```

```
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array2
x[0] = 1.2
x[1] = 2.3
x[2] = 3.3
total=6.8

> Terminated with exit code 0.
```

EXAMPLE : Create a string array and add array variables into one string

```
public class array3
{
public static void main(String arg[])
{
String x[]={"Ali "," Veli"," 49 "," elli"};
String s="";
for(int i=0;i<x.length;i++)
{s+=x[i];
System.out.println("x["+i+"] = "+x[i]);
}
System.out.println("s="+s);
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array3
x[0] = Ali
x[1] =  Veli
x[2] =  49
x[3] =  elli
s=Ali  Veli 49  elli

> Terminated with exit code 0.
```

EXAMPLE: Create a box3 class array and calculate volume of the boxes in array elements

```
public class box3
{
public double length;
public double width;
public double height;
public String color;
public box3(double l,double w,double h,String s)
{
length=l;
width=w;
height=h;
color=s;
}
```

```
public double volume()
{ return height*width*length;}

public String output()
{ return "volume = "+volume()+"color = "+color;}
}
```

```
public class array4
{
public static void main(String arg[])
{
box3 x[]=new box3[2];
x[0]=new box3(1.2,2.3,2.5,"yellow");
x[1]=new box3(1.1,1.3,1.5,"blue");
String s="";
for(int i=0;i<x.length;i++)
{
System.out.println("volume of x["+i+"] = "+x[i].volume());
}
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array4
volume of x[0] = 6.8999999999999995
volume of x[1] = 2.1450000000000005
> Terminated with exit code 0.
```

EXAMPLE: Create a complex class array and calculate sum of the two complex array element through the static method definition

```
public class complex
{
public double real;
public double imag;
public complex(double r,double i)
{real=r;imag=i;}
//static addition of two complex variable result returns as complex
public static complex add(complex z1,complex z2)
{complex z3=new complex((z1.real+z2.real),(z1.imag+z2.imag));
return z3;
}
//dynamic addition to a complex variable into the value of the class
public void add(complex z1)
{real+=z1.real;
imag+=z1.imag;
}

public String output()
{String s=real+" +i*"+imag;
return s;
}
}
```

```
public class array5
{
public static void main(String arg[])
{
complex x[]=new complex[2];
x[0]=new complex(1.2,2.3);
x[1]=new complex(1.1,1.3);
complex z=complex.add(x[0],x[1]);
System.out.println(z.output());
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array4
volume of x[0] = 6.8999999999999995
volume of x[1] = 2.1450000000000005
> Terminated with exit code 0.
```

EXAMPLE: Create a complex class array and calculate sum of the two complex array element through the dynamic method definition

```
public class array6
{
public static void main(String arg[])
{
complex x[]=new complex[2];
x[0]=new complex(1.2,2.3);
x[1]=new complex(1.1,1.3);
x[0].add(x[1]);
System.out.println(x[0].output());
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array4
volume of x[0] = 6.8999999999999995
volume of x[1] = 2.1450000000000005
> Terminated with exit code 0.
```

EXAMPLE : Create a polynomial function class and calculate polynomial values for x=1..10

```
public class f
{// General polynomial function
 // y=a[0]+a[1]*x+a[2]*x^2+...
 public double a[];
 public f(double ia[])
 {int n=ia.length;
  a=new double[n];
  for(int i=0;i<n;i++) {a[i]=ia[i];}
 }
 public double func(double x)
 {
  int n=a.length;
  double ff=a[n-1];
```

```
  for(int i=n-2;i>=0;i--)
  {ff=ff*x+a[i];}
  return ff;
 }
}
```

```
public class array7
{
public static void main(String arg[])
{
double a[]={1.0,2.0,3.0};
f z=new f(a);
for(double x=0;x<=10;x+=1)
{System.out.println("x="+x+"y="+z.func(x));}
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" array7
x=0.0y=1.0
x=1.0y=6.0
x=2.0y=17.0
x=3.0y=34.0
x=4.0y=57.0
x=5.0y=86.0
x=6.0y=121.0
x=7.0y=162.0
x=8.0y=209.0
x=9.0y=262.0
x=10.0y=321.0

> Terminated with exit code 0.
```

POLYMORPHISM, ABSTRACT CLASSES AND INTERFACE

ABSTRACT CLASS

It is seen in chapter 7 and 8 that when classes extended from each other, base class can be used as an indirect reference adres. This property is quite useful when a code is desired to generalise for different classes. In order to establishe this an emty class, called abtract class, can be used as base reference and programs and classes can be developed by using this class. When, in later stages, a new class is developed, this classes can be used with the new class due to this indirect reference possibility.

Definition of abstract class

abstact class class_name
{
class variable list
class method list
}

Class methods can also be abstract in an abstract class. In an abstract method no definition is given

As the first example class shape will be given

Program Abstract class shape

```
public abstract class shape
{ public double area(){return 0.0;}
  public double volume() {return 0.0;}
  public abstract String name();
}
```

In this class no abstract method is defined, all methods have same definition. As a second example class f_x is given

Program Abstract class f_x

```
abstract class f_x
{ abstract double func(double x);}
```

In this class, method of the class is defined also as abstract so no class definition is given. If a method definition is given as abstract, definition of the method must be given in the extended method with the same input and output structure. To see how extended classes can be established and used, classes point and circle and cylinder is created as extended from class shape.

Program class point extended from class shape

```
public class point extends shape
{   protected double x, y;
    public point(double a, double b)
    {  x=a;y=b;}
    public void enterpoint(double a, double b)
    {x=a;y=b;}
    public double read_x()
    {return x;}
    public double read_y()
    {return y;}
    public String toString()
    {return "["+x+","+y+"]";}
    public String name() {return "point";}
}
```

Program class circle extended from class point

```
public class circle extends point
{   protected double R;
    public circle(double r, double a, double b)
    {  super(a,b);enterR(r);}
    public void enterR(double r)
    {   if(r >= 0.0)  R=r;
       else          R=0.0;}
    public double read_R()
    { return R;}
    public double area()
    { return Math.PI*R*R;}
    public String toString()
    { return "Center =  "+super.toString()+"; Radius ="+R;}
    public String isim() {return "circle";}
}
```

Program class cylinder extended from class circle

```
public class cylinder extends circle
```

```
 {protected double H;
 public cylinder( double h, double r, int a, int b)
 {super(r, a, b );enterH( h);}
 public void enterH(double h )
 {if(h>=0) H=h;
  else H=0; }
 public double read_H(){return H;}
 public double area()
 { return 2* super.area()+2*Math.PI*R*H;}
 public double volume(){return super.area()*H;}
 public String toString()
 {return super.toString()+";Height = "+H;}
 public String isim(){return "Cylinder";}
}
```

When these classes needs to be used the abstract class shape can be used to get Access to any of these classes.

Program output example class H9Ex1

```
import javax.swing.*;
class H9Ex1 {
public static void main(String args[])
{point     n   = new point(7,11);
 circle    d   = new circle(3.5,22,8);
 cylinder  s   = new cylinder(10,3.3,10,10);
 shape     a[] = new shape[3];
 a[0]=n;
 a[1]=d;
 a[2]=s;
 String s2="direct call : \n";
       s2+=n.name()+": "+n.toString()+"\n";
       s2+=d.name()+": "+d.toString()+"\n";
       s2+=s.name()+": "+s.toString()+"\n";
       s2+="indirect call : \n";
 for(int i=0;i<a.length;i++)
 {s2+=a[i].name()+": "+a[i].toString()+"\n";}
 String s1="indirect adress use";
 JOptionPane.showMessageDialog(null,s2,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

As it is seen acces to this classes can be achived throyugh the abstract class shape. As another exampel class numerical is given. In this class two mathematical method is given one to get derivative, another to integrate a function. While programsa re written actual functions to be used is not defined, theredore method definitions are utilises abstract class f_x and all programs are written for this abstract class.

Program class numerical, uses abstract class f_x and abstract method func as a general function definition.

```
public class numerical
{

public static double derivative(f_x f,double x)
{  double h=0.00001;
  return (-f.func(x+2.0*h)+8.0*f.func(x+h)-8.0*f.func(x-h)+f.func(x-2.0*h))/(12.0*h);
}

public static double integral(f_x f_xnt,double a,double b)
{
//integral f(x)dx
```

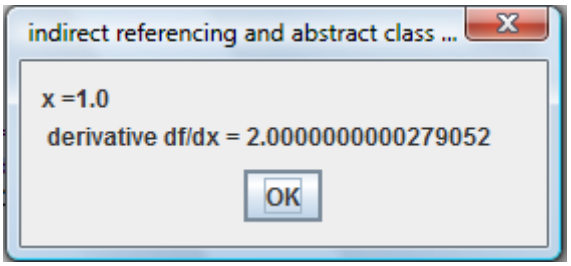
```
 double r[]={-0.973906528517171,-0.865063366688984,-0.679409568299024,-0.433395394129247,-0.148874338981631,
 0.148874338981631,0.433395394129247,0.679409568299024,0.865063366688984,0.973906528517171};
 double c[]={0.066671344308684,0.149451349150580,0.219086362515982,0.269266719309996,0.295524224714752,
 0.295524224714752,0.269266719309996,0.219086362515982,0.149451349150580,0.066671344308684};
double z=0,x,y;
double k1=(b-a)/2.0;
double k2=(b+a)/2.0;
for(int i=0;i<r.length;i++)
{
x=k2+k1*r[i];
y=f_xnt.func(x);
z+=k1*c[i]*y;
}
return z;
}
}
```

When the methods in the numerical class is needed to be used, the actual function is also known. Therefore it can be defined when user is needed to be use it and used previously defined class numerical.

Program An example class uses abstract class f_x and numerical and derive the new class for the fuction definition

```
import javax.swing.*;

class fb extends f_x
{ public double func(double x)
   { return x*x-5.0;}}

public class H9Ex3 {
public static void main(String args[])
{ fb ff=new fb();
  double x=1.0;
 String s="x ="+x+"\n derivative df/dx = "+numerical.derivative(ff,x);
 String s1=" indirect referencing and abstract class example :";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

In the example program class fb is defined as extended from class f_x and used with derivative static method of numerical class. The program output :



Program An example class uses abstract class f_x and numerical and derive the new class fort he fuction definition

```
import javax.swing.*;

class fb extends f_x
{ public double func(double x)
   { return x*x-5.0;}}
```

```
public class H9Ex3b {
public static void main(String args[])
{  double x=1.0;
   String s="x ="+x+"\n derivative df/dx = "+numerical.derivative(new fb(),x);
   String s1=" indirect referencing and abstract class example :";
   JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

In the second example instead of defining a new instance object of class fb, and then calling it into the derivative method, the new object directly created in the method call.

Program An example class uses abstract class f_x and numerical and derive the new class for the fuction definition

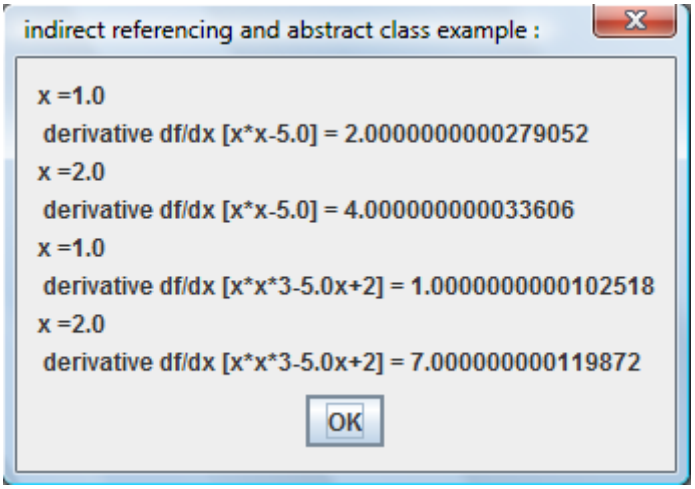
```
import javax.swing.*;

class fb extends f_x
{ public double func(double x)
   { return x*x-5.0;}}

public class H9Ex3b {
public static void main(String args[])
{  double x=1.0;
   String s="x ="+x+"\n derivative df/dx = "+numerical.derivative(new fb(),x);
   String s1=" indirect referencing and abstract class example :";
   JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

In the third example instead of one class for a fuction definition, two different class with two different function is defined.

Program An example class uses abstract class f_x and numerical and derive the new class for the fuction definition

```
import javax.swing.*;

class fb extends f_x
{ public double func(double x)
   { return x*x-5.0;}}

class fc extends f_x
{ public double func(double x)
   { return x*x*3-5.0*x+2;}}


public class H9Ex3c {

public static void main(String args[])
{
    double x=1.0;
            String s="x ="+x+"\n derivative df/dx [x*x-5.0] = "+numerical.derivative(new fb(),x)+"\n";
            x=2;
            s+="x ="+x+"\n derivative df/dx [x*x-5.0] = "+numerical.derivative(new fb(),x)+"\n";
            x=1;
            s+="x ="+x+"\n derivative df/dx [x*x*3-5.0x+2] = "+numerical.derivative(new fc(),x)+"\n";
            x=2;
    s+="x ="+x+"\n derivative df/dx [x*x*3-5.0x+2] = "+numerical.derivative(new fc(),x)+"\n";
```

```
   String s1=" indirect referencing and abstract class example :";
   JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```



In our final example, in order to avoid the lengthly definition of function call and output preperation each time a method is defined fort his duty.

Program An example class uses abstract class f_x and numerical and derive the new class for the fuction definition

```
import javax.swing.*;

class fb extends f_x
{ public double func(double x)
   { return x*x-5.0;}}

class fc extends f_x
{ public double func(double x)
   { return x*x*3-5.0*x+2;}}


public class H9Ex3d {

public static String toString(f_x f,String s,double x)
{ String s1="x ="+x+"\n derivative df/dx [s] = "+numerical.derivative(f,x)+"\n";
 return s1;
}

public static void main(String args[])
{
    String s=toString(new fb(),"x*x-5.0",1.0);
    s+=toString(new fb(),"x*x-5.0",2.0);
    s+=toString(new fc(),"x*x*3-5.0x+2",1.0);
    s+=toString(new fb(),"x*x*3-5.0x+2",2.0);
    String s1="indirect adress usage :";
    JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

INTERFACE

Extend statement can be used only once. One class can be extended only by another one class. From time to time more extensions might be needed. For this purpose, a definition similar to abstract class

definition is given in java. It is called interface. The basic difference of interface is not only the possibility of using for more than one classes, but also having all methods as empty methods(like abstract methods mentioned above but without word abstract). Not that interfaces can nat be considered as class by themselves. They are rather templates of classes to define how and with which method and variables derived classes should be defined.

Definition of interface

```
interface class_name
{
class variable list
class method list
}
```

Class methods can also be abstract in an abstract class. In an abstract method no definition is given. As the first example interface shape will be given

Program interface shape

```
public interface shape
{public double area();
 public double volume();
 public String name();
}
```

Compare abstract class and interface shape definitions. No method body is existed for area, volume or name methods (all methods of interface sre abstract type automatically) . As a second example interface f_x is given

Program interface f_x

```
public interface f_x
{ double func(double x);}
```

In this interface like all other interfaces, no method definition is given. To see how a method is written by using interface and used, classes point and circle and cylinder is created as extended from class shape. The word **implements** is used in interface connection istead of word **extend**.

Program class point implemented from interface shape

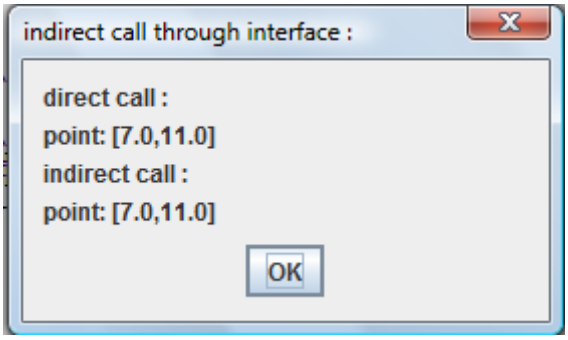
```
public class point implements shape
{ double x, y;
   public point(double a, double b) {  x=a;y=b;}
   public void enter_point(double a, double b) {x=a;y=b;}
   public double read_X() {return x;}
   public double read_Y() {return y;}
   public String toString(){ return "["+x+","+y+"]";}
   public double area() {return 0.0;}
   public double volume(){return 0.0;}
   public String name() {return "point";}
}
```

When these classes needs to be used the interface shape can be used to get Access to class point

Program output example class H9Ex2

```
import javax.swing.*;
class H9Ex2 {
public static void main(String args[])
```

```
{ point    n = new point(7,11);
  shape    a;
  a=n;
  String s="direct call : \n";
        s+=n.name()+": "+n.toString()+"\n";
        s+="indirect call : \n";
        s+=a.name()+": "+a.toString()+"\n";
  String s1="indirect call through interface : ";
  JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

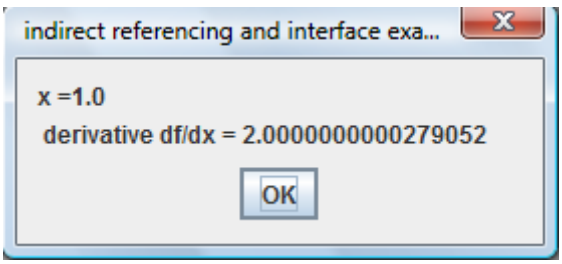


As it is seen acces to class point can be achived through the interface shape. As another example class numerical is given. Code in class numerical will be exatly same as the previous one, therefore class is not re-listed here, but usage of it will be different.

Program An example class uses interface f_x and numerical and derive the new class for the fuction definition

```
import javax.swing.*;

class fb implements f_x
{ public double func(double x)
   { return x*x-5.0;}}

public class H9Ex3 {
public static void main(String args[])
{ fb ff=new fb();
  double x=1.0;
 String s="x ="+x+"\n derivative df/dx = "+numerical.derivative(ff,x);
 String s1=" indirect referencing and interface example :";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

In the example program class fb is defined as **implemented** from class f_x and used with derivative static method of numerical class. The program output :



As it is seen from the example, usage of the interface is very similar to abstract class. Until Java 8 no defined methods or variables are allowed in the interface class and all methods and variable defined in interface must be defined also in the implemented class. But Java 8 on, it is possible to define methods in interfaces by adding word **default**. But in order to use it, interface should be implemented in another class.

Program An example class uses interface person with a default method sayHello

```
interface Person {
   //adds a java 8 default method
   default String sayHello() {
      return "Hello there!";
   }
}

class Sam implements Person {}

public class inherit1 {
   public static void main(String [] args) {
      Sam sam = new Sam();
      //calling sayHello method calls the method
      //defined in interface
      System.out.println(sam.sayHello());
   }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" inherit1
Hello there!

> Terminated with exit code 0.
```

INNER CLASSES AND LAMBDA EXPRESSIONS

Consider interface f_x and H9Ex3 defined in previous section. In H9Ex3 class fb is defined as implemented from interface f_x. In java classes can be written as innner classes instead of seperate outside classes.

Program An example class uses interface f_x and numerical and derive the new inner class for the function definition

```
import javax.swing.*;
public class H9Ex3_1 {
public static void main(String args[])
{    class fb implements f_x
     { public double func(double x) { return x*x-5.0;}}

 fb ff=new fb();
 double x=1.0;
 String s="x ="+x+"\n derivative df/dx = "+numerical.derivative(ff,x);
 String s1=" indirect referencing and interface example with inner class :";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

In java 8 on a new short-cut definition of the previous example can be written as:

Program An example class uses interface f_x and numerical and derive the new inner class in lambda expression form for the function definition

```
import javax.swing.*;
public class H9Ex3_2 {
public static void main(String args[])
{
 f_x ff = x -> x*x-5.0;
 double x=1.0;
 String s="x ="+x+"\n derivative df/dx = "+numerical.derivative(ff,x);
 String s1=" indirect referencing and interface example with inner class lambda definition:";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}}
```

This particular short form is called **lambda expressions.**
Let us give a second example to further explain the consept

Program An interface f_boolean

```
public interface f_boolean
{ boolean decision(double x,double y);}
```

Program An example class uses interface f_boolean and class fc

```
import javax.swing.*;
class fc implements f_boolean
   { public boolean decision(double x,double y)
     { return x>y;}
   }
public class H9Ex4_1 {
public static void main(String args[])
{
 fc ff=new fc();
 double x=1.0;
 double y=2.0;
 String s=x+" > "+y+" = "+ff.decision(x,y);
 String s1=" indirect referencing and interface example :";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

Program An example class uses interface f_boolean and inner class fc

```
import javax.swing.*;

public class H9Ex4_2 {
public static void main(String args[])
{   class fc implements f_boolean
   { public boolean decision(double x,double y)
     { return x>y;}
   }
 fc ff=new fc();
 double x=1.0;
 double y=2.0;
```

```
 String s=x+" > "+y+" = "+ff.decision(x,y);
 String s1=" indirect referencing and interface example :";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

Program An example class uses interface f_boolean and inner class fc as lambda expression

```
import javax.swing.*;

public class H9Ex4_3 {
public static void main(String args[])
{
 f_boolean ff=(x,y)->x>y;
 double x=1.0;
 double y=2.0;
 String s=x+" > "+y+" = "+ff.decision(x,y);
 String s1=" indirect referencing and interface example :";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}
}
```

It should be note that in order to lambda expressions work, there should be only one method definition in interface class new inner class connected. This kind of interfaces called **functional interfaces**. A functional interface is an interface with a single abstract method that is used as the type of a lambda expression. It is not required but function interfaces can be specified in Java 8 by using **@FunctionalInterface** word in front of the interface. In the next example such an interface is specified and used.

Program An example class uses WorkerInterface as lambda expression and Anonymous inner class

```
 //define a functional interface
@FunctionalInterface
interface WorkerInterface {
   public void doSomeWork();
}
public class WorkerInterfaceTest {

   public static void execute(WorkerInterface worker) {
      worker.doSomeWork();
   }

   public static void main(String [] args) {
      //invoke doSomeWork using Annonymous inner class
      execute(new WorkerInterface() {
         @Override
         public void doSomeWork() {
            System.out.println("Worker invoked using Anonymous class");
         }
      });
      //invoke doSomeWork using Lambda expression
      execute( () -> System.out.println("Worker invoked using Lambda expression") );
   }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" WorkerInterfaceTest
Worker invoked using Anonymous class
Worker invoked using Lambda expression

> Terminated with exit code 0.
```

Another version of the same program:

Program An example class uses WorkerInterface as lambda expression and Anonymous inner class

```
 //define a functional interface
@FunctionalInterface
interface WorkerInterface {
   public void doSomeWork();
}

public class WorkerInterfaceTest1 {
   public static void main(String [] args) {
      WorkerInterface worker=()->System.out.println("Worker invoked using Lambda expression ");
      worker.doSomeWork();
   }
}
```

As a last exercises, combining default interface methods usage with lambda expressions will be introduced. Previously, interface used together with numeric class to calculate integral and derivative of the function. This can be done by using default derivative and integral methods inside the interface class.

Program An example class defines interface inter_f_x

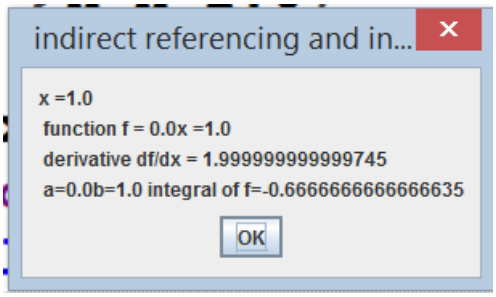
```
 import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;

@FunctionalInterface
interface inter_f_x
{
//Function
public double func(double x);

default double dfunc(double x)
{double h=1.0e-3;
 int n=1;
 return dfunc_n(x,n,h);
}
default double dfunc(double x,double h)
{int n=1;
 return dfunc_n(x,n,h);
}
default double dfunc(double x,int n)
{double h=1.0e-3;
 return dfunc_n(x,n,h);
}
```

```
default double dfunc_n(double x,int n,double h)
{
double hh=1/h;
double df=0;
if(n==0) df=func(x);
else if(n==1)
{df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-
168.0*func(x+2.0*h)+32.0*func(x+3.0*h)-3.0*func(x+4.0*h))/840.0*hh;}
else if(n==2)
{df=(-14350.0*func(x)-9.0*func(x-4*h)+128*func(x-3*h)-1008*func(x-2*h)+8064*func(x-h)+8064.0*func(x+h)-
1008.0*func(x+2.0*h)+128.0*func(x+3.0*h)-9.0*func(x+4.0*h))/5040.0*hh*hh;}
else if(n==3)
{df=(-7.0*func(x-4.0*h)+72.0*func(x-3.0*h)-338.0*func(x-2.0*h)+488.0*func(x-h)-
488.0*func(x+h)+338.0*func(x+2.0*h)-72.0*func(x+3.0*h)+7.0*func(x+4.0*h))/240.0*hh*hh*hh;}
else if(n==4)
{df=(2730.0*func(x)+7.0*func(x-4.0*h)-96.0*func(x-3.0*h)+676.0*func(x-2*h)-1952.0*func(x-h)-
1952.0*func(x+h)+676.0*func(x+2.0*h)-96.0*func(x+3.0*h)+7.0*func(x+4.0*h))/240.0*hh*hh*hh*hh;}
else if(n==5)
{df=(func(x-4.0*h)-9.0*func(x-3.0*h)+26.0*func(x-2.0*h)-29.0*func(x-h)+29.0*func(x+h)-
26.0*func(x+2.0*h)+9.0*func(x+3.0*h)-func(x+4.0*h))/6.0*hh*hh*hh*hh*hh;}
else if(n==6)
{df=(-150.0*func(x)-func(x-4.0*h)+12.0*func(x-3.0*h)-52.0*func(x-2.0*h)+116.0*func(x-h)+116.0*func(x+h)-
52.0*func(x+2.0*h)+12.0*func(x+3.0*h)-func(x+4.0*h))/4.0*hh*hh*hh*hh*hh*hh;}
else if(n==7)
{df=(-func(x-4.0*h)+6.0*func(x-3.0*h)-14.0*func(x-2.0*h)+14.0*func(x-h)-14.0*func(x+h)+14.0*func(x+2.0*h)-
6.0*func(x+3.0*h)+func(x+4.0*h))/2.0*hh*hh*hh*hh*hh*hh*hh;}
else if(n==8)
{df=(70.0*func(x)+func(x-4.0*h)-8.0*func(x-3.0*h)+28.0*func(x-2.0*h)-56.0*func(x-h)-
56.0*func(x+h)+28.0*func(x+2.0*h)-8.0*func(x+3.0*h)+func(x+4.0*h))*hh*hh*hh*hh*hh*hh*hh*hh;}
else df=0;
return df;
}
//INTEGRAL
default double integral(double a,double b)
{ //10 point Gauss-Legendre formula
  //integral f(x)dx
 double r[]={-0.973906528517171,-0.865063366688984,-0.679409568299024,-0.433395394129247,-
0.148874338981631,
 0.148874338981631,0.433395394129247,0.679409568299024,0.865063366688984,0.973906528517171};
 double c[]={0.066671344308684,0.149451349150580,0.219086362515982,0.269266719309996,0.295524224714752,
 0.295524224714752,0.269266719309996,0.219086362515982,0.149451349150580,0.066671344308684};
double z=0,x,y;
double k1=(b-a)/2.0;
double k2=(b+a)/2.0;
for(int i=0;i<r.length;i++)
{
x=k2+k1*r[i];
y=func(x);
z+=k1*c[i]*y;
}
return z;
}
}
```

Program An example class uses interface inter_f_x as lambda expression

```
import javax.swing.*;
```

```
public class inter_fx_Ex4{
public static void main(String args[])
{ inter_f_x ff=x->x*x-1.0;
 double x=1.0;
 String s="x ="+x+"\n function f = "+ff.func(x);
 s+="x ="+x+"\n derivative df/dx = "+ff.dfunc(x);
 double a=0;double b=1.0;
 s+="\n a="+a+"b="+b+" integral of f="+ff.integral(a,b);
 String s1=" indirect referencing and interface example with inner class lambda definition:";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}}
```



Program An example class uses interface inter_f_x as outside class definition

```
 import javax.swing.*;
  class ff1 implements inter_f_x
  {public double func(double x)
  {return x*x-1.0;}
  }
public class inter_fx_Ex1{
public static void main(String args[])
{ ff1 ff=new ff1();
 double x=1.0;
 String s="x ="+x+"\n function f = "+ff.func(x);
 s+="x ="+x+"\n derivative df/dx = "+ff.dfunc(x);
 double a=0;double b=1.0;
 s+="\n a="+a+"b="+b+" integral of f="+ff.integral(a,b);
 String s1=" indirect referencing and interface example with inner class lambda definition:";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}}
```

This concept can be used similarly if the input variable is an array instead of a single variable.

Program An example class defines interface if_xj

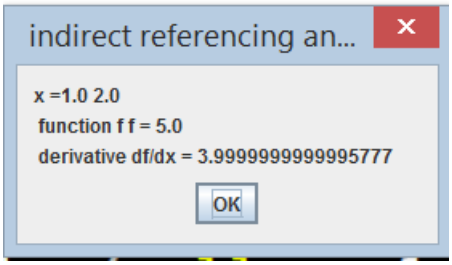
```
@FunctionalInterface
interface if_xj
{
// single function  multi independent variable
// a single value is returned indiced to equation_ref
public double func(double x[]);
default double dfunc(double x[],int x_ref)
{ // derivative of the function with respect to x_ref
double h0=0.0256;
int i,m;
int n=7;
double f1,f2;
double x1[];
```

```
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
f1=func(x1);
f2=func(x2);
T[i][0]=( f1 - f2)/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}

}
```

Program An example class uses interface if_xj as lambda expression

```
import javax.swing.*;
public class ifxj_Ex4 {
public static String array(double x[])
{   String s="";
```

```
            for(int i=0;i<x.length;i++) {s+=x[i]+" ";}
            return s;
}
public static void main(String args[])
{
 if_xj ff=x->x[0]*x[0]+x[1]*x[1];
 double x[]={1.0,2.0};
  String s="x ="+array(x)+"\n function f f = "+ff.func(x);
  s+="\n derivative df/dx = "+ff.dfunc(x,1);
 String s1=" indirect referencing and interface example with inner class lambda definition:";
 JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
}}
```



7 ADVANCED DATA STRUCTURES

14.1 GENERIC METHODS

In the classes and methods so far the type of the input and output variables are defined . Sometimes the code will be shorter if the types of the variable is defined as generic. In the class arrayprint 4 different data type is defined and an overloaded individual method is written for each data type, then the same name can be used to printout each data type.

14.1.1 arrayprint class to print 4 different data type by overloaded printArray methods

```
public class arrayprint
{
public static void printArray(Integer[] x)
{for(int i=0;i<x.length;i++)
  System.out.print(" "+x[i]);
 System.out.println();
}
public static void printArray(Double[] x)
{for(int i=0;i<x.length;i++)
  System.out.print(" "+x[i]);
 System.out.println();
}
public static void printArray(Character[] x)
{for(int i=0;i<x.length;i++)
  System.out.print(" "+x[i]);
 System.out.println();
}
public static void printArray(String[] x)
{for(int i=0;i<x.length;i++)
  System.out.print(" "+x[i]);
 System.out.println();
}
public static void main(String arg[])
{
Integer[] i={1,2,3,4,5,6};
```

```
Double[] d={1.1,2.2,3.3,4.4,5.5,6.6};
Character[] c={'a','b','c','d','f','g'};
String[] s={"ali","veli","49","elli"};
printArray(i);
printArray(d);
printArray(c);
printArray(s);
}

}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" arrayprint1
 1 2 3 4 5 6
 1.1 2.2 3.3 4.4 5.5 6.6
 a b c d f g
 ali veli 49 elli

> Terminated with exit code 0.
```

To make program code shorter array terms

14.1.2 arrayprint1 class to print 4 different data type by genereic printArray method

```
public class arrayprint1
{
public static<E> void printArray( E[] x)
{for(int i=0;i<x.length;i++)
  System.out.print(" "+x[i]);
 System.out.println();
}

public static void main(String arg[])
{
Integer[] i={1,2,3,4,5,6};
Double[] d={1.1,2.2,3.3,4.4,5.5,6.6};
Character[] c={'a','b','c','d','f','g'};
String[] s={"ali","veli","49","elli"};
printArray(i);
printArray(d);
printArray(c);
printArray(s);
}
}
```

The result will be exactly the same. In this second example a general class definition, called generic class, is used. In order to do this class definition is replaced with generic definition **<E>.** The same of general usage could be achived at least for printing function through the class Object. Because all the classes extended from class Object.

```
public class arrayprint2
{
public static void printArray( Object[] x)
{for(int i=0;i<x.length;i++)
  System.out.print(" "+x[i]);
 System.out.println();
}
```

```
public static void main(String arg[])
{
Integer[] i={1,2,3,4,5,6};
Double[] d={1.1,2.2,3.3,4.4,5.5,6.6};
Character[] c={'a','b','c','d','f','g'};
String[] s={"ali","veli","49","elli"};
printArray(i);
printArray(d);
printArray(c);
printArray(s);
}
}
```

14.2 GENERIC CLASSES

The generic class concept can also be applied to create new classes.

14.2.1 Generic matrix class

```
public class matrix<E>
{
public E[][] A;
//constructor methods
public matrix(E[][] Ai)
{   try{
            @SuppressWarnings("unchecked")
            final E[][] A1 =(E[][]) new Object[Ai.length][Ai[0].length];
            A=A1;
   }catch(EmptyMatrixException e){}
    catch(FullMatrixException f){};
 input_matrix(Ai);
}

public matrix(matrix<E> M)
{
            try{
           @SuppressWarnings("unchecked")
           final E[][] A1=(E[][])new Object[M.A.length][M.A[0].length];
           A=A1;
     }catch(EmptyMatrixException e){}
    catch(FullMatrixException f){};
            input_matrix(M.A);
}

public matrix(int n,int m)
{ try{
            @SuppressWarnings("unchecked")
            final E[][] A1=(E[][])new Object[n][m];
            A=A1;
             }catch(EmptyMatrixException e){}
    catch(FullMatrixException f){};
}

public void input_matrix(E[][] Ai)
{   for(int i=0;i<A.length;i++)
            { for(int j=0;j<A[0].length;j++)
              {A[i][j]=Ai[i][j];}
   }
}

public String toString(matrix< E > M)
{ int n=M.A.length;
```

```
  int m=M.A[0].length;
  String s="";
 for(int i=0;i<n;i++)
              { for(int j=0;j<m;j++)
                {s+=M.A[i][j]+" ";}
              s+="\n";
              }
return s;
}

public String toString()
{
  int n=A.length;
  int m=A[0].length;
  String s1="";
  for(int i=0;i<n;i++)
  {  for(int j=0;j<m;j++)
   { s1+=String.format(" %s ",A[i][j]);}
  s1+="\n";
  }
return s1;
} //end of method toString

}//end of class matrix
```

```
public class EmptyMatrixException extends RuntimeException
{private static final long serialVersionUID = 2097498L;
          public EmptyMatrixException()
 {super("Matrix is empty");}
public EmptyMatrixException(String exception)
 {super(exception);}
}
```

```
public class FullMatrixException extends RuntimeException
{private static final long serialVersionUID = 547638L;
          public FullMatrixException()
 {super("Matrix is full");}
public FullMatrixException(String exception)
 {super(exception);}
}
```

14.2.2 Generic matrix test class matrixtest

```
public class matrixtest
{public static void main(String arg[])
{Double[][] a={{1.1,2.2},{3.3,4.4}};
 Integer[][] b={{1,2},{3,4}};
 Character[][] c={{'a','b'},{'c','d'}};
 matrix<Double> A=new matrix<Double>(a);
 matrix<Integer> B=new matrix<Integer>(b);
 matrix<Character> C=new matrix<Character>(c);
 System.out.println(A.toString());
 System.out.println(B.toString());
 System.out.println(C.toString());
}
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" matrixtest
```

```
1.1  2.2
3.3  4.4

1  2
3  4

a  b
c  d
> Terminated with exit code 0.
```

It should be note that array structure is defined as Object an then transfermed to the generic E[][] . Generic classes will be used in advanced data structures such as lists, trees etc.

14.3 USING LAMBDA EXPRESSIONS WITH GENERIC CLASSES

Using of Lambda expressions are introduced at chapter 9 and referenced again in chapter 11 and 12. IN Java 8 several built in Functional interfaces are defined to be specifically used ias Lambda expressions. Some of the most comon interfaces are listed in the table below.

Table 14.3 Important functional interfaces in Java

Interface name	Argument	Returns
Predicate<T>	T	boolean
Consumer<T>	T	void
Function<T,R>	T	R
BiFunction<T,W,R>	(T,W)	R
Supplier<T>	None	T
UnaryOperator<T>	T	T
BinaryOperator<T>	(T,T)	T
Binary Function<T,W,R>	(T,W)	R

Now, let us see utilization of these interfaces with Lambda expressions with generic classes

14.3.1 Predicate Function with Lambda expression usage

```
import java.util.function.*;
public class  Predicate_L
{
  public static void main(String[] arg)
  {
  Predicate<Integer> atLeast5 = x -> x > 5;
  Boolean b=atLeast5.test(2);
  System.out.println(b);
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Predicate_L
false

> Terminated with exit code 0.
```

14.3.2 Consumer Function with Lambda expression usage

```
import java.util.function.*;
public class  Function_L
```

```
{ public static void main(String[] arg)
  { Function<Double,Double> y = x -> {return x*x-2.0*+5;};
    System.out.println(y.apply(2.0));
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Consumer_L
-6.0

> Terminated with exit code 0.
```

14.3.3 Function Function with Lambda expression usage

```
import java.util.function.*;
public class  H9E7
{ public static void main(String[] arg)
  { Function<Double,Double> y = x -> {return x*Math.sin(x);};
    System.out.println(y.apply(2.0));
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\javaw.exe" H9E7
1.8185948536513634

> Terminated with exit code 0.
```

14.3.4 Function Function with Lambda expression usage

```
import java.util.function.*;
public class  Function_L1
{ public static void main(String[] arg)
  { Function<Double,Boolean> y = x -> (x*x-2.0*x+5)>2;
    System.out.println(y.apply(2.0));
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Function_L1
true

> Terminated with exit code 0.
```

14.3.5 BiFunction Function with Lambda expression usage

```
import java.util.function.*;
public class  BiFunction_L
{ public static void main(String[] arg)
  { BiFunction<Double,Double,Boolean> z = (x,y) -> (x*x+y*y)>3;
    System.out.println(z.apply(2.0,3.0));
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" BiFunction_L
```

```
true

> Terminated with exit code 0.
```

14.4 STRINGTOKENIZER CLASS

If a sentence should be divided up to its words, StringTokenizer java library class is ideal fort his task. StringTokenizer class divides A string into a set of String array by using empty space between the words. Methods of the StringTokenizer class:

StringTokenizer(String str) : Constructs a string tokenizer for the specified string.

StringTokenizer(String str, String delim) : Constructs a string tokenizer for the specified string.

StringTokenizer(String str, String delim, boolean returnDelims) :Constructs a string tokenizer for the specified string.

İnt	countTokens() : Calculates the number of times that this tokenizer's nextToken method can be called before it generates an exception.
Boolean	hasMoreElements() : Returns the same value as the hasMoreTokens method.
Boolean	hasMoreTokens() : Tests if there are more tokens available from this tokenizer's string.
Object	nextElement() : Returns the same value as the nextToken method, except that its declared return value is Object rather than String.
String	nextToken() : Returns the next token from this string tokenizer.
String	nextToken(String delim) : Returns the next token in this string tokenizer's string.

14.3.1 StringTokenizerTest class

```
import java.io.*;
import java.util.*;

public class StringTokenizerTest
{
  public static void main(String arg[])
  {
  String s="Once upon a time in a country far far away";
  System.out.println("sentence : "+s);
  StringTokenizer t=new StringTokenizer(s);
  System.out.println("word count : "+t.countTokens());
  int i=0;
  while(t.hasMoreTokens())
  {System.out.println("word index : "+(i++)+" word : "+t.nextToken());}
  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" StringTokenizerTest
sentence : Once upon a time in a country far far away
word count : 10
word index : 0 word : Once
word index : 1 word : upon
word index : 2 word : a
word index : 3 word : time
word index : 4 word : in
word index : 5 word : a
```

```
word index : 6 word : country
word index : 7 word : far
word index : 8 word : far
word index : 9 word : away

> Terminated with exit code 0.
```

Our first example is an application of this class. String s is loaded into a StringTokenizer class object t. Then by using

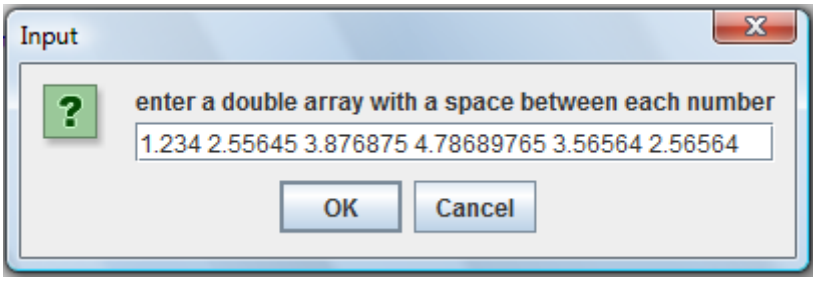
while(**t.hasMoreTokens()**)
{System.out.println("word index : "+(i++)+" word : "+**t.nextToken());}**
}
Token divided up into the words.

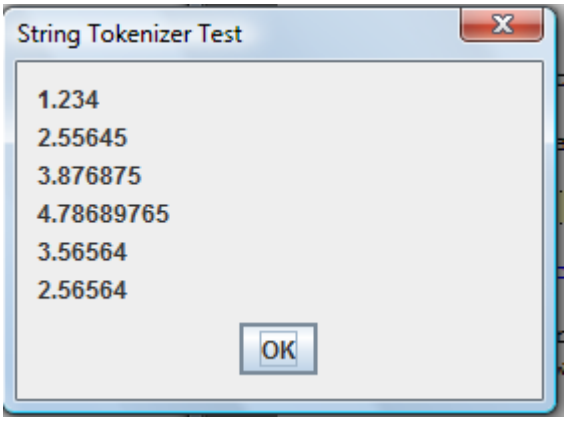
In the second example a double array is read by using JOptionPane class the size of the class is processed by using StringTokenizer class and loaded into array and result is printed by using print method. Otomatic array counting in an array input is one of the most important application of the StringTokenizer class.

14.3.2 StringTokenizerTest1 class

```
import java.io.*;
import java.util.*;
import javax.swing.*;

public class StringTokenizerTest1
{
    public static double[] enterdoublearray(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

    public static String print(double a[])
    {
    String s="";
    int n=a.length;
    int i=0;
    while(i<n) {s+=a[i]+"\n";}
    return s;
    }

  public static void main(String arg[])
```

```
  {
  double x[]=enterdoublearray("enter a double array with a space between each number");
  JOptionPane.showMessageDialog(null,print(x),"String Tokenizer
Test",JOptionPane.PLAIN_MESSAGE);
  }
}
```





Of course you can write your own program to do the same task as well if required

```
import javax.swing.*;
public class problem15C
{
public static int countcharacters(char a,String s)
{
int n=s.length();
int number=0;
for(int i=0;i<n;i++)
{if(s.charAt(i)==a) number++;}
return number;
}
public static String[] word(char a,String s)
{
int nn=countcharacters(a,s);
int n=s.length();
String s1[]=new String[nn+1];
String s2="";
int j=0;
for(int i=0;i<n;i++)
{if(s.charAt(i)==a) {s1[j]=s2;s2="";j++;}
 else              {s2+=s.charAt(i);}
}
```

```
s1[nn]=s2;
return s1;
}

public static String[] word(String s)
{
char a=' ';
return word(a,s);
}

public static String toString(String s[])
{int n=s.length;
 String ss="";
 for(int i=0;i<n;i++)
 {ss+="i = "+i+" "+s[i]+"\n";}
  return ss;
}

public static void main(String arg[])
{
String s="Once upon a time in a country far far away";
System.out.println("number of characters = "+countcharacters(' ',s));
String s1[]=word(s);
System.out.println(toString(s1));
}
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" problem15C
number of characters = 9
i = 0 Once
i = 1 upon
i = 2 a
i = 3 time
i = 4 in
i = 5 a
i = 6 country
i = 7 far
i = 8 far
i = 9 away

> Terminated with exit code 0.
```

14.4 STRINGBUFFER CLASS

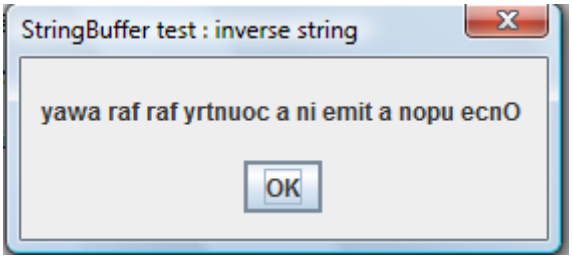
StringBuffer class is sister class of String. It will dynamically add parts to the strings or change the strings. Two examples are given in here in the first example, String is reversed by using StringBuffer class. In the seccond example a word is added up into the string.

14.4.1 StringBuffer class test

```
import java.io.*;
import java.util.*;
```

```
import javax.swing.*;

public class StringBufferTest
{
            public static String reverseIt(String source) {
               int i, len = source.length();
               StringBuffer dest = new StringBuffer(len);
               for (i = (len - 1); i >= 0; i--) {
                  dest.append(source.charAt(i));
               }
               return dest.toString();
            }

  public static void main(String arg[])
  {
  String s=JOptionPane.showInputDialog("enter a string");
  JOptionPane.showMessageDialog(null,reverseIt(s),"StringBuffer test : inverse string",JOptionPane.PLAIN_MESSAGE);
  }
}
```





14.4.2 StringBuffer class test

```
import java.io.*;

class javaCoffee{

    public static void main (String args[])
    {
    StringBuffer sb = new StringBuffer("Drink java Coffee!");
            sb.insert(6, "hot ");
            System.out.println(sb.toString());
    }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" javaCoffee
Drink hot java Coffee!

> Terminated with exit code 0.
```

Constructor Summary

StringBuffer()
Constructs a string buffer with no characters in it and an initial capacity of 16 characters.

StringBuffer(CharSequence seq)
Constructs a string buffer that contains the same characters as the specified CharSequence.

StringBuffer(int capacity)
Constructs a string buffer with no characters in it and the specified initial capacity.

StringBuffer(String str)
Constructs a string buffer initialized to the contents of the specified string.

Methods of the StringBuffer class is given belove:

Method Summary

StringBuffer	append(boolean b) Appends the string representation of the boolean argument to the sequence.
StringBuffer	append(char c) Appends the string representation of the char argument to this sequence.
StringBuffer	append(char[] str) Appends the string representation of the char array argument to this sequence.
StringBuffer	append(char[] str, int offset, int len) Appends the string representation of a subarray of the char array argument to this sequence.
StringBuffer	append(CharSequence s) Appends the specified CharSequence to this sequence.
StringBuffer	append(CharSequence s, int start, int end) Appends a subsequence of the specified CharSequence to this sequence.
StringBuffer	append(double d) Appends the string representation of the double argument to this sequence.
StringBuffer	append(float f) Appends the string representation of the float argument to this sequence.
StringBuffer	append(int i) Appends the string representation of the int argument to this sequence.
StringBuffer	append(long lng) Appends the string representation of the long argument to this sequence.
StringBuffer	append(Object obj) Appends the string representation of the Object argument.
StringBuffer	append(String str) Appends the specified string to this character sequence.
StringBuffer	append(StringBuffer sb) Appends the specified StringBuffer to this sequence.
StringBuffer	appendCodePoint(int codePoint) Appends the string representation of the codePoint argument to this sequence.
İnt	capacity() Returns the current capacity.
Char	charAt(int index) Returns the char value in this sequence at the specified index.
İnt	codePointAt(int index) Returns the character (Unicode code point) at the specified index.
İnt	codePointBefore(int index) Returns the character (Unicode code point) before the specified index.
İnt	codePointCount(int beginIndex, int endIndex) Returns the number of Unicode code points in the specified text range of this sequence.
StringBuffer	delete(int start, int end) Removes the characters in a substring of this sequence.

StringBuffer	deleteCharAt(int index) Removes the char at the specified position in this sequence.
Void	ensureCapacity(int minimumCapacity) Ensures that the capacity is at least equal to the specified minimum.
Void	getChars(int srcBegin, int srcEnd, char[] dst, int dstBegin) Characters are copied from this sequence into the destination character array dst.
İnt	indexOf(String str) Returns the index within this string of the first occurrence of the specified substring.
İnt	indexOf(String str, int fromIndex) Returns the index within this string of the first occurrence of the specified substring, starting at the specified index.
StringBuffer	insert(int offset, boolean b) Inserts the string representation of the boolean argument into this sequence.
StringBuffer	insert(int offset, char c) Inserts the string representation of the char argument into this sequence.
StringBuffer	insert(int offset, char[] str) Inserts the string representation of the char array argument into this sequence.
StringBuffer	insert(int index, char[] str, int offset, int len) Inserts the string representation of a subarray of the str array argument into this sequence.
StringBuffer	insert(int dstOffset, CharSequence s) Inserts the specified CharSequence into this sequence.
StringBuffer	insert(int dstOffset, CharSequence s, int start, int end) Inserts a subsequence of the specified CharSequence into this sequence.
StringBuffer	insert(int offset, double d) Inserts the string representation of the double argument into this sequence.
StringBuffer	insert(int offset, float f) Inserts the string representation of the float argument into this sequence.
StringBuffer	insert(int offset, int i) Inserts the string representation of the second int argument into this sequence.
StringBuffer	insert(int offset, long l) Inserts the string representation of the long argument into this sequence.
StringBuffer	insert(int offset, Object obj) Inserts the string representation of the Object argument into this character sequence.
StringBuffer	insert(int offset, String str) Inserts the string into this character sequence.
İnt	lastIndexOf(String str) Returns the index within this string of the rightmost occurrence of the specified substring.
İnt	lastIndexOf(String str, int fromIndex) Returns the index within this string of the last occurrence of the specified substring.
İnt	length() Returns the length (character count).
İnt	offsetByCodePoints(int index, int codePointOffset) Returns the index within this sequence that is offset from the given index by codePointOffset code points.
StringBuffer	replace(int start, int end, String str) Replaces the characters in a substring of this sequence with characters in the specified String.
StringBuffer	reverse() Causes this character sequence to be replaced by the reverse of the sequence.
Void	setCharAt(int index, char ch) The character at the specified index is set to ch.
Void	setLength(int newLength) Sets the length of the character sequence.

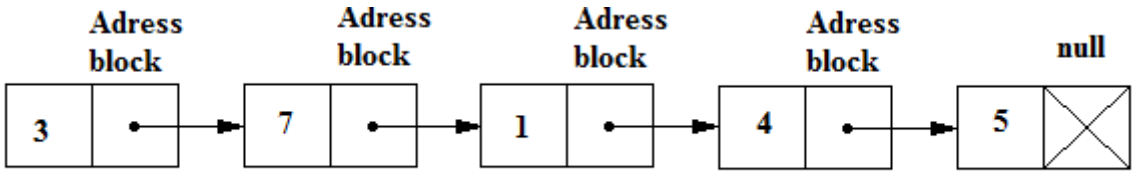
CharSequence	subSequence(int start, int end) Returns a new character sequence that is a subsequence of this sequence.
String	substring(int start) Returns a new String that contains a subsequence of characters currently contained in this character sequence.
String	substring(int start, int end) Returns a new String that contains a subsequence of characters currently contained in this sequence.
String	toString() Returns a string representing the data in this sequence.
Void	trimToSize() Attempts to reduce storage used for the character sequence.

Methods inherited from class java.lang. Object

clone, equals, finalize, getClass, hashCode, notify, notifyAll, wait, wait, wait

14.5 DYNAMIC MEMORY ALLOCATTED CLASSES : LIST, ARRAYLIST, VECTOR, LINKEDLIST,QUE CLASSES

So far if a set of variable is required, array structures are used. In the array structures memory allocation is static, when you declare the array, you also declare the memory adresses. This adresses can not be change during running of the program. In modern programming changing the array size while running the program (Dynamic allocation) is quite important. Dynamic array structures are created by a node structure. Each node carries a number or other type object information plus an adress to the next connection node block. Nodes can be connected in this fashion and the last element in the node simply does not carry any adres so array will be ended in this element. If you would like to add another element to this list, you simply add a new adres to the last element, and a new node can be added into the structure. Or you can add another node to the front and give the adres of the first node. This type of data structures called List in general



Java has built in List structures. But in order to understand the structure, a List structure program is created for you. Note that List class is defined a Linked List. It can be added or removed to the list from either end.

14.5.1 ListNode and List classes as a dynamic data allocation example

```
Class ListNode
{
  Object data;
  ListNode next;
  ListNode(Object o) {this(o,null);}
  ListNode(Object o,ListNode nextNode)
  {
  data=o;
  next=nextNode;
  }
  Object getObject() {return data;}
  ListNode getnext() {return next;}
```

```
}

public class List
{
//List
  private ListNode firstNode;
  private ListNode lastNode;
  private String name;
  public List(String s)
  {
  name=s;
  firstNode=lastNode=null;
  }
  public List(){this("List ");}
  public synchronized void insertAtFront(Object insertItem)
  {
  if( isEmpty() )
      firstNode = lastNode = new ListNode(insertItem);
  else
     firstNode = new ListNode(insertItem,firstNode);
  }
  public synchronized void insertAtBack(Object insertItem)
  {
  if(isEmpty())
      firstNode=lastNode=new ListNode(insertItem);
  else
     lastNode=lastNode.next=new ListNode(insertItem);
  }
  public synchronized Object removeFromFront()
  throws EmptyListException
  {
  Object removeItem=null;
  if(isEmpty())
     throw new EmptyListException(name);
  removeItem=firstNode.data;
  if(firstNode.equals(lastNode))
     firstNode=lastNode=null;
  else
     firstNode=firstNode.next;
  return removeItem;
  }
  public synchronized Object removeFromBack() throws EmptyListException
  {
  Object removeItem=null;
  if(isEmpty())
     throw new EmptyListException(name);
  removeItem=lastNode.data;
  if(firstNode.equals(lastNode))
     firstNode=lastNode=null;
  else
  {
     ListNode current=firstNode;
     while(current.next != lastNode)
        current=current.next;
     lastNode=current;
     current.next=null;
  }
  return removeItem;
  }
  public boolean isEmpty() {return firstNode==null; }
  public void print() throws EmptyListException
  {
```

```
    if(isEmpty())
    {  throw new EmptyListException(name); }
    System.out.print(" "+name+" : ");
    ListNode current=firstNode;
    while(current!=null)
    {
    System.out.print(current.data.toString()+" ");
    current=current.next;
    }
    System.out.println("\n");
  }
}
```

14.5.2 List class test program

```
Public class ListTest
{
  public static void main(String args[])
  {
    List objList=new List();
    Boolean b=new Boolean(true);
    Character c=new Character('$');
    Integer i=new Integer(34567);
    String s=new String("hello");
    objList.insertAtFront(b);
    objList.print();
    objList.insertAtFront(c);
    objList.print();
    objList.insertAtFront(i);
    objList.print();
    objList.insertAtFront(s);
    objList.print();
    Object removedObj;
    try{
      removedObj=objList.removeFromFront();
      System.out.print(removedObj.toString()+" removed");
      objList.print();
      removedObj=objList.removeFromFront();
      System.out.print(removedObj.toString()+" removed");
      objList.print();
      removedObj=objList.removeFromBack();
      System.out.print(removedObj.toString()+" removed");
      objList.print();
      removedObj=objList.removeFromBack();
      System.out.print(removedObj.toString()+" removed");
      objList.print();
      }
      catch(EmptyListException e) {System.out.println("\n"+e.toString());}
  }
}
```

Program output will be:

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" ListTest
List  : true
List  : $ true
List  : 34567 $ true
List  : hello 34567 $ true
hello removed List  : 34567 $ true
34567 removed List  : $ true
true removed List  : $
```

```
$ removed
EmptyListException:  List  List is empty

> Terminated with exit code 0.
```

As you see, a new elements can easily be added or removed from the structure.

In java structures to define Lists are available. ArrayList, LinkedList and Vector classes can be used for the same purpose. Therefore user defined Dynamic array structures are not needed. Example problems are given to show us how this classes can be used. Our first example will be use ArrayList class as dynamically allocated array structure. A Collection class is also used. This class is utilised to sort, reverse, mix type of operations of dynamic array structures.

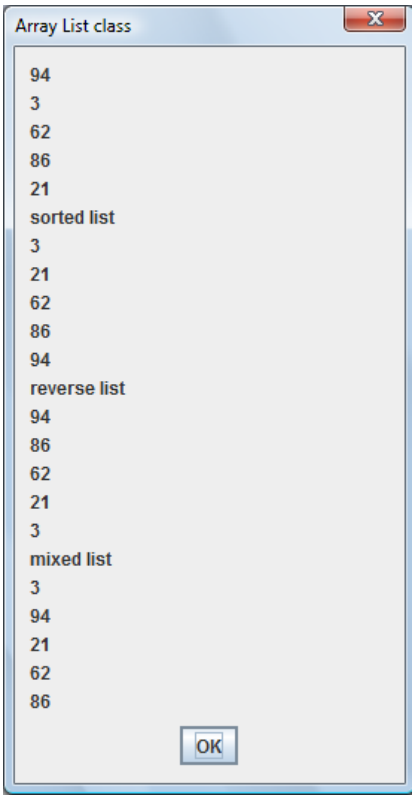
14.5.3 Java library class ArrayList test program

```
import java.util.*;
import javax.swing.*;

public class ArrayList1
{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }

  public static void main(String args[])
  {
  ArrayList<Integer> a=new ArrayList<Integer>();
  for(int i=0;i<5;i++)
  {int sayi=((int)(Math.random()*100));
   a.add(new Integer(sayi));
  }
  String s1="Array List class";
  String s=print(a);
  s+="sorted list\n";
  Collections.sort(a);
  s+=print(a);
  s+="reverse list\n";
  Collections.reverse(a);
  s+=print(a);
   s+="mixed list\n";
  Collections.shuffle(a);
  s+=print(a);
  JOptionPane.showMessageDialog(null,s,s1,
  JOptionPane.PLAIN_MESSAGE);
  }}
```

In the program ArrayList is defined by the constructor **ArrayList<Integer> a=new ArrayList<Integer>();** **<Integer>** java object variable type Integer. ArrayList uses **Generic** type input variables. Any java Object type can be used in ArrayList or other dynamically define java data structures. For example, if it is desired to define String type ArrayList constructor will be :

ArrayList<String> a=new ArrayList<String>( );

A class **Iterator** is used to collect data from the ArrayList class. The main advantages of this class is that it will stop automatically when the array is empty. The result of the program :



In the next program instead of using Iterator class, elements at the ArrayList is reached directly by using **size()** and get() methods of ArrayList class. Result will be similar.

14.5.4 Java library class ArrayListList test program

```
import java.util.*;
import javax.swing.*;

public class ArrayList2
{

  public static void main(String args[])
  {

  ArrayList<Integer> a=new ArrayList<Integer>();
  for(int i=0;i<5;i++)
  {int sayi=((int)(Math.random()*100));
   a.add(new Integer(sayi));
  }
  String s1="Array List class";
  String s="";
  for (int i=0, n=a.size(); i < n; i++) {s+=a.get(i)+"\n";};
  JOptionPane.showMessageDialog(null,s,s1,
  JOptionPane.PLAIN_MESSAGE);
  }
}
```

An example program showing using Lambda expressions together with ArrayList

14.5.4 Java library class ArrayList List test program with lambda expression usage (predicate Function)

```
import java.util.function.Predicate;
import java.util.*;
import javax.swing.*;
```

```
public class ArrayList3 {

   public static void main(String [] arg)  {

      ArrayList<Integer> list =new  ArrayList<Integer>();
      int sayi;
      for(int i=0;i<10;i++)
      {sayi=(int)(20*Math.random());
       list.add(new Integer(sayi));
      }
      System.out.println("Print all numbers:");
      evaluate(list, (n)->true);
      System.out.println("Print no numbers:");
      evaluate(list, (n)->false);
      System.out.println("Print even numbers:");
      evaluate(list, (n)-> n%2 == 0 );
      System.out.println("Print odd numbers:");
      evaluate(list, (n)-> n%2 == 1 );
      System.out.println("Print numbers greater than 5:");
      evaluate(list, (n)-> n > 5 );
   }

   public static void evaluate(ArrayList<Integer> list, Predicate<Integer> predicate) {
      for(Integer n: list)
      {if(predicate.test(n)) {
       System.out.println(n + " ");
      }
      }
   }

}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" ArrayList3
Print all numbers:
17
19
13
8
9
5
18
19
10
5
Print no numbers:
Print even numbers:
8
18
```

```
10
Print odd numbers:
17
19
13
9
5
19
5
Print numbers greater than 5:
17
19
13
8
9
18
19
10

> Terminated with exit code 0.
```

In the next example we define two sets as ArrayList<String>, then calculate union and intersections of the sets

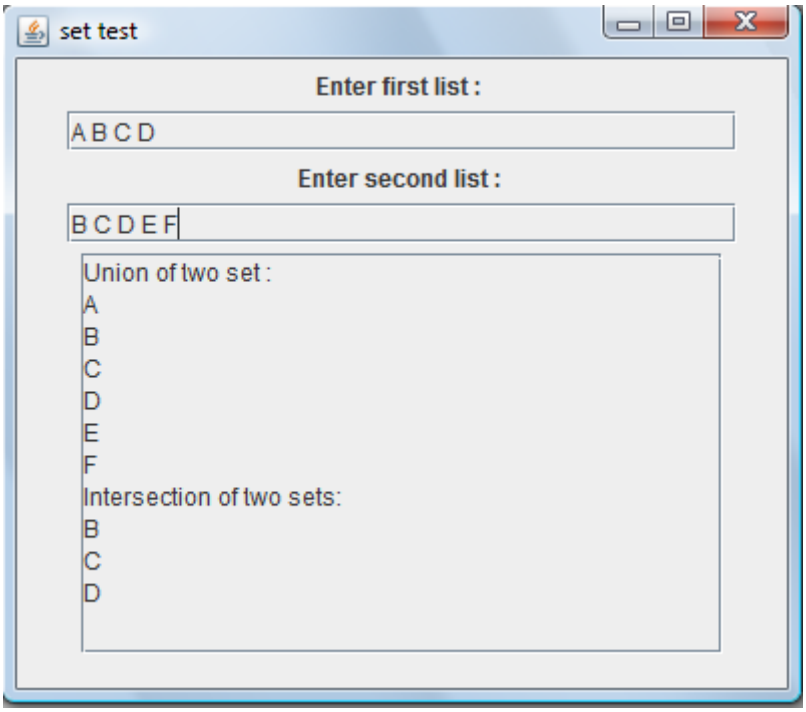
14.5.5 SetA.java a class example using ArrayList

```
 import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;

class SetA
{
   public ArrayList<String> union(ArrayList<String> list1, ArrayList<String> list2)
   {
      ArrayList<String> unionArray = new ArrayList<String>();
      String s1,s2;
      Iterator n1=list1.iterator();
      Iterator n2=list2.iterator();
      while(n1.hasNext())
      {  s1=(String)n1.next();
         unionArray.add(s1);
      }
      while(n2.hasNext())
      {
         s2=(String)n2.next();
         if(!unionArray.contains(s2))
            unionArray.add(s2);
      }
      return unionArray;
   }

   public ArrayList<String> intersection(ArrayList<String> list1, ArrayList<String> list2)
   {
      ArrayList<String> intersectionArray = new ArrayList<String>();
      String s;
      Iterator n = list1.iterator();
```

```
        while(n.hasNext())
        {
           s=(String)n.next();
           if(list2.contains(s))
              intersectionArray.add(s);
        }
        return intersectionArray;
    }
}

public class  SetTest extends JFrame implements ActionListener
{
    SetA    set = new SetA();   // SetA
    JLabel L1, P1, P2;
    JTextField T1,T2;
    JTextArea output;

//  dinamik boyutlu degisken sinifi Vector'u kullanarak
//  iki vector listesi yarat
    ArrayList<String> list1 = new ArrayList<String>();
    ArrayList<String>list2  = new ArrayList<String>();

//  Sonuclari yine vektor cinsinden de§iskenler
//  kullanarak aktar
    ArrayList<String> ansUn;
    ArrayList<String> ansInt;
    Container c;

    public SetTest()
    {
       super("set test");
       c=getContentPane();
       c.setLayout(new FlowLayout());
       P1 = new JLabel("Enter first list : ");
       T1 = new JTextField(30);
       T1.setBackground(c.getBackground());
       P2 = new JLabel("Enter second list : ");
       T2 = new JTextField(30);
       T2.setBackground(c.getBackground());
       output = new JTextArea();
       output.setEditable(false);
       output.setText("");
       output.setBackground(c.getBackground());
       c.add(P1);
       c.add(T1);
       T1.addActionListener(this);
       c.add(P2);
       c.add(T2);
       T2.addActionListener(this);
       JScrollPane jp=new  JScrollPane(output);
       jp.setPreferredSize(new Dimension(320,200));
       c.add(jp);
    }

    public void inputA(String string, ArrayList<String> list)
    {
       StringTokenizer tokens = new StringTokenizer(string);
       while(tokens.hasMoreTokens()) {
          String test = tokens.nextToken();
          list.add(test);
       }
    }
```

```
   public void outputA(ArrayList<String> v)
   {
      Iterator enm = v.iterator();
      while(enm.hasNext())
      {
         String ans = (String)enm.next();
         output.append(ans + "\n");
      }
   }

   public void actionPerformed(ActionEvent e)
   {
       if(e.getSource()==T1)
       {
         String stringToTokenize = T1.getText();
         inputA(stringToTokenize,list1);
       }
      else if(e.getSource()==T2)
      {
         String stringToTokenize = T2.getText();
         inputA(stringToTokenize,list2);
//       calculate union and interface sets for two input set
         ansUn = set.union(list1,list2);
         ansInt = set.intersection(list1,list2);
//       show results in the output field
         output.setText("");
         output.append("Union of two set : \n");
         outputA(ansUn);
         output.append("intersection of two set : \n");
         outputA(ansInt);
      }
   } //end of actionperformed method
    public static void main(String[] args)
    {
    SetTest pencere = new SetTest();
    pencere.setDefaultCloseOperation( JFrame.EXIT_ON_CLOSE );
    pencere.setSize(400,350);
    pencere.setVisible(true);
    }
   }
```

The program output:



Methods of the ArrayList class can be given as Table

Constructor Summary

ArrayList()
Constructs an empty list with an initial capacity of ten.

ArrayList(Collection c)
Constructs a list containing the elements of the specified collection, in the order they are returned by the collection's iterator.

ArrayList(int initialCapacity)
Constructs an empty list with the specified initial capacity.

Method Summary

void	add(int index, Object element) Inserts the specified element at the specified position in this list.
boolean	add(Object o) Appends the specified element to the end of this list.
boolean	addAll(Collection c) Appends all of the elements in the specified Collection to the end of this list, in the order that they are returned by the specified Collection's Iterator.
boolean	addAll(int index, Collection c) Inserts all of the elements in the specified Collection into this list, starting at the specified position.
void	clear() Removes all of the elements from this list.
Object	clone() Returns a shallow copy of this ArrayList instance.

boolean contains(Object elem)
Returns true if this list contains the specified element.

void ensureCapacity(int minCapacity)
Increases the capacity of this ArrayList instance, if necessary, to ensure that it can hold at least the number of elements specified by the minimum capacity argument.

Object get(int index)
Returns the element at the specified position in this list.

int indexOf(Object elem)
Searches for the first occurence of the given argument, testing for equality using the equals method.

boolean isEmpty()
Tests if this list has no elements.

int lastIndexOf(Object elem)
Returns the index of the last occurrence of the specified object in this list.

Object remove(int index)
Removes the element at the specified position in this list.

protected void removeRange(int fromIndex, int toIndex)
Removes from this List all of the elements whose index is between fromIndex, inclusive and toIndex, exclusive.

Object set(int index, Object element)
Replaces the element at the specified position in this list with the specified element.

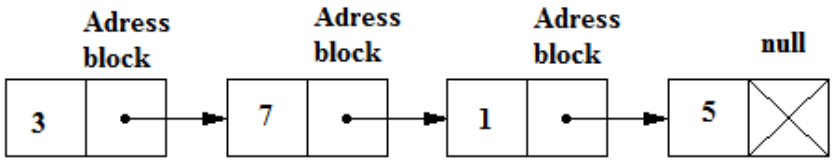
int size()
Returns the number of elements in this list.

Object[] toArray()
Returns an array containing all of the elements in this list in the correct order.

Object[] toArray(Object[] a)
Returns an array containing all of the elements in this list in the correct order; the runtime type of the returned array is that of the specified array.

void trimToSize()
Trims the capacity of this ArrayList instance to be the list's current size.

It should be noted that by using method ***a*.add(*i*, *e*)** a new node can be added into any plave in the dynamic array structure.



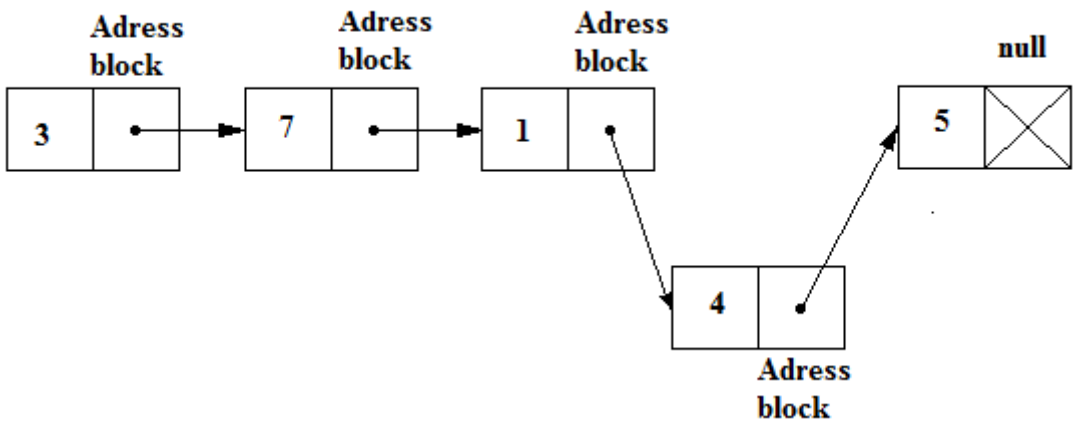


Figure 14.3.1 Adding up a new adress in between two elements in ArrayList or Vector classes

Vector class is very similar to ArrayList class. In fact they can replace each other in utilisation

14.5.6 Java library class Vector test program

```
İmport java.util.*;
import javax.swing.*;

public class Vector1
{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }

  public static void main(String args[])
  {

  Vector<Integer> a=new Vector<Integer>();
  for(int i=0;i<5;i++)
  {int sayi=((int)(Math.random()*100));
   a.add(new Integer(sayi));
  }
  String s1="Array List class";
  String s=print(a);
  s+="sorted list\n";
  Collections.sort(a);
  s+=print(a);
  JOptionPane.showMessageDialog(null,s,s1,
  JOptionPane.PLAIN_MESSAGE);
  }
}
```

A second example in the Vector class

14.5.7 Java library class Vector test program

```
import java.util.*;
import javax.swing.JOptionPane;
```

```
public class Planets1{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }

  public static void main (String args[]) {
    String names[] = {"Mercur", "Venus", "Earth",
      "Mars", "Jupiter", "Saturn", "Uranus",
      "Neptun", "Pluto"};
    Vector<String> Planets = new Vector<String> ();
    for(int i=0;i<names.length;i++)
    { Planets.add(names[i]);}
    String ss=print(Planets);
    JOptionPane.showMessageDialog(null,ss);
  }
}
```



Methods of the Vector class is given in the table belove

Field Summary

protected int	capacityIncrement The amount by which the capacity of the vector is automatically incremented when its size becomes greater than its capacity.
protected int	elementCount The number of valid components in this Vector object.
protected Object[]	elementData The array buffer into which the components of the vector are stored.

Fields inherited from class java.util. AbstractList

modCount

Constructor Summary

Vector()
Constructs an empty vector so that its internal data array has size 10 and its standard capacity increment is zero.

Vector(Collection c)
Constructs a vector containing the elements of the specified collection, in the order they are returned by the collection's iterator.

Vector(int initialCapacity)
Constructs an empty vector with the specified initial capacity and with its capacity increment equal to zero.

Vector(int initialCapacity, int capacityIncrement)
Constructs an empty vector with the specified initial capacity and capacity increment.

Method Summary

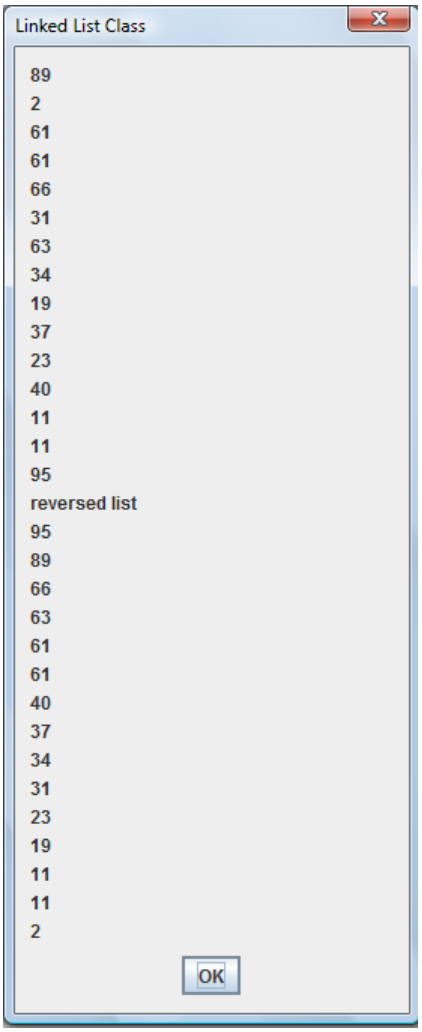
void	add(int index, Object element) Inserts the specified element at the specified position in this Vector.
boolean	add(Object o) Appends the specified element to the end of this Vector.
boolean	addAll(Collection c) Appends all of the elements in the specified Collection to the end of this Vector, in the order that they are returned by the specified Collection's Iterator.
boolean	addAll(int index, Collection c) Inserts all of the elements in in the specified Collection into this Vector at the specified position.
void	addElement(Object obj) Adds the specified component to the end of this vector, increasing its size by one.
int	capacity() Returns the current capacity of this vector.
void	clear() Removes all of the elements from this Vector.
Object	clone() Returns a clone of this vector.
boolean	contains(Object elem) Tests if the specified object is a component in this vector.
boolean	containsAll(Collection c) Returns true if this Vector contains all of the elements in the specified Collection.
void	copyInto(Object[] anArray) Copies the components of this vector into the specified array.
Object	elementAt(int index) Returns the component at the specified index.
Enumeration	elements() Returns an enumeration of the components of this vector.
void	ensureCapacity(int minCapacity) Increases the capacity of this vector, if necessary, to ensure that it can hold at least the number of components specified by the minimum capacity argument.
boolean	equals(Object o) Compares the specified Object with this Vector for equality.
Object	firstElement() Returns the first component (the item at index 0) of this vector.
Object	get(int index) Returns the element at the specified position in this Vector.
int	hashCode() Returns the hash code value for this Vector.
int	indexOf(Object elem) Searches for the first occurence of the given argument, testing for equality using the equals method.
int	indexOf(Object elem, int index) Searches for the first occurence of the given argument, beginning the search at index, and testing for equality using the equals method.

void	insertElementAt(Object obj, int index) Inserts the specified object as a component in this vector at the specified index.
boolean	isEmpty() Tests if this vector has no components.
Object	lastElement() Returns the last component of the vector.
int	lastIndexOf(Object elem) Returns the index of the last occurrence of the specified object in this vector.
int	lastIndexOf(Object elem, int index) Searches backwards for the specified object, starting from the specified index, and returns an index to it.
Object	remove(int index) Removes the element at the specified position in this Vector.
boolean	remove(Object o) Removes the first occurrence of the specified element in this Vector If the Vector does not contain the element, it is unchanged.
boolean	removeAll(Collection c) Removes from this Vector all of its elements that are contained in the specified Collection.
void	removeAllElements() Removes all components from this vector and sets its size to zero.
boolean	removeElement(Object obj) Removes the first (lowest-indexed) occurrence of the argument from this vector.
void	removeElementAt(int index) Deletes the component at the specified index.
protected void	removeRange(int fromIndex, int toIndex) Removes from this List all of the elements whose index is between fromIndex, inclusive and toIndex, exclusive.
boolean	retainAll(Collection c) Retains only the elements in this Vector that are contained in the specified Collection.
Object	set(int index, Object element) Replaces the element at the specified position in this Vector with the specified element.
void	setElementAt(Object obj, int index) Sets the component at the specified index of this vector to be the specified object.
void	setSize(int newSize) Sets the size of this vector.
int	size() Returns the number of components in this vector.
List	subList(int fromIndex, int toIndex) Returns a view of the portion of this List between fromIndex, inclusive, and toIndex, exclusive.
Object[]	toArray() Returns an array containing all of the elements in this Vector in the correct order.
Object[]	toArray(Object[] a) Returns an array containing all of the elements in this Vector in the correct order; the runtime type of the returned array is that of the specified array.
String	toString() Returns a string representation of this Vector, containing the String representation of each element.
void	trimToSize() Trims the capacity of this vector to be the vector's current size.

Utilisation of **LinkedList** is similar. There is one difference at ArrayList and Vector a new element can be added in any point in the array. In the Linked List it can be added only to the either end. Remember that the example program 14.3.1(List) was a LinkedList

14.5.8 Java library class LinkedListList test program

```
import java.util.*;
import javax.swing.*;

public class LinkedList1
{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }

  public static void main(String args[])
  {

  LinkedList<Integer> a=new LinkedList<Integer>();
  for(int i=0;i<15;i++)
  {int sayi=((int)(Math.random()*100));
   a.add(new Integer(sayi));
  }
  String s1="Linked List Class";
  String s=print(a);
  s+="tersten sıraya sokulmuş liste\n";
  Collections.sort(a,Collections.reverseOrder());
  s+=print(a);
  JOptionPane.showMessageDialog(null,s,s1,
  JOptionPane.PLAIN_MESSAGE);
  }
}
```



The methods of the LinkedList is given in the table belove

Constructor Summary

LinkedList()
Constructs an empty list.

LinkedList(Collection<? extends E> c)
Constructs a list containing the elements of the specified collection, in the order they are returned by the collection's iterator.

Method Summary

boolean	add(E o) Appends the specified element to the end of this list.
void	add(int index, E element) Inserts the specified element at the specified position in this list.
boolean	addAll(Collection<? extends E> c) Appends all of the elements in the specified collection to the end of this list, in the order that they are returned by the specified collection's iterator.
boolean	addAll(int index, Collection<? extends E> c) Inserts all of the elements in the specified collection into this list, starting at the specified position.
void	addFirst(E o) Inserts the given element at the beginning of this list.

void	addLast(E o) Appends the given element to the end of this list.
void	clear() Removes all of the elements from this list.
Object	clone() Returns a shallow copy of this LinkedList.
boolean	contains(Object o) Returns true if this list contains the specified element.
E	element() Retrieves, but does not remove, the head (first element) of this list.
E	get(int index) Returns the element at the specified position in this list.
E	getFirst() Returns the first element in this list.
E	getLast() Returns the last element in this list.
int	indexOf(Object o) Returns the index in this list of the first occurrence of the specified element, or -1 if the List does not contain this element.
int	lastIndexOf(Object o) Returns the index in this list of the last occurrence of the specified element, or -1 if the list does not contain this element.
ListIterator<E>	listIterator(int index) Returns a list-iterator of the elements in this list (in proper sequence), starting at the specified position in the list.
boolean	offer(E o) Adds the specified element as the tail (last element) of this list.
E	peek() Retrieves, but does not remove, the head (first element) of this list.
E	poll() Retrieves and removes the head (first element) of this list.
E	remove() Retrieves and removes the head (first element) of this list.
E	remove(int index) Removes the element at the specified position in this list.
boolean	remove(Object o) Removes the first occurrence of the specified element in this list.
E	removeFirst() Removes and returns the first element from this list.
E	removeLast() Removes and returns the last element from this list.
E	set(int index, E element) Replaces the element at the specified position in this list with the specified element.
int	size() Returns the number of elements in this list.
Object[]	toArray() Returns an array containing all of the elements in this list in the correct order.

<T> T[]

toArray(T[] a)

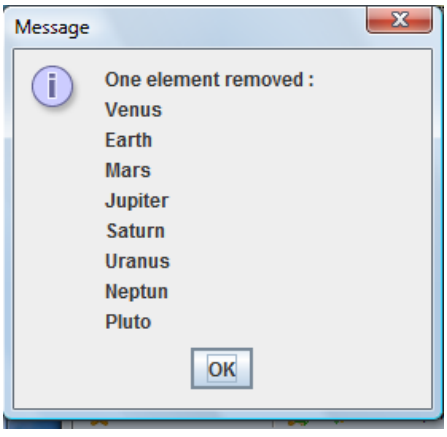
Returns an array containing all of the elements in this list in the correct order; the runtime type of the returned array is that of the specified array.

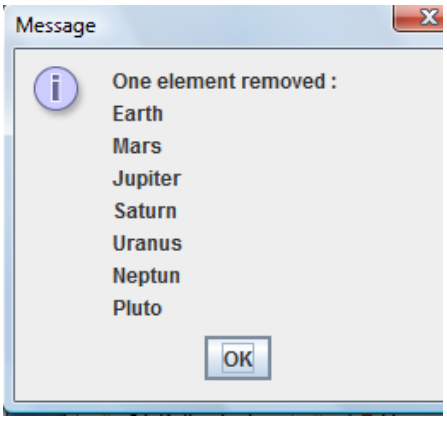
The Que dynamic constructor class **ArrayDeque** is also similar to the previous classes. Eventhough it has methods to add and remove elements to the both sides of the vector, usual interperetation of a que is to add element to the one end and remove the element from the other end, so that elements will remain in a que, and first come will be served first. Here is an example problem:

Program 14.5.9 Planets2.java an example of ArrayDeque class

```
import java.util.*;
import javax.swing.JOptionPane;

public class Planets2{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }

  public static void main (String args[]) {
    String names[] = {"Mercur", "Venus", "Earth",
    "Mars", "Jupiter", "Saturn", "Uranus",
    "Neptun", "Pluto"};
    ArrayDeque<String> Planets = new ArrayDeque<String> ();
    for(int i=0;i<names.length;i++)
    { Planets.add(names[i]);}
    String ss="Complete list : \n"+print(Planets);
    JOptionPane.showMessageDialog(null,ss);
    Planets.removeFirst();
    ss="One element removed :\n"+print(Planets);
    JOptionPane.showMessageDialog(null,ss);
    Planets.removeFirst();
    ss="One element removed :\n"+print(Planets);
    JOptionPane.showMessageDialog(null,ss);
  }
}
```







Methods of the **ArrayDeque** are given in the following table:

Constructor Summary

ArrayDeque()
Constructs an empty array deque with an initial capacity sufficient to hold 16 elements.

ArrayDeque(Collection<? extends E> c)
Constructs a deque containing the elements of the specified collection, in the order they are returned by the collection's iterator.

ArrayDeque(int numElements)
Constructs an empty array deque with an initial capacity sufficient to hold the specified number of elements.

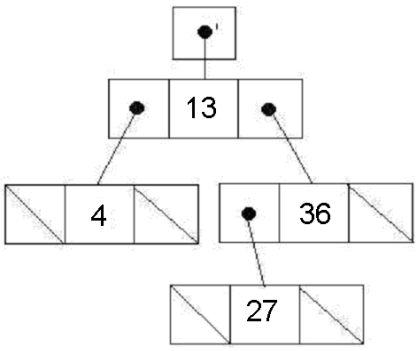
Method Summary

boolean	add(E e) Inserts the specified element at the end of this deque.
void	addFirst(E e) Inserts the specified element at the front of this deque.
void	addLast(E e) Inserts the specified element at the end of this deque.
void	clear() Removes all of the elements from this deque.
ArrayDeque<E>	clone() Returns a copy of this deque.
boolean	contains(Object o) Returns true if this deque contains the specified element.
Iterator<E>	descendingIterator() Returns an iterator over the elements in this deque in reverse sequential order.
E	element() Retrieves, but does not remove, the head of the queue represented by this deque.
E	getFirst() Retrieves, but does not remove, the first element of this deque.
E	getLast() Retrieves, but does not remove, the last element of this deque.
boolean	isEmpty() Returns true if this deque contains no elements.
Iterator<E>	iterator() Returns an iterator over the elements in this deque.
boolean	offer(E e) Inserts the specified element at the end of this deque.
boolean	offerFirst(E e) Inserts the specified element at the front of this deque.
boolean	offerLast(E e) Inserts the specified element at the end of this deque.
E	peek() Retrieves, but does not remove, the head of the queue represented by this deque, or returns null if this deque is empty.
E	peekFirst() Retrieves, but does not remove, the first element of this deque, or returns null if this deque is empty.
E	peekLast() Retrieves, but does not remove, the last element of this deque, or returns null if this deque is empty.
E	poll() Retrieves and removes the head of the queue represented by this deque (in other words, the first element of this deque), or returns null if this deque is empty.
E	pollFirst() Retrieves and removes the first element of this deque, or returns null if this deque is empty.
E	pollLast() Retrieves and removes the last element of this deque, or returns null if this deque is empty.

E	pop() Pops an element from the stack represented by this deque.
void	push(E e) Pushes an element onto the stack represented by this deque.
E	remove() Retrieves and removes the head of the queue represented by this deque.
boolean	remove(Object o) Removes a single instance of the specified element from this deque.
E	removeFirst() Retrieves and removes the first element of this deque.
boolean	removeFirstOccurrence(Object o) Removes the first occurrence of the specified element in this deque (when traversing the deque from head to tail).
E	removeLast() Retrieves and removes the last element of this deque.
boolean	removeLastOccurrence(Object o) Removes the last occurrence of the specified element in this deque (when traversing the deque from head to tail).
int	size() Returns the number of elements in this deque.
Object[]	toArray() Returns an array containing all of the elements in this deque in proper sequence (from first to last element).
<T> T[]	toArray(T[] a) Returns an array containing all of the elements in this deque in proper sequence (from first to last element); the runtime type of the returned array is that of the specified array.

14.6 DYNAMIC ARRAY STRUCTURES : TREES

Trees are a dynamic array structure with one datapoint and two self referencing adres. Data is entered to datapoint, the next data point is entered with a given rule, for example if the next data point is a number and smaller then the number in the node, it will go to the left node. If it is bigger than the node value, it will go to the right node. This way, when tree elements are called back, they will be automaticaly in sorted order. Such a tree structure is shown in the figure.



In order to understand this structure better, a treenode and tree class is created here and tested.

14.6.1 TreeNode and Tree classes

```
class TreeNode
{
  TreeNode left;
```

```
  int data;
  TreeNode right;
  public TreeNode(int d)
  {
  data=d;
  left=right=null;
  }
  public synchronized void enter(int d)
  {
  // not ayni deger iki kere enterilmeye calisilirsa
  // ikinci enteris dikkate alinmiyacaktir
    if(d<data)
    { if(left==null) {left=new TreeNode(d);}
      else        {left.enter(d);}
    }
    else if(d>data)
    { if(right==null) {right=new TreeNode(d);}
      else        {right.enter(d);}
    }
  }
}
public class Tree
{
  private TreeNode kok;
  public Tree() {kok=null;}
  public synchronized void enter(int d)
  {
    if(kok==null) kok=new TreeNode(d);
    else kok.enter(d);
  }
  public void node_left_right_sort(){node_left_right_sorter(kok);}
  public void left_node_right_sort(){left_node_right_sorter(kok);}
  public void left_right_node_sort(){left_right_node_sorter(kok);}
  public void node_right_left_sort(){node_right_left_sorter(kok);}
  public void right_node_left_sort(){right_node_left_sorter(kok);}
  public void right_left_node_sort(){right_left_node_sorter(kok);}
  public void node_left_right_sorter(TreeNode node)
  {
  if(node==null) return;
  System.out.print(node.data+" ");
  node_left_right_sorter(node.left);
  node_left_right_sorter(node.right);
  }
  public void node_right_left_sorter(TreeNode node)
  {
  if(node==null) return;
  System.out.print(node.data+" ");
  node_right_left_sorter(node.right);
  node_right_left_sorter(node.left);
  }
  public void left_node_right_sorter(TreeNode node)
  {
  if(node==null) return;
  left_node_right_sorter(node.left);
  System.out.print(node.data+" ");
  left_node_right_sorter(node.right);
  }
  public void right_node_left_sorter(TreeNode node)
  {
  if(node==null) return;
  right_node_left_sorter(node.right);
  System.out.print(node.data+" ");
```

```
  right_node_left_sorter(node.left);
  }
  public void left_right_node_sorter(TreeNode node)
  {
  if(node==null) return;
  left_node_right_sorter(node.left);
  left_node_right_sorter(node.right);
  System.out.print(node.data+" ");
  }
  public void right_left_node_sorter(TreeNode node)
  {
  if(node==null) return;
  right_node_left_sorter(node.right);
  right_node_left_sorter(node.left);
  System.out.print(node.data+" ");
  }
}
```

14.6.2 TreeTest class

```
public class TreeTest
{
  public static void main(String args[])
  {
  Tree tree=new Tree();
  int number;
  System.out.println("numbers originally entered to the tree");
  for(int i=0;i<=6;i++)
  {
   number=(int)(Math.random()*45)+1;
   System.out.print(number+" ");
   tree.enter(number);
  }
  System.out.println("\nleft tree - node - right tree sort : ");
  tree.left_node_right_sort();
  System.out.println("\nnode -left tree - right tree sort : ");
  tree.node_left_right_sort();
  System.out.println("\nleft tree - right tree - node sort : ");
  tree.left_right_node_sort();
  System.out.println("\nright tree - node - left tree sort : ");
  tree.right_node_left_sort();
  System.out.println("\nnode -right tree - left tree sort : ");
  tree.node_right_left_sort();
  System.out.println("\nright tree - left tree - node sort : ");
  tree.right_left_node_sort();
  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" TreeTest
numbers originally entered to the tree
32 42 24 18 39 4 39
left tree - node - right tree sort :
4 18 24 32 39 42
node -left tree - right tree sort :
32 24 18 4 42 39
left tree - right tree - node sort :
4 18 24 39 42 32
right tree - node - left tree sort :
42 39 32 24 18 4
node -right tree - left tree sort :
```

```
32 42 39 24 18 4
right tree - left tree - node sort :
42 39 24 18 4 32
> Terminated with exit code 0.
```

As a Java Library example TreeSet will be given. As a member of set this class will not take a repeated value. Note that the values entered are the same as the given figure:

14.6.3 TreeSet class example

```
import java.util.*;
import javax.swing.JOptionPane;

public class TreeTest1
{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }
  public static void main(String args[])
  {
  TreeSet<Integer> tree=new TreeSet<Integer>();
  int number;
  System.out.println("numbers originally entered to the tree");
  tree.add(new Integer(13));
  tree.add(new Integer(4));
  tree.add(new Integer(36));
  tree.add(new Integer(27));
  String s=print(tree);
  System.out.print(s);
  }
}
```

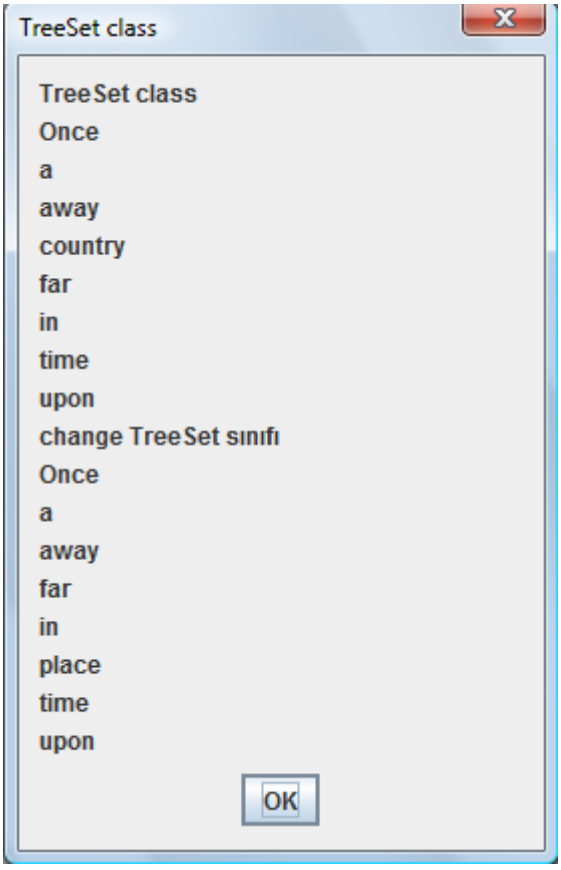
```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" TreeTest1
numbers originally entered to the tree
4
13
27
36
> Terminated with exit code 0.
```

Another example of TreeSet class. In this class

14.6.4 TreeSet class example

```
import java.util.*;
import javax.swing.*;

public class TreeSet2
{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
```

```
    return s;
  }
  public static void main(String args[])
  {
  TreeSet<String> a=new TreeSet<String>();
  String s[]={"Once","upon","a","time","in", "a","country","far","far","away"};
  for(int i=0;i<s.length;i++) a.add(s[i]);
  String s1="TreeSet class\n";
  String s2="TreeSet class";;
  s1+=print(a);
  a.remove("country");
  a.add("place");
  s1+="change TreeSet sınıfı\n";
  s1+=print(a);
  JOptionPane.showMessageDialog(null,s1,s2,
  JOptionPane.PLAIN_MESSAGE);
  }
}
```



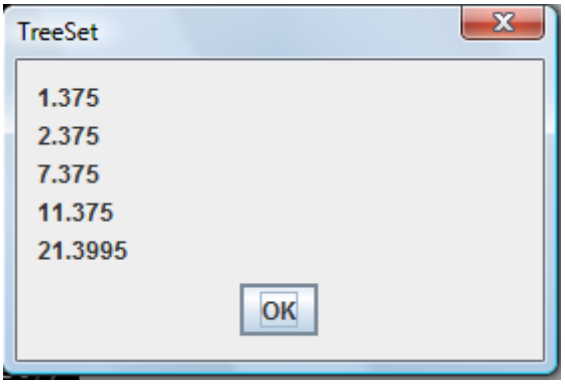
As it is seen from the table repeated values a and far is **listed only once**, not twice

Another Example of TreeSet :

14.6.5 TreeSet class example

```
class TreeSet3 {

        public static void main(String args[])
        {
        TreeSet<Double> a=new TreeSet<Double>();
        Double s1=new Double(1.375);
        Double s2=new Double(2.375);
```

```
        Double s3=new Double(21.3995);
        Double s4=new Double(1.375);
        Double s5=new Double(7.375);
        Double s6=new Double(11.375);
        a.add(s1);
        a.add(s2);
        a.add(s3);
        a.add(s4);
        a.add(s5);
        a.add(s6);
        Iterator i=a.iterator();
        String s="";
        while(i.hasNext())
        {s+=i.next()+"\n";}
        JOptionPane.showMessageDialog(null,s,"TreeSet",
        JOptionPane.PLAIN_MESSAGE);
        }
    }
```



As in the previous example continious value of 1.375 is listed only once in the TreeSet

Methods of TreeSet class is given belove

Constructor Summary

TreeSet()
Constructs a new, empty set, sorted according to the elements' natural order.

TreeSet(Collection c)
Constructs a new set containing the elements in the specified collection, sorted according to the elements' *natural order*.

TreeSet(Comparator c)
Constructs a new, empty set, sorted according to the specified comparator.

TreeSet(SortedSet s)
Constructs a new set containing the same elements as the specified sorted set, sorted according to the same ordering.

Method Summary

boolean	add(Object o) Adds the specified element to this set if it is not already present.
boolean	addAll(Collection c) Adds all of the elements in the specified collection to this set.
Void	clear() Removes all of the elements from this set.
Object	clone() Returns a shallow copy of this TreeSet instance.

Comparator	comparator() Returns the comparator used to order this sorted set, or null if this tree set uses its elements natural ordering.
boolean	contains(Object o) Returns true if this set contains the specified element.
Object	first() Returns the first (lowest) element currently in this sorted set.
SortedSet	headSet(Object toElement) Returns a view of the portion of this set whose elements are strictly less than toElement.
boolean	isEmpty() Returns true if this set contains no elements.
Iterator	iterator() Returns an iterator over the elements in this set.
Object	last() Returns the last (highest) element currently in this sorted set.
boolean	remove(Object o) Removes the specified element from this set if it is present.
İnt	size() Returns the number of elements in this set (its cardinality).
SortedSet	subSet(Object fromElement, Object toElement) Returns a view of the portion of this set whose elements range from fromElement, inclusive, to toElement, exclusive.
SortedSet	tailSet(Object fromElement) Returns a view of the portion of this set whose elements are greater than or equal to fromElement.

Methods inherited from class java.util. AbstractSet

equals, hashCode, removeAll

Methods inherited from class java.util. AbstractCollection

containsAll, retainAll, toArray, toArray, toString

Methods inherited from class java.lang. Object

finalize, getClass, notify, notifyAll, wait, wait, wait

Methods inherited from interface java.util. Set

containsAll, equals, hashCode, removeAll, retainAll, toArray, toArray

The next java library class in the Tree group will be TreeMap class. In this class two argument is required as input . First a simple example is given

14.6.6 TreeMap class example

```
import java.util.*;

public class TreeMapTest{
  public static void main(String[] args) {
    System.out.println("Tree Map Example!\n");
    TreeMap <Integer, String>tMap = new TreeMap<Integer, String>();
    //Addding data to a tree map
    tMap.put(1, "Sunday");
    tMap.put(2, "Monday");
    tMap.put(3, "Tuesday");
    tMap.put(4, "Wednesday");
    tMap.put(5, "Thursday");
    tMap.put(6, "Friday");
    tMap.put(7, "Saturday");
    //Rerieving all keys
    System.out.println("Keys of tree map: " + tMap.keySet());
    //Rerieving all values
```

```
    System.out.println("Values of tree map: " + tMap.values());
    //Rerieving the value from key with key number 5
    System.out.println("Key: 5 value: " + tMap.get(5)+ "\n");
    //Rerieving the First key and its value
    System.out.println("First key: " + tMap.firstKey() + " Value: "
                              + tMap.get(tMap.firstKey()) + "\n");
    //Rerieving the Last key and value
    System.out.println("Last key: " + tMap.lastKey() + " Value: "
                                + tMap.get(tMap.lastKey()) + "\n");
    //Removing the first key and value
    System.out.println("Removing first data: " + tMap.remove(tMap.firstKey()));
    System.out.println("Now the tree map Keys: " + tMap.keySet());
    System.out.println("Now the tree map contain: " + tMap.values() + "\n");
    //Removing the last key and value
    System.out.println("Removing last data: " + tMap.remove(tMap.lastKey()));
    System.out.println("Now the tree map Keys: " + tMap.keySet());
    System.out.println("Now the tree map contain: " + tMap.values());
  }
}
```

As a second example one of the previous example af array structure is taken and changed as a TreeMap class application. Application is a simple library class wich contains books. In the previous example in arrays (chapter 8) books defined as array of books in the library class. In this example books defined as **TreeMap.** Two seperate TreeMap classes are used to arrange it book name and book author base.

14.6.7 book class example

```
public class book
{
String name;
String author;
String clasification;
String language;

public book(String kname,String kauthor,String kclasification,String klanguage)
{name=kname;
 author=kauthor;
 clasification=kclasification;
 language=klanguage;
}

public book(book k)
{name=k.name;
 author=k.author;
 clasification=k.clasification;
language=k.language;
}

public String toString()
{
String s="name = "+name+"\n";
s+="author = "+author+"\n";
s+="book clasification = "+clasification+"\n";
s+="language = "+language+"\n\n";
return s;
}
}
```

14.6.8 fiction class extended from book

```
public class fiction extends book
{
public fiction(String kisim,String kauthor,String klanguage)
{ super(kisim,kauthor,"fiction",klanguage);}
}
```

14.6.9 encyclopedia class extended from book class

```
public class encyclopedia extends book
{
public encyclopedia(String kname,String klanguage)
{ super(kname,"editing board","encyclopedia",klanguage);}
}
```

14.6.10 encyclopedia class extended from book class

```
public class poetry extends book
{
public poetry(String kname,String kauthor,String klanguage)
{ super(kname,kauthor,"poetry",klanguage);}
}
```

14.6.11 library class with TreeMap class

```
import java.util.*;

public class library
{
String name;
int number_of_books;
TreeMap <String, book> book_name = new TreeMap<String, book>();
TreeMap <String, book> author_name = new TreeMap<String, book>();

public library(String kname)
{name=kname;
 number_of_books=0;
}

public void add(book b)
{
book_name.put(b.name,b);
author_name.put(b.author,b);
number_of_books=book_name.size();
}

public void remove(book b)
{
number_of_books--;
book_name.remove(b.name);
author_name.remove(b.author);
number_of_books=book_name.size();
}

public String list_with_name()
{String s=book_name.values().toString();
return s;
}

public String list_with_author()
{String s=author_name.values().toString();
return s;
}

public String get_with_name(String name)
{String s=book_name.get(name).toString();
return s;
}

 public String toString_with_name()
 {            String s="Library name : "+name+"\n";
    s+="Total number of books : "+number_of_books+"\n\n";
    s+=list_with_name();
    return s;
 }

 public String toString_with_author()
 {            String s="Library name : "+name+"\n";
    s+="Total number of books : "+number_of_books+"\n\n";
    s+=list_with_author();
    return s;
 }}
```

14.6.12 booktest1 class with TreeMap class

```
import javax.swing.*;
class booktest1
{
public static void outputwindow(String s)
{
String s1="library class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}
public static void main(String args[])
{
library l=new library("Library of congress");
book magazine=new book("Bilim ve Teknik","TÜBİTAK","magazine","Turkish");
fiction r1=new fiction("It is all quiet in the westen front ","Erich Maria Reamarque","English");
book r2=new book("The God Delusion","Richard Dawkins","fiction","English");
poetry  s1=new poetry("Human lanscapes from my country","Nazım Hikmet","English and Turkish");
encyclopedia a1=new encyclopedia("Encyclopedia Britannica","English");
l.add(magazine);
l.add(r1);
l.add(r2);
l.add(s1));
l.add(a1);
String s=s=l.toString_with_author();
outputwindow(s);
}
}
```

The TreeMap class set methods are as follows:

Nested classes/interfaces inherited from class java.util. AbstractMap

AbstractMap.SimpleEntry<K,V>, AbstractMap.SimpleImmutableEntry<K,V>

Constructor Summary

Constructor and Description

TreeMap()
Constructs a new, empty tree map, using the natural ordering of its keys.

TreeMap(Comparator<? super K> comparator)
Constructs a new, empty tree map, ordered according to the given comparator.

TreeMap(Map<? extends K,? extends V> m)
Constructs a new tree map containing the same mappings as the given map, ordered according to the *natural ordering* of its keys.

TreeMap(SortedMap<K,? extends V> m)
Constructs a new tree map containing the same mappings and using the same ordering as the specified sorted map.

Modifier and Type	Method and Description
Map.Entry<K,V>	ceilingEntry(K key) Returns a key-value mapping associated with the least key greater than or equal to the given key, or null if there is no such key.
K	ceilingKey(K key) Returns the least key greater than or equal to the given key, or null if there is no such key.
Void	clear() Removes all of the mappings from this map.

Object	clone() Returns a shallow copy of this TreeMap instance.
Comparator<? super K>	comparator() Returns the comparator used to order the keys in this map, or null if this map uses the natural ordering of its keys.
boolean	containsKey(Object key) Returns true if this map contains a mapping for the specified key.
boolean	containsValue(Object value) Returns true if this map maps one or more keys to the specified value.
NavigableSet<K>	descendingKeySet() Returns a reverse order NavigableSet view of the keys contained in this map.
NavigableMap<K,V>	descendingMap() Returns a reverse order view of the mappings contained in this map.
Set<Map.Entry<K,V>>	entrySet() Returns a Set view of the mappings contained in this map.
Map.Entry<K,V>	firstEntry() Returns a key-value mapping associated with the least key in this map, or null if the map is empty.
K	firstKey() Returns the first (lowest) key currently in this map.
Map.Entry<K,V>	floorEntry(K key) Returns a key-value mapping associated with the greatest key less than or equal to the given key, or null if there is no such key.
K	floorKey(K key) Returns the greatest key less than or equal to the given key, or null if there is no such key.
V	get(Object key) Returns the value to which the specified key is mapped, or null if this map contains no mapping for the key.
SortedMap<K,V>	headMap(K toKey) Returns a view of the portion of this map whose keys are strictly less than toKey.
NavigableMap<K,V>	headMap(K toKey, boolean inclusive) Returns a view of the portion of this map whose keys are less than (or equal to, if inclusive is true) toKey.
Map.Entry<K,V>	higherEntry(K key) Returns a key-value mapping associated with the least key strictly greater than the given key, or null if there is no such key.
K	higherKey(K key) Returns the least key strictly greater than the given key, or null if there is no such key.
Set<K>	keySet() Returns a Set view of the keys contained in this map.
Map.Entry<K,V>	lastEntry() Returns a key-value mapping associated with the greatest key in this map, or null if the map is empty.
K	lastKey() Returns the last (highest) key currently in this map.
Map.Entry<K,V>	lowerEntry(K key) Returns a key-value mapping associated with the greatest key strictly less than the given key, or null if there is no such key.
K	lowerKey(K key) Returns the greatest key strictly less than the given key, or null if there is no such key.
NavigableSet<K>	navigableKeySet() Returns a NavigableSet view of the keys contained in this map.

Map.Entry<K,V>	pollFirstEntry() Removes and returns a key-value mapping associated with the least key in this map, or null if the map is empty.
Map.Entry<K,V>	pollLastEntry() Removes and returns a key-value mapping associated with the greatest key in this map, or null if the map is empty.
V	put(K key, V value) Associates the specified value with the specified key in this map.
void	putAll(Map<? extends K,? extends V> map) Copies all of the mappings from the specified map to this map.
V	remove(Object key) Removes the mapping for this key from this TreeMap if present.
int	size() Returns the number of key-value mappings in this map.
NavigableMap<K,V>	subMap(K fromKey, boolean fromInclusive, K toKey, boolean toInclusive) Returns a view of the portion of this map whose keys range from fromKey to toKey.
SortedMap<K,V>	subMap(K fromKey, K toKey) Returns a view of the portion of this map whose keys range from fromKey, inclusive, to toKey, exclusive.
SortedMap<K,V>	tailMap(K fromKey) Returns a view of the portion of this map whose keys are greater than or equal to fromKey.
NavigableMap<K,V>	tailMap(K fromKey, boolean inclusive) Returns a view of the portion of this map whose keys are greater than (or equal to, if inclusive is true) fromKey.
Collection<V>	values() Returns a Collection view of the values contained in this map.

Methods inherited from class java.util. AbstractMap

equals, hashCode, isEmpty, toString

Methods inherited from class java.lang. Object

finalize, getClass, notify, notifyAll, wait, wait, wait

Methods inherited from interface java.util. Map

equals, hashCode, isEmpty

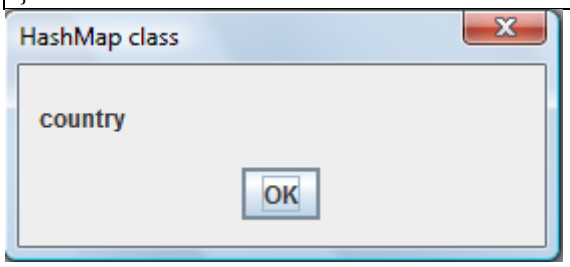
14.7 A DYNAMIC ARRAY STRUCTURES : HASHTABLES

HashTables are data structers that define as a couple of data together. When one of the couple is given the other one can be obtained from this data. As a java class example to hashtables HashMap class will be investigated in here.

14.7.1 HasMap1 class

```
import java.util.*;
import javax.swing.*;

public class HashMap1
{

  public static void main(String args[])
  {
  HashMap<String,String> a=new HashMap<String,String>();
  String s[]={"Once","upon","a","time","in", "a","country","far","far","away"};
  String w[]={"1","2","3","4","5","6","7","8","9","10"};
```

```
  for(int i=0;i<s.length;i++) a.put(w[i],s[i]);
  String s2="HashMap class\n";
  String s1=a.get("7");
  JOptionPane.showMessageDialog(null,s1,s2,
  JOptionPane.PLAIN_MESSAGE);
  }
}
```



A second example is also a derived from a previous example given in arrays (see chapter 8) a vehicle registry class

Program 14.7.2 vehicle class

```
import javax.swing.*;

public class vehicle
{
//valid vehicle types bus,truck,minitruck,minivan,auto
public String vehicletype;
public String brandName;
public double power; //KW olarak
public String color;
public double weight;          //yüksüz net ağırlık
public String registration_number;
//kurucu metodlar

public vehicle()
{vehicletype="not defined";
brandName="not defined";
power=0;
color="not defined";
weight=0;
}

public vehicle(String i_vehicletype,String i_brandName,double i_motor,String i_color,double i_weight,String regno)
{//vehicle type "bus","truck","mini truck","mini bus","auto","tır"
//if not define as "not defined"
if(i_vehicletype.equals("truck") ||
   i_vehicletype.equals("minitruck") ||
   i_vehicletype.equals("bus")||
   i_vehicletype.equals("minibus")||
   i_vehicletype.equals("auto") ||
   i_vehicletype.equals("motorcycle") ||
   i_vehicletype.equals("bicycle") ||
   i_vehicletype.equals("off road vehicle")
   )
{vehicletype=i_vehicletype;}
else
{vehicletype="not defined";}
brandName=i_brandName;
power=i_motor;
color=i_color;
weight=i_weight;
registration_number=regno;
}
public vehicle(vehicle A)
```

```
{//gerekli tanımları yap
vehicletype=A.vehicletype;
brandName=A.brandName;
power=A.power;
color=A.color;
weight=A.weight;
registration_number=A.registration_number;
}
public void define_vehicle(String i_vehicletype,String i_brandName,double i_motor,String i_color,double i_weight,String
regno)
{//vehicle type "bus","truck","mini truck","mini bus","auto","tır"
//if not define as "not defined"
if(i_vehicletype.equals("truck") ||
   i_vehicletype.equals("minitruck") ||
   i_vehicletype.equals("bus")||
   i_vehicletype.equals("minibus")||
   i_vehicletype.equals("auto") ||
   i_vehicletype.equals("motorcycle") ||
   i_vehicletype.equals("bicycle") ||
   i_vehicletype.equals("off road vehicle"))
   {vehicletype=i_vehicletype;}
else
{vehicletype="not defined";}
brandName=i_brandName;
power=i_motor;
color=i_color;
weight=i_weight;
registration_number=regno;
}
public String toString()
{String s1="";
s1+="Vehicle Type : "+vehicletype+"\n";
s1+="Brand Name : "+brandName+"\n";
s1+="Motor Power : "+power+" KW\n";
s1+="Color : "+color+"\n";
s1+="Weight : "+weight+" kg\n";
s1+="Registration number : "+registration_number+"\n\n";
return s1;//gerekli tanımları yap
}
public void outputwindow()
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,toString(),s1, JOptionPane.PLAIN_MESSAGE);
}}
```

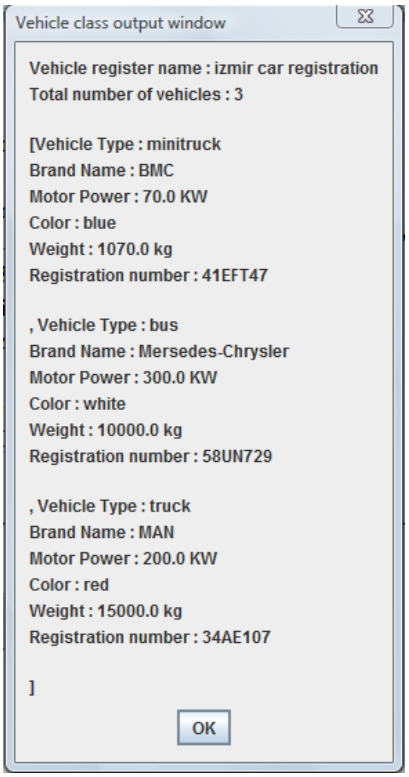
Program 14.7.3 vehicle_register class

```
İmport java.util.*;
public class vehicle_register
{
String name;
int number_of_vehicles;
HashMap<String,vehicle> v;
public vehicle_register(String kname)
{name=kname;
 number_of_vehicles=0;
 v=new HashMap<String,vehicle>();
}
 public String toString()
 {            String s="Vehicle register name : "+name+"\n";
    s+="Total number of vehicles : "+number_of_vehicles+"\n\n";
    s+=list();
```

```
    return s;
 }
public void add(vehicle b)
{
v.put(b.registration_number,b);
number_of_vehicles=v.size();
}
public void remove(vehicle b)
{v.remove(b.registration_number);
 number_of_vehicles=v.size();}
public String list()
{String s=v.values().toString();
return s;}
public String get(String regno)
{String s=v.get(regno).toString();
return s; }
}
```

14.7.4 vehicletest1 class

```
import javax.swing.*;
class vehicletest1
{
public static void outputwindow(String s)
{
String s1="Vehicle class output window";
JOptionPane.showMessageDialog(null,s,s1, JOptionPane.PLAIN_MESSAGE);
}
public static void main(String args[])
{
vehicle v1=new vehicle("truck","MAN",200.0,"red",15000,"34AE107");
vehicle v2=new vehicle("bus","Mersedes-Chrysler",300.0,"white",10000,"58UN729");
vehicle v3=new vehicle("minitruck","BMC",70,"blue",1070,"41EFT47");
vehicle_register vr=new vehicle_register("izmir car registration");
vr.add(v1);
vr.add(v2);
vr.add(v3);
String s=vr.toString()+"\n";
outputwindow(s);
}}
```



Methods of HashMap class is given belove:

Nested Class Summary

Nested classes/interfaces inherited from class java.util. AbstractMap

AbstractMap.SimpleEntry<K,V>, AbstractMap.SimpleImmutableEntry<K,V>

Constructor Summary

HashMap()
Constructs an empty HashMap with the default initial capacity (16) and the default load factor (0.75).

HashMap(int initialCapacity)
Constructs an empty HashMap with the specified initial capacity and the default load factor (0.75).

HashMap(int initialCapacity, float loadFactor)
Constructs an empty HashMap with the specified initial capacity and load factor.

HashMap(Map<? extends K,? extends V> m)
Constructs a new HashMap with the same mappings as the specified Map.

Method Summary

Void	clear() Removes all of the mappings from this map.
Object	clone() Returns a shallow copy of this HashMap instance: the keys and values themselves are not cloned.
Boolean	containsKey(Object key) Returns true if this map contains a mapping for the specified key.
Boolean	containsValue(Object value) Returns true if this map maps one or more keys to the specified value.
Set<Map.Entry<K,V>>	entrySet() Returns a Set view of the mappings contained in this map.

V	get(Object key) Returns the value to which the specified key is mapped, or null if this map contains no mapping for the key.
Boolean	isEmpty() Returns true if this map contains no key-value mappings.
Set<K>	keySet() Returns a Set view of the keys contained in this map.
V	put(K key, V value) Associates the specified value with the specified key in this map.
Void	putAll(Map<? extends K,? extends V> m) Copies all of the mappings from the specified map to this map.
V	remove(Object key) Removes the mapping for the specified key from this map if present.
İnt	size() Returns the number of key-value mappings in this map.
Collection<V>	values() Returns a Collection view of the values contained in this map.

Methods inherited from class java.util. AbstractMap

equals, hashCode, toString

Methods inherited from class java.lang. Object

finalize, getClass, notify, notifyAll, wait, wait, wait

Methods inherited from interface java.util. Map

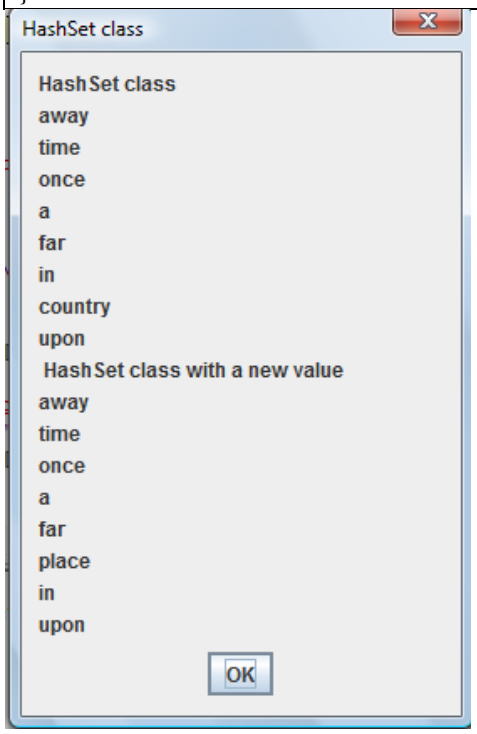
equals, hashCode

Another class to be introduced is HashSet. HashSet class is taken only one value anda a value can not be entered twice into a set (will be assumed the same value)

14.7.5 HasSet1 class

```
import java.util.*;
import javax.swing.*;
import java.util.*;
import javax.swing.*;

public class HashSet1
{
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }
  public static void main(String args[])
  {
  HashSet<String> a=new HashSet<String>();
  String s[]={"once","upon","a","time","in","a","country","far","far","away"};
  for(int i=0;i<s.length;i++) a.add(s[i]);

  String s1="HashSet class\n";
  String s2="HashSet class";;
  s1+=print(a);
  a.remove("country");
```

```
  a.add("place");
  s1+=" HashSet class with a new value\n";
  s1+=print(a);
  JOptionPane.showMessageDialog(null,s1,s2,
  JOptionPane.PLAIN_MESSAGE);
  }
}
```



Methods of HashSet class is given in the table below

Constructor Summary

HashSet()
Constructs a new, empty set; the backing HashMap instance has default initial capacity (16) and load factor (0.75).

HashSet(Collection<? extends E> c)
Constructs a new set containing the elements in the specified collection.

HashSet(int initialCapacity)
Constructs a new, empty set; the backing HashMap instance has the specified initial capacity and default load factor, which is 0.75.

HashSet(int initialCapacity, float loadFactor)
Constructs a new, empty set; the backing HashMap instance has the specified initial capacity and the specified load factor.

Method Summary

boolean	add(E o) Adds the specified element to this set if it is not already present.
void	clear() Removes all of the elements from this set.
Object	clone() Returns a shallow copy of this HashSet instance: the elements themselves are not cloned.
boolean	contains(Object o) Returns true if this set contains the specified element.
boolean	isEmpty() Returns true if this set contains no elements.

Iterator<E>	iterator() Returns an iterator over the elements in this set.
boolean	remove(Object o) Removes the specified element from this set if it is present.
int	size() Returns the number of elements in this set (its cardinality).

Methods inherited from class java.util. AbstractSet

equals, hashCode, removeAll

Methods inherited from class java.util. AbstractCollection

addAll, containsAll, retainAll, toArray, toArray, toString

Methods inherited from class java.lang. Object

finalize, getClass, notify, notifyAll, wait, wait, wait

Methods inherited from interface java.util. Set

addAll, containsAll, equals, hashCode, removeAll, retainAll, toArray, toArray

14.8 DYNAMIC ARRAY STRUCTURES : NAVIGABLESET

NavigableSets are data structers that subsets can be define with easy comparisons. Methods lower, floor, ceiling, and higher return elements respectively less than, less than or equal, greater than or equal, and greater than a given element, returning null if there is no such element. A NavigableSet may be accessed and traversed in either ascending or descending order.

descendingIterator() and descendingSet()

The first interesting navigation methods are the `descendingIterator()` and `descendingSet()` methods.

The `descendingSet()` method returns a `NavigableSet` in which the order of the elements is reversed compared to this one. The returned "view" is backed by the original `NavigableSet`, so changes to the descending set are also reflected in the original set.

Here is a simple example:

NavigableSet reverse=original.descendingSet();

The `descendingIterator()` method allows you to iterate the elements of the `NavigableSet` (which is also a `SortedSet`) in reverse order, without changing the order of the elements internally.

Iterator reverse=original.descendingIterator();

headSet(), tailSet() and subSet()

The `headSet()` method returns a view of the original `NavigableSet` which only contains elements that are "less than" the given element. Here is an example:

```
NavigableSet original=new TreeSet();

Original.add(“1”);

Original.add(“2”);

Original.add(“3”);

//this headset will contain “1” and “2”

SortedSet headset=original.headSet(“3”);

//This headset will contain “1”,”2” and “3” because “inclusive”=true

NavigableSet headset=original.headSet(“3”,true);
```

The `tailSet()` method works the same way, except it returns all elements that are **higher** than the given parameter element.

The `subSet()` allows you to pass two parameters demarcating the boundaries of the view set to return. Here is an example:

```
NavigableSet original = new TreeSet();
original.add("1");
original.add("2");
original.add("3");
original.add("4");
original.add("5");

//this subset will contain "3"
SortedSet    subset  = original.subSet("2", "4");

//this subset will contain "2", "3" and "4" because
//    fromInclusive=true, and toInclusive=true
NavigableSet subset = original.subSet("2", true, "4", true);
```

ceiling(), floor(), higher(), and lower()

The `ceiling()` method returns the least (smallest) element in this set that is greater than or equal to the element passed as parameter to the `ceiling()` method. Here is an example:

```
NavigableSet original = new TreeSet();
original.add("1");
original.add("2");
original.add("3");
```

```
//ceiling will be "2".
Object ceiling = original.ceiling("2");

//floor will be "2".
Object floor = original.floor("2");
```

The `floor()` method does the opposite of `ceiling()`

The `higher()` method returns the least (smallest) element in this set that is greater than (not equal too) the element passed as parameter to the `higher()` method. Here is an example:

```
NavigableSet original = new TreeSet();
original.add("1");
original.add("2");
original.add("3");
//higher will be "3".
Object higher = original.higher("2");
//lower will be "1"
Object lower = original.lower("2");
```

The `lower()` method does the opposite of the `higher()` method.

pollFirst() and pollLast()

The `pollFirst()` method returns and removes the "first" element in the `NavigableSet` or null if the set is empty. The `pollLast()` returns and removes the "last" element in the set or null if the set is empty. "First" means smallest element according to the sort order of the set. "Last" means largest according to teh element sorting order of the set.

Here are two examples:

```
NavigableSet original = new TreeSet();
original.add("1");
original.add("2");
original.add("3");
//first is "1"
Object first = original.pollFirst();
//last is "3"
Object last = original.pollLast();
```

14.8.1 NavigableSetTest

```
import java.util.*;
import java.util.concurrent.*;

public class NavigableSetTest{
  public static void main(String[] args) {
    System.out.println("Navigable set Example!\n");
    NavigableSet <Integer>nSet = new ConcurrentSkipListSet<Integer>();
    nSet.add(10);
    nSet.add(20);
    nSet.add(50);
```

```
    nSet.add(30);
    nSet.add(100);
    nSet.add(80);
    // Returns an iterator over the elements in navigable set, in ascending order.
    Iterator iterator = nSet.iterator();
    System.out.print("Ascending order navigable set: ");
    //Ascending order list
    while (iterator.hasNext()){
      System.out.print(iterator.next() + " ");
    }
    System.out.println();
    //Descending order list
    System.out.println("Descending order navigable set: " + nSet.descendingSet() + "\n");
    //Greater than or equal to the given element
    System.out.println("Least element in Navigable set greater than or equal to 35: " + nSet.ceiling(35));
    //Less than or equal to the given element
    System.out.println("Greatest element in Navigable set less than or equal to 35: " + nSet.floor(35) + "\n");
    //Viewing the portion of navigable set whose elements are strictly less than the given element
    System.out.println("Navigable set whose elements are strictly less than '40': " + nSet.headSet(40));
    //Viewing the portion of navigable set whose elements are greater than or equal to the given element
    System.out.println("Navigable set whose elements are greater than or equal to '40': " + nSet.tailSet(40) + "\n");
    //Removing first element from navigable set
    System.out.println("Remove element: "+nSet.pollFirst());
    //After removing the first element, now get navigable set
    System.out.println("Now navigable set: " + nSet.descendingSet() + "\n");
    //Removing last element from navigable set
    System.out.println("Remove element: " + nSet.pollLast());
    //After removing the last element, now get navigable set
    System.out.println("Now navigable set: " + nSet.descendingSet());
  }
}
```

The result will be;

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NavigableSetTest
Navigable set Example!

Ascending order navigable set: 10 20 30 50 80 100
Descending order navigable set: [100, 80, 50, 30, 20, 10]

Least element in Navigable set greater than or equal to 35: 50
Greatest element in Navigable set less than or equal to 35: 30

Navigable set whose elements are strictly less than '40': [10, 20, 30]
Navigable set whose elements are greater than or equal to '40': [50, 80, 100]

Remove element: 10
Now navigable set: [100, 80, 50, 30, 20]

Remove element: 100
Now navigable set: [80, 50, 30, 20]

> Terminated with exit code 0.
```

The methods of the navigableSet as follows

Method Summary

E	ceiling(E e) Returns the least element in this set greater than or equal to the given element, or null if there is no such element.

Iterator<E>	descendingIterator() Returns an iterator over the elements in this set, in descending order.
NavigableSet<E>	descendingSet() Returns a reverse order view of the elements contained in this set.
E	floor(E e) Returns the greatest element in this set less than or equal to the given element, or null if there is no such element.
SortedSet<E>	headSet(E toElement) Returns a view of the portion of this set whose elements are strictly less than toElement.
NavigableSet<E>	headSet(E toElement, boolean inclusive) Returns a view of the portion of this set whose elements are less than (or equal to, if inclusive is true) toElement.
E	higher(E e) Returns the least element in this set strictly greater than the given element, or null if there is no such element.
Iterator<E>	iterator() Returns an iterator over the elements in this set, in ascending order.
E	lower(E e) Returns the greatest element in this set strictly less than the given element, or null if there is no such element.
E	pollFirst() Retrieves and removes the first (lowest) element, or returns null if this set is empty.
E	pollLast() Retrieves and removes the last (highest) element, or returns null if this set is empty.
NavigableSet<E>	subSet(E fromElement, boolean fromInclusive, E toElement, boolean toInclusive) Returns a view of the portion of this set whose elements range from fromElement to toElement.
SortedSet<E>	subSet(E fromElement, E toElement) Returns a view of the portion of this set whose elements range from fromElement, inclusive, to toElement, exclusive.
SortedSet<E>	tailSet(E fromElement) Returns a view of the portion of this set whose elements are greater than or equal to fromElement.
NavigableSet<E>	tailSet(E fromElement, boolean inclusive) Returns a view of the portion of this set whose elements are greater than (or equal to, if inclusive is true) fromElement.

Methods inherited from interface java.util. SortedSet

comparator, first, last

Methods inherited from interface java.util. Set

add, addAll, clear, contains, containsAll, equals, hashCode, isEmpty, remove, removeAll, retainAll, size, toArray, toArray

SERİES SOLUTIONS OF FUNCTIONS

Digital computers can only calculate addition process, therefore most of the mathematical expressions are calculated by using series. Series solutions of a few functions will be listed in here.

$$\ln(x) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1} \quad [0 < x]$$

$$\exp(x) = \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$\exp(-x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}$$

$$\exp(-x^2) = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

$$\exp(x)(1+x) = \sum_{k=1}^{\infty} \frac{x^k (k+1)}{k!}$$

$$a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$$

$\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x}$ [a>1 and x<0 or 0<a<1 x>0]

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\sin h(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\cos\ h(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cos\ h(x)}$$

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \le 1$$

$$\operatorname{arcsinh}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \qquad x^2 \le 1$$

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)} \qquad x^2 \le 1$$

$$\operatorname{arctanh}(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)} \qquad x^2 \le 1$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{2}{8k+6}\right)$ Bailey, Borwein Plouffe(BBP) formula[90]

$J_n(z) = \left(\frac{1}{2z}\right)\sum_{k=1}^{\infty}(-1)^k \frac{\left(\frac{1}{2z}\right)^{2k}}{k!\,\Gamma(n++1)}$ Bessel function

$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{t/n}$ where $\gamma = 0.5772156649015328\ldots$ is the Euler-Mascheroni constant

$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}$ Bessel function

Now we would like to convert some of these mathematical series formulations into computer programming codes in popular computer languages. As a first series let us look at a simple form as: $f(n) = sum(n) = \sum_{k=1}^{n} k = 1 + 2 + \ldots + n$ can be approximated in computer environment as

Program sum(n) function series solution, Java Version

```
public class sum1
```

```
{ public static double sum(int n)
  {double total=0;
   for(int k=1;k<=n;k++)
   {total+=k;}
   return total;
  }
  public static void main(String arg[])
   {System.out.println(sum(100));}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" sum1
5050.0
> Terminated with exit code 0.
```

f(n) $= average(n) = \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n}(1 + 2 + .. + n)$ can be approximated in computer environment as

Program average(x) function series solution, java Version

```
public class average1
{ public static double average(int n)
  {double total=0;
   for(int k=1;k<=n;k++)
   {total+=k;}
   return total/n;
  }
  public static void main(String arg[])
   {System.out.println(average(100));}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" average1
50.5
> Terminated with exit code 0.
```

f(n) $= average(x_k) = \frac{\sum_{k=0}^{n} x_k}{n}$ can be approximated in computer environment as

Program 1.3.10 average(x_k) function series solution, java Version

```
public class average2
{ public static double average(double x[])
  {double total=0;
   int n=x.length;
   for(int k=0;k<n;k++)
   {total+=x[k];}
   return total/n;
  }
  public static void main(String arg[])
   {   double x[]={1.11,2.21,3.31,4.41,5.51,6.61};
                System.out.println(average(x));}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" average2
3.86
> Terminated with exit code 0.
```

The function f(x) = exp(x) $= \sum_{k=1}^{\infty} \frac{x^k}{k!}$ Or in open form:

f(x) $= e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$ can be approximated in computer environment as

Program exp(x) function series solution, Java Version

```
public class func
{
  public static double exp(double x)
   {double total=0;
    double factorial=1;
    double power=1;
    for(int i=1;i<=200;i++)
```

```
    {total=total+power/factorial;
     power=power*x;
     factorial=factorial*i;
    }
    return total;
   }
   public static void main(String arg[])
   {System.out.println(exp(1));}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" func
2.7182818284590455
> Terminated with exit code 0.
```

Program exp(x) function series solution, OpenOffice Calc (excel) version

	x=		1
n	x^n	n!	total
0	1	1	1
1	1	1	2
2	1	2	2,5
3	1	6	2,66666667
4	1	24	2,70833333
5	1	120	2,71666667
6	1	720	2,71805556
7	1	5040	2,71825397
8	1	40320	2,71827877
9	1	362880	2,71828153
10	1	3628800	2,7182818
11	1	39916800	2,71828183
12	1	479001600	2,71828183
13	1	6227020800	2,71828183
14	1	8,718E+010	2,71828183

function f(x)=ln(x) can be approximated as $\mathbf{ln}(\boldsymbol{x}) = \mathbf{2}\sum_{\boldsymbol{k=1}}^{\infty}\frac{\mathbf{1}}{\mathbf{2k-1}}\left(\frac{\boldsymbol{x-1}}{\boldsymbol{x+1}}\right)^{\mathbf{2k-1}}$ $[\mathbf{0} < \boldsymbol{x}]$

Program ln(x) function series solution, java Version

```
public class func2
{
   public static double ln(double x)
   {double total=0;
    double y=(x-1.0)/(x+1.0);
    double power=y;
    for(int k=1;k<=200;k++)
    {  total=total+power/(2.0*k-1.0);
       power=power*y*y;
    }
    return 2.0*total;
   }
   public static void main(String arg[])
   {   double r=2.7182818284590455;
                System.out.println(ln(r));
   }
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" func2
0.9999999999999999
> Terminated with exit code 0.
```

Some of the functions are defined as combination of more than one function (or series solution). For example power function is defined as:

$$f(x) = a^x = e^{(xln(a))} = \sum_{k=1}^{\infty} \frac{[xln(a)]^k}{k!}$$

Program $f(x) = a^x$ function series solution, Java Version

```
public class func3
{
  public static double ln(double x)
  {double total=0;
   double y=(x-1.0)/(x+1.0);
   double power=y;
   for(int k=1;k<=200;k++)
   {   total=total+power/(2.0*k-1.0);
       power=power*y*y;
   }
   return 2.0*total;
  }
  public static double exp(double x)
  {double total=0;
   double factorial=1;
   double power=1;
   for(int i=1;i<=200;i++)
   {total=total+power/factorial;
    power=power*x;
    factorial=factorial*i;
   }
   return total;
  }
  public static double pow(double a,double x)
  {return exp(x*ln(a));}

  public static void main(String arg[])
  {   System.out.println(pow(2.0,3.0));}}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" func3
7.999999999999997
> Terminated with exit code 0.
```

Program $f(x) = a^x$ function series solution,OpenOffice Calc (excel) version

ln(a)

k		a=	2
		x=	3
		y=(x-1)/(x+1)	0,3333333333
		power	total
	0	0,3333333333	0
	1	0,037037037	0,6666666667
	2	0,0041152263	0,6913580247
	3	0,0004572474	0,6930041152
	4	5,08053E-005	0,6931347573
	5	0,000005645	0,6931460474
	6	6,27225E-007	0,6931470738
	7	6,96917E-008	0,6931471703
	8	7,74352E-009	0,6931471795
	9	8,60392E-010	0,6931471805
	10	9,55991E-011	0,6931471805
	11	1,06221E-011	0,6931471806
	12	1,18024E-012	0,6931471806
	13	1,31137E-013	0,6931471806
	14	1,45708E-014	0,6931471806
	15	1,61898E-015	0,6931471806
	16	1,79887E-016	0,6931471806
	17	1,99874E-017	0,6931471806
	18	2,22082E-018	0,6931471806
	19	2,46758E-019	0,6931471806
	20	2,74175E-020	0,6931471806

exp(z) = exp(x*ln(a))

z=x*ln(a)	2,0794415417

power	factorial	total
1	1	1
2,0794415417	1	3,0794415417
4,3240771253	2	5,2414801043
8,9916656037	6	6,7400910383
18,697642985	24	7,519159496
38,880655555	120	7,8431649589
80,850050329	720	7,9554566955
168,1229533	5040	7,9888144243
349,6018532	40320	7,9974851052
726,9766166	362880	7,9994884579
1511,7053764	3628800	7,9999050434
3143,5029584	39916800	7,9999837947
6536,7306382	479001600	7,9999974413
13592,749236	6227020800	7,9999996242
28265,327427	87178291200	7,9999999484
58776,09604	1,3077E+012	7,9999999934
122221,45576	2,0923E+013	7,9999999992
254152,3724	3,5569E+014	7,9999999999
528495,00108	6,4024E+015	8
1098974,4598	1,2165E+017	8
2285253,145	2,4329E+018	8

Some of the series solutions might not be as clear as a directly given function. For example let us look at the **Russian peasant multiplication problem:**

Take two integer for example 21 and 37 then double the first one(left hand side) and divide by two the second one(right side) ignore remaining in division process and stop when the right hand side reached to 1 and then cross each line where the right side number is even add up remaining left hand side numbers for example. If the given integer numbers to be multiplied is 21 and 17, the process will follow the path given below:

```
21 37
42 18 cross out (18 is even)
84 9
168 4 cross out(4 is even)
336 2 cross out(2 is even)
672 1
then 21+84+672=777
21*37=777
```

Now let us convert this to an algorithmic structure:

Program Russian peasant multiplication, Java version

```
public  class russian
{
public static int multiply(int x1,int x2)
{ int y2=x2;
  int y1=x1;
  int total=0;
  while(y2>=1)
  {  if(y2%2 != 0) total+=y1;
     System.out.println("y1 = "+y1+" y2 = "+y2);
     y1=y1*2;y2=y2/2;
  }
  return total;
}
public static void main(String arg[])
{System.out.println(multiply(21,37));}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" russian
y1 = 21 y2 = 37
y1 = 42 y2 = 18
y1 = 84 y2 = 9
y1 = 168 y2 = 4
y1 = 336 y2 = 2
y1 = 672 y2 = 1
777

> Terminated with exit code 0.
```

Program 1.3.35 Java sin(x) function

```
public class sin
{
   public static double sin(double x)
   {double total=0;
    double factorial=1;
    double power=x;
    int plusminus=1;
    int i=0;
    for(int k=1;k<=200;k++)
    {total=total+power/factorial*plusminus;
     power=power*x*x;
     i=2*k+1;
     factorial=factorial*(i-1)*i;
     plusminus*=-1;
    }
    return total;
   }
   public static void main(String arg[])
   {   double x=Math.PI/3.0;
                System.out.println("sin = "+sin(x)+" Math.sin = "+Math.sin(x));}
   }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" sin
sin = 0.8660254037844385 Math.sin = 0.8660254037844386

> Terminated with exit code 0.
```

Program java version arcsin(x) function

```
public class asin
{
  public static double asin(double x)
  {double total=0;
   double factorial1=1;
   double factorial2=1;
   double power=x;
   double power2=1;
   int i=0;
   for(int k=1;k<=80;k++)
   {total=total+power*factorial1/(power2*factorial2*factorial2)/(2.0*(k-1)+1.0);
    power*=x*x;
    power2*=4;
    i=2*k;
    factorial1=factorial1*(i-1)*i;
    factorial2=factorial2*k;
   }
   return total;
  }
  public static void main(String arg[])
  {   double x=Math.sin(Math.PI/3.0);
              System.out.println("asin = "+asin(x)+" Math.asin = "+Math.asin(x));}
  }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" asin
asin = 1.047197551196467 Math.asin = 1.0471975511965976

> Terminated with exit code 0.
```

Program java version arcsinh(x) function

```
public class asinh
{  public static double asinh1(double x)
   {return Math.log(x + Math.sqrt(x*x + 1.0));}
   public static double asinh(double x)
   {double total=0;
    double factorial1=1;
    double factorial2=1;
    double power=x;
    double power2=1;
    double artieksi=1;
    int i=0;
    for(int k=1;k<=80;k++)
    {total=total+power*factorial1*artieksi/(power2*factorial2*factorial2)/(2.0*(k-1)+1.0);
     power*=x*x;
     power2*=4;
     i=2*k;
     factorial1=factorial1*(i-1)*i;
     factorial2=factorial2*k;
     artieksi*=-1;
    }
    return total;
   }
   public static void main(String arg[])
   {   double x=0.1;
              System.out.println("asinh = "+asinh(x)+"asinh1 = "+asinh1(x));}
   }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" asinh
asinh = 0.09983407889920756asinh1 = 0.09983407889920758

> Terminated with exit code 0.
```

Program java version erf(x) , error function

```
public class erf1
{  public static double erf(double x)
   {double A=2.0/Math.sqrt(Math.PI);
    double total=0;
    double plusminus=1;
```

```
    double power=x;
    double factorial=1;
    for(int n=1;n<200;n++)
    {total+=power*plusminus/factorial/(2.0*(n-1)+1.);
     power*=x*x;
     factorial*=n;
     plusminus*=-1;
     }
    return A*total;
   }
   public static void main(String arg[])
   {   double x=0.1;
                System.out.println("erf = "+erf(x)+"Mathd.erf="+Mathd.erf(x));
   }
   }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" erf1
erf = 0.1124629160182849Mathd.erf=0.11246291599933575

> Terminated with exit code 0.
```

other in structure. If you investigate Taylor series solutions with library based results, you might see small differences in further digits. This is due to approximation and accuracy of computer double numbers.

2 Root finding problems

FINDING THE ROOTS OF EQUATION NON-LINEAR FUNCTIONS WITH ONE VARIABLE

Root finding is one of the most used numerical math methods. For the one independent variable functions, it requires to find x value where the function is zero f(x)=0. If multidimensional (has more than one independent value) functions are involved, it requires to find $x_1,x_2,x_3,\ldots,x_n$ roots of n equation $f_i(x_1,x_2,x_3,\ldots,x_n)=0$, i=1..n
Some of the root finding methods will be investigated. As a starting point, we will start a simpler form of a function y=f(x). In this equation you have one root value..

BISECTION METHOD

In the bisection method root of the functions is searched in a given region $a \leq x \leq b$ If a single root is existed in the given region equation $f(a)f(b) < 0$ (2.1.1) will be satisfied. If this condition is not satisfied, it can be assumed that there is no root in the given region. Of course the reason will be existance of double root in the given region. If the root existed region is divide to two equal parts as

$$p = \frac{a+b}{2} \quad (2.1.2)$$

and function is evaluated in the middle. If

$f(a) * f(p) < 0$ (2.1.3)

root should be located in a-p else it is located in p-b region. Rarely it is also possible that

$f(a) * f(p) = 0$ (2.1.4)

can happen, that will indicate the root is located at point p. The new root region is used in further iterations. In order to use iterative process an exceptible criteria for error limit can be assumed

$\left|\frac{(b-a)}{(b+a)}\right| < \epsilon$ or $|f(p)| < \epsilon$ (2.1.5)

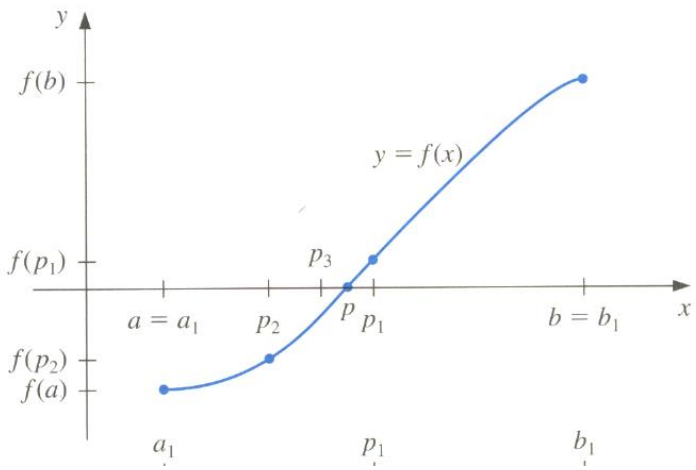


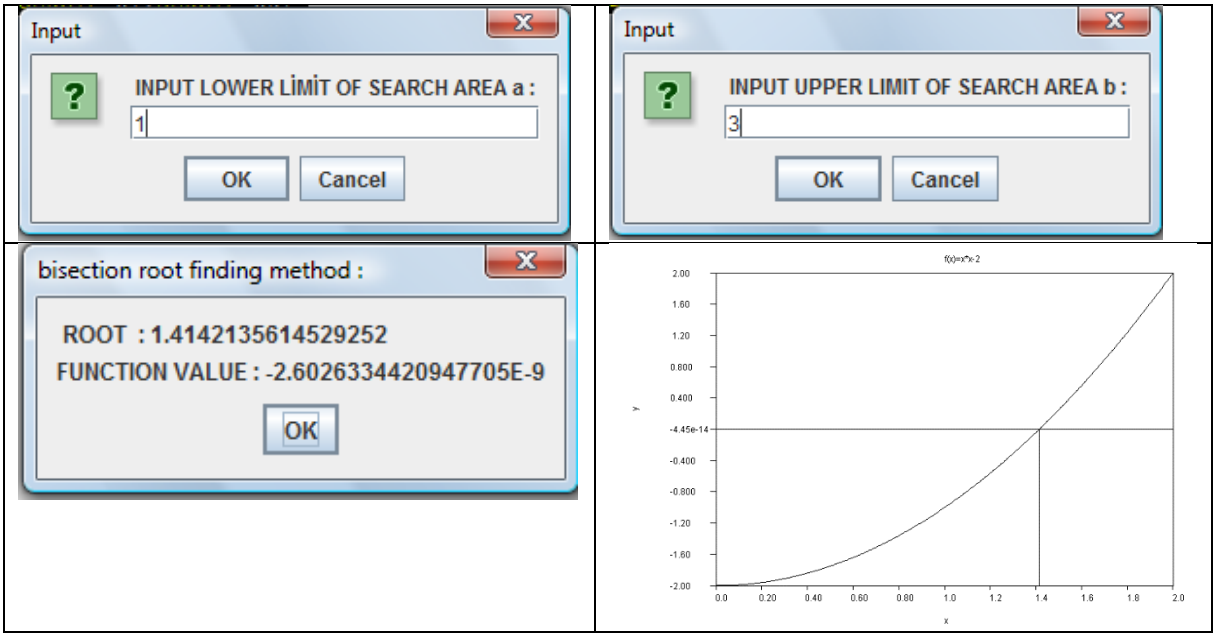
Figure Convergence of bisection Method

An example code is created for the Bisection method. In order to define function as a general definition abstract class f_x defined in the following code. There are also defined methods for the derivatives of to function in this class. The details of the numerical derivatives will be defined in related chapter.

Program Bisection method NA1

```
 import javax.swing.*;

class f1 extends f_x
{
public double func(double x)
{return x*x-2;}
}
public class NA1
{

public static double bisection(f_x f,double xl,double xu)
{
//bisection root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr,fxu;
double xold=0;
int maxit=100,iter=0;
es=0.0000001;
ea=1.1*es;
String s="";
fxl= f.func(xl);
fxu= f.func(xu);
if(fxl*fxu>0) System.out.println("xl="+xu+"xr="+xr+"fxl="+fxl+"fxu="+fxu+"error");
while((ea>es)&&(iter<maxit))
{   xold=xr;
    xr=(xl+xu)/2.0;
    fxr= f.func(xr);
    iter++;
    if(xr!=0)
       { ea=Math.abs((xr-xold)/xr)*100;}
    test= fxl*fxr;
    if(test==0.0)      ea=0;
    else if(test<0.0) {xu=xr;fxu=fxr;}
    else if(test>0)   {xl=xr;fxl=fxr;}
    else              {ea=0;}
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " result might not be valid","MAKSİMUM NUMBER OF ITERATION
WARNING",JOptionPane.WARNING_MESSAGE);
```

```
//JOptionPane.showMessageDialog(null,s,
//     "bisection root finding method : ",JOptionPane.PLAIN_MESSAGE);
  return xr;
}
    public static void main (String args[])
    {
    double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" INPUT LOWER LİMİT OF SEARCH AREA a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" INPUT UPPER LIMIT OF SEARCH AREA b : "));
    double r;
    f1 f=new f1();
    r= bisection(f,a,b);
    JOptionPane.showMessageDialog(null," ROOT  : "+r+"\nFUNCTION VALUE : "+f.func(r),
    "bisection root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Bisection function definitions can also be given as **lambda expressions.** Lambda expressions are defined by using an empty referance class called **interface**. An interface class equivalent of the abstract class f_x is listed below

Program Bisection method NA1a (lambda definitions)

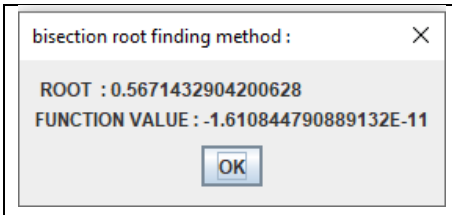
```
 import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;

public class NA1a
{
public static double bisection(if_x f,double xl,double xu)
{
//bisection root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr,fxu;
double xold=0;
int maxit=100,iter=0;
es=0.0000001;
```

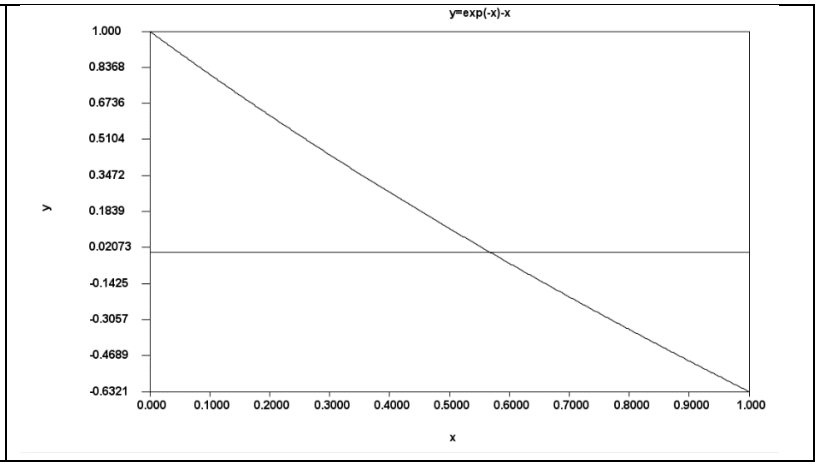
```
ea=1.1*es;
String s="";
fxl= f.func(xl);
fxu= f.func(xu);
if(fxl*fxu>0) System.out.println("xl="+xu+"xr="+xr+"fxl="+fxl+"fxu="+fxu+"error");
while((ea>es)&&(iter<maxit))
{   xold=xr;
    xr=(xl+xu)/2.0;
    fxr= f.func(xr);
    iter++;
    if(xr!=0)
       { ea=Math.abs((xr-xold)/xr)*100;}
    test= fxl*fxr;
    if(test==0.0)      ea=0;
    else if(test<0.0) {xu=xr;fxu=fxr;}
    else if(test>0)   {xl=xr;fxl=fxr;}
    else              {ea=0;}
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " result might not be valid","MAKSİMUM NUMBER OF ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
//JOptionPane.showMessageDialog(null,s,
//     "bisection root finding method : ",JOptionPane.PLAIN_MESSAGE);
  return xr;
}
    public static void main (String args[]) throws IOException
    {
    double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" INPUT LOWER LİMİT OF SEARCH AREA a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" INPUT UPPER LIMIT OF SEARCH AREA b : "));
    double r;
    if_x f1=x->x*x-2.0;
    r= bisection(f1,a,b);
    JOptionPane.showMessageDialog(null," ROOT  : "+r+"\nFUNCTION VALUE : "+f1.func(r),
    "bisection root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    } }
```

Inthe next example root of $f(x) = e^{-x} - 1$ will be found by using bisectionmethod and lambda variable willbe used, but exp(x) function defined inside of the code and bisection method is called from NA1a class above.

```
import javax.swing.*;

public class NA1x
{
   public static double exp(double x)
   {double total=0;
    double factorial=1;
    double power=1;
    for(int i=1;i<=200;i++)
    {total=total+power/factorial;
     power=power*x;
     factorial=factorial*i;
    }
    return total;
   }

     public static void main (String args[])
     {
     double a,b;
     a=Double.parseDouble(JOptionPane.showInputDialog(" INPUT LOWER LİMİT OF SEARCH
AREA a : "));
```

```
    b=Double.parseDouble(JOptionPane.showInputDialog(" INPUT UPPER LIMIT OF SEARCH
AREA b : "));
    double r;
    if_x f1=x->exp(-x)-x;
    r= NA1a.bisection(f1,a,b);
    JOptionPane.showMessageDialog(null," ROOT  : "+r+"\nFUNCTION VALUE : "+f1.func(r),
    "bisection root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    } }
```





Steps of bisection method is shown in detail in the following table(a spreadsheet solution) for function **$f(x)=x^2-2$**

Program Bisection method in excel environment

xl	xu	xr	fxl	fxu	fxr	fxr*fxl
1	2	1.5	-1	2	0.25	-0.25
1	1.5	1.25	-1	0.25	-0.4375	0.4375
1.25	1.5	1.375	-0.4375	0.25	-0.10938	0.047852
1.375	1.5	1.4375	-0.10938	0.25	0.066406	-0.00726
1.375	1.4375	1.40625	-0.10938	0.066406	-0.02246	0.002457
1.40625	1.4375	1.421875	-0.02246	0.066406	0.021729	-0.00049
1.40625	1.421875	1.4140625	-0.02246	0.021729	-0.00043	9.6E-06
1.4140625	1.421875	1.41796875	-0.00043	0.021729	0.010635	-4.5E-06
1.4140625	1.41796875	1.416015625	-0.00043	0.010635	0.0051	-2.2E-06
1.4140625	1.416015625	1.415039063	-0.00043	0.0051	0.002336	-1E-06
1.4140625	1.415039063	1.414550781	-0.00043	0.002336	0.000954	-4.1E-07
1.4140625	1.414550781	1.414306641	-0.00043	0.000954	0.000263	-1.1E-07
1.4140625	1.414306641	1.41418457	-0.00043	0.000263	-8.2E-05	3.5E-08
1.41418457	1.414306641	1.414245605	-8.2E-05	0.000263	9.06E-05	-7.4E-09

In order to see the difference in the code if different languages is used an Octave (Matlab) code of bisection method is given below:

Program Bisection method in Matlab(Octave) environment

```
function xr=bisection(f,xl,xu)
% bisection root finding method
maxit=100;
iter=0;
es=0.0000001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
   xr=(xl+xu)/2.0;
   iter=iter+1;
   if xr~=0 ea=abs((xu-xl)/(xu+xl))*100;end
   fxl= f(xl);
   fxr= f(xr);
   test= fxl*fxr;
   if test == 0.0      ea=0;
   elseif test < 0.0  xu=xr;
   else               xl=xr;
   end
end
```

```
if(iter>=maxit) fprintf('Maximum number of iteration is exceeded result might not be valid',"MAKSİMUM NUMBER OF ITERATION WARNING');
end
end
```

To use the function defined above in Matlab editor, for the function f(x)=x*x-2. Note that function f1 is given as lambda definition also in matlab code

```
>> f1=@(x) x*x-2;
>> y=bisection(f1,0,2)

y =1.414213561452925
```

As it is seen function definition here is similar to lambda expressions in Java language.The same program given in C++ language. **Abstract class f_x** is also given as a part of the code.

Program Bisection method in C++ language (lambda variable version)

```
  #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;
class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};
double bisection(f_x &f,double xl,double xu)
{
//bisection root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr;
int maxit=100;
int iter=0;
es=0.00000001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
{
   xr=(xl+xu)/2.0;
   iter++;
   if(xr!=0)
      { ea=fabs((xu-xl)/(xu+xl))*100.0;}
   fxl= f.func(xl);
   fxr= f.func(xr);
   test= fxl*fxr;
   if(test==0.0)      ea=0;
   else if(test<0.0)  xu=xr;
   else               {xl=xr;}
}
if(iter>=maxit) cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return xr;
}
int main()
{  double a,b,r;
  class f1:public f_x{public:double func(double x){ return x*x-2; };};
  f1 f;
  cout << " INPUT LOWER LIMIT OF SEARCH AREA a : ";
  cin >>a;
  cout << " INPUT UPPER LIMIT OF SEARCH AREA b : ";
  cin >>b;
  r=bisection(f,a,b);
```

```
   cout<<setprecision(20);
   cout<<"r="<<r;
   return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" bisection

D:\okul\SCO1>bisection
 INPUT LOWER LIMIT OF SEARCH AREA a : 1
 INPUT UPPER LIMIT OF SEARCH AREA b : 3
r=1.4142135622678325
> Terminated with exit code 0.
```

Pyton definition of abstract if_x class

Program Interface(abstract class) if_x in python language

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;
class if_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};
double bisection(if_x &f,double a,double b)
  {double b1=1.1*b;
   double r=(a+b)/2.0;
   double eps=1.0e-6;
   int nmax=100;
   int i=0;
   while(abs(f.func(r))>eps && i<nmax)
   {if(f.func(a)*f.func(r)<0) b=r;
    else a=r;
    r=(a+b)/2.0;;
    i++;
   }
   if(i>=nmax) r=bisection(f,a,b1);
   return r;
  }
int main()
{  double a,b,r;
   class f1:public if_x{public:double func(double x){ return x*x-2; };};
   f1 f;
   cout << " INPUT LOWER LIMIT OF SEARCH AREA a : ";
   cin >>a;
   cout << " INPUT UPPER LIMIT OF SEARCH AREA b : ";
   cin >>b;
   r=bisection(f,a,b);
   cout<<setprecision(20);
   cout<<"r="<<r;
   return 0;
}
```

Program Bisection method in python language class f1 is defined as seperate entity

```
 from math import *
from if_x import *;

class f1(if_x):
  def func(self,x):
      y=x*x-2;
      return y;

def bisection(f,xl,xu):
   # bisection root finding method
```

```
  maxit=100
  iter=0
  es=0.0000001
  ea=1.1*es
  s=""
  fxl= f.func(xl)
  fxu= f.func(xu)
  xr=xl
  while ea>es and iter<maxit:
    xold=xr
    xr=(xl+xu)/2.0;
    fxr= f.func(xr);
    iter=iter+1;
    if xr!=0:
      ea=abs((xr-xold)/xr)*100
    test= fxl*fxr;
    if test==0.0 :      ea=0;
    elif test<0.0: xu=xr;fxu=fxr;
    elif test>0:    xl=xr;fxl=fxr;
    else :          ea=0;
    if iter>=maxit: print("Maximum number of iteration is exceeded \n result might not be valid")
  return xr

f=f1();
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=bisection(f,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

Pyton version of the bisection program: Note that function f1 is given as lambda definition but not as a separate class

Program Bisection method in python language

```
from math import *

def bisection(f,xl,xu):
  # bisection root finding method
  maxit=100
  iter=0
  es=0.0000001
  ea=1.1*es
  s=""
  fxl= f(xl)
  fxu= f(xu)
  xr=xl
  while ea>es and iter<maxit:
    xold=xr
    xr=(xl+xu)/2.0;
    fxr= f(xr);
    iter=iter+1;
    if xr!=0:
      ea=abs((xr-xold)/xr)*100
    test= fxl*fxr;
    if test==0.0 :      ea=0;
    elif test<0.0: xu=xr;fxu=fxr;
    elif test>0:    xl=xr;fxl=fxr;
    else :          ea=0;
    if iter>=maxit: print("Maximum number of iteration is exceeded \n result might not be valid")
  return xr

f1 = lambda x: x*x-2;
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=bisection(f1,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r));x = np.arange(0, 3, 0.1);
```

```
runfile('D:/okul/SCO1/bisection.py', wdir='D:/okul/SCO1')
enter lower limit : 1
enter upper limit : 3
ROOT  :  1.4142135614529252
FUNCTION VALUE :  -2.6026334420947705e-09
```

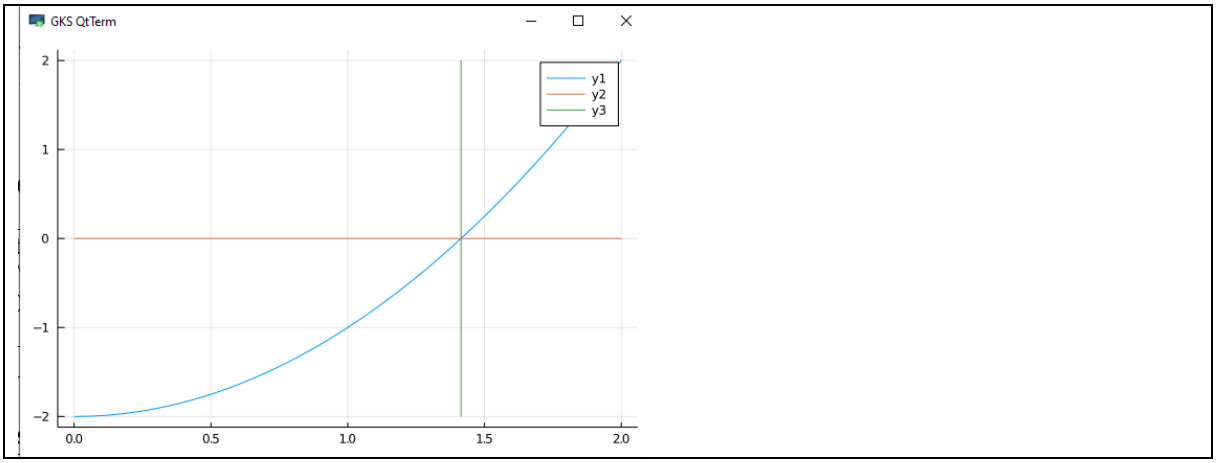
Program Bisection method in python language class f1 is defined internally with lambda variable

```
class f1(if_x):func=lambda self,x:x*x-2;
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=bisection(f,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

Bisection method bisection.jl in julia language

```
 function bisection(f,xl,xu)
test=1.0
xr=0.0
fx=0.0
fxr=0.0
fxu=0.0
xold=0.0
maxit=100
iter=0
es=0.00001
global ea=1.5*es
s=""
fxl= f(xl)
fxu= f(xu)
while ea>es && iter<maxit
   xold=xr
   xr=(xl+xu)/2.0
   fxr= f(xr)
   iter=iter+1
   ea=abs(fxr)
   test= fxl*fxr
   if(test==0.0)
     ea=0.0
   elseif(test<0.0)
     xu=xr
     fxu=fxr
   elseif(test>0)
     xl=xr
     fxl=fxr
   else
      ea=0.0
   end
end # while
if(iter>=maxit)
  print("Maximum number of iteration is exceeded \nresult might not be valid")
end
   return xr
end

using Plots
f(x)=x*x-2.0
a=0.0
b=2.0
x=bisection(f,a,b)
print("x = $x")
f1(x)=0.0
x1=[x,x]
y1=[-2.0,2.0]
p=plot([f, f1],0.0:0.01:2.0)
plot!(p,x1,y1)
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" bisection.jl
x = 1.414215087890625
> Terminated with exit code 0.
```



PROBLEM: Following equation is given:

$$f(x) = -xJ_1(x) - BiJ_0(x) = 0$$

Where $J_0(x)$ and $J_1(x)$ first and second order Bessel functions

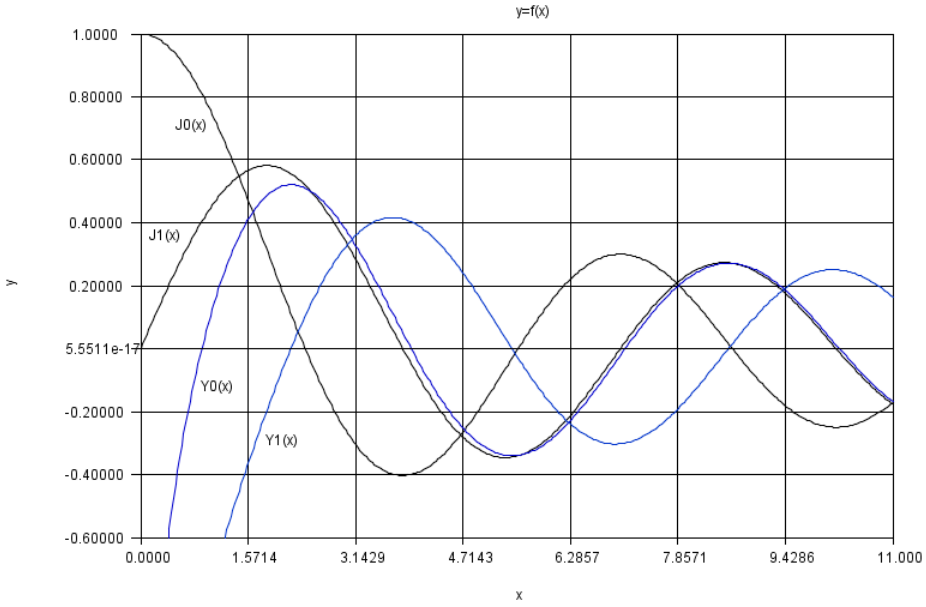
Bessel function $J_\nu(z)$ can be defined by using the following series:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{2\kappa}}{k!\,\Gamma(\nu+k+1)}$$

Where gamma function defined as

$$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

Where γ=0.57721566490153286... is the Euler-Mascheroni constant. The Bessel function $Y_\nu(z)$ is given by:



For Bi=0.1 Find the multiple roots of the given equation, Julia version

```
function bisection(f,a,b)
    b1=1.1*b
    r=(a+b)/2.0
    fr=f(r)
    fa=f(a)
    eps=1.0e-6
    nmax=100
    i=1
    while abs(fr)>eps && i<nmax
```

```
        if fa*fr<0
            b=r
        else
            a=r
            fa=fr
        end
        r=(a+b)/2.0
        fr=f(r)
        i=i+1
    end
    if i>=nmax
      r=bisection(f,a,b1)
    end
    return r
end


function enlarge(f,x0,dx)
    # enlarge region untill a root is existed
    x1=x0
    x2=x1+dx
    NTRY=200
    FACTOR=1.001
    j=0
    if x1 == x2
      print("input variables are wrong")
    end
    f1=f(x1)
    f2=f(x2)
    for j=1:NTRY+1
        if f1*f2 < 0.0
            break
        else
            x2=x2+dx
            f2=f(x2)
        end
    end
    return x2
end


using SpecialFunctions
Bi=0.1
f(x)=x*besselj1(x)-Bi*besselj0(x)
global x0=0.0
global dx=0.001
for n=1:20
    x1=enlarge(f,x0,dx)
    #print("x1 = $x1")
    ksi=bisection(f,x0,x1)
    global x0=ksi+dx
    ff=f(ksi)
    println("Bi = $Bi n = $n x = $ksi f = $ff")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" bisection1.jl
Bi = 0.1 n = 1 x = 0.44168142409737154 f = -1.5856229017285894e-7
Bi = 0.1 n = 2 x = 3.85771028940425 f = -5.972990407415724e-7
Bi = 0.1 n = 3 x = 7.029825369592585 f = 2.8627071236089763e-7
Bi = 0.1 n = 4 x = 10.183292774584878 f = -5.330633232672466e-7
Bi = 0.1 n = 5 x = 13.331195186460318 f = 2.2249596180398679e-7
Bi = 0.1 n = 6 x = 16.476700564557774 f = -9.5064303932621e-7
Bi = 0.1 n = 7 x = 19.620955665797776 f = -1.9191641553728545e-7
Bi = 0.1 n = 8 x = 22.7644775014558 f = 3.185612201840604e-7
Bi = 0.1 n = 9 x = 25.907532057621463 f = -7.29784770465286e-7
Bi = 0.1 n = 10 x = 29.050270873930266 f = 6.881174789922367e-7
Bi = 0.1 n = 11 x = 32.19278651300452 f = 7.975333959336223e-7
Bi = 0.1 n = 12 x = 35.335137694139746 f = 2.7911729158849874e-8
Bi = 0.1 n = 13 x = 38.47736544533629 f = 9.803157654426908e-7
Bi = 0.1 n = 14 x = 41.61949703048528 f = -1.6925118888518054e-7
Bi = 0.1 n = 15 x = 44.76155293117235 f = -9.534498347195386e-7
Bi = 0.1 n = 16 x = 47.90354857359345 f = -6.401504185019752e-7
Bi = 0.1 n = 17 x = 51.04549425511897 f = -1.7453986554671896e-9
Bi = 0.1 n = 18 x = 54.18739917737421 f = -3.4937047258161436e-7
Bi = 0.1 n = 19 x = 57.32926974134999 f = -1.8724345816900634e-7
```

```
Bi = 0.1 n = 20 x = 60.47111154822296 f = 2.581444502602892e-9

> Terminated with exit code 0.
```

The same program is given with user defined J bessel function:

```
# Bessel function J
function J(v,x)
JJ=0
a=1.0
b=0.0
fact=1
if  x<=20
            x1=0.0
            x2=0.0
            x3=0.0
            x4=0.0
            x5=1
            lna=0.0
            x1=v*log(0.5*x);
            lna1=log(0.25*x*x);
            for k=1:100
                x2=lna
                x3=loggamma(k)
                x4=loggamma(v+k)
                        b=x1+x2-x3-x4;
                        b=x5*exp(b);
                        lna+=lna1;
                        x5*=-1;
                        JJ+=b;
            end
elseif v==0 && x==0
  JJ=1.0
else #x>30
  JJ=sqrt(2.0/pi/x)*cos(x-0.5*v*pi-0.25*pi)
end
return JJ;
end # function



function bisection(f,a,b)
   b1=1.1*b
   r=(a+b)/2.0
   fr=f(r)
   fa=f(a)
   eps=1.0e-6
   nmax=100
   i=1
   while abs(fr)>eps && i<nmax
      if fa*fr<0
         b=r
      else
         a=r
         fa=fr
      end
      r=(a+b)/2.0
      fr=f(r)
      i=i+1
   end
   if i>=nmax
     r=bisection(f,a,b1)
   end
   return r
end


function enlarge(f,x0,dx)
   # enlarge region untill a root is existed
   x1=x0
   x2=x1+dx
   NTRY=200
   FACTOR=1.001
   j=0
   if x1 == x2
```

```
      print("input variables are wrong")
   end
   f1=f(x1)
   f2=f(x2)
   for j=1:NTRY+1
      if f1*f2 < 0.0
         break
      else
         x2=x2+dx
         f2=f(x2)
      end
   end
   return x2
end

using SpecialFunctions
Bi=0.1
f(x)=x*J(1,x)-Bi*J(0,x)
global x0=0.0
global dx=0.001
for n=1:20
   x1=enlarge(f,x0,dx)
   #print("x1 = $x1")
   ksi=bisection(f,x0,x1)
   global x0=ksi+dx
   ff=f(ksi)
   println("Bi = $Bi n = $n x = $ksi f = $ff")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" bisection3.jl
Bi = 0.1 n = 1 x = 0.44168142409737154 f = -1.5856229017285894e-7
Bi = 0.1 n = 2 x = 3.85771028940425 f = -5.972990452032811e-7
Bi = 0.1 n = 3 x = 7.029825369592585 f = 2.862704970921226e-7
Bi = 0.1 n = 4 x = 10.183292774584878 f = -5.330583249078813e-7
Bi = 0.1 n = 5 x = 13.331195186460318 f = 2.22557060280959e-7
Bi = 0.1 n = 6 x = 16.476700564557774 f = -9.280136813206996e-7
Bi = 0.1 n = 7 x = 19.620955665797776 f = 5.308665928481504e-7
Bi = 0.1 n = 8 x = 22.780936100092504 f = 9.348724384315721e-7
Bi = 0.1 n = 9 x = 25.921997117309406 f = 6.854885873860961e-8
Bi = 0.1 n = 10 x = 29.0631727331714 f = 3.4206956301199853e-7
Bi = 0.1 n = 11 x = 32.204429688792445 f = -7.399280716711976e-7
Bi = 0.1 n = 12 x = 35.34574648462637 f = 2.630303112139548e-7
Bi = 0.1 n = 13 x = 38.4871082430234 f = -1.496648539909412e-7
Bi = 0.1 n = 14 x = 41.628504916998224 f = -3.162006329150008e-7
Bi = 0.1 n = 15 x = 44.76992896512047 f = 6.961250890134829e-8
Bi = 0.1 n = 16 x = 47.91137518640969 f = -1.943174872936354e-7
Bi = 0.1 n = 17 x = 51.052839435342094 f = 3.540394542290992e-7
Bi = 0.1 n = 18 x = 54.19431840018331 f = 4.93537519649298e-7
Bi = 0.1 n = 19 x = 57.33580996009236 f = -4.643536838065676e-7
Bi = 0.1 n = 20 x = 60.4773121109086 f = -1.1275709955842206e-7

> Terminated with exit code 0.
```

Julia built-in bisection method:

```
Pkg.add("Roots")
```

```
using Roots
f(x)=x*x-2.0
x=find_zero(f,(0.0,3.0),Bisection())
print("x = $x")
```

```
---------- Capture Output ----------
> "D:\co\Julia\bin\julia.exe" bisection_j.jl
x = 1.4142135623730951
> Terminated with exit code 0.
```

fzero (similar to matlab-octave) is another built-in root finding method for Julia (in this case of upper and lower bound it uses bisection method) , Julia code

```
using Roots
f(x)=x*x-2.0
x=fzero(f,1.0,3.0)
print("x = $x")
```

```
---------- Capture Output ----------
> "D:\co\Julia\bin\julia.exe" newton_raphson_j.jl
x = 1.4142135623730951
```

```
> Terminated with exit code 0.
```

```
julia> using Roots
julia> f(x)=x*x-2
f (generic function with 1 method)
julia> x=fzero(f,1.0,3.0)
1.4142135623730951
```

Built-in function **fzeros (calculates muliple roots), Julia code**

```
using Roots
using SpecialFunctions
Bi=0.1
f(x)=x*besselj1(x)-Bi*besselj0(x)
x=fzeros(f,0.0,61.0)
print("x = $x")
```

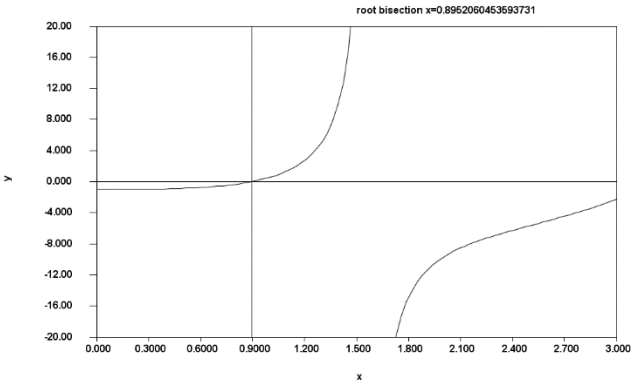
```
---------- Capture Output ----------
> "D:\co\Julia\bin\julia.exe" bessel_rootsj.jl
x = [0.4416817828748415, 3.8577099051034023, 7.029825233917619, 10.183292564960094, 13.331195110029427,
16.476700270892078, 19.620955720117923, 22.764477585157593, 25.907532237354236, 29.050271033965462, 32.19278633681301,
35.33513770002532, 38.477365247246496, 41.61949699760189, 44.76155310979392, 47.90354845766682, 51.04549425542516,
54.187399117887956, 57.32926977234529, 60.47111154863902]
> Terminated with exit code 0.
```

From program in different programming languages you might notice that numerical analysis algorithms are almost independent of computer languages, and beside in all popular languages the code is very similar to each other.

Example Problem:

Use Bisection Method to Find the smallest positive root of $y = f(x) = x^2 \tan(x) - 1$, Java code

```
import java.io.*;
class PlotT01
{
public static double bisection(if_x f,double a,double b)
  {double b1=1.1*b;
   double r=(a+b)/2.0;
   double eps=1.0e-8;
   int nmax=100;
   int i=0;
   while(Math.abs(f.func(r))>eps && i<nmax)
   {if(f.func(a)*f.func(r)<0) b=r;
    else a=r;
    r=(a+b)/2.0;;
    i++;
   }
   if(i>=nmax) r=bisection(f,a,b1);
   return r;
  }
public static void main (String args[]) throws java.io.IOException
{  double ymin=-20;
   double ymax=20;
   double xmin=0;
   double xmax=3;
   int N=100;
   if_x f1=x->x*x*Math.tan(x)-1.0;
   if_x f2=x->0.0;
   Plot p=new Plot(f1,xmin,xmax,N);
   p.addFunction(f2,xmin,xmax,N);
   double x=bisection(f1,xmin,xmax);
   double xx[]={x,x};
   double yy[]={ymin,ymax};
   p.addData(xx,yy);
   p.setPlabel("root bisection x="+x);
   p.setYMinMax(ymin,ymax);
   p.plot();
}
}
```



FALSE POSITION METHOD (REGULA FALSI)

In bisection method search region was devided by two in each time. It is caused a relatively long interpolation period. In order to improve interpolation time, two points can be joined with a line and root of the line can be taken as the next iteration point .

$$p = b - f(b)\frac{(a-b)}{f(a)-f(b)} = \frac{bf(a)-af(b)}{f(a)-f(b)} \quad (2.2.1)$$

Region selection process is same as bisection process. If f(a)*f(p) < 0 root should be located in a-p else it is located in p-b region. The new root region is used in further iterations. In order to use iterative process an exceptible criteria for error limit can be assumed

$$\left|\frac{(b-a)}{(b+a)}\right| < \epsilon \quad \text{or} \quad |f(p)| < \epsilon \quad (2.2.2)$$

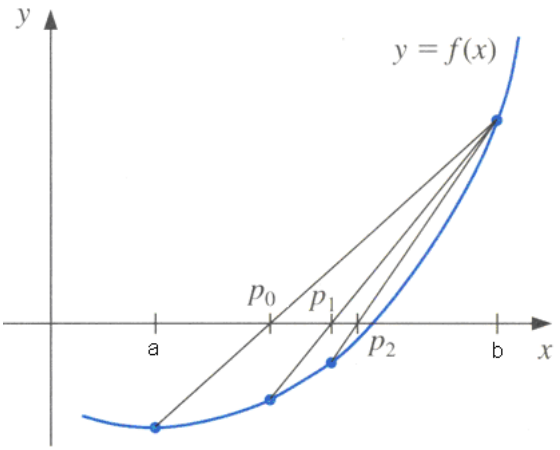
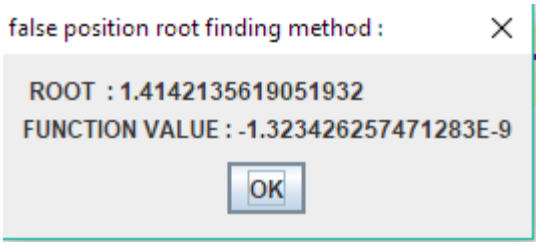


Figure 2.2 Convergence of the False position Method

Program False position (Regula falsi) NA2

```
import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;

class f1 extends f_x
{
public double func(double x)
{ return x*x-2;}        }

public class NA2
{

public static double false_position(f_x f,double xl,double xu)
{
```

```
//false position root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr,fxu;
double xold=0;
int maxit=100,iter=0;
es=0.0000001;
ea=1.1*es;
String s="";
fxl= f.func(xl);
fxu= f.func(xu);
if(fxl*fxu>0) System.out.println("xl="+xu+"xr="+xr+"fxl="+fxl+"fxu="+fxu+"error");
while((ea>es)&&(iter<maxit))
{   xold=xr;
   xr=xu-fxu*(xl-xu)/(fxl-fxu);
   fxr= f.func(xr);
   iter++;
   if(xr!=0)
      { ea=Math.abs((xr-xold)/xr)*100;}
   test= fxl*fxr;
   if(test==0.0)     ea=0;
   else if(test<0.0) {xu=xr;fxu=fxr;}
   else if(test>0)   {xl=xr;fxl=fxr;}
   else              {ea=0;}
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
  " result might not be valid","MAKSIMUM NUMBER OF ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
  return xr;
}
    public static void main (String args[]) throws IOException
    {
    double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" INPUT LOWER LİMİT OF SEARCH AREA a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" INPUT UPPER LIMIT OF SEARCH AREA b : "));
    double r;
    f1 f=new f1();
    r= false_position(f,a,b);
    JOptionPane.showMessageDialog(null," ROOT  : "+r+"\nFUNCTION VALUE : "+f.func(r),
    "false position root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Plot of the iterations through this example is shown in the plot.

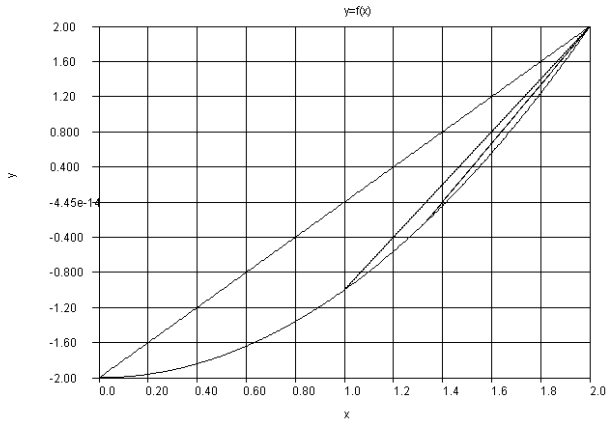


Figure Convergence of the False position Method example program function $f(x)=x^2-2$

Steps of False position method is shown in detail in the following open Office calc (excel) program for function **$f(x)=x^2-2$**

Program 2.2-2 False position (Regula falsi) open Office calc (excel)

xl	xu	xr	fxl	fxu	fxr	fxr*fxl
1	2	1.333333333	-1	2	-0.22222	0.222222
1.333333333	2	1.4	-0.22222	2	-0.04	0.008889
1.4	2	1.411764706	-0.04	2	-0.00692	0.000277
1.411764706	2	1.413793103	-0.00692	2	-0.00119	8.23E-06
1.413793103	2	1.414141414	-0.00119	2	-0.0002	2.43E-07
1.414141414	2	1.414201183	-0.0002	2	-3.5E-05	7.14E-09
1.414201183	2	1.414211438	-3.5E-05	2	-6E-06	2.1E-10
1.414211438	2	1.414213198	-6E-06	2	-1E-06	6.19E-12
1.414213198	2	1.4142135	-1E-06	2	-1.8E-07	1.82E-13
1.4142135	2	1.414213552	-1.8E-07	2	-3E-08	5.37E-15
1.414213552	2	1.414213561	-3E-08	2	-5.2E-09	1.58E-16
1.414213561	2	1.414213562	-5.2E-09	2	-8.9E-10	4.65E-18
1.414213562	2	1.414213562	-8.9E-10	2	-1.5E-10	1.37E-19
1.414213562	2	1.414213562	-1.5E-10	2	-2.6E-11	4.03E-21
1.414213562	2	1.414213562	-2.6E-11	2	-4.5E-12	1.19E-22
1.414213562	2	1.414213562	-4.5E-12	2	-7.7E-13	3.49E-24
1.414213562	2	1.414213562	-7.7E-13	2	-1.3E-13	1.03E-25
1.414213562	2	1.414213562	-1.3E-13	2	-2.3E-14	3.04E-27
1.414213562	2	1.414213562	-2.3E-14	2	-4.2E-15	9.65E-29

As it is seen from the plot and table for iteration steps 2.0 is always stays and root search goes on between this two points. Even if solution is close to first point iterations did not cut from the left side of the search area. It may cause a long number of iteration steps.

Program False position (Regula falsi) in C++

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

double false_position(f_x &f,double xa,double xu)
```

```
{
//Modified False position root finding
double test;
double p=0;
double es,ea;
double fa,fp,fu;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
int ia=0,iu=0;
fa=f.func(xa);
fu=f.func(xu);
p=xu;
double xpold;
while(iter<maxit && ea>es)
{
xpold=p;
p=xu-fu*(xa-xu)/(fa-fu);
fp=f.func(p);
iter++;
if(p!=0) ea=fabs((p-xpold)/p)*100.0;
test=fa*fp;
if(test<0) {xu=p;fu=f.func(xu);iu=0;ia++;if(ia>=2) fa/=2.0;}
else if(test>0) {xa=p;fa=f.func(xa);ia=0;iu+=1;if(iu>=2)fu/=2.0;}
else ea=0;
}
if(iter>=maxit) std::cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return p;
}

int main()
{  double a,b,r;
   class f1:public f_x{public:double func(double x){ return x*x-2; };};
   f1 f;
   cout << " INPUT LOWER LIMIT OF SEARCH AREA a : ";
   cin >>a;
   cout << " INPUT UPPER LIMIT OF SEARCH AREA b : ";
   cin >>b;
   r=false_position(f,a,b);
   cout <<setprecision(20)<<"r="<<r;
   return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" false_position

D:\okul\SCO1>false_position
 INPUT LOWER LIMIT OF SEARCH AREA a : 1
 INPUT UPPER LIMIT OF SEARCH AREA b : 3
r=1.4142135623730951
> Terminated with exit code 0.
```

I will try the same code in Python language now

Program False position (Regula falsi) false_position.py python code

```
 from math import *

def false_position(f,xl,xu):
   # bisection root finding method
   maxit=100
   iter=0
   es=0.0000001
   ea=1.1*es
   s=""
   fxl= f(xl)
   fxu= f(xu)
   xr=xl
   while ea>es and iter<maxit:
      xold=xr
      xr=xu-fxu*(xl-xu)/(fxl-fxu);
      fxr= f(xr);
      iter=iter+1;
      if xr!=0:
         ea=abs((xr-xold)/xr)*100
```

```
    test= fxl*fxr;
    if test==0.0 :      ea=0;
    elif test<0.0: xu=xr;fxu=fxr;
    elif test>0:    xl=xr;fxl=fxr;
    else :          ea=0;
    if iter>=maxit: print("Maximum number of iteration is exceeded \n result might not be valid")
  return xr

f1 = lambda x: x*x-2;
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=false_position(f1,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

```
---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" false_position.py
enter lower limit : 1
enter upper limit : 3
 ROOT  :  1.4142135617510516
FUNCTION VALUE :  -1.759404621992644e-09

> Terminated with exit code 0.
```

Program False position (Regula falsi) fortran 90 code

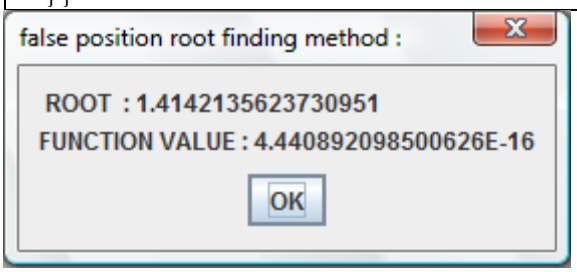
```
program false_position
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) a,b
a=1.0
b=2.15
print *,'false position(',a,',',b,') = ',false(a,b)

CONTAINS
  FUNCTION func(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x-2.0
  END FUNCTION func

  FUNCTION false(a,b)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) false,a,b,b1,r,eps,fa,fb,fr
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  i=0;
  fa=func(a)
  fb=func(b)
  DO i=1,nmax
    r=b-fb*(a-b)/(fa-fb)
    fr=func(r)
    IF (fr*fa<0) THEN
      b=r
      fb=fr
    ELSE
      a=r
      fa=fr
    END IF
    IF (abs(fr)<eps) EXIT
  END DO
  false=r
  END FUNCTION false
END PROGRAM false_position
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" false_position

E:\okul\SCO1>false_position
 false position(   1.0000000000000000      ,   2.1500000953674316      ) =    1.4142135594660559

> Terminated with exit code 0.
```

In order to overcome the difficulty of one sided cut of the region a bisection step can be put in with some regular intervals. For example once in every 4 false position steps. This is called **modified false position** method. The next example looks the root of $f(x)=x^2-2$ with modified false position method.

Program Modified False position (Regula falsi) NA3 java code

```
 import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;
public class NA3
{
public static double false_position(f_x f,double xl,double xu)
{
//false position root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr,fxu;
double xold=0;
int maxit=100,iter=0;
es=0.0000001;
ea=1.1*es;
String s="";
fxl= f.func(xl);
fxu= f.func(xu);
if(fxl*fxu>0) System.out.println("xl="+xu+"xr="+xr+"fxl="+fxl+"fxu="+fxu+"error");
while((ea>es)&&(iter<maxit))
{   xold=xr;
   if(iter%4!=0)
   {xr=xu-fxu*(xl-xu)/(fxl-fxu);}
   else
   {xr=(xl+xu)/2.0;}
   fxr= f.func(xr);
   iter++;
   if(xr!=0)
      { ea=Math.abs((xr-xold)/xr)*100;}
   test= fxl*fxr;
   if(test==0.0)      ea=0;
   else if(test<0.0) {xu=xr;fxu=fxr;}
   else if(test>0)   {xl=xr;fxl=fxr;}
   else              {ea=0;}
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
  " result might not be valid","MAKSIMUM NUMBER OF ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
  return xr;
}
    public static void main (String args[]) throws IOException
    {
    double a,b;
    class f1 extends f_x{public double func(double x){return x*x-2;}};
    f1 f=new f1();
    a=Double.parseDouble(JOptionPane.showInputDialog(" INPUT LOWER LİMİT OF SEARCH AREA a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" INPUT UPPER LIMIT OF SEARCH AREA b : "));
    double r;
    r= false_position(f,a,b);
    JOptionPane.showMessageDialog(null," ROOT  : "+r+"\nFUNCTION VALUE : "+f.func(r),
    "false position root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    } }
```

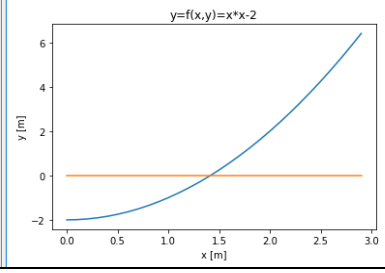


Program Modified False position (Regula falsi) in python code

```
import numpy as np
import matplotlib.pyplot as plt
from math import *

def f1(x):
  return x*x-2.0

def false_position(xl,xu):
  # bisection root finding method
  maxit=100
  iter=0
  es=0.0000001
  ea=1.1*es
  s=""
  fxl= f1(xl)
  fxu= f1(xu)
  xr=xl
  while ea>es and iter<maxit:
    xold=xr
    if iter%4!=0:
      xr=xu-fxu*(xl-xu)/(fxl-fxu);
    else:
      xr=(xl+xu)/2.0;
    fxr= f1(xr);
    iter=iter+1;
    if xr!=0:
      ea=abs((xr-xold)/xr)*100
    test= fxl*fxr;
    if test==0.0 :      ea=0;
    elif test<0.0: xu=xr;fxu=fxr;
    elif test>0:    xl=xr;fxl=fxr;
    else :          ea=0;
    if iter>=maxit: print("Maximum number of iteration is exceeded \n result might not be valid")
  return xr

x = np.arange(0, 3, 0.1);
y = f1(x)
y1=0.0*x
plt.title('y=f(x,y)=x*x-2')
plt.xlabel('x [m]')
plt.ylabel('y [m]')
plt.plot(x, y,x,y1)
print(" Modified false position: ")
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=false_position(a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

Modified false position:

```
enter upper limit : 3
 ROOT  :  1.414213562372946
FUNCTION VALUE :  -4.2166270475263445e-13
```

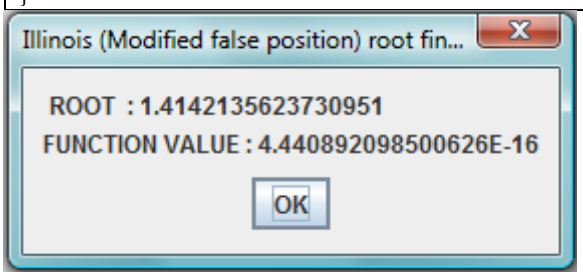


As it is seen no problem arise finding the root of this function in modified . Another modification of false position method is the **Illinois method**. In this method function value is halved (divided by two) whenever end point is reused.

Program Modified False position Illinois method (Regula falsi) NA3A (java code)

```
 import java.util.*;
import java.awt.*;
```

```
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============== Function to be define ================

class fb extends f_x
{
  public double func (double x)
  { return x*x-2.0;}
}
//============================================================
public class NA3A
{
public static double illinois(f_x f,double xa,double xu)
{
//Modified False position root finding: Illinois method
double test;
double p=0;
double es,ea;
double fa,fp,fu;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
double xold;
int ia=0,iu=0;
fa=f.func(xa);
fu=f.func(xu);
p=xu;
double xpold;
while(iter<maxit && ea>es)
{
xpold=p;
p=xu-fu*(xa-xu)/(fa-fu);
fp=f.func(p);
if(Math.signum(fp)!=Math.signum(fu))
{xa=xu;fa=fu;}
else
{fa=fa/2;}
xu=p;fu=fp;
iter++;
if(p!=0) ea=Math.abs((p-xpold)/p)*100.0;
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
  " result might not be valid","MAKSİMUM NUMBER OF ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return p;
}

    public static void main (String args[])
    {
   double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" INPUT LOWER LİMİT OF SEARCH AREA a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" INPUT UPPER LIMIT OF SEARCH AREA b : "));
    double r;
    fb f=new fb();
    r= illinois(f,a,b);
    JOptionPane.showMessageDialog(null," ROOT  : "+r+"\nFUNCTION VALUE : "+f.func(r),
    "Illinois (Modified false position) root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Modified fixed position method code in matlab is given below to compare:

Program Modified False position (Regula falsi) Matlab (Octave) code: false_positon.m

```
function p=false_position(f,xa,xu)
% modified false position root finding method
maxit=100;
iter=0;
es=0.000001;
ea=1.1*es;
ia=0;
iu=0;
fa=f(xa);
fu=f(xu);
p=xu;
double xpold;
while(iter<maxit && ea>es)
xpold=p;
p=xu-fu*(xa-xu)/(fa-fu);
fp=f(p);
iter=iter+1;
if p~=0 ea=abs((p-xpold)/p)*100.0;
test=fa*fp;
if test<0
   xu=p;
   fu=f(xu);
   iu=0;ia=ia+1;
   if ia >= 2 fa=fa/2.0;end
elseif test>0  xa=p;fa=f(xa);ia=0;iu=iu+1;
      if iu>=2 fu=fu/2.0; end
elseif test==0 ea = 0; end
end
if iter>=maxit fprintf('Maximum number of iteration is exceeded result might not be valid'); end
end
```

```
>> f1=@(x) x*x-2;
>> false_position(f1,0,3)
ans =1.414213562373074
>>
```

Modified false position method given as c++ code is listed below:

Program Modified False position (Regula falsi) c++ code false_position.cpp

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x)
  { double ff=x*x-2.0;
                return ff;
  }
};

double false_position(f_x &f,double xa,double xu)
{
//Modified False position root finding
double test;
```

```
double p=0;
double es,ea;
double fa,fp,fu;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
int ia=0,iu=0;
fa=f.func(xa);
fu=f.func(xu);
p=xu;
double xpold;
while(iter<maxit && ea>es)
{
xpold=p;
p=xu-fu*(xa-xu)/(fa-fu);
fp=f.func(p);
iter++;
if(p!=0) ea=fabs((p-xpold)/p)*100.0;
test=fa*fp;
if(test<0) {xu=p;fu=f.func(xu);iu=0;ia++;if(ia>=2) fa/=2.0;}
else if(test>0) {xa=p;fa=f.func(xa);ia=0;iu+=1;if(iu>=2)fu/=2.0;}
else ea=0;
}
if(iter>=maxit) std::cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return p;
}

int main()
{  double a,b,r;
   f1 f;
   cout << " INPUT LOWER LIMIT OF SEARCH AREA a : ";
   cin >>a;
   cout << " INPUT UPPER LIMIT OF SEARCH AREA b : ";
   cin >>b;
   r=false_position(f,a,b);
   cout<<setprecision(20);
   cout<<"r="<<r;
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" false_position1

D:\okul\SCO1>false_position1
 INPUT LOWER LIMIT OF SEARCH AREA a : 1
 INPUT UPPER LIMIT OF SEARCH AREA b : 3
r=1.4142135623730951
> Terminated with exit code 0.
```

Program Modified False position (Regula falsi) Fortran 90 code

```
program modified_false_position
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) a,b
a=1.0
b=2.15
print *,'false position(',a,',',b,') = ',false(a,b)

CONTAINS
  FUNCTION func(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x-2.0
  END FUNCTION func

  FUNCTION false(a,b)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) false,a,b,b1,r,eps,fa,fb,fr,remain
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  i=0;
  fa=func(a)
```

```
  fb=func(b)
  DO i=1,nmax
    remain=i-(i/4)*4
    if(remain==0) then
      r=(a+b)/2.0
    else
      r=b-fb*(a-b)/(fa-fb)
    end if
    fr=func(r)
    IF (fr*fa<0) THEN
      b=r
      fb=fr
    ELSE
      a=r
      fa=fr
    END IF
    IF (abs(fr)<eps) EXIT
  END DO
  false=r
  END FUNCTION false
END PROGRAM modified_false_position
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" modified_false_position

E:\okul\SCO1>modified_false_position
 false position(   1.0000000000000000     ,   2.1500000953674316     ) =    1.4142135609373101

> Terminated with exit code 0.
```

Program False position (Regula falsi) in julia code

```
 function false_position(f,xl,xu)
test=1.0
xr=0.0
fx=0.0
fxr=0.0
fxu=0.0
xold=0.0
maxit=100
iter=0
es=0.00001
global ea=1.5*es
s=""
fxl= f(xl)
fxu= f(xu)
while ea>es && iter<maxit
   xold=xr
   xr=xu-fxu*(xl-xu)/(fxl-fxu)
   fxr= f(xr)
   iter=iter+1
   ea=abs(fxr)
   test= fxl*fxr
   if(test==0.0)
     ea=0.0
   elseif(test<0.0)
     xu=xr
     fxu=fxr
   elseif(test>0)
     xl=xr
     fxl=fxr
   else
      ea=0.0
   end
end # while
if(iter>=maxit)
  print("Maximum number of iteration is exceeded \nresult might not be valid")
end
   return xr
end

using Plots
```

```
f(x)=x*x-2.0
a=0.0
b=2.0
x=false_position(f,a,b)
print("x = $x")
f1(x)=0.0
x1=[x,x]
y1=[-2.0,2.0]
p=plot([f, f1],0.0:0.01:2.0)
plot!(p,x1,y1)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" bisection.jl
x = 1.41421143847487
> Terminated with exit code 0.
```

FIXED POSITION METHOD WITH AITKEN EXTRAPOLATION (STEPHENSEN METHOD)

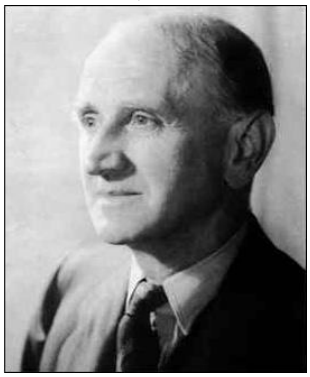


Figure 2.3.1 Alexander Craig Aitken

Fixed position method is relatively simple. When a function f(x) is given to find the root, the function f(x) converted with algebraic manupulations and converted to f(x)=g(x) – x form. Since this equation can be written in as g(x)=x, starting from an initial guess, iteration continued till x and g(x) gives the same x value. Even though the relative simplicity, this method is not used much due to high failure possibility. The equation can diverge from the root sometimes. In order to overcome this problem aitken extrapolation process will be combined with fixed point , iteration. Method is called Stephensen method.

Aitken extrapolation process is a error reduction method. For a very big k number

$$\frac{x_{k+2}-a}{x_{k+1}-a} \approx \frac{x_{k+1}-a}{x_k-a} \quad k \gg 1 \qquad (2.3.1)$$

Relation can be valid. If the value of fourth point desired to be calculated from this equation

$$x = \frac{x_k x_{k+2} - x_{k+1}^2}{2x_k + 2x_{k+1} + 2x_{k+2}} \qquad (2.3.2)$$

When 3 points are previously known, a third one can be evaluated

$$x_k = x_{k+1} - x_k \qquad (2.3.3)$$

$$\Delta^2 x_k = x_{k+2} - 2x_{k+1+} x_k \qquad (2.3.4)$$

$$x_e = x_{k+2} - \frac{(\Delta x_k)^2}{\Delta^2 x_k} \qquad (2.3.5)$$

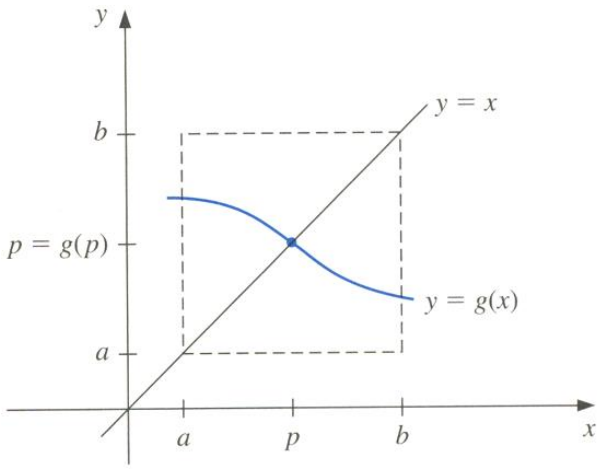


Figure Convergence of the Fixed iteration Method

Stephensen method evaluates equation by using an aitken extrapolation step after two step of fixed point iteration.

Program: Fixed point iteration (Stephensen method) : (java code)

```
// Stephensen method x=g(x)

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Tanimi gereken fonksiyon ================
class fb extends f_x
{  //x = g(x) fonction
   //f(x)=x-Math.exp(-x)
   public double func (double x)
   { return Math.exp(-x);}
}

//============================================================
public class NA4
{

public static double stephensen(f_x f,double x0)
{
//Stephensen root finding method
double es,ea;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
double x,x1,x2;
x=0;
while(iter<maxit && ea>es)
{
x1=f.func(x0);
x2=f.func(x1);
x=x0-(x1-x0)*(x1-x0)/(x2-2*x1+x0);
ea=Math.abs(x-x0)*100.0;
x0=x;
iter++;
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " result might not be valid","MAKSİMUM NUMBER OF ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
    public static void main (String args[])
    {
    fb g=new fb();
    double xa,xu;
    xa=Double.parseDouble(JOptionPane.showInputDialog(" first guess for the root value : "));
    double r= stephensen(g,xa);
    double fr=g.func(r);
    String s=" root value  : "+r+"\ng(x) = "+fr+"\nf(x)=g(x)-x = "+(fr-r);
```

```
    JOptionPane.showMessageDialog(null,s,
    "Stephensen root finding method : ",JOptionPane.PLAIN_MESSAGE);
    } }
```

Stephensen Method is simple enough to calculate in excel format: The example problem is :

$$f(x) = x - \sqrt{\frac{5}{x+1}} = x - g(x) \qquad\qquad g(x) = \sqrt{\frac{5}{x+1}}$$

Program: Fixed point iteration (Stephensen method) with excel:

Fixedİteration

f(x)=x-(5/(x+1)^0.5

Aitken		dxk	d2xk	xe=xk-(dxk)^2/d2xk	g(x)=(5/(x+1))^0.5
	x				
0	0.5				1.825741858
0	1.825741858	1.325741858			1.330205563
0	1.330205563	-0.495536296	-1.821278154	1.465031877	1.464832223
1	1.465031877	0.134826314	0.63036261	1.436194292	1.424209021
0	1.424209021	-0.040822856	-0.17564917	1.433696715	1.436150556
0	1.436150556	0.011941535	0.052764391	1.433447971	1.432626367
0	1.432626367	-0.003524189	-0.015465724	1.433429427	1.433663727
1	1.433429427	0.00080306	0.004327249	1.433280394	1.433427145
0	1.433427145	-2.28264E-06	-0.000805343	1.433427151	1.433427817
0	1.433427817	6.72303E-07	2.95494E-06	1.433427664	1.433427619
0	1.433427619	-1.98012E-07	-8.70315E-07	1.433427664	1.433427677
1	1.433427664	4.50514E-08	2.43064E-07	1.433427656	1.433427664
0	1.433427664	-7.32747E-15	-4.50514E-08	1.433427664	1.433427664
0	1.433427664	1.9984E-15	9.32587E-15	1.433427664	1.433427664
0	1.433427664	0	-2.44249E-15	1.433427664	1.433427664

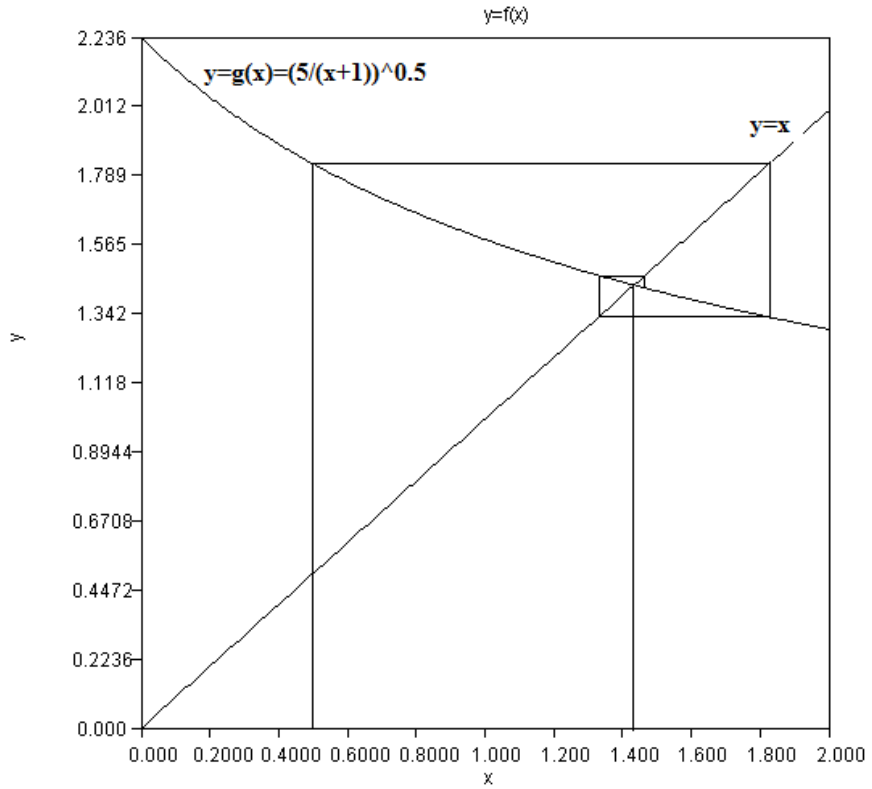


Figure Convergence of the Fixed position Method

If the same problem is worked through the Matlab:

Program: Fixed point iteration (Stephensen method) with Matlab(Octave):

```
function x = stephensen(f,x0)
%Fixed point iteration with Aitken extrapolation
%Stephensen Method
maxit=100;
iter=0;
es=0.000001;
ea=1.1*es;
x=0;
```

```
while iter<maxit && ea>es
x1=f(x0);
x2=f(x1);
x=x0-(x1-x0)*(x1-x0)/(x2-2*x1+x0);
ea=abs(x-x0)*100.0;
x0=x;
iter=iter+1;
end
if iter>=maxit  fprintf('Maximum number of iteration is exceeded  result might not be valid');end
end
```

```
>> g=@(x) sqrt(5/(x+1));
>> stephensen(g,1)
ans =
   1.433427663863820
```

C++ code of the stephensen method is given below:

Program: Fixed point iteration (Stephensen method) in c++ language

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-
168.0*func(x+2.0*h) +32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}};

double stephensen(f_x &f,double x0)
{
//Stephensen root finding method
double es,ea;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
double x,x1,x2;
x=0;
while(iter<maxit && ea>es)
{
x1=f.func(x0);
x2=f.func(x1);
x=x0-(x1-x0)*(x1-x0)/(x2-2*x1+x0);
ea=fabs(x-x0)*100.0;
x0=x;
iter++;
}
if(iter>=maxit) std::cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;
}

int main()
{  double xa,r;
   class f1:public f_x{ public: double func(double x) {return sqrt(5.0/(x+1.0));}};
   f1 f;
   cout << " first guess for the root value : ";
   cin >>xa;
```

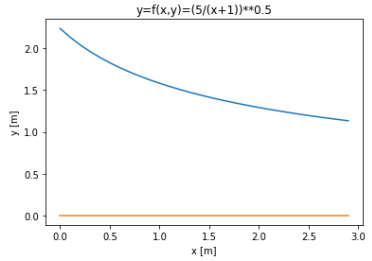
```
  r=stephensen(f,xa);
  cout<<setprecision(20);
  cout<<"root value : = "<<r<<" fr = "<<f.func(r)-r;
  return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" stephensen
D:\okul\SCO1>stephensen
 first guess for the root value : 1
root value : = 1.4334276638638197 fr = 0
> Terminated with exit code 0.
```

Program: Fixed point iteration (Stephensen method) in Python code

```
 import numpy as np
import matplotlib.pyplot as plt
from math import *

def f1(x):
  return (5.0/(x+1.0))**0.5

def stephensen(x0):
  # Fixed point iteration with Aitken extrapolation
  # Stephensen Method
  maxit=100;
  iter=0;
  es=0.000001;
  ea=1.1*es;
  x=0;
  while iter<maxit and ea>es:
    x1=f1(x0);
    x2=f1(x1);
    x=x0-(x1-x0)*(x1-x0)/(x2-2*x1+x0);
    ea=abs(x-x0)*100.0;
    x0=x;
    iter=iter+1;
    if iter>=maxit:  print("Maximum number of iteration is exceeded  result might not be valid")
  return x

x = np.arange(0, 3, 0.1);
y = f1(x)
y1 = 0.0*x
plt.title('y=f(x,y)=(5/(x+1))**0.5')
plt.xlabel('x [m]')
plt.ylabel('y [m]')
plt.plot(x, y,x,y1)
print(" Stephensen method: ")
a = float(input("enter initial value : "))
rr=stephensen(a)
print(" ROOT  : ",rr,"\nFUNCTION VALUE : ",f1(rr))
```

```
In [21]: runfile('D:/okul/SCO1/stephensen.py', wdir='D:/okul/SCO1')
 Stephensen method:

enter initial value : 1
 ROOT  :  1.4334276638638197
FUNCTION VALUE :  1.4334276638638197
```

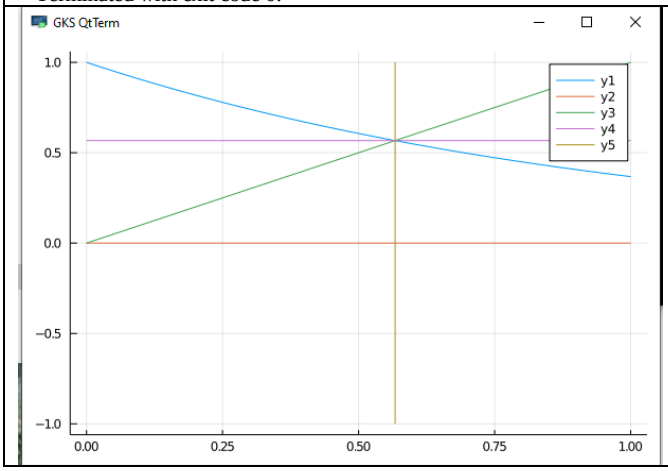



```
In [22]:
```

Program: Fixed point iteration (Stephensen method) :

```
 # Stephensen method x=g(x)
function  stephensen(f,x0)
maxit=100
iter=0
es=0.000001
ea=1.1*es
x=0.0
x1=0.5
x2=1.0
while iter<maxit && ea>es
            x1=f(x0)
            x2=f(x1)
            x=x0-(x1-x0)*(x1-x0)/(x2-2*x1+x0)
            ea=abs(x-x0)
            x0=x
            iter=iter+1
            if iter>=maxit
              print("Maximum number of iteration is exceeded \nresult might not be valid")
            end
end
return x
end

using Plots
f(x)=exp(-x)
x0=1.0
x2=stephensen(f,x0)
print("x = $x2")
f1(x)=0.0
f2(x)=x
f3(x)=x2
x1=[x2,x2]
y1=[-1.0,1.0]
p=plot([f, f1,f2,f3],0.0:0.01:1.0)
plot!(p,x1,y1)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" stephensen.jl
x = 0.5671432904097838
> Terminated with exit code 0.
```



NEWTON-RAPHSON METHOD



Isaac Newton



Joseph Raphson

Taylor Formula can be written as

$$f(x_{n+1}) = f(x_n) + \frac{f'(x_n)}{1!}(x_{n+1} - x_n) + \frac{f''(x_n)}{2!}(x_{n+1} - x_n)^2 + \frac{f^{(3)}(x_n)}{3!}(x_{n+1} - x_n)^3 + \cdots \quad (2.4.1)$$

We would like to find $f(x_{n+1}) = 0$. If We substitude it into the Taylor Formula, and also cancel out terms above the first degree , The equation becomes

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n) \quad (2.4.2)$$

If x_{n+1} taken to left hand side Newton-Raphson Formula is obtained

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (2.4.3)$$

We can find the root iteratively starting from a first guess. Newton Formula aproached to the root quickly if the first guess is close to the actual root. But if the first gues is far away from the root, it might take longer or it might fail to reach to the root. Another difficulty in Newton-Raphson Formula is the requirement of knowing the actual derivative of the function

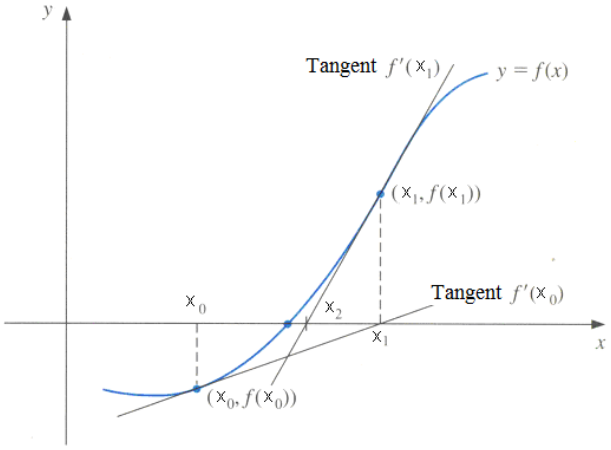
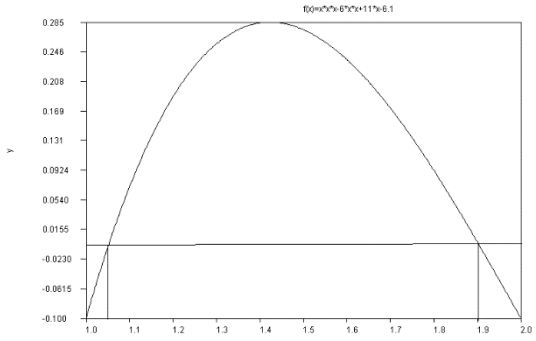


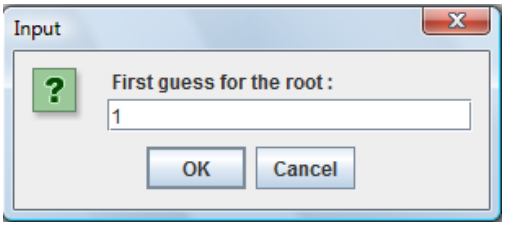
Figure 2.5 Convergence of the Newton-Raphson Method

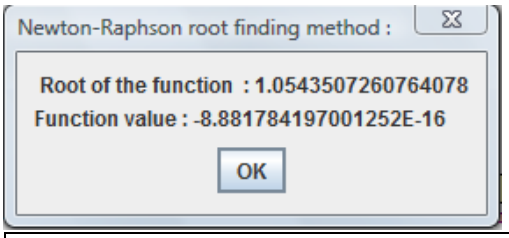
PROGRAM Newton-Raphson method (java code)

```
// Newton-Raphson root finding
//Numerical Analysis
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= The function ================

class fb extends f_x
```

```
{
   public double func (double x)
   { return x*x*x-6*x*x+11*x-6.1;}
}
//============= Derivative of the function ================
class dfb extends f_x
{
   public double func (double x)
   { return 3.0*x*x-12.0*x+11;}
}
public class NA5
{
public static double newton(f_x f,f_x df,double x)
{
int nmax=100;
double tolerance=1.0e-10;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=df.func(x);
  x-=fx/dfx;
  System.out.println("i="+i+"x="+x+"fx="+fx+"dfx = "+dfx);
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
   " çözüm geçerli olmiyabilir","MAXİMUM ITERATION NUMBER
WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
     public static void main (String args[])
     { fb f=new fb();
        dfb df=new dfb();
       double x0;
       x0=Double.parseDouble(JOptionPane.showInputDialog(" First guess for the root : "));
       double r= newton(f,df,x0);
       JOptionPane.showMessageDialog(null," Root of the function  : "+r+"\nFunction value : "+f.func(r),
       "Newton-Raphson root finding method : ",JOptionPane.PLAIN_MESSAGE);
     } }
```








```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA5
i=0x=1.0499999999999998fx=-0.09999999999999964dfx = 2.0
i=1x=1.054319180087848fx=-0.007375000000005755dfx = 1.7075000000000014
i=2x=1.0543507243988341fx=-5.30870767248004E-5dfx = 1.68293663944916
i=3x=1.0543507260764082fx=-2.82295076203809E-9dfx = 1.6827576573350296
i=4x=1.0543507260764078fx=8.881784197001252E-16dfx = 1.6827576478166488
> Terminated with exit code 0.
```

In order to see step calculations of Newton-Raphson method, root of $f(x)=x^2-2$ is calculated in the spreadsheet environment as well

PROGRAM Newton-Raphson method in excel

Newton-Raphson root finding $f(x)=x^2-a$

a	2	
x	f(x)=x*x-a	fx/dx=2*x
1.00000000000000000	-1	2
1.50000000000000000	0.25	3
1.41666666666667000	0.006944444	2.833333333
1.41421568627451000	6.0073E-06	2.828431373
1.41421356237469000	4.51061E-12	2.828427125
1.41421356237310000	0	2.828427125
1.41421356237310000	0	2.828427125
1.41421356237310000	0	2.828427125

PROGRAM Newton-Raphson method in Matlab(Octave)

```
function p=newton_raphson(f,df,x,tolerance,nmax)
% secant method for root finding
if nargin<5 nmax=100;end
if nargin<4 tolerance=1e-10;end
i=0;
fx=f(x);
while (abs(fx) > tolerance) && i<nmax
   fx=f(x);
   dfx=df(x);
   x=x-fx/dfx;
   i=i+1;
end
p=x;
if i>=nmax fprintf('Maximum number of iterations is exceeded the result may not be valid');end
end
```

```
>> f=@(x) x*x-2;
>> df=@(x) 2*x;
>> newton_raphson(f,df,1)
ans =
  1.414213562373095
```

PROGRAM Newton-Raphson method in C++

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x){ double h=0.0001;double hh=1.0/h;
 double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;}};
```

```
class f1:public f_x
{ public: double func(double x) { double ff=x*x-2.0; return ff;}};

class df1:public f_x
{ public: double func(double x) { double ff=2.0*x;return ff;}};

double newton(f_x &f,f_x &df,double x)
{
int nmax=100;
double tolerance=1.0e-8;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=df.func(x);
  x-=fx/dfx;
  cout <<"i="<<i<<"x="<<x<<"fx="<<fx<<"dfx="<<dfx<<"\n";
  if(fabs(fx)<tolerance) { return x;}
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;
}

int main()
{  double xa,r;
   cout << " first guess for the root value : ";
   cin >>xa;
   f1 f;
   df1 df;
   cout<<setprecision(20);
   r=newton(f,df,xa);
   cout<<"root value : ="<<r;
   return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" newton

D:\okul\SCO1>newton
 first guess for the root value : 1
i=0x=1.5fx=-1dfx=2
i=1x=1.4166666666666667fx=0.25dfx=3
i=2x=1.4142156862745099fx=0.006944444444444654dfx=2.8333333333333335
i=3x=1.4142135623746899fx=6.007304882971447 5e-006dfx=2.8284313725490198
i=4x=1.4142135623730951fx=4.5108350648498385e-012dfx=2.8284271247493797
root value : =1.4142135623730951
> Terminated with exit code 0.
```

Python version of the Newton-Raphson method.

PROGRAM Newton-Raphson method in Python

```
 from math import *

def f1(x):
   return x*x-2.0

def df1(x):
   return 2.0*x

def newton(x):
   # Newton-Raphson root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
      fx= f1(x);
      dfx=df1(x)
      x=x-fx/dfx
      print("fx",fx,"dfx",dfx,"x=",x)
      if abs(fx)<tolerance: return x
  return x
```

```
a=float(input("enter initial guess : "))
r=newton(a)
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" NA5.py
enter initial guess : 1
fx -1.0 dfx 2.0 x= 1.5
fx 0.25 dfx 3.0 x= 1.4166666666666667
fx 0.006944444444444642 dfx 2.8333333333333335 x= 1.4142156862745099
fx 6.007304882871267e-06 dfx 2.8284313725490198 x= 1.4142135623746899
fx 4.510614104447086e-12 dfx 2.8284271247493797 x= 1.4142135623730951
 ROOT  :  1.4142135623730951
FUNCTION VALUE :  4.440892098500626e-16

> Terminated with exit code 0.
```

PROGRAM Newton-Raphson method in Fortran 90

```
program newton_raphson
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) a
a=1.0
print *,'Newton_Raphson(',a,') = ',newton(a)

CONTAINS
  FUNCTION func(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x-2.0
  END FUNCTION func

  FUNCTION dfunc(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) dfunc,x
  dfunc=2.0*x
  END FUNCTION dfunc

  FUNCTION newton(a)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) newton,a,x,eps,fx,dfx
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  x=a
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x)
    x=x-fx/dfx
    IF (abs(fx)<eps) EXIT
  END DO
  newton=x
  END FUNCTION newton
END PROGRAM newton_raphson
```

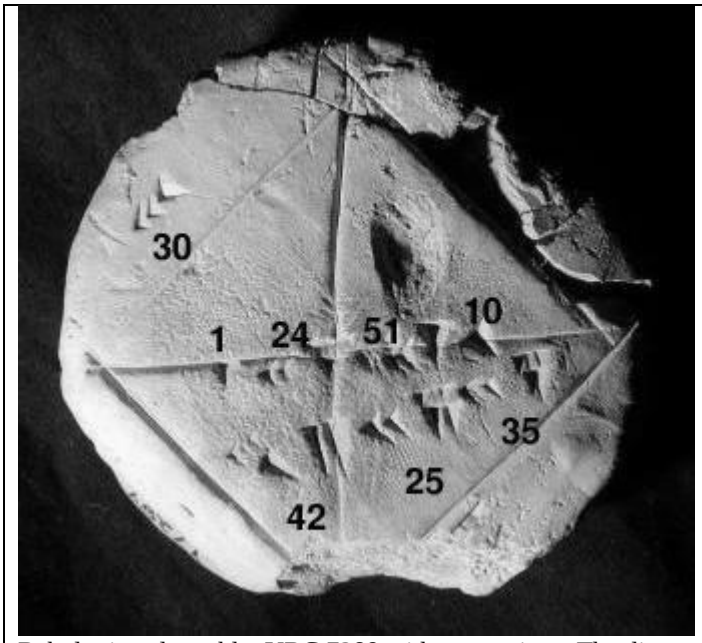
```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" newton_raphson

E:\okul\SCO1>newton_raphson
 Newton_Raphson(   1.0000000000000000      ) =    1.4142135623730951

> Terminated with exit code 0.
```

Another program for Python version of the Newton-Raphson method. Please refer to **Program 2.1-1e Abstract class f_x in python language.** In this program function defined as a class function which is derived from the abstract class f_x

```
 from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
      y=x*x-2;
      return y;
```

```
class df1(f_x):
  def func(self,x):
      y=2.0*x;
      return y;

def newton(f,df,x):
   # Newton-Raphson root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
      fx= f.func(x);
      dfx=df.func(x);
      x=x-fx/dfx
      print("fx",fx,"dfx",dfx,"x=",x)
      if abs(fx)<tolerance: return x
  return x


f=f1();
df=df1();
a=float(input("enter initial guess : "))
r=newton(f,df,a);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

As it is seen from the results, convergence is quite fast for this problem.



Babylonian clay tablet YBC 7289 with annotations. The diagonal displays an approximation of the square root of 2 in four sexagesimal figures, 1 24 51 10, which is good to about six decimal digits.
$1 + 24/60 + 51/60^2 + 10/60^3 = 1.41421296...$
Babylonian algorithm is actually newton-raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_n) = x^2 - 2$$
$$f'(x_n) = 2x$$
$$x = x - \frac{x^2 - 2}{2x} = \frac{x}{2} + \frac{1}{x}$$

Babylonian algorithm in java format:

```
public class sqrt1
{
//Babylonian algorithm
public static double sqrt(double xi)
```

```
{
  double x = xi;
  double y =x*x-2;
  double e = 1.0e-10;
  while(Math.abs(y) > e)
  { x =x /2.0+ 1.0/x;
     y =x*x-2.0;
    System.out.println("x="+x+"y="+y);
  }
  return x;
}
Public
public static void main(String arg[])
{
System.out.println(sqrt(2.0));
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" sqrt1
x=1.5y=0.25
x=1.4166666666666665y=0.006944444444444198
x=1.4142156862745097y=6.007304882427178E-6
x=1.4142135623746899y=4.510614104447086E-12
1.4142135623746899

> Terminated with exit code 0.
```

PROGRAM Newton-Raphson method (julia code)

```
 function newton(f,df,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(x)
  x-=fx/dfx
  println("i = $i x = $x fx = $fx dfx = $dfx")
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

f(x)=x*x-2.0
df(x)=2.0*x
x=1.0
x=newton(f,df,x)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton.jl
i = 0 x = 1.5 fx = -1.0 dfx = 2.0
i = 1 x = 1.4166666666666667 fx = 0.25 dfx = 3.0
i = 2 x = 1.4142156862745099 fx = 0.006944444444444642 dfx = 2.8333333333333335
i = 3 x = 1.4142135623746899 fx = 6.007304882871267e-6 dfx = 2.8284313725490198
i = 4 x = 1.4142135623730951 fx = 4.510614104447086e-12 dfx = 2.8284271247493797
x = 1.4142135623730951
> Terminated with exit code 0.
```

SECANT METHOD

One of the difficulty of the Newton-Raphson method is the requirement to find the derivative. If difference equation (Numerical derivative) is used instead of derivative (tangent to the function), a secant to the function is drawn to the function, so the method is called secant method. If derivative is approximated with a first degree difference equation

$$f'(x_n) \cong \frac{f(x_n)-f(x_{n-1})}{(x_n-x_{n-1})} \quad (2.5.1)$$

Newton –Raphson Formula is converted to

$$x_{n+1} = x_n - \frac{f(x_n)(x_n-x_{n-1})}{f(x_n)-f(x_{n-1})} \quad (2.5.2)$$

Two first estimation is needed to start iteration by using this formula $f(x_0)$ and $f(x_{-1})$

If we would like to establish a one estimation secant Formula, the previous difference equation can be written for a small Δx as (central difference Formula)

$$f'(x_n) \cong \frac{f(x_n+\Delta x)-f(x_n-\Delta x)}{2\Delta x} \quad (2.5.3)$$

Second equation becomes:

$$x_{n+1} = x_n - \frac{2f(x_n)\Delta x}{f(x_n+\Delta x)-f(x_n-\Delta x)} \quad (2.5.4)$$

In order to minimize error Δx should be relatively small, to have a further approximation a second or higher derivative (difference) formulas can also be used

$$f'(x_n) \cong \frac{-f(x_n+2\Delta x)+8f(x_n+\Delta x)-8f(x_n-\Delta x)+f(x_n-2\Delta x)}{12\Delta x} \quad (2.5.5)$$

$$f'(x_n) \cong \frac{f(x_n+3\Delta x)-9f(x_n+2\Delta x)+45f(x_n+\Delta x)-45f(x_n-\Delta x)+9f(x_n-2\Delta x)-f(x_n-3\Delta x)}{60\Delta x} \quad (2.5.6)$$

$$f'(x_n) \cong \frac{-3f(x_n+4\Delta x)+32f(x_n+3\Delta x)-168f(x_n+2\Delta x)+672f(x_n+\Delta x)-672f(x_n-\Delta x)+168f(x_n-2\Delta x)-32f(x_n-3\Delta x)+3f(x_n-4\Delta x)}{840\Delta x}$$

(2.5.7)

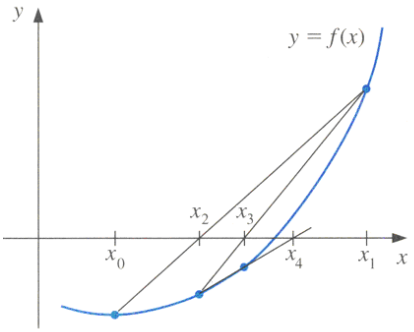
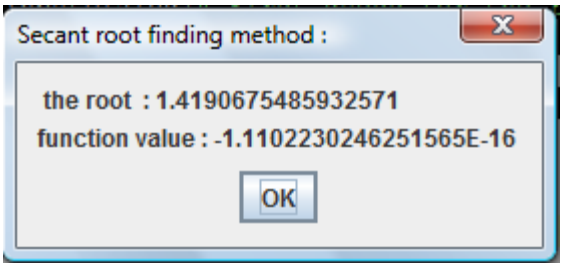


Figure Second method approximation with linear difference

PROBLEM NA7 : Secant method with two initial root estimation (java code)

```
// Numerical Analysis
//
//
// Root finding with secant method
// two initia root estimation
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//============= Function definition ================

class fb extends f_x
{
  public double func (double x)
  { return Math.sin(Math.sqrt(x))-x;}
```

```
}

public class NA7
{
public static double secant(f_x f,double x0,double x1)
{
int nmax=500;
double tolerance=1.0e-10;
double fx0,fx1;
double x;
fx0=f.func(x0);
for(int i=0;i<nmax;i++)
{
  fx1=f.func(x1);
  x=x1-fx1*(x1-x0)/(fx1-fx0);
  x0=x1;
  fx0=fx1;
  x1=x;
  if(Math.abs(fx1)<tolerance) { return x1;}
}
JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
   " çözüm geçerli olmiyabilir","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x1;
}

    public static void main (String args[])
    {
                fb f=new fb();
    double x0,x1;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" First guess for the root : "));
    x1=Double.parseDouble(JOptionPane.showInputDialog(" Second guess for the root : "));
    double r= secant(f,x0,x1);
    JOptionPane.showMessageDialog(null," the root  : "+r+"\nfunction value : "+f.func(r),
    "Secant root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Re-write the program to find root of f(x)=tan(x-1)

In order to see calculation steps of the method a spreadsheet program is introduced, for the function $f(x)=x^2-2$, initial values $x_{-1}=0$, $x_0=1$ and $f'(x_n) \cong \frac{f(x_n)-f(x_{n-1})}{(x_n-x_{n-1})}$

PROBLEM Secant method with two initial root estimation (octave-excel code)

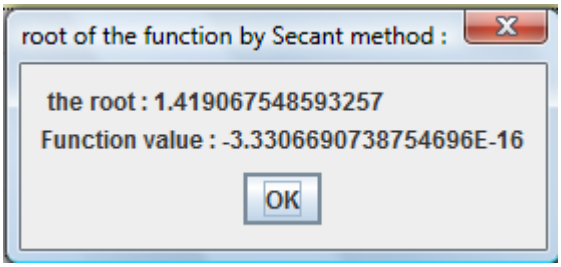
Secant 1 $f(x)=x^2-a$

a	2	
x	f(x)=x*x-a	fx/dx=(f(x_n)-f(x_n-1))/(x_n-x_{n-1})
0.00000000000000000	-2	
1.00000000000000000	-1	1
2.00000000000000000	2	3
1.33333333333333000	-0.222222222	3.333333333
1.40000000000000000	-0.04	2.733333333
1.41463414634146000	0.001189768	2.814634146
1.41421143847487000	-6.00729E-06	2.828845585
1.41421356205732000	-8.93146E-10	2.828425001
1.41421356237310000	0	2.828424244
1.41421356237310000	0	

In the next java example $f'(x_n) \cong \frac{f(x_n+\Delta x)-f(x_n-\Delta x)}{2\Delta x}$ approach is used to estimate the derivative

PROBLEM NA8 : Secant method with one initial root estimation, linear difference Formula (java code)

```
///Numerical Analysis
// Secant method with one root estimation
// Linear difference formula
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//============= function definition ================
class fb extends f_x
{
  public double func (double x)
  { return x*x-2;}
}
public class NA8
{
public static double secant(f_x f,double x)
{
int nmax=100;
double tolerance=1.0e-10;
double fx,dfx;
double dx=0.01;
for(int i=0;i<nmax;i++)
{
 fx=f.func(x);
 dfx=(f.func(x+dx)-f.func(x-dx))/(2.0*dx);
 x-=fx/dfx;
 if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
  " çözüm geçerli olmiyabilir","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

    public static void main (String args[])
    {
            fb f=new fb();
    double x0;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" First estimation for the root : "));
    double r= secant(f,x0);
    JOptionPane.showMessageDialog(null," the root : "+r+"\nFunction value : "+f.func(r),
    "root of the function by Secant method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Find roos of f(x)=x*x*x-2

to see calculation steps of the method a spreadsheet program is introduced, for the function $f(x)=x^2-2$, initial values $x_0=1$, $\Delta x=0.1$ and $f'(x_n) \cong \frac{f(x_n+\Delta x)-f(x_n-\Delta x)}{2\Delta x}$

PROBLEM Secant method with one initial root estimation, linear difference Formula (octave-excel code)

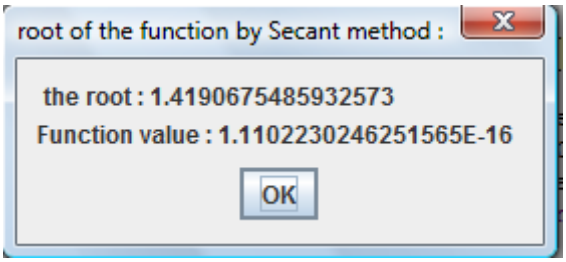
Secant 2 f(x)=x*x-a

a	2					

x	$f(x)=x^2-a$	$x+\Delta x$	$x-\Delta x$	$f(x+\Delta x)$	$f(x-\Delta x)$	$f'(x)$
Δx	0.1					
1	-1	1.1	0.9	-0.79	-1.19	2
1.5	0.25	1.6	1.4	0.56	-0.04	3
1.416666667	0.00694	1.51667	1.31667	0.30028	-0.26639	2.83333
1.414215686	6E-06	1.51422	1.31422	0.29285	-0.27284	2.82843
1.414213562	4.5E-12	1.51421	1.31421	0.29284	-0.27284	2.82843
1.414213562	0	1.51421	1.31421	0.29284	-0.27284	2.82843

PROBLEM NA9 : Secant method with one initial root estimation, quadratic difference Formula (java code)

```
///Numerical Analysis
//
//
// Secant method with one root estimation
// quadratic difference formula
//

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//============= function definition ================
class fb extends f_x
{
   public double func (double x)
   { return Math.log(x*x)-0.7;}
}
//============================================================
public class NA9
{

public static double df(f_x f,double x,double h)
{ // derivative formula
  double hh=1.0/h;
  return (-f.func(x+2.0*h)+8.0*f.func(x+h)-8.0*f.func(x-h)+f.func(x-2.0*h))/12.0*hh;
}

public static double secant(f_x f,double x)
{
int nmax=500;
double tolerance=1.0e-15;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=df(f,x,0.00001);
  x-=fx/dfx;
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " çözüm geçerli olmiyabilir","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

    public static void main (String args[])
    {
              fb f=new fb();
    double x0;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" First estimation for the root : "));
    double r= secant(f,x0);
    JOptionPane.showMessageDialog(null," the root : "+r+"\nFunction value : "+f.func(r),
    "root of the function by Secant method : ",JOptionPane.PLAIN_MESSAGE);
    } }
```



PROBLEM NA9 : Secant method with one initial root estimation, An alternative formula for the difference equation (4th order) (java code)

```
public static double df(f_x f,double x,double h)
{ // derivative formula
  double hh=1.0/h;
  double df=(3.0*f.func(x-4.0*h)-32.0*f.func(x-3.0*h)+168.0*f.func(x-2.0*h)-672.0*f.func(x-h)+672.0*f.func(x+h)-
168.0*f.func(x+2.0*h) +32.0*f.func(x+3.0*h) - 3.0*f.func(x+4.0*h))/840.0*hh;
  return df;
}
```

C++ version of the secant method:

PROBLEM secant2.cpp : Secant method with one initial root estimation, An alternative formula for the difference equation (4th order) (C++ code)

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;
class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x) { double h=0.0001;double hh=1.0/h;
double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
return df;}};
class f1:public f_x{ public: double func(double x)
  { double ff=x*x-2.0;return ff;}};
double secant(f_x &f,double x){
int nmax=100;
double tolerance=1.0e-8;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=f.dfunc(x);
  x-=fx/dfx;
  cout <<"i="<<i<<"x="<<x<<"fx="<<fx<<"dfx="<<dfx<<"\n";
  if(fabs(fx)<tolerance) { return x;}
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;}
    int main()
    {
    double x0;
    cout << "root estimation x0 : ";
    cin >>x0;
    double r;
    f1 f;
    cout<<setprecision(20);
    r= secant(f,x0);
    cout<<" Value of the root  : "<<r<<"\nFunction value : "<<f.func(r);
    }
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" secant1

D:\okul\SCO1>secant1
root estimation x0 : 1
i=0x=1.4999999999999951fx=-1dfx=2.0000000000000191
i=1x=1.4166666666666554fx=0.24999999999998535dfx=2.999999999999599
```

```
i=2x=1.4142156862745094fx=0.006944444444444125685dfx=2.8333333333330035
i=3x=1.4142135623746899fx=6.0073048817152908e-006dfx=2.8284313725487062
i=4x=1.4142135623730951fx=4.5108350648498385e-012dfx=2.8284271247490693
 Value of the root  : 1.4142135623730951
Function value : 2.7343578790084422e-016
> Terminated with exit code 0.
```

Utilize the method to find root of f(x)=x*x*x-3

In the next example secant solution of the function $f(x)=x^2-2$ will be inroduced with initial values $x_0=1$, $\Delta x=0.1$ and $f'(x_n) \cong \frac{-f(x_n+2\Delta x)+8f(x_n+\Delta x)-8f(x_n-\Delta x)+f(x_n-2\Delta x)}{12\Delta x}$ in the spreadsheet environment (excel)

PROBLEM Secant method with one initial root estimation, quadratic difference Formula in excel (or open ofice calc) spreadsheet

Secant 3 $f(x)=x^2-a$

a	2									
Δx	0.1									
x	f(x)=x*x-a	x+2Δx	x+Δx	x-Δx	x-2Δx	f(x+2Δx)	f(x+Δx)	f(x-Δx)	f(x-2Δx)	f'(x)
1	-1	1.2	1.1	0.9	0.8	-0.56	-0.79	-1.19	-1.36	2.99167
1.334261838	-0.21975	1.53426	1.43426	1.23426	1.13426	0.35396	0.05711	-0.4766	-0.71345	2.08121
1.439846995	0.07316	1.63985	1.53985	1.33985	1.23985	0.6891	0.37113	-0.20481	-0.46278	1.71622
1.397218859	-0.04778	1.59722	1.49722	1.29722	1.19722	0.55111	0.24166	-0.31722	-0.56667	1.86805
1.422795983	0.02435	1.6228	1.5228	1.3228	1.2228	0.63347	0.31891	-0.25021	-0.50477	1.77768
1.409099259	-0.01444	1.6091	1.5091	1.3091	1.2091	0.5892	0.27738	-0.28626	-0.53808	1.82635
1.417005353	0.0079	1.61701	1.51701	1.31701	1.21701	0.61471	0.30131	-0.2655	-0.5189	1.79833
1.412610076	-0.00453	1.61261	1.51261	1.31261	1.21261	0.60051	0.28799	-0.27705	-0.52958	1.81393
1.415108941	0.00253	1.61511	1.51511	1.31511	1.21511	0.60858	0.29556	-0.27049	-0.52351	1.80507
1.413705498	-0.00144	1.61371	1.51371	1.31371	1.21371	0.60405	0.2913	-0.27418	-0.52692	1.81005
1.414499268	0.00081	1.6145	1.5145	1.3145	1.2145	0.60661	0.29371	-0.27209	-0.52499	1.80724
1.414052077	-0.00046	1.61405	1.51405	1.31405	1.21405	0.60516	0.29235	-0.27327	-0.52608	1.80882
1.414304575	0.00026	1.6143	1.5143	1.3143	1.2143	0.60598	0.29312	-0.2726	-0.52546	1.80793
1.414162185	-0.00015	1.61416	1.51416	1.31416	1.21416	0.60552	0.29269	-0.27298	-0.52581	1.80843
1.414242539	8.2E-05	1.61424	1.51424	1.31424	1.21424	0.60578	0.29293	-0.27277	-0.52562	1.80815
1.414197211	-4.6E-05	1.6142	1.5142	1.3142	1.2142	0.60563	0.29279	-0.27289	-0.52573	1.80831
1.414222786	2.6E-05	1.61422	1.51422	1.31422	1.21422	0.60572	0.29287	-0.27282	-0.52566	1.80822
1.414208358	-1.5E-05	1.61421	1.51421	1.31421	1.21421	0.60567	0.29283	-0.27286	-0.5257	1.80827
1.414216498	8.3E-06	1.61422	1.51422	1.31422	1.21422	0.60569	0.29285	-0.27283	-0.52568	1.80824

Now if the same problem is solved by uisng Matlab functions:

PROBLEM Secant method with one initial root estimation in Matlab(octave) environment

```
function p=df(f,x,dx)
% derivative of a function
if nargin<3
   dx=0.001;
end
p = ( -3*f(x+4*dx)+32*f(x+3*dx)-168*f(x+2*dx)+672*f(x+dx)-672*f(x-dx) + 168*f(x-2*dx)-32*f(x-3*dx)+3*f(x-4*dx) )/(dx*840);
end
```

```
function p=secant(f,x,tolerance,nmax)
% secant method for root finding
if nargin<4 nmax=100;end
if nargin<3 tolerance=1e-10;end
i=0;
fx=f(x);
while (abs(fx) > tolerance) && i<nmax
   fx=f(x);
   dfx=df(f,x);
   x=x-fx/dfx;
   i=i+1;
end
p=x;
if i>=nmax fprintf('Maximum number of iterations is exceeded the result may not be valid');end
end
```

```
f=@(x) x*x-2;
>> secant(f,1)
ans =1.414213562373095
```

Secant method given as python code:

PROBLEM Secant method with one initial root estimation, quadratic difference Formula in python language

```
 from math import *

#def df(f,x):
#  h=0.001
#   return (f(x+dx)-f(x-dx))/(2.0*h)
def df(f,x):
  h=0.001
  return (-f(x+2.0*h)+8.0*f(x+h)-8.0*f(x-h)+f(x-2.0*h))/(12.0*h);

def secant(f,x):
  # secant root finding method
 nmax=100
 tolerance=1.0e-10
 for i in range(0,nmax):
    fx= f(x);
    dfx=df(f,x)
    x=x-fx/dfx
    print("fx",fx,"dfx",dfx,"x=",x)
    if abs(fx)<tolerance: return x
 return x

f1 = lambda x: x*x-2
a=float(input("enter initial guess : "))
r=secant(f1,a)
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

```
 ---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" secant.py
enter initial guess : 1
fx -1.0 dfx 1.9999999999998537 x= 1.5000000000000366
fx 0.25000000000011013 dfx 2.9999999999995217 x= 1.4166666666666532
fx 0.00694444444440645 dfx 2.8333333333329103 x= 1.4142156862745094
fx 6.0073048815389996e-06 dfx 2.828431372548642 x= 1.4142135623746899
fx 4.510614104447086e-12 dfx 2.828427124748826 x= 1.4142135623730951
 ROOT  :  1.4142135623730951
FUNCTION VALUE :  4.440892098500626e-16

> Terminated with exit code 0.
```

Note: lambda variable type function is used to define function in the last python code.

Second version of the python secant method (utilising class f_x as abstract root class)

```
from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
    y=x*x-2;
    return y;

def secant(f,x):
  # secant root finding method
 nmax=100
 tolerance=1.0e-10
 for i in range(0,nmax):
    fx= f.func(x);
    dfx=f.dfunc(x,1);
    x=x-fx/dfx
    print("fx",fx,"dfx",dfx,"x=",x)
    if abs(fx)<tolerance: return x
 return x

f=f1();
a=float(input("enter initial guess : "))
r=secant(f,a);
```

```
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

PROBLEM Secant method with one initial root estimation, Fortran 90 version

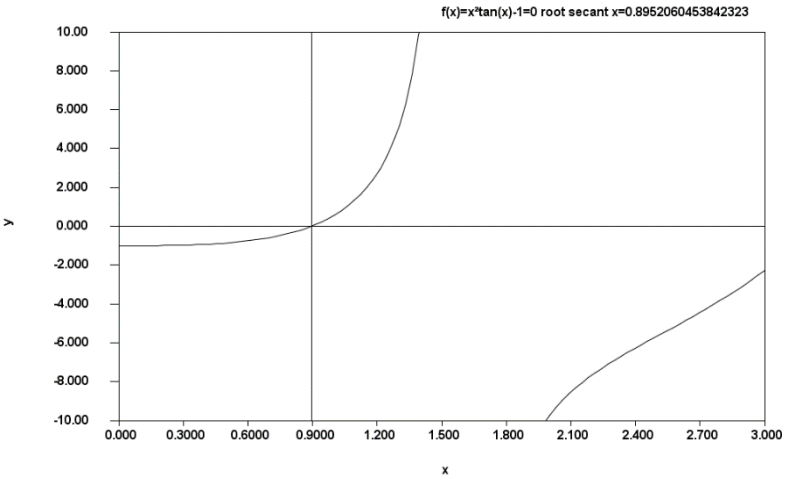
```
 program secant1
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) a
a=1.0
print *,'secant(',a,') = ',secant(a)

CONTAINS
 FUNCTION func(x)
 REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
 func=x*x-2.0
 END FUNCTION func

 FUNCTION dfunc(x)
 REAL(KIND = SELECTED_REAL_KIND(10,34)) dfunc,x,h,df,f1,f2,f3,f4,f5,f6,f7,f8
 h=1.12314e-4
 f1=3.0*func(x-4.0*h)
 f2=-32.0*func(x-3.0*h)
 f3=168.0*func(x-2.0*h)
 f4=-672.0*func(x-h)
 f5=672.0*func(x+h)
 f6=-168.0*func(x+2.0*h)
 f7=32.0*func(x+3.0*h)
 f8=-3.0*func(x+4.0*h)
 dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(840.0*h)
 END FUNCTION dfunc

 FUNCTION secant(a)
 REAL(KIND = SELECTED_REAL_KIND(10,34)) secant,a,x,eps,fx,dfx
 INTEGER nmax,i
 eps=1.0e-8;
 nmax=100;
 x=a
 DO i=1,nmax
  fx=func(x)
  dfx=dfunc(x)
  x=x-fx/dfx
  IF (abs(fx)<eps) EXIT
 END DO
 secant=x
 END FUNCTION secant
END PROGRAM secant1
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" secant1
E:\okul\SCO1>secant1
 Newton_Raphson(   1.0000000000000000      ) =    1.4142135623730951
> Terminated with exit code 0.
```

Example problem: Use Secant method to find the smallest positive root of $f(x) = x^2 \tan(x) - 1$

```
 import java.io.*;
class PlotT01
{
public static double secant(if_x f,double x)
{ double eps=1.0e-10;
 double y=f.dfunc(x);
 double dy=f.dfunc2(x);
 int miter=100;
 int i=0;
 while(Math.abs(y)>eps && i<miter)
 {x=x-y/dy;
  y=f.func(x);
  dy=f.dfunc(x);
  i++;
 }
 return x;
}
public static void main (String args[]) throws java.io.IOException
{  double ymin=-10;
  double ymax=10;
  double xmin=0;
```

```
   double xmax=3;
   int N=100;
   if_x f1=x->x*x*Math.tan(x)-1.0;
   if_x f2=x->0.0;
   Plot p=new Plot(f1,xmin,xmax,N);
   p.addFunction(f2,xmin,xmax,N);
   double x=secant(f1,(xmin+xmax)/2.0);+
   double xx[]={x,x};
   double yy[]={ymin,ymax};
   p.addData(xx,yy);
   p.setPlabel("f(x)=x"+'\u00B2'+"tan(x)-1=0 root secant x="+x);
   p.setYMinMax(ymin,ymax);
   p.plot();
} }
```



PROBLEM : Secant method with initial root estimation (juliacode)

```
 function secant(f,df,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x-=fx/dfx
  println("i = $i x = $x fx = $fx dfx = $dfx")
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function df(f,x)
 dx=0.001
 ff=(-3.0f(x+3dx)+32.0f(x+2dx)-168.0f(x+2dx)+672.0f(x+dx)-672.0f(x-dx)+168.0f(x-2dx)-32f(x-3dx)+3f(x-4dx))/(840.0dx)
 return ff
end

f(x)=x*x-2.0
x=1.0
x=secant(f,df,x)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton.jl
i = 0 x = 1.5179235458526263 fx = -1.0 dfx = 1.9307869047617066
i = 1 x = 1.4141688176043983 fx = 0.3040918910538104 dfx = 2.930872608776783
i = 2 x = 1.4142151659128144 fx = -0.0001265553153781962 dfx = 2.7305271692308506
i = 3 x = 1.4142135049334956 fx = 4.535497808877409e-6 dfx = 2.7306166656073354
i = 4 x = 1.4142135644306453 fx = -1.6246371781214464e-7 dfx = 2.730613458335259
i = 5 x = 1.4142135622993912 fx = 5.819630999326364e-9 dfx = 2.730613573221527
i = 6 x = 1.4142135623757353 fx = -2.0846591120005087e-10 dfx = 2.730613569106061
i = 7 x = 1.4142135623730003 fx = 7.467804152838653e-12 dfx = 2.7306135692534137
x = 1.4142135623730003
> Terminated with exit code 0.
```

Let us try a complex root now (Julia code):

```
 function secant(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x-=fx/dfx
  println("i = $i x = $x fx = $fx dfx = $dfx")
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function df(f,x)
 dx=0.001
 ff=(-3.0f(x+3dx)+32.0f(x+2dx)-168.0f(x+2dx)+672.0f(x+dx)-672.0f(x-dx)+168.0f(x-2dx)-32f(x-3dx)+3f(x-4dx))/(840.0dx)
 return ff
end

f(x)=x*x+2.0
x=1.0+1im
x=secant(f,x)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton.jl
i = 0 x = -0.035802703577893835 + 1.0000443843233529im fx = 2.0 + 2.0im dfx = 1.9307869047618762 + 1.9309523809520521im
i = 1 x = 0.0198149936939985 + 1.5165224545909592im fx = 1.0011930629668127 - 0.07160858531333268im dfx = -
0.06929879190867586 + 1.931038084967228im
i = 2 x = 0.0006250391365980393 + 1.4140140224387467im fx = -0.29944772130349495 + 0.06009976574905397im dfx =
0.03809633306177515 + 2.9283326444601654im
i = 3 x = -2.2429148695638293e-5 + 1.4142206082908915im fx = 0.0005647350205177926 + 0.0017676282074452696im dfx =
0.0010414446185980615 + 2.730393743328152im
i = 4 x = 8.01470657986998e-7 + 1.414213308849167im fx = -1.9928411592040618e-5 - 6.343952862358489e-5im dfx = -
0.0002087858083918558 + 2.7307926507712215im
i = 5 x = -2.86434804586515e-8 + 1.414213571488776im fx = 7.170745328632933e-7 + 2.266900942354623e-6im dfx = -
0.00016392858874727945 + 2.7307785558968445im
i = 6 x = 1.023670395186448e-9 + 1.4142135620453358im fx = -2.578303837097451e-8 - 8.101599759859701e-8im dfx = -
0.00016553149967005586 + 2.7307790630414224im
i = 7 x = -3.6584161715999224e-11 + 1.4142135623848795im fx = 9.27043309893633e-10 + 2.8953771118739662e-9im dfx = -
0.00016547421375779 86 + 2.7307790448065887im
i = 8 x = 1.3074484980161337e-12 + 1.4142135623726713im fx = -3.33315597345063e-11 - 1.0347563533449558e-10im dfx = -
0.00016547626107514074 + 2.7307790454622314im
i = 9 x = -4.672556362748515e-14 + 1.4142135623731102im fx = 1.198596777385319e-12 + 3.69802279599639e-12im dfx = -
0.00016547618804546 32 + 2.730779045438658im
x = -4.672556362748515e-14 + 1.4142135623731102im
> Terminated with exit code 0.
```

Julia built-in secant method:

```
using Roots
f(x)=x*x-2.0
x=find_zero(f,1.0)
print("x = $x")
```

```
---------- Capture Output ----------
> "D:\co\Julia\bin\julia.exe" secant_j.jl
x = 1.4142135623730951
> Terminated with exit code 0.
```

MULLER'S METHOD



Figure Werner Muller, German Mathematician

In false position method roots were found by using the root of the linear line passing through the two interval points. The same concept can be use to fit a quadratic curve and find the roots of the quadratic function. But quadratic function fitting is required 3 estimation point. The method consist of deriving the coefficients of a quadratic curve(parabola) that goes through three points. If function evaluation of these 3 points are $f(x_0)$, $f(x_1)$ and $f(x_2)$, enough information will be available to make a quadratic curve root estimation. Assuming quadratic function defined b the following equation :

$f(x) = a(x - x_2)^2 + b(x - x_2) + c$ (2.6.1)

Evaluation of the function in 3 given points will be

$f(x_0) = a(x_0 - x_2)^2 + b(x_0 - x_2) + c$ (2.6.2)

$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$ (2.6.3)

$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c = c$ (2.6.4)

Considering the difference of the functions:

$f(x_0) - f(x_2) = a(x_0 - x_2)^2 + b(x_0 - x_2)$ (2.6.5)

$f(x_1) - f(x_2) = a(x_1 - x_2)^2 + b(x_1 - x_2)$ (2.6.6)

$h_0 = (x_1 - x_0)$ (2.6.7)

$h_1 = (x_2 - x_0)$ (2.6.8)

$d_0 = \frac{f(x_1) - f(x_0)}{h_0}$ (2.6.9)

$d_1 = \frac{f(x_2) - f(x_0)}{h_1}$ (2.6.10)

When these are substituted back into the main equation

$(h_0 + h_1)b - (h_0 + h_1)^2 a = h_0 d_0 + h_1 d_1$ (2.6.11)

$h_1 b - (h_1)^2 a = h_1 d_1$ (2.6.12)

is obtained. a and b can be solved from here.

$a = \frac{d_1 - d_0}{h_1 - h_0}$ (2.6.13)

$b = a h_1 + d_1$ (2.6.14)

$c = f(x_2)$ (2.6.16)

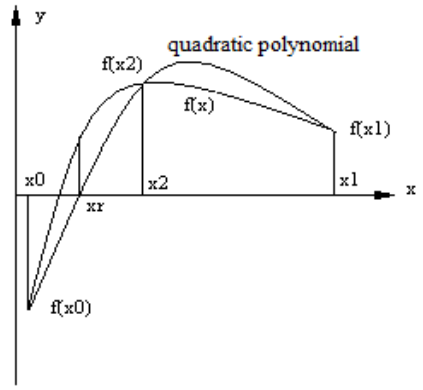


Figure graphic representation of Muller root finding method.

to find root root :

$\Delta = \sqrt{b^2 - 4ac}$ (2.6.17)

if

$|b + \Delta| > |b + \Delta| \quad then\ e = b + \Delta$ (2.6.18)

Else

$e = b + \Delta$ (2.6.19)

$x_r = x_2 - \frac{2c}{e}$ (2.6.20)

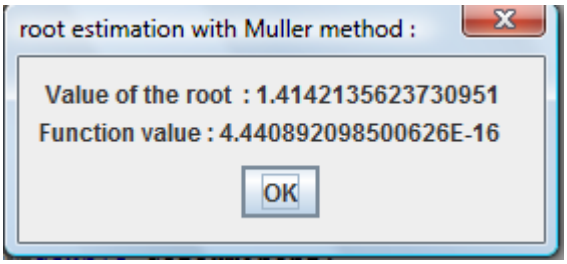
$x_0 = x_1 \quad x_1 = x_2 \quad x_2 = x_r$ (2.6.21)

x_2 will be substituted with x_r and iteration continues. It should be noted that in a quadratic formula complex roots can be located as well as the real roots, therefore this method can be used to find complex roots as well. The program NA10 is developed to calculate real roots by

Program NA10 : Muller method root finding with quadratic formula

```
//Numerical Analysis
// Muller method for root finding
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Function definition ================
class fb extends f_x
{public double func (double x)
  { return x*x-2;}
}
//============================================================
public class NA10
{public static double muller(f_x f,double x0,double x1,double x2)
{ //Finding real roots by using Muller method
  //orta xr noktasindan +/- h kadar giderek diğer 2 noktaya ulaşiyoruz.
int maxit=5;
double xr=x0;
int iter=0;
double es1=0.001;
double es=es1;
double ea=1.1*es;
double h0,h1;
double d0,d1;
double a,b,c;
double fx0,fx1,fx2;
double determinant;
double den;
double dxr;
while(iter<maxit && ea>es)
{
   fx0=f.func(x0);
   fx1=f.func(x1);
   fx2=f.func(x2);
   h0=x1-x0;
   h1=x2-x1;
   d0=(fx1-fx0)/h0;
   d1=(fx2-fx1)/h1;
   a=(d1-d0)/(h1+h0);
   b=a*h1+d1;
   c=fx2;
   determinant=b*b-4.0*a*c;
   if(determinant<=0) determinant=0;//we do not want to calculate complex roots here
   else determinant=Math.sqrt(determinant);
   if(Math.abs(b+determinant)>Math.abs(b-determinant)) {den=b+determinant;;}
   else {den=b-determinant;}
   dxr=-2*c/den;
   xr=x2+dxr;
   System.out.println("a="+a+"b="+b+"c="+c+"disc="+determinant+"xr="+xr+"iter="+iter);
   ea=Math.abs((xr-x2)/xr)*100;
   iter++;
   x0=x1;
   x1=x2;
   x2=xr;
```

```
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
   " çözüm geçerli olmiyabilir","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return xr;
}

    public static void main (String args[])
    {
                fb f=new fb();
    double x0,x1,x2;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x0 : "));
    x1=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x1 : "));
    x2=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x2 : "));
    //h=Double.parseDouble(JOptionPane.showInputDialog(" h=(küçük bir rakam örneğin 0.1"));
    double r;
    r= muller(f,x0,x1,x2);
    JOptionPane.showMessageDialog(null," Value of the root  : "+r+"\nFunction value : "+f.func(r),
    "root estimation with Muller method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Program Muller method root finding with quadratic formula in C++

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x){ double ff=x*x-2.0;return ff;}};

double muller(f_x &f,double x0,double x1,double x2)
{
//Finding real roots by using Muller method
//orta xr noktasından +/- h kadar giderek diğer 2 noktaya ulaşıyoruz.
int maxit=5;
double xr=x0;
int iter=0;
double es1=0.001;
double es=es1;
double ea=1.1*es;
double h0,h1;
double d0,d1;
double a,b,c;
double fx0,fx1,fx2;
double determinant;
double den;
double dxr;
while(iter<maxit && ea>es)
{
```

```
    fx0=f.func(x0);
    fx1=f.func(x1);
    fx2=f.func(x2);
    h0=x1-x0;
    h1=x2-x1;
    d0=(fx1-fx0)/h0;
    d1=(fx2-fx1)/h1;
    a=(d1-d0)/(h1+h0);
    b=a*h1+d1;
    c=fx2;
    determinant=b*b-4.0*a*c;
    if(determinant<=0) determinant=0;//we do not want to calculate complex roots here
    else determinant=sqrt(determinant);
    if(fabs(b+determinant)>fabs(b-determinant)) {den=b+determinant;;}
    else {den=b-determinant;}
    dxr=-2*c/den;
    xr=x2+dxr;
    ea=fabs((xr-x2)/xr)*100;
    iter++;
    x0=x1;
    x1=x2;
    x2=xr;
}
if(iter>=maxit) cout<<"Warning : Maximum number of iteration is exceeded \n";
return xr;
}

     int main()
     {  f1 f;
     double x0,x1,x2;
     cout << "root estimation x0 : ";
     cin >>x0;
     cout << "root estimation x1 : ";
     cin >>x1;
     cout << "root estimation x2 : ";
     cin >>x2;
     double r;
     cout<<setprecision(20);
     r= muller(f,x0,x1,x2);
     cout<<" Value of the root  : "<<r<<"\nFunction value : "<<f.func(r);
     }
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" NA10

D:\okul\SCO1>NA10
root estimation x0 : 1
root estimation x1 : 3
root estimation x2 : 4
 Value of the root  : 1.41421
Function value : 2.73436e-016
> Terminated with exit code 0.
```

Program Muller method root finding with quadratic formula in Matlab (Octave)

```
function xr=muller(f,x0,x1,x2)
% Finding real roots by using Muller method
%
iter=0;
es1=0.001;
es=es1;
ea=1.1*es;
maxit = 100;
xr=x0;
while (iter<maxit) && (ea>es)
    fx0=f(x0);
    fx1=f(x1);
    fx2=f(x2);
    h0=x1-x0;
    h1=x2-x1;
    d0=(fx1-fx0)/h0;
    d1=(fx2-fx1)/h1;
    a=(d1-d0)/(h1+h0);
    b=a*h1+d1;
    c=fx2;
```

```
    determinant=b*b-4.0*a*c;
    if determinant<=0  determinant=0;
    else determinant=sqrt(determinant);end
    if abs(b+determinant)>  abs(b-determinant) den=b+determinant;
    else den=b-determinant;end
    dxr=-2*c/den;
    xr=x2+dxr;
    ea=abs((xr-x2)/xr)*100;
    iter=iter+1;
    x0=x1;
    x1=x2;
    x2=xr;
end
if iter>=maxit  fprintf('Warning : Maximum number of iteration is exceeded result may not be valis');end
end
```

```
>> f=@(x) x*x-2;
>> muller(f,0,1,2)
ans =1.414213562373095
```

Program Muller method root finding with quadratic formula in Python code

```
# Muller method for root finding
# python version

from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
      y=x*x-2;
      return y;

def muller(f,x0,x1,x2):
  # Finding real roots by using Muller method
  iter=0;
  es1=0.001;
  es=es1;
  ea=1.1*es;
  maxit = 100;
  xr=x0;
  for iter in range(0,maxit):
    fx0=f.func(x0);
    fx1=f.func(x1);
    fx2=f.func(x2);
    h0=x1-x0;
    h1=x2-x1;
    d0=(fx1-fx0)/h0;
    d1=(fx2-fx1)/h1;
    a=(d1-d0)/(h1+h0);
    b=a*h1+d1;
    c=fx2;
    determinant=b*b-4.0*a*c;
    if determinant<=0:  determinant=0;
    else: determinant=sqrt(determinant);
    if abs(b+determinant)>  abs(b-determinant): den=b+determinant;
    else: den=b-determinant;
    dxr=-2*c/den;
    xr=x2+dxr;
    ea=abs((xr-x2)/xr)*100;
    x0=x1;
    x1=x2;
    x2=xr;
    if ea<es: return xr
  if iter>=maxit:  print('Warning : Maximum number of iteration is exceeded result may not be valis');
  return xr;


f=f1();
x0=float(input("enter first number : "));
x1=float(input("enter second number : "));
x2=float(input("enter third number : "));
r=muller(f,x0,x1,x2);
print("muller method root  : ",r);
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" muller.py
enter first number : 0
enter second number : 1
enter third number : 3
muller method root  :  1.4142135623730951

> Terminated with exit code 0.
```

Program Muller method root finding with quadratic formula in Fortran 90

```
program muller1
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) x0,x1,x2
x0=1.0
x1=1.15
x2=3
print *,'muller(',x0,',',x1,',',x2,') = ',muller(x0,x1,x2)

CONTAINS
  FUNCTION func(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x-2.0
  END FUNCTION func

  FUNCTION muller(x0,x1,x2)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) muller,eps,x0,x1,x2,xr,dxr,fx0,fx1,fx2,h0,h1,d0,d1,a,b,c,determinant,den
  INTEGER maxit,i
  eps=1.0e-8;
  maxit=100
  i=0;
  fx0=func(x0)
  fx1=func(x1)
  fx2=func(x2)
  DO i=1,maxit
    h0=x1-x0
    h1=x2-x1
    d0=(fx1-fx0)/h0
    d1=(fx2-fx1)/h1
    a=(d1-d0)/(h1+h0)
    b=a*h1+d1
    c=fx2
    determinant=b*b-4.0*a*c
    IF (determinant<=0) THEN
      determinant=0
    ELSE
      determinant=sqrt(determinant)
    END IF
    IF (abs(b+determinant)>abs(b-determinant)) THEN
      den=b+determinant
    ELSE
      den=b-determinant
    END IF
    dxr=-2.0*c/den
    xr=x2+dxr
    x0=x1
    x1=x2
    x2=xr
    fx0=func(x0)
    fx1=func(x1)
    fx2=func(x2)
    IF (abs(fx2)<eps) EXIT
  END DO
  muller=xr
  END FUNCTION muller
END PROGRAM muller1
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" muller1

E:\okul\SCO1>muller1
 muller(   1.0000000000000000     ,   1.1499999761581421     ,   3.0000000000000000     ) =    1.4142135623730949
```

```
> Terminated with exit code 0.
```

Program Muller method root finding with quadratic formula

```
 function  muller(f,x0,x1,x2)
 # Finding real roots by using Muller method
maxit=5
xr=x0
iter=0
es1=0.001
es=es1
ea=1.1*es
h0=0.0
h1=0.0
d0=0.0
d1=0.0
a=0.0
b=0.0
c=0.0
fx0=0.0
fx1=0.0
fx2=0.0
determinant=0.0
den=0.0
dxr=0.0
while iter<maxit && ea>es
   fx0=f(x0)
   fx1=f(x1)
   fx2=f(x2)
   h0=x1-x0
   h1=x2-x1
   d0=(fx1-fx0)/h0
   d1=(fx2-fx1)/h1
   a=(d1-d0)/(h1+h0)
   b=a*h1+d1
   c=fx2
   determinant=b*b-4.0*a*c
   if determinant<=0.0
     determinant=0
   else
     determinant=sqrt(determinant)
   end
   if abs(b+determinant)>abs(b-determinant)
     den=b+determinant
   else
     den=b-determinant
   end
   dxr=-2*c/den
   xr=x2+dxr
   println("a= $a b= $b c= $c disc= $determinant xr= $xr iter= $iter")
   ea=abs((xr-x2)/xr)*100
   iter+=1
   x0=x1
   x1=x2
   x2=xr
end
if iter>=maxit
  print("Maximum number of iteration is exceeded \nresult might not be valid")
end
return xr
end

f(x)=x*x-2.0
x0=0.0
x1=1.0
x2=2.0
x=muller(f,x0,x1,x2)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" muller.jl
```

```
a= 1.0 b= 4.0 c= 2.0 disc= 2.8284271247461903 xr= 1.414213562373095 iter= 0
a= 1.0 b= 2.82842712474619 c= -4.440892098500626e-16 disc= 2.8284271247461903 xr= 1.4142135623730951 iter= 1
x = 1.4142135623730951
> Terminated with exit code 0.
```

Complex roots:

```
function  muller(f,x0,x1,x2)
 # Finding real roots by using Muller method
maxit=5
xr=x0
iter=0
es1=0.001
es=es1
ea=1.1*es
h0=0.0
h1=0.0
d0=0.0
d1=0.0
a=0.0
b=0.0
c=0.0
fx0=0.0
fx1=0.0
fx2=0.0
determinant=0.0
den=0.0
dxr=0.0
while iter<maxit && ea>es
   fx0=f(x0)
   fx1=f(x1)
   fx2=f(x2)
   h0=x1-x0
   h1=x2-x1
   d0=(fx1-fx0)/h0
   d1=(fx2-fx1)/h1
   a=(d1-d0)/(h1+h0)
   b=a*h1+d1
   c=fx2
   determinant=sqrt(b*b-4.0*a*c)
   if abs(b+determinant)>abs(b-determinant)
     den=b+determinant
   else
     den=b-determinant
   end
   dxr=-2*c/den
   xr=x2+dxr
   println("a= $a b= $b c= $c disc= $determinant xr= $xr iter= $iter")
   ea=abs((xr-x2)/xr)*100
   iter+=1
   x0=x1
   x1=x2
   x2=xr
end
if iter>=maxit
  print("Maximum number of iteration is exceeded \nresult might not be valid")
end
return xr
end

f(x)=x*x+2.0
x0=0.0+1im
x1=1.0+1im
x2=2.0+1im
x=muller(f,x0,x1,x2)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" muller.jl
a= 1.0 + 0.0im b= 4.0 + 2.0im c= 5.0 + 4.0im disc= 0.0 + 2.8284271247461903im xr= 0.0 + 1.414213562373095im iter= 0
a= 1.0 - 0.0im b= 0.0 + 2.82842712474619im c= 4.440892098500626e-16 + 0.0im disc= 0.0 + 2.8284271247461903im xr= 0.0 +
1.4142135623730951im iter= 1
x = 0.0 + 1.4142135623730951im
> Terminated with exit code 0.
```

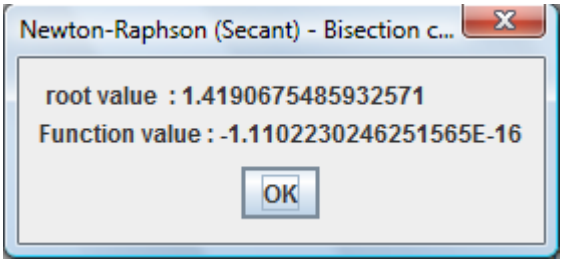
NEWTON-RAPHSON (SECANT) – BISECTION COMBINED METHOD

Bisection method is good in getting closer to the root. But when it get close to root it takes a long time to achieve the final accuracy. On the other hand, Newton-Raphson method is very efficient when the estimation is close to the actual root but rather poor performer when the root estimation is not very close to the actual root. By combining two methods advantages of both methods will be maximized while inefficiencies will be eliminated. Therefore a combine method is a good candidate for an efficient algorithm.

Program NA11 : Newton-Raphson (Secant)-Bisection combined method

```
//
// Newton-Raphson (Secant)-Bisection combined root finding method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Function definition ================
class fb extends f_x
{
  public double func (double x)
  { return Math.log(x*x)-0.7;}
}
//===========================================================

public class NA11
{
public static double dfx(f_x f,double x,double h)
{ //Derivative formula
 double hh=1.0/h;
 return (-f.func(x+2.0*h)+8.0*f.func(x+h)-8.0*f.func(x-h)+f.func(x-2.0*h))/12.0*hh;
}

public static double newton_bisection(f_x fx, double x1, double x2)
{
double xacc=1.0e-10; //accuracy
double h=0.0001; //derivative function step
int MAXIT=100;  //Maximum number of iterations
int j;
double dfr,dx,dxold,fr,fh,fl;
double temp,b,a,r;
fl=fx.func(x1);
fh=fx.func(x2);
//Check if a root is existed in the given region
if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
System.out.println("There are no root in the given region or double root?");
if (fl == 0.0) return x1;
if (fh == 0.0) return x2;
if (fl < 0.0)   { a=x1;b=x2;}
else           { b=x1;a=x2;}
r=0.5*(x1+x2); //midpoint value
dxold=Math.abs(x2-x1);
dx=dxold;
fr=fx.func(r);   //function value at midpoint
dfr=dfx(fx,r,h); //derivative of function value at midpoint
for (j=1;j<=MAXIT;j++)
{
if ((((r-b)*dfr-fr)*((r-a)*dfr-fr) > 0.0) || (Math.abs(2.0*fr) > Math.abs(dxold*dfr)))
   { //bisection step
                dxold=dx;
     dx=0.5*(b-a);
     r=a+dx;
     if (a == r) return r; //çözüm!!!!
   }
else{ //Newton-Raphson (Secant) step
```

```
                dxold=dx;
      dx=fr/dfr;
      temp=r;
      r -= dx;
      if (temp == r) return r;  //solution!!!!!
    }
if (Math.abs(dx) < xacc) return r; //solution!!!!!
fr=fx.func(r);
dfr=dfx(fx,r,h);
if (fr < 0.0)
a=r;
else
b=r;
}
System.out.println("Maximum number of iterations are exceeded");
return 0.0; //program should never reach here
          //dummy return
}
    public static void main (String args[])
    {   fb f=new fb();
    double x0,x1;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" lower root limit : "));
    x1=Double.parseDouble(JOptionPane.showInputDialog(" higher root limit : "));
    double r= newton_bisection(f,x0,x1);
    JOptionPane.showMessageDialog(null," root value  : "+r+"\nFunction value : "+f.func(r),
    "Newton-Raphson (Secant) - Bisection combined root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



C++ version of the code:

Program Newton-Raphson (Secant)-Bisection combined method C++ version

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x)
  { double ff=log(x*x)-0.7;return ff;}};

double newton_bisection(f_x &f, double x1, double x2)
{
double xacc=1.0e-10; //accuracy
double h=0.0001; //derivative function step
int MAXIT=100;  //Maximum number of iterations
int j;
double dfr,dx,dxold,fr,fh,fl;
double temp,b,a,r;
```

```
fl=f.func(x1);
fh=f.func(x2);
//Check if a root is existed in the given region
if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
cout<<"There are no root in the given region or double root?";
if (fl == 0.0) return x1;
if (fh == 0.0) return x2;
if (fl < 0.0)   { a=x1;b=x2;}
else            { b=x1;a=x2;}
r=0.5*(x1+x2); //midpoint value
dxold=fabs(x2-x1);
dx=dxold;
fr=f.func(r);   //function value at midpoint
dfr=f.dfunc(r); //derivative of function value at midpoint
for (j=1;j<=MAXIT;j++)
{
if ((((r-b)*dfr-fr)*((r-a)*dfr-fr) > 0.0) || (fabs(2.0*fr) > fabs(dxold*dfr)))
    { //bisection step
                dxold=dx;
      dx=0.5*(b-a);
      r=a+dx;
      if (a == r) return r; //çözüm!!!!
    }
else{ //Newton-Raphson (Secant) step
                dxold=dx;
      dx=fr/dfr;
      temp=r;
      r -= dx;
      if (temp == r) return r;  //solution!!!!!
    }
if (fabs(dx) < xacc) return r; //solution!!!!!
fr=f.func(r);
dfr=f.dfunc(r);
if (fr < 0.0)
a=r;
else
b=r;
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return 0.0; //program should never reach here
          //dummy return
}

int main()
{  double x1,x2,r;
   cout << " first guess for the root value : ";
   cin >>x1;
   cout << " second guess for the root value : ";
   cin >>x2;
   f1 f;
   cout<<setprecision(20);
   r=newton_bisection(f,x1,x2);
   cout<<"root value : ="<<r;
   return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" newton_bisection
D:\okul\SCO1>newton_bisection
 first guess for the root value : 1
 second guess for the root value : 3
root value : =1.4190675485932571
> Terminated with exit code 0.
```

Program Newton-Raphson (Secant)-Bisection combined method Fortran 90 version

```
program newton_bisection1
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) a,b
a=1.0
b=2.15
print *,'newton_bisection(',a,',',b,') = ',newton_bisection(a,b)

CONTAINS
  FUNCTION func(x)
```

```
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x-2.0
  END FUNCTION func
  FUNCTION dfunc(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) dfunc,x,dx,xp,xn
  dx=1.12314e-4
  xp=x+dx
  xn=x-dx
  dfunc=(func(xp)-func(xn))/(2.0*dx)
  END FUNCTION dfunc
  FUNCTION newton_bisection(a,b)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) newton_bisection,a,b,b1,r,eps,fa,fb,fr,dfr,remain
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  i=0;
  fa=func(a)
  fb=func(b)
  DO i=1,nmax
   remain=i-(i/4)*4
   if(remain==0) then
    fr=func(r)
    dfr=dfunc(r)
    r=r-fr/dfr
   else
    r=(a+b)/2.0
    fr=func(r)
    IF (fr*fa<0) THEN
      b=r
      fb=fr
    ELSE
      a=r
      fa=fr
    END IF
   end if
   IF (abs(fr)<eps) EXIT
  END DO
  newton_bisection=r
  END FUNCTION newton_bisection
END PROGRAM newton_bisection1
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" newton_bisection1

E:\okul\SCO1>newton_bisection1
 newton_bisection(   1.0000000000000000     ,   2.1500000953674316     ) =    1.4142135609714437

> Terminated with exit code 0.
```

Another problem will be defined as an engineering problem from heat transfer:
Time dependent temperature of a flat wall will be given with the differential equation :
In this equation T is temperature (degree Kelvin), x is the distance(meter), t is the time(second)

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} \quad (2.7.1)$$

$$\alpha = \frac{k}{\rho C_p} \quad (2.7.2)$$

And α is thermal diffusivity, where k is thermal conductivity(W/(mK)), ρ is density (kg/m3) and C_p is specific heat(J/(kgK))

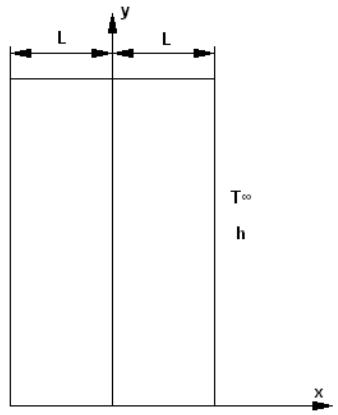


Figure geometry of wall in time dependent heat transfer problemT_∞

Boundary conditions:

$T(x,0) = T_i$ (2.7.3)

$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = 0$ (2.7.4)

$\left.-k\frac{\partial T(x,t)}{\partial x}\right|_{x=L} = [\![hT(L,t) - T_\infty]\!]$ (2.7.5)

Converted dimensionless form of the equation and boundary conditions

Boundary conditions:

$\theta(\bar{x},0) = 1$ (2.7.6)

$\left.\frac{\partial \theta(\bar{x},t)}{\partial \bar{x}}\right|_{\bar{x}=0} = 0$ (2.7.7)

$\left.\frac{\partial \theta(\bar{x},\bar{t})}{\partial \bar{x}}\right|_{\bar{x}=L} = -Bi * \theta(1,Fo)$ (2.7.8)

$\bar{x} = \frac{x}{L} \quad \bar{t} = Fo = \frac{\alpha t}{L^2} \qquad \theta(\bar{x},\bar{t}) = \frac{T(x,t)-T_\infty}{T_i-T_\infty}$ (2.7.9)

Where Bi is called Biot Number, and Fo is caled Fourier number, t is time α is thermal diffusivity coefficient, h is thermal convectivity coefficient, k is thermal conductivity coefficient, T_i is initial time and T_∞ is environmental temperature

$T = T(x,t,T_i,T_\infty,L,k,h,\alpha) \quad \theta = T(\bar{x},Fo,Bi)$ (2.7.10)

Analytic solution of this differential equation :

$\theta = \sum_{n=1}^{\infty} C_n e^{-\zeta_n^2 Fo}\cos(\zeta_n \bar{x})$ (2.7.11)

Where

$C_n = \frac{4\sin(\zeta_n)}{2\zeta_n + \sin(2\zeta_n)}$ (2.7.12)

And

$\zeta_n \tan(\zeta_n) = Bi$ (2.7.13)

As it is seen from the last equation, this equation can only be solved by using numerical root finding methods. After finding the roots coefficients C_n's can be evaluated and heat transfer equation is solved. Program calculated the heat transfer is given as a whole below:

Program HT_wall_x_t : Newton-Raphson (Secant)-Bisection combined method

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class HT_duvar
{

public static double fx(double lambda,double Bi)
{return lambda*Math.tan(lambda)-Bi;}
```

```
public static double dfx(double x,double h,double Bi)
{ //Türev hesabi
  return (-fx((x+2.0*h),Bi)+8.0*fx((x+h),Bi)-8.0*fx((x-h),Bi)+fx((x-2.0*h),Bi))/(12.0*h);
}

public static double Cn(double ln,double Bi)
{
 return 4.0*Math.sin(ln)/(2.0*ln+Math.sin(2.0*ln));
}

public static double lambda(double n,double Bi)
{
 //linear olmayan denklemin kökü : ikiyebölme-Newton Raphson metodu
 //lambda*tan(lambda)-Bi=0 denkleminin köklerini bulur
 double x1=(n-1)*Math.PI;
 double x2=           x1+Math.PI/2-0.000001;
double xacc=1.0e-10; //iterasyonlar arasindaki fark, bu rakamdan
// küçük olduğunda çözüme ulaştiğimizi kabul edeceğiz
double h=0.0001; //türev step büyüklüğü
int MAXIT=100;  //Maksimum iterasyon sayisi
int j;
double dfr,dx,dxold,fr,fh,fl;
double temp,b,a,r;
fl=fx(x1,Bi);
fh=fx(x2,Bi);
//System.out.println("x1 ="+x1+"fl="+fl+"x2="+x2+"fh="+fh);
//Bölgemizde kök değeri varmi kontrol et
if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
System.out.println("n = "+n+"x1="+x1+"x2="+x2+"verilen sinirlar içinde kök değeri yok");
if (fl == 0.0) return x1;
if (fh == 0.0) return x2;
if (fl < 0.0)   { a=x1;b=x2;}
else           { b=x1;a=x2;}
r=0.5*(x1+x2); //orta nokta değeri
dxold=Math.abs(x2-x1);
dx=dxold;
fr=fx(r,Bi);   //fonksiyonun r noktasindaki değeri
dfr=dfx(r,h,Bi); //fonksiyonun türevinin r noktasindaki değeri
for (j=1;j<=MAXIT;j++)
{
if ((((r-b)*dfr-fr)*((r-a)*dfr-fr) > 0.0) || (Math.abs(2.0*fr) > Math.abs(dxold*dfr)))
    { //ikiye bölme
              dxold=dx;
      dx=0.5*(b-a);
      r=a+dx;
      if (a == r) return r; // solution!!!!!
    }
else{ //Newton-Raphson
              dxold=dx;
      dx=fr/dfr;
      temp=r;
      r -= dx;
      if (temp == r) return r;  //solution!!!!!
    }
if (Math.abs(dx) < xacc) return r; // solution!!!!!
fr=fx(r,Bi);
dfr=dfx(r,h,Bi);
if (fr < 0.0)
a=r;
else
b=r;
}
```

```
System.out.println("Maximum iteration number is exceeded");
return 0.0; //program never reach here
}

public static double teta(double Fo,double Bi,double x)
{double tt=0;double ln=0;
 for(int n=1;n<20;n++)
 {ln=lambda(n,Bi);tt+=Cn(ln,Bi)*Math.exp(-ln*ln*Fo)*Math.cos(ln*x);}
 return tt;
}

    public static void main (String args[])
    { fb f=new fb();
      double Bi,Fo,x;
      Fo=Double.parseDouble(JOptionPane.showInputDialog(" Forier sayisi Fo : "));
      Bi=Double.parseDouble(JOptionPane.showInputDialog(" Biot sayisi   Bi : "));
      x=Double.parseDouble(JOptionPane.showInputDialog("        x* = x/L : "));
      JOptionPane.showMessageDialog(null," teta = "+teta(Fo,Bi,x),
      "duvarda zamana bağli isi transferi  : ",JOptionPane.PLAIN_MESSAGE);
      System.exit(0);
    }
}
```

Program Newton-Raphson (Secant)-Bisection combined method Julia code

```
 function newton_bisection(f,x1,x2)
xacc=1.0e-10 #accuracy
h=0.0001    #derivative function step
MAXIT=100   #Maximum number of iterations
j=0
dfr=0.0
dx=0.0
dxold=0.0
fr=0.0
fh=0.0
fl=0.0
temp=.0
b=0.0
a=0.0
r=0.0
fl=f(x1)
fh=f(x2)
#Check if a root is existed in the given region
if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
  print("There are no root in the given region or root?")
end
if fl == 0.0
  return x1
elseif fh == 0.0
  return x2
elseif  fl < 0.0
  a=x1
  b=x2
else
  b=x1a=x2
end
r=0.5*(x1+x2) #midpovalue
dxold=abs(x2-x1)
dx=dxold
fr=f(r)   #function value at midpoint
dfr=df(f,r) #derivative of function value at midpo
for j=1:MAXIT
if ((r-b)*dfr-fr)*((r-a)*dfr-fr) > 0.0 || (abs(2.0*fr) > abs(dxold*dfr))
   #bisection step
              dxold=dx
   dx=0.5*(b-a)
   r=a+dx
```

```
    if a == r
      return r
    end
else
    #Newton-Raphson (Secant) step
                dxold=dx
    dx=fr/dfr
    temp=r
    r -= dx
    if (temp == r)
       return r  #solution!!!!!
    end
end
if (abs(dx) < xacc)
   return r #solution!!!!!
end
fr=f(r)
dfr=df(f,r)
if (fr < 0.0)
a=r
else
b=r
end
end
return 0.0
end

function df(f,x)
 dx=0.001
 ff=(-3.0f(x+3dx)+32.0f(x+2dx)-168.0f(x+2dx)+672.0f(x+dx)-672.0f(x-dx)+168.0f(x-2dx)-32f(x-3dx)+3f(x-4dx))/(840.0dx)
 return ff
end

f(x)=x*x-2.0
x1=1.0
x2=3.0
x=newton_bisection(f,x1,x2)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" secant.jl
x = 1.4142135623713463
> Terminated with exit code 0.
```

INVERSE QUADRATIC LAGRANGE INTERPOLATION

Lagrange interpolation will be investigated further in the interpolation section. In here, the Formula for the quadratic inverse lagrange polinomial will be introduced to utilise in finding roots of a function.

$$x = \frac{[y-f(x_1)][y-f(x_2)]x_3}{[f(x_3)-f(x_1)][f(x_3)-f(x_2)]} + \frac{[y-f(x_2)][y-f(x_3)]x_1}{[f(x_1)-f(x_2)][f(x_1)-f(x_3)]} + \frac{[y-f(x_3)][y-f(x_1)]x_2}{[f(x_2)-f(x_3)][f(x_2)-f(x_1)]} \quad (2.8.1)$$

In this equation if y=0 is taken, the value x will give us a quadratic approximation to the root of the function

$$x = \frac{f(x_1)f(x_2)x_3}{[f(x_3)-f(x_1)][f(x_3)-f(x_2)]} + \frac{f(x_2)f(x_3)x_1}{[f(x_1)-f(x_2)][f(x_1)-f(x_3)]} + \frac{f(x_3)f(x_1)x_2}{[f(x_2)-f(x_3)][f(x_2)-f(x_1)]} \quad (2.8.2)$$

In order to make the equation simpler further definations will apply as:

$R = \frac{f(x_2)}{f(x_3)}$ (2.8.4) $S = \frac{f(x_2)}{f(x_1)}$ (2.8.5) $T = \frac{f(x_1)}{f(x_3)}$ (2.8.6)

$$P = S[T(R-T)(x_3-x_2) - (1-R)(x_2-x_1)] \quad (2.8.7)$$

$$Q = (T-1)(R-1)(S-1) \quad (2.8.8)$$

Now the root can be written as:

$$x = x_2 + \frac{P}{Q} \qquad (2.8.9)$$

When a new root is found the neigbour region of the root is left for the next iteration and the region further away to the root is eliminated.

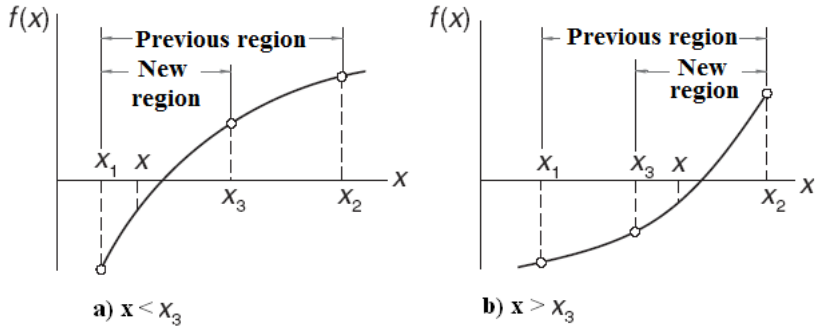
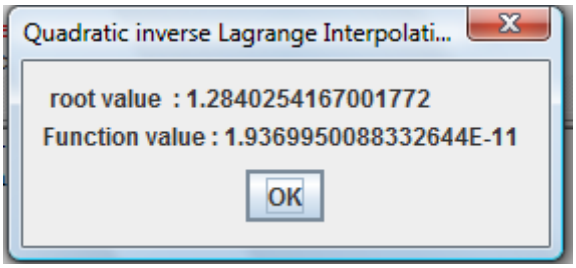


Figure Quadratic inverse Lagrange interpolation region selection

Program NA12 : quadratic inverse lagrange interpolation formula method (java code)

```
public class NA12
{public static double inverseinter2Droot(f_x f,double x1,double x3)
{double x2=(x1+x3)/2.0;
return inverseinter2Droot(f,x1,x2,x3);}
public static double inverseinter2Droot(f_x f,double x1,double x2,double x3)
{// Inverse quadratic Lagrange interpolation
double test;
double p=0;
double es,ea;
double f1,f2,f3,fp;
int maxit=100,iter=0;
double tol=1.0e-10;
es=0.00000001;
ea=0.001;
f1=f.func(x1);
f2=f.func(x2);
f3=f.func(x3);
if(f1==0) return x1;
else if(f2==0) return x2;
else if(f3==0) return x3;
if(f1*f3>0) System.out.println("Root is not existed in the given region");
while((ea>es)&&(iter<maxit))
{               p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=f.func(p);
   ea=Math.abs(x3-p);
               if(Math.abs(f3)<tol) return x3;
   if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
   else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   iter++;}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
  " çözüm geçerli olmiyabilir","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return p;}
    public static void main (String args[])
    {
                fc f=new fc();
    double x1,x2,x3;
    x1=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x1 : "));
    x3=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x2 : "));
    double r;
    r= inverseinter2Droot(f,x1,x3);
    JOptionPane.showMessageDialog(null," root value  : "+r+"\nFunction value : "+f.func(r),
    "Quadratic inverse Lagrange Interpolation method : ",JOptionPane.PLAIN_MESSAGE);
```

```
    } }
```



Program quadratic inverse lagrange interpolation formula method in C++

```
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x)
  { double ff=x*x-2.0;return ff;}};

double inverseinter2Droot(f_x &f,double x1,double x2,double x3)
{
// Inverse quadratic Lagrange interpolation
double test;
double p=0;
double es,ea;
double f1,f2,f3,fp;
int maxit=100,iter=0;
double tol=1.0e-10;
if(x1==0) x1=1.0e-10;
if(x2==0) x2=1.0e-10;
if(x3==0) x3=1.0e-10;
es=0.00000001;
ea=0.001;
f1=f.func(x1);
f2=f.func(x2);
f3=f.func(x3);
if(f1==0) return x1;
else if(f2==0) return x2;
else if(f3==0) return x3;
if(f1*f3>0) cout<<"Root is not existed in the given region";
while((ea>es)&&(iter<maxit))
{               p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=f.func(p);
   ea=fabs(x3-p);
               if(fabs(f3)<tol) return x3;
   if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
   else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   iter=iter+1;
   cout<<"iter="<<iter<<"p="<<p<<"fp="<<fp<<endl;
}
if(iter>=maxit) cout<<"Warning : MAximum number of iteration is exceeded \n";
return p;
}
    int main ()
    { double x1,x2,x3,r;
      cout << " first guess for the root value : ";
      cin >>x1;
```

```
      cout << " second guess for the root value : ";
      cin >>x3;
      x2=(x1+x3)/2.0;
      f1 f;
      cout<<setprecision(20);
      r=inverseinter2Droot(f,x1,x2,x3);
      cout<<"root value : ="<<r;
      return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" inverseinter2Droot

D:\okul\SCO1>inverseinter2Droot
 first guess for the root value : 1
 second guess for the root value : 3
iter=1p=1.3666666666666667fp=-0.13222222222222213
iter=2p=1.419336630469842fp=0.014516470593484942
iter=3p=1.4141618310219912fp=-0.00014631568052914416
iter=4p=1.4142135655958628fp=9.1153636039027069e-009
iter=5p=1.4142135623730949fp=-3.546425306200085e-016
root value : =1.4142135623730949
> Terminated with exit code 0.
```

Program quadratic inverse lagrange interpolation formula method in Matlab(Octave)

```
function p = inverseinter2Droot(f,x1,x2,x3)
%Inverse quadratic Lagrange erpolation
p=1;
maxit=100,iter=0;
tol=1.0e-10;
es=0.00000001;
ea=0.001;
f1=f(x1);
f2=f(x2);
f3=f(x3);
if f1==0 p=x1;
elseif f2==0 p=x2;
elseif f3==0 p=x3;end
if f1*f3>0 fprintf('Root is not existed in the given region');end
while (ea>es) && (iter<maxit) && f1~=0 && f2~=0 && f3~=0
   p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=f(p);
   ea=abs(x3-p);
   if abs(f3)<tol  p=x3; end
   if p>x3  x1=x3;f1=f3;x3=p;f3=fp;
   elseif(p<x3) x2=x3;f2=f3;x3=p;f3=fp;end
   iter=iter+1;
   end
if(iter>=maxit) fprintf('Warning : Maximum number of iteration is exceeded \nresult may not be valid');end
end
```

Program quadratic inverse lagrange interpolation formula method in Python language

```
 from math import *

from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
     y=x*x-2;
     return y;

def inverseinter2Droot(f,x1,x2,x3):
  #Inverse quadratic Lagrange erpolation
  p=1;
  maxit=100
  iter=0;
  tol=1.0e-10;
  es=0.00000001;
  ea=0.001;
  f1=f.func(x1);
  f2=f.func(x2);
  f3=f.func(x3);
```

```
  if f1==0: p=x1;
  elif f2==0: p=x2;
  elif f3==0: p=x3;
  if f1*f3>0: print("Root is not existed in the given region");
  while (ea>es) and (iter<maxit) and f1!=0 and f2!=0 and f3!=0:
    p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
    fp=f.func(p);
    ea=abs(x3-p);
    if abs(f3)<tol:  p=x3;
    if p>x3:  x1=x3;f1=f3;x3=p;f3=fp;
    elif p<x3: x2=x3;f2=f3;x3=p;f3=fp;
    iter=iter+1;
  if iter>=maxit:  print("Warning : Maximum number of iteration is exceeded");
  return p;

f=f1();
r=inverseinter2Droot(f,1.0,2.0,3.0);
print("r=",r);
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" inverseinter2Droot.py
r= 1.4142135623730954
> Terminated with exit code 0.
```

```
program quadratic_inverse_lagrange
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) x1,x2,x3
x1=1.0
x2=1.15
x3=2.5
print *,'inverse quadratic lagrange(',x1,',',x2,',',x3,') = ',quad_lagrange(x1,x2,x3)

CONTAINS
  FUNCTION func(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x-2.0
  END FUNCTION func
  FUNCTION dfunc(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) dfunc,x,h,df,f1,f2,f3,f4,f5,f6,f7,f8
  h=1.12314e-4
  f1=3.0*func(x-4.0*h)
  f2=-32.0*func(x-3.0*h)
  f3=168.0*func(x-2.0*h)
  f4=-672.0*func(x-h)
  f5=672.0*func(x+h)
  f6=-168.0*func(x+2.0*h)
  f7=32.0*func(x+3.0*h)
  f8=-3.0*func(x+4.0*h)
  dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(840.0*h)
  END FUNCTION dfunc
  FUNCTION quad_lagrange(x1,x2,x3)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) quad_lagrange,x1,x2,x3,test,p,es,ea,f1,f2,f3,fp,tol
  INTEGER maxit,iter,i
  p=0
  maxit=100
  iter=0
  if(x1==0) then
    x1=1e-10
  end if
  if(x2==0) then
    x2=1e-10
  end if
  if(x2==0) then
    x2=1e-10
  end if
  es=1e-8
  ea=0.00001
  f1=func(x1)
  f2=func(x2)
  f3=func(x3)
  DO i=1,maxit
    p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1))
    fp=func(p)
    ea=abs(x3-p)
```

```
    if(abs(f3)<tol) EXIT
    if(p>x3) then
      x1=x3
      f1=f3
      x3=p
      f3=fp
    else if(p<x3) then
      x2=x3
      f2=f3
      x3=p
      f3=fp
    end if
    IF (ea<es) EXIT
  END DO
  quad_lagrange=p
  END FUNCTION quad_lagrange
END PROGRAM quadratic_inverse_lagrange
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" quadratic_inverse_lagrange

E:\okul\SCO1>quadratic_inverse_lagrange
 inverse quadratic lagrange(   1.0000000000000000     ,   1.1499999761581421     ,   2.5000000000000000     ) =
1.4142135623730951

> Terminated with exit code 0.
```

Program quadratic inverse lagrange interpolation formula method (Julia code)

```
 function inverseinter2Droot(f,x1,x3)
# Inverse quadratic Lagrange interpolation
x2=(x1+x3)/2.0154254
test=0.0
global p=0.0
f1=0.0
f2=0.0
f3=0.0
fp=0.0
maxit=100
iter=0
tol=1.0e-10
es=0.00000001
ea=0.001
f1=f(x1)
f2=f(x2)
f3=f(x3)
if f1==0
 return x1
elseif f2==0
 return x2
elseif f3==0
 return x3
end
if f1*f3>0
 print("Root is not existed in the given region")
end
while ea>es && iter<maxit
            p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1))
            println("i = $iter p = $p")
   fp=f(p)
   ea=abs(x3-p)
            if abs(f3)<tol
              return x3
            end
   if p>x3
      x1=x3
      f1=f3
      x3=p
      f3=fp
   elseif p<x3
      x2=x3
```

```
      f2=f3
      x3=p
      f3=fp
    end
    iter+=1
end
if iter>=maxit
  print("Maximum number of iteration is exceeded \nresult might not be valid")
end
return p
end

f(x)=x*x-2.0
x0=0.0
x2=2.0
x=inverseinter2Droot(f,x0,x2)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" inverse_quadratic_lagrange_int_root.jl
i = 0 p = 1.6735267838105297
i = 1 p = 1.3253225124221881
i = 2 p = 1.4358421880222398
i = 3 p = 1.4141003048715692
i = 4 p = 1.4142136188326897
i = 5 p = 1.414213562373061
i = 6 p = 1.4142135623730951
x = 1.414213562373061
> Terminated with exit code 0.
```

BRENT METHOD (INVERSE QUADRATIC LAGRANGE INTERPOLATION – BISECTION AND SECANT COMBINED METHOD)



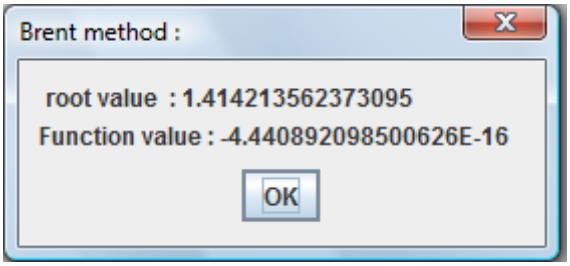
Figure 2.9.1 Richard P Brent

Being a quadratic method, Inverse quadratic Lagrange interpolation is an efficient method by combining it with bi-section and secant method one of the most used root finding method is created. Two versions of the program code and algorithm of the code is given below.

Program NA13.java : BRENT METHOD (java language version)

```
/ Numerical Analysis package
// Brent root finding method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Function Definition ================
class fc extends f_x
{
public double func(double x)
{return x*x-2;}
}
//============================================================
```

```
public class NA13
{

public static double brent(f_x f,double a,double b)
{
double test;
double p=0;
double es,ea;
double f1,f2,f3,fp;
int maxit=500,iter=0;
double tol=1.0e-15;
es=0.0000001;
double x1=a;f1=f.func(x1);
double x2=b;f2=f.func(x2);
double x3=(x1+x2)/2.0;f3=f.func(x3);
if(f1==0) return x1;
else if(f2==0) return x2;
else if(f3==0) return x3;
if(f1*f2>0) System.out.println("No root is existed in the given region");
p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
fp=f.func(p);
ea=Math.abs(x3-p);
while((ea>es)&&(iter<maxit))
{               if(Math.abs(f3)<tol) return x3;
   if((p<x1) && (p>x2))
   {p=(x1+x2)/2.0;
    if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
    else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   }
   else
   {
   if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
   else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=f.func(p);
   ea=Math.abs(x3-p);
   }
   iter++;
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return p;
}

    public static void main (String args[])
    {
               fc f=new fc();
    double x1,x2,x3;
    x1=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x1 : "));
    x2=Double.parseDouble(JOptionPane.showInputDialog(" root estimation x2 : "));
    double r;
    r= brent(f,x1,x2);
     JOptionPane.showMessageDialog(null," root value  : "+r+"\nFunction value : "+f.func(r),
    "Brent method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Program brent.cpp : BRENT METHOD (C++ version)

```
 #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x)
  { double ff=x*x-2.0;
                return ff;
  }
};

double brent(f_x &f,double a,double b)
{
double test;
double p=0;
double es,ea;
double f1,f2,f3,fp;
int maxit=500,iter=0;
double tol=1.0e-15;
es=0.0000001;
double x1=a;f1=f.func(x1);
double x2=b;f2=f.func(x2);
double x3=(x1+x2)/2.0;f3=f.func(x3);
if(f1==0) return x1;
else if(f2==0) return x2;
else if(f3==0) return x3;
if(f1*f2>0) cout<<"No root is existed in the given region";
p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
fp=f.func(p);
ea=fabs(x3-p);
while((ea>es)&&(iter<maxit))
{               if(fabs(f3)<tol) return x3;
   if((p<x1) && (p>x2))
   {p=(x1+x2)/2.0;
    if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
    else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   }
   else
   {
   if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
   else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=f.func(p);
   ea=fabs(x3-p);
   }
   iter++;
}
if(iter>=maxit) cout<<"Warning : MAximum number of iteration is exceeded \n";
return p;
}
    int main ()
    {
    double x1,x2,x3;
    cout << "root estimation x1 : ";
    cin >>x1;
    cout << "root estimation x2 : ";
    cin >>x2;
    double r;
    f1 f;
    cout<<setprecision(20);
    r= brent(f,x1,x2);
```

```
    cout<<" root value  : "<<r<<"\nFunction value : "<<f.func(r);
    }
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" brent

D:\okul\SCO1>brent
root estimation x1 : 1
root estimation x2 : 3
 root value  : 1.4142135623730951
Function value : 2.7343578790084422e-016
> Terminated with exit code 0.
```

The same code given in python language

Program brent.py : BRENT METHOD (FIRST VERSION) python code

```
from math import *

def brent(f,a,b):
    #brent root finding method
    maxit=500;
    iter=0;
    tol=1.0e-15;
    es=0.0000001;
    x1=a;f1=f(x1);
    x2=b;f2=f(x2);
    x3=(x1+x2)/2.0;f3=f(x3);
    if f1==0:    return x1;
    elif f2==0:  return x2;
    elif f3==0:  return x3;
    if f1*f2>0: print("No root is existed in the given region");
    p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
    fp=f(p);
    ea=abs(x3-p);
    while (ea>es) and (iter<maxit):
      if abs(f3)<tol: return x3;
      if (p<x1) and (p>x2):
        p=(x1+x2)/2.0;
        if p>x3:   x1=x3;f1=f3;x3=p;f3=fp;
        elif p<x3: x2=x3;f2=f3;x3=p;f3=fp;
      else:
        if p>x3: x1=x3;f1=f3;x3=p;f3=fp;
        elif p<x3: x2=x3;f2=f3;x3=p;f3=fp;
        p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
        fp=f(p);
        ea=abs(x3-p);
      iter=iter+1;
    if iter>=maxit: print("Warning : Maximum number of iteration is exceeded");
    return p;

f1 = lambda x: x*x-2
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=brent(f1,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

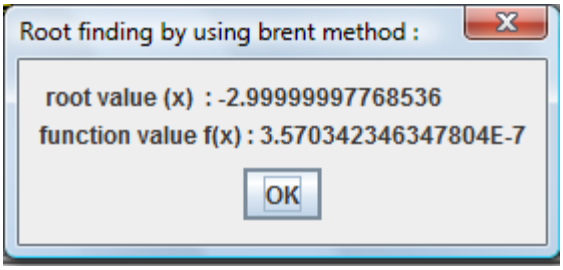
```
runfile('D:/okul/SCO1/brent.py', wdir='D:/okul/SCO1')
enter lower limit : 1
enter upper limit : 3
 ROOT  :  1.414213562373095
FUNCTION VALUE :  -4.440892098500626e-16
```

Program NA13A.java : BRENT METHOD (SECOND VERSION) java language

```
// Brent root finding method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fc extends f_x
{
```

```
public double func(double x)
{return (x+3)*(x-1)*(x-1);}
}

public class NA13A
{

public static double brent(f_x f,double x1,double x2)
{
   // roots of the nonlinear equation: Brent method
   // referance : Numerical Recipes in C, second edition, William H. Press,
   // Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery
   // cambridge university press
            double ITMAX=100;
           double EPS=2.22044604925031e-16;
            int iter;
            double a=x1,b=x2,c,d,e,min1,min2;
            double tol=1.0e-5;
            c=0;
            d=0;
            e=0;
            double fa=f.func(a),fb=f.func(b),fc,p,q,r,s,tol1,xm;
            if (fb*fa > 0.0) JOptionPane.showMessageDialog(null,"Root limit selection is uncorrect \n result may be wrong"
            ,"ROOT LIMITS WARNING",JOptionPane.WARNING_MESSAGE);
            fc=fb;
            for (iter=1;iter<=ITMAX;iter++) {
                        if (fb*fc > 0.0) {
                                    c=a;
                                    fc=fa;
                                    e=d=b-a;
                        }
                        if (Math.abs(fc) < Math.abs(fb)) {
                                    a=b;
                                    b=c;
                                    c=a;
                                    fa=fb;
                                    fb=fc;
                                    fc=fa;
                        }
                        tol1=2.0*EPS*Math.abs(b)+0.5*tol;
                        xm=0.5*(c-b);
                        if (Math.abs(xm) <= tol1 || fb == 0.0) return b;
                        if (Math.abs(e) >= tol1 && Math.abs(fa) > Math.abs(fb)) {
                                    s=fb/fa;
                                    if (a == c) {
                                                p=2.0*xm*s;
                                                q=1.0-s;
                                    } else {
                                                q=fa/fc;
                                                r=fb/fc;
                                                p=s*(2.0*xm*q*(q-r)-(b-a)*(r-1.0));
                                                q=(q-1.0)*(r-1.0)*(s-1.0);
                                    }
                                    if (p > 0.0)  q = -q;
                                    p=Math.abs(p);
                                    min1=3.0*xm*q-Math.abs(tol1*q);
                                    min2=Math.abs(e*q);
                                    if (2.0*p < (min1 < min2 ? min1 : min2)) {
                                                e=d;
                                                d=p/q;
                                    } else {
                                                d=xm;
                                                e=d;
                                    }
                        } else {
                                    d=xm;
                                    e=d;
                        }
                        a=b;
                        fa=fb;
```

```
                        if (Math.abs(d) > tol1)
                                b += d;
                        else
                                b += (xm > 0.0 ? Math.abs(tol1) : -Math.abs(tol1));
                        fb=f.func(b);
                }
JOptionPane.showMessageDialog(null,"Maximum number of iterations are exceeded \n"+
  " Result mAy not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return b;
}
    public static void main (String args[])
    {
    double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" Fonksiyonun alt sinirini giriniz a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" Fonksiyonun üst sinirini giriniz b : "));
    double r;
    fc f=new fc();
    r= brent(f,a,b);
    JOptionPane.showMessageDialog(null," root value (x)  : "+r+"\nfunction value f(x) : "+f.func(r),
    "Root finding by using brent method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



A third version of the Brent method is given below

Program NA13A : BRENT METHOD (THIRD VERSION)

```
/
//
//
// Brent root finding method
//
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fc extends f_x
{
public double func(double x)
{return (x+3)*(x-1)*(x-1);}
}
public class NA13F
{
public static double brent(f_x f,double xl,double xu)
{
   // roots of the nonlinear equation: Brent method
   // referances : Numerical Methods for Engineers,
   // Steven C. Chapra, Raymond P. Chanale, sixth edition
   // Mc-Graw-Hill
   // Moler,C.B. Numerical Computing with MATLAB, SIAM,
   // Philedelphia, 2004
   double eps=2.22044604925031e-16;
   double tol=0.000001;
   double s,p,q,r;
   double m;
```

```
double a=xl;double b=xu;double fa=f.func(a);double fb=f.func(b);
double c=a;double fc=fa;double d=b-c;double e=d;
int MAXITER=100;
for(int i=0;i<MAXITER;i++)
{
if(fb==0) return b;
if(Math.signum(fa)==Math.signum(fb))
{a=c;fa=fc;d=b-c;e=d;
if(Math.abs(fa)<Math.abs(fb))
{c=b;b=a;a=c;fc=fb;fb=fa;fa=fc;}
m=0.5*(a-b);
tol=2*eps*Math.max(Math.abs(b),1);
if(Math.abs(m)<=tol || fb==0) return b; //exit
if(Math.abs(e)>=tol && Math.abs(fc)>Math.abs(fb))
{s=fb/fc;
 if(a==c) {p=2*m*s;q=1-s;} //secant method
 else     {q=fc/fa;r=fb/fa;p=s*(2*m*q*(q-r)-(b-c)*(r-1));//İnverse quadratic interpolation
          q=(q-1)*(r-1)*(s-1);
          }
 if(p>0) q=-q;
 else    p=-p;
 if(2*p<(3*m*q-Math.abs(tol*q)) && p<Math.abs(0.5*e*q))
 {e=d;d=p/q;}
 else
 {d=m;e=m;}
 }
 c=b;fc=fb;
 if(Math.abs(d)>tol) {b=b+d;}
 else {b=b-Math.signum(b-a)*tol;}
 fb=f.func(b);
}
}
return b;
}
 public static void main (String args[])
 {
 double a,b;
 a=Double.parseDouble(JOptionPane.showInputDialog(" Enter lower limit of the function a : "));
 b=Double.parseDouble(JOptionPane.showInputDialog(" Enter upper limit of the function  b : "));
 double r;
 fc f=new fc();
 r= brent(f,a,b);
 JOptionPane.showMessageDialog(null," root value (x)  : "+r+"\nfunction value f(x) : "+f.func(r),
 "Root finding by using brent method : ",JOptionPane.PLAIN_MESSAGE);
 System.exit(0);
 } }
```

BRENT METHOD (julia language version)

```
 function brent(f,a,b)
test=0.0
p=0.0
f1=0.0
f2=0.0
f3=0.0
fp=0.0
maxit=500
iter=0
tol=1.0e-15
es=0.0000001
x1=a
f1=f(x1)
x2=b
f2=f(x2)
x3=(x1+x2)/2.0
f3=f(x3)
if(f1==0)
  return x1
elseif(f2==0)
  return x2
```

```
elseif(f3==0)
  return x3
end
if(f1*f2>0)
  print("No root is existed in the given region")
end
p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1))
fp=f(p)
ea=abs(x3-p)
while (ea>es)&&(iter<maxit)
                if abs(f3)<tol
                   return x3
                end
    if (p<x1) && (p>x2)
      p=(x1+x2)/2.0
       if p>x3
         x1=x3
         f1=f3
         x3=p
         f3=fp
       elseif p<x3
         x2=x3
         f2=f3
         x3=p
         f3=fp
       end
    else  #(p<x2) && (p>x3)
       if p>x3
         x1=x3
         f1=f3
         x3=p
         f3=fp
       elseif p<x3
         x2=x3
         f2=f3
         x3=p
         f3=fp
       end # if p>x3
    end #if else
    p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1))
    fp=f(p)
    ea=abs(x3-p)
    iter+=1
end  # while
if iter>=maxit
  print("Maximum number of iteration is exceeded \nresult might not be valid")
end #if
return p
end # function

f(x)=x*x-2.0
a=0.0
b=2.0
x=brent(f,a,b)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" brent.jl
x = 1.4142135623730954
> Terminated with exit code 0.
```

Algorithm
Input a,b and a function f
Calculate f(a) and f(b)
if(f(a)*f(b)<0 then give no root error
if |f(a)| < |f(b)| then swap(a,b)
c=a
repeat untill f(b)=0 or s=0 or |b-a| is small enough (convergence)
if(f(a)≠f(c) and f(b)≠f(c) then

$$s = \frac{af(b)f(c)}{(f(a)-f(b))(f(a)-f(c))} + \frac{bf(a)f(c)}{(f(b)-f(a))(f(b)-f(c))} + \frac{cf(a)f(b)}{(f(c)-f(a))(f(c)-f(b))}$$

(inverse quadratic interpolation)
else

$$s = b - f(b)\frac{b-a}{(f(b)-f(a))}$$ (secant rule)

end if
if *(condition 1) s* is not between (3*a* + *b*)/4 and *b*
or *(condition 2)* (mflag is set and |*s*−*b*| ≥ |*b*−*c*| / 2)
or *(condition 3)* (mflag is cleared and |*s*−*b*| ≥ |*c*−*d*| / 2)
or *(condition 4)* (mflag is set and |*b*−*c*| < |δ|)
or *(condition 5)* (mflag is cleared and |*c*−*d*| < |δ|)
then

$$s := \frac{a+b}{2}$$ *(bisection method)*

set mflag
else
clear mflag
end if
calculate *f*(*s*)
d := *c (d is assigned for the first time here; it won't be used above on the first iteration because mflag is set)*
c := *b*
if *f*(*a*) *f*(*s*) < 0 then *b* := *s* else *a* := *s* end if
if |*f*(*a*)| < |*f*(*b*)| then swap (*a*,*b*) end if
end repeat
output *b or s (return the root*

ROOTS OF A COMPLEX FUNCTION BY NEWTON_RAPHSON METHOD

Roots of a complex function can be found by Newton Raphson method. Complex variables are not defined in standart java libraries, therefore should be user defined. A complex class defined by author will be listed here and used for definition of Newton-Raphson method and other further complex methods. Before giving the complex code let us define basic mathematics of the complex numbers

$z = a + bi$ where $i = \sqrt{-1}$

$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i$

$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$

$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$

Cojugate$(z) = a - bi$

$z_1 * z_2 = (a_1 * a_2 - b_1 * b_2) + (a_1 * b_2 + b_1 * a_2)i$

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)} = \frac{(a_1 * a_2 + b_1 * b_2) + (b_1 * a_2 - a_1 * b_2)i}{a_2^2 + b_2^2}$$

$z = a + bi = r(\cos(\theta) + i\sin(\theta))$
$r = \sqrt{a^2 + b^2} \quad \theta = \text{atan}\left(\frac{b}{a}\right)$

$z^n = (a + bi)^n = r^n(\cos(n\theta) + i\sin(n\theta))$
$\sqrt{z} = (a + bi)^{0.5} = \sqrt{r}(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right))$

Program complex class

```
import java.io.*;
```

```
import java.text.*;
import java.util.Locale;
import java.util.*;

import javax.swing.*;

// class c
//   c number abstraction

class c {
   // constructors
public c()
{nr=0;ni=0;}
public c(double nre,double nim)
{nr=nre;ni=nim;}
public static c toComplex(double nre,double nim)
{c c=new c(nre,nim);return c;}
public c(int nre,int nim)
{nr=(double)nre;ni=(double)nim;}
public static c toComplex(int nre,int nim)
{c c=new c(nre,nim);return c;}
public c(double numer)
{nr=numer;ni=0;}

public static c toComplex(double nre)
{c c=new c(nre,0.0);return c;}

public c(c value)
{nr=value.real();ni=value.imaginary();}

public static c p(String s,c x1)
{ c z=new c();
if(s.equals("sin")) {z=sin(x1);}
else if(s.equals("cos")) {z=cos(x1);}
else if(s.equals("tan")) {z=tan(x1);;}
if(s.equals("asin")) {z=asin(x1);}
else if(s.equals("acos")) {z=acos(x1);}
else if(s.equals("atan")) {z=atan(x1);;}
else if(s.equals("exp")) {z=exp(x1);}
else if(s.equals("sqrt")) {z=sqrt(x1);}
else if(s.equals("cbrt")) {z=cbrt(x1);}
else if(s.equals("log")) {z=log(x1);}
return z;
}
public static c p(String s,double x1,double x2)
{ c z=new c();
if(s.equals("+")) {z=(new c((x1+x2)));}
else if(s.equals("-")) {z=(new c((x1-x2)));}
else if(s.equals("*")) {z=(new c((x1*x2)));}
else if(s.equals("/")) {z=(new c((x1/x2)));}
else if(s.equals("pow")) {z=(new c(Math.pow(x1,x2)));}
return z;
}
public static c p(double x1,String s,double x2)
{ return p(s,x1,x2);}

public static c p(String s,c x1,double x2)
{c z=new c();
if(s.equals("+")) {z=c.add(x1,new c(x2));}
else if(s.equals("-")) {z=c.substract(x1,new c(x2));}
else if(s.equals("*")) {z=c.multiply(x1,new c(x2));}
else if(s.equals("/")) {z=c.divide(x1,new c(x2));}
```

```
return z;
}
public static c p(c x1,String s,double x2)
{ return p(s,x1,x2);}
public static c p(String s,double x1,c x2)
{ c z=new c();
if(s.equals("+")) {z=c.add(new c(x1),x2);}
else if(s.equals("-")) {z=c.substract(new c(x1),x2);}
else if(s.equals("*")) {z=c.multiply(new c(x1),x2);}
else if(s.equals("/")) {z=c.divide(new c(x1),x2);}
return z;
}
public static c p(double x1,String s,c x2)
{ return p(s,x1,x2);}
public static c p(String s,c x1,c x2)
{ c z=new c();
if(s.equals("+")) {z=c.add(x1,x2);}
else if(s.equals("-")) {z=c.substract(x1,x2);}
else if(s.equals("*")) {z=c.multiply(x1,x2);}
else if(s.equals("/")) {z=c.divide(x1,x2);}
return z;
}
public static c p(c x1,String s,c x2)
{ return p(s,x1,x2);}
// accessor functions
public double real()
{//return real part of the c number
return nr;}

public double imaginary()
{//return imaginary part of the c number
return ni;}
public double R()
{// Return radius of polar coordinate equivalent of c number
return Math.sqrt(nr*nr+ni*ni);}
public double theta()
{// Return angle of polar coordinate equivalent of c number in radian
return Math.atan2(ni,nr);}
public double dtheta()
{// Return angle of polar coordinate equivalent of c number in degree
return Math.atan2(ni,nr)*180.0/Math.PI;}
   // assignments
public void assign(c right)
{nr=right.real();ni=right.imaginary();}
public void add(c  right)
{ni = ni + right.imaginary();nr = nr + right.real();}
public void add(double  nr1)
{nr = nr+nr1;}
public static int sgn(c left)
{int i=(int)(left.real()/Math.abs(left.real()));
 return i;}
public void add(double  nr1,double ni1)
{nr = nr+nr1;ni = ni + ni1;}
public void substract(c right)
{ni = ni - right.imaginary();nr = nr - right.real();}
public void multiply(c right )
{nr = nr*right.real() - ni*right.imaginary();
ni = nr*right.imaginary() +  ni*right.real();}
public void divide(c right )
{double a=nr*nr+ni*ni;
nr = ( nr*right.real() +  ni*right.imaginary())/a;
ni = (-nr*right.imaginary() +  ni*right.real())/a;}
```

```
public static c add(c left, c right)
{ // return sum of two c numbers
 double r1=(left.real() + right.real());
 double i1=(left.imaginary() + right.imaginary());
 c result;
 result=new c(r1,i1);
 return result;}
public static c add(c left, double right)
{ // return sum of two c numbers
 double r1=(left.real() + right);
 double i1=left.imaginary();
 c result;
 result=new c(r1,i1);
 return result;
}
public static c conjugate(c z)
{c z1=new c(z.real(),-z.imaginary());return z1;}
public static c substract(c left, c right)
{ // return substraction of two c numbers
 c result;
 result=new c((left.real() - right.real()),
              (left.imaginary() - right.imaginary()));
 return result;
}
public static c multiply(c left, c right)
{ // return multiplication of two c numbers
 c result;
 result=new c
 ((left.real()*right.real() - left.imaginary()*right.imaginary()),
  (left.real()*right.imaginary() + left.imaginary()*right.real()));
  //System.out.println("left="+left.toString()+"right="+right.toString()+"result="+result.toString());
return result;}
public static c multiply(c left, double  right)
{c result;
 result=new c((left.real()*right),(left.imaginary()*right));
 return result;}
public static c multiply(double  right,c left )
{c result;
 result=new c((left.real()*right),(left.imaginary()*right));
 return result;}
public static c divide(c left, c  right)
{ // return division of two c numbers
 double a=right.real()*right.real()+right.imaginary()*right.imaginary();
 c result;
 result=new c
 ((left.real()*right.real()  + left.imaginary()*right.imaginary())/a,
  (-left.real()*right.imaginary() + left.imaginary()*right.real())/a);
 return result;}
public static c divide(c left, double  right)
{ // return division of a c number to a real number
 c result;
 result=new c((left.real()/right),(left.imaginary()/right));
 return result;
}
public static c pow(c left, double right)
{ // return sum of two c numbers
 double Rad,th;
 Rad=Math.pow(left.R(),right);
 th=right*left.theta();
 c result=new c((Rad*Math.cos(th)),(Rad*Math.sin(th)));
 return result;
}
```

```
public static c pow(double left, c right)
{ double p1=Math.log(left);
  c p2=c.multiply(right,p1);
  c p3=exp(p2);
  return p3;
}

public static c pow(c left, c right)
{ c p1=log(left);
  c p2=c.multiply(right,p1);
  c p3=exp(p2);
  return p3;
}

public static c exp(c left)
{ // exp(x+i*y)
 c result;
 result =new c((Math.exp(left.real())*Math.cos(left.imaginary()) ),
                (Math.exp(left.real())*Math.sin(left.imaginary()) ));
 return result;
}

public static c exp(double left)
{ // exp(i*left) imaginary exponent
 c result;
 result =new c(Math.cos(left),
                Math.sin(left) );
 return result;}
public static c sqrt(c z)
{return pow(z,0.5);}
public static c cbrt(c z)
{return pow(z,(1.0/3.0));}
public static double abs(c z)
{return z.R();}
public static c log(c z)
{ c z1;
  double x=Math.log(Math.sqrt(z.nr*z.nr+z.ni*z.ni));
  double y=Math.atan2(z.ni,z.nr);
  z1=new c(x,y);
  return z1;
}
public static c sin(c z)
{ c z1,z2,z3;
  z1=c.multiply(z,new c(0.0,1.0));
  z2=c.multiply(z,new c(0.0,-1.0));
  z3=c.substract(exp(z1),exp(z2));
  z3=c.divide(z,new c(0.0,2.0));
  return z3;
}
public static c asin(c z)
{ c z1,z2,z3,z4,z5,z6;
  z1=c.multiply(z,new c(0.0,1.0));//iz
  z2=c.substract(new c(1.0),c.multiply(z,z));//(1-z^2)
  z3=c.sqrt(z2);//(1-z*z)^.5
  z4=c.add(z1,z3);
  z5=log(z4);
  z6=c.multiply(z5,new c(0.0,-1.0));
  return z6;
}
public static c sinh(c left)
{ c z,z1,z2;
```

```
  z1=c.multiply(left,1.0);
  z2=c.multiply(left,1.0);
  z=c.substract(exp(z1),exp(z2));
  z=c.divide(z,2.0);
  return z;
}
public static c cos(c left)
{ c z,z1,z2;
  z1=c.multiply(left,new c(0.0,1.0));
  z2=c.multiply(left,new c(0.0,-1.0));
  z=c.add(exp(z1),exp(z2));
  z=c.divide(z,new c(2.0,0.0));
  return z;
}
public static c acos(c z)
{ c z1,z2,z3,z4,z5,z6,z7;
  z1=c.multiply(z,new c(0.0,1.0));//iz
  z2=c.substract(new c(1.0),c.multiply(z,z));//(1-z^2)
  z3=c.sqrt(z2);//(1-z*z)^.5
  z4=c.add(z1,z3);
  z5=log(z4);
  z6=c.multiply(z5,new c(0.0,1.0));
  z7=c.add(z6,Math.PI/2.0);
  return z7;
}

public static c cosh(c z)
{ c z1,z2,z3;
  z1=c.multiply(z,1.0);
  z2=c.multiply(z,-1.0);
  z3=c.add(exp(z1),exp(z2));
  z3=c.divide(z,2.0);
  return z3;
}

public static c tan(c z)
{return c.divide(sin(z),cos(z));}

public static c atan(c z)
{ c z1,z2,z3,z4,z5,z6;
  z1=c.multiply(z,new c(0.0,-1.0));
  z2=c.substract(new c(1.0),z1);
  z3=c.multiply(z,new c(0.0,1.0));
  z4=c.substract(new c(1.0),z3);
  z5=c.substract(log(z2),log(z4));
  z6=c.multiply(z5,new c(0.0,0.5));
  return z6;
}
public static c tanh(c left)
{return c.divide(sinh(left),cosh(left));}
public boolean smaller(c left,c right)
{return (left.R() < right.R());}
public boolean smaller_equal(c left,c right)
{return (left.R() <= right.R());}
public boolean greater(c left,c right)
{// greater then comparison of two c numbers
return left.R() > right.R();}
public boolean greater_equal(c left,c  right)
{// greater then and equal comparison of two c numbers
return left.R() >= right.R();}
public boolean equal(c left,c right)
{// equal comparison of two c numbers
```

```
return left.R() == right.R();}
public boolean not_equal(c left,c  right)
{// not equal comparison of two c numbers
return left.R() != right.R();}
public static String toString(double left, int w, int d)
// converts a double to a string with given width and decimals.
{   NumberFormat df=NumberFormat.getInstance(Locale.US);
            df.setMaximumFractionDigits(d);
   df.setMinimumFractionDigits(d);
   df.setGroupingUsed(false);
   String s = df.format(left);
   while (s.length() < w)
     s = " " + s;
   if (s.length() > w)
   {s = "";for (int i=0; i<w; i++) s = s + "-";
   }
   return s;
}

public static String toString(double left)
{// converts a double to a string with a constant width and constant decimals.
  return toString(left,15,10);}
public static String toString(c  value)
{ return value.toString();}

public String toString()
{ return toString("%10.5f");}

public String toString(String f)
{String b="";
 String s1="("+f+" + "+f+"*i ) ";
 String s2="("+f+" - "+f+"*i ) ";
 StringBuilder sb = new StringBuilder();
 Formatter fo = new Formatter(sb, Locale.US);
if(Math.abs(imaginary())!=1)
  {
  if(imaginary() >= 0)
    b=String.format(Locale.US,s1,real(),imaginary());
  else
    b=String.format(Locale.US,s2,real(),-imaginary());
  }
return b;
}

public double toDouble()
{ if(        imaginary()==0) return nr;
  else               return R();
}
public double[] toDoubleArray()
{
double a[]={nr,ni};
return a;
}

public static double[] toDoubleArray(complex value)
{ double a[]={value.nr,value.ni};
  return a;
}

public static double toDouble(c w)
{
if(w.imaginary()==0) return w.real();
```

```
else              return w.R();
}

    public static c toComplex(String s)
    {
              //bu metod compleks sayıyı ekrandan okur.
              //StringTokanizer kütüphane sınıfı bir stringi cümlelere ayırır
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    c b=new c(a[0],a[1]);
    return b;
    }
    public static c toComplex()
    {return toComplex("input c number:");}
   // data areas
   public double   nr;
   public double   ni;
   //Matrix an vector input system for complex variable
   public static int countcharacters(char a,String s)
{
int n=s.length();
int number=0;
for(int i=0;i<n;i++)
{if(s.charAt(i)==a) number++;}
return number;
}

public static String[] word(char a,String s)
{
int nn=countcharacters(a,s);
int n=s.length();
String s1[]=new String[nn+1];
String s2="";
int j=0;
for(int i=0;i<n;i++)
{if(s.charAt(i)==a) {s1[j]=s2;s2="";j++;}
 else             {s2+=s.charAt(i);}
}
s1[nn]=s2;
return s1;
}
public static String[] word(char a1,char a2,char a3,String s)
{
int nn1=countcharacters(a1,s);
int nn2=countcharacters(a2,s);
if(nn1!=nn2) System.out.println("char couple is not match");
int n=s.length();
String s1[]=new String[nn1+1];
String s2="";
int j=0;
for(int i=0;i<n;i++)
{if(s.charAt(i)==a1) {s1[j]=s2;s2="";}
 else if(s.charAt(i)!=a2) {s2+=s.charAt(i);}
```

```
 else if(s.charAt(i)==a2) {j++;}
}
s1[nn1]=s2;
String s3[]=new String[nn1];
int n1=0;
for(int i=0;i<nn1;i++)
{n1=s1[i+1].length();
 if(s1[i+1].charAt(n1-1)==a3) s1[i+1]=s1[i+1].substring(0,n1-1);
 s3[i]=s1[i+1];
}
return s3;
}

public static String[] word(String s)
{
char a=' ';
return word(a,s);
}

public static String toString(String s[])
{int n=s.length;
 String ss="";
 for(int i=0;i<n;i++)
 {ss+="i = "+i+" "+s[i]+"\n";}
  return ss;
}
public static String toString(String s[][])
{int n=s.length;
 int m=s[0].length;
 String ss="";
 for(int i=0;i<n;i++)
 {
 for(int j=0;j<m;j++)
 {ss+="  "+s[i][j].toString();}
 ss+="\n";
 }
 return ss;
}

public static String toString(c s[][])
{int n=s.length;
 int m=s[0].length;
 String ss="";
 for(int i=0;i<n;i++)
 {
 for(int j=0;j<m;j++)
 {ss+="  "+s[i][j].toString();}
 ss+="\n";
 }
 return ss;
}

public static String toString(c s[])
{int n=s.length;
 String ss="";
 for(int i=0;i<n;i++)
 {ss+="  "+s[i];}
 return ss;
}

public static c[][] toComplexMatrix(String s)
{String s1[]=word('{','}',',',s);
```

```
int n=s1.length;
String s2[][]=new String[n][];
for(int i=0;i<n;i++)
  s2[i]=word('(',')',',',s1[i]);
int m=s2[0].length;
String s3[][][]=new String[n][m][];
c z[][]=new c[n][m];
for(int i=0;i<n;i++)
 {for(int j=0;j<m;j++)
  {s3[i][j]=word(',',s2[i][j]);}
 }
int k=s3[0][0].length;
double d[][][]=new double[n][m][k];
for(int i=0;i<n;i++)
 {for(int j=0;j<m;j++)
  {for(int l=0;l<k;l++){d[i][j][l]=Double.parseDouble(s3[i][j][l]);}}
 }
for(int i=0;i<n;i++)
 {for(int j=0;j<m;j++)
  {z[i][j]=toComplex(d[i][j][0],d[i][j][1]);}
 }
return z;
}

public static c[] toComplexVector(String s)
{
c[][] z1=toComplexMatrix(s);
return z1[0];
}
    public static void main(String arg[])
    {c a=c.toComplex();
     c b=toComplex();
     c c1=p("+",a,b);
     System.out.println("a="+a+"\nb="+b+"\nc="+c1);
    }}
```

Program abstract class cf_cx general function and derivatives definitions of a complex variable

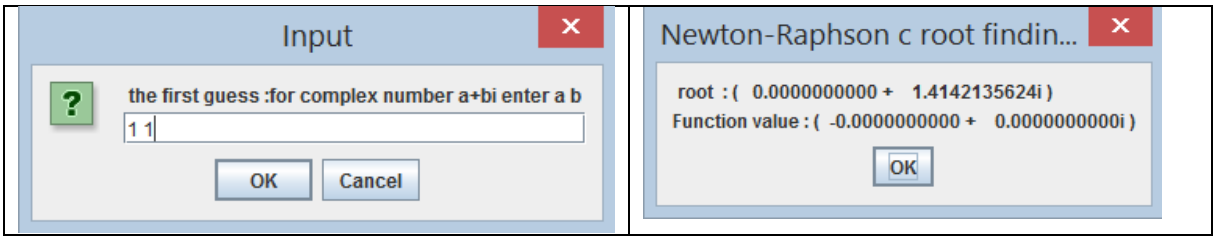
```
abstract class cf_cx
{
  abstract c func(c x);
 //complex derivative
  public c dfunc(c x)
  { // c numerical derivative
   double hh=0.001;
   c h=new c(hh,hh);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.p(x,"+",h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(xe2h),"*",(1.0/12.0));
   c feh=c.p(func(xeh),"*",(-8.0/12.0));
   c fah=c.p(func(xah),"*",(8.0/12.0));
   c fa2h=c.p(func(xa2h),"*",(-1.0/12.0));
   c df=c.p(fe2h,"+",c.p(feh,"+",c.p(fah,"+",fa2h)));
   df=c.p(df,"/",h);
   return df;
  }
//complex second derivative
  public c d2func(c x)
  { // c sayısal ikinci türev fonksiyonu
   double hh=0.001563465314;
   c h=new c(hh,hh);
```

```
   c h_2=c.multiply(h,h);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.p(x,"+",h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(xe2h),"*",(-1.0/12.0));
   c feh=c.p(func(xeh),"*",(16.0/12.0));
   c f=c.p(func(x),"*",(-30.0/12.0));
   c fah=c.p(func(xah),"*",(16.0/12.0));
   c fa2h=c.p(func(xa2h),"*",(-1.0/12.0));
   c df=c.p(fe2h,"+",c.p(feh,"+",c.p(f,"+",c.p(fah,"+",fa2h))));
   df=c.p(df,"/",h_2);
   return df;
  }
}
```

Now a complex definition of Newton-Raphson method can be given. Our example function is $f(x)=x^2+2$

Program NA5A1 complex Newton-Raphson method

```
 // complex number newton-raphson root finding
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
class fa extends cf_cx
{  //one function, one independent variable
  // example function f=x*x-2
c func(c x)
{ c f=c.p(c.p(x,"*",x),"+",2.0);
  return f;
}}

public class NA5A1
{

public static c newton(cf_cx f,c x)
{
int nmax=50;
double tolerance=1.0e-8;
c fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=f.dfunc(x);
  //x=c.add(x,c.multiply(c.divide(fx,dfx),-1.0));
  x=c.p(x,"-",c.p(fx,"/",dfx));
  if(c.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
    public static void main (String args[])
    { fa f=new fa();
    double x0[]=NA5A.enterdata(" the first guess :for complex number a+bi enter a b ");
    c x1=new c(x0[0],x0[1]);
    c r= newton(f,x1);
    c ff=f.func(r);
    JOptionPane.showMessageDialog(null," root  : "+r+"\nFunction value : "+ff.toString(),
    "Newton-Raphson c root finding method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



In order to make a lambda expression version of the program, an interface version of the abstract function cf_cx is created as icf_x

Program icf_x interface class

```
 import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;
  // single function single independent variable
  // example f=x*x
  // includes full set of derivatives
  // Reference :"Generation of Finite Difference Formulas on Arbitrary Spaced Grids",
  // Bength Fornberg, Mathematics of Computation, Volume 51, Number 184, October 1988
  // pages 699-706
  //interface version
@FunctionalInterface
interface icf_x
{  public c func(c x);
// nth order derivative
default c func(double x)
  {c x1=new c(x,0.0);
   return func(x1);
  }
  default c dfunc(c x)
  { // c sayısal türev fonksiyonu
   double hh=0.001563465314;
   c h=new c(hh,hh);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.add(x,h);
   c xa2h=c.add(x,h2);
   c xeh=c.add(x,he);
   c xe2h=c.add(x,he2);
   c fe2h=c.multiply(func(xe2h),(1.0/12.0));
   c feh=c.multiply(func(xeh),(-8.0/12.0));
   c fah=c.multiply(func(xah),(8.0/12.0));
   c fa2h=c.multiply(func(xa2h),(-1.0/12.0));
   c df=c.add(fe2h,c.add(feh,c.add(fah,fa2h)));
   df=c.divide(df,h);
   return df;
  }
  default c dfunc(double x)
  { c x1=new c(x,0.0);
   return dfunc(x1);
  }
  default c d2func(c x)
  { // c sayısal ikinci türev fonksiyonu
   double hh=0.001563465314;
   c h=new c(hh,hh);
   c h_2=c.multiply(h,h);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.add(x,h);
   c xa2h=c.add(x,h2);
   c xeh=c.add(x,he);
   c xe2h=c.add(x,he2);
   c fe2h=c.multiply(func(xe2h),(-1.0/12.0));
   c feh=c.multiply(func(xeh),(16.0/12.0));
   c f=c.multiply(func(x),(-30.0/12.0));
   c fah=c.multiply(func(xah),(16.0/12.0));
```

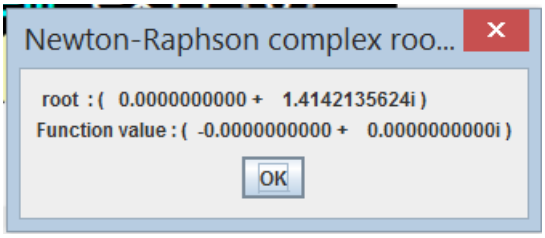
```
   c fa2h=c.multiply(func(xa2h),(-1.0/12.0));
   c df=c.add(fe2h,c.add(feh,c.add(f,c.add(fah,fa2h))));
   df=c.divide(df,h_2);
   return df;
}
default c d2func(double x)
{ c x1=new c(x,0.0);
 return d2func(x1);
}}
```

Now we can use an lambda expression version of complex Newton-Raphson root finding method.

Program complex Newton-Raphson method (lambda expression version) NA5A2.java

```
 import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class NA5A2
{
public static c newton(icf_x f,c x)
{
int nmax=500;
double tolerance=1.0e-10;
c fx,dfx;
for(int i=0;i<nmax;i++)
{
  fx=f.func(x);
  dfx=f.dfunc(x);
  x=c.p(x,"-",c.p(fx,"/",dfx));
  if(c.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double[] enterdata(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }
     public static void main (String args[])
     { icf_x f=(c x)->c.p(c.p(x,"*",x),"+",2.0);
     double x0[]=enterdata(" the first guess for the function : ");
     c x1;
     if(x0.length==1) x1=new c(x0[0],0.0);
     else
                    x1=new c(x0[0],x0[1]);
     c r= newton(f,x1);
     c ff=f.func(r);
     JOptionPane.showMessageDialog(null," root  : "+r+"\nFunction value : "+ff,
     "Newton-Raphson c root finding method : ",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);
     }
```

```
}
```



In C++ in most of the compilers complex class is existed so we can use it directly in our programs similar to double variable

Program complex Newton-Raphson method C++ version cnewton.cpp

```
 #include <iostream>
#include <math.h>
#include <complex>
#include <iomanip>
using namespace std;
typedef complex<double> cdouble;

class cf_x{
public: virtual cdouble func(cdouble x)=0;
public: cdouble dfunc(cdouble x)
{ cdouble h=0.0001;
  cdouble hh=1.0/h;
  cdouble df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public cf_x
{ public: cdouble func(cdouble x)
  { cdouble ff=x*x+2.0;return ff;}};

cdouble newton(cf_x &f,cdouble x)
{
int nmax=100;
double tolerance=1.0e-8;
cdouble fx,dfx;
for(int i=0;i<nmax;i++)
{ fx=f.func(x);
  dfx=f.dfunc(x);
  x-=fx/dfx;
  cout <<"i="<<i<<"x="<<x<<"fx="<<fx<<"dfx="<<dfx<<"\n";
  if(abs(fx)<tolerance) { return x;}
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;
}
//cnewton.cpp
int main()
{  f1 f;
    cdouble xa,r;
   cout << " first guess for the root value : ";
   cin >>xa;
   cout<<setprecision(20);
   r=newton(f,xa);
   cout<<"root value : ="<<r;
   return 0;
}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" cnewton

D:\okul\SCO1>cnewton
 first guess for the root value : (1,1)
i=0x=(-1.7097434579227411e-013,0.99999999999857003)fx=(2,2)dfx=(2.000000000002518,1.9999999999967979)
```

```
i=1x=(-1.7182023000249511e-013,1.500000000000715)fx=(1.0000000000028599,-3.4194869158405925e-
013)dfx=(6.872809200060493e-013,1.9999999999971401)
i=2x=(2.8783559897908822e-014,1.4166666666667063)fx=(-0.25000000000214495,-5.1546069000773101e-
013)dfx=(1.0362081563168128e-012,3.0000000000014291)
i=3x=(0,1.4142156862745099)fx=(-0.0069444444445565523,8.1553419710743948e-014)dfx=(0,2.8333333333334125)
i=4x=(0,1.4142135623746899)fx=(-6.0073048828712672e-006,0)dfx=(0,2.8284313725490202)
i=5x=(0,1.4142135623730951)fx=(-4.510614104447086e-012,0)dfx=(0,2.8284271247493797)
root value : =(0,1.4142135623730951)
> Terminated with exit code 0.
```

In Python language, in order to make a complex variable code you only change the input variable. Function will remain exactly the same.

Program complex Secant method in Python (lambda version) secant.py

```
 from math import *

#def df(f,x):
#  h=0.001
#   return (f(x+dx)-f(x-dx))/(2.0*h)
def df(f,x):
  h=0.001
  return (-f(x+2.0*h)+8.0*f(x+h)-8.0*f(x-h)+f(x-2.0*h))/(12.0*h);

def secant(f,x):
  # secant root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
     fx= f(x);
     dfx=df(f,x)
     x=x-fx/dfx
     print("fx",fx,"dfx",dfx,"x=",x)
     if abs(fx)<tolerance: return x
  return x

f1 = lambda x: x*x+2
a=complex(1,1);
r=secant(f1,a)
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

```
 ---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" secant.py
fx (2+2j) dfx (2.0000000000000018+1.9999999999998908j) x= (-2.6867397195928788e-14+0.9999999999999722j)
fx (1.0000000000000555-5.373479439185609e-14j) dfx (-1.4802973661668753e-13+1.9999999999999445j) x= (3.700743415417599e-
14+1.5000000000000138j)
fx (-0.25000000000000413+1.11022302462529e-13j) dfx (-3.700743415417188e-14+3.0000000000000275j) x= (-
1.0279842820604793e-15+1.4166666666666674j)
fx (-0.006944444444446418-2.912622132504693e-15j) dfx 2.833333333333335j x= 1.4142156862745099j
fx (-6.007304882871267e-06+0j) dfx 2.8284313725490198j x= 1.4142135623746899j
fx (-4.510614104447086e-12+0j) dfx 2.82842712474938j x= 1.4142135623730951j
 ROOT  :  1.4142135623730951j
FUNCTION VALUE :  (-4.440892098500626e-16+0j)

> Terminated with exit code 0.
```

If functions defined by using abstract class f_x, code will be

Program complex Secant method in Python (abstract class f_x derived class function version)

```
rom math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
     y=x*x+2;
     return y;

def secant(f,x):
  # secant root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
     fx= f.func(x);
     dfx=f.dfunc(x,1);
     x=x-fx/dfx
```

```
    print("fx",fx,"dfx",dfx,"x=",x)
    if abs(fx)<tolerance: return x
  return x

f=f1();
a=complex(1,1);
#a=complex(input("enter initial guess : "))
r=secant(f,a);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

Similar to the Python , you do not have to specify code as complex in Octave(matlab) code to calculate complex roots

Program complex Secant method in Octave(Matlab) secant.m

```
function p=secant(f,x,tolerance,nmax)
% secant method for root finding
if nargin<4 nmax=100;end
if nargin<3 tolerance=1e-10;end
 dx=0.0001;
i=0;
fx=f(x);
dfx=df(f,x);
while (abs(fx) > tolerance) && i<nmax
   fx=f(x);
   dfx=df(f,x);
   x=x-fx/dfx;
   i=i+1;
end
p=x;
if i>=nmax fprintf('Maximum number of iterations is exceeded the result may not be valid\n');end
endfunction
```

```
>> f=@(x)x*x+2;
>> x=0+1j;
>> secant(f,x)
ans =  0.000000000000000 + 1.414213562373095i
```

Program complex Secant method in Fortran 90

```
program secant2
IMPLICIT NONE
COMPLEX(KIND = 8) a
a=cmplx(1,1,KIND=8)
print *,'secant(',a,') = ',secant(a)

CONTAINS
  FUNCTION func(x)
  COMPLEX(KIND = 8) func,x
  func=x*x+2.0
  END FUNCTION func

  FUNCTION dfunc(x)
  COMPLEX(KIND = 8) dfunc,x,h,df,f1,f2,f3,f4,f5,f6,f7,f8
  h=dcmplx(1.12314e-4,1.12314e-4)
  f1=3.0*func(x-4.0*h)
  f2=-32.0*func(x-3.0*h)
  f3=168.0*func(x-2.0*h)
  f4=-672.0*func(x-h)
  f5=672.0*func(x+h)
  f6=-168.0*func(x+2.0*h)
  f7=32.0*func(x+3.0*h)
  f8=-3.0*func(x+4.0*h)
  dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(840.0*h)
  END FUNCTION dfunc

  FUNCTION secant(a)
  COMPLEX(KIND = 8) secant,a,x,fx,dfx
  REAL(KIND=8) eps
  INTEGER nmax,i
  eps=1.12314e-15
  nmax=100;
  x=a
  DO i=1,nmax
```

```
    fx=func(x)
    dfx=dfunc(x)
    x=x-fx/dfx
    IF (abs(fx).LT.eps) EXIT
  END DO
  secant=x
  END FUNCTION secant
END PROGRAM secant2
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" secant2

E:\okul\SCO1>secant2
 secant( (  1.0000000000000000    ,  1.0000000000000000    ) ) =  (  3.2208930371605715E-033,  1.4142135623730951    )

> Terminated with exit code 0.
```

HIGHER DERIVATIVE ORDER METHODS

Consider the Taylor Formula, but this time assume that two term is remaining

$$f(x_{n+1}) = 0 = f(x_n) + \frac{f'(x_n)}{1!}(x_{n+1} - x_n) + \frac{f"(x_n)}{2!}(x_{n+1} - x_n)^2 + \frac{f^{(3)}(x_n)}{3!}(x_{n+1} - x_n)^3 + \cdots \qquad (2.11.1)$$

When second term of the Taylor Formula is also taken into acount the Newton-Raphson Formula takes the form:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)f'(x_n) - f(x_n)f"(x_n)} \qquad (2.11.2)$$

This method is called **Halley's method**.



Figure Edmond Halley, British Mathematician and Astronomer

The iterative methods with higher-order convergence are presented in some literature . **Olver's method** is also cubically convergent second order method. It is faster than the Newton-Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{f(x_n)f(x_n)f"(x_n)}{f'(x_n)f'(x_n)f'(x_n)} \qquad (2.11.3)$$

In [48], Abbasbandy gives a third order iterative method, called **Abbasbandy's method**(AM) provisionally, which is expressed as



Figure Daeid Abbasbandy, Iranian Mathematician

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{f(x_n)f(x_n)f"(x_n)}{f'(x_n)f'(x_n)f'(x_n)} - \frac{1}{6}\frac{f(x_n)f(x_n)f(x_n)f'''(x_n)}{f'(x_n)f'(x_n)f'(x_n)f'(x_n)} \qquad (2.11.4)$$

Li and Wang[49] suggested two fourth order methods and called them **Thiele Method 1** (TM1) and **Thiele Method 2** (TM2) due to fact that equations derived using Thiele's continued fraction method. Equations has a general form of

$$x_{n+1} = x_n - \frac{P_1(x_n)}{Q_1(x_n)} \quad (2.11.5)$$

$$P_1(x) = 4f(x)f'(x)[3f'(x)f'(x)f"(x)\text{-}3f(x)f"(x)f"(x) + f(x)f'(x)f'''(x)] \quad (2.11.6)$$

$$Q_1(x) = 12f'(x)f'(x)f'(x)f'(x)f(x)\text{-}18f(x)f^{'}(x)f^{'}(x)f"(x)f"(x) + 3f(x)f(x)\, f"(x)f"(x)f"(x) + 4f(x)f^{'}(x)f^{'}(x)f^{'}(x)f'''(x) \quad (2.11.7)$$

$$x_{n+1} = x_n - \frac{P_2(x_n)}{Q_2(x_n)} \quad (2.11.8)$$

$$P_2(x) = f(x)[6f'(x)f'(x)\text{-}3f(x)f"(x)f"(x) + 2f(x)f'(x)f'''(x)] \quad (2.11.9)$$

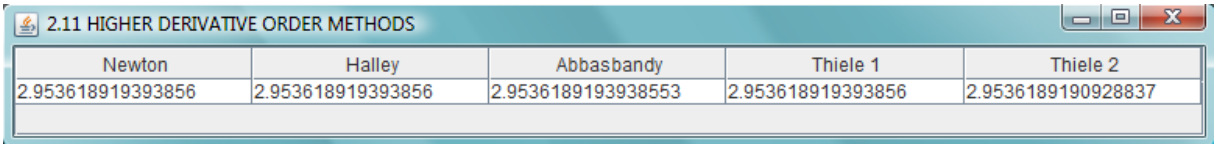
$$Q_2(x) = 2f'(x)\left[3f'(x)f'(x)f"(x) - 3f(x)f"(x)f"(x) + f(x)f'(x)f^{'''}(x)\right] \quad (2.11.10)$$

Program Higher order methods, Newton, Halley, Abbasbandy Thiele1 and Thiele2 (Java code)

```
 /  High order methods Halley, Abbasbandy hiele1 and Thiele2
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= function to be defined ================

class fb extends f_x
{
  public double func (double x)
  { return x*x*x+3.6*x-36.4;}
}
//============================================================
public class NA5B
{
public static double newton(f_x f,double x)
{
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
  dfx=f.dfunc(x);
  x-=fx/dfx;
 if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Newton Method\n"+
  " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double halley(f_x f,double x)
{
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,fxm,dfxm,d2fx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
  dfx=f.dfunc(x);
  d2fx=f.dfunc(x,2);
  x-=2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx);
 if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Halley method\n"+
  " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double abbasbandy(f_x f,double x)
{
```

```
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx,d3fx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   x-=(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx));
  if(Math.abs(fx)<tolerance) { return x;}
}
  JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Abbasbandy\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
   return x;
}

public static double TM1(f_x f,double x)
{ //Thiele 1 method
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx,d3fx;
double P1,Q1;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
   Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx;
   x-=P1/Q1;
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Thiele 1\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

public static double TM2(f_x f,double x)
{ //Thiele 2 method
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx,d3fx;
double P2,Q2;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   P2=fx*(6.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+2*fx*dfx*d3fx);
   Q2=2.0*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
   x-=P2/Q2;
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Thiele 2 method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
    public static void main (String args[])
    {         fb f=new fb();
              dfb df=new dfb();
    double x0;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" first guess for x : "));
```

```
    double r[]=new double[5];
    String s[]={"Newton","Halley","Abbasbandy","Thiele 1","Thiele 2"};
    r[0]=newton(f,x0);
    r[1]= halley(f,x0);
    r[2]= abbasbandy(f,x0);
    r[3]= TM1(f,x0);
    r[4]= TM2(f,x0);
    Text.print(r,s,"2.11 HIGHER DERIVATIVE ORDER METHODS");
    }
}
```



We will write the same program in C++ now

Program Higher order methods, Newton, Halley, Abbasbandy Thiele1 and Thiele2 (C++ code)

```
 //  High order methods Halley, Abbasbandy Thiele1 and Thiele2
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
public: double dfunc(double x,int n,double dx)
{       double df=0;
        if(n==0) df=func(x);
        else if( n==1) df=(3.0*func(x-4.0*dx)-32.0*func(x-3.0*dx)+168.0*func(x-2.0*dx)-672.0*func(x-dx)+672.0*func(x+dx)-
168.0*func(x+2.0*dx)+32.0*func(x+3.0*dx)-3.0*func(x+4.0*dx))/840.0/dx;
        else if( n==2) df=(-14350.0*func(x)-9.0*func(x-4.0*dx)+128*func(x-3.0*dx)-1008*func(x-2.0*dx)+8064*func(x-
dx)+8064.0*func(x+dx)-1008.0*func(x+2.0*dx)+128.0*func(x+3.0*dx)-9.0*func(x+4.0*dx))/5040.0/dx/dx;
        else if( n==3) df=(-7.0*func(x-4.0*dx)+72.0*func(x-3.0*dx)-338.0*func(x-2.0*dx)+488.0*func(x-dx)-
488.0*func(x+dx)+338.0*func(x+2.0*dx)-72.0*func(x+3.0*dx)+7.0*func(x+4.0*dx))/240.0/dx/dx/dx;
        else if( n==4) df=(2730.0*func(x)+7.0*func(x-4.0*dx)-96.0*func(x-3.0*dx)+676.0*func(x-2*dx)-1952.0*func(x-dx)-
1952.0*func(x+dx)+676.0*func(x+2.0*dx)-96.0*func(x+3.0*dx)+7.0*func(x+4.0*dx))/240.0/dx/dx/dx/dx;
        else if( n==5) df=(func(x-4.0*dx)-9.0*func(x-3.0*dx)+26.0*func(x-2.0*dx)-29.0*func(x-dx)+29.0*func(x+dx)-
26.0*func(x+2.0*dx)+9.0*func(x+3.0*dx)-func(x+4.0*dx))/6.0/dx/dx/dx/dx/dx;
        else if( n==6) df=(-150.0*func(x)-func(x-4.0*dx)+12.0*func(x-3.0*dx)-52.0*func(x-2.0*dx)+116.0*func(x-dx)+116.0*func(x+dx)-
52.0*func(x+2.0*dx)+12.0*func(x+3.0*dx)-func(x+4.0*dx))/4.0/dx/dx/dx/dx/dx/dx;
        else if( n==7) df=(-func(x-4.0*dx)+6.0*func(x-3.0*dx)-14.0*func(x-2.0*dx)+14.0*func(x-dx)-
14.0*func(x+dx)+14.0*func(x+2.0*dx)-6.0*func(x+3.0*dx)+func(x+4.0*dx))/2.0/dx/dx/dx/dx/dx/dx/dx;
        else if( n==8) df=(70.0*func(x)+func(x-4.0*dx)-8.0*func(x-3.0*dx)+28.0*func(x-2.0*dx)-56.0*func(x-dx)-
56.0*func(x+dx)+28.0*func(x+2.0*dx)-8.0*func(x+3.0*dx)+func(x+4.0*dx))/dx/dx/dx/dx/dx/dx/dx/dx;
        else df=0;
        return df;
}
public: double dfunc(double x,int n)
{       double dx=0.0001;
       return dfunc(x,n,dx);
}
};

class fb:public f_x
{ public: double func(double x)
  { double ff=x*x*x+3.6*x-36.4;
                return ff;
  }
};

double newton(f_x &f,double x)
{
```

```
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   x-=fx/dfx;
  if(fabs(fx)<tolerance) { return x;}
}
cout<<"Maximum number of iteration is exceeded in Newton Method\n";
return x;
}
double halley(f_x &f,double x)
{
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,fxm,dfxm,d2fx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   x-=2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx);
  if(fabs(fx)<tolerance) { return x;}
}
cout<<"Maximum number of iteration is exceeded in Newton Method\n";return x;
}
double abbasbandy(f_x &f,double x)
{
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx,d3fx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   x-=(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx));
  if(fabs(fx)<tolerance) { return x;}
}
cout<<"Maximum number of iteration is exceeded in Newton Method\n";
return x;
}

double TM1(f_x &f,double x)
{ //Thiele 1 method
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx,d3fx;
double P1,Q1;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
   Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx;
   x-=P1/Q1;
  if(fabs(fx)<tolerance) { return x;}
}
cout<<"Maximum number of iteration is exceeded in Newton Method\n";
return x;
}

double TM2(f_x &f,double x)
{ //Thiele 2 method
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx,d3fx;
double P2,Q2;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   P2=fx*(6.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+2*fx*dfx*d3fx);
   Q2=2.0*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
```

```
   x-=P2/Q2;
  if(fabs(fx)<tolerance) { return x;}
}
cout<<"Maximum number of iteration is exceeded in Newton Method\n";
return x;
}

    int main ()
    {
              fb f;
              double x0;
              cout<<setprecision(20);
              cout << " first guess for x ";
    cin >>x0;
    double r[5];
    string s[]={"Newton    ","Halley    ","Abbasbandy","Thiele 1  ","Thiele 2  "};
    r[0]=newton(f,x0);
    r[1]= halley(f,x0);
    r[2]= abbasbandy(f,x0);
    r[3]= TM1(f,x0);
    r[4]= TM2(f,x0);
    cout<<"HIGHER DERIVATIVE ORDER METHODS\n";
    for(int i=0;i<5;i++)
      {cout<<s[i]<<" = "<<r[i]<<endl;}}
```

```
 ---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" NA5B

D:\okul\SCO1>NA5B
 first guess for x 1
HIGHER DERIVATIVE ORDER METHODS
Newton     = 2.9536189193938553
Halley     = 2.9536189193938553
Abbasbandy = 2.9536189193938553
Thiele 1   = 2.9536189193938553
Thiele 2   = 2.9536189193938553

> Terminated with exit code 0.
```

Program Higher order methods, Halley, Abbasbandy in Fortran 90

```
 program halley1
IMPLICIT NONE
REAL(KIND = SELECTED_REAL_KIND(10,34)) a
a=1.0
print *,'halley(',a,') = ',halley(a)
a=1.0
print *,'olver(',a,') = ',olver(a)
a=1.0
print *,'abbasbandy(',a,') = ',abbasbandy(a)
a=1.0
print *,'TM1(',a,') = ',TM1(a)
a=1.0
print *,'TM2(',a,') = ',TM2(a)

CONTAINS
  FUNCTION func(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) func,x
  func=x*x*x+3.6*x-36.4
  END FUNCTION func

  FUNCTION dfunc(x,n)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) dfunc,x,a,h,df,f1,f2,f3,f4,f5,f6,f7,f8,f9
  INTEGER n
  h=1.12314e-4
  if(n==1) then
    f1=3.0*func(x-4.0*h)
    f2=-32.0*func(x-3.0*h)
    f3=168.0*func(x-2.0*h)
    f4=-672.0*func(x-h)
    f5=672.0*func(x+h)
    f6=-168.0*func(x+2.0*h)
    f7=32.0*func(x+3.0*h)
    f8=-3.0*func(x+4.0*h)
    dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(840.0*h)
```

```
  else if(n==2) then
    f1=-14350.0*func(x)
    f2=-9.0*func(x-4.0*h)
    f3=128.0*func(x-3.0*h)
    f4=-1008.0*func(x-2.0*h)
    f5=8064.0*func(x-h)
    f6=8064.0*func(x+h)
    f7=-1008.0*func(x+2.0*h)
    f8=128.0*func(x+3.0*h)
    f9=-9.0*func(x+4.0*h)
    dfunc=(f1+f2+f3+f4+f5+f6+f7+f8+f9)/(5040.0*h*h)
  else if(n==3) then
    f1=-7.0*func(x-4.0*h)
    f2=72.0*func(x-3.0*h)
    f3=-338.0*func(x-2.0*h)
    f4=488.0*func(x-h)
    f5=-488.0*func(x+h)
    f6=338.0*func(x+2.0*h)
    f7=-72.0*func(x+3.0*h)
    f8=7.0*func(x+4.0*h)
    dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(240.0*h*h*h);
  end if
  END FUNCTION dfunc

  FUNCTION olver(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) olver,x,eps,fx,dfx,d2fx
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    x=x-fx/dfx-1.0/2.0*fx*fx*d2fx/(dfx*dfx*dfx)
    IF (abs(fx)<eps) EXIT
  END DO
  olver=x
  END FUNCTION olver
    FUNCTION halley(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) halley,x,eps,fx,dfx,d2fx
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    x=x-2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx)
    IF (abs(fx)<eps) EXIT
  END DO
  halley=x
  END FUNCTION halley
  FUNCTION abbasbandy(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) abbasbandy,x,eps,fx,dfx,d2fx,d3fx
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  x=a
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    d3fx=dfunc(x,3)
    x=x-(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx));
    IF (abs(fx)<eps) EXIT
  END DO
  abbasbandy=x
  END FUNCTION abbasbandy
  FUNCTION TM1(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) TM1,x,eps,fx,dfx,d2fx,d3fx,P1,Q1
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  x=a
  DO i=1,nmax
    fx=func(x)
```

```
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    d3fx=dfunc(x,3)
    P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx)
    Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx
    x=x-P1/Q1;
    IF (abs(fx)<eps) EXIT
  END DO
  TM1=x
  END FUNCTION TM1

  FUNCTION TM2(x)
  REAL(KIND = SELECTED_REAL_KIND(10,34)) TM2,x,eps,fx,dfx,d2fx,d3fx,P2,Q2
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  x=a
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    d3fx=dfunc(x,3)
    P2=fx*(6.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+2*fx*dfx*d3fx)
    Q2=2.0*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx)
    x=x-P2/Q2;
    IF (abs(fx)<eps) EXIT
  END DO
  TM2=x
  END FUNCTION TM2
END PROGRAM halley1
```

```
 ---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" halley1

E:\okul\SCO1>halley1
 halley(   1.0000000000000000      ) =    2.9536189801080357
 olver(   1.0000000000000000      ) =    2.9536189801080357
 abbasbandy(   1.0000000000000000      ) =    2.9536189801080357
 TM1(   1.0000000000000000      ) =    2.9536189801080357
 TM2(   1.0000000000000000      ) =    2.9536189801080357
> Terminated with exit code 0.
```

Program Higher order methods, Newton, Halley, Abbasbandy Thiele1 and Thiele2

```
 function newton(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x-=fx/dfx
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function halley(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x-=2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx)
  if abs(fx)<tolerance
    return x
  end #f
end #for
```

```
return x
end #function

function abbasbandy(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  d3fx=d2f(f,x)
     x-=(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx))
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function TM1(f,x)
# Thiele 1
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  d3fx=d2f(f,x)
  P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx)
  Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx
  x-=P1/Q1;
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function TM2(f,x)
# Thiele 2
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  d3fx=d2f(f,x)
  P2=fx*(6.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+2*fx*dfx*d3fx)
  Q2=2.0*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx)
  x-=P2/Q2;
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

#first derivative
function df(f,x)
 dx=0.001
 c=[1.0/280.0 -4.0/105.0  1.0/5.0  -4.0/5.0  0  4.0/5.0  -1.0/5.0  4.0/105.0 -1.0/280.0]
 ff=0.0
```

```
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end
#second derivative
function d2f(f,x)
 dx=0.001
 c=[-1.0/560.0         8.0/315.0 -1.0/5.0 8.0/5.0 -205.0/72.0 8.0/5.0 -1.0/5.0 8.0/315.0 -1.0/560.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx)
end
#third derivative
function d3f(f,x)
 dx=0.001
 c=[-7.0/240.0         3.0/10.0   -169.0/120.0          61.0/30.0 0.0          -61.0/30.0              169.0/120.0          -3.0/10.0
             7.0/240.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx*dx)
end

f(x)=x*x*x+3.6*x-36.4
x=1.0
s=["Newton" "Halley" "Abbasbandy" "Thiele1" "Thiele2"]
r=zeros(Float64,5)
r[1]=newton(f,x)
r[2]=halley(f,x)
r[3]=abbasbandy(f,x)
r[4]=TM1(f,x)
r[5]=TM2(f,x)
for i=1:5
  s1=s[i]
  r1=r[i]
  f1=f(r1)
  println("$s1 x = $r1 y = $f1")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton2.jl
Newton x = 2.953618919393856 y = 7.105427357601002e-15
Halley x = 2.9536189193938553 y = -7.105427357601002e-15
Abbasbandy x = 2.953618919393856 y = 7.105427357601002e-15
Thiele1 x = 2.9536189193938553 y = -7.105427357601002e-15
Thiele2 x = 2.9536189193938553 y = -7.105427357601002e-15

> Terminated with exit code 0.
```

This methods can also be used for complex roots

```
function newton(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x-=fx/dfx
  if abs(fx)<tolerance
    return x
  end #f
```

```
end #for
return x
end #function

function halley(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x-=2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx)
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function abbasbandy(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  d3fx=d2f(f,x)
     x-=(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx))
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function TM1(f,x)
# Thiele 1
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  d3fx=d2f(f,x)
  P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx)
  Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx
  x-=P1/Q1;
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function TM2(f,x)
# Thiele 2
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  d3fx=d2f(f,x)
  P2=fx*(6.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+2*fx*dfx*d3fx)
  Q2=2.0*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx)
  x-=P2/Q2;
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

#first derivative
function df(f,x)
 dx=0.001
```

```
 c=[1.0/280.0 -4.0/105.0  1.0/5.0  -4.0/5.0  0  4.0/5.0  -1.0/5.0  4.0/105.0  -1.0/280.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end
#second derivative
function d2f(f,x)
 dx=0.001
 c=[-1.0/560.0  8.0/315.0 -1.0/5.0 8.0/5.0 -205.0/72.0 8.0/5.0 -1.0/5.0 8.0/315.0 -1.0/560.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx)
end
#third derivative
function d3f(f,x)
 dx=0.001
 c=[-7.0/240.0  3.0/10.0  -169.0/120.0  61.0/30.0  0.0  -61.0/30.0 169.0/120.0  -3.0/10.0  7.0/240.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx*dx)
end

f(x)=x*x*-3.6*x-36.4
x=1.0+1im
s=["Newton" "Halley" "Abbasbandy" "Thiele1" "Thiele2"]
r=zeros(ComplexF64,5)
r[1]=newton(f,x)
r[2]=halley(f,x)
r[3]=abbasbandy(f,x)
r[4]=TM1(f,x)
r[5]=TM2(f,x)
for i=1:5
  s1=s[i]
  r1=r[i]
  f1=f(r1)
  println("$s1 x = $r1 y = $f1")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton3.jl
Newton x = 1.0811923527860583 + 1.8726800877803869im y = 0.0 + 3.552713678800501e-15im
Halley x = 1.0811923527860583 + 1.8726800877803869im y = 0.0 + 3.552713678800501e-15im
Abbasbandy x = 1.0811923527860583 + 1.8726800877803869im y = 0.0 + 3.552713678800501e-15im
Thiele1 x = 1.0811923527860583 + 1.8726800877803869im y = 0.0 + 3.552713678800501e-15im
Thiele2 x = 1.0811923527860585 + 1.872680087780387im y = 1.4210854715202004e-14 + 0.0im

> Terminated with exit code 0.
```

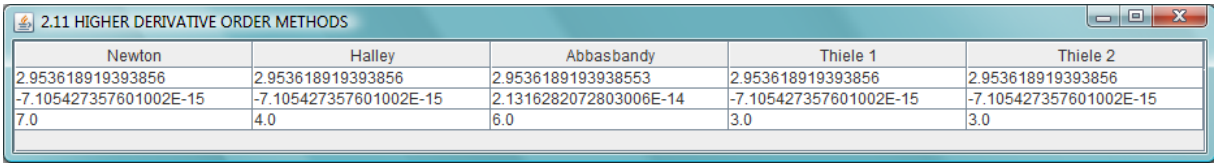
In this stage, it will be interesting to compare iteration efficiencies of the higher order methods. In order to achive that a version of the program with 3 output values namely root x, function value at the root f(x) and number of iteration is developed.

Program Higher order methods, Newton, Halley, Abbasbandy Thiele1 and Thiele2 with 3 output values root x, function f(x) and number of iteration n

```
 //  High order methods Halley, Abbasbandy hiele1 and Thiele2
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= function to be defined ================
class fb extends f_x
{  public double func (double x)
   { return x*x*x+3.6*x-36.4;}
}
```

```
public class NA5B1
{public static double[] newton(f_x f,double x)
{
double a[]=new double[3];
int nmax=500;
double tolerance=1.0e-10;
double fx=0,dfx;
int i;
for(i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   x-=fx/dfx;
  if(Math.abs(fx)<tolerance) { a[0]=x;a[1]=fx;a[2]=i;return a;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Newton Method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
a[0]=x;a[1]=fx;a[2]=i;return a;
}

public static double[] halley(f_x f,double x)
{
              double a[]=new double[3];
int nmax=500;
double tolerance=1.0e-10;
double fx=0,dfx,fxm,dfxm,d2fx;
int i;
for(i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   x-=2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx);
  if(Math.abs(fx)<tolerance) { a[0]=x;a[1]=fx;a[2]=i;return a;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Halley method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
a[0]=x;a[1]=fx;a[2]=i;return a;
}

public static double[] abbasbandy(f_x f,double x)
{
              double a[]=new double[3];
int nmax=500;
double tolerance=1.0e-10;
double fx=0,dfx,d2fx,d3fx;
int i;
for(i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   x-=(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx));
  if(Math.abs(fx)<tolerance) { a[0]=x;a[1]=fx;a[2]=i;return a;}
}
  JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Abbasbandy\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
  a[0]=x;a[1]=fx;a[2]=i;return a;
}

public static double[] TM1(f_x f,double x)
{ //Thiele 1 method
double a[]=new double[3];
int nmax=500;
double tolerance=1.0e-10;
double fx=0,dfx,d2fx,d3fx;
double P1,Q1;
int i;
for(i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
```

```
   d3fx=f.dfunc(x,3);
   P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
   Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx;
   x-=P1/Q1;
  if(Math.abs(fx)<tolerance) { a[0]=x;a[1]=fx;a[2]=i;return a;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Thiele 1\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
{ a[0]=x;a[1]=fx;a[2]=i;return a;}
}

public static double[] TM2(f_x f,double x)
{ //Thiele 2 method
double a[]=new double[3];
int nmax=500;
double tolerance=1.0e-10;
double fx=0,dfx,d2fx,d3fx;
double P2,Q2;
int i;
for(i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   d3fx=f.dfunc(x,3);
   P2=fx*(6.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+2*fx*dfx*d3fx);
   Q2=2.0*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
   x-=P2/Q2;
  if(Math.abs(fx)<tolerance) { a[0]=x;a[1]=fx;a[2]=i;return a;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in Thiele 2 method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
 a[0]=x;a[1]=fx;a[2]=i;return a;
}

    public static void main (String args[])
    {
              fb f=new fb();
              dfb df=new dfb();
    double x0;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" first guess for x : "));
    double r[][]=new double[5][3];
    String s[]={"Newton","Halley","Abbasbandy","Thiele 1","Thiele 2"};
    r[0]=newton(f,x0);
    r[1]= halley(f,x0);
    r[2]= abbasbandy(f,x0);
    r[3]= TM1(f,x0);
    r[4]= TM2(f,x0);
    Text.printT(r,s,"2.11 HIGHER DERIVATIVE ORDER METHODS");
    }
}
```



Newton	Halley	Abbasbandy	Thiele 1	Thiele 2
2.953618919393856	2.953618919393856	2.9536189193938553	2.953618919393856	2.953618919393856
-7.105427357601002E-15	-7.105427357601002E-15	2.1316282072803006E-14	-7.105427357601002E-15	-7.105427357601002E-15
7.0	4.0	6.0	3.0	3.0

Program Higher order numerical derivatives in Matlab(Octave) code

```
function df=dnf(f,x,n,dx)
% degree nth derivative of a function f
if nargin<4
   dx=0.001;
end
if nargin<3
   dx=0.001;n=1;
end
dxh=1.0/dx;
if n==0 df=f(x);
```

```
elseif n==1
df=(3.0*f(x-4.0*dx)-32.0*f(x-3.0*dx)+168.0*f(x-2.0*dx)-672.0*f(x-dx)+672.0*f(x+dx)-168.0*f(x+2.0*dx)+32.0*f(x+3.0*dx)-
3.0*f(x+4.0*dx))/840.0*dxh;
elseif n==2
df=(-14350.0*f(x)-9.0*f(x-4.0*dx)+128*f(x-3.0*dx)-1008*f(x-2.0*dx)+8064*f(x-dx)+8064.0*f(x+dx)-
1008.0*f(x+2.0*dx)+128.0*f(x+3.0*dx)-9.0*f(x+4.0*dx))/5040.0*dxh*dxh;
elseif n==3
df=(-7.0*f(x-4.0*dx)+72.0*f(x-3.0*dx)-338.0*f(x-2.0*dx)+488.0*f(x-dx)-488.0*f(x+dx)+338.0*f(x+2.0*dx)-
72.0*f(x+3.0*dx)+7.0*f(x+4.0*dx))/240.0*dxh*dxh*dxh;
elseif n==4
df=(2730.0*f(x)+7.0*f(x-4.0*dx)-96.0*f(x-3.0*dx)+676.0*f(x-2*dx)-1952.0*f(x-dx)-1952.0*f(x+dx)+676.0*f(x+2.0*dx)-
96.0*f(x+3.0*dx)+7.0*f(x+4.0*dx))/240.0*dxh*dxh*dxh*dxh;
elseif n==5
df=(f(x-4.0*dx)-9.0*f(x-3.0*dx)+26.0*f(x-2.0*dx)-29.0*f(x-dx)+29.0*f(x+dx)-26.0*f(x+2.0*dx)+9.0*f(x+3.0*dx)-
f(x+4.0*dx))/6.0*dxh*dxh*dxh*dxh*dxh;
elseif n==6
df=(-150.0*f(x)-f(x-4.0*dx)+12.0*f(x-3.0*dx)-52.0*f(x-2.0*dx)+116.0*f(x-dx)+116.0*f(x+dx)-52.0*f(x+2.0*dx)+12.0*f(x+3.0*dx)-
f(x+4.0*dx))/4.0*dxh*dxh*dxh*dxh*dxh*dxh;
elseif n==7
df=(-f(x-4.0*dx)+6.0*f(x-3.0*dx)-14.0*f(x-2.0*dx)+14.0*f(x-dx)-14.0*f(x+dx)+14.0*f(x+2.0*dx)-
6.0*f(x+3.0*dx)+f(x+4.0*dx))/2.0*dxh*dxh*dxh*dxh*dxh*dxh*dxh;
elseif n==8
df=(70.0*f(x)+f(x-4.0*dx)-8.0*f(x-3.0*dx)+28.0*f(x-2.0*dx)-56.0*f(x-dx)-56.0*f(x+dx)+28.0*f(x+2.0*dx)-
8.0*f(x+3.0*dx)+f(x+4.0*dx))*dxh*dxh*dxh*dxh*dxh*dxh*dxh*dxh;

else df=0;end
end
```

Program Thiele2 method in Matlab(Octave) code

```
function xx = TM2(f,x)
%Thiele 2 Method
nmax=500;
tolerance=1.0e-8;
i=0;
f_x=1;
while   abs(f_x)>=tolerance && i<=nmax
   f_x=f(x);
   df_x=dnf(f,x,1);
   d2f_x=dnf(f,x,2);
   d3f_x=dnf(f,x,3);
   P2=f_x*(6.0*df_x*df_x*d2f_x-3.0*f_x*d2f_x*d2f_x+2*f_x*df_x*d3f_x);
   Q2=2.0*df_x*(3.0*df_x*df_x*d2f_x-3.0*f_x*d2f_x*d2f_x+f_x*df_x*d3f_x);
   x=x-P2/Q2;
   i=i+1;
  end
if(i>=nmax) fprintf('Maximum number of iteration is exceeded in Thiele 2 method\nresults may not be valid');end
xx=x;
end
```

```
>> f1=@(x)x*x-2;
>> TM2(f1,1)
ans =  1.41421356237310
>>
```

Program Thiele1 method in python code

```
from math import *

def df(f,x):
    h=0.001
    return (-f(x+2.0*h)+8.0*f(x+h)-8.0*f(x-h)+f(x-2.0*h))/(12.0*h);

# second derivative
def d2f(f,x):
    dx=0.001
    return (-1.0/12.0*f(x-2.0*dx)+4.0/3.0*f(x-dx)-5.0/2.0*f(x)+4.0/3.0*f(x+dx)-1.0/12.0*f(x+2.0*dx))/dx

# nth derivative
def dnf(f,x,n):
    dx=0.001
    dxh=1.0/dx
    df=0.0;
```

```
    if n==0:  df=f(x);
    elif n==1: df=(3.0*f(x-4.0*dx)-32.0*f(x-3.0*dx)+168.0*f(x-2.0*dx)-672.0*f(x-dx)+672.0*f(x+dx)-
168.0*f(x+2.0*dx)+32.0*f(x+3.0*dx)-3.0*f(x+4.0*dx))/840.0*dxh;
    elif n==2: df=(-14350.0*f(x)-9.0*f(x-4.0*dx)+128*f(x-3.0*dx)-1008*f(x-2.0*dx)+8064*f(x-dx)+8064.0*f(x+dx)-
1008.0*f(x+2.0*dx)+128.0*f(x+3.0*dx)-9.0*f(x+4.0*dx))/5040.0*dxh*dxh;
    elif n==3:
      df=(-7.0*f(x-4.0*dx)+72.0*f(x-3.0*dx)-338.0*f(x-2.0*dx)+488.0*f(x-dx)-488.0*f(x+dx)+338.0*f(x+2.0*dx)-
72.0*f(x+3.0*dx)+7.0*f(x+4.0*dx))/240.0*dxh*dxh*dxh;
    elif n==4:
      df=(2730.0*f(x)+7.0*f(x-4.0*dx)-96.0*f(x-3.0*dx)+676.0*f(x-2*dx)-1952.0*f(x-dx)-1952.0*f(x+dx)+676.0*f(x+2.0*dx)-
96.0*f(x+3.0*dx)+7.0*f(x+4.0*dx))/240.0*dxh*dxh*dxh*dxh;
    elif n==5:
      df=(f(x-4.0*dx)-9.0*f(x-3.0*dx)+26.0*f(x-2.0*dx)-29.0*f(x-dx)+29.0*f(x+dx)-26.0*f(x+2.0*dx)+9.0*f(x+3.0*dx)-
f(x+4.0*dx))/6.0*dxh*dxh*dxh*dxh*dxh;
    return df;

def TM1(f,x):
    #Thiele 1 method
    a=[0.0 for i in range(3)];
    nmax=500;
    tolerance=1.0e-10;
    fx=0;
    for i in range(0,nmax):
      fx=f(x);
      dfx=df(f,x);
      d2fx=d2f(f,x);
      d3fx=dnf(f,x,3);
      P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
      Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx;
      x-=P1/Q1;
      if abs(fx)<tolerance: a[0]=x;a[1]=fx;a[2]=i;return a;
    print("Maximum number of iteration is exceeded in Thiele 1\n");
    a[0]=x;a[1]=fx;a[2]=i;
    return a;

f1 = lambda x: x*x-2
x=float(input("enter estimation : "))
print(x)
r=Mathd.TM1(f1,x);
print(r)
```

```
runfile('D:/okul/SCO1/TM1_2.py', wdir='D:/okul/SCO1')
enter estimation : 1
1.0
[1.4142135623730951, 4.429789868254375e-12, 4]
```

Program Thiele1 method in python code abstract f_x class based class function func is used

```
from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
    y=x*x+2;
    return y;

def TM1(f,x):
    #Thiele 1 method
    a=[0.0 for i in range(3)];
    nmax=500;
    tolerance=1.0e-10;
    fx=0;
    for i in range(0,nmax):
      fx=f.func(x);
      dfx=f.dfunc(x,1);
      d2fx=f.dfunc(x,2);
      d3fx=f.dfunc(x,3);
      P1=4.0*fx*dfx*(3.0*dfx*dfx*d2fx-3.0*fx*d2fx*d2fx+fx*dfx*d3fx);
      Q1=12.0*dfx*dfx*dfx*dfx*d2fx-18.0*fx*dfx*dfx*d2fx*d2fx+3.0*fx*fx*d2fx*d2fx*d2fx+4.0*fx*dfx*dfx*dfx*d3fx;
      x-=P1/Q1;
      if abs(fx)<tolerance: a[0]=x;a[1]=fx;a[2]=i;return a;
    print("Maximum number of iteration is exceeded in Thiele 1\n");
    a[0]=x;a[1]=fx;a[2]=i;
    return a;
```

```
f =  f1();
x=complex(input("enter estimation example: 1+2j : "));
r=TM1(f,x);
print(r)
```

```
---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" TM1_2.py
enter estimation example: 1+2j : 1+1j
[(9.341989763213027e-46+1.414213562373095j), (-4.440892098500626e-16+0j), 3]

> Terminated with exit code 0.
```

A complex version of Halley's method is given here:

Program Haley's method complex lambda version (java)

```
// compleks Halley's method to find roots of function
// xn+1=xn-2(f(xn)*f(xn)*f'(xn))/(2f'(xn)*f'(xn)-f(xn)*f"(xn))
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class Halley1
{
public static c Halley(icf_cx f,c x)
{
double eps=1.0e-15;
int m=100;
int k=0;
c f1,df,d2f;
c b1,b2,b3,b4,b5;
do{
  f1=f.func(x);
  df=f.dfunc(x);
  d2f=f.d2func(x);
  b1=c.p(c.p(f1,"*",df),"*",2.0);
  b2=c.p(c.p(df,"*",df),"*",2.0);
  b3=c.p(f1,"*",d2f);
  b4=c.p(b2,"-",b3);
  b5=c.p(b1,"/",b4);
  x=c.p(x,"-",b5);
  if(c.abs(f1)<eps) return x;
  k++;
}while(c.abs(f1)>eps && k<m);
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

public static c Halley(cf_cx f,c x)
{
double eps=1.0e-15;
int m=100;
int k=0;
c f1,df,d2f;
c b1,b2,b3,b4,b5;
do{
  f1=f.func(x);
  df=f.dfunc(x);
  d2f=f.d2func(x);
  b1=c.p(c.p(f1,"*",df),"*",2.0);
  b2=c.p(c.p(df,"*",df),"*",2.0);
  b3=c.p(f1,"*",d2f);
  b4=c.p(b2,"-",b3);
  b5=c.p(b1,"/",b4);
  x=c.p(x,"-",b5);
  if(c.abs(f1)<eps) return x;
  k++;
}while(c.abs(f1)>eps && k<m);
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
```

```
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}
    public static double[] input(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

    public static void main (String args[])
    { icf_cx f=(c x)->c.p(c.p(x,"*",x),"+",2.0);
      c x1=new c(1.0,1.0);
      c y=Halley(f,x1);
      System.out.println("root = "+y);
    }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Halley1
root = (   0.0000000000 +    1.4142135624i )

> Terminated with exit code 0.
```

Program Halley's method complex C++ code

```
#include <iostream>
#include <math.h>
#include <complex>
#include <iomanip>
using namespace std;
typedef complex<double> cdouble;

class cf_x{
public: virtual cdouble func(cdouble x)=0;
public: cdouble dfunc(cdouble x)
{ cdouble h=0.0001;
  cdouble hh=1.0/h;
  cdouble df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
public: cdouble dfunc(cdouble x,int n,cdouble dx)
{       cdouble df=0;
       if(n==0) df=func(x);
       else if( n==1) df=(3.0*func(x-4.0*dx)-32.0*func(x-3.0*dx)+168.0*func(x-2.0*dx)-672.0*func(x-dx)+672.0*func(x+dx)-
168.0*func(x+2.0*dx)+32.0*func(x+3.0*dx)-3.0*func(x+4.0*dx))/840.0/dx;
       else if( n==2) df=(-14350.0*func(x)-9.0*func(x-4.0*dx)+128.0*func(x-3.0*dx)-1008.0*func(x-2.0*dx)+8064.0*func(x-
dx)+8064.0*func(x+dx)-1008.0*func(x+2.0*dx)+128.0*func(x+3.0*dx)-9.0*func(x+4.0*dx))/5040.0/dx/dx;
       else if( n==3) df=(-7.0*func(x-4.0*dx)+72.0*func(x-3.0*dx)-338.0*func(x-2.0*dx)+488.0*func(x-dx)-
488.0*func(x+dx)+338.0*func(x+2.0*dx)-72.0*func(x+3.0*dx)+7.0*func(x+4.0*dx))/240.0/dx/dx/dx;
       else if( n==4) df=(2730.0*func(x)+7.0*func(x-4.0*dx)-96.0*func(x-3.0*dx)+676.0*func(x-2.0*dx)-1952.0*func(x-dx)-
1952.0*func(x+dx)+676.0*func(x+2.0*dx)-96.0*func(x+3.0*dx)+7.0*func(x+4.0*dx))/240.0/dx/dx/dx/dx;
       else if( n==5) df=(func(x-4.0*dx)-9.0*func(x-3.0*dx)+26.0*func(x-2.0*dx)-29.0*func(x-dx)+29.0*func(x+dx)-
26.0*func(x+2.0*dx)+9.0*func(x+3.0*dx)-func(x+4.0*dx))/6.0/dx/dx/dx/dx/dx;
       else if( n==6) df=(-150.0*func(x)-func(x-4.0*dx)+12.0*func(x-3.0*dx)-52.0*func(x-2.0*dx)+116.0*func(x-dx)+116.0*func(x+dx)-
52.0*func(x+2.0*dx)+12.0*func(x+3.0*dx)-func(x+4.0*dx))/4.0/dx/dx/dx/dx/dx/dx;
       else if( n==7) df=(-func(x-4.0*dx)+6.0*func(x-3.0*dx)-14.0*func(x-2.0*dx)+14.0*func(x-dx)-
14.0*func(x+dx)+14.0*func(x+2.0*dx)-6.0*func(x+3.0*dx)+func(x+4.0*dx))/2.0/dx/dx/dx/dx/dx/dx/dx;
       else if( n==8) df=(70.0*func(x)+func(x-4.0*dx)-8.0*func(x-3.0*dx)+28.0*func(x-2.0*dx)-56.0*func(x-dx)-
56.0*func(x+dx)+28.0*func(x+2.0*dx)-8.0*func(x+3.0*dx)+func(x+4.0*dx))/dx/dx/dx/dx/dx/dx/dx/dx;
       else df=0;
```

```
        return df;
}
public: cdouble dfunc(cdouble x,int n)
{       cdouble dx=0.0001;
       return dfunc(x,n,dx);
}};

class f1:public cf_x
{ public: cdouble func(cdouble x)
  { cdouble ff=x*x+2.0;return ff;}};
cdouble halley(cf_x &f,cdouble x)
{
int nmax=500;
double tolerance=1.0e-8;
cdouble fx,dfx,fxm,dfxm,d2fx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   x-=2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx);
  if(fabs(fx)<tolerance) { return x;}
}
cout<<"Maximum number of iteration is exceeded in halley Method\n";return x;
}
cdouble newton(cf_x &f,cdouble x)
{
int nmax=100;
double tolerance=1.0e-8;
cdouble fx,dfx;
for(int i=0;i<nmax;i++)
{ fx=f.func(x);
  dfx=f.dfunc(x);
  x-=fx/dfx;
  cout <<"i="<<i<<"x="<<x<<"fx="<<fx<<"dfx="<<dfx<<"\n";
  if(abs(fx)<tolerance) { return x;}
}
cout <<("Maximum number of iteration is exceeded \n result might not be valid");
return x;
}

int main()
{  f1 f;
              cdouble xa,r;
   cout << " first guess for the root value : ";
   cin >>xa;
   cout<<setprecision(20);
   r=halley(f,xa);
   cout<<"root value : ="<<r;
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" challey
D:\okul\SCO1>challey
 first guess for the root value : (1,1)
root value : =(-2.5849394142282115e-026,1.4142135623730949)
> Terminated with exit code 0.
```

Program Halley's and Abasbandy methods complex Fortran 90 code

```
program halley2
IMPLICIT NONE
COMPLEX(KIND = 8) a
a=cmplx(1,1,KIND=8)
print *,'halley(',a,') = ',halley(a)
a=cmplx(1,1,KIND=8)
print *,'abbasbandy(',a,') = ',abbasbandy(a)

CONTAINS
  FUNCTION func(x)
  COMPLEX(KIND = 8) func,x
  func=x*x+2
  END FUNCTION func

  FUNCTION dfunc(x,n)
```

```
  COMPLEX(KIND = 8) dfunc,x,a,h,df,f1,f2,f3,f4,f5,f6,f7,f8,f9
  INTEGER n
  h=cmplx(1.12314e-4,1.12314e-4,KIND=8)
  if(n==1) then
    f1=3.0*func(x-4.0*h)
    f2=-32.0*func(x-3.0*h)
    f3=168.0*func(x-2.0*h)
    f4=-672.0*func(x-h)
    f5=672.0*func(x+h)
    f6=-168.0*func(x+2.0*h)
    f7=32.0*func(x+3.0*h)
    f8=-3.0*func(x+4.0*h)
    dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(840.0*h)
  else if(n==2) then
    f1=-14350.0*func(x)
    f2=-9.0*func(x-4.0*h)
    f3=128.0*func(x-3.0*h)
    f4=-1008.0*func(x-2.0*h)
    f5=8064.0*func(x-h)
    f6=8064.0*func(x+h)
    f7=-1008.0*func(x+2.0*h)
    f8=128.0*func(x+3.0*h)
    f9=-9.0*func(x+4.0*h)
    dfunc=(f1+f2+f3+f4+f5+f6+f7+f8+f9)/(5040.0*h*h)
  else if(n==3) then
    f1=-7.0*func(x-4.0*h)
    f2=72.0*func(x-3.0*h)
    f3=-338.0*func(x-2.0*h)
    f4=488.0*func(x-h)
    f5=-488.0*func(x+h)
    f6=338.0*func(x+2.0*h)
    f7=-72.0*func(x+3.0*h)
    f8=7.0*func(x+4.0*h)
    dfunc=(f1+f2+f3+f4+f5+f6+f7+f8)/(240.0*h*h*h);
  end if
  END FUNCTION dfunc

  FUNCTION halley(x)
  COMPLEX(KIND = 8) halley,x,fx,dfx,d2fx
  REAL(KIND=8) eps
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    x=x-2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx)
    IF (abs(fx)<eps) EXIT
  END DO
  halley=x
  END FUNCTION halley

  FUNCTION abbasbandy(x)
  COMPLEX(KIND = 8) abbasbandy,x,fx,dfx,d2fx,d3fx
  REAL(KIND=8) eps
  INTEGER nmax,i
  eps=1.0e-8;
  nmax=100;
  x=a
  DO i=1,nmax
    fx=func(x)
    dfx=dfunc(x,1)
    d2fx=dfunc(x,2)
    d3fx=dfunc(x,3)
    x=x-(fx/dfx+fx*fx*d2fx/(2.0*dfx*dfx*dfx)+fx*fx*fx*d3fx/(6.0*dfx*dfx*dfx*dfx));
    IF (abs(fx)<eps) EXIT
  END DO
  abbasbandy=x
  END FUNCTION abbasbandy
 END PROGRAM halley2
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" halley2
```

```
E:\okul\SCO1>halley2
 halley( (  1.0000000000000000    ,  1.0000000000000000    ) ) =  ( -3.6628389551222020E-026,  1.4142135623730951    )
 abbasbandy( (  1.0000000000000000    ,  1.0000000000000000    ) ) =  (  3.2208545212530678E-033,  1.4142135623730951    )

> Terminated with exit code 0.
```

Halley Method in Python language: Not that a graphic input window is used to input variables in this code.

Program Haley's method complex Python code

```
 #halley.py
from math import *
from graphics import *
from f_x import *;

class f1(f_x):
  def func(self,x):
      y=x*x+2;
      return y;

def halley(f,x):
   # halley root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
      fx= f.func(x);
      dfx=f.dfunc(x,1);
      d2fx=f.dfunc(x,2);
      x=x-2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx)
      print("fx",fx,"dfx",dfx,"x=",x)
      if abs(fx)<tolerance: return x
  return x

def read(s1,s2):
    # reads x1 and x2 values as float then
    # converts to a float array and returned
    s="read data "+s1+" and "+s2+" as float push mouse to exit";
    win = GraphWin(s, 450, 100)
    win.setCoords(0.0, 0.0, 3.0, 4.0)
    # Draw the interface
    Text(Point(1,2), s1).draw(win)
    Text(Point(1,1),s2).draw(win)
    i1 = Entry(Point(2,2), 20)
    i1.setText("")
    i1.draw(win);
    i2 = Entry(Point(2,1), 20)
    i2.setText("")
    i2.draw(win);
    # wait for a mouse click
    win.getMouse()
    # convert input
    x1 = float(i1.getText());
    x2 = float(i2.getText());
    x=[x1,x2];
    return x;
def main():
    x1=read("x1:","x2:");
    x=complex(x1[0],x1[1]);
    f=f1();
    r=halley(f,x);
    print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
main();
```

```
---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" halley.py
fx (2+2j) dfx (2.0000000000002363+1.9999999999999245j) x= (-0.199999999825631+1.3999999997473398j)
fx (0.08000000063770107-0.5599999994107027j) dfx (-0.39999999965142174+2.7999999994947347j) x=
(0.0010152284274246581+1.4142131979720642j)
fx (2.0613703868299638e-06+0.002871498882040751j) dfx (0.0020304568549186653+2.8284263959441294j) x= (-
1.3079811626154347e-10+1.414213562373095j)
fx (4.440892098500626e-16-3.699529398998553e-10j) dfx (-2.615318015063481e-10+2.8284271247461894j) x=
1.4142135623730951j
fx (-4.440892098500626e-16+0j) dfx 2.8284271247461903j x= (4.4398356754174374e-46+1.414213562373095j)
 ROOT  :  (4.4398356754174374e-46+1.414213562373095j)
```

```
FUNCTION VALUE :  (4.440892098500626e-16+1.25577516537665e-45j)

> Terminated with exit code 0.
```

Hasan[50] proposed several higher order methods that is relatively more economical to use (relatively less function calculations) Some of them are listed here.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)} \qquad (2.11.11)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)} \qquad (211.12)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'\left(x_n - \alpha\frac{f(x_n)}{f'(x_n)}\right) + f'\left(x_n - \beta\frac{f(x_n)}{f'(x_n)}\right)} \quad where\ \alpha + \beta = 1 \qquad (2.11.13)$$

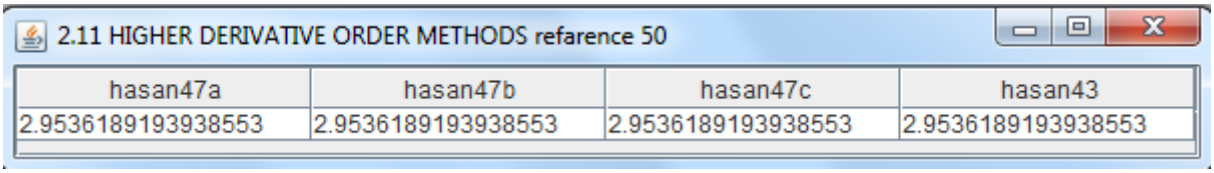
$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + \frac{1}{r}f'(x_n)log\left(\frac{f'(x_n)f'(x_n) - f(x_n)f''(x_n)}{f'(x_n)f'(x_n)}\right)} \qquad (2.11.14)$$

Program Higher order methods, proposed by Hasan[50] java code

```
//  High order methods proposed by Hasan
// 50. Hasan, Mohammed A., On the Derivation of Higher Order Root-finding Methods,
//  Proceedings of the 2007 American Control Conference, ThA07.3,
//  New York City, USA, July 11-13 2007
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= function to be defined ================

class fb extends f_x
{
  public double func (double x)
  { return x*x*x+3.6*x-36.4;}
}
//============================================================
public class NA5B2
{
public static double hasan47a(f_x f,double x)
{ // third order
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx;
double x1=0,dfx1=0;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   x1=x-fx/(2.0*dfx);
   dfx1=f.dfunc(x1);
   x-=fx/dfx1;
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in hasan47a Method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double hasan47b(f_x f,double x)
{ //third order
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx;
double x1=0,dfx1=0;
```

```
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   x1=x-fx/dfx;
   dfx1=f.dfunc(x1);
   x-=2.0*fx/(dfx+dfx1);
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in hasan47b Method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double hasan47c(f_x f,double x,double alfa)
{ //third order
int nmax=500;
double beta=1-alfa;
double tolerance=1.0e-8;
double fx,dfx;
double x1=0,dfx1=0;
double x2=0,dfx2=0;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   x1=x-alfa*fx/dfx;
   dfx1=f.dfunc(x1);
   x2=x-beta*fx/dfx;
   dfx2=f.dfunc(x2);
   x-=2.0*fx/(dfx1+dfx2);
  if(Math.abs(fx)<tolerance) { return x;}
}

JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in hasan47c Method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

public static double hasan43(f_x f,double x)
{ //fourth order
int r=3;
int nmax=500;
double tolerance=1.0e-8;
double fx,dfx,d2fx;
for(int i=0;i<nmax;i++)
{  fx=f.func(x);
   dfx=f.dfunc(x);
   d2fx=f.dfunc(x,2);
   x-=fx/(dfx+1.0/r*dfx*Math.log((dfx*dfx-fx*d2fx)/(dfx*dfx)));
  if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded in hasan43 Method\n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
    public static void main (String args[])
    {
            fb f=new fb();
            dfb df=new dfb();
    double x0;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" first guess for x : "));
    double r[]=new double[4];
    String s[]={"hasan47a","hasan47b","hasan47c","hasan43"};
    r[0]=hasan47a(f,x0);
```

```
    r[1]= hasan47b(f,x0);
    r[2]= hasan47c(f,x0,0.5);
    r[3]= hasan43(f,x0);
    Text.print(r,s,"2.11 HIGHER DERIVATIVE ORDER METHODS referance 50");
    }
}
```



Program Higher order methods, proposed by Hasan[50] python code

```
 #halley.py
from math import *
from graphics import *
from f_x import *;

class f1(f_x):
  def func(self,x):
     y=x*x-2;
     return y;
def hasan(f,x):
      # finding real roots by using hasan method fourth order
      nmax=500;
      r=3;
      tolerance=1.0e-15;
      for iter in range(0,nmax):
         fx=f.func(x);
         dfx=f.dfunc(x,1);
         d2fx=f.dfunc(x,2);
         x=x-fx/(dfx+1.0/r*dfx*log((dfx*dfx-fx*d2fx)/(dfx*dfx)));
         if abs(fx)<tolerance: return x;
      return x;
f=f1();
x=float(input("enter initial estimation : "))
r=hasan(f,x);
print(" hasan1 method ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

```
> "D:\co\anaconda\pythonw.exe" hasan2.py
enter initial estimation : 1
 hasan1 method ROOT  :  1.4142135623730951
FUNCTION VALUE :  4.440892098500626e-16

> Terminated with exit code 0.
```

A complex version of Hasan formula is also given here:

Program 2.11-13 Complex Hasan method[50] (java version) Hasan1.java

```
// compleks Hasan's method to find roots of the polynomials
// xn+1=xn-f(xn)/(f'(xn)+1/3*f'(xn)*log((f'(xn)*f'(xn)*-f(xn)*f''(xn))/(f'(xn)*f'(xn))
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class Hasan1
{
public static c Hasan(icf_cx f,c x)
{
double eps=1.0e-15;
int m=100;
int k=0;
c f1,df,d2f;
c b1,b2,b3,b4,b5,b6,b7;
```

```
double r=3.0;
do{
  f1=f.func(x);
  df=f.dfunc(x);
  d2f=f.dfunc(x);
  b1=c.p(c.p(df,"*",df),"-",c.p(f1,"*",d2f));
  b2=c.p(df,"*",df);
  b3=c.p(b1,"/",b2);
  b4=c.p(df,"*",c.log(b3));
  b5=c.p(b4,"/",r);
  b6=c.p(df,"+",b5);
  b7=c.p(f1,"/",b6);
  x=c.p(x,"-",b7);
  if(c.abs(f1)<eps) return x;
  k++;
}while(c.abs(f1)>eps && k<m);
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static String toString(c x[])
{ int n=x.length;
  String s="";
  for(int i=n-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
  return s;
}
public static c Hasan(cf_cx f,c x)
{
double eps=1.0e-15;
int m=100;
int k=0;
c f1,df,d2f;
c b1,b2,b3,b4,b5,b6,b7;
double r=3.0;
do{
  f1=f.func(x);
  df=f.dfunc(x);
  d2f=f.dfunc(x);
  b1=c.p(c.p(df,"*",df),"-",c.p(f1,"*",d2f));
  b2=c.p(df,"*",df);
  b3=c.p(b1,"/",b2);
  b4=c.p(df,"*",c.log(b3));
  b5=c.p(b4,"/",r);
  b6=c.p(df,"+",b5);
  b7=c.p(f1,"/",b6);
  x=c.p(x,"-",b7);
  if(c.abs(f1)<eps) return x;
  k++;
}while(c.abs(f1)>eps && k<m);
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
    public static double[] input(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

    public static void main (String args[])
    { icf_cx f=(c x)->c.p(c.p(x,"*",x),"+",2.0);
      c x1=new c(1.0,1.0);
      c y=Hasan(f,x1);
      System.out.println("root = "+y);
}}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Hasan1
root = (   0.0000000000 -    1.4142135624i
> Terminated with exit code 0.
```

In order to change python code to complex version, changing input variable to complex and adding the complex math class (log function should be defined as complex) is enough

Program Higher order methods, proposed by Hasan[50] complex python code hasan2.py

```
#halley.py
from math import *
from cmath import *
from graphics import *
from f_x import *;

class f1(f_x):
  def func(self,x):
      y=x*x+2;
      return y;
def hasan(f,x):
       # finding real roots by using hasan method fourth order
       nmax=500;
       r=3;
       tolerance=1.0e-15;
       for iter in range(0,nmax):
           fx=f.func(x);
           dfx=f.dfunc(x,1);
           d2fx=f.dfunc(x,2);
           x=x-fx/(dfx+1.0/r*dfx*log((dfx*dfx-fx*d2fx)/(dfx*dfx)));
           if abs(fx)<tolerance: return x;
       return x;
f=f1();
x=complex(input("enter initial estimation : "))
r=hasan(f,x);
print(" hasan1 method ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

```
---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" hasan2.py
enter initial estimation : 1+1j
 hasan1 method ROOT  :  1.414213562373095j
FUNCTION VALUE :  (4.440892098500626e-16+0j)
> Terminated with exit code 0.
```

Program Higher order methods, proposed by Hasan[50] julia code

```
 function newton(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x-=fx/dfx
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan47a(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x1=x-fx/(2.0*dfx)
```

```
  dfx1=df(f,x1)
  x-=fx/dfx1
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan47b(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x1=x-fx/dfx
  dfx1=df(f,x1)
  x-=2.0*fx/(dfx+dfx1)
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan47c(f,x,alfa)
nmax=100
tolerance=1.0e-10
beta=1.0-alfa
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x1=x-alfa*fx/dfx
  dfx1=df(f,x1)
  x2=x-beta*fx/dfx
  dfx2=d2f(f,x2)
  x-=2.0*fx/(dfx1+dfx2)
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan43(f,x)
nmax=100
tolerance=1.0e-10
r=3.0 #fourth order
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  yx=(dfx*dfx-fx*d2fx)/(dfx*dfx)
  dyx=dfx+1.0/r*dfx*log(yx)
  x-=fx/dyx
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function
```

```
#first derivative
function df(f,x)
 dx=0.001
 c=[1.0/280.0 -4.0/105.0  1.0/5.0  -4.0/5.0  0  4.0/5.0  -1.0/5.0  4.0/105.0 -1.0/280.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end
#second derivative
function d2f(f,x)
 dx=0.001
 c=[-1.0/560.0  8.0/315.0 -1.0/5.0 8.0/5.0 -205.0/72.0 8.0/5.0 -1.0/5.0 8.0/315.0 -1.0/560.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx)
end
#third derivative
function d3f(f,x)
 dx=0.001
 c=[-7.0/240.0  3.0/10.0  -169.0/120.0  61.0/30.0 0.0  -61.0/30.0  169.0/120.0  -3.0/10.0
      7.0/240.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx*dx)
end

f(x)=x*x*x+3.6*x-36.4
x=1.0
s=["Newton" "Hasan47a" "Hasan47b" "Hasan47c" "Hasan43" ]
r=zeros(Float64,5)
r[1]=newton(f,x)
r[2]=hasan47a(f,x)
r[3]=hasan47b(f,x)
r[4]=hasan47c(f,x,0.5)
r[5]=hasan43(f,x)
for i=1:5
  s1=s[i]
  r1=r[i]
  f1=f(r1)
  println("$s1 x = $r1 y = $f1")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" hasan.jl
Newton x = 2.953618919393856 y = 7.105427357601002e-15
Hasan47a x = 2.953618919393856 y = 7.105427357601002e-15
Hasan47b x = 2.9536189193938553 y = -7.105427357601002e-15
Hasan47c x = 2.9536189193934064 y = -1.3372414287005085e-11
Hasan43 x = 2.9536189193938553 y = -7.105427357601002e-15

> Terminated with exit code 0.
```

A complex version of Hasan formula is also given here. In order to change julia code to complex version, changing input variable to complex is enough

Program Complex Hasan method[50] (julia version) hasan1.jl

```
 function newton(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x-=fx/dfx
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan47a(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x1=x-fx/(2.0*dfx)
  dfx1=df(f,x1)
  x-=fx/dfx1
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan47b(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x1=x-fx/dfx
  dfx1=df(f,x1)
  x-=2.0*fx/(dfx+dfx1)
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan47c(f,x,alfa)
nmax=100
tolerance=1.0e-10
beta=1.0-alfa
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
  x1=x-alfa*fx/dfx
  dfx1=df(f,x1)
  x2=x-beta*fx/dfx
  dfx2=d2f(f,x2)
  x-=2.0*fx/(dfx1+dfx2)
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

function hasan43(f,x)
nmax=100
tolerance=1.0e-10
r=3.0 #fourth order
for i=0:nmax
  fx=f(x)
  dfx=df(f,x)
```

```
  d2fx=d2f(f,x)
  yx=(dfx*dfx-fx*d2fx)/(dfx*dfx)
  dyx=dfx+1.0/r*dfx*log(yx)
  x-=fx/dyx
  if abs(fx)<tolerance
    return x
  end #f
end #for
return x
end #function

#first derivative
function df(f,x)
 dx=0.001
 c=[1.0/280.0 -4.0/105.0  1.0/5.0  -4.0/5.0  0  4.0/5.0  -1.0/5.0  4.0/105.0  -1.0/280.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end
#second derivative
function d2f(f,x)
 dx=0.001
 c=[-1.0/560.0  8.0/315.0 -1.0/5.0 8.0/5.0 -205.0/72.0 8.0/5.0 -1.0/5.0 8.0/315.0 -1.0/560.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx)
end
#third derivative
function d3f(f,x)
 dx=0.001
 c=[-7.0/240.0  3.0/10.0  -169.0/120.0  61.0/30.0  0.0  -61.0/30.0 169.0/120.0  -3.0/10.0  7.0/240.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx*dx)
end

f(x)=x*x*x+3.6*x-36.4
x=1.0+1im
s=["Newton" "Hasan47a" "Hasan47b" "Hasan47c" "Hasan43" ]
r=zeros(ComplexF64,5)
r[1]=newton(f,x)
r[2]=hasan47a(f,x)
r[3]=hasan47b(f,x)
r[4]=hasan47c(f,x,0.5)
r[5]=hasan43(f,x)
for i=1:5
  s1=s[i]
  r1=r[i]
  f1=f(r1)
  println("$s1 x = $r1 y = $f1")
end
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" hasan1.jl
Newton x = 2.9536189193938553 - 8.519697776386927e-29im y = -7.105427357601002e-15 - 2.53644984590039e-27im
Hasan47a x = 2.953618919393856 - 4.353526120688459e-29im y = 7.105427357601002e-15 - 1.296114128431739e-27im
Hasan47b x = -1.4768094596969277 - 3.1847917578314275im y = -7.105427357601002e-15 - 1.7763568394002505e-15im
Hasan47c x = 2.9536189193942644 + 5.003957697275287e-13im y = 1.2164491636212915e-11 + 1.48975797772156e-11im
Hasan43 x = 2.9536189193938553 + 1.0349855076499675e-27im y = -7.105427357601002e-15 + 3.081316849834548e-26im

> Terminated with exit code 0.
```

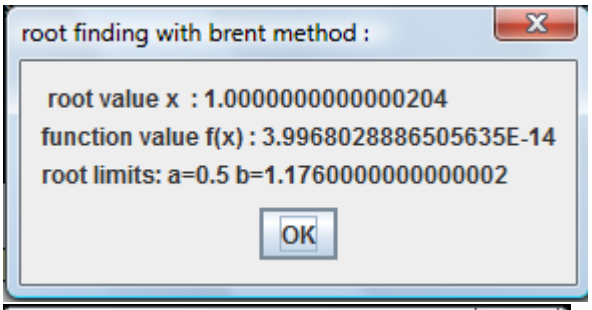
FINDING THE ROOTS LIMITS

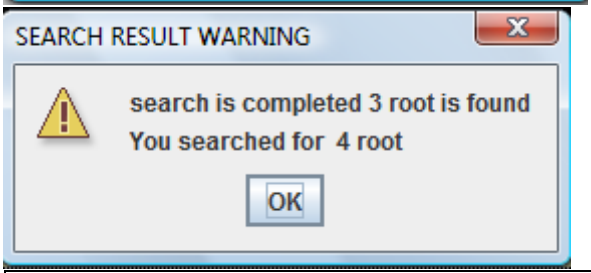
In a one variable function, if a root to be find and if only one root existed in the given region boundaries of the function should carry inverse sign. To further process this, if given region is divided up to smaller region and function values at the limits are cheched, the root area can be narrowed down. region can be enlarged to the side that it is estimated to have a root to ensure that at least one root is existed in a given region.

Root finding NA13B

```
//
// root findings in a given region
// Dr. M. Turhan Çoban

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fx extends f_x
{
public double func(double x)
{return x*x*x-6.0*x*x+11.0*x-6.0;}
}

public class NA13B
{
public static double[] enlarge(f_x f,double x1)
{return enlarge(f,x1,(x1+0.1));}

public static double[] enlarge(f_x f,double x1,double x2)
{
	//kök olan bir bölge bulana kadar verilen siniri genişletir
	int NTRY=50;
	double a[]=new double[2];
	double FACTOR=1.6;
	int j;
	double f1,f2;
	if (x1 == x2) System.out.println("Bad initial range in ZBRAC");
	f1=f.func(x1);
	f2=f.func(x2);
	for (j=1;j<=NTRY;j++) {
		if (f1*f2 < 0.0) {a[0]=x1;a[1]=x2;return a;}
		if (Math.abs(f1) < Math.abs(f2))
			f1=f.func(x1 += FACTOR*(x1-x2));
		else
			f2=f.func(x2 += FACTOR*(x2-x1));
	}
	{a[0]=x1;a[1]=x2;return a;}
}

public static double[][] rootlimits(f_x f,double x1,double x2,int n,int nbb)
{
	// find root limits
	// n : number of subdivisions
	// x1,x2 : limit values of the root
	// nbb = number of roots in the serched region
	//
	int nb;
	int i;
	double x,fp,fc,dx;
	double xb[][]=new double[2][nbb];
	nb=0;
	dx=(x2-x1)/n;
	x=x1;
	fp=f.func(x1);
	for (i=1;i<=n;i++)
	{
		x+=dx;
```

```
                        fc=f.func(x);
                        // eğer kök olan bölge bulunduysa.....
                        if (fc*fp < 0.0 || fp==0) {
                                xb[0][nb]=x-dx;
                                xb[1][nb]=x;
                                nb++;
                        }
                        fp=fc;
                        if (nbb == nb) return xb;
                }
if( nb == 0)
                JOptionPane.showMessageDialog(null,"search is completed no root is found"
   ,"SEARCH RESULT WARNING",JOptionPane.WARNING_MESSAGE);
else if(nb<nbb)
  {             JOptionPane.showMessageDialog(null,"search is completed "+nb+" root is found \n"+
    "You searched for  "+nbb+" root","SEARCH RESULT WARNING",JOptionPane.WARNING_MESSAGE);
   double xc[][]=new double[2][nb];
   for (i=0;i<nb;i++) {xc[0][i]=xb[0][i];xc[1][i]=xb[1][i];}
   return xc;
  }
return xb;
}
    public static void main (String args[])
    {
    double a[],x0,r;
    double dx=0.1;
    fx f=new fx();
    a=enlarge(f,Double.parseDouble(JOptionPane.showInputDialog(" first guess x0 : ")));
    System.out.println(a[0]+" "+a[1]);
    r= NA13.brent(f,a[0],a[1]);
    JOptionPane.showMessageDialog(null," root value x  : "+r+"\nfunction value f(x) : "+f.func(r)+"\nroot limits:
a="+a[0]+" b="+a[1],
    "root finding with brent method : ",JOptionPane.PLAIN_MESSAGE);
    double b[][]=rootlimits(f,0.0,10.0,10,4);
    System.out.println(Matrix.toString(b));
    b=rootlimits(f,b[0][0],b[1][0],10,4);
    System.out.println(Matrix.toString(b));
    System.exit(0);
    }
}
```






```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA13B
```

```
0.5 1.1760000000000002
      1.000000000000000      2.000000000000000      3.000000000000000
      2.000000000000000      3.000000000000000      4.000000000000000

      1.000000000000000
      1.100000000000000

> Terminated with exit code 0.
```

RIDDER METHOD

False position (Ragula Falsi) method was previously investigated. Ridder method is a modified form False position method. If a root existed between x1 and x2 Ridder method first evaluate the midpoint

$$x_3 = \frac{x_1+x_2}{2} \qquad (2.13.1)$$

and than solve a special conversion conversion factor e^Q. This convertion linearize the equation. It is basically find the root of the quadratic function

$$f(x_1) - 2\,f(x_3)e^Q + f(x_1) - 2\,f(x_2)e^Q = 0 \qquad (2.13.2)$$

The root of this function can be calculated as
of the quadratic function

$$e^Q = \frac{f(x_3)+sign[f(x_2)]\sqrt{f(x_3)^2-f(x_1)f(x_2)}}{f(x_2)} \qquad (2.13.3)$$

Function sign(x) will be given the sign of x. If x is smaller than 0 it will be -1, if x is bigger than 0 it will be +1. Now the false position method is applied, not to the values f(x1), f(x3), f(x2), but to the values $f(x_1)$, $f(x_3)e^Q$, $f(x_2)e^{2Q}$, yielding a new guess for the root, x_4. The overall formula is:

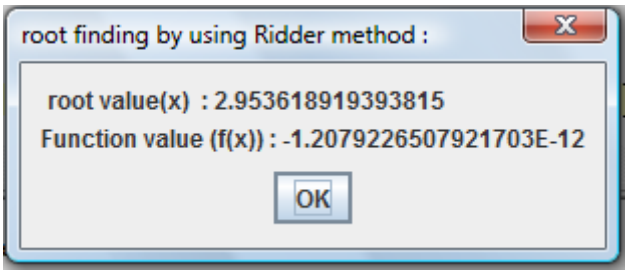
$$x_4 = x_3 + (x_3 - x_1)\frac{f(x_3)+sign[f(x_1)-f(x_2)]f(x_3)}{\sqrt{f(x_3)^2-f(x_1)f(x_2)}} \qquad (2.13.4)$$

Because the equation is quadratic, it supplies a quadratic approximation. In addition the root x_4 should be in between x_1 and x_2.

Program Root finding NA13C Ridder Method

```
// Ridder method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//=============Function to be defined ================
class fb extends f_x
{
  public double func (double x)
  { return x*x-2;}
}
public class NA13C
{
// Ridder method for solving roots of non-linear equations

public static double SIGN(double a,double b)
{
return Math.abs(a)*b/Math.abs(b);
}
public static double ridder(f_x f, double x1, double x2)
{double xacc=1.0e-9;
 return ridder(f, x1,x2,xacc);
}
public static double ridder(f_x f, double x1, double x2, double xacc)
{
```

```
        int NR;
        int MAXITER=200;
        double ABC=-1.0e30;
        int j;
        double answer,f_high,f_low,f_mid,fnew,s,x_high,x_low,x_mid,xnew;
        f_low=f.func(x1);
        f_high=f.func(x2);
        if ((f_low > 0.0 && f_high < 0.0) || (f_low < 0.0 && f_high > 0.0)) {
                x_low=x1;
                x_high=x2;
                answer=ABC;
                for (j=0;j<MAXITER;j++) {
                        x_mid=0.5*(x_low+x_high);
                        f_mid=f.func(x_mid);
                        s=Math.sqrt(f_mid*f_mid-f_low*f_high);
                        if (s == 0.0) return answer;
                        xnew=x_mid+(x_mid-x_low)*((f_low >= f_high ? 1.0 : -1.0)*f_mid/s);
                        if (Math.abs(xnew-answer) <= xacc) return answer;
                        answer=xnew;
                        fnew=f.func(answer);
                        if (fnew == 0.0) return answer;
                        if (SIGN(f_mid,fnew) != f_mid) {
                                x_low=x_mid;
                                f_low=f_mid;
                                x_high=answer;
                                f_high=fnew;
                        } else if (SIGN(f_low,fnew) != f_low) {
                                x_high=answer;
                                f_high=fnew;
                        } else if (SIGN(f_high,fnew) != f_high) {
                                x_low=answer;
                                f_low=fnew;
                        } else System.out.println("we should not reach to this point");
                        if (Math.abs(x_high-x_low) <= xacc) return answer;
                }
                JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
    " result may not be valid","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
        }
        else {
                if (f_low == 0.0) return x1;
                if (f_high == 0.0) return x2;
                JOptionPane.showMessageDialog(null,"Warning : No roots existed between the given limits ","NO ROOTS
WARNING",JOptionPane.WARNING_MESSAGE);
        }
        return 0.0;
}
    public static void main (String args[])
    {
             fb f=new fb();
    double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" Enter lower limit of the function a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" Enter upper limit of the function b : "));
    double r;
    r= ridder(f,a,b);
    JOptionPane.showMessageDialog(null," root value(x)  : "+r+"\nFunction value (f(x)) : "+f.func(r),
    "root finding by using Ridder method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Program Root finding Ridder Method C++ version

```
//Ridder method C++
#include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
using namespace std;

class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-h)+672.0*func(x+h)-168.0*func(x+2.0*h)
+32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
};

class f1:public f_x
{ public: double func(double x)
  { double ff=x*x-2.0;
                return ff;
  }
};
double SIGN(double a,double b)
{
return fabs(a)*b/fabs(b);
}
double ridder(f_x &f, double x1, double x2, double xacc)
{   int NR;
            int MAXITER=200;
            double ABC=-1.0e30;
            int j;
            double answer,f_high,f_low,f_mid,fnew,s,x_high,x_low,x_mid,xnew;
            f_low=f.func(x1);
            f_high=f.func(x2);
            if ((f_low > 0.0 && f_high < 0.0) || (f_low < 0.0 && f_high > 0.0)) {
                        x_low=x1;
                        x_high=x2;
                        answer=ABC;
                        for (j=0;j<MAXITER;j++) {
                                    x_mid=0.5*(x_low+x_high);
                                    f_mid=f.func(x_mid);
                                    s=sqrt(f_mid*f_mid-f_low*f_high);
                                    if (s == 0.0) return answer;
                                    xnew=x_mid+(x_mid-x_low)*((f_low >= f_high ? 1.0 : -1.0)*f_mid/s);
                                    if (fabs(xnew-answer) <= xacc) return answer;
                                    answer=xnew;
                                    fnew=f.func(answer);
                                    if (fnew == 0.0) return answer;
                                    if (SIGN(f_mid,fnew) != f_mid) {
                                                x_low=x_mid;
                                                f_low=f_mid;
                                                x_high=answer;
                                                f_high=fnew;
                                    } else if (SIGN(f_low,fnew) != f_low) {
                                                x_high=answer;
                                                f_high=fnew;
                                    } else if (SIGN(f_high,fnew) != f_high) {
                                                x_low=answer;
                                                f_low=fnew;
                                    } else cout<<"we should not reach to this point";
                                    if (fabs(x_high-x_low) <= xacc) return answer;
                        }
            cout<<"Warning : Maximum number of iteration is exceeded \n";
            }
            else {
                        if (f_low == 0.0) return x1;
                        if (f_high == 0.0) return x2;
                        cout<<"Warning : No roots existed between the given limits ";
            }
            return 0.0;
}
```

```
int main ()
{
double x1,x2,x3;
cout << "root estimation x1 : ";
cin >>x1;
cout << "root estimation x2 : ";
cin >>x2;
double r;
f1 f;
cout<<setprecision(20);
r= ridder(f,x1,x2,1.0e-9);
cout<<" root value  : "<<r<<"\nFunction value : "<<f.func(r);
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" ridder

D:\okul\SCO1>ridder
root estimation x1 : 1
root estimation x2 : 3
 root value  : 1.4142135621695027
Function value : -5.7584620860758051e-010
> Terminated with exit code 0.
```

Program Root finding Ridder Method python version

```
from math import *

def SIGN(a,b):
    return abs(a)*b/abs(b);

def ridder(f,x1,x2):
    xacc=1.0e-9;
    MAXITER=200;ABC=-1.0e30;
    f_low=f(x1);
    f_high=f(x2);
    b1=f_low > 0.0 and f_high < 0.0;
    b2=f_low < 0.0 and f_high > 0.0;
    b3=b1 or b2;
    if b3 :
      x_low=x1;
      x_high=x2;
      answer=ABC;
      for j in range(0,MAXITER):
         x_mid=0.5*(x_low+x_high);
         f_mid=f(x_mid);
         s=sqrt(f_mid*f_mid-f_low*f_high);
         if s == 0.0: return answer;
         if f_low>=f_high: xx=1;
         else: xx=-1;
         xnew=x_mid+(x_mid-x_low)*xx*f_mid/s;
         if abs(xnew-answer) <= xacc: return answer;
         answer=xnew;
         fnew=f(answer);
         if fnew == 0.0: return answer;
         if SIGN(f_mid,fnew) != f_mid:x_low=x_mid;f_low=f_mid;x_high=answer;f_high=fnew;
         elif SIGN(f_low,fnew) != f_low:x_high=answer;f_high=fnew;
         elif SIGN(f_high,fnew) != f_high:x_low=answer;f_low=fnew;
         else: print("we should not reach to this point");
      if abs(x_high-x_low) <= xacc: return answer;
      print("Warning : Maximum number of iteration is exceeded \n");
    else:
       if f_low == 0.0: return x1;
       if f_high == 0.0: return x2;
       print("Warning : No roots existed between the given limits ");
    return 0.0;

f1 = lambda x: x*x-2
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=ridder(f1,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

```
runfile('D:/okul/SCO1/ridder2.py', wdir='D:/okul/SCO1')
```

```
Reloaded modules: Mathd

enter lower limit : 1

enter upper limit : 3
 ROOT  :  1.4142135621695027
FUNCTION VALUE :  -5.758462595650826e-10
```

Program : Root finding Ridder Method (Julia version)

```
 unction SIGN(a,b)
  return abs(a)*b/abs(b)
end

function ridder(f,x1,x2)
   xacc=1.0e-9
   NR=0
            MAXITER=200
            ABC=-1.0e30
            j=0
            answer=0.0
            f_high=0.0
            f_low=0.0
            f_mid=0.0
            fnew=0.0
            s=0.0
            x_high=0.0
            x_low=0.0
            x_mid=0.0
            xnew=0.0
            f_low=f(x1)
            f_high=f(x2)
            if (f_low > 0.0 && f_high < 0.0) || (f_low < 0.0 && f_high > 0.0)
                        x_low=x1
                        x_high=x2
                        answer=ABC
                        for j=1:MAXITER
                                    x_mid=0.5*(x_low+x_high)
                                    f_mid=f(x_mid)
                                    s=sqrt(f_mid*f_mid-f_low*f_high)
                                    if s == 0.0
                                      return answer
                                    end
                                    xnew=x_mid+(x_mid-x_low)*((f_low >= f_high ? 1.0 : -1.0)*f_mid/s)
                                    if abs(xnew-answer) <= xacc
                                       return answer
                                    end
                                    answer=xnew
                                    fnew=f(answer)
                                    if fnew == 0.0
                                      return answer
                                    end
                                    if SIGN(f_mid,fnew) != f_mid
                                                x_low=x_mid
                                                f_low=f_mid
                                                x_high=answer
                                                f_high=fnew
                                    elseif SIGN(f_low,fnew) != f_low
                                                x_high=answer
                                                f_high=fnew
                                    elseif SIGN(f_high,fnew) != f_high
                                                x_low=answer
                                                f_low=fnew
                                    else
                                        print("we should not reach to this point")
                                    end # if SIGN
                                    if (abs(x_high-x_low) <= xacc)
                                      return answer
                                    end
```

```
                        end #for MAXITER
    print("Maximum number of iteration is exceeded \nresult might not be valid")
            else
                        if  f_low == 0.0
                          return x1
                        elseif f_high == 0.0
                          return x2
                        end
  end #if(f_low >
  print("no roots existed between given limits")
            return 0.0
end # function

f(x)=x*x-2.0
a=0.0
b=2.0
x=ridder(f,a,b)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" ridder.jl
x = 1.4142135624398362
> Terminated with exit code 0.
```

ROOTS OF THE SECOND AND THIRD DEGREE POLYNOMIALS

The roots of quadratic polynomials are well known.

$f(x) = ax^2 + bx + c = 0$ (2.14.1)

$\Delta = b^2 - 4ac$ (2.14.2)

$x_0 = \frac{-b}{2a} + \frac{\sqrt{\Delta}}{2a}$ (2.14.3)

$x_1 = \frac{-b}{2a} - \frac{\sqrt{\Delta}}{2a}$ (2.14.4)

Of course if $\Delta = b^2 - 4ac$ is negative, roots will be complex numbers.

Program Root of quadratic polynomial (analytical formula)

```
public class square_roots
{
public static double[][] square_roots(double d[])
{
  double x[][]=new double[2][2];
  double a=d[2];
  double b=d[1];
  double c=d[0];
  double delta=b*b-4*a*c;
  if(delta>=0)
  {
  x[0][0]=(-b+Math.sqrt(delta))/(2*a);
  x[1][0]=(-b-Math.sqrt(delta))/(2*a);
  }
  else
  {x[0][0]=-b/(2*a);
   x[1][0]=-b/(2*a);
   x[0][1]=Math.sqrt(-delta)/(2*a);
   x[1][1]=Math.sqrt(-delta)/(2*a);
  }
  return x;
}
public static void main(String arg[])
{
double d[]={1,-2,1};
double x[][]=square_roots(d);
String s="";
for(int i=0;i<x.length;i++)
{ s+="x["+i+"] = "+x[i][0]+" + "+x[i][1]+"*i\n";}
System.out.println(s);
}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" square_roots
x[0] = 1.0 + 0.0*i
x[1] = 1.0 + 0.0*i


> Terminated with exit code 0.
```

Program Root of quadratic polynomial (analytical formula) python version

```
from math import *

def square_roots(d):
   x= [0 for i in range(2)];
   a=d[2];
   b=d[1];
   c=d[0];
   delta=b*b-4*a*c;
   x[0]=(-b+sqrt(delta))/(2*a);
   x[1]=(-b-sqrt(delta))/(2*a);
   return x;

a=[1.0,-2.0,1.0]  # y(x)=x^2-2*x++1
print("Roots of the polynomial : ",square_roots(a))
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" square_roots.py
Roots of the polynomial :  [1.0, 1.0]
```

```
program quadratic1
IMPLICIT NONE
REAL(KIND=8) a,b,c
COMPLEX(KIND=8), DIMENSION(0:1) :: x
print*,'ax^2+bx+c=0'
print*,'a'
read*, a
print*,'b'
read*, b
print*,'c'
read*, c
call quadratic(a,b,c,x)
print *,'x(0) = ',x(0)
print *,'x(1) = ',x(1)

CONTAINS
  SUBROUTINE quadratic(a,b,c,x)
  REAL(KIND=8) a,b,c,determinant
  COMPLEX(KIND=8), DIMENSION(0:1) :: x
  determinant=b*b-4.0*a*c
  x(0)=(-b+sqrt(determinant))/(2.0*a)
  x(1)=(-b-sqrt(determinant))/(2.0*a)
  END SUBROUTINE quadratic
end program quadratic1
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" quadratic1
E:\okul\SCO1>quadratic1
 ax^2+bx+c=0
 a
1
 b
-2
 c
1
 x(0) =  (  1.0000000000000000     ,  0.0000000000000000     )
 x(1) =  (  1.0000000000000000     ,  0.0000000000000000     )
> Terminated with exit code 0.
```

Program Root of quadratic polynomial (analytical formula) Julia version

```
function square_roots(d)
   x=zeros(ComplexF64,2)
   a=d[3]
```

```
    b=d[2];
    c=d[1];
    delta=b*b-4*a*c
    delta1=Complex(delta)^0.5
    x[1]=(-b+delta1)/(2*a)
    x[2]=(-b-delta1)/(2*a);
    return x
end

d=[1 -2 1]
x=square_roots(d)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" square_roots.jl
x = ComplexF64[1.0 + 0.0im, 1.0 - 0.0im]
> Terminated with exit code 0.
> Terminated with exit code 0.
```

```
d=[2 0 1]
x=square_roots(d)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" square_roots.jl
x = ComplexF64[0.0 + 1.4142135623730951im, 0.0 - 1.4142135623730951im]
> Terminated with exit code 0.
```

Similar to quadratic polynomials, third degree polynomials can also be solved by using analytical formulations. The basic difference is that the roots should be calculated as complex variables. Assuming a third degree polynomial of



Figure 2.141 NiccoloTartaglia

$$f(x) = d_3x^3 + d_2x^2 + d_1x + d_0 = 0 \qquad (2.14.1)$$

$$a = \frac{d_2}{d_3} \quad b = \frac{d_1}{d_3} \quad c = \frac{d_0}{d_3} \qquad (2.14.2)$$

$$f(x) = x^3 + ax^2 + bx + c = 0 \qquad (2.14.3)$$

$Q = \frac{a^2-3b}{9}$ (2.14.4) $\quad y_1 = 2a^3$ (2.14.5) $y_2 = -9ab$ (2.14.6) $y_3 = 27c$ (2.14.7)

$y_4 = y_1 + y_2 + y_3$ (2.14.8)

$R = \frac{y_4}{54} = \frac{2a^3-9ab+27c}{54} \quad \theta = cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right)$ (2.14.9)

If $R^2 < Q^3$ three real roots are existed

$x_0 = \left(-2\sqrt{Q}\right)cos\left(\frac{\theta}{3}\right) - \frac{a}{3}$ (2.14.10)

$x_1 = \left(-2\sqrt{Q}\right)cos\left(\frac{\theta-2\pi}{3}\right) - \frac{a}{3}$ (2.14.11)

$x_2 = \left(-2\sqrt{Q}\right)cos\left(\frac{\theta+2\pi}{3}\right) - \frac{a}{3}$ (2.14.12)

(This equation first appears in Chapter VI of François Viete's treatise "De emendatione," published in 1615!, Original equation created by Niccolo Tartaglia in 1539)

If $R^2 \geq Q^3$

$$A = (R + \sqrt{R^2 - Q^3})^{1/3} \qquad (2.14.13)$$

If A=0 $\Rightarrow$B=0 (2.14.14)

If A< >0 $\Rightarrow B = \frac{Q}{A}$ (2.14.15)

$x_0 = (A+B) - \frac{a}{3}$ (2.14.16) real root

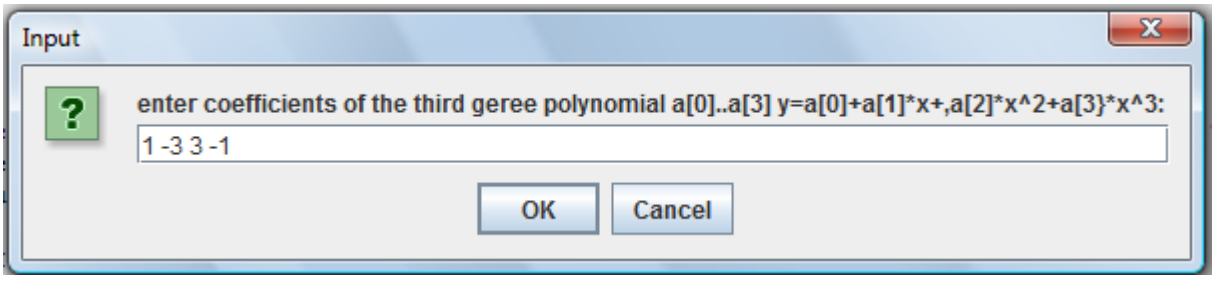
$x_1 = \left[-\frac{(A+B)}{2} - \frac{a}{3}\right] + \left[-\frac{\sqrt{3}(A-B)}{2}\right] i$ (2.14.17) complex root

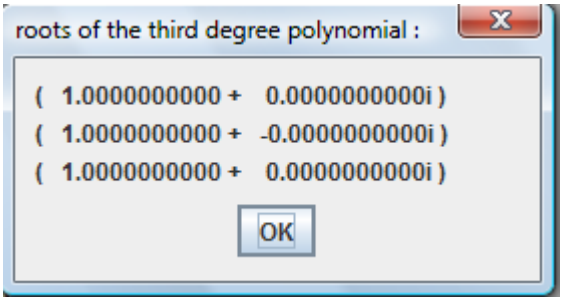
$x_2 = \left[-\frac{(A+B)}{2} - \frac{a}{3}\right] - \left[-\frac{\sqrt{3}(A-B)}{2}\right] i$ (2.14.18) complex root

Program Root of third degree polynomial (analytical formula-java version)

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;
import java.text.*;
import java.util.Locale;
//import Numeric;

class NA13D
{
public static double[][] cubic_roots(double d[])
{
// roots of a degree 3 polynomial
// P(x)=a[3]*x^3+a[2]*x^2+a[1]*x+a[0]=0;
// assuming all a is all real;
// if more than 4 values are entered the remaining ones will be ignored
double x[][]=new double[2][3];
double a,b,c;
a=d[2]/d[3];
b=d[1]/d[3];
c=d[0]/d[3];
double Q,R,theta;
Q=(a*a-3.0*b)/9.0;
double y1,y2,y3,y4;
y1=2.0*a*a*a;
y2=-9*a*b;
y3=27*c;
y4=y1+y2+y3;
R=y4/54.0;
double Q3=Q*Q*Q;
double R2=R*R;
double qq;
double A,B;
if(R2<Q3)
{
qq=-2.0*Math.sqrt(Q);
theta=Math.acos(R/Math.sqrt(Q3));
x[0][0]=qq*Math.cos(theta/3.0)-a/3.0;
x[0][1]=qq*Math.cos((theta-2.0*Math.PI)/3.0)-a/3.0;
x[0][2]=qq*Math.cos((theta+2.0*Math.PI)/3.0)-a/3.0;
}
else
{
A=-Math.pow((R+Math.sqrt(R2-Q3)),(1.0/3.0));
if(A==0) B=0;
else     B=Q/A;
x[0][0]=(A+B)-a/3.0;
x[0][1]=-0.5*(A+B)-a/3;
x[1][1]=Math.sqrt(3.0)/2.0*(A-B);
x[0][2]=-0.5*(A+B)-a/3;
x[1][2]=-Math.sqrt(3.0)/2.0*(A-B);
}
return x;
}
```

```
public static c[] cubic_rootsC(double c[])
{
// roots of a degree 3 polynomial
//return 3 complex roots
double a[][]=new double[2][3];
a=cubic_roots(c);
c e[]=new c[3];
for(int i=0;i<3;i++)
  e[i]=new c(a[0][i],a[1][i]);
return e;
}

    public static double[] enterdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

  public static void main(String args[]) throws IOException
  {
  String s="enter coefficients of the third degree polynomial a[0]..a[3] y=a[0]+a[1]*x+,a[2]*x^2+a[3}*x^3: ";
  double [] x0=enterdata(s);
  c z[]=cubic_rootsC(x0);
  String s1="";
  for(int i=0;i<3;i++)
    {s1+=c.toString(z[i])+"\n";}
  JOptionPane.showMessageDialog(null,s1,
  "roots of the third degree polynomial : ",JOptionPane.PLAIN_MESSAGE);
  }
}
```





Excel version of the program

Program Root of third degree polynomial (analytical formula) excel version

Root of third degree polynomial (analytical formula) Excel version

$f(x)=d_3x^3+d_2x^2+d_1x+d_0=0$ $\quad f(x)=x^3+ax^2+bx+c=0$

Enter poynomial coefficients

d3	d2	d1	d0
1	3	3	-1

	a	b	c
1	3	3	-1

Q	0
y1	54
y2	-81
y3	-27
y4	-54
R	-1
θ	#SAYI/0!

true/false
0

true if R^2<Q^3

x0	x1	x2
0	0	0

true if R^2>=Q^3

A	0	
B	0	

true/false
1

xo real	xo imaginary
-1	0

x1 real	x1 imaginary
-1	0

x2 real	x2 imaginary
-1	0

Python version of the program:

Program Root of third degree polynomial (analytical formula) Python version

```
from math import *

def cubic_roots(d):
  x=[[0 for i in range(3)] for j in range(2)]
  a=d[2]/d[3]
  b=d[1]/d[3]
  c=d[0]/d[3]
  Q=(a*a-3.0*b)/9.0
  y1=2.0*a*a*a
  y2=-9*a*b
  y3=27*c
  y4=y1+y2+y3
  R=y4/54.0
  Q3=Q*Q*Q
  R2=R*R
  if R2<Q3:
    qq=-2.0*sqrt(Q);
    theta=acos(R/Math.sqrt(Q3));
    x[0][0]=qq*cos(theta/3.0)-a/3.0;
    x[0][1]=qq*cos((theta-2.0*pi)/3.0)-a/3.0;
    x[0][2]=qq*cos((theta+2.0*pi)/3.0)-a/3.0;
  else:
    A=-pow((R+sqrt(R2-Q3)),(1.0/3.0));
    if A==0:  B=0
    else   :  B=Q/A
    x[0][0]=(A+B)-a/3.0
    x[0][1]=-0.5*(A+B)-a/3
    x[1][1]=sqrt(3.0)/2.0*(A-B)
    x[0][2]=-0.5*(A+B)-a/3
    x[1][2]=-sqrt(3.0)/2.0*(A-B)
  return x

a=[1.0,3.0,3.0,1.0]
print("Roots of the polynomial : ",cubic_roots(a))
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" NA13D.py
Roots of the polynomial :  [[-1.0, -1.0, -1.0], [0, -0.0, 0.0]]
```

Program Root of third degree polynomial (analytical formula) fortran version cubic1.f90

```
 program cubic1
 IMPLICIT NONE
 REAL(KIND=8), DIMENSION(0:3) :: d
 COMPLEX(KIND=8), DIMENSION(0:2) :: x
 d=(/1,-3,3,-1/)
 call cubic(d,x)
 print *,'x(0) = ',x(0)
 print *,'x(1) = ',x(1)
 print *,'x(2) = ',x(2)

 CONTAINS
  SUBROUTINE cubic(d,x)
  REAL(KIND=8), DIMENSION(0:3) :: d
  COMPLEX(KIND=8), DIMENSION(0:2) :: x
  REAL(KIND=8) a,b,c,Q,R,tetha,A1,B1,xr,xi,PI
  PI=4.0*atan(1.0)
  a=d(2)/d(3)
  b=d(1)/d(3)
  c=d(0)/d(3)
  Q=(a*a-3.0*b)
  R=(2.0*a*a*a-9.0*a*b+27.0*c)/54
  tetha=acos(R/sqrt(Q*Q*Q))
  if(R*R<Q*Q*Q) then
    x(0)=(-2.0*sqrt(Q))*cos(tetha/3.0)-a/3.0
    x(1)=(-2.0*sqrt(Q))*cos((tetha-2.0*PI)/3.0)-a/3.0
    x(2)=(-2.0*sqrt(Q))*cos((tetha+2.0*PI)/3.0)-a/3.0
  else
    A1=(R+sqrt(R*R-Q*Q*Q))**(1.0/3.0)
    if(A1==0) THEN
      B1=0
    ELSE
      B1=Q/A
    END IF
    x(0)=(A1+B1)-a/3.0
    xr=-(A1+B1)/2.0-a/3
    xi=-sqrt(3.0)*(A1-B1)/2.0
    x(1)=cmplx(xr,xi)
    x(2)=cmplx(xr,-xi)
  end if
  END SUBROUTINE cubic
end program cubic1
```

```
- ---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" cubic1

E:\okul\SCO1>cubic1
 x(0) =  (  1.0000000000000000     ,  0.0000000000000000     )
 x(1) =  (  1.0000000000000000     , -0.0000000000000000     )
 x(2) =  (  1.0000000000000000     ,  0.0000000000000000     )

> Terminated with exit code 0.
```

Program Root of third degree polynomial (analytical formula-julia version)

```
 function cubic_roots(d)
        # roots of a degree 3 polynomial
        # P(x)=a[3]*x^3+a[2]*x^2+a[1]*x+a[0]=0
        # assuming all a is all real
        # if more than 4 values are entered the remaining ones will be ignored
        x=zeros(Float64,2,3)
        x1=zeros(Float64,3)
        x2=zeros(ComplexF64,3)
        a=d[3]/d[4];
        b=d[2]/d[4];
        c=d[1]/d[4];
        R=0.0
        theta=0.0
        Q=(a*a-3.0*b)/9.0
```

```
            y1=2.0*a*a*a
            y2=-9*a*b
            y3=27*c
            y4=y1+y2+y3
            R=y4/54.0
            Q3=Q*Q*Q
            R2=R*R
            qq=0.0
            A=0.0
            B=0.0
if(R2<=Q3)
             qq=-2.0*sqrt(Q)
             if(R==0)
               theta=acos(R)
             else
               theta=acos(R/sqrt(Q3))
             end
             x1[1]=qq*cos(theta/3.0)-a/3.0
             x1[2]=qq*cos((theta-2.0*pi)/3.0)-a/3.0
             x1[3]=qq*cos((theta+2.0*pi)/3.0)-a/3.0
             return x1
else
             A=-(R+sqrt(R2-Q3))^(1.0/3.0)
             if(A==0)
                      B=0.0
    else
                      B=Q/A
             end
                      x[1,1]=(A+B)-a/3.0
                      x[1,2]=-0.5*(A+B)-a/3
                      x[2,2]=sqrt(3.0)/2.0*(A-B)
                      x[1,3]=-0.5*(A+B)-a/3
                      x[2,3]=-sqrt(3.0)/2.0*(A-B)
                      x2[1]=complex(x[1,1],0.0)
                      x2[2]=complex(x[1,2],x[2,2])
                      x2[3]=complex(x[1,3],x[2,3])
                      return x2
end
end


d=[1.0 -3.0 3.0 -2.0]
x=cubic_roots(d)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" cubic_roots.jl
x = ComplexF64[0.5 + 0.0im, 0.5 - 0.8660254037844386im, 0.5 + 0.8660254037844386im]
> Terminated with exit code 0.
```

```
d=[1.0 -3.0 3.0 -1.0]
x=cubic_roots(d)
print("x = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" cubic_roots.jl
x = [1.0, 1.0, 1.0]
> Terminated with exit code 0.
```

Tartaglia's Poem

Italian:	English Translation:
Quando chel cubo con le cose appresso *Se agguaglia à qualche numero discreto* *Trouan dui altri differenti in esso.*	01) *When the cube with the cose beside it <x^3+px >* 02) *Equates itself to some other whole number, <=q>*

Dapoi terrai questo per consueto *Che'llor produtto sempre sia eguale* *Alterzo cubo delle cose neto,* *El residuo poi suo generale* *Delli lor lati cubi ben sottratti* *Varra la tua cosa principale.* *In el secondo de cotestiatti* *Quando che'l cubo restasse lui solo* *Tu osseruarai quest'altri contratti,* *Del numer farai due tal part'à uolo* *Che l'una in l'altra si produca schietto* *El terzo cubo delle cose in stolo* *Delle qual poi, per communprecetto* *Torrai li lati cubi insieme gionti* *Et cotal somma sara il tuo concetto.* *El terzo poi de questi nostri conti* *Se solue col secondo se ben guardi* *Che per natura son quasi congionti.* *Questi trouai, & non con paβi tardi* *Nel mille cinquecentè, quatroe trenta* *Con fondamenti ben sald'è gagliardi* *Nella citta dal mar'intorno centa.*	03) *Find two other, of which it is the difference. <u–v=q>* 04) *Hereafter you will consider this customarily* 05) *That their product always will be equal <uv=>* 06) *To the third of the cube of the cose net.* 3/3, instead of (*p*/3)³>* 07) *Its general remainder then* 08) *Of their cube sides , well subtracted,* <u−−√3−v√3u3−v3> 09) *Will be the value of your principal unknown. <=x>* 10) *In the second of these acts,* 11) *When the cube remains solo , <x³=px+q>* 12) *You will observe these other arrangements:* 13) *Of the number <q> you will quickly make two such parts, <q=u+v>* 14) *That the one times the other will produce straightforward <uv=>* 15) *The third of the cube of the cose in a multitude, <p³/3,* instead of (*p*/3)³>* 16) *Of which then, per common precept,* 17) *You will take the cube sides joined together.* <u−−√3+v√3u3+v3 18) *And this sum will be your concept. <=x>* 19) *The third then of these our calculations <x³+q=px>* 20) *Solves itself with the second, if you look well after,* 21) *That by nature they are quasi conjoined.* 22) *I found these, & not with slow steps,* 23) *In thousand five hundred, four and thirty* 24) *With very firm and strong foundations* 25) *In the city girded around by the sea.*

COMPLEX ROOTS OF THE POLYNOMIALS BY UTILISING SYNTHETIC DIVISION PROCESS

Horners method utilises synthetic division process to solve roots of polynomials. Synthetic division process use recursive process to obtain a division for a polynomial. If

$P(x) = f_n(x) = a_0 + a_1x + a_1x^2 + \cdots + a_nx^n$ (2.15.1)

and if this polynomial divides with $(x - x_0)$ to form polynomial

$Q(x) = f_{n-1}(x) = b_1 + b_2x + b_3x^2 + \cdots + b_{n-1}x^{n-1}$ (2.15.2)

Where $P(x) = f_n(x) = b_0 + (x - x_0)f_{n-1}(x)$ or $P(x) = b_0 + (x - x_0)Q(x)$ (2.15.3)

$R = b_0$ is the remainder of the synthetic division process. Of course if the remainder is equal to zero x_0 would be the root of this polynomial. Synthetic division can be calculated as following iterative equation:

$b_n = a_n$ (2.15.4)

$b_i = a_i + b_{i+1}x_0$ (2.15.5)

For example if a polynomial

$f_n(x) = 2x^4 - 3x^2 + 3x - 4$ is given, and x_0 is -2 {divide by (x+2)} synthetic division is applied as

x_0	a_4	a_3	a_2	a_1	a_0
-2	**2**	**0**	**-3**	**3**	**-4**
	b_4	b_3	b_2	b_1	b_0
	2	**-4**	**5**	**-7**	**10**

$Q(x) = f_{n-1}(x) = 2x^3 - 4x^2 + 5x - 7$

$P(x) = f_n(x) = 10 + (x + 2)Q(x)$

Differentiating of $f_n(x)$with respect to x gives

$f'_n(x) = f_{n-1}(x) + (x - x_0)f'_{n-1}(x)$ (2.15.6)

and at $x=x_0$

$f_n(x) = b_0$

$f'_n(x) = f_{n-1}(x) = Q(x)$ (2.15.7)

Therefore Newton - Raphson iterative method can be applied to find the roots by using iterative process

$x_n = x_{n-1} - \frac{f_n(x_{n-1})}{f'_n(x_{n-1})} = x_{n-1} - \frac{b_0}{f'_n(x_{n-1})}$ (2.15.8)

$x_n = x_{n-1} - \frac{b_0}{Q(x_{n-1})}$ (2.15.9)

xo	a4	a3	a2	a1	a0
-2	2	0	-3	3	-4
	2	-4	5	-7	10

Q	-49
x1	-1.7959184

x1	a4	a3	a2	a1	a0
-1.7959184	2	0	-3	3	-4
	2	-3.5918367	3.4506456	-3.1970777	1.7416906

Q	-32.563821
x2	-1.7424329

x2	a4	a3	a2	a1	a0
-1.7424329	2	0	-3	3	-4
	2	-3.4848658	3.0721449	-2.3530065	0.0999559

Q	-28.866623
x3	-1.7389702

x3	a4	a3	a2	a1	a0
-1.7389702	2	0	-3	3	-4
	2	-3.4779405	3.048035	-2.3004421	0.0004003

Q	-28.63559
x4	-1.7389563

x4	a4	a3	a2	a1	a0
-1.7389563	2	0	-3	3	-4
	2	-3.4779125	3.0479377	-2.3002304	6.505E-09

Q	-28.634659
x5	-1.7389563

x5	a4	a3	a2	a1	a0
-1.7389563	2	0	-3	3	-4
	2	-3.4779125	3.0479377	-2.3002304	0

Q	-28.634659
x5	-1.7389563

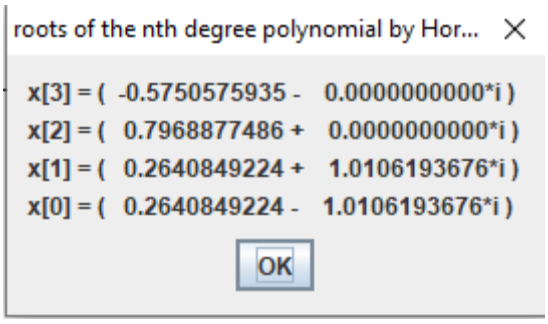
Note that roots of a polynomial can be complex numbers, therefore it is better to calculate the roots as complex numbers. It should be also note that after reducing polynomial to a third, second or first degree remaining roots can be calculated by using standard root finding equations. An example program for Horner's method is listed below (complex class in java language was defined previously) :

Program Complex Horner method example to find the roots of a polynomial horner.java (Java version)

```
 import javax.swing.*;
import java.util.*;
public class horner
{
c CA[];//complex polynomial coefficient
public static c f_poly(c c1[],c x)
{
// this function calculates the value of
// c least square curve fitting function
int n=c1.length;
c ff;
if(n!=0.0)
  { ff=new c(c1[n-1]);
  for(int i=n-2;i>=0;i--)
   { ff=c.p(c.p(ff,"*",x),"+",c1[i]); }
  }
else
  ff=new c(0,0);
return ff;
}
public double[] toDouble()
{int n=CA.length;
 double d[]=new double[n];
 for(int i=0;i<n;i++)
    {   d[i]=c.toDouble(CA[i]);}
    return d;}
public  c horner(double a[], c x)
{   int n=a.length;
                c c1[]=new c[n];
    for(int i=0;i<n;i++) c1[i]=new c(a[i]);
    return horner(c1,x);
}
public  c horner(c c3[], c x)
{   int n=c3.length;
    int iter=0;
    int nmax=200;
    boolean b=true;
    int j=0;
                c c1[]=new c[n];
                int n1=n-1;
                c cp[]=new c[n1];
                c1[n-1]=new c(c3[n-1]);
    for(int i=n-2;i>=0;i--)
    {   c c2=new c(c3[i]);
                    c1[i]=c.p(c2,"+",c.p(x,"*",c1[i+1]));
    }
                c P=c1[0];
                for(int i=n1-1;i>=0;i--)
                {cp[i]=c1[i+1];}
                CA=cp;
                c Q=f_poly(cp,x);
                //Newton's method
                x=c.p(x,"-",c.p(P,"/",Q));//x=x-P/Q
                iter++;
                if(c.abs(P)<1.0e-25 || c.abs(P)==0 ||iter>nmax)
                {return x;}
                else
                {
                while(b)
                {
                c1[n-1]=new c(c3[n-1]);
    for(int i=n-2;i>=0;i--)
    {   c c2=new c(c3[i]);
                    c1[i]=c.p(c2,"+",c.p(x,"*",c1[i+1]));
                }
                P=c1[0];
                for(int i=n1-1;i>=0;i--)
                {cp[i]=c1[i+1];}
                CA=cp;
    Q=f_poly(cp,x);
                if(c.abs(P)<1.0e-25||iter>nmax)
                {b=false;
                 return x;
                }
                //Newton's method
                x=c.p(x,"-",c.p(P,"/",Q));
```

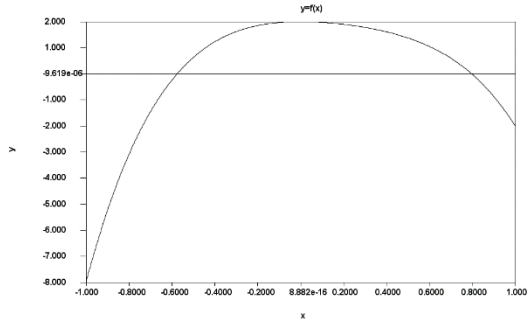
```
                iter++;
   }
}
   return x;
}

public String toString()
{int n=CA.length;
 String s="";
 for(int i=n-1;i>=0;i--) {s+=CA[i]+" ";}
 return s;
}
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}
public c[] root(double a[])
{   int n=a.length;
    c y[]=new c[n-1];
                c x=new c(1.0,1.0);
                c x1=horner(a, x);
                int i=n-2;
    y[i]=x1;
    i--;
    int nn=CA.length;
    while(i>=0)
    {  if(nn!=CA.length)
       {c D[]=new c[nn];
       for(int j=0;j<nn;j++) {D[j]=CA[j];}
       CA=D;
       }
     {c z=horner(CA,x);
      y[i]=z;
     i--;
     nn--;
     }
     }
    return y;
}
     public static double[] enterdata(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }
public static void calculate_Horner()
{
                String s="enter coefficients of the n degree polynomial a[0] a[1]..a[n] y=a[0]+a[1]*x+a[2]*x^2+a[3]*x^3+..+a[n]*x^n ";
    double [] x0=enterdata(s);
    c x=new c(1.0,1.0);
    horner h=new horner();
    int n=x0.length;
    String s1[]=new String[n];
    String s2=" coefficients of the n degree polynomial y=a[0]+a[1]*x+a[2]*x^2+a[3]*x^3+..+a[n]*x^n ";
    for(int i=0;i<n;i++)
    {s1[i]="a["+i+"]";}
    JOptionPane.showMessageDialog(null,toString(h.root(x0)),
    "roots of the nth degree polynomial by Horner's synthetic division & Newton-Raphson method : ",JOptionPane.PLAIN_MESSAGE);

}
public static void main(String arg[])
{   calculate_Horner();}
}
```

$$f(x) = -4x^4 + 3x^3 - 3x^2 + 2$$




```
 public class gr1
{
public static void main(String arg[])
{double a[]={2,0,-3,3,-4};
 if_poly f=new if_poly(a);
 f.plot(-1,1);
 System.out.println(f.print_roots());
}}
```

```
 ---------- Capture Output ----------
> "E:\co\java\bin\java.exe" gr1
 coefficients of the n degree polynomial y=a[0]+a[1]*x+a[2]*x^2+a[3]*x^3+..+a[n]*x^n
x[0](   0.2640849224 -    1.0106193676*i )
x[1](   0.2640849224 +    1.0106193676*i )
x[2](   0.7968877486 +    0.0000000000*i )
x[3](  -0.5750575935 -    0.0000000000*i )
```



Horner Method in python language:

Program Complex Horner method example to find the roots of a polynomial (python language)

```
from math import *

def f_poly(c1,x):
    # this function calculates the value of
    # c least square curve fitting function
    n=len(c1);
    if n!=0.0:
      ff=c1[n-1];
      for i in range(n-2,-1,-1):
        ff=ff*x+c1[i];
    else:
      ff=0,0;
    return ff;

def horner(a):
  # roots of a polynomial by using synthetic division process
    n=len(a);
    x1=complex(1.56435,0.1);
    n1=n;
    nmax=150;
    tolerance=1.0e-10;
    n1=n+1;
    ref=3.1;
    y=[complex(0.0,0.0) for i in range(n1-2)];
    n2=n1-2;
```

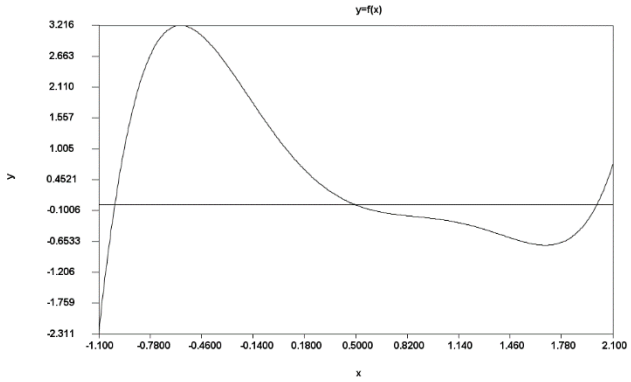
```
    for k in range(0,n2):
        n=len(a);
        x=x1;
        xold=3.4878;
        ref=abs(xold-x);
        b=[complex(0.0,0) for i in range(n)];
        c=[complex(0.0,0.0) for i in range(n-1)];
        b[n-1]=a[n-1];
        ii=0;
        while ref>tolerance and ii<nmax:
          for i in range(n-2,-1,-1):
             b[i]=a[i]+b[i+1]*x;
          for i in range(n-2,-1,-1):  c[i]=b[i+1];
          Q=f_poly(c,x);
          P=b[0];
          x=x-P/Q;
          ref=abs(xold-x);
          ii=ii+1;
          xold=x;
        a=c;
        n=len(a);
        y[k]=x;
    return y;

a=[-4,3,-3,0,2];
print(horner(a));
```

```
runfile('D:/okul/SCO1/Mathd.py', wdir='D:/okul/SCO1')
[(1.254881884834291+1.34106353887572e-29j),
 (0.24203718580880437-0.9262454872675325j),
(0.24203718584251482+0.9262454872691118j),
 (-1.738956256536873-9.031220216115798e-12j)]
```

$$f(x) = x^5 - 3.5x^4 + 2.75x^3 + 2.125x^2 - 3.875x + 1.25$$

```
public class gr1
{
public static void main(String arg[])
{double a[]={1.25,-3.875,2.125,2.75,-3.5,1};
 if_poly f=new if_poly(a);
 f.plot(-1.1,2.1);
 System.out.println(f.print_roots());
}}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" gr1
 coefficients of the n degree polynomial y=a[0]+a[1]*x+a[2]*x^2+a[3]*x^3+..+a[n]*x^n
x[0](  -1.0000000000 +    0.0000000000*i )
x[1](   2.0000000000 +    0.0000000000*i )
x[2](   1.0000000000 -    0.5000000000*i )
x[3](   0.5000000000 -    0.0000000000*i )
x[4](   1.0000000000 +    0.5000000000*i )
```



Same function results in python code:

```
a=[1.25,-3.875,2.125,2.75,-3.5,1];
print(horner(a));
```

```
runfile('D:/okul/SCO1/horner1.py', wdir='D:/okul/SCO1')
[(1.0000000000000002-0.5j), (0.5000000000000011+1.0217019574503208e-15j), (1.9999999999895546-4.1185274253612687e-13j),
(1.0000000000176792+0.49999999997300376j), (-0.9999999999955536-3.5171310308612647e-12j)]
```

Now let us find roots of $f(x) = x^3 + 9813.18x^2 + 8571.08x + 0.781736 = 0$ by using horner method

```
public static void main(String arg[])
{   double c1[]={0.781736,8571.08,9813.18,1};
   horner h=new horner();
            System.out.println(h.toString(h.root(c1)));
}
```

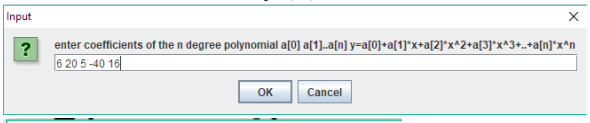
```
---------- Capture Output ----------
> "C:\java\bin\java.exe" horner
x[2] = (-9.121576846347157E-5)
x[1] = (-0.8734118615166153)
x[0] = (-9812.306496922716)
```

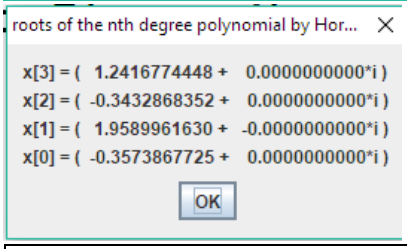
Same function results in python code:

```
a=[0.781736,8571.08,9813.18,1];
print(horner(a));
```

```
runfile('D:/okul/SCO1/horner1.py', wdir='D:/okul/SCO1')
[(-9.121576846347157e-05+4.9325894681659426e-26j), (-0.8734118615171431-4.424838606974589e-13j), (-9812.306496922753-
6.1173554997356664e-12j)]
```

Now let us find roots of $f(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6 = 0$ by using horner method






```
a=[6,20,5,-40,16];
print(Mathd.horner(a));
```

```
runfile('D:/okul/SCO1/Mathd.py', wdir='D:/okul/SCO1')
[(1.2416774447647838+0j), (1.9704460787298799-4.672866064934229e-20j), (-0.35606176190945776+0.16275838255939531j), (-
0.3560617615461311-0.16275838258481318j)]
```

Le Guerre method is also specificaly developed to find roots of a polynomial. Method formulations is as follows:

$$g(x) = [f'(x)]^2 - \frac{nf(x)f''(x)}{n-1} \qquad (2.15.10)$$

$$x_{k+1} = x_k - \frac{nf(x_k)}{f'(x_k) \pm (n-1)\sqrt{g(x_k)}} \qquad (2.15.11)$$

In this definition, n is the degree of polynomial. The term $\sqrt{g(x_k)}$ in the equation can be complex (roots of polynomials can be complex numbers). Therefore the method will be written as complex variable method. By combining LeGuerre root finding method with synthetic division process, the remaining roots can also be found.

Program Complex LeGuerre with synthetic division example to find the roots of a polynomial (Java version-complex)

```
 //
//
// compleks Lequerre method to find roots of the polynomials
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class Leguerre
```

```
{ public static c func(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return func(a,x);
  }
  public static c func(c a[],c x)
  {  // c polynomial
  int n=a.length;
  c y=new c(a[n-1]);
  for(int i=n-2;i>=0;i--)
  {y=c.p(c.p(y,"*",x),"+",a[i]);}
  return y;
  }
  public static c dfunc(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return dfunc(a,x);
  }
  static c dfunc(c a[],c x)
  {//c numeric derivative of a polynomial
   double hh=0.001;
   c h=new c(hh,0.0);
   c he=new c(-hh,0.0);
   c h2=new c(2.0*hh,0.0);
   c he2=new c(-2.0*hh,0.0);
   c xah=c.p(x,"+",h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(a,xe2h),"*",(1.0/12.0));
   c feh=c.p(func(a,xeh),"*",(-8.0/12.0));
   c fah=c.p(func(a,xah),"*",(8.0/12.0));
   c fa2h=c.p(func(a,xa2h),"*",(-1.0/12.0));
   c df=c.p(fe2h,"+",c.p(feh,"+",c.p(fah,"+",fa2h)));
   df=c.p(df,"/",h);
   return df;
  }
  public static c d2func(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return d2func(a,x);
  }
  static c d2func(c a[],c x)
  {//c second numeric derivative of a polynomial
   double hh=0.001;
   c h=new c(hh,hh);
   c hkare=c.pow(h,2);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.p(x,"+",h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(a,xe2h),"*",(-1.0/12.0));
   c feh=c.p(func(a,xeh),"*",(4.0/3.0));
   c ff=c.p(func(a,x),"*",(-5.0/2.0));
   c fah=c.p(func(a,xah),"*",(4.0/3.0));
   c fa2h=c.p(func(a,xa2h),"*",(-1.0/12.0));
   c d2f=c.p(fe2h,"+",c.p(feh,"+",c.p(ff,"+",c.p(fah,"+",fa2h))));
   d2f=c.p(d2f,"/",hkare);
   return d2f;
  }

public static c Leguerre(double ai[],c x)
{int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return Leguerre(a,x);
}

public static c Leguerre(c a[],c x)
{
```

```
int n=a.length-1;
double eps=1.0e-15;
int m=50;
int k=0;
c f0=func(a,x);
c f1=new c();
c f2=new c();
c g=new c();
c d1=new c();
c d2=new c();
c x0=x;
c x1=c.p(a[0],"/",a[1]);
x1=c.p(x1,"*",-1.0);
do{
  f1=dfunc(a,x0);
  f2=d2func(a,x0);
  g=c.p(c.p(f1,"*",f1),"+",c.p(c.p(f0,"*",f2),"*",(-(double)n/(double)(n-1))));
  d1=c.p(f1,"+",c.p(c.pow(g,0.5),"*",(double)(n-1)));
  d2=c.p(f1,"+",c.p(c.pow(g,0.5),"*",(double)(1-n)));
  if(c.abs(d1)>=c.abs(d2)) x1=c.p(x0,"+",c.p(c.p(f0,"/",d1),"*",(double)(-n)));
  else x1=c.p(x0,"+",c.p(c.p(f0,"/",d2),"*",(double)(-n)));
  x0=c.p(x1,"+",0.0);
  f0=func(a,x0);
  k++;
}while(c.abs(f0)>eps && k<m);
if(c.abs(f0)<eps) { return x1;}
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
" result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x1;
}

public static c[] synthetic_division(double ci[],c x)
{
  int n=ci.length;
  c c[]=new c[n];
  for(int i=0;i<n;i++) c[i]=new c(ci[i]);
  return synthetic_division(c,x);
}
public static c[] synthetic_division(c c4[],c x)
{ int n=c4.length;
  c c1[]=new c[n];
  c1[n-1]=new c(c4[n-1]);
    for(int i=n-2;i>=0;i--)
    {   c c2=new c(c4[i]);
                c1[i]=c.p(c2,"+",c.p(x,"*",c1[i+1]));
              }
  c c3[]=new c[n-1];
  for(int i=n-1;i>=1;i--)
{   c c2=new c(c1[i]);
              c3[i-1]=c1[i];
              }
  return c3;
}

public static c[] quadratic(c C[])
{ c z[]=new c[2];
     c z1=new c(C[0]);
    c ac4=c.p(z1,"*",C[2]);
    ac4=c.p(ac4,"*",4.0);
    z1=new c(C[1]);
    c b2=c.p(z1,"*",z1);
    c DD=c.sqrt(c.p(b2,"-",ac4));
    z1=new c(C[1]);
    z[0]=c.p(DD,"-",z1);
    z[0]=c.p(z[0],"/",2.0);
    z[0]=c.p(z[0],"/",C[2]);
    z[1]=c.p(DD,"+",C[1]);
    z[1]=c.p(z[1],"/",2.0);
    z[1]=c.p(z[1],"/",C[2]);
    z[1]=c.p(z[1],"*",-1.0);
    return z;
}
public static c[] root(double C1[])
{ int n=C1.length;
c C[]=new c[n];
for(int i=0;i<n;i++)
```

```
{C[i]=new c(C1[i]);}
return root(C);
}

public static c[] root(c C[])
{   int n=C.length;
   c y[]=new c[n-1];
              c x0=new c(1.0,0.0);
              int i=n-2;
              int nn=C.length;
   while(i>=0)
   {
    if(nn>3)
    { c z=Leguerre(C,x0);
      y[i]=z;
      C=synthetic_division(C,z);
      nn=C.length;
      i--;
    }
    else if(nn==3)
    { c z[]=quadratic(C);
      y[i]=z[0];
                 y[i-1]=z[1];
      i-=2;
      nn-=2;
    }
    else if(nn==2)
    {c z;
    c z1=new c(C[0]);
               z=c.p(z1,"*",-1.0);
               z=c.p(z,"/",C[1]);
               y[i]=z;
               i-=2;
    nn-=2;
               }
    }
   return y;
}
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}

    public static double[] input(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

    public static void main (String args[])
    { double c[]={-4,3,-3,0,2};
      c y[]=root(c);
      System.out.println(toString(y));
    }
}
```

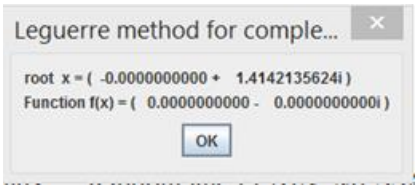
```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Leguerre
x[3] = (   1.2548818848 +    0.0000000000i )
x[2] = (   0.2420371858 -    0.9262454873i )
x[1] = (   0.2420371858 +    0.9262454873i )
x[0] = (  -1.7389562565)
> Terminated with exit code 0.
```

Of course method can be used to find roots of any general function as well, the next program will utilise LeGuerre method for general function with lambda expressions and general interface defination icf_cx defined previously.

Program Complex Lequerre method example NA13E1 general complex function defined with lambda expressions

```
 //
//  compleks Lequerre method to find roots of the polynomials
//  Lambda expression version for a general function

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class NA13E1
{
public static c LeguerreC(icf_cx f,c x)
{
int n=10; //degree of polynomial
double eps=1.0e-15;
int m=50;
int k=0;
c f0=f.func(x);
c f1=new c();
c f2=new c();
c g=new c();
c d1=new c();
c d2=new c();
c x0=x;
c x1=new c();
do{
  f1=f.dfunc(x0);
  f2=f.d2func(x0);
  g=c.p(c.p(f1,"*",f1),"+",c.p(c.p(f0,"*",f2),"*",(-(double)n/(double)(n-1))));
  d1=c.p(f1,"+",c.p(c.pow(g,0.5),"*",(double)(n-1)));
  d2=c.p(f1,"+",c.p(c.pow(g,0.5),"*",(double)(1-n)));
if(c.abs(d1)>=c.abs(d2)) x1=c.p(x0,"+",c.p(c.p(f0,"/",d1),"*",(double)(-n)));
else                     x1=c.p(x0,"+",c.p(c.p(f0,"/",d2),"*",(double)(-n)));
  x0=c.p(x1,"+",0.0);
  f0=f.func(x0);
  k++;
}while(c.abs(f0)>eps && k<m);

if(c.abs(f0)<eps) { return x1;}
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x1;
}
    public static double[] verigir(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
```

```
    public static void main (String args[])
    {
    double x0[]=verigir(" first guess for the root give c number with a space in between 1.0 0.5 ");
    c x1=new c(x0[0],x0[1]);
    icf_cx f=(c x)->c.p(c.p(x,"*",x),"+",2.0);
    c r=LeguerreC(f,x1);
    String s="root  x = "+r.toString()+"\n"+"Function f(x) = "+f.func(r).toString();
    JOptionPane.showMessageDialog(null,s,
    "Leguerre method for c root of polynomials : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Muller's method was also previously investigated as a general real function root finding method. In Horners's method the basic method used to find the roots was Newton's method. In Muller's method a quadratic polynomial root finding is actually used to determine the roots. As Horner's method, synthetic division process can also be utilised to find the remaing roots also.

Program Complex Muller's method example to find the roots of a polynomial muller.java

```
 import javax.swing.*;
public class muller
{ //This version of the program is written for
  //polynomials only
c C[];
public static c f_poly(c c1[],c x)
{
// this function calculates the value of
// c least square curve fitting function
int n=c1.length;
c ff;
if(n!=0.0)
  { ff=new c(c1[n-1]);
  for(int i=n-2;i>=0;i--)
   { ff=c.p(c.p(ff,"*",x),"+",c1[i]); }
  }
else
  ff=new c(0,0);
return ff;
}
public static c muller(c c1[],double p0i,double p1i,double p2i)
{
  c p0=new c(p0i);
  c p1=new c(p1i);
  c p2=new c(p2i);
  return muller(c1,p0,p1,p2);
}
public static c muller(double ci[],double p0i,double p1i,double p2i)
{ int n=ci.length;
  c c1[]=new c[n];
  for(int i=0;i<n;i++) c1[i]=new c(ci[i]);
  c p0=new c(p0i);
  c p1=new c(p1i);
  c p2=new c(p2i);
  return muller(c1,p0,p1,p2);
}

public static c muller(c Ci[],c p0,c p1,c p2)
{
int nmax=300;
double tolerance=1.0e-20;
c fp0,fp1,fp2,fp3;
```

```
c p3=new c(0.0);
c a,b,ca;
c b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11,b12;
c a1,a2,a3;
c c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11;
fp0=f_poly(Ci,p0);
fp1=f_poly(Ci,p1);
fp2=f_poly(Ci,p2);
for(int i=0;i<nmax;i++)
{               ca=fp2;
   b1=c.p(p0,"-",p2);
   b2=c.p(p1,"-",p2);
   b3=c.p(p0,"-",p1);
   b4=c.p(b1,"*",b1);
   b5=c.p(b2,"*",b2);
   b6=c.p(fp1,"-",fp2);
   b7=c.p(fp0,"-",fp2);
   b8=c.p(b1,"*",b2);
   b8=c.p(b8,"*",b3);
   b9=c.p(b8,"*",b3);
   b10=c.p(b4,"*",b6);
   b11=c.p(b5,"*",b7);
   b12=c.p(b10,"-",b11);
   b=c.p(b12,"/",b8);
   a1=c.p(b2,"*",b7);
   a2=c.p(b1,"*",b6);
   a3=c.p(a1,"-",a2);
   a=c.p(a3,"/",b8);
   c1=c.p(ca,"*",-2.0);
   c2=c.p(b,"*",b);
   c3=c.p(a,"*",ca);
   c4=c.p(-4.0,"*",c3);
   c5=c.p(c2,"-",c4);
   c6=c.sqrt(c5);
   c7=c.p(c.sgn(b),"*",c6);
   c8=c.p(b,"+",c7);
   c9=c.p(c1,"/",c8);
   p3=c.p(p2,"+",c9);
   fp3=f_poly(Ci,p3);
   if(c.abs(fp3)<tolerance) {return p3;}
   else{p0=p1;p1=p2;p2=p3;fp0=fp1;fp1=fp2;fp2=fp3;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
  " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return p3;
}

public double[] toDouble()
{int n=C.length;
 double d[]=new double[n];
 for(int i=0;i<n;i++)
   {   d[i]=c.toDouble(C[i]);}
   return d;}


public static c[] synthetic_division(double ci[],c x)
{
  int n=ci.length;
  c c1[]=new c[n];
  for(int i=0;i<n;i++) c1[i]=new c(ci[i]);
  return synthetic_division(c1,x);
}
public static c[] synthetic_division(c ci[],c x)
{ int n=ci.length;
  c c1[]=new c[n];
  c1[n-1]=new c(ci[n-1]);
    for(int i=n-2;i>=0;i--)
   {   c c2=new c(ci[i]);
                c1[i]=c.p(c2,"+",c.p(x,"*",c1[i+1]));
               }
  c c3[]=new c[n-1];
  for(int i=n-1;i>=1;i--)
{   c c2=new c(c1[i]);
              c3[i-1]=c1[i];
              }
  return c3;
```

```
}

public static c[] quadratic(c Ci[])
{ c z[]=new c[2];
      c z1=new c(Ci[0]);
     c ac4=c.p(z1,"*",Ci[2]);
     ac4=c.p(ac4,"*",4.0);
     z1=new c(Ci[1]);
     c b2=c.p(z1,"*",z1);
     c DD=c.sqrt(c.substract(b2,ac4));
     z1=new c(Ci[1]);
     z[0]=c.p(DD,"-",z1);
     z[0]=c.p(z[0],"/",2.0);
     z[0]=c.p(z[0],"/",Ci[2]);
     z[1]=c.p(DD,"+",Ci[1]);
     z[1]=c.p(z[1],"/",2.0);
     z[1]=c.p(z[1],"/",Ci[2]);
     z[1]=c.p(z[1],"*",-1.0);
     return z;
}
public static c[] root(double C1[])
{ int n=C1.length;
c Ci[]=new c[n];
for(int i=0;i<n;i++)
{Ci[i]=new c(C1[i]);}
return root(Ci);
}

public static c[] root(c Ci[])
{    int n=Ci.length;
    c y[]=new c[n-1];
                c x0=new c(0.0,0.0);
                c x1=new c(1.0,0.0);
                c x2=new c(2.0,0.0);
                int i=n-2;
                int nn=Ci.length;
    while(i>=0)
    {
     if(nn>3)
     { c z=muller(Ci,x0,x1,x2);
       y[i]=z;
       Ci=synthetic_division(Ci,z);
       nn=Ci.length;
       i--;
     }
     else if(nn==3)
     { c z[]=quadratic(Ci);
       y[i]=z[0];
                  y[i-1]=z[1];
       i-=2;
       nn-=2;
     }
     else if(nn==2)
     {c z;
     c z1=new c(Ci[0]);
                z=c.p(z1,"*",-1.0);
                z=c.p(z,"/",Ci[1]);
                y[i]=z;
                i-=2;
     nn-=2;
                }
     }
    return y;
}
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}
public static void main(String arg[])
{    double c1[]={-4,3,-3,0,2};
    c y[]=root(c1);
    System.out.println(toString(y));
}}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" muller
x[3] = (   1.2548818848)
x[2] = (  -1.7389562565 +    0.0000000000i )
x[1] = (   0.2420371858 +    0.9262454873i )
x[0] = (   0.2420371858 -    0.9262454873i )
> Terminated with exit code 0.
```

Inverse quadratic Lagrange interpolation formula is a quadratic formula relatively similar to Mullers formula. The detailed formulations are previously given for general non-linear function root solving. In this subchapter the method will be utilised to find roots of polynomial.

$$x = \frac{f(x_1)f(x_2)x_3}{[f(x_3)-f(x_1)][f(x_3)-f(x_2)]} + \frac{f(x_2)f(x_3)x_1}{[f(x_1)-f(x_2)][f(x_1)-f(x_3)]} + \frac{f(x_3)f(x_1)x_2}{[f(x_2)-f(x_3)][f(x_2)-f(x_1)]} \qquad (2.8.2)$$

Program Complex Inverse Lagrange interpolation method example to find the roots of a polynomial inverse_quadratic.java

```
 import javax.swing.*;
//Inverse quadratic interpolation formula for c polynomial
public class inverse_quadratic
{ //This version of the program is written for
  //polynomials only
c C[];
public static c f_poly(c c1[],c x)
{
// this function calculates the value of
// c least square curve fitting function
int n=c1.length;
c ff;
if(n!=0.0)
  { ff=new c(c1[n-1]);
  for(int i=n-2;i>=0;i--)
   { ff=c.p(c.p(ff,"*",x),"+",c1[i]); }
  }
else
  ff=new c(0,0);
return ff;
}
public static c inverse_quadratic(c c1[],double p0i,double p1i,double p2i)
{
  c p0=new c(p0i);
  c p1=new c(p1i);
  c p2=new c(p2i);
  return inverse_quadratic(c1,p0,p1,p2);
}
public static c inverse_quadratic(double ci[],double p0i,double p1i,double p2i)
{ int n=ci.length;
  c c1[]=new c[n];
  for(int i=0;i<n;i++) c1[i]=new c(ci[i]);
  c p0=new c(p0i);
  c p1=new c(p1i);
  c p2=new c(p2i);
  return inverse_quadratic(c1,p0,p1,p2);
}

public static c inverse_quadratic(c Ci[],c p0,c p1,c p2)
{
int nmax=300;
double tolerance=1.0e-20;
c fp0,fp1,fp2,fp3;
c R,S,T,P,Q;
c p3=new c(0.0);
c a,b,ca;
c b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11,b12;
c a1,a2,a3;
c c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11;
fp0=f_poly(Ci,p0);
fp1=f_poly(Ci,p1);
fp2=f_poly(Ci,p2);
for(int i=0;i<nmax;i++)
{               ca=fp2;
   R=c.p(fp1,"/",fp2);
   S=c.p(fp1,"/",fp0);
   T=c.p(fp0,"/",fp2);
   b1=c.p(R,"-",T);//R-T
```

```
    b2=c.p(p2,"-",p1);//x3-x2
    b3=c.p(R,"-",1.0);//R-1
    b4=c.p(p1,"-",p0);//x2-x1
    b5=c.p(b4,"*",b3);//(R-1)(x2-x1)
    b6=c.p(b1,"*",T);//T(R-T)
    b7=c.p(b6,"*",b2);//T(R-T)(x3-x2)
    b8=c.p(b7,"+",b5);//T(R-T)(x3-x2)+(R-1)(x2-x1)
    P=c.p(S,"*",b8);//P=S*[T(R-T)(x3-x2)+(R-1)(x2-x1)]
    b9=c.p(T,"-",1.0);
    b10=c.p(S,"-",1.0);;
    b11=c.p(b3,"*",b9);
    Q=c.p(b11,"*",b10);//Q=(T-1)(S-1)(R-1)
    p3=c.p(p1,"+",c.p(P,"/",Q));//x3=x1+P/Q
    fp3=f_poly(Ci,p3);
    if(c.abs(fp3)<tolerance) {return p3;}
    else{p0=p1;p1=p2;p2=p3;fp0=fp1;fp1=fp2;fp2=fp3;}
}
JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " results may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return p3;
}
public double[] toDouble()
{int n=C.length;
 double d[]=new double[n];
 for(int i=0;i<n;i++)
    {   d[i]=c.toDouble(C[i]);}
    return d;}
public static c[] synthetic_division(double ci[],c x)
{
  int n=ci.length;
  c c1[]=new c[n];
  for(int i=0;i<n;i++) c1[i]=new c(ci[i]);
  return synthetic_division(c1,x);
}
public static c[] synthetic_division(c ci[],c x)
{ int n=ci.length;
  c c1[]=new c[n];
  c1[n-1]=new c(ci[n-1]);
    for(int i=n-2;i>=0;i--)
    {   c c2=new c(ci[i]);
                c1[i]=c.add(c2,c.multiply(x,c1[i+1]));
              }
  c c3[]=new c[n-1];
  for(int i=n-1;i>=1;i--)
{   c c2=new c(c1[i]);
              c3[i-1]=c1[i];
              }
  return c3;
}

public static c[] quadratic(c Ci[])
{ c z[]=new c[2];
      c z1=new c(Ci[0]);
     c ac4=c.multiply(z1,Ci[2]);
     ac4=c.multiply(ac4,4.0);
     z1=new c(Ci[1]);
     c b2=c.multiply(z1,z1);
     c DD=c.sqrt(c.substract(b2,ac4));
     z1=new c(Ci[1]);
     z[0]=c.substract(DD,z1);
     z[0]=c.divide(z[0],2.0);
     z[0]=c.divide(z[0],Ci[2]);
     z[1]=c.add(DD,Ci[1]);
     z[1]=c.divide(z[1],2.0);
     z[1]=c.divide(z[1],Ci[2]);
     z[1]=c.multiply(z[1],-1.0);
     return z;
}
public static c[] root(double C1[])
{ int n=C1.length;
c Ci[]=new c[n];
for(int i=0;i<n;i++)
{Ci[i]=new c(C1[i]);}
return root(Ci);
}
```

```
public static c[] root(c Ci[])
{   int n=Ci.length;
   c y[]=new c[n-1];
              c x0=new c(0.5,0.0);
              c x1=new c(1.0,0.0);
              c x2=new c(2.0,0.0);
              int i=n-2;
              int nn=Ci.length;
   while(i>=0)
   {
    if(nn>3)
    { c z=inverse_quadratic(Ci,x0,x1,x2);
      y[i]=z;
      Ci=synthetic_division(Ci,z);
      nn=Ci.length;
      i--;
    }
    else if(nn==3)
    { c z[]=quadratic(Ci);
      y[i]=z[0];
                 y[i-1]=z[1];
      i-=2;
      nn-=2;
    }
    else if(nn==2)
    {c z;
    c z1=new c(Ci[0]);
              z=c.multiply(z1,-1.0);
              z=c.divide(z,Ci[1]);
              y[i]=z;
              i-=2;
    nn-=2;
              }
    }
   return y;
}
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}
public static void main(String arg[])
{   double c1[]={-4,3,-3,0,2};
   c y[]=root(c1);
   System.out.println(toString(y));
}}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" inverse_quadratic
x[3] = (   1.2548818848)
x[2] = (  -1.7389562565)
x[1] = (   0.2420371858 -    0.9262454873i )
x[0] = (   0.2420371858 +    0.9262454873i )


> Terminated with exit code 0.
```

Halley Method of root finding is a second order Newton type method that previously investigated to find roots of real functions. A complex polynomial version of the method with sysnthetic division to find all the roots are given next

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)f'(x_n) - f(x_n)f''(x_n)} \qquad (2.11.2)$$

Program Complex Halley's method example to find the roots of a polynomial

```
 // compleks Halley's method to find roots of the polynomials
// xn+1=xn-2(f(xn)*f(xn)*f'(xn))/(2f'(xn)*f'(xn)-f(xn)*f''(xn))
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
```

```
public class Halley
{ public static c func(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return func(a,x);
  }
  public static c func(c a[],c x)
  {  // c polynomial
  int n=a.length;
  c y=new c(a[n-1]);
  for(int i=n-2;i>=0;i--)
  {y=c.p(c.p(y,"*",x),"+",a[i]);}
  return y;
  }
  public static c dfunc(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return dfunc(a,x);
  }
  static c dfunc(c a[],c x)
  {//c numeric derivative of a polynomial
   double hh=0.001;
   c h=new c(hh,0.0);
   c he=new c(-hh,0.0);
   c h2=new c(2.0*hh,0.0);
   c he2=new c(-2.0*hh,0.0);
   c xah=c.add(x,h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(a,xe2h),"*",(1.0/12.0));
   c feh=c.p(func(a,xeh),"*",(-8.0/12.0));
   c fah=c.p(func(a,xah),"*",(8.0/12.0));
   c fa2h=c.p(func(a,xa2h),"*",(-1.0/12.0));
   c df=c.p(fe2h,"+",c.p(feh,"+",c.p(fah,"+",fa2h)));
   df=c.p(df,"/",h);
   return df;
  }
  public static c d2func(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return d2func(a,x);
  }
  static c d2func(c a[],c x)
  {//c second numeric derivative of a polynomial
   double hh=0.001;
   c h=new c(hh,hh);
   c hkare=c.pow(h,2);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.p(x,"+",h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(a,xe2h),"*",(-1.0/12.0));
   c feh=c.p(func(a,xeh),"*",(4.0/3.0));
   c ff=c.p(func(a,x),"*",(-5.0/2.0));
   c fah=c.p(func(a,xah),"*",(4.0/3.0));
   c fa2h=c.p(func(a,xa2h),"*",(-1.0/12.0));
   c d2f=c.p(fe2h,"+",c.p(feh,"+",c.p(ff,"+",c.p(fah,"+",fa2h))));
   d2f=c.p(d2f,"/",hkare);
   return d2f;
  }

 public static c Halley(double ai[],c x)
 {int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return Halley(a,x);
 }

 public static c Halley(c a[],c x)
```

```
{
int n=a.length-1;
double eps=1.0e-15;
int m=100;
int k=0;
c f,df,d2f;
c b1,b2,b3,b4,b5;
do{
  f=func(a,x);
  df=dfunc(a,x);
  d2f=d2func(a,x);
  b1=c.p(c.p(f,"*",df),"*",2.0);
  b2=c.p(c.p(df,"*",df),"*",2.0);
  b3=c.p(f,"*",d2f);
  b4=c.p(b2,"-",b3);
  b5=c.p(b1,"/",b4);
  x=c.p(x,"-",b5);
  if(c.abs(f)<eps) return x;
  k++;
}while(c.abs(f)>eps && k<m);
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static c[] synthetic_division(double ci[],c x)
{
  int n=ci.length;
  c c1[]=new c[n];
  for(int i=0;i<n;i++) c1[i]=new c(ci[i]);
  return synthetic_division(c1,x);
}
public static c[] synthetic_division(c ci[],c x)
{ int n=ci.length;
  c c1[]=new c[n];
  c1[n-1]=new c(ci[n-1]);
    for(int i=n-2;i>=0;i--)
    {   c c2=new c(ci[i]);
                c1[i]=c.p(c2,"+",c.p(x,"*",c1[i+1]));
              }
  c c3[]=new c[n-1];
  for(int i=n-1;i>=1;i--)
{   c c2=new c(c1[i]);
              c3[i-1]=c1[i];
              }
  return c3;
}

public static c[] quadratic(c Ci[])
{ c z[]=new c[2];
     c z1=new c(Ci[0]);
    c ac4=c.p(z1,"*",Ci[2]);
    ac4=c.p(ac4,"*",4.0);
    z1=new c(Ci[1]);
    c b2=c.p(z1,"*",z1);
    c DD=c.sqrt(c.p(b2,"-",ac4));
    z1=new c(Ci[1]);
    z[0]=c.p(DD,"-",z1);
    z[0]=c.p(z[0],"/",2.0);
    z[0]=c.p(z[0],"/",Ci[2]);
    z[1]=c.p(DD,"+",Ci[1]);
    z[1]=c.p(z[1],"/",2.0);
    z[1]=c.p(z[1],"/",Ci[2]);
    z[1]=c.p(z[1],"*",-1.0);
    return z;
}
public static c[] root(double C1[])
{ int n=C1.length;
c Ci[]=new c[n];
for(int i=0;i<n;i++)
{Ci[i]=new c(C1[i]);}
return root(Ci);
}

public static c[] root(c Ci[])
{   int n=Ci.length;
   c y[]=new c[n-1];
```

```
                c x0=new c(1.0,0.0);
                int i=n-2;
                int nn=Ci.length;
    while(i>=0)
    {
     if(nn>3)
     { c z=Halley(Ci,x0);
       y[i]=z;
       Ci=synthetic_division(Ci,z);
       nn=Ci.length;
       i--;
     }
     else if(nn==3)
     { c z[]=quadratic(Ci);
       y[i]=z[0];
                  y[i-1]=z[1];
       i-=2;
       nn-=2;
     }
     else if(nn==2)
     {c z;
     c z1=new c(Ci[0]);
                z=c.multiply(z1,-1.0);
                z=c.divide(z,Ci[1]);
                y[i]=z;
                i-=2;
     nn-=2;
                }
     }
    return y;
}
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i].toString("%10.7f")+"\n";}
 return s;
}
     public static double[] input(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }

     public static void main (String args[])
     { double c1[]={-4,3,-3,0,2};
       c y[]=root(c1);
       System.out.println(toString(y));
     }
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\javaw.exe" Halley
x[3] = ( 1.2548819 +  0.0000000*i )
x[2] = (-1.7389563 +  0.0000000*i )
x[1] = ( 0.2420372 +  0.9262455*i )
x[0] = ( 0.2420372 -  0.9262455*i )
> Terminated with exit code 0.
```

Hasan Method of root finding is one of the most efficient method to find roots, a complex polynomial version of the method with sysnthetic division to find all the roots are given next

Program Complex Hasan's method example to find the roots of a polynomial

```
 // compleks Hasan's method to find roots of the polynomials
// xn+1=xn-f(xn)/(f'(xn)+1/3*f'(xn)*log((f'(xn)*f'(xn)*-f(xn)*f''(xn))/(f'(xn)*f'(xn))
 import java.util.*;
```

```
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class Hasan
{ public static c func(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return func(a,x);
  }
  public static c func(c a[],c x)
  {  // c polynomial
  int n=a.length;
  c y=new c(a[n-1]);
  for(int i=n-2;i>=0;i--)
  {y=c.p(c.p(y,"*",x),"+",a[i]);}
  return y;
  }
  public static c dfunc(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return dfunc(a,x);
  }
  static c dfunc(c a[],c x)
  {//c numeric derivative of a polynomial
   double hh=0.001;
   c h=new c(hh,0.0);
   c he=new c(-hh,0.0);
   c h2=new c(2.0*hh,0.0);
   c he2=new c(-2.0*hh,0.0);
   c xah=c.add(x,h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(a,xe2h),"*",(1.0/12.0));
   c feh=c.p(func(a,xeh),"*",(-8.0/12.0));
   c fah=c.p(func(a,xah),"*",(8.0/12.0));
   c fa2h=c.p(func(a,xa2h),"*",(-1.0/12.0));
   c df=c.p(fe2h,"+",c.p(feh,"+",c.p(fah,"+",fa2h)));
   df=c.p(df,"/",h);
   return df;
  }
  public static c d2func(double ai[],c x)
  {   int n=ai.length;
                c a[]=new c[n];
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return d2func(a,x);
  }
  static c d2func(c a[],c x)
  {//c second numeric derivative of a polynomial
   double hh=0.001;
   c h=new c(hh,hh);
   c hkare=c.pow(h,2);
   c he=new c(-hh,-hh);
   c h2=new c(2.0*hh,2.0*hh);
   c he2=new c(-2.0*hh,-2.0*hh);
   c xah=c.p(x,"+",h);
   c xa2h=c.p(x,"+",h2);
   c xeh=c.p(x,"+",he);
   c xe2h=c.p(x,"+",he2);
   c fe2h=c.p(func(a,xe2h),"*",(-1.0/12.0));
   c feh=c.p(func(a,xeh),"*",(4.0/3.0));
   c ff=c.p(func(a,x),"*",(-5.0/2.0));
   c fah=c.p(func(a,xah),"*",(4.0/3.0));
   c fa2h=c.p(func(a,xa2h),"*",(-1.0/12.0));
   c d2f=c.p(fe2h,"+",c.p(feh,"+",c.p(ff,"+",c.p(fah,"+",fa2h))));
   d2f=c.p(d2f,"/",hkare);
   return d2f;
  }

public static c Hasan(double ai[],c x)
{int n=ai.length;
                c a[]=new c[n];
```

```
                for(int i=0;i<n;i++) a[i]=new c(ai[i]);
                return Hasan(a,x);
}

public static c Hasan(c a[],c x)
{
int n=a.length-1;
double eps=1.0e-15;
int m=100;
int k=0;
c f,df,d2f;
c b1,b2,b3,b4,b5,b6,b7;
double r=3.0;
do{
  f=func(a,x);
  df=dfunc(a,x);
  d2f=dfunc(a,x);
  b1=c.p(c.p(df,"*",df),"-",c.p(f,"*",d2f));
  b2=c.p(df,"*",df);
  b3=c.p(b1,"/",b2);
  b4=c.p(df,"*",c.log(b3));
  b5=c.p(b4,"/",r);
  b6=c.p(df,"+",b5);
  b7=c.p(f,"/",b6);
  x=c.p(x,"-",b7);
  if(c.abs(f)<eps) return x;
  k++;
}while(c.abs(f)>eps && k<m);
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}


public static c[] synthetic_division(double ci[],c x)
{
  int n=ci.length;
  c c1[]=new c[n];
  for(int i=0;i<n;i++) c1[i]=new c(ci[i]);
  return synthetic_division(c1,x);
}
public static c[] synthetic_division(c ci[],c x)
{ int n=ci.length;
  c c1[]=new c[n];
  c1[n-1]=new c(ci[n-1]);
    for(int i=n-2;i>=0;i--)
    {   c c2=new c(ci[i]);
                  c1[i]=c.p(c2,"+",c.p(x,"*",c1[i+1]));
               }
  c c3[]=new c[n-1];
  for(int i=n-1;i>=1;i--)
{   c c2=new c(c1[i]);
               c3[i-1]=c1[i];
               }
  return c3;
}

public static c[] quadratic(c Ci[])
{ c z[]=new c[2];
     c z1=new c(Ci[0]);
    c ac4=c.p(z1,"*",Ci[2]);
    ac4=c.p(ac4,"*",4.0);
    z1=new c(Ci[1]);
    c b2=c.p(z1,"*",z1);
    c DD=c.sqrt(c.p(b2,"-",ac4));
    z1=new c(Ci[1]);
    z[0]=c.p(DD,"-",z1);
    z[0]=c.p(z[0],"/",2.0);
    z[0]=c.p(z[0],"/",Ci[2]);
    z[1]=c.p(DD,"+",Ci[1]);
    z[1]=c.p(z[1],"/",2.0);
    z[1]=c.p(z[1],"/",Ci[2]);
    z[1]=c.p(z[1],"*",-1.0);
    return z;
}
public static c[] root(double C1[])
```

```
{ int n=C1.length;
c Ci[]=new c[n];
for(int i=0;i<n;i++)
{Ci[i]=new c(C1[i]);}
return root(Ci);
}

public static c[] root(c Ci[])
{   int n=Ci.length;
   c y[]=new c[n-1];
              c x0=new c(1.0,0.0);
              int i=n-2;
              int nn=Ci.length;
   while(i>=0)
   {
    if(nn>3)
    { c z=Hasan(Ci,x0);
      y[i]=z;
      Ci=synthetic_division(Ci,z);
      nn=Ci.length;
      i--;
    }
    else if(nn==3)
    { c z[]=quadratic(Ci);
      y[i]=z[0];
                y[i-1]=z[1];
      i-=2;
      nn-=2;
    }
    else if(nn==2)
    {c z;
    c z1=new c(Ci[0]);
              z=c.multiply(z1,-1.0);
              z=c.divide(z,Ci[1]);
              y[i]=z;
              i-=2;
    nn-=2;
              }
    }
   return y;
}
public static String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}
    public static double[] input(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

    public static void main (String args[])
    { double c1[]={-4,3,-3,0,2};
      c y[]=root(c1);
      System.out.println(toString(y));
}}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" Hasan
x[3] = (   1.2548818848)
x[2] = (   0.2420371858 +    0.9262454873i )
x[1] = (   0.2420371858 -    0.9262454873i )
x[0] = (  -1.7389562565 +    0.0000000000i )
```

```
> Terminated with exit code 0.
```

Program Complex Horner method example to find the roots of a polynomial horner.jl

```
 function f_poly(c1,x)
    # this function calculates the value of
    # c least square curve fitting function
    n=size(c1)[1]
    if n!=0.0
      ff=c1[n]
      for i=n-1:-1:1
        ff=ff*x+c1[i]
      end
    else
      ff=0.0
    end
    return ff
end

function horner(a)
  # roots of a polynomial by using synthetic division process
    n=size(a)[2]
    x1=-2.0
    n1=n
    nmax=20
    tolerance=1.0e-10
    n2=n-1
    ref=3.1
    ii=0
    global y=zeros(ComplexF64,n2)
    for k=1:n2
          x=x1
          xold=3.4878
          ref=abs(xold-x)
          b=zeros(ComplexF64,n1)
          global c=zeros(ComplexF64,n2)
          b[n]=a[n]
          xx=b[n]
          ii=0
          # find the root in while
          while ref>tolerance && ii<nmax
            for i=n2:-1:1
               b[i]=a[i]+b[i+1]*x
               xx=b[i]
            end # for
            for i=n2:-1:1
              c[i]=b[i+1]
            end # for
            Q=f_poly(c,x)
            P=b[1]
            x=x-P/Q
            ref=abs(xold-x)
            ii=ii+1
            xold=x
          end # while
          n-=1
          a=c
          y[k]=x
    end # for
return y
end  # function

a=[-4 3 -3 0 2]
print(horner(a))
```

$$f(x) = -4x^4 + 3x^3 - 3x^2 + 2$$

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" horner.jl
ComplexF64[-1.738956256451892 + 0.0im, 1.254881884834291 - 0.0im, -0.6819503221875776 + 0.0im, 2.474337965776697 - 0.0im]
> Terminated with exit code 0.
```

$$f(x) = x^3 - 3x^2 + 3x - 1$$

```
a=[1 -3 3 -1]
```

```
print(horner(a))
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" horner.jl
ComplexF64[1.0000030176099868 + 0.0im, 0.9999963232971536 + 0.0im, 1.0000040546033393 + 0.0im]
> Terminated with exit code 0.
```

$$f(x) = x^5 - 3.5x^4 + 2.75x^3 + 2.125x^2 - 3.875x + 1.25$$

```
a=[1.25 -3.875 2.125 2.75 -3.5 1]
print(horner(a))
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" horner.jl
ComplexF64[-1.0 - 0.0im, 0.5000000000002416 + 0.0im, 1.999999999999984 + 0.0im, 1.37720103480294 + 0.0im,
1.2491220612554996 - 0.0im]
> Terminated with exit code 0.
```

COMPLEX ROOTS OF THE POLYNOMIALS BY BAIRSTOW'S METHOD

Bairstow's method is an iterative approach, like Horner's method it also uses synthetic division as basic tool. But instead of a single step division, a two step division is taken and utilised. The basic reason for that is to get a better utilisation of complex roots conjugate effect. Remember from the previous section that sythetic division recursion calculations has the form:

$f_n(x) = a_0 + a_1x + a_1x^2 + \cdots + a_nx^n$ (2.16.1)

and if this polynomial divides with $(x - x_0)$ to form polynomial

$f_{n-1}(x) = b_1 + b_2x + b_3x^2 + \cdots + b_{n-1}x^{n-1}$ (2.16.2)

Where $f_n(x) = b_0 + (x - x_0)f_{n-1}(x)$ (2.16.3) and $R = b_0$ is the remainder of the synthetic division process. Of course if the remainder is equal to zero x_0 would be the root of this polynomial. Synthetic division can be calculated as following iterative equation:

$b_n = a_n$ (2.16.4)

$b_i = a_i + b_{i-1}x_0$ (2.16.5)

Now if a second degree synthetic division is desired, the formulation will change a little. If second degree division equation is $x^2 - rx - s$ the result is a new polynomial in the form of

$f_{n-2}(x) = b_2 + b_3x + b_4x^2 + \cdots + b_nx^{n-2}$ (2.16.6) with the remainder

$R = b_1(x - r) + b_0$ (2.16.7) As with the normal synthetic division a simple recurance relation can be used as: $b_n = a_n$ (2.16.8)

$b_{n-1} = a_{n-1} + rb_n$ (2.16.9)

$b_i = a_i + rb_{i+1} + sb_{i+2}$ for i = n-2 to 0 (2.16.10)

As seen from the equation we have two roots (r and s) to solve simultaneously in one step calculation now. Consider that b coefficients are function of r and s, First degree Taylor series expansion of these coefficients can be written as

$b_1^n = b_1^{n-1} + \frac{\partial b_1^{n-1}}{\partial r}\Delta r + \frac{\partial b_1^{n-1}}{\partial s}\Delta s$ (2.16.11)

$b_0^n = b_0^{n-1} + \frac{\partial b_0^{n-1}}{\partial r}\Delta r + \frac{\partial b_0^{n-1}}{\partial s}\Delta s$ (2.16.12) Higher order terms of the Taylor series are ignored. The roots are defined as remainder terms of sysnthetic division as zero (b_0 and b_1 should be equal to zero) so if we assume the left hand side of the equation is zero, this equation becomes

$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1$ (2.16.13)

$\frac{\partial b_0}{\partial r}\Delta r + \frac{\partial b_0}{\partial s}\Delta s = -b_0$ (2.16.14) If partial derivatives of b is determined this set of equation can be easily solved. Bairstow showed that the partial derivatives can be obtained by synthetic division of b values. Then the partial derivatives can be substituted in above equation

$c_n = b_n$ (2.16.15)

$c_{n-1} = b_{n-1} + rc_n$ (2.16.16)

$c_i = b_i + rc_{i+1} + sc_{i+2}$ for i = n-2 to 1 (2.16.17)

So the equation becomes:

$c_2\Delta r + c_3\Delta s = -b_1$ (2.16.18)

$c_1\Delta r + c_2\Delta s = -b_0$ (2.16.19)

Solution of this equation by Gauss elimination process will give the following solution:

$W = \frac{c_1}{c_2}$ (2.16.20)

$\Delta s = \frac{b_0 - W b_1}{c_2 - W c_3}$ (2.16.21)

$\Delta r = \frac{-b_1 - c_3 \Delta s}{c_2}$ (2.16.22)

$r^{n+1} = r^n + \Delta r^{n+1}$ (2.16.23)

$s^{n+1} = s^n + \Delta s^{n+1}$ (2.16.24)

This equation can be solved iteratively until the error term shrink to a minimum value

$|\varepsilon| = \left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta s}{s}\right| < \varepsilon_s$ (2.16.25) when the error level drops under a specified stopping criterion, iteration stops, and roots can be calculated by using quadratic equation

$x = \frac{r \pm \sqrt{r^2 + 4s}}{2}$ (2.16.26)

Note that all these procedure can be calculated with complex numbers. It should be also note that after reducing polynomial to a third, second or first degree remaining roots can be calculated by using standard root finding equations. An example program is listed below:

$f(x) = x^5 - 3.5x^4 + 2.75x^3 + 2.125x^2 - 3.875x + 1.25$ with r=s=-1

Program Complex Bairstow's method example

```
 public class bairstow
{
c CA[];
public double[] toDouble()
{int n=CA.length;
 double d[]=new double[n];
 for(int i=0;i<n;i++)
    {   d[i]=c.toDouble(CA[i]);}
    return d;
}
public double[] toDouble(c w[])
{int n=w.length;
 double d[]=new double[n];
 for(int i=0;i<n;i++)
    {   d[i]=c.toDouble(w[i]);}
    return d;
}

public c[] bairstow(double a[], c r,c s)
{   int n=a.length;
    boolean bb=true;
    int j=0;
                c b[]=new c[n];
                int n1=n-2;
                c cp[]=new c[n1];
                c ci[]=new c[n];
                c ds,dr;
                while(bb)
                {
                b[n-1]=new c(a[n-1]);
                b[n-2]=c.p(c.p(r,"*",b[n-1]),"+",a[n-2]);
    for(int i=n1-1;i>=0;i--)
    {   c c2=new c(a[i]);
                    b[i]=c.p(c2,"+",c.p(r,"*",b[i+1]));
                    b[i]=c.p(b[i],"+",c.p(s,"*",b[i+2]));
    }
    ci[n-1]=b[n-1];
    ci[n-2]=c.p(b[n-2],"+",c.p(r,"*",ci[n-1]));
    for(int i=n1-1;i>=0;i--)
    {   ci[i]=c.p(b[i],"+",c.p(r,"*",ci[i+1]));
                    ci[i]=c.p(ci[i],"+",c.p(s,"*",ci[i+2]));
    }
                for(int i=n1-1;i>=0;i--)
                {cp[i]=b[i+2];}
                CA=cp;
```

```
                // Gauss elimination formula to solve system of equation
                c CC=c.p(ci[1],"/",ci[2]);
                c b0=c.p(CC,"*",b[1]);
                b0=c.p(b0,"-",b[0]);
                c c0=c.p(CC,"*",ci[3]);
                c0=c.p(ci[2],"-",c0);
                ds=c.p(b0,"/",c0);
                c d0=c.p(ds,"*",ci[3]);
                d0=c.p(d0,"+",b[1]);
                d0=c.p(d0,"*",-1.0);
                dr=c.p(d0,"/",ci[2]);
                r=c.p(r,"+",dr);
                s=c.p(s,"+",ds);
                double de=c.abs(c.p(dr,"/",r))+c.abs(c.p(ds,"/",s));
                if(de<1e-15) bb=false;
    }
    c x[]=new c[2];
    c DD=c.sqrt(c.p(c.p(r,"*",r),"+",c.p(s,"*",4.0)));
    x[0]=c.p(r,"+",DD);
    x[0]=c.p(x[0],"/",2.0);
    x[1]=c.p(r,"-",DD);
    x[1]=c.p(x[1],"/",2.0);
   return x;
}
public c[] root(double a[])
{int n=a.length;
 c x[]=new c[n-1];
 c r=new c(1.0,0.0);
 c s=new c(1.0,0.0);
 c y[]=bairstow(a,r,s);
 int i=n-2;
 x[i]=y[0];
 x[i-1]=y[1];
 i-=2;
 int nn=CA.length;
 while(i>=0)
 {  if(nn!=CA.length)
    {c D[]=new c[nn];
     for(int j=0;j<nn;j++) {D[j]=CA[j];}
     CA=D;
     }
                y=bairstow(CA,r,s);
    if(nn==2) x[i]=y[0];
    else {x[i]=y[0];x[i-1]=y[1];}
    nn-=2;
    i-=2;
 }
return x;
}

public c[] bairstow(c a[], c r,c s)
{   int n=a.length;
    c x[];
    if(n>3)
    {
                x=new c[2];
    boolean bb=true;
    int j=0;
                c b[]=new c[n];
                int n1=n-1;
                c cp[]=new c[n1];
                c ci[]=new c[n];
                c ds,dr;
                while(bb)
                {
                b[n-1]=a[n-1];
                c z1=c.p(r,"*",b[n-1]);
                b[n-2]=c.add(z1,a[n-2]);
    for(int i=n-3;i>=0;i--)
    {   c c2=new c(a[i]);
                    b[i]=c.p(c2,"+",c.p(r,"*",b[i+1]));
                    b[i]=c.p(b[i],"+",c.p(s,"*",b[i+2]));
    }
    ci[n-1]=b[n-1];
    ci[n-2]=c.p(b[n-2],"+",c.p(r,"*",ci[n-1]));
    for(int i=n-3;i>=0;i--)
```

```
    {   ci[i]=c.p(b[i],"+",c.p(r,"*",ci[i+1]));
                ci[i]=c.p(ci[i],"+",c.p(s,"*",ci[i+2]));
    }
                for(int i=n1-2;i>=0;i--)
                {cp[i]=b[i+2];}
                CA=cp;
                c CC=c.p(ci[1],"/",ci[2]);
                c b0=c.p(CC,"*",b[1]);
                b0=c.p(b0,"-",b[0]);
                c c0=c.p(CC,"*",ci[3]);
                c0=c.p(ci[2],"-",c0);
                ds=c.p(b0,"/",c0);
                c d0=c.p(ds,"*",ci[3]);
                d0=c.p(d0,"+",b[1]);
                d0=c.p(d0,"*",-1.0);
                dr=c.p(d0,"/",ci[2]);
                r=c.p(r,"+",dr);
                s=c.p(s,"+",ds);
                double de=c.abs(c.p(dr,"/",r))+c.abs(c.p(ds,"/",s));
                if(de<1e-15) bb=false;
    }

    c DD=c.sqrt(c.p(c.p(r,"*",r),"+",c.p(s,"*",4.0)));
    x[0]=c.p(r,"+",DD);
    x[0]=c.p(x[0],"/",2.0);
    x[1]=c.p(r,"-",DD);
    x[1]=c.p(x[1],"/",2.0);
    }
    else if(n==3)
    { x=new c[2];
     c ac4=c.p(a[0],"*",a[2]);
     ac4=c.p(ac4,"*",4.0);
     c b2=c.p(a[1],"*",a[1]);
     c DD=c.sqrt(c.p(b2,"-",ac4));
     x[0]=c.p(DD,"-",a[1]);
     x[0]=c.p(x[0],"/",2.0);
     x[0]=c.p(x[0],"/",a[2]);
     x[1]=c.p(DD,"+",a[1]);
     x[1]=c.p(x[1],"/",2.0);
     x[1]=c.p(x[1],"/",a[2]);
     x[1]=c.p(x[1],"*",-1.0);
     }
     else //(n==2)
     { x=new c[1];
                x[0]=c.p(a[0],"*",-1.0);
                x[0]=c.p(x[0],"/",a[1]);
                }
   return x;
}

public String toString()
{int n1=CA.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+=CA[i]+" ";}
 return s;
}
public String toString(c x[])
{int n1=x.length;
 String s="";
 for(int i=n1-1;i>=0;i--) {s+="x["+i+"] = "+x[i]+"\n";}
 return s;
}

public static void main(String arg[])
{   double c1[]={-4,3,-3,0,2};
                bairstow h=new bairstow();
                c x[]=h.root(c1);
                System.out.println(h.toString(x));
}
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" bairstow
x[3] = (   1.2548818848)
x[2] = (  -1.7389562565)
```

```
x[1] = (   0.2420371858 -    0.9262454873i )
x[0] = (   0.2420371858 +    0.9262454873i )
```

For: **double a[]={24,-50,35,-10,1};** $f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" bairstow
x[3] = (   4.0000000000)
x[2] = (   1.0000000000)
x[1] = (   3.0000000000)
x[0] = (   2.0000000000)
```

For: **double a[]={-1 3 -3 1};** $f(x) = x^3 - 3x^2 + 3x - 1$

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" bairstow
x[2] = (   1.0000003728 +    0.0000006458i )
x[1] = (   1.0000003728 -    0.0000006458i )
x[0] = (   0.9999992543)

> Terminated with exit code 0.
```

ROOT FINDING BY USING SOFTWARE PACKAGES

In order to find roots, some built in functions already existed in package math programs. In Matlab-Octave function **fzero** is existed to find roots of a function and **roots** to find roots of a polynomial

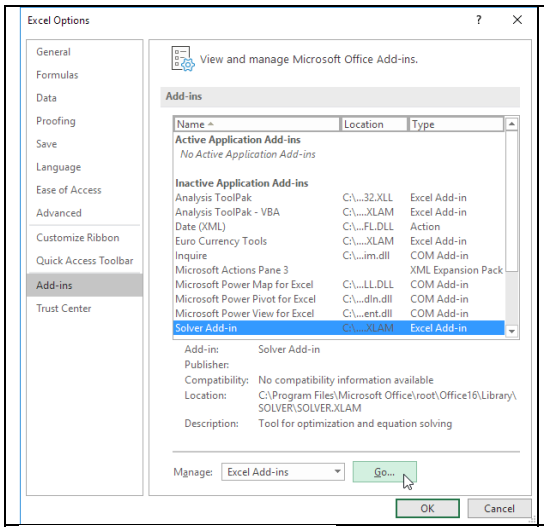
```
>> format long
>> x=2;
>> y=@(x)x*x-2;
>> fzero(y,x)
ans =  1.41421356237310
>>
```

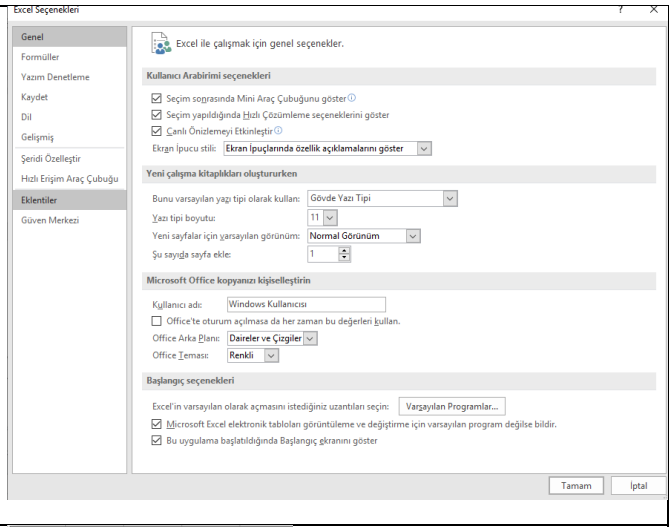
```
>> x=1;
>> y=@(x)x^5-3.5*x^4+2.75*x^3+2.125*x^2-3.875*x+1.25;
>> fzero(y,x)
ans =  0.500000000000001
```

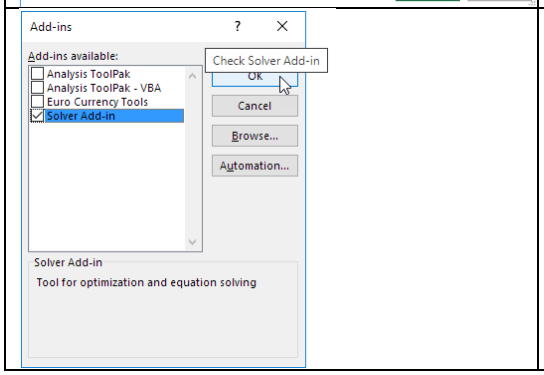
```
>> a=[1 -3.5 2.75 2.125 -3.875 1.25];
>> roots(a)
ans =
   2.000000000000003 + 0.000000000000000i
  -1.000000000000002 + 0.000000000000000i
   1.000000000000000 + 0.500000000000000i
   1.000000000000000 - 0.500000000000000i
   0.499999999999999 + 0.000000000000000i
```

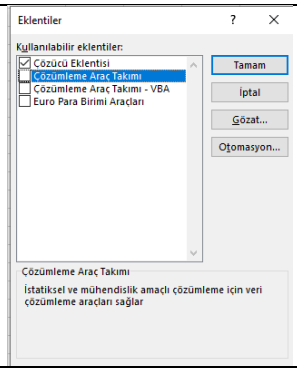
```
>> y=@(x)x*x+2;
>> x=1+1i;
>> fsolve(y,x)
ans =
 -0.000000000000016 + 1.414213562374690i
```

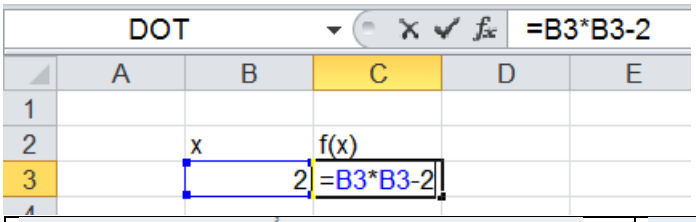
In excel, solver is existed to find real roots: We can use solver add in. In order to open solver:
On the File tab, click Options. (in turkish version dosya, seçenekler)
Under Add-ins(in turkish version eklentiler), select Solver Add-in (in turkish version Çözücü eklentisi), and click on the Go(Git..) button. You can find the Solver on the Data tab(Veri), in the Analyze group.

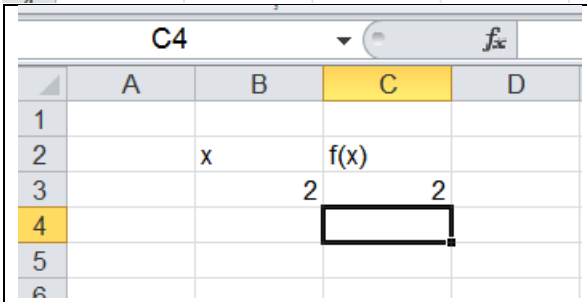

Excel Options
General
Formulas
Data
Proofing
Save
Language
Ease of Access
Advanced
Customize Ribbon
Quick Access Toolbar
Add-ins
Trust Center
View and manage Microsoft Office Add-ins.
Add-ins
Name
Location
Type
Active Application Add-ins
No Active Application Add-ins
Inactive Application Add-ins
Analysis ToolPak C:\...32.XLL Excel Add-in
Analysis ToolPak - VBA C:\....XLAM Excel Add-in
Date (XML) C:\...FL.DLL Action
Euro Currency Tools C:\....XLAM Excel Add-in
Inquire C:\...im.dll COM Add-in
Microsoft Actions Pane 3 XML Expansion Pack
Microsoft Power Map for Excel C:\...LL.DLL COM Add-in
Microsoft Power Pivot for Excel C:\...dln.dll COM Add-in
Microsoft Power View for Excel C:\...ent.dll COM Add-in
Solver Add-in C:\....XLAM Excel Add-in
Add-in: Solver Add-in
Publisher:
Compatibility: No compatibility information available
Location: C:\Program Files\Microsoft Office\root\Office16\Library\SOLVER\SOLVER.XLAM
Description: Tool for optimization and equation solving
Manage: Excel Add-ins
Go...
OK
Cancel

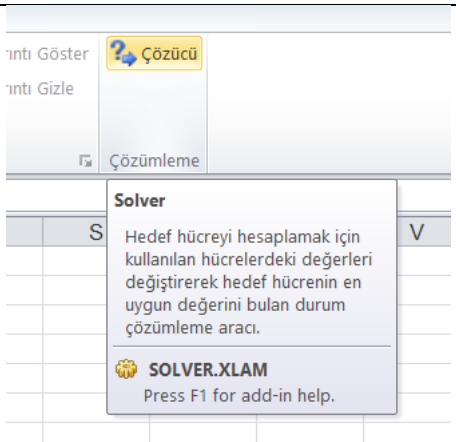

Excel Seçenekleri
Genel
Formüller
Yazım Denetleme
Kaydet
Dil
Gelişmiş
Şeridi Özelleştir
Hızlı Erişim Araç Çubuğu
Eklentiler
Güven Merkezi
Excel ile çalışmak için genel seçenekler.
Kullanıcı Arabirimi seçenekleri
Seçim sonrasında Mini Araç Çubuğunu göster
Seçim yapıldığında Hızlı Çözümleme seçeneklerini göster
Canlı Önizlemeyi Etkinleştir
Ekran İpucu stili: Ekran İpuçlarında özellik açıklamalarını göster
Yeni çalışma kitapları oluştururken
Bunu varsayılan yazı tipi olarak kullan: Gövde Yazı Tipi
Yazı tipi boyutu: 11
Yeni sayfalar için varsayılan görünüm: Normal Görünüm
Şu sayıda sayfa ekle: 1
Microsoft Office kopyanızı kişiselleştirin
Kullanıcı adı: Windows Kullanıcısı
Office'te oturum açılmasa da her zaman bu değerleri kullan.
Office Arka Planı: Daireler ve Çizgiler
Office Teması: Renkli
Başlangıç seçenekleri
Excel'in varsayılan olarak açmasını istediğiniz uzantıları seçin: Varsayılan Programlar...
Microsoft Excel elektronik tabloları görüntüleme ve değiştirme için varsayılan program değilse bildir.
Bu uygulama başlatıldığında Başlangıç ekranını göster
Tamam
İptal

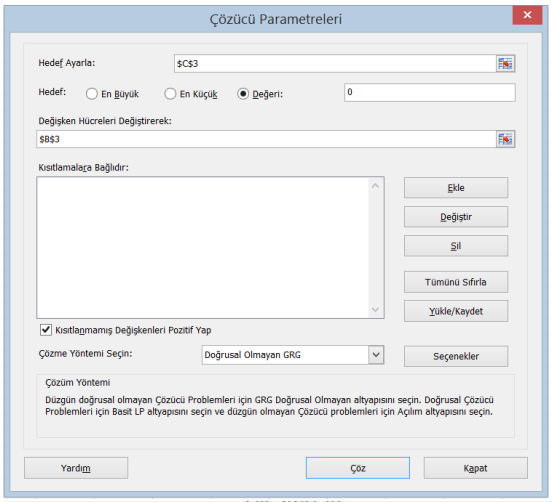

Add-ins
Add-ins available:
Analysis ToolPak
Analysis ToolPak - VBA
Euro Currency Tools
Solver Add-in
Check Solver Add-in
OK
Cancel
Browse...
Automation...
Solver Add-in
Tool for optimization and equation solving

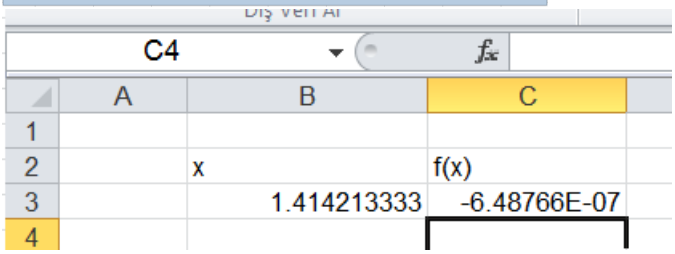

Eklentiler
Kullanılabilir eklentiler:
Çözücü Eklentisi
Çözümleme Araç Takımı
Çözümleme Araç Takımı - VBA
Euro Para Birimi Araçları
Tamam
İptal
Gözat...
Otomasyon...
Çözümleme Araç Takımı
İstatiksel ve mühendislik amaçlı çözümleme için veri çözümleme araçları sağlar


DOT
=B3*B3-2
A
B
C
D
E
1
2
x
f(x)
3
2
=B3*B3-2


C4
A
B
C
D
1
2
x
f(x)
3
2
2
4
5


Çözücü
Çözümleme
Solver
Hedef hücreyi hesaplamak için kullanılan hücrelerdeki değerleri değiştirerek hedef hücrenin en uygun değerini bulan durum çözümleme aracı.
SOLVER.XLAM
Press F1 for add-in help.



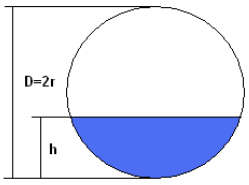


PROBLEMS

PROBLEM 1 Find the roots of $\boldsymbol{f(x) = x^2 - 0.9x - 8.5}$ by using bisection method

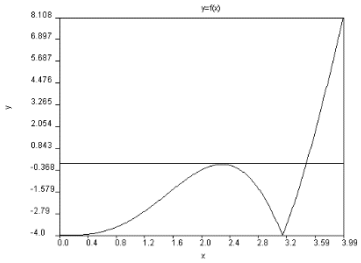
PROBLEM 2 Calculate $\sqrt{7}$ by using one of the root finding method

PROBLEM 3

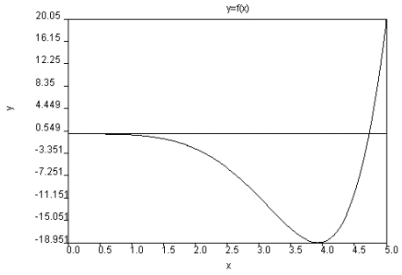


A spherical tank of radius r is filled with a liquid of height h. The Volume of liquid (V) in the tank is given by he following eqaution $\boldsymbol{V = \pi h^2(3r - h)/3}$
If r = 1 m and V= 0.5 m^3 find the liquid height
Note : due to geometry of a sphere h can not be less than 0 and can not be bigger than D=2r. The maximum amount of liquid ths spherical container can be take is 4.18879 m^3 (you do not need this data to solve problem)

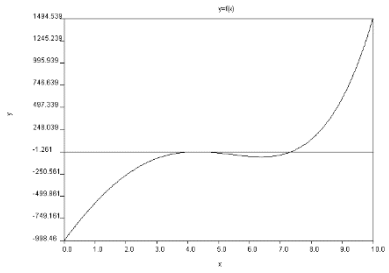
PROBLEM 4 Find the roots of $f(x) = x^2|sin(x)| - 4.$ by using one of the root finding method



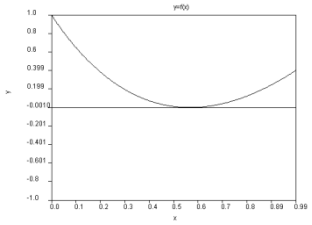
PROBLEM 5 Find the roots of $f(x) = \cos(x)\cosh(x) - 1$ by using one of the root finding method



PROBLEM 6 Find the roots of $f(\boldsymbol{x}) = x^4 - 8.6x^3 - 35.5x^2 + 464.4x - 998.46$ by using one of the root finding method Try x_0=4 and $\boldsymbol{7 \leq x \leq 8}$

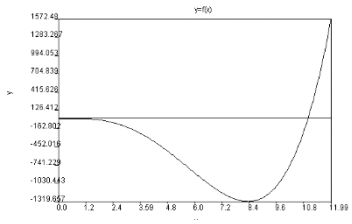


PROBLEM 7 Function $\boldsymbol{f(x)} = x^2 - 2e^{-x} + e^{-2x}$ has only one real root. Find the root bu using Newton-Raphson method. Use as starting value x_0=1

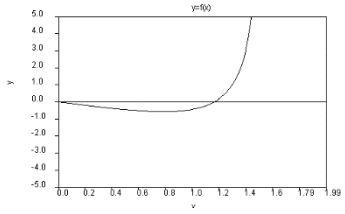


PROBLEM 8

Find all real roots of function $f(\boldsymbol{x}) = x^4 - 7.223x^3 + 13.4475x^2 - 0.672x - 10.223$



PROBLEM 9 Find roots of functionf $(\boldsymbol{x}) = \tan(x) - 2x$.



PROBLEM 10 For Turbulent flow inside of a pipe (Re>2300) friction coefficient inside of a pipe can be calculated by using Colebrook equation.

$$\frac{\mathbf{1}}{\sqrt{\boldsymbol{f}}} = -2.0log_{10}\left[\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right]$$

Re is Reynolds number, ε is surface rougness. Write a program to calculate total pressure drop in the pipe for a given Re number, pipe diameter, pipe length and local pressure drop coefficient.

f : friction coefficient

D pipe diameter

Hint : Change the colebrook equation into the form

$$F(x) = X + 2.0log_{10}\left[\frac{\varepsilon/D}{3.7} + \frac{2.51X}{Re}\right] = 0$$

Then find the root of the equation where $X = \frac{1}{\sqrt{f}}$

Use Haaland equation to obtaing the first estimate of the solution which has the form of

$$\frac{\mathbf{1}}{\sqrt{\boldsymbol{f}}} = -1.8log_{10}\left[\frac{(\varepsilon/D)^{1.11}}{3.7} + \frac{6.9}{Re}\right]$$

```
import java.io.*;
import javax.swing.*;

class f1 extends f_x
{ public double eod;
  public double Re;
  public f1(double eod,double Re)
  {this.eod=eod;
   this.Re=Re;
  }
  public double func(double X)
  {double F=X+2.0*Math.log10(eod/3.7+2.51*X/Re);
        return F;
  }
  public double Haaland()
  {double X=-1.8*Math.log10(Math.pow(eod,1.11)/3.7+6.9/Re);
   return X;
  }
}
class colebrook1
{
public static double secant(f_x f,double x)
{
int nmax=500;
double tolerance=1.0e-10;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
fx=f.func(x);
dfx=f.dfunc(x);
x-=fx/dfx;
if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   "result might not be valid","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static void main (String args[]) throws java.io.IOException
{  f1 f=new f1(0.01,2500);
   double a=f.Haaland();
   double r=secant(f,a);
   double f1=1.0/(r*r);
   System.out.println("f colebrook = "+f1);
}}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\java.exe" colebrook1
f colebrook = 0.053919466488887825

> Terminated with exit code 0.
```

```
from math import *
from f_x import *;

class f1(f_x):
    def __init__(self,eod,Re):
        self.name = 'mathd',
        self.eod=eod;
        self.Re=Re;
    def func(self,X):
        F=X+2.0*log10(self.eod/3.7+2.51*X/self.Re);
        return F;
    def Haaland(self):
```

```
        X=-1.8*log10(pow(self.eod,1.11)/3.7+6.9/self.Re);
        return X;

def secant(f,x):
   # secant root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
     fx= f.func(x);
     dfx=f.dfunc(x,1);
     x=x-fx/dfx
     if abs(fx)<tolerance: return x
  return x


f=f1(0.01,2500);
a=f.Haaland();
r=secant(f,a);
f1=1.0/(r*r);
print(" f colebrook  : ",f1);
```

```
runfile('D:/okul/SCO1/secant1.py', wdir='D:/okul/SCO1')
Reloaded modules: f_x
 f colebrook  :  0.053919466488887825
```

PROBLEM 11

In round smooth tubes for turbulent flow region Fanning friction factor is given as:

$$\frac{\mathbf{1}}{\sqrt{\boldsymbol{f}}} = 1.737\ln\left[Re\sqrt{f}\right] - 0.4$$

This equation is valid for $4 * 10^3 < Re < 3 * 10^6$ By using one of the root finding methods, calculate friction factor for Re=5000. You can utilize any root finding method you learned so far.

Note : You can assume that f is changing between 0,001 and 0.01

PROBLEM 12

van der Waals equation of state for the gases :

$P = \frac{RT}{v-b} - \frac{a}{v^2}$ In this equation

$a = \frac{27}{64}\frac{R^2T_c^2}{P_c}$

$b = \frac{RT_c}{8P_c}$

Redlich ve Kwong equation of states

$P = \frac{RT}{v-b} - \frac{a}{v(v+b)T^{1/2}}$ In this equation

$a = 0.42748\frac{R^2T_c^{2.5}}{P_c}$

$b = 0.08664\frac{RT_c}{P_c}$

O2, for oxygen P_c=5.043 MPa T_c=154.58 K M=31.999 kg/kmol.

N2, for nitrogen P_c=3.39 MPa T_c=126.2 K M=28.013 kg/kmol.

R=8314.5 J/kmolK

Benedict -Webb-Rubin (BWR) equation of state

$$P = \frac{RT}{v} + \frac{RTB_0 - A_0 - C_0/T^2}{v^2} + \frac{RTb - a}{v^3} + \frac{a\alpha}{v^6} + \frac{c}{v^3}\left(1 + \frac{\gamma}{v^2}\right)e^{-\frac{\gamma}{v^2}}$$

Gas	A_0	B_0	$C_0.10^{-6}$	A	B	$c.10^{-6}$	$\alpha.10^3$	$\gamma.10^3$
O_2	1.49880	0.046524	0.0038617	-0.040507	-0.00027963	-0.00020376	0.008641	0.359
N_2	1.1925	0.0458	0.0058891	0.01490	0.00198154	0.000548064	0.291545	0.75

Develop a program to calculate v when T and P is given. By using

A) van der Waals equation of state : gas : oxygen and nitrogen
B) Redlich Kwong equation of state : gas : oxygen and nitrogen
C) Benedict-webb rubin equation of state : gas : oxygen
D) Benedict-webb equation of state : gas : nitrogen

E) Compare the result with ideal gas equation $P = \frac{RT}{v}$

```
 import java.io.*;
import javax.swing.*;

class f2 extends f_x
{ public String name;
  public double R;
  public double a,b;
  public double T,P;
  public f2(String n,double Pc,double Tc,double M,double Pi,double Ti)
  {   name=n;
        R=8314.5/M; //J/kmolK
        a=27.0/64.0*R*R*Tc*Tc/Pc;
        b=R*Tc/(8.0*Pc);
        P=Pi;
        T=Ti;
  }
  public double func(double v)
  { double x=R*T/(v-b)-a/(v*v)-P;
    return x;
  }
  public void setT(double Ti)
  {T=Ti;}
  public void setP(double Pi)
  {P=Pi;}

}
class vanderWaals1
{
public static double secant(f_x f,double x)
{
int nmax=500;
double tolerance=0.01;
double fx,dfx;
double dx[];
for(int i=0;i<nmax;i++)
{
fx=f.func(x);
dfx=f.dfunc(x);
x-=fx/dfx;
if(Math.abs(fx)<tolerance) { return x;}
}
JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration is exceeded \n"+
   "result might not be valid","MAXİMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public static void main (String args[]) throws java.io.IOException
{  double T=300.0;  //deg K
   double P=0.2e6;  //Pa
   f2 f=new f2("oxygen",5.043e6,154.58,31.999,P,T);
   double v1=f.R*T/P; //ideal gas;
   double P1=f.func(v1);
   double v=secant(f,v1);
   System.out.println("ideal gas v = "+v1+" m^3/kg");
   System.out.println("van der Waals v = "+v+" m^3/kg");
}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\java.exe" vanderWaals1
ideal gas v = 0.3897543673239789 m^3/kg
van der Waals v = 0.3890197822476924 m^3/kg
> Terminated with exit code 0.
```

```
from math import *
from f_x import *;

class f2(f_x):
    def __init__(self,name,Pc,Tc,M,P,T):
        self.name = name,
        self.T=T;
        self.P=P;
        self.R=8314.5/M;
        self.a=27.0/64.0*self.R*self.R*Tc*Tc/Pc;
```

```
        self.b=self.R*Tc/(8.0*Pc)

    def func(self,v):
        x=self.R*self.T/(v-self.b)-self.a/(v*v)-self.P
        return x;
    def setT(T):
        self.T=T;
    def setP(P):
        self.P=P;

    def Haaland(self):
        X=-1.8*log10(pow(self.eod,1.11)/3.7+6.9/self.Re);
        return X;

def secant(f,x):
  # secant root finding method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
      fx= f.func(x);
      dfx=f.dfunc(x,1);
      x-=fx/dfx
      if abs(fx)<tolerance: return x
  return x


T=300.0;  #deg K
P=0.2e6;  #Pa
f=f2("oxygen",5.043e6,154.58,31.999,P,T);
v1=f.R*T/P; #ideal gas;
v=secant(f,v1);
print("ideal gas v = ",v1," m^3/kg");
print("van der Waals v = ",v," m^3/kg");
```

```
runfile('D:/okul/SCO1/vanderWaals.py', wdir='D:/okul/SCO1')
Reloaded modules: f_x
ideal gas v =  0.3897543673239789  m^3/kg
van der Waals v =  0.3890197822476924  m^3/kg
```

PROBLEM 13

Find the root of function f(x)=x $e^{-x/2}$ + x^3 + 1 with the limits of x_1=-1, x_2=0, x_3=1 ny using muller method.

PROBLEM 14

Find the root of function f(x)=x^4-20x^3-25x^2+100x+130 by using Leguerre method.

PROBLEM 15

Find the root of fuction f(x)=x^2+1.

PROBLEM 16

For a paralel flow heat exchanger Number of transfer unit(NTU) and effectiveness (ε) equation given as follows

$$\varepsilon = \frac{1 - exp[-NTU(1 + C^*)]}{(1 + C^*)}$$

After the experiments for a specific heat exchanger NTU=1.989 and ε = 0.574 is found from the experiment

a) Determine the range C^* . For this start C^*=0.1 and increase 0.1 in each step
b) Calculate C^* by using false position method.

PROBLEM 17

Two end of 5 cm long plate with crossection of 0.1cm x 20 cm plate is held at 80 °C ve 60 °C. The environmental temperature is T_∞ = 10 °C, convective heat transfer coefficient h=15 W/m^2 K. Bar cross sectional area A=2×10^{-4} m^2, and perimeter P=0.402 m.In the experiment mid point(x=0.025 m) temperature is found as T=67.5 °C. In this conditions, temperature distribution in the bar is given as

$$T - T_\infty = 68.7e^{-0.025m} + 11.3e^{0.025m}$$

In this equation m is defined as: $m = (\frac{hP}{kA})^{0.5}$

a) Find the range of m, stating from m=5, and increasing 5 at each step.
b) Solve m by using Newton-Raphson method
c) Determine thermal conductivity coefficient k from this value.

PROBLEM 18

The cost of the heat exchanger(M), is defined as a function of the length of heat excahger, L, diameter of the heat exchanger, D as: $M=900+1100\ D^{2,5}\ L+320\ D\ L$ In the optimization study the length of the heat exchanger is found as L=1.3 m and the cost as M=1900 TL. Find the diameter of the heat exchanger.

PROBLEM 19

An instrument 20000 TL worth is purcahased with no first payment and 4000 TL annual payment for 6 years. Calculate the interest rate applied by using secant method.

$$A = P\frac{i(i+1)^n}{(i+1)^n - 1}$$

In the equation P is the present worth, A is the annual payment, i is the interestrate, n is the number of payment years.

PROBLEM 20

In order to heat a substance to a required temperature the energy equation is given as

$Q = \int mC_p dT$ eşitliği ile bulunmaktadir.

A substance with an initial temperature of 30 °C, with a mass m=0.2 kg, is heated to some temperature and total energy spend is calculated as 3069.7 kJ . The specific heat of the substance is given with the equation $C_p = 41.87 + 1.6748 * T + 0.007537 * T^2$ (kJ/kg K). Find the final temperature by using bi-section method.

PROBLEM 21

Find the roots of $f(x) = x^3-6x^2+11x-6.1$ by using Legurre method

PROBLEM 22 Find the root(s) of the following equation in the range of [-3.5,3.5]

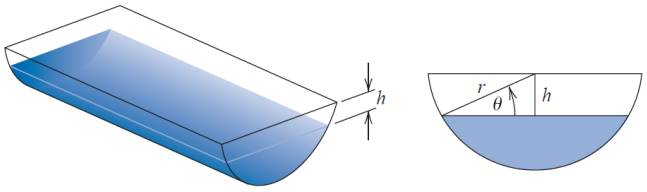
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2})x^2} + \frac{1}{10}\sin(\pi x)$$

PROBLEM 23 Find the root(s) of the following equation

$$f(x) = \left[10x\left(1 - e^{-\frac{0.0004}{2000x}}\right) - 0.00001\right]$$

PROBLEM 24 In celestial mechanics, **Kepler's equation** is important. It reads $\boldsymbol{x = y - \varepsilon \sin y}$, in which x is a planet's mean anomaly, y its eccentric anomaly, and ε the eccentricity of its orbit. Taking $\varepsilon = 0.9$, construct a table of y for 30 equally spaced values of x in the interval 0 to π. Use Newton's method to obtain each value of y. The y corresponding to an x can be used as the starting point for the iteration when x is changed slightly.

PROBLEM 25 A trough of length L has a cross section in the shape of a semicircle with radius r. When filled with water to within a distance h of the top, the volume V of water is



$$V = L\left[0.5\pi r^2 - r^2 \arcsin\left(\frac{h}{r}\right) - h(r^2 - h^2)^{1/2}\right]$$

Suppose L=1 m, r=0.25 m and V=0.05 m^3 Find the depth of water, h

PROBLEM 26 Following functions and root values are given. Investigate number of iteration they are taken when Newton, Halley, Abbasbandy, Thiele 1 and Thiele 2 methods are used.

$f(x) = x^3 + 4x^2 - 25 \qquad x^* = 2.035268481182$

$f(x) = x^3 - e^x - 3x + 2 \qquad x^* = 0.257530285439$

$f(x) = x^3 - 10 \qquad x^* = 2.154434690032$

$f(x) = \cos(x) - x \qquad x^* = 0.739085133215$

$f(x) = \sin^2(x) - x^2 + 1 \qquad x^* = 1.404491648215$

$f(x) = x^2 + \sin(x/5) - 1/4 \qquad x^* = 0.409992017989$

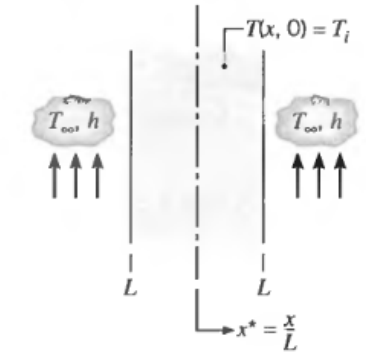
PROBLEM 27 (a) Use bisection to find the smallest positive zero of the function $f(x) = \cot(3x) - \dfrac{x^2 - 1}{2x}$

```
from math import *

def bisection(f,xl,xu):
  # bisection root finding method
  maxit=100
  iter=0
  es=0.0000001
  ea=1.1*es
  s=""
  fxl= f(xl)
  fxu= f(xu)
  xr=xl
  while ea>es and iter<maxit:
    xold=xr
    xr=(xl+xu)/2.0;
    fxr= f(xr);
    iter=iter+1;
    if xr!=0:
      ea=abs((xr-xold)/xr)*100
    test= fxl*fxr;
    if test==0.0 :      ea=0;
    elif test<0.0: xu=xr;fxu=fxr;
    elif test>0:    xl=xr;fxl=fxr;
    else :          ea=0;
    if iter>=maxit: print("Maximum number of iteration is exceeded \n result might not be valid")
  return xr

f1 = lambda x: 1.0/tan(3*x)-(x*x-1)/(2*x);
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=bisection(f1,a,b);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f1(r))
```

```
---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" bisection.py
enter lower limit : .1
enter upper limit : 3
 ROOT  :  0.6588271550484934
FUNCTION VALUE :  -5.676126790810088e-10

> Terminated with exit code 0.
```

(b) Did you remember to plot the function before using bisection?
c) repeat using the root by Newton-Raphson, Illinois, Halley, Abbasbandy , Thiele1 and Thiele 2 methods. Find the number of iteration it takes in each case.

PROBLEM 28 In time dependent heat transfer, heat transfer of a Wall where both sides are exposed to convective heat transfer with convective heat transfer coefficient h and temperature T_∞ is shown in the figure (reference Fundamentals of heat and mass transfer Incropera, DeWitt, Bergman**)**



Time dependent temperature difference for such a wall is given by the equation

$$\theta_0^* = \frac{T_0 - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} C_n \exp(-\xi_n^2 Fo)\cos(\xi_n x^*)$$

Where $Fo = \frac{\alpha t}{L^2}$ is Fourier number t is time(second), $\alpha = \frac{k}{\rho C_p}$ is thermal diffisuvity coefficient, L is the half length of the Wall. In order to solve coefficients ξ_n (eigenvalues) the following equation should be solved.

$\tan \xi_n - \frac{Bi}{\xi_n} = 0$ where $Bi = \frac{hL}{k}$ is Biot number, h is thermal covection coefficient and k is thermal conductivity of the Wall material. Assume Bi number is given and developed an algorithm to solve first 4 roots of the ξ_n. In order to help you The first 4 roots of the equation for various Bi number are given in the following table.

$Bi = \frac{hL}{k}$	ξ_1	ξ_2	ξ_3	ξ_4
0	0	3.1416	6.2832	9.4248
0.001	0.0316	3.1419	6.2833	9.4249
0.002	0.0447	3.1422	6.2835	9.4250
0.004	0.0632	3.1429	6.2838	9.4252
0.006	0.0774	3.1435	6.2841	9.4254
0.008	0.0893	3.1441	6.2845	9.4256
0.01	0.0998	3.1448	6.2848	9.4258
0.02	0.1410	3.1479	6.2864	9.4269
0.04	0.1987	3.1543	6.2895	9.4290
0.06	0.2425	3.1606	6.2927	9.4311
0.08	0.2791	3.1668	6.2959	9.4333
0.1	0.3111	3.1731	6.2991	9.4354
0.2	0.4328	3.2039	6.3148	9.4459
0.3	0.5218	3.2341	6.3305	9.4565
0.4	0.5932	3.2636	6.3461	9.4670
0.5	0.6533	3.2923	6.3616	9.4775
0.6	0.7051	3.3204	6.3770	9.4879
0.7	0.7506	3.3477	6.3923	9.4983
0.8	0.7910	3.3744	6.4074	9.5087
0.9	0.8274	3.4003	6.4224	9.5190
1.0	0.8603	3.4256	6.4373	9.5293
1.5	0.9882	3.5422	6.5097	9.5801
2.0	1.0769	3.6436	6.5783	9.6296
3.0	1.1925	3.8088	6.7040	9.7240
4.0	1.2646	3.9352	6.8140	9.8119
5.0	1.3138	4.0336	6.9096	9.8928

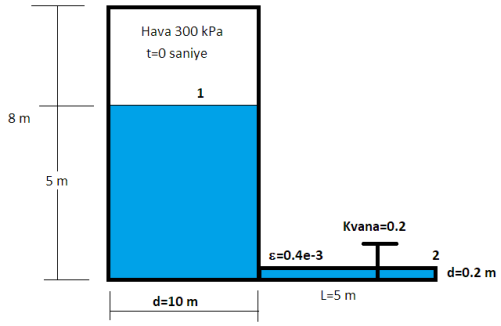
PROBLEM 29 Solve equation $3^x x^3 = 1$

PROBLEM 30 Calculate all roots of the following polynomials

$$f(x) = x^3 - x^2 + 2x - 2$$

$$f(x) = 3.704x^3 + 16.3x^2 - 21.97x + 9.34$$

PROBLEM 31A water container filled to H=5 m level. Total height of the container is HT=8 meters. At the top of water air with 300 kPa and 27 C existed. The top of the storage tank is closed. Diameter of the container is 10 m. Water is discharged through a pipe of d=0.2 m in diameter and L=5 m in length. There is a globe valve at this pipe (KL=0.2). Calculate velocity of the discharge water as a function of time




```
import javax.swing.JOptionPane;
import java.util.Locale;
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class sudeposu_boru_akis
{  public liquid_CS su;   // soğuk sıvı (boru içi)
   public Gas hava;       // deponun tepesindeki su
   public double T;       // sistem sıcaklığı
   public double Phava[]; // depo hava basıncı zamana bağlı
   public double mhava;   // depoda bulunan hava miktarı kg
   public double Patm;    // atmosfer basıncı
   public double Vhava[]; // depo hava hacmi zamana bağlı m^3
   public double eps;     // yüzey pürüzlülüğü
   public double d;       // boru iç çapı
   public double Do;      // depo çapı
   double L;       // boşaltma borusu boyu
   double eod;     // eps/di
   double H;       // depo yüksekliği
   double h[];     // zamana bağlı depo su seviyesi
   double V[];     // zamana bağlı hız
   double V2[];    // zamana bağlı depo içi su hızı
   double KL;      // yerel basınç düşümü
   int n_finite;   // sonlu farklar zaman step sayısı
   double Ahava;   // depo kesit alanı
   double A;       // boru kesit alanı
   double Aratio;  // A/Ahava
   double rohava,rosu;
   double g;  //yerçekimi sabiri 9.81 m/s^2
   double gammasu; //rosu*g
   public if_x ff;
   public double dt; //zaman adımı
 public sudeposu_boru_akis(double Ti,double Pi,double Patmi,double epsi,double di,double Doi, double Li,double Hi,double H1i,int
n_finitei,double KLi)
 {    n_finite=n_finitei;
              su=new liquid_CS("su");
   hava=new Gas("air");
   T=Ti;
   double rohava=hava.M/hava.v((T+273.15),Pi*1e-2);
   double rosu=su.ro(T);
   eps=epsi;
   d=di;
   Do=Doi;
   eod=eps/d;
   L=Li;
   H=Hi;
   h=new double[n_finite+1];
   h[0]=H1i;
   Vhava=new double[n_finite+1];

   V=new double[n_finite+1];
   V2=new double[n_finite+1];
```

```
  Phava=new double[n_finite+1];
  Phava[0]=Pi;
  Patm=Patmi;
  KL=KLi;
  Ahava=Math.PI*Do*Do/4.0;
  A=Math.PI*di*di/4.0;
  Aratio=A/Ahava;
  Vhava[0]=Ahava*(H-h[0]);
  mhava=Vhava[0]*rohava;
  System.out.println("mhava = "+mhava+" kg");
  System.out.println("rohava = "+rohava+" kg/m^3");
  System.out.println("Vhava = "+Vhava[0]+" m^3");
  g=9.81; //m/s^2
  gammasu=rosu*g;
  dt=1;
 }
 // boru içi
 public double f(double Re)
 {
 //Goudar equation
 double a=2/Math.log(10);
 double b=eod/3.7;
 double d=Math.log(10.0)/5.02*Re;
 double s=b*d+Math.log(d);
 double q=Math.pow(s,(s/(s+1)));
 double g=b*d+Math.log(d/q);
 double z=Math.log(q/g);
 double dLA=(g/(g+1))*z;
 double dCFA=dLA*(1+z/2.0/((g+1)*(g+1)+(z/3)*(2.0*g-1)));
 double f1=a*(Math.log(d/q)+dCFA);
 f1=1.0/(f1*f1);
 return f1;
 }
// boru dışı ısı transferi
public double Reout(double t,double v)
{
//======= Reynold sayısı boru dışı ==========
//x : % karışım miktarı
//t sıvı sıcaklığı
//boru geçiş alanı
double Dh=d;
double ro=su.ro(t);
double vis=su.vis(t);
//System.out.println("vis="+vis/ro);
double res=v*ro*Dh/vis;
return res;
}
 public static double secant(if_x f,double x0)
 { int nmax=500;
   double tolerance=1.0e-10;
   double fx1,dfx1;
   double x=x0;
   for(int i=0;i<nmax;i++)
   { fx1=f.func(x);
     dfx1=f.dfunc(x);
     x=x-fx1/dfx1;
     if(Math.abs(fx1)<tolerance) { return x;}
   }
   JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n","MAXİMUM ITERATION
NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
 return x;
 }
 public void calculate()
 {
                for(int i=0;i<n_finite;i++)
    {double pp=Phava[i]*1e3;
     double vv=10.0;
     double hh=h[i];

     ff=v->pp/gammasu+Aratio*v*Aratio*v/(2.0*g)+hh-(Patm*1e3/gammasu+v*v/(2.0*g)+(f(Reout(T,v))*L/d+KL)*v*v/(2.0*g));
     V[i]=secant(ff,vv);
     V2[i]=Aratio*V[i];
     double Re=Reout(T,V[i]);
     double ff=f(Re);
     System.out.println("t="+i*dt+"s V="+V[i]+"m/s h = "+h[i]+" m f = "+ff+" Re = "+Re);
     Vhava[i+1]=Vhava[i]+dt*V2[i]*Ahava;
```

```
      rohava=mhava/Vhava[i+1];
      Phava[i+1]=mhava*(8314.5/28.97)*(T+273.15)/Vhava[i+1]*1e-3;
      h[i+1]=h[i]-dt*V2[i];
     }
 }
 public static void main(String arg[])
 { sudeposu_boru_akis sdba=new sudeposu_boru_akis(20.0,300.0,100.0,0.4e-3,0.2,10.0,5.0,8.0,5.0,120,0.2);
   sdba.calculate();
 } }
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" sudeposu_boru_akis
mhava = 839.9778757312198 kg
rohava = 3.5649768269454247 kg/m^3
Vhava = 235.61944901923448 m^3
t=0.0s V=16.708041102062847m/s h = 5.0 m f = 0.023476798315720825 Re = 3342389.2009554342
t=1.0s V=16.681425413399687m/s h = 4.993316783559175 m f = 0.023476887797448397 Re = 3337064.8191311196
t=2.0s V=16.656929405513416m/s h = 4.986644213393815 m f = 0.02347697040454885 Re = 3332164.472541986
t=3.0s V=16.63253191594394m/s h = 4.979981441631609 m f = 0.023477052920333044 Re = 3327283.8342212336
t=4.0s V=16.608232150184605m/s h = 4.973328428865232 m f = 0.023477135345636377 Re = 3322422.745230393
t=5.0s V=16.58402932285497m/s h = 4.966685136005158 m f = 0.02347721768128694 Re = 3317581.0484566656
t=6.0s V=16.559922657564837m/s h = 4.960051524276016 m f = 0.02347729992810565 Re = 3312758.5885857213
t=7.0s V=16.53591138678078m/s h = 4.953427555212991 m f = 0.023477382086906338 Re = 3307955.2120749974
t=8.0s V=16.51199475169512m/s h = 4.946813190658278 m f = 0.023477464158495845 Re = 3303170.767127491
t=9.0s V=16.48817200209729m/s h = 4.9402083927576 m f = 0.023477546143674078 Re = 3298405.10366602
t=10.0s V=16.464442396247446m/s h = 4.933613123956761 m f = 0.02347762804323416 Re = 3293658.0733079454
t=11.0s V=16.440805200752475m/s h = 4.927027346998262 m f = 0.02347770985796245 Re = 3288929.529340364
t=12.0s V=16.417259690444155m/s h = 4.920451024917961 m f = 0.023477791588638663 Re = 3284219.326695734
t=13.0s V=16.393805148259506m/s h = 4.913884121041783 m f = 0.02347787323603596 Re = 3279527.3219279437
t=14.0s V=16.370440865123307m/s h = 4.907326598982479 m f = 0.02347795480092101 Re = 3274853.3731888034
t=15.0s V=16.347166139832648m/s h = 4.90077842263643 m f = 0.023478036284054097 Re = 3270197.3402049546
t=16.0s V=16.323980278943615m/s h = 4.894239556180497 m f = 0.023478117686189164 Re = 3265559.0842552017
t=17.0s V=16.300882596659886m/s h = 4.88770996406892 m f = 0.023478199008073942 Re = 3260938.468148224
t=18.0s V=16.27787241472335m/s h = 4.881189611030256 m f = 0.023478280250449957 Re = 3256335.3562006964
t=19.0s V=16.254949062306622m/s h = 4.874678462064367 m f = 0.02347836141405269 Re = 3251749.614215784
t=20.0s V=16.232111875907446m/s h = 4.868176482439444 m f = 0.023478442499611594 Re = 3247181.1094620223
t=21.0s V=16.209360199244927m/s h = 4.861683637689081 m f = 0.02347852350785017 Re = 3242629.7106525563
t=22.0s V=16.18669338315761m/s h = 4.855199893609384 m f = 0.023478604439486083 Re = 3238095.287924752
t=23.0s V=16.16411078550325m/s h = 4.8487252162561205 m f = 0.023478685295231182 Re = 3233577.712820144
t=24.0s V=16.141611771060393m/s h = 4.842259571941919 m f = 0.023478766075791614 Re = 3229076.858264749
t=25.0s V=16.119195711431583m/s h = 4.835802927233495 m f = 0.023478846781867867 Re = 3224592.598549703
t=26.0s V=16.09686198494829m/s h = 4.8293552489489215 m f = 0.023478927414154842 Re = 3220124.809312242
t=27.0s V=16.074609976577428m/s h = 4.822916504154942 m f = 0.023479007973341934 Re = 3215673.3675170005
t=28.0s V=16.052439077829458m/s h = 4.816486660164311 m f = 0.023479088460113064 Re = 3211238.151437627
t=29.0s V=16.0303486866681m/s h = 4.810065684533179 m f = 0.023479168875146828 Re = 3206819.040638726
t=30.0s V=16.008338207421517m/s h = 4.803653545058512 m f = 0.023479249219116427 Re = 3202415.915958087
t=31.0s V=15.98640705069504m/s h = 4.797250209775544 m f = 0.023479329492689875 Re = 3198028.6594892256
t=32.0s V=15.964554633285358m/s h = 4.790855646955266 m f = 0.023479409696529972 Re = 3193657.154564218
t=33.0s V=15.942780378096115m/s h = 4.784469825101952 m f = 0.02347948983129437 Re = 3189301.285736819
t=34.0s V=15.921083714054966m/s h = 4.778092712950714 m f = 0.02347956989763568 Re = 3184960.938765866
t=35.0s V=15.899464076031977m/s h = 4.771724279465092 m f = 0.02347964989620149 Re = 3180636.000598957
t=36.0s V=15.877920904759419m/s h = 4.765364493834679 m f = 0.02347972982763444 Re = 3176326.3593564034
t=37.0s V=15.856453646752842m/s h = 4.759013325472775 m f = 0.023479809692572292 Re = 3172031.9043154437
t=38.0s V=15.835061754233509m/s h = 4.7526707440140745 m f = 0.02347988949164796 Re = 3167752.5258947266
t=39.0s V=15.813744685052056m/s h = 4.7463367193123815 m f = 0.02347996922548958 Re = 3163488.115639038
t=40.0s V=15.792501902613447m/s h = 4.740011221438361 m f = 0.023480048894720596 Re = 3159238.566204288
t=41.0s V=15.771332875803136m/s h = 4.733694220677315 m f = 0.02348012849995975 Re = 3155003.771342742
t=42.0s V=15.75023707891443m/s h = 4.727385687526994 m f = 0.02348020804182119 Re = 3150783.625888488
t=43.0s V=15.72921399157705m/s h = 4.721085592695428 m f = 0.023480287520914515 Re = 3146578.025743147
t=44.0s V=15.708263098686835m/s h = 4.714793907098797 m f = 0.023480366937844808 Re = 3142386.8678618087
t=45.0s V=15.687383890336601m/s h = 4.708510601859323 m f = 0.023480446293212703 Re = 3138210.0502392026
t=46.0s V=15.666575861748093m/s h = 4.702235648303188 m f = 0.023480525587614428 Re = 3134047.4718960826
t=47.0s V=15.645838513205042m/s h = 4.695969017958489 m f = 0.02348060482164186 Re = 3129899.0328658377
t=48.0s V=15.6251713499873m/s h = 4.689710682553207 m f = 0.023480683995882597 Re = 3125764.634181312
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```

```
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> Terminated with exit code 0.
```

3 Linear system of equations & solution Methods

3.1 LINEAR SYSTEM OF EQUATIONS GENERAL INFORMATIONS

Linear system of equations is a collection of linear equations of the same set of variables. For example

$$3x+2y-z=1$$
$$2x-2y+4z=-2$$
$$-x+0.5y-z=0$$

Is system of three equation in 3 variables x,y and z. A solution to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied. A solution to the system above is given by

x=1,y=-2,z=-2

This system of equations can be written in matrix multiplication form as:

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & 0.5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

In more general form linear system of equations are in the following form:

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 + \ldots + a_{1n} X_n = b_1$$
$$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 + \ldots + a_{2n} X_n = b_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 + \ldots + a_{3n} X_n = b_3$$
.....
$$a_{n1} X_1 + a_{n2} X_2 + a_{n3} X_3 + \ldots + a_{nn} X_n = b_n$$

The same set of equation can be given in matrix notation

$[A]\{X\}=\{B\}$ or

$\vec{A}\vec{x}=\vec{B}$

If matrix notation is written in open form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \ldots \\ X_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

In some sources the same system of equation can also be given as:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right] \text{ or } \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{bmatrix}$$

In this set A and B is known values and X is unknown vector. Finding of X values are colled solution of system of equations.

3.2 GAUSS ELIMINATION METHOD

Gauss elimination method based on the properties of the linear system of equations. Changing the matrix rows, adding one row to another, or multiplying one row with a constant will not change the solution values of the system of equations. Therefore a process such as

(row i)- constant * (row j) can be easily applied without changing the solution set of the linear system of equations. The purpose of this manipulations is to change the A matrix to a upper diagonal matrix. For example if n=5, system of equation will take the following upper diagonal form

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & a'_{14} & a'_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a'_{33} & a'_{34} & a'_{35} \\ 0 & 0 & 0 & a'_{44} & a'_{45} \\ 0 & 0 & 0 & 0 & a'_{55} \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 \\ X_1 \\ X_1 \\ X_1 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \\ b'_5 \end{bmatrix}$$

A systematic elimination processs is applied for this purpose. For example in order to change element a_{21} to 0 in the equation

$a'_{21} = 0 = a_{21} - C_1 * a_{11}$

a constant with the value $C_1=(a_{21}/a_{11})$ is required. Then each member of the second row will be multiply with the same constant and take a new value.

$a'_{22} = a_{22} - C_1 * a_{12}$

$a'_{23} = a_{23} - C_1 * a_{13}$

.....

$a'_{25} = a_{25} - C_1 * a_{15}$

$b'_2 = b_2 - C_1 * b_1$

At the end of this process a'_{21} become zero. The same process repeat first repeat the whole rows under element a_{11} and all first column rows except the first one become zero. Then the same process repeat for the rows under the diagonal element a_{22} . A for structure doing this process will look like as follows

```
for(k=0;k<(n-1);k++)
 {   for(i=k+1;i<n;i++)
       {  C[k]=a[i][k]/a[k][k];
          a[i][k]=0;
          for(j=k+1;j<n;j++)  {   a[i][j]-= C[k]*a[k][j]; }
          b[i]    =b[i]  - C[k]*b[k];
       }}
```

After the linear system of equation converted into upper diagonal form, unnowns are solved by using back-substitution method. The set of the back-substitution for the set given above(n=5) will be:

$X_5 = b'_5 / a'_{55}$

$X_4 = (b'_4 – a'_{45}*X_5)/a'_{44}$

$X_3 = (b'_3 – a'_{34}*X_4 – a'_{35}*X_5)/a'_{33}$

$X_2 = (b'_2 – a'_{23}*X_3 – a'_{24}*X_3 – a'_{25}*X_5)/a'_{22}$

$X_1 = (b'_1 – a'_{12}*X_2 – a'_{13}*X_3 – a'_{14}*X_4 – a'_{15}*X_5)/a'_{11}$

Java code equivalent for an n by n matrix

```
 x[n-1]=b[n-1]/a[n-1][n-1];
 for(i=n-2;i>=0;i--)
 {
  toplam=0;
  for(j=i+1;j<n;j++)
  {  toplam+=a[i][j]*x[j];}
  x[i]=(b[i]-toplam)/a[i][i];
 }
```

A matrix output class is also created in here to get easy outputs from the matrices.

PROGRAM matris output program matrixoutput.java to format the system of equation (java code)

```
import java.io.*;
import java.text.*;
import java.util.Locale;

public class matrixoutput
{
public static String toString(double left, int w, int d)
// convert double into a w digit string with d digits in real part
{
            NumberFormat df=NumberFormat.getInstance(Locale.US);
            df.setMaximumFractionDigits(d);
   df.setMinimumFractionDigits(d);
   df.setGroupingUsed(false);
   String s = df.format(left);
   while (s.length() < w)
     s = " " + s;
   if (s.length() > w)
   {
      s = "";
      for (int i=0; i<w; i++)
        s = s + "-";
   }
   return s;
}

public static String toString(double left)
{return toString(left,25,15);}

public static String toString(double[][] left,int w,int d)
{
//string equivalent of a double matris
int n,m;
```

```
String b;
b="";
n=left.length;
m=left[0].length;
for(int i=0;i<n;i++)
  {
  for(int j=0;j<m;j++)
    {
    b=b+toString(left[i][j],w,d);
    }
    b=b+"\n";
  }
return b;
}

public static String toString(double[][] a,double b[],int w,int d)
{//string equivalent of a lineear system of equation
int n,m;
String s;
s="";
n=a.length;
m=a[0].length;
for(int i=0;i<n;i++)
  {
  s+=" |";
  for(int j=0;j<m;j++)
    {
    s+=toString(a[i][j],w,d);
    }
    s+="| "+toString(b[i],w,d)+"| \n";
  }
return s;
}

public static String toString(double[][] a,double b[])
{//string equivalent of a lineear system of equation
return toString(a,b,25,15);
}

public static String toStringT(double[][] left,int w,int d)
{ return toString(Matrix.Transpose(left),w,d);}

public static String toStringT(double[][] left)
{ return toString(Matrix.Transpose(left),15,10);}


public static String toStringT(double[] left)
{
// returns a vertical string representation
// of a double vector
int n,m;
String b;
b="";
n=left.length;
for(int i=0;i<n;i++)
  {
    b=b+toString(left[i])+"\n";
  }
return b;
}

public static String toString(double[] left)
{
// returns a horizontal string representation
// of a double vector
int n,m;
String b;
b="";
n=left.length;
for(int i=0;i<n;i++)
```

```
  {
    b=b+toString(left[i])+"\t ";
  }
  b=b+"\n";
return b;
}}
```

Now A complete Gauss elimination code can be developed

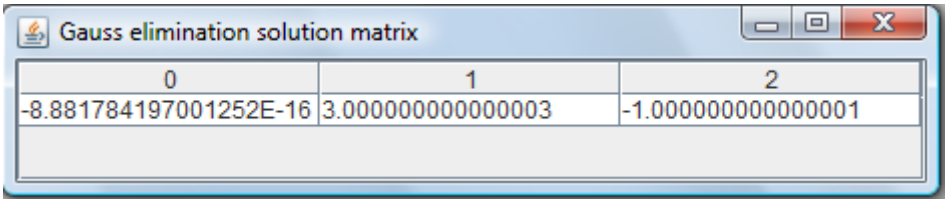
PROGRAM Gauss elimination (java code)

```
import java.io.*;

class NA14
{
  public static double[] gauss(double a[][],double b[])
  {
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  //gauss eleme
  int i,j,k;
  System.out.println("Original matrix :\n"+Matrix.toString(a));
  for(k=0;k<(n-1);k++)
  {
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  System.out.println("Eliminated matrix :\n"+Matrix.toString(a));
  //back substituting
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{1,2,5},{2,1,2},{3,1,1}};
  double b[]={1,1,2};

  double x[]=gauss(a,b);
  Text.print(x,"Gauss elimination solution matrix");
  }
}
```

output Gauss elimination process

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA14
Original matrix :
       1.000000000000000       2.000000000000000       5.000000000000000
       2.000000000000000       1.000000000000000       2.000000000000000
       3.000000000000000       1.000000000000000       1.000000000000000
Eliminated matrix :
       1.000000000000000       2.000000000000000       5.000000000000000
       0.000000000000000      -3.000000000000000      -8.000000000000000
       0.000000000000000       0.000000000000000      -0.666666666666666
       0.000000000000000       0.000000000000000      -0.666666666666666
```

> Terminated with exit code 0.



In the program following set of system of equation is taken :

$$\begin{bmatrix}1 & 2 & 5\\ 2 & 1 & 2\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$$

The solution set of this set of equation is 0,3 ve -1. As it is seen from the results, a small error is remained in the solution set. If the same elinination process is carried out by hand:

column 0(column indices starts from 0), row 1(raw indices starts from 0) multiplier $C_1=a_{10}/a_{00} = 2/1=2$
$a_{10} = a_{10} - C_1 * a_{00} = a_{10} - (a_{10}/a_{00})* a_{00} = 2 - 2*1 = 0$
$a_{11} = a_{11} - C_1 * a_{01} = a_{11} - (a_{10}/a_{00})* a_{01} = 1 - 2*2 = -3$
$a_{12} = a_{12} - C_1 * a_{02} = a_{12} - (a_{10}/a_{00})* a_{02} = 2 - 2*5 = -8$
$b_2 = b_2 - C_1 * b_1 = b_2 - (a_{10}/a_{00})* b_1 = 1 – 2*1= -1$
column0 raw 1 elimination is completed
colun0 raw 2 elimination:
$C_1=a_{20}/a_{00} = 3/1=3$
$A_{20} = a_{20} – C_2 * a_{00} = a_{20} - (a_{20}/a_{00})* a_{00} = 3 - 3*1 = 0$
$a_{21} = a_{21} – C_2 * a_{01} = a_{21} - (a_{20}/a_{00})* a_{01} = 1 - 3*2 = -5$
$a_{22} = a_{22} – C_2 * a_{02} = a_{22} - (a_{20}/a_{00})* a_{02} = 2 - 3*5 = -14$
$b_2 = b_2 - C_2 * b_1 = b_2 - (a_{10}/a_{00})* b_1 = 1 – 3*1= -1$
column 1, row 2 multiplier C
$C_3=a_{21}/a_{11} = -5/-3=(5/3)$
$a_{21} = a_{21} – C_3 * a_{11} = a_{21} - (a_{21}/a_{11})* a_{11} = -5 – (5/3)*(-3) = 0$
$a_{22} = a_{22} - C_3 * a_{12} = a_{11} - (a_{21}/a_{11})* a_{01} = -14 – (5/3)*(-8) = -0.6666666666666$
$b_2 = b_2 - C_3 * b_1 = b_2 - (a_{21}/a_{11})* b_1 = -1 – (5/3)*(-1) = 0.6666666666666$
system of equation becomes:

$$\begin{bmatrix}1 & 2 & 5\\ 0 & -3 & -8\\ 0 & 0 & -0.66666\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 0.66666\end{bmatrix}$$

Back-substitution process :
$X_2 = b_2/a_{22} = 0.6666666666/(-0.6666666666) = -1$
$X_1 = (b_1 - a_{12}*X_2)/a_{11} = (-1 - (-8)*(-1))/(-3) =(-9)/(-3)=3$
$X_0 = (b_0 - a_{01}*X_1 - a_{02}*X_2)/a_{00} = (1 – 2*3-5*(-1))/1=0$

$$\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0\\ 3\\ -1\end{bmatrix}$$

Another example for Gauss elimination with detailed steps :

$$\begin{bmatrix}7 & 1 & 2\\ -1 & 4 & -1\\ 3 & 15 & 20\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}47\\ 19\\ 87\end{bmatrix}$$

	A			B
	7	1	2	47
	-1	4	-1	19
	3	15	20	87

c				
	7	1	2	47
-0.14286	0	4.142857	-0.71429	25.71429
0.428571	0	14.57143	19.14286	66.85714

c				
3.517241	7	1	2	47
	0	4.142857	-0.71429	25.71429
	0	0	21.65517	-23.5862

X_0	**6.165605**
X_1	**6.019108**
X_2	**-1.08917**

In excel or Octave gauss elimination can also be carried out by using MMULTIPLY and MINVERSE functions (DİZEY_TERS and DÇARP in turkish version of excel)

= MINVERSE (B3:D5) ctrl-shift-enter = MMULTIPLY(B7:D9;E3:E5) ctrl-shift-enter	=DİZEY_TERS(B3:D5) ctrl-shift-enter =DÇARP(B7:D9;E3:E5) ctrl-shift-enter

$$\begin{bmatrix} -8 & 1 & 2 \\ 5 & 7 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Input Matrix

-8.000E+00	1.000E+00	2.000E+00	0.000E+00
5.000E+00	7.000E+00	-3.000E+00	1.000E+01
2.000E+00	1.000E+00	-2.000E+00	-2.000E+00

Inverse Matrix

-1.486E-01	5.405E-02	-2.297E-01
5.405E-02	1.622E-01	-1.892E-01
-1.216E-01	1.351E-01	-8.243E-01

turkish fx	english fx	
DİZEY_TERS	MINVERSE	ctrl-shift-enter

Solution

1.000E+00
2.000E+00
3.000E+00

DÇARP	MMULTIPLY	ctrl-shift-enter

The same program can be given as Matlab(Octave) code as:

PROGRAM 3 Gauss elimination in Octave(Matlab)

```
function x=gauss(a,b)
% Gauss elimination
n=length(b);
carpan=0;
toplam=0;
 %gauss eleme
 for k=1:n
  for i=k+1:n
   carpan=a(i,k)/a(k,k);
    a(i,k)=0;
   for j=k+1:n
     a(i,j)=a(i,j)-carpan*a(k,j);
   end
      b(i)   =b(i)   -carpan*b(k);
  end
 end
 %back substituting
 x(n)=b(n)/a(n,n);
 for i=n-1:-1:1
  toplam=0;
  for j=i+1:n
   toplam=toplam+a(i,j)*x(j);end
  x(i)=(b(i)-toplam)/a(i,i);
 end
end
```

```
>> a=[7 1 2;-1 4 -1;3 15 20]
a =
    7    1    2
   -1    4   -1
    3   15   20
>> b=[47 19 87]'
b =
   47
```

```
    19
    87
>> x=gauss(a,b)
x =
    6.1656    6.0191   -1.0892

>> xx=a\b
xx =
    6.1656
    6.0191
   -1.0892
>>
```

The same program can be given as Python code as:

PROGRAM Gauss Elimination process in Python language:

```
from math import *

def gauss(a,b):
  n=len(a)
  x=[]
  print("origial matrix :\na =",a,"\nb =",b)
  for i in range(n):
    x.append(0.0)
  for k in range(len(a)):
    for i in range(k+1,n):
      carpan=a[i][k]/a[k][k]
      a[i][k]=0;
      for j in range(k+1,n):
        a[i][j]=a[i][j]-carpan*a[k][j]
      b[i]   =b[i]-carpan*b[k]
  print("eliminated matrix :\na =",a,"\nb =",b)
  x[n-1]=b[n-1]/a[n-1][n-1]
  for i in range(n-1,0,-1):
    toplam=0;
    for j in range(i+1,n):
      toplam=toplam+a[i][j]*x[j]
      x[i]=(b[i]-toplam)/a[i][i]
  return x;

a=[[1,2,5],[2,1,2],[3,1,1]]
b=[1,1,2]
x=gauss(a,b);
print("Gauss elimination solution matrix :\nx = ",x)
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" NA14.py
origial matrix :
a = [[1, 2, 5], [2, 1, 2], [3, 1, 1]]
b = [1, 1, 2]
eliminated matrix :
a = [[1, 2, 5], [0, -3.0, -8.0], [0, 0, -0.6666666666666661]]
b = [1, -1.0, 0.6666666666666667]
Gauss elimination solution matrix :
x =  [0.0, 3.000000000000003, -1.000000000000001]

> Terminated with exit code 0.
```

PROGRAM Gauss elimination in C++ using pointers to pass matrix and vector to method

```
#include <iostream>
#include <cmath>
#include <vector>

using namespace std;

void print(double *a,double *b,int n,int m)
{
   //int n = a.size();
   for (int i=0; i<n; i++) {
//       for (int j=0; j<m; j++) {cout << *(a+i*n+j) << "\t";}
       for (int j=0; j<m; j++) {cout << *(a+i*n+j) << "\t";}
       cout << "| "<<*(b+i)<<"\n";
```

```
    }
    cout<<"\n";
}

void print(double *b,int m) {
    for (int j=0; j<m; j++) {
        cout <<*(b+j)<< "\t";
    }
    cout<<"\n";
}
double* pivotlugaus(double *a,double *b,int n)
{ //gauss elimination with partial pivoting
  double x[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy;
  //gauss eleme
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //Solving gauss elimination
    for(i=k+1;i<n;i++)
    {  carpan=(*(a+i*n+k))/(*(a+k*n+k));
       *(a+i*n+k)=0;
       for(j=k+1;j<n;j++)
       {   *(a+n*i+j)-=carpan**(a+n*k+j); }
          *(b+i)   -=carpan**(b+k) ;
    }
  }
  //back substitution
  cout<<"elimination matrix\n";
  print(a,b,n,n);
  *(x+n-1)=(*(b+n-1))/(*(a+n*(n-1)+(n-1)));
  cout<<"i="<<(n-1)<<"x="<<(*(x+n-1))<<endl;
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=(*(a+n*i+j))*(*(x+j));}
    *(x+i)=((*(b+i))-toplam)/(*(a+i*n+i));
    cout<<"i="<<i<<"x="<<(*(x+i))<<endl;
  }
  print(x,n);
  return x;
  }
int main()
{
                int n=3;//row
    int m=3;//column
    double a[n][m]={{1,2,5},{2,1,2},{3,1,1}};
    double b[n]={1,1,2};
    double *x=pivotlugaus(&a[0][0],&b[0],n);
    cout<<"x[0]="<<x[0]<<" x[1]= "<<x[1]<<endl;
}
```

```
---------- Capture Output ----------
> "D:\SCO1\run.bat" gauss

D:\co\SCO_CPP>gauss
elimination matrix
1           2           5           | 1
0           -3          -8          | -1
0           0           -0.666667 | 0.666667
i=2x=-1
i=1x=3
i=0x=0
> Terminated with exit code -1073741819.
```

Gauss elimination can be applied also to complex variables. A complex version of the Gauss elimination routine is given below:

PROGRAM 3.2-6 complex Gauss elimination

```
  import java.io.*;
```

```
class NA14C
{
  public static c[] c_gauss(c a[][],c b[])
  {
  int n=b.length;
  c x[]=new c[n];
  c carpan=new c(1.0,0.0);
  c toplam=new c(0.0,0.0);
  //gauss eleme
  int i,j,k;
  for(k=0;k<(n-1);k++)
  {
    for(i=k+1;i<n;i++)
    {  carpan=c.p(a[i][k],"/",a[k][k]);
       a[i][k]=new c(0.0,0.0);
       for(j=k+1;j<n;j++)
       {   a[i][j]=c.p(a[i][j],"-",c.p(carpan,"*",a[k][j])); }
           b[i]=c.p(b[i],"-",c.p(carpan,"*",b[k]));
                }
  System.out.println("k = "+k+" :\n"+c.toString(a));
  }
  //back substituting
  x[n-1]=c.p(b[n-1],"/",a[n-1][n-1]);
  for(i=n-2;i>=0;i--)
  {
    toplam=new c(0.0,0.0);
    for(j=i+1;j<n;j++)
    {  toplam=c.p(toplam,"+",c.p(a[i][j],"*",x[j]));}
    x[i]=c.p(c.p(b[i],"-",toplam),"/",a[i][i]);
  }
  return x;
  }


  public static void main(String args[]) throws IOException
  {
  String s1="{(1.2,2.3),(1.1,-1.543),(3.77,0)},{(-1,1),(3.25,0),(1,0)},{(-1.75,0),(1.25,0.5),(1,0.7)}";
  String s2="{(1.0,2.0),(1.1,0),(1.2,1.3)}";
  c a[][]=c.toComplexMatrix(s1);
  c b[]=c.toComplexVector(s2);
  System.out.println("Original matrix :\n"+c.toString(a));
  c x[]=c_gauss(a,b);
  System.out.println("x = \n"+c.toString(x));
  }
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\javaw.exe" NA14C
Original matrix :
  (  1.200000 +   2.300000*i )   (  1.100000 -   1.543000*i )   (  3.770000 +   0.000000*i )
  ( -1.000000 +   1.000000*i )   (  3.250000 +   0.000000*i )   (  1.000000 +   0.000000*i )
  ( -1.750000 +   0.000000*i )   (  1.250000 +   0.500000*i )   (  1.000000 +   0.700000*i )

k = 0 :
  (  1.200000 +   2.300000*i )   (  1.100000 -   1.543000*i )   (  3.770000 +   0.000000*i )
  (  0.000000 +   0.000000*i )   (  2.267756 -   0.319866*i )   (  0.383804 -   1.960624*i )
  (  0.000000 +   0.000000*i )   (  0.670420 -   0.639346*i )   (  2.176374 -   1.554718*i )

k = 1 :
  (  1.200000 +   2.300000*i )   (  1.100000 -   1.543000*i )   (  3.770000 +   0.000000*i )
  (  0.000000 +   0.000000*i )   (  2.267756 -   0.319866*i )   (  0.383804 -   1.960624*i )
  (  0.000000 +   0.000000*i )   (  0.000000 +   0.000000*i )   (  2.511972 -   0.819554*i )

x =
  ( -0.809176 -   0.009645*i )   ( -0.088646 -   0.064709*i )   (  0.569280 +   1.009835*i )

> Terminated with exit code 0.
```

3.3 GAUSS ELIMINATION WITH PARTIAL PIVOTING

As it is seen from the previous example some error existed in the solution set. It is called round off error. When gauss elimination process is used as it is defined above some error might be created by the process. In order to minimize the round off error, absolute value of the multiplier terms should be as big as possible. In the

calculation process of the multiplier if divisor is zero term will go to infinity, if divisior is relatively small it will be a very big number, and this will cause an extra error in digital computers. In order to prevent this in a degree, elimination raw is interchanged with the raw with maximum absolute value for the multiplier term (in the diagonal). This process is called partial pivoting. Partial pivoting is shown in the example

$$\begin{bmatrix}1 & 2 & 5\\ 2 & 1 & 2\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$$

Before starting the gauss elimination the column 0 elements are compared and raw with the largest absolute value of this column is replace with elimination raw (raw 0)

$$\begin{bmatrix}3 & 1 & 1\\ 2 & 1 & 2\\ 1 & 2 & 5\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}$$

If this process is converted to computer code

```
double buyuk, dummy;
for(k=0;k<(n-1);k++)
  { //pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }}
```

After pivoting gauss elimination process is carried out as above

PROGRAM 3.3-1 Gauss elimination with partial pivoting

```
import java.io.*;
class NA15
{
  public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss eleme
  int i,j,k,p,ii,jj;
  System.out.println("Original matrix :\n"+Matrix.toString(a));
  for(k=0;k<(n-1);k++)
  { //pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
```

```
                b[k]=dummy;
               }
                System.out.println("pivoted matrix : k="+k+"\n"+Matrix.toString(a));
              //Solving gauss elimination
   for(i=k+1;i<n;i++)
   {  carpan=a[i][k]/a[k][k];
     a[i][k]=0;
     for(j=k+1;j<n;j++)
     {   a[i][j]-=carpan*a[k][j]; }
        b[i]   =b[i]   -carpan*b[k];
              }
  }
  System.out.println("final upper triangular matrix : \n"+Matrix.toString(a));
  //back substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
   toplam=0;
   for(j=i+1;j<n;j++)
   {  toplam+=a[i][j]*x[j];}
   x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{1,2,5},{2,1,2},{3,1,1}};
  double b[]={1,1,2};
  //double a[][]={{9.915,2.205},{2.205,0.52}};
  //double b[]={1.046,-2.042};
  double x[]=pivotlugauss(a,b);
  Text.print(x,"Pivotlu Gauss eliminasyon metodu çözüm matrisi");
  }
}
```

Output 3.3-1 gauss elimination with partial pivoting

```
---------- Capture Output ---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA15
Original matrix :
      1.000000000000000      2.000000000000000      5.000000000000000
      2.000000000000000      1.000000000000000      2.000000000000000
      3.000000000000000      1.000000000000000      1.000000000000000
pivoted matrix : k=0
      3.000000000000000      1.000000000000000      1.000000000000000
      2.000000000000000      1.000000000000000      2.000000000000000
      1.000000000000000      2.000000000000000      5.000000000000000
pivoted matrix : k=1
      3.000000000000000      1.000000000000000      1.000000000000000
      0.000000000000000      1.666666666666667      4.666666666666667
      0.000000000000000      0.333333333333333      1.333333333333334
final upper triangular matrix :
      3.000000000000000      1.000000000000000      1.000000000000000
      0.000000000000000      1.666666666666667      4.666666666666667
      0.000000000000000      0.000000000000000      0.400000000000000
```

As it is seen error shown in the previous non-pivoted version is disapeared here. In solution of small matrices Gauss elimination is a good solution, but when the matrix size is big, due to enourmous calculation size required, it is not a practical method.

PROGRAM 3.3-2 Gauss elimination with partial pivoting in Matlab(Octave) code

```
function x=gauss1(a,b)
% Gauss elimination
n=length(b);
carpan=0;
toplam=0;
 for k=1:n
  %pivoting
```

```
    p=k;
    buyuk=abs(a(k,k));
    for ii=k+1:n
     dummy=abs(a(ii,k));
      if dummy > buyuk buyuk=dummy;p=ii;end
    end
    if p~=k
     for jj=k:n
      dummy=a(p,jj);
      a(p,jj)=a(k,jj);
      a(k,jj)=dummy;
      end
     dummy=b(p);
     b(p)=b(k);
     b(k)=dummy;
    end
    %gauss eleme
    for i=k+1:n
     carpan=a(i,k)/a(k,k);
      a(i,k)=0;
      for j=k+1:n
        a(i,j)=a(i,j)-carpan*a(k,j);
      end
         b(i)   =b(i)   -carpan*b(k);
    end
  end
  %back substituting
  x(n)=b(n)/a(n,n);
  for i=n-1:-1:1
   toplam=0;
   for j=i+1:n
    toplam=toplam+a(i,j)*x(j);end
   x(i)=(b(i)-toplam)/a(i,i);
  end
 end
```

PROGRAM 3.3.3 Gauss elimination with partial pivoting in Python code

```
# Gauss Elimination with pivoting
# python version

from math import *

def gauss(a,b):
  n=len(a)
  x=[0 for i in range(n)]
  #print("origial matrix :\na =",a,"\nb =",b)
  for k in range(n):
    p=k
    buyuk=abs(a[k][k])
    for ii in range(k+1,n):
      dummy=abs(a[ii][k])
      if dummy>buyuk:
        buyuk=dummy
        p=ii
    if p!=k:
      for jj in range(k,n):
        dummy=a[p][jj]
        a[p][jj]=a[k][jj]
        a[k][jj]=dummy
      dummy=b[p]
      b[p]=b[k]
      b[k]=dummy
    #print("pivoted matrix :\na =",a,"\nb =",b)
    for i in range(k+1,n):
      carpan=a[i][k]/a[k][k]
      a[i][k]=0;
      for j in range(k+1,n):
        a[i][j]=a[i][j]-carpan*a[k][j]
      b[i]   =b[i]-carpan*b[k]
  #print("eliminated matrix :\na =",a,"\nb =",b)
  x[n-1]=b[n-1]/a[n-1][n-1]
  for i in range(n-1,-1,-1):
    toplam=0;
    for j in range(i+1,n):
      toplam=toplam+a[i][j]*x[j]
```

```
   x[i]=(b[i]-toplam)/a[i][i];
  return x;

a=[[1,2,5],[2,1,2],[3,1,1]]
b=[1,1,2]
x=gauss(a,b);
print("Gauss elimination solution vector :\nx = ",x)
```

```
Gauss elimination solution vector :
x =  [7.401486830834377e-17, 2.9999999999999996, -0.9999999999999998]
Reloaded modules: fi_xi, gauss
Gauss elimination solution vector :
x =  [7.401486830834377e-17, 2.9999999999999996, -0.9999999999999998]
```

PROGRAM Pivoted Gauss elimination in C++ using vector class (c++ std 11 version is used to compile)

```
#include <iostream>
#include <cmath>
#include <vector>
using namespace std;

void print(vector< vector<double> > a,vector<double> b) {
   int n = a.size();
   for (int i=0; i<n; i++) {
      for (int j=0; j<n; j++) {cout << a[i][j] << "\t";}
      cout << "| "<<b[i]<<"\n";
   }
   cout<<"\n";
}
void print(vector<double> b) {
   int n = b.size();
   for (int i=0; i<n; i++) {
      cout <<b[i]<< "\t";
   }
   cout<<"\n";
}
vector<double> pivotlugauss(vector< vector<double> > a,vector<double> b)
{//gauss elimination with partial pivoting
  int n=a.size();
  int m=b.size();
  vector<double> x(n);
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss eleme
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
                p=k;
                buyuk=abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;

                }
                //Solving gauss elimination
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]-carpan*b[k];
    }
```

```
  }
  //back substitution
  cout<<"elimination matrix\n";
  print(a,b);
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }

int main() {
    int n=3;
    vector< vector<double> > a{{1,2,5},{2,1,2},{3,1,1}};
    vector<double> b{1,1,2};
    vector<double> x(n);
    x = pivotlugauss(a,b);
    // Print result
    cout<<"x = \n";
    print(x);
}
```

```
---------- Capture Output ----------
> "D:\SCO1\run.bat" gauss1

D:\okul\SCO1>gauss1
elimination matrix
3           1           1           | 2
0           1.66667     4.66667     | 0.333333
0           0           0.4         | -0.4

x =
2.22045e-016           3           -1

> Terminated with exit code 0.
```

Pivoted Gauss elimination in Julia language

```
function gauss_pivot(A1,b)
            n=size(A1)[1]
            carpan=0.0
            toplam=0.0
            global x=Array{Float64}(undef,n)
            global a=Array{Float64}(undef,n,n)
            for i=1:n
              for j=1:n
                a[i,j]=A1[i,j]
              end
            end
            for k=1:n
            #pivoting
               p=k
               buyuk=abs(a[k,k])
               for ii=k+1:n
                 dummy=abs(a[ii,k])
                 if dummy > buyuk
                    buyuk=dummyp=ii
                 end #if dummy
               end #for ii
               if p!=k
                 for jj=k:n
                   dummy=a[p,jj]
                   a[p,jj]=a[k,jj]
                   a[k,jj]=dummy
                   end #for jj

                 dummy=b[p]
                 b[p]=b[k]
                 b[k]=dummy
               end #if p!
            # gauss elimination
   for i=k+1:n
```

```
                carpan=a[i,k]/a[k,k]
                a[i,k]=0.0
                for j=k+1:n
                  a[i,j]=a[i,j]-carpan*a[k,j]
                end #for j
                b[i]=b[i]-carpan*b[k]
              end #for i
              end #k
                          # back substituting
               x[n]=b[n]/a[n,n]
               n1=n-1
               for i=n1:-1:1
                 toplam=0.0
                 for j=i+1:n
                   toplam=toplam+a[i,j]*x[j]
                 end #for j
                 x[i]=(b[i]-toplam)/a[i,i]
               end #for i
              return x
end #function

A=[7.0 1 2;-1 4 -1;3 15 20]
B=[47.0 19 87]
X=gauss_pivot(A,B)
print("X=$X")
```

Gauss elimination can also be applied by using complex variables, we are given here a complex equivalent of the partial pivoted Gauss elimination code below

PROGRAM Gauss elimination with partial pivoting in complex variables (java version)

```
 import java.io.*;

class NA15C
{ public static c[] c_gauss_pivot(c a[][],c b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  c x[]=new c[n];
  c carpan=new c();
  c toplam=new c();
  c buyuk;
  c dummy=new c();
  //gauss eleme
  int i,j,k,p,ii,jj;
  System.out.println("Original matrix :\n"+c.toString(a));
  for(k=0;k<(n-1);k++)
  { //pivoting
              p=k;
              buyuk=a[k][k];
              for(ii=k+1;ii<n;ii++)
              {  dummy=a[ii][k];
                 if(dummy.R() > buyuk.R()) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
  }
  //gauss eleme
  System.out.println("Matrix after pivoting = \n"+c.toString(a));
  for(k=0;k<(n-1);k++)
  {
    for(i=k+1;i<n;i++)
    {  carpan=c.p(a[i][k],"/",a[k][k]);
       a[i][k]=new c(0.0,0.0);
       for(j=k+1;j<n;j++)
       {   a[i][j]=c.p(a[i][j],"-",c.p(carpan,"*",a[k][j])); }
          b[i]=c.p(b[i],"-",c.p(carpan,"*",b[k]));
```

```
                }
                System.out.println("k = "+k+" :\n"+c.toString(a));
  }
  //back substituting
  x[n-1]=c.p(b[n-1],"/",a[n-1][n-1]);
  for(i=n-2;i>=0;i--)
  {
    toplam=new c(0.0,0.0);
    for(j=i+1;j<n;j++)
    {  toplam=c.p(toplam,"+",c.p(a[i][j],"*",x[j]));}
    x[i]=c.p(c.p(b[i],"-",toplam),"/",a[i][i]);
  }
  return x;
  }

  public static c[] c_gauss(c a[][],c b[])
  {
  int n=b.length;
  c x[]=new c[n];
  c carpan=new c(1.0,0.0);
  c toplam=new c(0.0,0.0);
  //gauss eleme
  int i,j,k;
  for(k=0;k<(n-1);k++)
  {
    for(i=k+1;i<n;i++)
    {  carpan=c.p(a[i][k],"/",a[k][k]);
       a[i][k]=new c(0.0,0.0);
       for(j=k+1;j<n;j++)
       {   a[i][j]=c.p(a[i][j],"-",c.p(carpan,"*",a[k][j])); }
          b[i]=c.p(b[i],"-",c.p(carpan,"*",b[k]));
                }
  }
  //back substituting
  x[n-1]=c.p(b[n-1],"/",a[n-1][n-1]);
  for(i=n-2;i>=0;i--)
  {
    toplam=new c(0.0,0.0);
    for(j=i+1;j<n;j++)
    {  toplam=c.p(toplam,"+",c.p(a[i][j],"*",x[j]));}
    x[i]=c.p(c.p(b[i],"-",toplam),"/",a[i][i]);
  }
  return x;
  }


  public static void main(String args[]) throws IOException
  {
  String s1="{(-1.75,0),(1.25,0.5),(1,0.7)},{(1.2,2.3),(1.1,-1.543),(3.77,0)},{(-1,1),(3.25,0),(1,0)}";
  String s2="{(1.0,2.0),(1.1,0),(1.2,1.3)}";
  c a[][]=c.toComplexMatrix(s1);
  c b[]=c.toComplexVector(s2);
  c x[]=c_gauss_pivot(a,b);
  System.out.println("x = \n"+c.toString(x));
  }
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\javaw.exe" NA15C
Original matrix :
 ( -1.750000 +   0.000000*i )  (  1.250000 +   0.500000*i )  (  1.000000 +   0.700000*i )
 (  1.200000 +   2.300000*i )  (  1.100000 -   1.543000*i )  (  3.770000 +   0.000000*i )
 ( -1.000000 +   1.000000*i )  (  3.250000 +   0.000000*i )  (  1.000000 +   0.000000*i )

Matrix after pivoting =
 (  1.200000 +   2.300000*i )  (  1.100000 -   1.543000*i )  (  3.770000 +   0.000000*i )
 ( -1.750000 +   0.000000*i )  (  3.250000 +   0.000000*i )  (  1.000000 +   0.000000*i )
 ( -1.000000 +   1.000000*i )  (  1.250000 +   0.500000*i )  (  1.000000 +   0.700000*i )

k = 0 :
 (  1.200000 +   2.300000*i )  (  1.100000 -   1.543000*i )  (  3.770000 +   0.000000*i )
 (  0.000000 +   0.000000*i )  (  2.670420 -   1.139346*i )  (  2.176374 -   2.254718*i )
 (  0.000000 +   0.000000*i )  (  0.267756 +   0.180134*i )  (  0.383804 -   1.260624*i )

k = 1 :
```

```
  (  1.200000 +   2.300000*i )   (  1.100000 -   1.543000*i )   (  3.770000 +   0.000000*i )
  (  0.000000 +   0.000000*i )   (  2.670420 -   1.139346*i )   (  2.176374 -   2.254718*i )
  (  0.000000 +   0.000000*i )   (  0.000000 +   0.000000*i )   (  0.041909 -   1.327228*i )

 x =
  (  0.747087 -   1.487163*i )   (  1.054138 -   0.592099*i )   ( -0.918547 +   0.621788*i )

 > Terminated with exit code 0.
```

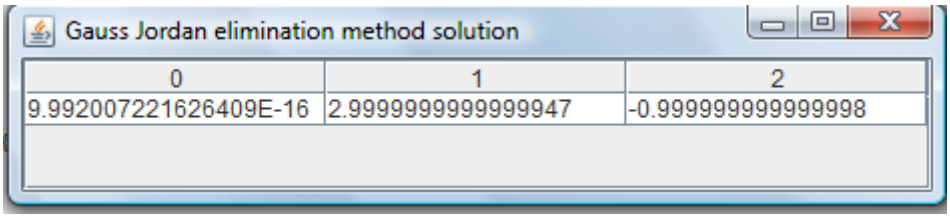
3.4 GAUSS-JORDAN METHOD

Gauss Jordan elimination method is similar to Gauss elimination with exception of a normalisation process before elimination and application of the elimination for all raws (instead of only lower lows of diagonal reference row in gauss elimination) except the diagonal reference raw. The result is conversion of the right hand side into the solution without any back substitution process. Example program is given below

PROGRAM Gauss_Jordan elimination method

```
import java.io.*; import java.io.*;

clasS NA16
{
//Gauss-Jordan elimination process
 public static double[] gaussjordan(double a[][],double b[])
 { //NO PIVOTİNG
 int n=b.length;
 int np1=n+1;
 double C[][]=new double[n][np1];
 double x[]=new double[n];
 double carpan=0;
 int i,j,k;

 for(i=0;i<n;i++)
 {for(j=0;j<n;j++){C[i][j]=a[i][j];};C[i][n]=b[i];}

 for(k=0;k<n;k++)
 {
            //gauss jordan elimination
            for(j=k+1;j<np1;j++)
            { C[k][j]/=C[k][k]; }
            for(i=0;i<n;i++)
  {  if(i!=k)
    {  carpan=C[i][k];
      C[i][k]=0;
      for(j=k+1;j<np1;j++)
      {   C[i][j]-=carpan*C[k][j]; }
    }
            }
 }
 for(i=0;i<n;i++)
 {x[i]=C[i][n];}
 return x;
 }

 public static void main(String args[]) throws IOException
 {
 double a[][]={{1,2,5},{2,1,2},{3,1,1}};
 double b[]={1,1,2};
 double x[]=gaussjordan(a,b);
 Text.print(x,"Gauss Jordan elimination method solution");
 }
}
```

output 1 Gauss_Jordan Elimination solution matrix



To enter a second example into the code :

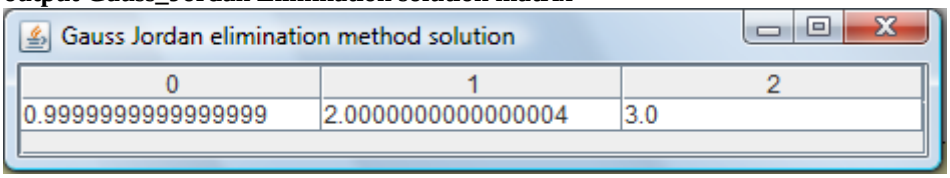
$$\begin{bmatrix}3 & 1 & -1\\ 1 & 4 & 1\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}2\\ 12\\ 10\end{bmatrix}$$

Following change can be apply to the program code

```
double a[][]={{3,1,-1},{1,4,1},{2,1,2}};
double b[]={2,12,10};
```

solution set will be:

output Gauss_Jordan Elimination solution matrix



Now if the same system of equation is solved by hand:

$$\begin{bmatrix}1 & 0.3333 & -0.3333\\ 1 & 4 & 1\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0.6667\\ 12\\ 10\end{bmatrix}$$

$$\begin{bmatrix}1 & 0.3333 & -0.3333\\ 0 & 3.6667 & 1.3333\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0.6667\\ 11.333\\ 10\end{bmatrix}$$

$$\begin{bmatrix}1 & 0.3333 & -0.3333\\ 0 & 3.6667 & 1.3333\\ 0 & 0.3333 & 2.6667\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0.6667\\ 11.333\\ 8.6667\end{bmatrix}$$

$$\begin{bmatrix}1 & 0.3333 & -0.3333\\ 0 & 1 & 0.3636\\ 0 & 0.3333 & 2.6667\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0.6667\\ 3.0909\\ 8.6667\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -0.4545\\ 0 & 1 & 0.3636\\ 0 & 0.3333 & 2.6667\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}-0.3636\\ 3.0909\\ 8.6667\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -0.4545\\ 0 & 1 & 0.3636\\ 0 & 0 & 2.5455\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}-0.3636\\ 3.0909\\ 7.6365\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -0.4545\\ 0 & 1 & 0.3636\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}-0.3636\\ 3.0909\\ 3.0000\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0.3636\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}1.0000\\ 3.0909\\ 3.0000\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix}=\begin{bmatrix}1.0000\\ 2.0001\\ 3.0000\end{bmatrix}$$

Solution vector

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 2.0001 \\ 3.0000 \end{bmatrix}$$

Exact solution :

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 2.0000 \\ 3.0000 \end{bmatrix}$$

Another example for Gauss-Jordan elimination with detailed steps :

$$\begin{bmatrix} 7 & 1 & 2 \\ -1 & 4 & -1 \\ 3 & 15 & 20 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 47 \\ 19 \\ 87 \end{bmatrix}$$

Gauss-Jordan

A				B
7	1	2	X_0	47
-1	4	-1	X_1	19
3	15	20	X_2	87
Normalisation				
1	0.142857	0.285714	X_0	6.714286
-1	4	-1	X_1	19
3	15	20	X_2	87
Elimination				
1	0.142857	0.285714	X_0	6.714286
0	4.142857	-0.71429	X_1	25.71429
0	14.57143	19.14286	X_2	66.85714
Normalisation				
1	0.142857	0.285714	X_0	6.714286
0	1	-0.17241	X_1	6.206897
0	14.57143	19.14286	X_2	66.85714
Elimination				
1	0	0.310345	X_0	5.827586
0	1	-0.17241	X_1	6.206897
0	0	21.65517	X_2	-23.5862
Normalisation				
1	0	0.310345	X_0	5.827586
0	1	-0.17241	X_1	6.206897
0	0	1	X_2	-1.08917
Elimination				
1	0	0	X_0	**6.165605**
0	1	0	X_1	**6.019108**
0	0	1	X_2	**-1.08917**

PROGRAM Gauss_Jordan elimination in Matlab(Octave) code

```
function x=gaussjordan(a,b)
 n=length(b);
 np1=n+1;
 carpan=0;
 C=a;
 for i=1:n
    C(i,n+1)=b(i);
 end
 for k=1:n
  %gauss jordan elimination
  for j=k+1:np1
   C(k,j)=C(k,j)/C(k,k);
  end
  for i=1:n
   if i~=k
     carpan=C(i,k);
      C(i,k)=0;
      for j=k+1:np1
        C(i,j)=C(i,j)-carpan*C(k,j); end
    end
  end
 end
 for i=1:n
  C(i,i)=1;
  x(i)=C(i,np1);
 end
 end
```

0	0	1	X_2	**-1.08917**

PROGRAM Gauss_Jordan elimination in Julia code

```
 function gauss_jordan(a,b)
n=size(b)[1]
np1=n+1
C=zeros(Float64,n,np1)
for i=1:n
  for j=1:n
    C[i,j]=a[i,j]
  end
  C[i,np1]=b[i]
end
for k=1:n
  for j=k+1:np1
    C[k,j]/=C[k,k]
  end
  C[k,k]=1.0
  for i=1:n
    if i!=k
      carpan=C[i,k]
      C[i,k]=0.0
      for j=k+1:np1
        C[i,j]-=carpan*C[k,j]
      end # for
    end # if
  end # for i
end # for k
for i=1:n
 x[i]=C[i,np1]
end # for i
return x
end # function

global a=[7 1 2;-1 4 -1;3 15 20]
global b=[47 19 87]
b1=b'
x=a\b1
print("Julia x = $x")
x1=gauss_jordan(a,b1)
print("\nGauss-Jordan x = $x1")
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_jordan.jl
Julia x = [6.165605095541401; 6.019108280254777; -1.0891719745222923]
Gauss-Jordan x = [6.165605095541401; 6.019108280254777; -1.0891719745222925]
> Terminated with exit code 0.
```

PROGRAM Gauss_Jordan elimination in Python code

```
from math import *

def gaussjordan(a,b):
  n=len(a)
  np1=n+1
  x=[]
  C=[[0 for x in range(np1)] for y in range(n)]
  print("origial matrix :\na =",a,"\nb =",b)
  for i in range(n):
    x.append(0.0)
  for i in range(n):
    for j in range(n):
      C[i][j]=a[i][j]
    C[i][n]=b[i]
  for k in range(n):
    for j in range(k+1,np1):
      C[k][j]=C[k][j]/C[k][k]
    for i in range(n):
      if i!=k:
        carpan=C[i][k]
        C[i][k]=0.0
        for j in range(k+1,np1):
          C[i][j]=C[i][j]-carpan*C[k][j]
```

```
  print("eliminated matrix :\nC =",C)
  for i in range(n):
    x[i]=C[i][n]
  return x;

a=[[1,2,5],[2,1,2],[3,1,1]]
b=[1,1,2]
x=gaussjordan(a,b);
print("Gauss - Jordan elimination solution matrix :\nx = ",x)
```

```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" NA16.py
origial matrix :
a = [[1, 2, 5], [2, 1, 2], [3, 1, 1]]
b = [1, 1, 2]
eliminated matrix :
a = [[1, 0.0, 0.0, 9.992007221626409e-16], [0.0, -3.0, 0.0, 2.9999999999999947], [0.0, 0.0, -0.6666666666666679, -
0.9999999999999998]]
Gauss elimination solution matrix :
x =  [9.992007221626409e-16, 2.9999999999999947, -0.9999999999999998]

> Terminated with exit code 0.
```

3.5 GAUSS-JORDAN METHOD WITH PARTIAL PIVOTING

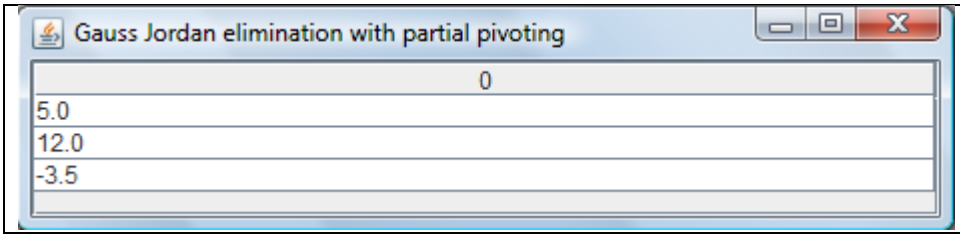
Pivoting process can be applied to Gauss-Jordan elimination similar to Gauss elimination

PROGRAM 3.5-1 Gauss_Jordan Elimination with partial pivoting

```
import java.io.*;

class NA17
{ public static double[][] C(double a[][],double b[])
  {int n=a.length;int m=a[0].length;double c[][]=new double[n][m+1];
  for(int i=0;i<n;i++){for(int j=0;j<m;j++){c[i][j]=a[i][j];};c[i][m]=b[i];}
  return c;
  }

  public static double[][] C(double a[][],double b[][])
  {int n=a.length;int o=b[0].length;int m=n+o;double c[][]=new double[n][m];
  for(int i=0;i<n;i++)
  {for(int j=0;j<n;j++){c[i][j]=a[i][j];};
   for(int j=n;j<m;j++){c[i][j]=b[i][j];}
  }
  return c;
  }

  public static double[][] gaussjordanwithpivot(double a[][])
  { //pivotlu
  int n=a.length;
  int m=a[0].length;
  int o=m-n;
  double x[][]=new double[n][o];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  int i,j,k,p,ii,jj;
  i=0;
  for(k=0;k<n;k++)
  { //pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=0;ii<n;ii++)
              {  if(i!=k)
                 {
                 dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
     }
              }
              if(p!=k)
              { for(jj=k;jj<m;jj++)
                { dummy=a[p][jj];
```

```
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                }
                //gauss jordan elimination
                for(j=k+1;j<m;j++)
                { a[k][j]/=a[k][k]; }
                a[k][k]=1.0;
    for(i=0;i<n;i++)
    {  if(i!=k)
      {
                      carpan=a[i][k];
        a[i][k]=0;
        for(j=k+1;j<m;j++)
          {a[i][j]-=carpan*a[k][j]; }
      }
                }
  }
  for(jj=0;jj<o;jj++)
  {for(i=0;i<n;i++) {x[i][jj]=a[i][n+jj];}}
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{3,0,2,8},{0,0,2,-7},{6,2,8,26}};
  //double a[][]=Text.readDouble_Table();
  double x[][]=gaussjordanwithpivot(a);
  Text.print(x,"Gauss Jordan elimination with partial pivoting");
  }
}
```

output 3.5-1Gauss_Jordan Elimination with partial pivoting



PROGRAM 3.5-2 Gauss_Jordan Elimination with partial pivoting in Matlab(Octave)

```
function x=gaussjordan1(a,b)
  n=length(b);
  np1=n+1;
  carpan=0;
  C=a;
  for i=1:n
     C(i,np1)=b(i);
  end
  for k=1:n
    %pivoting
    p=k;
    buyuk=abs(C(k,k));
    for ii=k+1:n
      dummy=abs(C(ii,k));
       if dummy > buyuk buyuk=dummy;p=ii;end
    end
    if p~=k
      for jj=k:np1
        dummy=C(p,jj);
        C(p,jj)=C(k,jj);
        C(k,jj)=dummy;
      end
    end
    %gauss jordan elimination
    for j=k+1:np1
      C(k,j)=C(k,j)/C(k,k);
```

```
    end
    for i=1:n
     if i~=k
        carpan=C(i,k);
         C(i,k)=0;
         for j=k+1:np1
           C(i,j)=C(i,j)-carpan*C(k,j); end
      end
    end
  end
  for i=1:n
   C(i,i)=1;
   x(i)=C(i,np1);
  end
  end
```

```
>> a=[3 0 2;0 0 2;6 2 8]
a =
     3     0     2
     0     0     2
     6     2     8
>> b=[8,-7,26]'
b =
     8
    -7
     26
>> gaussjordan1(a,b)
ans =
    5.0000    12.0000   -3.5000
```

3.6 LU-DECOMPOSITION- GAUSS (DOLİTTLE) METHOD

In Gauss or Gauss-Jordan elimination processes left hand matrix A and right hand vector b is eliminated together. Therefore if a new left hand side vector b1 will be considered to be used in as second set with the same A matrix, Matrix A is reprocessed. This will spend valuable calculation time for a repeated process. In order to avoid this repetation, Matrix A is decomposed into an upper triangle(U) and a lower triangle (L) matrices so that multiplication of this two matricess will give the original A matrix
For example for N=5

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{bmatrix}$$

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{bmatrix}$$

To solve system of equation with a B vector following process applies
[A]=[L][U]
[A]{X}={B}
[L][U]{X}={B}
[L]{D}={B}
[U]{X}={D}

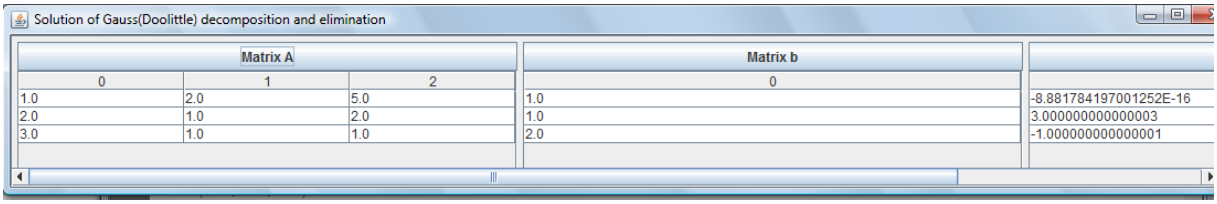
In Gauss(Doolittle) decomposition process matrix U is the same as Gauss elimination upper triangualar matrix. L matrix is consist of the multipliers calculated in Gauss elimination process. Due to diagonal element of matrix L is 1, LU matrix can be represented in one matrix if space saving is desired. Example code is given below

PROGRAM 3.6-1 Gauss (Dolittle) Elemimination & decomposition

```
import java.io.*;
```

```
import javax.swing.*;
class NA18
{
  public static double[][] gaussLU(double ai[][])
  {
  int n=ai.length;
  double x[]=new double[n];
  double a[][]=new double[n][n];
  double carpan=0;
  double toplam=0;
  //gauss eleme
  int i,j,k;
  for(i=0;i<n;i++)
     for(j=0;j<n;j++){a[i][j]=ai[i][j];};
  for(k=0;k<(n-1);k++)
  {
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=carpan;///L elemani
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
                }
  }
  return a;
  }


  public static double[] yerine_koyma(double ai[][],double bi[])
  { //gauss LU çözümüyle LU indirgenmesi yapilmiş Matris kullanarak linear
    //denklem sistemi kökleri
    int n=ai.length;
    double toplam;
    double x[]=new double[n];
     double a[][]=new double[n][n];
    double b[]=new double[n];
    int i,j;
  for(i=0;i<n;i++)
     {for(j=0;j<n;j++){a[i][j]=ai[i][j];};b[i]=bi[i];}
  System.out.println("a=\n "+Matrix.toString(a));
  System.out.println("b=\n "+Matrix.toStringT(b));
  for(i=1;i<n;i++)
     {toplam=b[i];for(j=0;j<i;j++){toplam-=a[i][j]*b[j];};b[i]=toplam;}
  System.out.println("d=\n "+Matrix.toStringT(b));
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=(n-2);i>=0;i--)
     {toplam=0;
      for(j=i+1;j<n;j++)
       {toplam+=a[i][j]*x[j];};
       x[i]=(b[i]-toplam)/a[i][i];
     }
  System.out.println("d=\n "+Matrix.toStringT(x));
  return x;
  }

  public static double[][] yerine_koyma(double ai[][],double bi[][])
  {
    int n=ai.length;
    int m=bi[0].length;
    double toplam;
    double x[][]=new double[n][m];
    double a[][]=new double[n][n];
    double b[][]=new double[n][m];
    int i,j,k;
  for(i=0;i<n;i++)
     {for(j=0;j<n;j++){a[i][j]=ai[i][j];};for(k=0;k<m;k++) {b[i][k]=bi[i][k];}}
  System.out.println("a=\n "+Matrix.toString(a));
  System.out.println("b=\n "+Matrix.toStringT(b));
  for(k=0;k<m;k++)
  {
  for(i=1;i<n;i++)
```

```
    {toplam=b[i][k];for(j=0;j<i;j++){toplam-=a[i][j]*b[j][k];};b[i][k]=toplam;}
  System.out.println("d=\n "+Matrix.toStringT(b));
  x[n-1][k]=b[n-1][k]/a[n-1][n-1];
  for(i=(n-2);i>=0;i--)
     {toplam=0;
      for(j=i+1;j<n;j++)
       {toplam+=a[i][j]*x[j][k];};
       x[i][k]=(b[i][k]-toplam)/a[i][i];
     }
  }//kin sonu
  System.out.println("d=\n "+Matrix.toStringT(x));
  return x;
  }

  public static double [] AXB(double a[][],double b[])
  {
  //denklem sistemi çözümü b vektörü için
  int n=a.length;
  double c[][]=gaussLU(a);
  Matrix.toString(c);
  double s[]=yerine_koyma(c,b);
  return s;
  }

  public static double [][] AXB(double a[][],double b[][])
  {//denklem sistemi çözümü b matrisi için
  int n=a.length;
  double c[][]=gaussLU(a);
  Matrix.toString(c);
  double s[][]=yerine_koyma(c,b);
  return s;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{1,2,5},{2,1,2},{3,1,1}};
  double b[]={1,1,2};
  JPanel y=new JPanel();
  Text.add(y,a,"Matrix A");
  Text.addT(y,b,"Matrix b");
  double x[]=AXB(a,b);
  Text.addT(y,x,"Solution Matrix");
  Text.print(y,"Solution of Gauss(Doolittle) decomposition and elimination");
  }
}
```



PROGRAM 3.6-2 Gauss (Dolittle) LU decomposition with partial pivoting

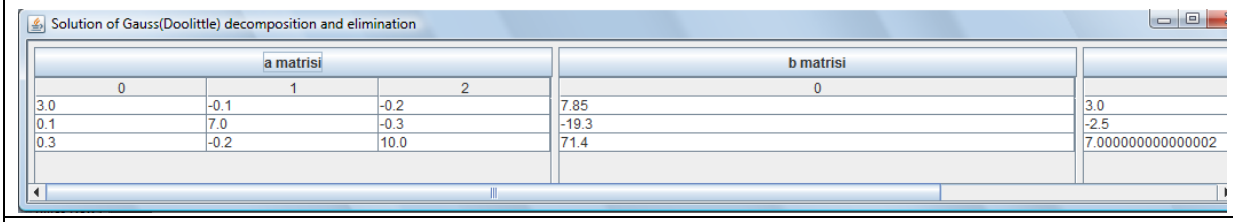
```
import java.io.*;
import javax.swing.*;
class NA19
{
  public static void pivot(double a[][],int o[],double s[],int k)
  {// Gauss(Dolittle) LU decomposition with partial pivoting
    int n=a.length;
    int dummy1=0;
    double buyuk;
    double dummy=0;
    //pivotlama
                int p=k;
                buyuk=Math.abs(a[o[k]][k]);
```

```
            for(int ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[o[ii]][k]/s[o[ii]]);
              if(dummy > buyuk) {buyuk=dummy;p=ii;}
            }
            dummy1=o[p];
            o[p]=o[k];
            o[k]=dummy1;
}

public static int[] sirali_indeks(int n)
{   // sirali rakamlari taşiyan bir referans vektörü oluşturur
              int o[]=new int[n];
              for(int i=0;i<n;i++) {o[i]=i;}
              return o;
}

public static double[][] gaussLU(double ai[][],int o[])
{ //kismi pivotlu gauss eleme yöntemi
int n=ai.length;
double a[][]=new double[n][n];
double s[]=new double[n];
double carpan=0;
double toplam=0;
double absaij;

//gauss eleme
int i,j,k,p,ii,jj;
//System.out.println("Orijinal Matris :\n"+Matrix.toString(ai));
for(i=0;i<n;i++)
{for(j=0;j<n;j++){a[i][j]=ai[i][j];}}
for(i=0;i<n;i++)
{ o[i]=i;s[i]=Math.abs(a[i][0]);
  for(j=1;j<n;j++)
    { absaij=Math.abs(a[i][j]);
      if(absaij>s[i]) {s[i]=absaij;}
    }
}
for(k=0;k<(n-1);k++)
{ pivot(a,o,s,k);
            //pivot referansini değiştir
            //System.out.println("pivotlu Matris : k="+k+"\n"+Matrix.toString(a));
            //gauss elemeyi çözme
  for(i=k+1;i<n;i++)
  {  carpan=a[o[i]][k]/a[o[k]][k];
     a[o[i]][k]=carpan;
     for(j=k+1;j<n;j++)
     {   a[o[i]][j]-=carpan*a[o[k]][j]; }
            }
}
return a;
}

public static double[] yerine_koyma(double ai[][],double bi[])
{   int n=ai.length;
              return yerine_koyma(ai,bi,sirali_indeks(n));
}

public static double[][] yerine_koyma(double ai[][],double bi[][])
{   int n=ai.length;
              return yerine_koyma(ai,bi,sirali_indeks(n));
}

public static double[] yerine_koyma(double ai[][],double bi[],int o[])
{ //gauss LU çözümüyle LU indirgenmesi yapilmiş Matris kullanarak linear
  //denklem sistemi kökleri
  int n=ai.length;
  double toplam;
  double x[]=new double[n];
   double a[][]=new double[n][n];
  double b[]=new double[n];
```

```
  int i,j;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};b[i]=bi[i];}
//System.out.println("a=\n "+Matrix.toString(a));
//System.out.println("b=\n "+Matrix.toStringT(b));
for(i=1;i<n;i++)
    {toplam=b[o[i]];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]];};b[o[i]]=toplam;}
 //System.out.println("d=\n "+Matrix.toStringT(b));
x[n-1]=b[o[n-1]]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {toplam=0;
     for(j=i+1;j<n;j++)
      {toplam+=a[o[i]][j]*x[j];};
      x[i]=(b[o[i]]-toplam)/a[o[i]][i];
    }
//System.out.println("d=\n "+Matrix.toStringT(x));
return x;
}

public static double[][] yerine_koyma(double ai[][],double bi[][],int o[])
{
  int n=ai.length;
  int m=bi[0].length;
  double toplam;
  double x[][]=new double[n][m];
  double a[][]=new double[n][n];
  double b[][]=new double[n][m];
  int i,j,k;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};for(k=0;k<m;k++) {b[i][k]=bi[i][k];}}
//System.out.println("a=\n "+Matrix.toString(a));
//System.out.println("b=\n "+Matrix.toStringT(b));
for(k=0;k<m;k++)
{
for(i=1;i<n;i++)
    {toplam=b[o[i]][k];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]][k];};b[o[i]][k]=toplam;}
 //System.out.println("d=\n "+Matrix.toStringT(b));
x[n-1][k]=b[o[n-1]][k]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {toplam=0;
     for(j=i+1;j<n;j++)
      {toplam+=a[o[i]][j]*x[j][k];};
      x[i][k]=(b[o[i]][k]-toplam)/a[o[i]][i];
    }
}//kin sonu
//System.out.println("d=\n "+Matrix.toStringT(x));
return x;
}

public static double [] AXB(double a[][],double b[])
{
//denklem sistemi çözümü b vektörü için
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[]=yerine_koyma(c,b,q);
return s;
}

public static double [][] AXB(double a[][],double b[][])
{//denklem sistemi çözümü b matrisi için
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[][]=yerine_koyma(c,b,q);
return s;
}

public static void main(String args[]) throws IOException
{
```

```
  double a[][]={{3,-0.1,-0.2},{0.1,7,-0.3},{0.3,-0.2,10}};
  JPanel y=new JPanel();
  Text.add(y,a,"a matrisi");
  double b[]={7.85,-19.3,71.4};
  Text.addT(y,b,"b matrisi");
  double x[]=AXB(a,b);
  Text.addT(y,x,"Solution Matrix");
  Text.print(y,"Solution of Gauss(Doolittle) decomposition and elimination");
  }
}
```

Output 3.6-1 Kismi Pivotlu LU Gauss (Dolittle) Eleme yöntemi



GaussLU in python language

```
# Gauss Elimination with pivoting
# python version
from math import *

def gaussLU(ai):
   #Gauss elimination with pivoting
  n=len(ai)
  x=[0 for i in range(n)]
  a=[[0 for i in range(n)] for j in range(n)]
  carpan=0
  toplam=0
  for i in range(n):
     for j in range(n):
        a[i][j]=ai[i][j]
  for k in range(n-1):
     for i in range(k+1,n):
        carpan=a[i][k]/a[k][k];
        a[i][k]=carpan
        for j in range(k+1,n):
           a[i][j]-=carpan*a[k][j]
  return a


def I(n):
   #unit matrix of dimension n
   B = [[0 for x in range(n)] for y in range(n)];
   for i in range(0,n):
      for j in range(0,n):
         if i==j: B[i][j]=1.0;
   return B;

def yerine_koyma(ai,bi):
   #gauss LU linear system of  equation solution
   n=len(ai)
   x=[0 for i in range(n)]
   a=[[0 for i in range(n)] for j in range(n)]
   b=[0 for i in range(n)]
   for i in range(n):
      for j in range(n):
         a[i][j]=ai[i][j]
      b[i]=bi[i]
   for i in range(n):
      toplam=b[i]
      for j in range(i):
         toplam-=a[i][j]*b[j]
      b[i]=toplam
   x[n-1]=b[n-1]/a[n-1][n-1];
```

```
    for i in range((n-2),-1,-1):
        toplam=0
        for j in range(i+1,n):
            toplam+=a[i][j]*x[j]
        x[i]=(b[i]-toplam)/a[i][i]
    return x

def AXB(a,b):
    n=len(a)
    c=gaussLU(a)
    s=yerine_koyma(c,b)
    return s;

a=[[1,2,5],[2,1,2],[3,1,1]]
b=[1,1,2]
x=AXB(a,b);
print("GaussLU elimination solution vector :\nx = ",x)
```

```
runfile('E:/okul/SCO1/gaussLU.py', wdir='E:/okul/SCO1')
GaussLU elimination solution vector :
x =  [-8.881784197001252e-16, 3.000000000000003, -1.000000000000001]
```

GaussLU in Julia Language

```
function gaussLU(ai)
  # Gauss elimination with pivoting
  # Julia version
  n=size(ai)[1]
  x=zeros(Float64,n)
  a=zeros(Float64,n,n)
  carpan=0.0
  toplam=0.0
  for i=1:n
     for j=1:n
        a[i,j]=ai[i,j]
     end
  end
  for k=1:n-1
     for i=k+1:n
        carpan=a[i,k]/a[k,k];
        a[i,k]=carpan
        for j=k+1:n
           a[i,j]-=carpan*a[k,j]
        end # for j
     end # for i
  end # for k
  return a
end # function gaussLU

function I(n)
    # unit matrix of dimension n
    B =zeros(Float64,n,n)
    for i=1:n
       for j=1:n
          if i==j
            B[i,j]=1.0
          end # if
       end # for j
    end # for i
    return B
end # function I

function yerine_koyma(ai,bi)
    # gauss LU linear system of  equation solution
    n=size(ai)[1]
    x=zeros(Float64,n)
     a=zeros(Float64,n,n)
    b=zeros(Float64,n)
    for i=1:n
       for j=1:n
          a[i,j]=ai[i,j]
       end # for j
       b[i]=bi[i]
    end # for i
```

```
    for i=1:n
       toplam=b[i]
       for j=1:i-1
          toplam-=a[i,j]*b[j]
       end # for j
       b[i]=toplam
    end # for j
    print("\nd = $b")
    x[n]=b[n]/a[n,n]
    for i=(n-1):-1:1
       toplam=0.0
       for j=(i+1):n
          toplam+=a[i,j]*x[j]
       end #for j
       x[i]=(b[i]-toplam)/a[i,i]
    end # for i
    return x
end # funtion yerine koyma

function AXB(A,b)
    n=size(a)[1]
    c=gaussLU(a)
    s=yerine_koyma(c,b)
    return s
end # function AXB


a=[1 2 5;2 1 2;3 1 1]
b=[1 1 2]
c=gaussLU(a)
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), c)
x=AXB(a,b)
print("\nx = $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gaussLU.jl
3X3 Matrix{Float64}:
 1.0   2.0       5.0
 2.0  -3.0      -8.0
 3.0   1.66667  -0.666667
d = [1.0, -1.0, 0.6666666666666667]
x = [-8.881784197001252e-16, 3.000000000000003, -1.000000000000001]
> Terminated with exit code 0.
```

3.7 CRAUT LU DECOMPOSITION METHOD

In the Gauss (Dolittle) LU decomposition diagonal of matrix L was setted as 1, In the Craut LU decompositon diagonal of matrix U is given as 1. For example for a 5 by 5 system

$$[L] = \begin{bmatrix} l_{11} & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} \end{bmatrix}$$

$$[U] = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & 1 & u_{23} & u_{24} & u_{25} \\ 0 & 0 & 1 & u_{34} & u_{35} \\ 0 & 0 & 0 & 1 & u_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution process is similar

[A]=[L][U]
[A]{X}={B}
[L][U]{X}={B}
[L]{D}={B}
[U]{X}={D}
Algorithm of Craut is different compare to Dolittle

$l_{i,1} = a_{i,1}$ loop i=1,2,….,n

$u_{1,j} = \frac{a_{1,j}}{l_{11}}$ loop i=2,3,….,n

loop j=2,3,….,n – 1

$l_{i,j} = a_{i,j} - \sum_{k=1}^{j-1} l_{i,k} u_{k,j}$ loop k=j,j+2,….,n

$u_{i,j} = \frac{a_{i,j} - \sum_{i=1}^{j-1} l_{j,i} u_{i,k}}{l_{j,j}}$ loop k=j+1,j+2,….,n

End of loop $l_{n,n} = a_{n,n} - \sum_{k=1}^{j-1} l_{n,k} u_{k,n}$

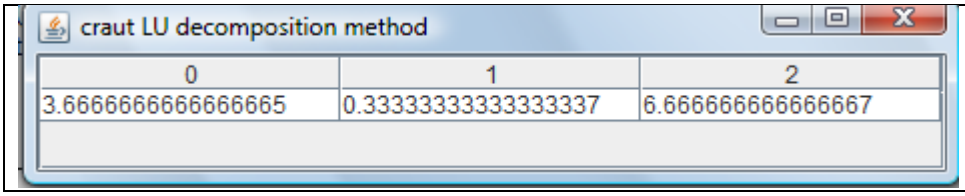
Algorithm is converted to programming code in the following example

PROGRAM 3.7-1 LU Craut Decomposition method

```
import java.io.*;

class NA20
{
 public static void pivot(double a[][],int o[],double s[],int k)
 {// Craut LU decomposition
  int n=a.length;
  int dummy1=0;
  double buyuk;
  double dummy=0;
  //pivoting
          int p=k;
          buyuk=Math.abs(a[o[k]][k]);
          for(int ii=k+1;ii<n;ii++)
          {  dummy=Math.abs(a[o[ii]][k]/s[o[ii]]);
            if(dummy > buyuk) {buyuk=dummy;p=ii;}
          }
          dummy1=o[p];
          o[p]=o[k];
          o[k]=dummy1;
 }

 public static int[] index(int n)
 {
           int o[]=new int[n];
           for(int i=0;i<n;i++) {o[i]=i;}
           return o;
 }

 public static double[] back_substitution(double ai[][],double bi[])
 {  int n=ai.length;
          return back_substitution(ai,bi,index(n));
 }

 public static double[][] back_substitution(double ai[][],double bi[][])
 {  int n=ai.length;
          return back_substitution(ai,bi,index(n));
 }

 public static double[] back_substitution(double ai[][],double bi[],int o[])
 {
  int n=ai.length;
  double toplam;
  double x[]=new double[n];
   double a[][]=new double[n][n];
  double b[]=new double[n];
  int i,j;
```

```
  for(i=0;i<n;i++)
     {for(j=0;j<n;j++){a[i][j]=ai[i][j];};b[i]=bi[i];}
  for(i=1;i<n;i++)
     {toplam=b[o[i]];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]];};b[o[i]]=toplam;}
  x[n-1]=b[o[n-1]]/a[o[n-1]][n-1];
  for(i=(n-2);i>=0;i--)
     {toplam=0;
      for(j=i+1;j<n;j++)
       {toplam+=a[o[i]][j]*x[j];};
       x[i]=(b[o[i]]-toplam)/a[o[i]][i];
     }
  return x;
  }

  public static double[][] back_substitution(double ai[][],double bi[][],int o[])
  {
    int n=ai.length;
    int m=bi[0].length;
    double toplam;
    double x[][]=new double[n][m];
    double a[][]=new double[n][n];
    double b[][]=new double[n][m];
    int i,j,k;
  for(i=0;i<n;i++)
     {for(j=0;j<n;j++){a[i][j]=ai[i][j];};for(k=0;k<m;k++) {b[i][k]=bi[i][k];}}
  for(k=0;k<m;k++)
  {
  for(i=1;i<n;i++)
     {toplam=b[o[i]][k];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]][k];};b[o[i]][k]=toplam;}
  x[n-1][k]=b[o[n-1]][k]/a[o[n-1]][n-1];
  for(i=(n-2);i>=0;i--)
     {toplam=0;
      for(j=i+1;j<n;j++)
       {toplam+=a[o[i]][j]*x[j][k];};
       x[i][k]=(b[o[i]][k]-toplam)/a[o[i]][i];
     }
  }
  return x;
  }

public static double[][] crautLU(double c[][])
{
//craut LU decomposition without pivoting

int n=c.length;
double sum;
int i,j,k;
double d[][]=new double[n][n];
for(i=0;i<n;i++)
{for(j=0;j<n;j++) d[i][j]=c[i][j];}

for(j=1;j<=n;j++)
{
  for(i=1;i<j;i++)
    {
    sum=d[i-1][j-1];
    for(k=1;k<i;k++)
              { sum-=d[i-1][k-1]*d[k-1][j-1];}
    d[i-1][j-1]=sum;
    }
  for(i=j;i<=n;i++)
    {
```

```
    sum=d[i-1][j-1];
    for(k=1;k<j;k++)
      {sum-=d[i-1][k-1]*d[k-1][j-1];}
    d[i-1][j-1]=sum;
    }
  if(j != n)
  { for(i=j+1;i<=n;i++)
      d[i-1][j-1]/=d[j-1][j-1];
  }
}
System.out.println(Matrix.toString(d));
return d;
}



  public static void main(String args[]) throws IOException
  {
  double a[][]={{2,-6,1},{-1,7,-1},{1,-3,2}};
  double b[]={12,-8,16};
  double c[][]= crautLU(a);
  double x[]=back_substitution(c,b);
  Text.print(x,"craut LU decomposition method");
  }
}
```



3.8 CHOLESKY SYMMETRIC MATRIX LU DECOMPOSITION METHOD

Cholesky method is similar to Dolittle(Gauss) LU method. This method is applied to symmetric matrix. Therefore, instead of LU, it decomposes to $[L][L]^T$ form.

$[A]=[L][L]^T$

For the decomposition process following steps are used

$l_{k,i} = \frac{a_{k,i}-\sum_{j=1}^{i-1} l_{i,j}l_{k,j}}{l_{i,i}}$ k=1,2,….,k-1

And

$$l_{k,k} = \sqrt{a_{k,k} - \sum_{j=0}^{n} l_{k,j}l_{k,j}}$$

Backsubstitution is similar to Dolittle method.

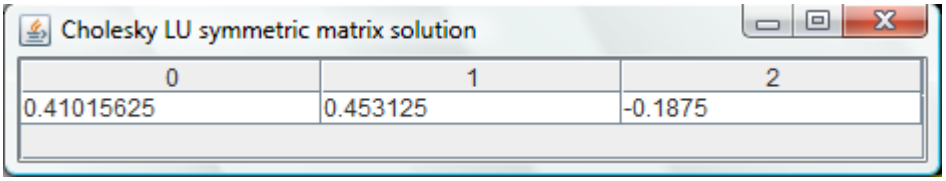
PROGRAM 3.8-1 Cholesky symmetric matrix decomposition method java version

```
import java.io.*;

class NA22
{
public static double[][] choleskyLU(double c[][])
{
//cholesky symmetric matrix LU
//symmetric matrices only
```

```
double factor=0;
int i,j,k,p,ii,jj;
int n=c.length;
double d[][]=new double[n][n];
double dummy;
double sum;
for(i=0;i<n;i++)
{for(j=0;j<n;j++) d[i][j]=c[i][j];}
for(k=1;k<=n;k++)
{
 for(i=1;i<=(k-1);i++)
    {sum=0;;
     for(j=1;j<=(i-1);j++)
     { sum+=d[i-1][j-1]*d[k-1][j-1]; }
       d[k-1][i-1]=(d[k-1][i-1]-sum)/d[i-1][i-1];

    }
 sum=0.0;
 for(j=1;j<=(k-1);j++)
 { sum+=d[k-1][j-1]*d[k-1][j-1]; }
 d[k-1][k-1]=Math.sqrt((d[k-1][k-1]-sum));
 for(j=k+1;j<=n;j++) d[k-1][j-1]=0.0;
}
return d;
}
  public static double[] substitutecholesky(double c[][],double b[])
  {
  //solving of system of equation by using cholesky decomposed matrix
  int i,j;
  double U[][]=T(c);
  double L[][]=c;
  int n=b.length;
  double sum=0;
  double x[]=new double[n];
  double d[]=new double[n];
  //forward substituting
  double toplam=0;
  for(i=0;i<n;i++)
  {toplam=0;
   for(j=0;j<i;j++)
   {toplam+=L[i][j]*d[j];}
   d[i]=(b[i]-toplam)/L[i][i];
  }
  //backward substitution
  x[n-1]=d[n-1]/U[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=U[i][j]*x[j];}
    x[i]=(d[i]-toplam)/U[i][i];
  }
  return x;
  }
  public static double[][] T(double [][] left)
  {
  //transpose matrix (if A=a(i,j) T(A)=a(j,i)
  int i,j;
  int n=left.length;
  int m=left[0].length;
  double b[][];
  b=new double[m][n];
```

```
  for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
  return b;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{4,-1,1},{-1,4.25,2.75},{1,2.75,3.5}};
  double b[]={1,1,1};
  double L[][]=choleskyLU(a);
  System.out.println("Matris L : \n"+Matrix.toString(L));
  double x[]=substitutecholesky(L,b);
  Text.print(x,"Cholesky LU symmetric matrix solution");
  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA22
Matris L :
       2.000000000000000       0.000000000000000       0.000000000000000
      -0.500000000000000       2.000000000000000       0.000000000000000
       0.500000000000000       1.500000000000000       1.000000000000000
```



PROGRAM 3.8-2 Cholesky symmetric matrix decomposition method python version

```
 # -*- coding: utf-8 -*-
"""
Created on Thu Aug 23 09:46:37 2018

@author: Mustafa Turhan Çoban
"""
from math import *
def choleskyLU(c):
#cholesky symmetric matrix LU
#symmetric matrices only
    factor=0;
    n=len(c);
    d=[[0.0 for i in range(n)] for j in range(n)];
    for i in range(n):
        for j in range(n):
            d[i][j]=c[i][j];
    for k in range(1,n+1):
        for i in range(1,k):
            sum=0;
            for j in range(1,i):
                sum+=d[i-1][j-1]*d[k-1][j-1];
            d[k-1][i-1]=(d[k-1][i-1]-sum)/d[i-1][i-1];
        sum=0.0;
        for j in range(1,k):
            sum+=d[k-1][j-1]*d[k-1][j-1];
        d[k-1][k-1]=sqrt((d[k-1][k-1]-sum));
        for j in range(k+1,n+1):
            d[k-1][j-1]=0.0;
    return d;
```

```
def T(A):
    #transpose matrix
    n=len(A);
    m=len(A[0]);
    B = [[0 for x in range(m)] for y in range(n)];
    for i in range(n):
        for j in range(m):
            B[i][j]=A[j][i];
    return B;

def substitutecholesky(c,b):
  #solving of system of equation by using cholesky decomposed matrix
  U=T(c);
  L=c;
  n=len(b);
  sum=0;
  x=[0.0 for j in range(n)];
  d=[0.0 for j in range(n)];
  #forward substituting
  toplam=0;
  for i in range(n):
      toplam=0;
      for j in range(i):
          toplam+=L[i][j]*d[j];
      d[i]=(b[i]-toplam)/L[i][i];
  #backward substitution
  x[n-1]=d[n-1]/U[n-1][n-1];
  for i in range(n-2,-1,-1):
      toplam=0;
      for j in range(i+1,n):
          toplam+=U[i][j]*x[j];
      x[i]=(d[i]-toplam)/U[i][i];
  return x;

a=[[4.0,-1.0,1.0],[-1.0,4.25,2.75],[1.0,2.75,3.5]];
b=[1.0,1.0,1.0];
L=choleskyLU(a);
print("L=\n",L);
x=substitutecholesky(L,b);
print("x=\n",x);
```

```
runfile('D:/okul/SCO1/choleskyLU.py', wdir='D:/okul/SCO1')
L=
 [[2.0, 0.0, 0.0], [-0.5, 2.0, 0.0], [0.5, 1.5, 1.0]]
x=
 [0.41015625, 0.453125, -0.1875]
```

PROGRAM Cholesky symmetric matrix decomposition method Julia version

```
 function  choleskyLU(c,B)
# cholesky symmetric matrix LU
# symmetric matrices only
A=cholesky(c)
print("\n cholesky L\n")
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), A.L)
print("\n cholesky U\n")
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), A.U)
D=A.L\B'
x=A.U\D
return x
end # for function

using LinearAlgebra
a=[4.0 -1 1;-1 4.25 2.75;1 2.75 3.5]
B=[1.0 1 1]
x=choleskyLU(a,B)
print("\n cholesky x = $x\n")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" choleskyLU.jl

 cholesky L
3X3 LowerTriangular{Float64, Matrix{Float64}}:
  2.0   â‹…    â‹…
```

```
 -0.5  2.0   â‹…
  0.5  1.5  1.0
 cholesky U
3X3 UpperTriangular{Float64, Matrix{Float64}}:
 2.0  -0.5  0.5
  â‹…    2.0  1.5
  â‹…     â‹…   1.0
 cholesky x = [0.41015625; 0.453125; -0.1875]

> Terminated with exit code 0.
```

Let us solve the same problem step by step by hand:

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{-1}{2} = -0.5$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{4.25 - (-0.5)^2} = 2$$

$$l_{32} = \frac{a_{32} - l_{21} - l_{31}}{l_{22}} = \frac{2.75 - (-0.5) * 0.5}{2} = 1.5$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{3.5 - (0.5)^2 - (1.5)^2} = 1$$

so cholesky decomposed L matrix :

$$[L] = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix}$$

$$[U] = [L]^T = \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[L]\{D\} = \{B\}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

d_1=1/2=0.5

d_2=(1-(-0.5)*0.5)/2=0.625

d_3=(1-0.5*0.5-1.5*0.625)/1=-0.1875

$$[U]\{x\} = \{D\}$$

$$\begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.625 \\ -0.1875 \end{Bmatrix}$$

x_3=-0.1875

x_2=(0.625-1.5*(-0.1875))/2=0.453125

x_1=(0.5-(-0.5)*0.453125-0.5*(-0.1875))/2=0.41015625

3.9 JACOBI GAUSS -SEIDEL AND SUCCESSIVE RELAXATION METHODS

In this section, iterative solution methods will be described. Iterative methods could be important in solution of big matrices. Gauss elimination requires $n^3/3$, Gauss-Jordan method requires $n^3/2$ process. When n is big, the total number of process will climb up. In this case the total number of processes in iterative methods will be much smaller. The simplest of the iterative methods are Jacobi method. In this method, a set of x solution vector is selected and substituted into the system of equation in the following form :

$$x_i^{k+1} = \frac{1}{A_{ij}}\left(b_i - \sum_{j \neq i} A_{ij} x_j^k\right) \quad 1 \leq i \leq n \qquad (3.9.1)$$

When a new set of x values is found, it will be substituted in the the equation as the next iteration step. To ensure the convergence matrix should be diagonally dominant

$$\sum_{i=1}^{n}|A_{ii}| \geq \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}|A_{i,j}| \qquad (3.9.2)$$

Diagonal dominancy is the important condition for all iterative formulations. Unfortunately, very big matrices solved in engineering problems are in general fits to this definition.

In order to reduce number of iteration steps, every new calculated x value in the x vector set can be substited into the equation as soon as it is calculated and is not waited the complete set to be calculated as in Jacobi method. This form of iterative process is called Gauss-Seidel iteration.

$$x_i^{k+1} = \frac{1}{A_{ij}}\left(b_i - \sum_{j=1}^{i-1} A_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} A_{ij}x_j^{k}\right) \quad 1 \leq i \leq n \qquad (3.9.3)$$

Gauss-Seidel and Jacobi methods can be merged to define an intermediate method between two by defining a relaxation coefficient, α, in the equation.

$$x_i^{k+1} = (1-\alpha)x_i^{k+1} + \frac{\alpha}{A_{ij}}\left(b_i - \sum_{j=1}^{i-1} A_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} A_{ij}x_j^{k}\right) \quad 1 \leq i \leq n \quad (3.9.4)$$

When α is equal to one equation coverged to the Gauss-Seidel method.

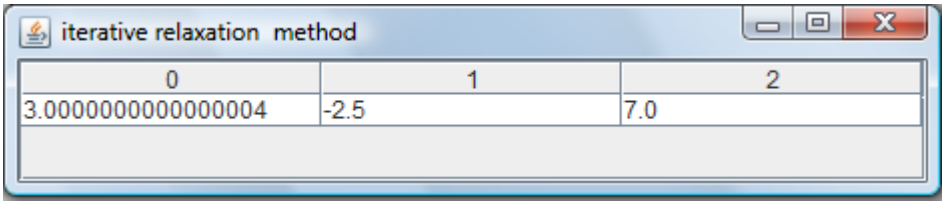
PROGRAM Iterative relaxation method

```
import java.io.*;

class NA21
{
  public static double [] gauss_seidel(double a[][],double b[])
  {
  double lambda=1.2;
  int n=a.length;
  //starting x vector is zero
  double x[]=new double[n];
  return gauss_seidel(a,b,x,lambda);
  }

  public static double [] gauss_seidel(double a[][],double b[],double x[])
  {
  double lambda=1.2;
  int n=a.length;
  return gauss_seidel(a,b,x,lambda);
  }

  public static double [] gauss_seidel(double a[][],double b[],double x[],double lambda)
  {
  //lambda is the relaxation coefficient
  int n=a.length;
  int imax=500;//maximum number of iterations
  double es=1.0e-20;
  double dummy=0;
  double ea=1e-1;
  double sum=0;
  double old=0;
  int iter=0;
  int sentinel=0;
  System.out.print("iter = "+(iter+1)+" ");
  for(int i=0;i<n;i++)
  {  dummy=a[i][i];
     for(int j=0;j<n;j++)
       {a[i][j]/=dummy;}
     b[i]/=dummy;
  }
  for(int i=0;i<n;i++)
  {  sum=b[i];
```

```
    for(int j=0;j<n;j++)
      {if(i!=j)sum-=a[i][j]*x[j];}
    x[i]=sum;
    System.out.print(x[i]+ "  ");
  }
   System.out.println();
  iter++;
  do
  {
  sentinel=1;
  System.out.print("iter = "+(iter+1)+" ");
  for(int i=0;i<n;i++)
    {
              old=x[i];
              sum=b[i];
              for(int j=0;j<n;j++)
                 {if(i!=j) sum-=a[i][j]*x[j];}
                 x[i]=lambda*sum+(1.0-lambda)*old;
                 if(sentinel==1 && x[i]!=0.0 ) {ea=Math.abs((x[i]-old)/x[i])*100.0;}
                 if(ea<es) sentinel=0;
               System.out.print(x[i]+ "  ");
    }
  System.out.println();
  iter++;
  }while(sentinel==1 && iter<imax);
  if(iter>=imax) System.out.println("Maximum number of iteration is exceed, result might not be valid iter = "+iter);
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{3,-0.1,-0.2},{0.1,7,-0.3},{0.3,-0.2,10}};
  double b[]={7.85,-19.3,71.4};

  double x[]={0,0,0};
  double lambda=0.0;
  Text.print(gauss_seidel(a,b),"iterative relaxation  method");
  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA21
iter = 1 2.6166666666666667  -2.7945238095238096  7.005609523809524
iter = 2 3.0653344761904764  -2.4419267678911565  6.997919811665852
iter = 3 2.9890896189795266  -2.5115345924328882  7.000531981165179
iter = 4 3.0017632509999936  -2.49769594964207  6.999885423939555
iter = 5 2.999730345729483  -2.5004620799100574  7.000021532847987
iter = 6 3.00003717028554  -2.4999071138192726  6.999996584568461
iter = 7 2.999996008155598  -2.5000186844552923  7.000000378365779
iter = 8 3.0000000812599312  -2.4999962450431576  7.000000011520452
iter = 9 3.000000134867924  -2.500000752710909  6.999999974775603
iter = 10 2.999999940900027  -2.4999998497419305  7.000000010778672
iter = 11 3.000000018692611  -2.500000029817727  6.999999996455708
iter = 12 2.9999999947852256  -2.4999999941293365  7.0000000010374865
iter = 13 3.00000000136078  -2.500000001144104  6.999999999716058
iter = 14 2.9999999996593645  -2.499999999779943  7.0000000000743325
iter = 15 3.000000000082876  -2.500000000041609  6.999999999981153
iter = 16 2.9999999999802522  -2.4999999999923093  7.000000000004666
iter = 17 3.0000000000046305  -2.500000000001377  6.999999999998867
iter = 18 2.9999999999989284  -2.4999999999997646  7.000000000000272
```

```
iter = 19 3.000000000000245  -2.500000000000037  6.999999999999938
iter = 20 2.9999999999999445  -2.4999999999999956  7.000000000000015
iter = 21 3.0000000000000124  -2.5000000000000004  6.999999999999997
iter = 22 2.999999999999997  -2.5  7.000000000000001
iter = 23 3.0000000000000004  -2.5  7.0
```



PROGRAM Iterative relaxation method, Python code

```
 # -*- coding: utf-8 -*-
"""
Created on Thu Aug 30 09:51:34 2018
@author: Mustafa Turhan Çoban
"""
def gauss_seidel(a,b,x,lembda):
    lamd=1.2;
    n=len(a);
    imax=500;#maximum number of iterations
    es=1.0e-20;
    dummy=0;
    ea=1e-1;
    sum=0;
    old=0;
    iter=0;
    sentinel=0;
    for i in range(n):
        dummy=a[i][i];
        for j in range(n):
            a[i][j]/=dummy;
        b[i]/=dummy;
    for i in range(n):
        sum=b[i];
        for j in range(n):
            if i!=j:sum-=a[i][j]*x[j];
        x[i]=sum;
    iter=iter+1;
    while iter<imax:
      sentinel=1;
      for i in range(n):
          old=x[i];
          sum=b[i];
          for j in range(n):
              if i!=j:
                  sum-=a[i][j]*x[j];
          x[i]=lamd*sum+(1.0-lamd)*old;
          if sentinel==1 and x[i]!=0.0: ea=abs((x[i]-old)/x[i])*100.0;
          if ea<es: return x;
      iter+=1;
    if iter>=imax: print("Maximum number of iteration is exceed, result might not be valid iter = ",iter);
    return x;

a=[[3,-0.1,-0.2],[0.1,7,-0.3],[0.3,-0.2,10]];
b=[7.85,-19.3,71.4];
x=[0,0,0];
lamd=0.0;
print("iterative relaxation  method",gauss_seidel(a,b,x,lamd));
```

```
runfile('D:/okul/SCO1/gauss_seidel.py', wdir='D:/okul/SCO1')
iterative relaxation  method [3.0000000000000004, -2.5, 7.000000000000001]
```

```
 function  gauss_seidel(ai,bi,xi,lambda)
    n=size(ai)[2]
    lamd=lambda
```

```
    imax=500#maximum number of iterations
    es=1.0e-20
    dummy=0.0
    ea=1.0e-1
    sum=0.0
    old=0.0
    sentinel=0.0
    i=0
    j=0
    n=size(ai)[1]
    x=zeros(Float64,n)
    b=zeros(Float64,n)
    a=zeros(Float64,n,n)

      for i=1:n
      for j=1:n
         a[i,j]=ai[i,j]
      end
         b[i]=bi[i]
         x[i]=xi[i]
      end

    for i=1:n
       dummy=a[i,i]
       for j=1:n
          a[i,j]/=dummy
       end # for j
       b[i]/=dummy
     end # for i
    for i=1:n
       sum=b[i]
       for j=1:n
           if i!=j
             sum=sum-a[i,j]*x[j]
           end # if
        end # for j
       x[i]=sum
    end #
    iter1=0
    while iter1<imax
      sentinel=1
      for i =1:n
         old=x[i]
         sum=b[i]
         for j =1:n
            if i!=j
               sum-=a[i,j]*x[j]
            end # if
         end # for j
         x[i]=lamd*sum+(1.0-lamd)*old
         if sentinel==1 && x[i]!=0.0
            ea=abs((x[i]-old)/x[i])*100.0
         end # if
         if ea<es
           return x
         end # if
      iter1+=1
      end # for i

    end # while
    if iter1>=imax
      print("Maximum number of iteration is exceed, result might not be valid iter = ",iter1)
    end # if
end #function



a=[3.0 -0.1 -0.2;0.1 7.0 -0.3;0.3 -0.2 10.0]
b=[7.85 -19.3 71.4]
x=[0,0,0]
lambda=0.5
x=gauss_seidel(a,b,x,lambda)
print("iterative relaxation  method $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_seidel.jl
```

```
iterative relaxation  method [2.9999999999999982, -2.499999999999999, 7.0000000000000036]
> Terminated with exit code 0.
```

```
function  gauss_seidel(ai,bi,xi,lambda)
  n=size(ai)[2]
  lamd=lambda
  imax=500#maximum number of iterations
  es=1.0e-20
  dummy=0.0
  ea=1.0e-1
  sum=0.0
  old=0.0
  sentinel=0.0
  i=0
  j=0
  n=size(ai)[1]
  x=zeros(Float64,n)
  b=zeros(Float64,n)
  a=zeros(Float64,n,n)

    for i=1:n
    for j=1:n
       a[i,j]=ai[i,j]
    end
       b[i]=bi[i]
       x[i]=xi[i]
    end

  for i=1:n
     dummy=a[i,i]
     for j=1:n
        a[i,j]/=dummy
     end # for j
     b[i]/=dummy
   end # for i
  for i=1:n
     sum=b[i]
     for j=1:n
         if i!=j
           sum=sum-a[i,j]*x[j]
         end # if
      end # for j
     x[i]=sum
  end #
  iter1=0
  while iter1<imax
    sentinel=1
    for i =1:n
       old=x[i]
       sum=b[i]
       for j =1:n
          if i!=j
             sum-=a[i,j]*x[j]
          end # if
       end # for j
       x[i]=lamd*sum+(1.0-lamd)*old
       if sentinel==1 && x[i]!=0.0
          ea=abs((x[i]-old)/x[i])*100.0
       end # if
       if ea<es
         return x
       end # if
    iter1+=1
    end # for i

  end # while
  if iter1>=imax
    print("Maximum number of iteration is exceed, result might not be valid iter = ",iter1)
  end # if
end #function



a=[3.0 -0.1 -0.2;0.1 7.0 -0.3;0.3 -0.2 10.0]
```

```
b=[7.85 -19.3 71.4]
x=[0,0,0]
lambda=0.5
x=gauss_seidel(a,b,x,lambda)
print("iterative relaxation  method $x")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_seidel.jl
iterative relaxation  method [2.9999999999999982, -2.499999999999999, 7.0000000000000036]
> Terminated with exit code 0.
```

3.10 METHODS FOR BAND MATRICES

A three-band matrix can be shown as:

$$\begin{bmatrix} f_1 & g_1 & & & & & & \\ e_2 & f_2 & g_1 & & & & & \\ & e_3 & f_3 & g_1 & & & & \\ & & e_4 & f_4 & g_1 & & & \\ & & & e_5 & f_5 & g_1 & & \\ & & & & . & . & . & \\ & & & & & & f_{n-1} & g_{n-1} \\ & & & & & & e_n & f_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ . \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ . \\ r_{n-1} \\ r_n \end{Bmatrix}$$

This kind of matrix can be solved by using decomposition back and front substitution processes. Following form of basic solition process is called Thomas algorithm

Decomposition step :

```
for(int k=1;k<n;k++)
  {e[k]=e[k]/f[k-1];
   f[k]=f[k]-e[k]*g[k-1];
  }
```

Forward substitution :

```
for(int k=1;k<n;k++)
  {r[k]=r[k]-e[k]*r[k-1];
  }
```

Backward substitution:

```
x[n-1]=r[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
  {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
```

The whole Thomas algorithm is defined in the following code

Program 3.10-1 Thomas algorithm for band matrices (Java version)

```
import java.io.*;

class NA23
{
  public static double [] thomas(double a[][],double r[])
  {
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
    {e[k]=e[k]/f[k-1];
     f[k]=f[k]-e[k]*g[k-1];
    }
```

```
  for(int k=1;k<n;k++)
    {r[k]=r[k]-e[k]*r[k-1];
    }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
    {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

  public static double [] thomas(double f[],double e[],double g[],double r[])
  {
  int n=f.length;
  double x[]=new double[n];
  for(int k=1;k<n;k++)
    {e[k]=e[k]/f[k-1];
     f[k]=f[k]-e[k]*g[k-1];
    }
  for(int k=1;k<n;k++)
    {r[k]=r[k]-e[k]*r[k-1];
    }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
    {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double f[]={2.04,2.04,2.04,2.04};
  double g[]={-1,-1,-1,0};
  double e[]={0,-1,-1,-1};
  double b[]={40.8,0.8,0.8,200.8};
  double x[]=thomas(f,e,g,b);
  Text.print(x,"thomas algorithm for 3 band matrix");
  System.out.println("thomas : \n"+Matrix.toStringT(x));
  double a[][]={{2.04,-1,0,0},{-1,2.04,-1,0 },{0,-1,2.04,-1},{0,0,-1,2.04}};
  double d[]={40.8,0.8,0.8,200.8};
  System.out.println("thomas : \n"+Matrix.toStringT(thomas(a,d)));
  }
}
```

```
thomas :
      65.969834366776620
      93.778462108224330
     124.538228334001000
     159.479523693137740
```

Another example with detailed calculation steps:

$$\begin{bmatrix} f_1 & g_1 & 0 \\ e_2 & f_2 & g_2 \\ 0 & e_3 & f_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

$$\begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 41 \\ 25 \\ 105 \end{Bmatrix}$$

Decomposition:

$$e_k = \frac{e_k}{f_{k-1}}$$

$$f_k = f_k - e_k g_{k-1}$$

$$e_2 = \frac{e_2}{f_1} = \frac{-0.4}{0.8} = -0.5$$

$$f_2 = f_2 - e_2 g_1 = 0.8 - (-0.5)(-0.4) = 0.6$$

$$e_3 = \frac{e_3}{f_3} = \frac{-0.4}{0.6} = -0.666666666$$

$$f_3 = f_3 - e_3 g_3 = 0.8 - (-0.666666666)(-0.4) = 0.533333333$$

Matrix took the form of

$$\begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.6 & -0.4 \\ 0 & -0.666666666 & 0.533333333 \end{bmatrix}$$

LU decomposed form of the matrix :

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.66666 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.6 & -0.4 \\ 0 & 0 & 0.53333 \end{bmatrix}$$

Forward substitution

$$[L]\{D\} = \{B\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.66666 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 41 \\ 25 \\ 105 \end{Bmatrix}$$

d_1=41

d_2=25-(-0.5)*41=45.5

d_3=105-(-0.66666)*45.5=135.3333333

Backward substitution

$$[U]\{x\} = \{D\}$$

$$\begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.6 & -0.4 \\ 0 & 0 & 0.53333 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 41 \\ 45.5 \\ 135.33333 \end{Bmatrix}$$

X_3=135.33333/0.53333=253.75

X_2=(45.5-(-0.4)* 253.75)/0.6=245

X_1=(41-(-0.4)*245)/0.8=173.75

Python version of the Thomas algorithm is given below:

Program 3.10-2 Thomas algorithm for band matrices (python version)

```
# -*- coding: utf-8 -*-
"""
Created on Sun Jun  3 21:38:38 2018

@author: Mustafa Turhan Çoban
"""
def thomas(a,r):
    n=len(a);
    f=[0.0 for i in range(n)];
    e=[0.0 for i in range(n)];
    g=[0.0 for i in range(n)];
    x=[0.0 for i in range(n)];
    for i in range(0,n): f[i]=a[i][i];
    for i in range(0,n-1): g[i]=a[i][i+1];
    for i in range(0,n-1): e[i+1]=a[i+1][i];
    for k in range(0,n): e[k]=e[k]/f[k-1];f[k]=f[k]-e[k]*g[k-1];
    for k in range(1,n): r[k]=r[k]-e[k]*r[k-1];
    x[n-1]=r[n-1]/f[n-1];
    for k in range(n-2,-1,-1):  x[k]=(r[k]-g[k]*x[k+1])/f[k];
    return x;

def thomas_n(f,e,g,r):
    n=len(f);
    x=[0.0 for i in range(n)];
    for k in range(0,n): e[k]=e[k]/f[k-1];f[k]=f[k]-e[k]*g[k-1];
    for k in range(1,n): r[k]=r[k]-e[k]*r[k-1];
    x[n-1]=r[n-1]/f[n-1];
    for k in range(n-2,-1,-1):  x[k]=(r[k]-g[k]*x[k+1])/f[k];
    return x;

a=[[0.8,-0.4,0],[-0.4,0.8,-0.4],[0,-0.4,0.8]];
b=[41,25,105];
x=thomas(a,b);
print("x=",x);
```

```
runfile('D:/okul/SCO1/thomas.py', wdir='D:/okul/SCO1')
x= [173.75, 244.99999999999997, 253.74999999999997]
```

Program Thomas algorithm for band matrices (julia version)

```
function thomas(a,b)
  n=size(a)[1]
```

```
  global f=zeros(Float64,n)
  global e=zeros(Float64,n)
  global g=zeros(Float64,n)
  global x=zeros(Float64,n)
  global r=zeros(Float64,n)
  i=0
  j=0
  k=0
    for i=1:n
    r[i]=b[i]
  end
  for i=1:n
    f[i]=a[i,i]
  end

  for i=1:n-1
    g[i]=a[i,i+1]
  end

  for i=1:n-1
    e[i+1]=a[i+1,i]
  end

  for k=2:n
    e[k]=e[k]/f[k-1]
    f[k]=f[k]-e[k]*g[k-1]
  end #for

  for k=2:n
    r[k]=r[k]-e[k]*r[k-1]
  end #for

  x[n]=r[n]/f[n]
  n1=n-1
  for k=n1:-1:1
    x[k]=(r[k]-g[k]*x[k+1])/f[k]
  end #for
  return x
  end #function

function thomas(f,e,g,r)
  n=size(f)[1]
  global x=Array{Float64}(undef,n)
for k=2:n
    e[k]=e[k]/f[k-1]
    f[k]=f[k]-e[k]*g[k-1]
  end #for

  for k=2:n
    r[k]=r[k]-e[k]*r[k-1]
  end #for

  x[n]=r[n]/f[n]
  n1=n-1
  for k=n1:-1:1
    x[k]=(r[k]-g[k]*x[k+1])/f[k]
  end #for
  return x
  end #function



a=[0.8 -0.4 0;-0.4 0.8 -0.4;0 -0.4 0.8]
b=[41 25 105]
x=thomas(a,b)
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), x)
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" thomas.jl
3-element Vector{Float64}:
 173.75
 244.99999999999997
 253.74999999999997
> Terminated with exit code 0.
```

Program Thomas algorithm for band matrices (Octave-Matlab)

```
 ## Author: M. Turhan ÇOBAN
## Created: 2021-06-13

function retval = thomas(a,r)
   n=length(a);
   f=zeros(1,n);
   e=zeros(1,n);
   g=zeros(1,n);
   x=zeros(1,n);
   for i=1:n
     f(i)=a(i,i);
   endfor
   for i=1:n-1
     g(i)=a(i,i+1);
   endfor
   for i=1:n-1
     e(i+1)=a(i+1,i);
   endfor
   for k=2:n
     e(k)=e(k)/f(k-1);
     f(k)=f(k)-e(k)*g(k-1);
   endfor
   for k=2:n
     r(k)=r(k)-e(k)*r(k-1);
   endfor
   x(n)=r(n)/f(n);
   for k=n-1:-1:1
     x(k)=(r(k)-g(k)*x(k+1))/f(k);
   endfor
   retval=x;
endfunction
```

```
>> A=[0.8 -0.4 0; -0.4 0.8 -0.4;0 -0.4,0.8]
A =

   0.800000000000000  -0.400000000000000                   0
  -0.400000000000000   0.800000000000000  -0.400000000000000
                   0  -0.400000000000000   0.800000000000000

>> B=[41 25 105]
B =

    41    25   105

>> x=thomas(A,B)
x =

   173.7500000000000   245.0000000000000   253.7500000000000

>>
```

Five band matrix equation can be written as :

$$\begin{bmatrix} d_1 & c_1 & f_1 & & & & & \\ a_1 & d_2 & c_2 & f_2 & & & & \\ e_1 & a_2 & d_3 & c_3 & f_3 & & & \\ & e_2 & a_3 & d_4 & c_4 & f_4 & & \\ & & e_3 & a_4 & d_5 & c_5 & f_5 & \\ & & & . & . & . & . & f_{n-2} \\ & & & & e_{n-3} & a_{n-2} & d_{n-1} & c_{n-1} \\ & & & & & e_{n-2} & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ . \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ . \\ b_{n-1} \\ b_n \end{bmatrix}$$

It should be noted that bands are written as vectors, instead of a complete matrix. It will give us storage advantage as well as calculation advantage. A five band algorithm is given in the following program for the system of equation:

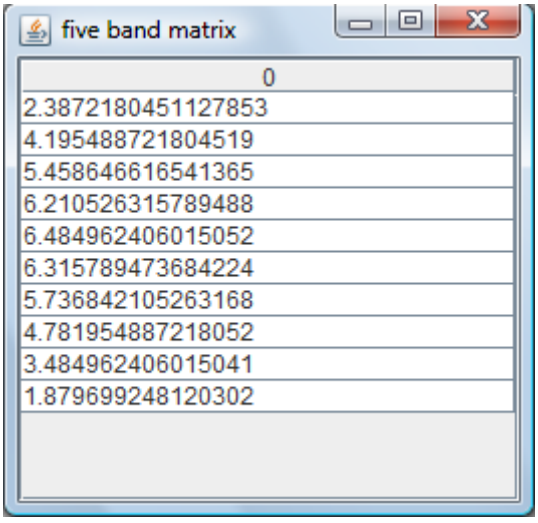
$$\mathrm{A} = \begin{bmatrix} 6 & -4 & 1 & 0 & 0 & \cdots \\ -4 & 6 & -4 & 1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ & \ddots & \ddots & \ddots & \ddots & \\ \cdots & 0 & 1 & -4 & 6 & -4 \\ \cdots & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 4 \end{bmatrix}$$

Program five band matrix algorithm (Java)

```
import java.io.*;

class NA24
{
 public static double [] fiveband(double A[][],double b[])
 {
 //
 int n=A.length;
 double f[]=new double[n-1];
 double c[]=new double[n-1];
 double d[]=new double[n];
 double a[]=new double[n-1];
 double e[]=new double[n-2];
 double x[]=new double[n];

 for(int i=0;i<n;i++)    {d[i]=A[i][i];}
 for(int i=0;i<(n-1);i++) {c[i]=A[i][i+1];}
 for(int i=0;i<(n-1);i++) {a[i]=A[i+1][i];}
 for(int i=0;i<(n-2);i++) {f[i]=A[i][i+2];}
 for(int i=0;i<(n-2);i++) {e[i]=A[i+2][i];}
 int i;
 double xmult;
 for(i=1;i<(n-1);i++)
 {xmult=a[i-1]/d[i-1];
  d[i]+=-xmult*c[i-1];
  c[i]+=-xmult*f[i-1];
  b[i]+=-xmult*b[i-1];
  xmult=e[i-1]/d[i-1];
  a[i]+=-xmult*c[i-1];
  d[i+1]+=-xmult*f[i-1];
  b[i+1]+=-xmult*b[i-1];
  //System.out.println(i+"d="+d[i]+"c="+c[i]+"b="+b[i]);
 }
 xmult=a[n-2]/d[n-2];
 d[n-1]+=-xmult*c[n-2];
 x[n-1]=(b[n-1]-xmult*b[n-2])/d[n-1];
 x[n-2]=(b[n-2]-c[n-2]*x[n-1])/d[n-2];
 for(i=n-3;i>=0;i--)
 {x[i]=(b[i]-f[i]*x[i+2]-c[i]*x[i+1])/d[i];}
 return x;
```

```
  }

   public static double [] fiveband(double f[],double c[],double d[],double a[],double e[],double b[])
  {
  int n=d.length;
  double x[]=new double[n];
  int i;
  double xmult;
  for(i=1;i<(n-1);i++)
  {xmult=a[i-1]/d[i-1];
   d[i]+=-xmult*c[i-1];
   c[i]+=-xmult*f[i-1];
   b[i]+=-xmult*b[i-1];
   xmult=e[i-1]/d[i-1];
   a[i]+=-xmult*c[i-1];
   d[i+1]+=-xmult*f[i-1];
   b[i+1]+=-xmult*b[i-1];
   //System.out.println(i+"d="+d[i]+"c="+c[i]+"b="+b[i]);
  }
  xmult=a[n-2]/d[n-2];
  d[n-1]+=-xmult*c[n-2];
  x[n-1]=(b[n-1]-xmult*b[n-2])/d[n-1];
  x[n-2]=(b[n-2]-c[n-2]*x[n-1])/d[n-2];
  for(i=n-3;i>=0;i--)
  {x[i]=(b[i]-f[i]*x[i+2]-c[i]*x[i+1])/d[i];}
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double d[]={6,6,6,6,6,6,6,6,6,7};
  double c[]={-4,-4,-4,-4,-4,-4,-4,-4,-4};
  double f[]={1,1,1,1,1,1,1,1};
  double a[]={-4,-4,-4,-4,-4,-4,-4,-4,-4};
  double e[]={1,1,1,1,1,1,1,1};

  double b[]={3,0,0,0,0,0,0,0,0,4};
  double x[]=fiveband(f,c,d,a,e,b);
  Text.printT(x,"five band matrix");
  }
}
```

The solution vector is :



3.11 CONJUGATE GRADIENT METHOD

If vector x^* is a solution of positive and well defined system of equation Ax = b, this solution is valid when the vector x^* is defined as minimum of function $g(x) = x^T Ax - 2b^T x$. The x vector is defined as a different vector then the solution vector x* . In addition a v vector with values different than 0 and a real number α is defined.

$$g(x + \alpha v) = g(x) + 2\alpha[v^T(Ax - B)] + \alpha^2[v^T(Av)]$$

In this equation, if v and x vector is known, for some value of α function will be minimum (One dimensional minimisation problem). If it is assumed that only independent variable of the function is α

$h(\alpha) = g(x + \alpha v)$ can be written. The minimum of the function, can be defined as the root of the derivative of the function with respect to α

$$h'^{(\alpha)} = 2[v^T(Ax - B)] + 2\alpha[v^T(Av)] = 0$$

From this equation

$$\alpha = \frac{v^T(Ax - B)}{v^T(Av)}$$

To simplify the equation a little bit further definition $r = b - Ax$

r = b – A x can be applied. If this definition is substituted into the previous equation:

$$\alpha = \frac{v^T r}{v^T(Av)}$$

If α is substituted into the iterative process iterative process

$$x_k = x_{k-1} + \alpha_k v_k$$

and for $r_k = b - Ax_k$ value would be

$$\alpha_k = \frac{v^T r_k}{v_k^T(Av_k)}$$

New x_k vector should be give solution to the $Ax_k = b$ equation . When a solution is obtained, it is known that equation will converged to

$$Ax_k = A(x_{k-1} + \alpha_k v_k) = b$$

But we still do not know the search direction vector, v_k . As a starting search direction **steepest descent** optimization direction can be selected. This direction is defined as

$$\nabla g(x) = 2(ax - b) = -2r$$

In this case v_{k+1} can be selected as

$$v_{k+1} = -\nabla g(x_k) = r_k$$

This direction gives the maximum negative change in $g(x_k)$. An alternative selection to the direction vector can be carried out such a vector that for this vector condition $v_i^T A v_j = 0 \quad i \neq j$ will be valid for a non-zero v vector. This type of vectors are called orthogonal vectors. If some orthogonal v vectors can be found, it will simplify our solution. In orthogonal sets terms out of diagonal will be zero so only diagonal terms will remain in the equation. Therefore process of solving matrix system will be converted to simple multiplication and division processes. The orthogonal set suggested by Hesteness and Stiefel has the following form :

$$p_k = \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$$

$$v_{k+1} = r_k + p_k v_k$$

As it is seen from the equation, values are derived from the previous sets. In summary this method can be applied as:

In the first iteration steepest descent value is taken.

$r_0 = b - Ax_0$ and $v_0 = r_0$(steepest descent value) .

then for the next iteration sets calculates the values as

for k=1 to n

$$\alpha_k = \frac{r_{k-1}^T r_k}{v_k^T A v_k}$$

$$x_k = x_{k-1} + \alpha_k v_k$$

$$r_k = b - Ax_k$$

$$p_k = \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$$

$$v_{k+1} = r_k + p_k v_k$$

As it is seen process is quite a complex process compared to Gauss elimination type of solution. It is not preffered to Gauss elimination in solution of the small systems, but when the system becomes larger, it will be prefferable.

Conjugate-gradient method to solve system of equations (java)

```
import java.util.*;
import java.awt.*;
```

```
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;
//CONJUGATE GRADIENT METHOD
public class NA25
{
public static double VT_V(double [] left)
{
//multiplies a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
              aa[i][j]=left[i]*left[j];
              }
  }
return aa;
}

public static double  VT_X(double [] left,double [] right)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}

public static double[] multiply(double[] left,double[][] right)
{
//multiplication of one vector with one matrix
int ii,jj,i,j,k;
int m2=right[0].length;
int n2=right.length;
int m1=left.length;
double[] b;
b=new double[m1];
 if(n2 != m1)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n2;ii++)
   {
     b[ii]=0;
   }
```

```
   return b;
 }
 for(i=0;i<m2;i++)
   {
     b[i]=0;
     for(k=0;k<m1;k++)
               b[i]+=right[i][k]*left[k];
   }
return b;
//end of multiply of a vector and a matrix
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
     b[ii]=0;
   }
   return b;
 }
 for(i=0;i<m1;i++)
   {
     b[i]=0;
     for(k=0;k<n1;k++)
               b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}
public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {
  b[i]=left*right[i];
  }
return b;
}
public static double[] substract(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
```

```
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]-right[i];
  }
return b;
//end of vector substraction method
}
public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

public static double[] equals(double[] left)
{
//addition of two vectors
int n1=left.length;
double b[];
b=new double[n1];
for(int i=0;i<n1;i++)
  {
    b[i]=left[i];
  }
return b;
//end of vector equal method
}

public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}
public static double VT_G_V(double V[],double G[][])
{
  double x1[]=multiply(V,G);
  return VT_X(V,x1);
}
```

```
public static double[] conjugate_gradient(double A[][],double b[])
{int n=A.length;
 double x[]=new double[n];
 return conjugate_gradient(x,A,b);
}


public static double[] conjugate_gradient(double x[],double A[][],double b[])
{ // iterative system of equation solution
double eps=1.0e-15;
int maxiter=500;
int n=x.length;
double r[]=new double[n];
double re[]=new double[n];
double v[]=new double[n];
double alpha;
double p;
r=substract(b,multiply(A,x));
v=equals(r);
int k=0;
do
  {
  re=equals(r);
  alpha=VT_V(re)/VT_G_V(v,A);
  x=add(x,multiply(alpha,v));
  r=substract(r,multiply(multiply(alpha,v),A));
  p=VT_V(r)/VT_V(re);
  v=add(r,multiply(p,v));
  k++;
  }while(norm(r)>eps && k<maxiter);
System.out.println("k = "+k+" norm(r)="+norm(r));
return x;
}

  public static void main(String args[]) throws IOException
  {
    double A[][]={{0.2,0.1,1,1,0},{0.1,4,-1,1,-1},{1,-1,60,0,-2},{1,1,0,8,4},{0,-1,-2,4,700}};
  double b[]={1,2,3,4,5};
  double x[]=conjugate_gradient(A,b);
  System.out.println("conjugate gradient method  : \n"+Matrix.toStringT(x));
  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA25
k = 9 norm(r)=5.161554370481792E-18
conjugate gradient method  :
        7.859713075445860
        0.422926408295008
       -0.073592239024046
       -0.540643016894627
        0.010626162854036

> Terminated with exit code 0.
```

Numerical example

To illustrate the conjugate gradient method, we will complete a simple example. Considering the linear system Ax = b given by

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we will perform two steps of the conjugate gradient method beginning with the initial guess

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

in order to find an approximate solution to the system.

Solution

k= 0 alpha = 0.25 p=0.0625		
r1 =	-0.500000000000000	0.250000000000000
v1 =	-0.437500000000000	0.375000000000000
x1 =	0.250000000000000	0.500000000000000
k= 1 alpha = 0.36363636363636365 p=2.465190328815662E-33		
r2 =	0.000000000000000	0.000000000000000
v2 =	-0.000000000000000	0.000000000000000
x2 =	0.090909090909091	0.636363636363636

The result, $\mathbf{x}_2$, is a "better" approximation to the system's solution than $\mathbf{x}_1$ and $\mathbf{x}_0$. If exact arithmetic were to be used in this example instead of limited-precision, then the exact solution would theoretically have been reached after $n = 2$ iterations (n being the order of the system).

We will now extend the conjugate gradient method to include ***preconditioning*.** If the matrix *A* is ill-conditioned, the conjugate gradient method is highly susceptible to rounding errors. So, although the exact answer should be obtained in *n* steps, this is not usually the case. As a direct method the conjugate gradient method is not as good as Gaussian elimination with pivoting. The main use of the conjugate gradient method is as an iterative method applied to a better-conditioned system. To apply the method to a better-conditioned system, we want to select a nonsingular conditioning matrix *C* so that system of equation can be converted to a better conditioned system.

$A^* = C^{-1}AC^{-1} \quad x^* = C^{-1}x \quad b^* = C^{-1}b \quad A^*x^* = b^*$

As a C conditioning matrix, a matrix with the rootsquare values of the diagonal elements and 0 values for the rest of the elements can be selected. For such a matrix, inverse matrix can be produced just by dividing numbers in the matrix by 1. Another possibility for the conditioning matrix could be unit matrix.

If the whole procedure is given as a recipe:

Input : Matrix A,b and x , C^{-1} inverse of positive conditioning matrix

Step 1 : $r_0 = b - A x$

$w_0 = C^{-1}r_0$

$v_0 = C^{-1} w_0$

$$\gamma_0 = \sum_{j=1}^{n} w_j^2$$

Step 2 : k=1

Step 3 : while(k<=N) do adim 4-5

Step 4 : $u=Av_k$

$$\alpha_k = \frac{\gamma_{k-1}}{\sum_{j=1}^{n} v_j u_j}$$

$x_k=x_{k-1} + \alpha_k v_k$

$r_k=r_{k-1} + \alpha_k u_k$

$w_k = C^{-1}r_k$

$$\gamma_k = \sum_{j=1}^{n} w_j^2$$

Step 5 : $p_k= \beta_k/\gamma_k$

$v_{k+1} = C^{-1} w_k + p_k v_k$

$\gamma_k = \beta_k$

k=k+1

Preconditioned conjugate - gradient program using above recipe is given below

PROBLEM Conditioned conjugate-gradient method to solve system of equations (java version)

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;
//reconditioned Conjugate Gradient
public class NA26
{
public static double VT_V(double [] left)
{
//multiplies a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
                aa[i][j]=left[i]*left[j];
                }
  }
return aa;
}

public static double  VT_X(double [] left,double [] right)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}
public static double[][] multiply(double left,double[][] right)
{
//multiplying a matrix with a constant
int i,j;
int n=right.length;
int m=right[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=right[i][j]*left;
  }
return b;
//end of multiplying a matrix with a constant double
}
public static double[] multiply(double[] left,double[][] right)
{
//multiplication of one vector with one matrix
```

```
int ii,jj,i,j,k;
int m2=right[0].length;
int n2=right.length;
int m1=left.length;
double[] b;
b=new double[m1];
 if(n2 != m1)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n2;ii++)
   {
    b[ii]=0;
   }
  return b;
 }
 for(i=0;i<m2;i++)
   {
    b[i]=0;
    for(k=0;k<m1;k++)
                b[i]+=right[i][k]*left[k];
   }
return b;
//end of multiply of a vector and a matrix
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
    b[ii]=0;
   }
  return b;
 }
 for(i=0;i<m1;i++)
   {
    b[i]=0;
    for(k=0;k<n1;k++)
                b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}
public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {
  b[i]=left*right[i];
  }
return b;
}
public static double[] substract(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
```

```
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]-right[i];
  }
return b;
//end of vector substraction method
}
public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}
public static double[] equals(double[] left)
{
//addition of two vectors
int n1=left.length;
double b[];
b=new double[n1];
for(int i=0;i<n1;i++)
  {
    b[i]=left[i];
  }
return b;
//end of vector equal method
}
public static double norm(double v[])
{return Math.sqrt(normkare(v));}

public static double normkare(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return total;
}

public static double VT_G_V(double V[],double G[][])
{
  double x1[]=multiply(V,G);
  return VT_X(V,x1);
}
```

```
public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=1.0;
return b;
}

public static double[][] D(double A[][])
{
//Diagonal elements of matrix
int n=A.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=A[i][i];
return b;
}

public static double[][] invsqrtD(double A[][])
{
//square root of Diagonal elements of a matrix
int n=A.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=1.0/Math.sqrt(A[i][i]);
return b;
}

public static double[] conjugate_gradient(double x[],double A[][],double b[])
{   int n=A.length;
            double C[][]=new double[n][n];
            C=invsqrtD(A);
   return conjugate_gradient(x,A,b,C);
}

public static double[] conjugate_gradient(double A[][],double b[])
{   int n=A.length;
            double C[][]=new double[n][n];
            C=invsqrtD(A);
            double x[]=new double[n];
            //başlangıç x değerleri 0 olarak alındı
   return conjugate_gradient(x,A,b,C);
}

public static double[] conjugate_gradient(double x[],double A[][],double b[],double C[][])
{ // iterative system of equation solution
  // x is the initial guess
  // C is the conditioning matrix
double eps=1.0e-15;
int maxiter=500;
int n=x.length;
double r[]=new double[n];
double re[]=new double[n];
double v[]=new double[n];
double w[]=new double[n];
double u[]=new double[n];
double alpha;
double gamma,beta,p;
int k=0;
r=substract(b,multiply(A,x));
w=multiply(C,r);
v=multiply(C,w);
gamma=normkare(w);
do
  {
  k++;
```

```
  if(norm(v)<eps) return x;
  re=equals(r);
  u=multiply(A,v);
  alpha=gamma/VT_X(v,u);
  x=add(x,multiply(alpha,v));
  r=substract(r,multiply(alpha,u));
  w=multiply(C,r);
  beta=normkare(w);
  if(beta<eps)
  {if(norm(r)<eps) return x;}
  p=beta/gamma;
  v=add(multiply(C,w),multiply(p,v));
  gamma=beta;
  }while(norm(r)>eps && beta>eps && k<maxiter );
  //System.out.println("k="+k);
return x;
}
  public static void main(String args[]) throws IOException
  {
  double A[][]={{0.2,0.1,1,1,0},{0.1,4,-1,1,-1},{1,-1,60,0,-2},{1,1,0,8,4},{0,-1,-2,4,700}};
  double b[]={1,2,3,4,5};
  double x[]=conjugate_gradient(A,b);
  System.out.println("conditioned conjugate gradient method : \n"+Matrix.toStringT(x));
  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA26
conditioned conjugate gradient method :
       7.859713075445862
       0.422926408295008
      -0.073592239024046
      -0.540643016894627
       0.010626162854036
> Terminated with exit code 0.
```

PROBLEM Conditioned conjugate-gradient method to solve system of equations (python version)

```
# -*- coding: utf-8 -*-
"""
Created on Thu Aug 23 11:49:55 2018

@author: Mustafa Turhan Çoban
"""
from math import *
def VT_V(left):
    #multiplies a vector transpose with a vector
    n=len(left);
    tot=0;
    for i in range(n): tot+=left[i]*left[i];
    return tot;

def V_VT(left):
    #multiplys a vector with a vector             transpose
    n=len(left);
    aa = [[0 for x in range(n)] for y in range(n)];
    for i in range(n):
        for j in range(n): aa[i][j]=left[i]*left[j];
    return aa;

def  VT_X(left,right):
    #multiplys a vector transpose with a vector
    n=len(left);
    tot=0;
    for i in range(n):tot+=left[i]*right[i];
    return tot;

def multiply_c_m(left,right):
    #multiplying a matrix with a constant
     n=len(rigth);
     m=len(right[0]);
     b = [[0 for x in range(m)] for y in range(n)];
     for i in range(n):
```

```
        for j in range(m):
            b[i][j]=right[i][j]*left;
     return b;

def multiply_v_m(left,right):
    #multiplication of one vector with one matrix
    m2=len(right[0]);
    n2=len(right);
    m1=len(left);
    b = [0.0 for i in range(m1)];
    if n2 != m1:
        print("inner matrix dimensions must agree");
        for ii in range(n2):b[ii]=0;
        return b;
    for i in range(m2):
        b[i]=0;
        for k in range(m1):
            b[i]+=right[i][k]*left[k];
    return b;

def multiply_m_v(left,right):
    #multiplication of one matrix with one vector
    m1=len(left[0]);
    n1=len(left);
    m2=len(right);
    b = [0.0 for i in range(m2)];
    if n1 != m2:
        print("inner matrix dimensions must agree");
        for ii in range(n1):b[ii]=0;
        return b;
    for i in range(m1):
        b[i]=0;
        for k in range(n1):
            b[i]+=left[i][k]*right[k];
    return b;

def multiply_c_v(left,right):
    #multiplying a vector with a constant
    n=len(right);
    b = [0.0 for i in range(n)];
    for i in range(n):
        b[i]=left*right[i];
    return b;

def substract(left,right):
    #addition of two vectors
    n1=len(left);
    n2=len(right);
    if n1>=n2: nMax=n1;
    else     : nMax=n2;
    b = [0.0 for i in range(nMax)];
    for i in range(n1):
        b[i]=b[i]+left[i];
    for i in range(n2):
        b[i]=b[i]-right[i];
    return b;

def add(left,right):
    #addition of two vectors
    n1=len(left);
    n2=len(right);
    if n1>=n2: nMax=n1;
    else     : nMax=n2;
    b = [0.0 for i in range(nMax)];
    for i in range(n1):
        b[i]=b[i]+left[i];
    for i in range(n2):
        b[i]=b[i]+right[i];
    return b;

def equals(left):
    #addition of two vectors
     n1=len(left);
     b = [0.0 for i in range(n1)];
     for i in range(n1):
         b[i]=left[i];
```

```
    return b;

def norm(v):
   return sqrt(normkare(v));

def normkare(v):
   #vector norm
   total=0;
   n=len(v);
   for i in range(n):
      total+=v[i]*v[i];
   return total;

def VT_G_V(V,G):
  x1=multiply_v_m(V,G);
  return VT_X(V,x1);

def I(n):
   #unit matrix
   b = [0.0 for i in range(n)];
   for i in range(n):
      b[i][i]=1.0;
   return b;

def D(A):
   #Diagonal elements of matrix
   n=len(A);
   b = [[0 for x in range(n)] for y in range(n)];
   for i in range(n):
      b[i][i]=A[i][i];
   return b;

def invsqrtD(A):
   #square root of Diagonal elements of a matrix
   n=len(A);
   b = [[0 for x in range(n)] for y in range(n)];
   for i in range(n):
      b[i][i]=1.0/sqrt(A[i][i]);
   return b;

def conjugate_gradient(x,A,b):
   n=len(A);
   C = [[0 for x in range(n)] for y in range(n)];
   C=invsqrtD(A);
   return conjugate_gradient1(x,A,b,C);

def conjugate_gradient(A,b):
   n=len(A);
   C = [[0 for x in range(n)] for y in range(n)];
   C=invsqrtD(A);
   x = [0.0 for i in range(n)];
   return conjugate_gradient1(x,A,b,C);

def conjugate_gradient1(x,A,b,C):
   #iterative system of equation solution
   #x is the initial guess
   #C is the conditioning matrix
   eps=1.0e-15;
   maxiter=500;
   n=len(x);
   x = [0.0 for i in range(n)];
   r = [0.0 for i in range(n)];
   v = [0.0 for i in range(n)];
   w = [0.0 for i in range(n)];
   u = [0.0 for i in range(n)];
   k=0;
   r=substract(b,multiply_m_v(A,x));
   w=multiply_m_v(C,r);
   v=multiply_m_v(C,w);
   beta=0;
   gamma=normkare(w);
   nr=norm(r);
   while (nr>eps and k<maxiter):
      k=k+1;
      if norm(v)<eps: return x;
      re=equals(r);
```

```
        u=multiply_m_v(A,v);
        alpha=gamma/VT_X(v,u);
        x=add(x,multiply_c_v(alpha,v));
        r=substract(r,multiply_c_v(alpha,u));
        nr=norm(r);
        w=multiply_m_v(C,r);
        beta=normkare(w);
        if beta<eps:
           nr=norm(r);
           if nr<eps: return x;
        p=beta/gamma;
        v=add(multiply_m_v(C,w),multiply_c_v(p,v));
        gamma=beta;
    return x;

A=[[0.2,0.1,1,1,0],[0.1,4,-1,1,-1],[1,-1,60,0,-2],[1,1,0,8,4],[0,-1,-2,4,700]];
b=[1,2,3,4,5];
x=conjugate_gradient(A,b);
print("conditioned conjugate gradient method : \n",x);
```

```
runfile('D:/okul/SCO1/conjugate_gradient.py', wdir='D:/okul/SCO1')
conditioned conjugate gradient method :
 [7.859713075445861, 0.4229264082950077, -0.07359223902404635, -0.5406430168946268, 0.010626162854036319]
```

3.12 THE RESIDUAL CORRECTION METHOD

We assume that Ax=b has ben solved for an approximate solution $x=x^{(0)}$. Also the LU decomposition along with a record of all row or column interchanges should been stored. Calculate residual

$r^{(0)}=b-Ax^{(0)}$

Define $e^{(0)}=x-x^{(0)}$. Then solve

$Ae^{(0)}=r^{(0)}$

Solve this system using the stored LU decomposition .If the resulting approximate solution is $e^{(0)}$. Define a new approximate solution to Ax=b by

$x^{(1)}=x^{(0)}+e^{(0)}$

define also error term $\varepsilon = \|x\|_e = \sqrt{\sum_{i=0}^{n} x_i^2} < \varepsilon_0$

convert iterations until error term reduce below the given range. As an example Hilbert matrix is taken as one of the ill conditioning matrix example. The n x n Hilbert matrix defined by:

$$H_{ij}^{(n)} = \frac{1}{i+j-1}$$

Hilbert Matrix defined in Mathd class

```
//The Hilbert Matrix
public static double[][] Hilbert(int n)
{
double H[][]=new double[n][n];
for(int i=0;i<n;i++)
{for(int j=0;j<n;j++)
 {H[i][j]=1.0/(i+j+1.0);}
}
return H;
}
```

Problem The residual correction by using LU decomposition

```
import java.io.*;
import javax.swing.*;
class residual
{
  public static void pivot(double a[][],int o[],double s[],int k)
  {// Gauss(Dolittle) LU decomposition with partial pivoting
    int n=a.length;
    int dummy1=0;
    double buyuk;
    double dummy=0;
```

```
  //pivotlama
            int p=k;
            buyuk=Math.abs(a[o[k]][k]);
            for(int ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[o[ii]][k]/s[o[ii]]);
              if(dummy > buyuk) {buyuk=dummy;p=ii;}
            }
            dummy1=o[p];
            o[p]=o[k];
            o[k]=dummy1;
}

public static int[] sirali_indeks(int n)
{   // sıralı rakamları taşıyan bir referans vektörü oluşturur
              int o[]=new int[n];
              for(int i=0;i<n;i++) {o[i]=i;}
              return o;
}

public static double[][] gaussLU(double ai[][],int o[])
{ //kısmi pivotlu gauss eleme yöntemi
int n=ai.length;
double a[][]=new double[n][n];
double s[]=new double[n];
double carpan=0;
double toplam=0;
double absaij;

//gauss eleme
int i,j,k,p,ii,jj;
//System.out.println("Orijinal Matris :\n"+Matrix.toString(ai));
for(i=0;i<n;i++)
{for(j=0;j<n;j++){a[i][j]=ai[i][j];}}
for(i=0;i<n;i++)
{ o[i]=i;s[i]=Math.abs(a[i][0]);
  for(j=1;j<n;j++)
    { absaij=Math.abs(a[i][j]);
      if(absaij>s[i]) {s[i]=absaij;}
    }
}
for(k=0;k<(n-1);k++)
{ pivot(a,o,s,k);
            //pivot referansını değiştir
            //System.out.println("pivotlu Matris : k="+k+"\n"+Matrix.toString(a));
            //gauss elemeyi çözme
  for(i=k+1;i<n;i++)
  {  carpan=a[o[i]][k]/a[o[k]][k];
     a[o[i]][k]=carpan;
     for(j=k+1;j<n;j++)
     {   a[o[i]][j]-=carpan*a[o[k]][j]; }
            }
}
return a;
}

public static double[] yerine_koyma(double ai[][],double bi[])
{   int n=ai.length;
              return yerine_koyma(ai,bi,sirali_indeks(n));
}

public static double[][] yerine_koyma(double ai[][],double bi[][])
{   int n=ai.length;
```

```
            return yerine_koyma(ai,bi,sirali_indeks(n));
}

public static double[] yerine_koyma(double ai[][],double bi[],int o[])
{ //gauss LU çözümüyle LU indirgenmesi yapılmış Matris kullanarak linear
  //denklem sistemi kökleri
  int n=ai.length;
  double toplam;
  double x[]=new double[n];
   double a[][]=new double[n][n];
  double b[]=new double[n];
  int i,j;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};b[i]=bi[i];}
//System.out.println("a=\n "+Matrix.toString(a));
//System.out.println("b=\n "+Matrix.toStringT(b));
for(i=1;i<n;i++)
    {toplam=b[o[i]];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]];};b[o[i]]=toplam;}
 //System.out.println("d=\n "+Matrix.toStringT(b));
x[n-1]=b[o[n-1]]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {toplam=0;
     for(j=i+1;j<n;j++)
      {toplam+=a[o[i]][j]*x[j];};
      x[i]=(b[o[i]]-toplam)/a[o[i]][i];
    }
//System.out.println("d=\n "+Matrix.toStringT(x));
return x;
}

public static double[][] yerine_koyma(double ai[][],double bi[][],int o[])
{
  int n=ai.length;
  int m=bi[0].length;
  double toplam;
  double x[][]=new double[n][m];
  double a[][]=new double[n][n];
  double b[][]=new double[n][m];
  int i,j,k;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};for(k=0;k<m;k++) {b[i][k]=bi[i][k];}}
//System.out.println("a=\n "+Matrix.toString(a));
//System.out.println("b=\n "+Matrix.toStringT(b));
for(k=0;k<m;k++)
{
for(i=1;i<n;i++)
    {toplam=b[o[i]][k];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]][k];};b[o[i]][k]=toplam;}
 //System.out.println("d=\n "+Matrix.toStringT(b));
x[n-1][k]=b[o[n-1]][k]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {toplam=0;
     for(j=i+1;j<n;j++)
      {toplam+=a[o[i]][j]*x[j][k];};
      x[i][k]=(b[o[i]][k]-toplam)/a[o[i]][i];
    }
}//kin sonu
//System.out.println("d=\n "+Matrix.toStringT(x));
return x;
}

public static double [] AXB(double a[][],double b[])
{
```

```
  //denklem sistemi çözümü b vektörü için
  int n=a.length;
  int q[]=new int[n];
  double c[][]=gaussLU(a,q);
  double s[]=yerine_koyma(c,b,q);
  return s;
  }

  public static double [][] AXB(double a[][],double b[][])
  {//denklem sistemi çözümü b matrisi için
  int n=a.length;
  int q[]=new int[n];
  double c[][]=gaussLU(a,q);
  double s[][]=yerine_koyma(c,b,q);
  return s;
  }


public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int n1=left.length;
int m1=left[0].length;

int m2=right.length;
double[] b;
b=new double[m2];
 for(i=0;i<n1;i++)
   {
     b[i]=0;
     for(k=0;k<m1;k++)
               {b[i]+=left[i][k]*right[k];
               }
   }
return b;
//end of multiply of a matrix and a vector
}

public static double[] substract(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]-right[i];
  }
return b;
//end of vector substraction method
}
public static double[] add(double[] left,double[] right)
```

```
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

 public static double error(double q[])
 { double err=0;
   for(int i=0;i<q.length;i++)
   {err+=q[i]*q[i];}
   return Math.sqrt(err);
 }

  public static double[] residual_correction(double a[][],double b[],double eps)
  {
  double eps1;
  System.out.println("a\n"+Matrix.toString(a));
  int n=a.length;
  int q[]=new int[n];
  double c[][]=gaussLU(a,q);
  double x[]=yerine_koyma(c,b,q);
  System.out.println("x\n"+Matrix.toString(x));
  double e[]=new double[x.length];
  double r[]=new double[x.length];
  do
  {
  r= substract(b,multiply(a,x));
  e=yerine_koyma(c,r,q);
  System.out.println("r\n"+Matrix.toString(r));
  System.out.println("e\n"+Matrix.toString(e));
  x=add(x,e);
  System.out.println("x\n"+Matrix.toString(x));
  eps1=error(e);
  } while(eps1>eps);
  return x;
  }

  public static void main(String args[]) throws IOException
  {
  double eps=1e-10;
  double a[][]=Mathd.Hilbert(4);

  double b[]={1,0,0,0};
  double x[]=residual_correction(a,b,eps);
  Text.print(x,"Solution of Gauss(Doolittle) decomposition and elimination with residual correction method");
  }
```

```
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" residual
a
      1.000000000000000      0.500000000000000      0.333333333333333      0.250000000000000
      0.500000000000000      0.333333333333333      0.250000000000000      0.200000000000000
      0.333333333333333      0.250000000000000      0.200000000000000      0.166666666666667
      0.250000000000000      0.200000000000000      0.166666666666667      0.142857142857143

x
     15.999999999998870000000000          -119.999999999986880000000000          239.99999999968000000000000
139.99999999979020000000000

r
      0.000000000000000000000000          -0.000000000000035527136788          -0.000000000000035527136788
0.000000000000000000000000

e
     -0.000000000000426325641456          0.000000000005329070518199          -0.000000000013429257705863
0.000000000008952838470575

x
     15.999999999984440000000000          -119.999999999815500000000000          239.99999999954580000000000          -
139.99999999970070000000000

> Terminated with exit code 0.
```

3.13 NORMS

Norms are an important tool to evaluate matrices: Several form of norms are defined. General definitions of norms are as follows:

Infinity norm : Maximum value of the sum of the columns

$$\|A\|_\infty = \sum_{i=1}^{n} max\left(\sum_{j=1}^{n}|a_{ij}|\right)$$

Norm: Maximum value of the sum of the raws

$$\|A\| = \sum_{j=1}^{n} max\left(\sum_{i=1}^{n}|a_{ij}|\right)$$

Euclidian norm: square root of the sum of the square of the elements

$$\|A\|_e = \|A\|_2 = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$$

m norm: mth root of the sum of the mth power of the elements

$$\|A\|_m = \sqrt[m]{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^m}$$

Of course the same norm definitions can be apply to the vectors as well, please be reminded that vectors are one dimensional matrices.

$$\|x\|_\infty = \sum_{i=1}^{n} max(|x_i|)$$

$$\|x\| = \sum_{i=1}^{n} max(|x_i|)$$

$$\|x\|_e = \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\|x\|_m = \sqrt[m]{\sum_{i=1}^{n} x_i^m}$$

Basic norm definitions are given in the norm class.

3.13-1 Matrix norm definitions : class norm

```
public class norm
{
//*************norm methods definition*****************
public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

public static double norm(double v[],int p)
  {
   //p vector norm
  if(p==0) p=1;
  double total=0;
  double x,y;
  for(int i=0;i<v.length;i++)
    {
    x=Math.abs(v[i]);
    total+=Math.pow(x,p);
    }
  return Math.pow(total,(1.0/p));
  }

public static double norminf(double v[])
    {
     //infinite vector norm
    double x;
    double max=0;
    for(int i=0;i<v.length;i++)
      {
      x=Math.abs(v[i]);
      if(x>max) max=x;
      }
    return max;
    }

public static double[] vek(double mat[][],int n)
{ double a[]=new double[mat.length];
  for(int i=0;i<mat.length;i++) {a[i]=mat[i][n];}
  return a;
}

public static double norminf(double a[][])
       {
         //infinite matrix norm
         double x;
         double max = 0;
         double total;
         int i,j;
         int n=a.length;
         for(i=0;i<n;i++)
         {
         total=0;
         for(j=0;j<n;j++)
         {total+=Math.abs(a[i][j]);}
         x=total;
         if(x>max) max=x;
         }
```

```
        return max;
       }

public static double norm(double a[][])
      {
      //matrix norm
            double x;
            double max = 0;
            double total;
            int i,j;
            int n=a.length;
            for(j=0;j<n;i++)
              {
              total=0;
              for(i=0;i<n;j++)
              {total+=Math.abs(a[i][j]);}
              x=total;
              if(x>max) max=x;
              }
            return max;
     }

public static double normE(double a[][])
           {
           //Euclidian matrix norm
           double x;
           double total;
           int i,j;
           total=0;
           int n=a.length;
           for(j=0;j<n;i++)
           {for(i=0;i<n;j++) {total+=a[i][j]*a[i][j];}}
           return Math.sqrt(total);
          }
}
```

3.14 UNIT MATRIX, TRANPOSE MATRIX, DETERMINANT AND MATRIX INVERSION

Unit matrix is a matrix with all diagonal elements are one and all the remaining elements are zero

$$a_{ii} = 1 \quad a_{ij, i \neq j} = 0$$

As an example for n=4

$$[I] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transpose matrix is the matrix with rows and colums are interchanged. Mathematically, it can be shown with a T sign at the top of the matrix. Transpose of a matrix can be taken as in the following method:

```
public static double[][] Transpose(double [][] left)
{
//transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
return b;
```

```
}
```

Inverse matrix is the matrix when multiplied with the original matrix will give the unit matrix

$$[A][A]^{-1} = [I]$$

Determinant of a matrix can be defined as follows: if a matrix A is given as

$$[A] = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

Determinat of this matrix determined as:

$$D[A] = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} = a_{00} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - a_{01} \begin{vmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{vmatrix} + a_{02} \begin{vmatrix} 0 & a_{11} \\ a_{20} & a_{21} \end{vmatrix}$$

$$D[A] = a_{00}(a_{11}a_{22} - a_{12}a_{21}) - a_{01}(a_{10}a_{22} - a_{12}a_{20}) + a_{02}(a_{10}a_{22} - a_{12}a_{20})$$

If matrix is converted to a upper triangular form Determinant will simply be the product of diagonals

$$[A] = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{bmatrix}$$

$$D[A] = a_{00}a_{11}a_{11}$$

Inverse matrix can be obtained by using any of the previous system of equation solution methods.

Problem 3.14-1 Inverse matrix program with partially pivoted Gauss LU method

```
import java.io.*;
import java.text.*;
import java.util.Locale;

class NA28
{
  public static void pivot(double a[][],int o[],double s[],int k)
  {// GaussLU  pivoting process
    int n=a.length;
    int dummy1=0;
    double buyuk;
    double dummy=0;
    //pivotlama
              int p=k;
              buyuk=Math.abs(a[o[k]][k]);
              for(int ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[o[ii]][k]/s[o[ii]]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              dummy1=o[p];
              o[p]=o[k];
              o[k]=dummy1;
  }

  public static int[] pivot_index(int n)
  {   // indexed the pivoted matrix
               int o[]=new int[n];
               for(int i=0;i<n;i++) {o[i]=i;}
               return o;
  }

  public static double[][] gaussLU(double ai[][],int o[])
  { //Gauss LU with pivoting
  int n=ai.length;
  double a[][]=new double[n][n];
  double s[]=new double[n];
  double carpan=0;
  double sum=0;
  double absaij;

  //gauss elimination
```

```
int i,j,k,p,ii,jj;
for(i=0;i<n;i++)
{for(j=0;j<n;j++){a[i][j]=ai[i][j];}}
for(i=0;i<n;i++)
{ o[i]=i;s[i]=Math.abs(a[i][0]);
  for(j=1;j<n;j++)
    { absaij=Math.abs(a[i][j]);
      if(absaij>s[i]) {s[i]=absaij;}
    }
}
for(k=0;k<(n-1);k++)
{ pivot(a,o,s,k);
              //change pivort reference
              //solve gauss elimination
  for(i=k+1;i<n;i++)
  {  carpan=a[o[i]][k]/a[o[k]][k];
     a[o[i]][k]=carpan;
     for(j=k+1;j<n;j++)
     {   a[o[i]][j]-=carpan*a[o[k]][j]; }
              }
}
return a;
}

public static double[] back_substiution(double ai[][],double bi[])
{   int n=ai.length;
              return back_substiution(ai,bi,pivot_index(n));
}

public static double[][] back_substiution(double ai[][],double bi[][])
{   int n=ai.length;
              return back_substiution(ai,bi,pivot_index(n));
}

public static double[] back_substiution(double ai[][],double bi[],int o[])
{
  int n=ai.length;
  double sum;
  double x[]=new double[n];
  double a[][]=new double[n][n];
  double b[]=new double[n];
  int i,j;
  for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};b[i]=bi[i];}
for(i=1;i<n;i++)
    {sum=b[o[i]];for(j=0;j<i;j++){sum-=a[o[i]][j]*b[o[j]];};b[o[i]]=sum;}
x[n-1]=b[o[n-1]]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {sum=0;
     for(j=i+1;j<n;j++)
      {sum+=a[o[i]][j]*x[j];};
      x[i]=(b[o[i]]-sum)/a[o[i]][i];
    }
return x;
}

public static double[][] back_substiution(double ai[][],double bi[][],int o[])
{
  int n=ai.length;
  int m=bi[0].length;
  double sum;
  double x[][]=new double[n][m];
  double a[][]=new double[n][n];
  double b[][]=new double[n][m];
  int i,j,k;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};for(k=0;k<m;k++) {b[i][k]=bi[i][k];}}
for(k=0;k<m;k++)
{
for(i=1;i<n;i++)
```

```
    {sum=b[o[i]][k];for(j=0;j<i;j++){sum-=a[o[i]][j]*b[o[j]][k];};b[o[i]][k]=sum;}
x[n-1][k]=b[o[n-1]][k]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {sum=0;
     for(j=i+1;j<n;j++)
      {sum+=a[o[i]][j]*x[j][k];};
      x[i][k]=(b[o[i]][k]-sum)/a[o[i]][i];
    }
}
return x;
}

public static double [] AXB(double a[][],double b[])
{
//solution of the system of equation
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[]=back_substiution(c,b,q);
return s;
}

public static double [][] AXB(double a[][],double b[][])
{//solution of the set of the system  of equations
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[][]=back_substiution(c,b,q);
return s;
}

public static double[][] I(int n)
{
//Inverse matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
  b[i][i]=1.0;
return b;
}

public static double[][] inverse(double a[][])
{
//Inverse matrix
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[][]=back_substiution(c,I(n),q);
return s;
}

// Matrix output classes
public static String toString(double[][] left)
{
//return a string representation of a matrix
return toString(left,25,15);
}
public static String toString(double[][] left,int w,int d)
{
//return a string representation of a matrix
int n,m;
String b;
b="";
n=left.length;
m=left[0].length;
for(int i=0;i<n;i++)
{
for(int j=0;j<m;j++)
  {
  b=b+toString(left[i][j],w,d);
```

```
    }
    b=b+"\n";
  }
  return b;
  }

  public static String toString(double left, int w, int d)
  // converts a double to a string with given width and decimals.
  {
              Locale us=new Locale("US");
              String s="%"+w+"."+d+"f";
              String s1=String.format(us,s,left);
   return s1;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{3.556,-1.778,0},{-1.778,3.556,-1.778},{0,-1.778,3.556}};
  System.out.println("Matrix a : \n"+toString(a));
  System.out.println("Inverse Matrix of a : \n"+toString(inverse(a)));
  }
}
```

Inverse matrix class NA28 output

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA28
Matrix a :
       3.556000000000000      -1.778000000000000       0.000000000000000
      -1.778000000000000       3.556000000000000      -1.778000000000000
       0.000000000000000      -1.778000000000000       3.556000000000000

Inverse Matrix of a :
       0.421822272215973       0.281214848143982       0.140607424071991
       0.281214848143982       0.562429696287964       0.281214848143982
       0.140607424071991       0.281214848143982       0.421822272215973


> Terminated with exit code 0.
```

Problem 3.14-2 Inverse matrix program with partially pivoted Gauss elimination method in Python

```
# Gauss Elimination with pivoting
# python version

from math import *

def gauss(a,b):
  n=len(a)
  x=[0 for i in range(n)]
  for k in range(n):
    p=k
    buyuk=abs(a[k][k])
    for ii in range(k+1,n):
      dummy=abs(a[ii][k])
      if dummy>buyuk:
        buyuk=dummy
        p=ii
    if p!=k:
      for jj in range(k,n):
        dummy=a[p][jj]
        a[p][jj]=a[k][jj]
        a[k][jj]=dummy
      dummy=b[p]
      b[p]=b[k]
      b[k]=dummy
    for i in range(k+1,n):
      carpan=a[i][k]/a[k][k]
      a[i][k]=0;
      for j in range(k+1,n):
        a[i][j]=a[i][j]-carpan*a[k][j]
      b[i]   =b[i]-carpan*b[k]
  x[n-1]=b[n-1]/a[n-1][n-1]
```

```
  for i in range(n-1,-1,-1):
    toplam=0;
    for j in range(i+1,n):
      toplam=toplam+a[i][j]*x[j]
    x[i]=(b[i]-toplam)/a[i][i];
  return x;

def I(n):
    #unit matrix of dimension n
    B = [[0 for x in range(n)] for y in range(n)];
    for i in range(0,n):
        for j in range(0,n):
            if i==j: B[i][j]=1.0;
    return B;

def invA(A):
    #inverse matrix (n=m)
    n=len(A);
    Im= I(n);
    B = [[0 for x in range(n)] for y in range(n)];
    for i in range(n):
        B[i]=gauss(A,Im[i]);
    return B;

A=[[3.556,-1.778,0],[-1.778,3.556,-1.778],[0,-1.778,3.556]];
print("matrix = \n",A);
AI=invA(A);
print("inverse matrix = \n",AI);
```

```
runfile('D:/okul/SCO1/gauss.py', wdir='D:/okul/SCO1')
matrix =
 [[3.556, -1.778, 0], [-1.778, 3.556, -1.778], [0, -1.778, 3.556]]
inverse matrix =
 [[0.421822272215973, 0.281214848143982, 0.140607424071991], [0.18747656542932134, 0.3749531308586427, 0.0],
[0.140607424071991, 0.281214848143982, 0.421822272215973]]
```

If inverse matrix is determined solution of x unknown vector of a system of equation can be solved by multiplication of inverse matrix and b. $[A]\{x\} = \{b\}$

$$\{x\} = [A]^{-1}\{b\}$$

Let us give an example of this method by an excell spreadsheet.

$$\begin{bmatrix} -8 & 1 & 2 \\ 5 & 7 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ -2 \end{bmatrix}$$

	B	C	D	E	F
1		A			
2	-8	1	2	x0	0
3	5	7	-3	x1	10
4	2	1	-2	x2	-2
5		A'			
6	-0.14865	0.054054	-0.22973		1
7	0.054054	0.162162	-0.18919	x=	2
8	-0.12162	0.135135	-0.82432		3

In order to solve these system of equation an inverse matrix function **MINVERSE**(If you are using a turkish version function name: **DIZEY_TERS**) and matrix multiplication function **MMULTIPLY**((If you are using a turkish version function name: **DÇARP**) can be used. While using these functions instead of enter key, Ctrl-shift-enter key (ctrl ⇑ ↵) should be used. The cells that inverse matrix will be created is selected first and then MINVERSE(A2:D4) is entered (a2:D is the original space matrix A is located), and then ctrl-shift-enter keys are pushed. Inverse matrix appears at the selected cell section. And than by using **MMULTIPLY(B6:D8;F2:F4)** ,

where B7:D9 is the space of the inverse matrix and F2:F4 is the space of vector b. We have to use ctrl-shift-enter again with this function to get the result.

3.5 CRAMER'S RULE

Another method to solve linear system of equations is Cramer's rule. The rule based on solutions of determinants. This solution method is not preferred much as a computer solution method, but it is practical for hand solutions of small systems. If a system of equation given as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{Bmatrix}$$

then solution can be given as:

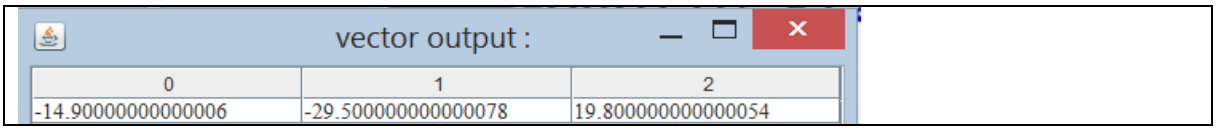
$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad x_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix} \frac{1}{D} \quad x_1 = \begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix} \frac{1}{D} \quad \dots.$$

Let us try the method by using an example problem:

Problem 3.17-Cramer' rule for solving system of equations

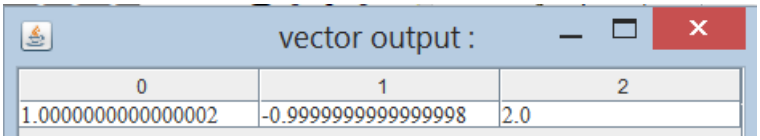
```
public class cramer
{
public static double[][] insert(double a[][],double b[],int nj)
{
int n=a.length;
int m=a[0].length;
double c[][]=new double[n][m];
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++) {if(j==nj) c[i][j]=b[i];
                else     c[i][j]=a[i][j];
               }
}
return c;
}
//LU decomposition method
public static double[][] LU(double c[][],int indx[],int d[])
{
//returns LU decomposition of matrix c and index indx
double a[][];
int n=c.length;
a=new double[n][n];
double vv[];
vv=new double[n];
double sum,dum,big,temp;
int i,j,k;
int imax;
int nmax=100;
double tiny=1.0e-40;
imax=0;
for(i=1;i<=n;i++)
{
  for(j=1;j<=n;j++)
    a[i-1][j-1]=c[i-1][j-1];
}
d[0]=1;
for(i=1;i<=n;i++)
  {
  big=0.0;
  for(j=1;j<=n;j++)
    {
    if(Math.abs(a[i-1][j-1])>big) big=Math.abs(a[i-1][j-1]);
    }
  if(big==0) {System.out.println("singular matrix");return a;}
  vv[i-1]=1.0/big;
  }
for(j=1;j<=n;j++)
{
  for(i=1;i<j;i++)
    {
      sum=a[i-1][j-1];
      for(k=1;k<i;k++)
              {
```

```
                sum-=a[i-1][k-1]*a[k-1][j-1];
               }
    a[i-1][j-1]=sum;
    }
  big=0;
  for(i=j;i<=n;i++)
    {
    sum=a[i-1][j-1];
    for(k=1;k<j;k++)
      {
      sum-=a[i-1][k-1]*a[k-1][j-1];
      }
    a[i-1][j-1]=sum;
    dum=vv[i-1]*Math.abs(sum);
    if(dum>=big)
      {
      imax=i;
      big=dum;
      }
    } //end of i=0
if(j != imax)
    {
    for(k=1;k<=n;k++)
      {
      dum=a[imax-1][k-1];
      a[imax-1][k-1]=a[j-1][k-1];
      a[j-1][k-1]=dum;
      }
    d[0]=-d[0];
    vv[imax-1]=vv[j-1];
    } //end of if
indx[j-1]=imax;
if(a[j-1][j-1]==0) a[j-1][j-1]=tiny;
if(j!=n)
  {
  dum=1.0/a[j-1][j-1];
  for(i=j+1;i<=n;i++)
    a[i-1][j-1]*=dum;
  }//endif
} //end for j=
return a;
}
public static double D(double a[][])
{
//determinant of a matrix
int n=a.length;
int indx[]=new int[n];
int d[]={1};
double e;
double b[][]=new double[n][n];
b=LU(a,indx,d);
e=d[0];
for(int i=0;i<n;i++)
  e=e*b[i][i];
return e;
}  //end of det

public static double[] cramer(double A[][],double B[])
{double D=D(A);
 int n=B.length;
 double x[]=new double[n];
 double E[][]=A;
 for(int i=0;i<n;i++)
 { E=insert(A,B,i);
   x[i]=D(E)/D;
 }
 return x;
}
public static void main(String arg[])
{double A[][]={{0.3,0.52,1},{0.5,1,1.9},{0.1,0.3,0.5}};
 double B[]={-0.01,0.67,-0.44};
 double x[]=cramer(A,B);
 Text.print(x);
}}
```



Another example

```
public static void main(String arg[])
{double A[][]={{6,-3,1},{2,1,-8},{1,-7,1}};
 double B[]={11,-15,10};
 double x[]=cramer(A,B);
 Text.print(x);
}
```



Even though it is very simple method to solve by hand, it is not recommended as a computer solution method due to heavier calculation load compare to Gauss Elimination and similar methods.

3.16 PROBLEMS

PROBLEM 1

Solve system of equation by using Gauss elimination method.

$$\begin{bmatrix} 7 & 1 & 2 \\ -1 & 4 & -1 \\ 3 & 15 & 20 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 47 \\ 19 \\ 87 \end{Bmatrix}$$

PROBLEM 2

Solve system of equation by using

a) By using Gauss elimination
b) By using Gauss-Jordan elimination

$$\begin{bmatrix} 1.1348 & 3.8226 & 1.1651 & 3.4017 \\ 0.5301 & 1.7875 & 2.5330 & 1.5435 \\ 3.4129 & 4.9317 & 8.7643 & 1.3142 \\ 1.2371 & 4.9998 & 10.6721 & 0.0147 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 9.5342 \\ 6.3941 \\ 18.4231 \\ 16.9237 \end{Bmatrix}$$

Exact solution: $\begin{Bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$

PROBLEM 3

Solve system of equation by using

a) By using Gauss elimination
b) By using Gauss-Jordan elimination

$$\begin{bmatrix} 4 & 3 & -1 \\ 7 & -2 & 3 \\ 5 & -18 & 13 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 9 \\ 3 \end{Bmatrix}$$

PROBLEM 4

Solve system of equation by using

a) By using Gauss elimination
b) By using Gauss-Jordan elimination
c) LU decomposition (Dolittle)
d) LU Craut Decomposition

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/4 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 137/60 \\ 87/60 \\ 459/60 \\ 743/840 \\ 1879/2520 \end{bmatrix}$$

PROBLEM 5

Solve system of equation by using

a) By using Gauss elimination
b) Gauss-Seidel iteration
c) Conjugate gradient method

$$\begin{bmatrix} 8 & 2 & 3 \\ 3 & -9 & 2 \\ 2 & 3 & 6 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 30 \\ 1 \\ 31 \end{Bmatrix}$$

PROBLEM 6 Solve system of equation by using

a) By using Gauss elimination
b) Gauss-Seidel with relaxation
c) Conjugate gradient method

$$\begin{bmatrix} 3 & -5 & 47 & 20 \\ 11 & 16 & 17 & 10 \\ 56 & 22 & 11 & -18 \\ 17 & 66 & -12 & 7 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 18 \\ 26 \\ 34 \\ 82 \end{Bmatrix}$$

PROBLEM 7 Solve system of equation by using Thomas algorithm (3 band matrix method)

$$\begin{bmatrix} -2 & 1 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & & \cdots & & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \\ X_7 \\ X_8 \\ X_9 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1.5 \\ -1.5 \\ \cdots \\ -1.5 \\ -1.5 \\ 0.5 \end{bmatrix}$$

PROBLEM 8 Solve system of equation by using Thomass algorithm (3 band matrix method)

$$\begin{bmatrix} -4 & 1 & & & & & & & \\ 1 & -4 & 1 & & & & & & \\ & 1 & -4 & 1 & & & & & \\ & & & \cdots & & & & & \\ & & & 1 & -4 & 1 & & & \\ & & & & 1 & -4 & 1 & & \\ & & & & & 1 & -4 & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \\ X_7 \\ X_8 \\ X_9 \end{bmatrix} = \begin{bmatrix} -27 \\ -15 \\ -15 \\ \cdots \\ -15 \\ -15 \\ -15 \end{bmatrix}$$

PROBLEM 9

Solve the following system of equation

$$\begin{bmatrix} 3 & -5 & 6 & 4 & -2 & -3 & 8 \\ 1 & 1 & -9 & 15 & 1 & -9 & 2 \\ 2 & -1 & 7 & 5 & -1 & 6 & 11 \\ -1 & 1 & 3 & 2 & 7 & -1 & -2 \\ 4 & 3 & 1 & -7 & 2 & 1 & 1 \\ 2 & 9 & -8 & 11 & -1 & -4 & -1 \\ 7 & 2 & -1 & 2 & 7 & -1 & 9 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} 47 \\ 17 \\ 24 \\ 8 \\ 13 \\ -10 \\ 34 \end{bmatrix}$$

PROBLEM 10

Solve the following system of equation

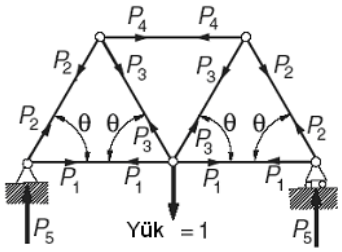
$$\begin{bmatrix} 1 & -1 & 2 & 5 & -7 & -8 \\ 3 & -9 & 1 & -1 & 8 & 1 \\ -1 & 1 & 9 & -9 & 2 & 3 \\ 1 & 7 & 2 & -3 & -1 & 4 \\ 7 & 1 & -2 & 4 & 1 & -1 \\ 2 & 3 & -9 & 12 & -2 & 7 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} -12 \\ 8 \\ 22 \\ 41 \\ 15 \\ 50 \end{bmatrix}$$

PROBLEM 11

Solve the following system of equation

$$\begin{bmatrix} 10 & -2 & -1 & 2 & 3 & 1 & -4 & 7 \\ 5 & 11 & 3 & 10 & -3 & 3 & 3 & -4 \\ 7 & 12 & 1 & 5 & 3 & -12 & 2 & 3 \\ 8 & 7 & -2 & 1 & 3 & 2 & 2 & 4 \\ 2 & -15 & -1 & 1 & 4 & -1 & 8 & 3 \\ 4 & 2 & 9 & 1 & 12 & -1 & 4 & 1 \\ -1 & 4 & -7 & -1 & 1 & 1 & -1 & -3 \\ -1 & 3 & 4 & 1 & 3 & -4 & 7 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ -5 \\ 3 \\ -25 \\ -26 \\ 9 \\ -7 \end{bmatrix}$$

PROBLEM 12

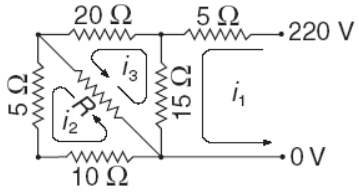


Load Matrix of the figure can be given with the following system of eqautions

$$\begin{bmatrix} c & 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 1 \\ 0 & 0 & 2s & 0 & 0 \\ 0 & -c & c & 1 & 0 \\ 0 & s & s & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Where c=cos(θ),s=sin(θ) . For θ =60º find the load distribution profile

PROBLEM 13



For the given electrical circuit current density-Voltage eqautions can be given as:

$$5i_1 + 15(i_1 - i_3) = 220\ V$$
$$5i_1 + 10i_2 + R(i_2 - i_3) = 0$$
$$20i_3 + R(i_3 - i_2) + 15(i_3 - i_1) = 0$$

Calculate current densities for R=10 Ω.

PROBLEM 14

$-x_1 + 3x_2 + 5x_3 + x_4 = 8$
$x_1 + 9x_2 + 3x_3 + 4x_4 = 10$
$x_2 + x_4 = 2$
$3x_1 + x_2 + x_3 - x_4 = -4$

Solve the given system of equations. Determine the best method to solve it.

PROBLEM 15 Solve the following system of equation by using Doolittle LU method

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 11 & 16 & 17 & 10 \\ 0 & 0 & 2 & 5 \\ -2 & -6 & -3 & 1 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{Bmatrix}$$

PROBLEM 16 Solve system of equation by using

a) By using Gauss elimination
b) By using Gauss-Jordan elimination
c) LU decomposition (Dolittle)

d) LU Craut Decomposition

$3x_1 + 5x_2 + 4x_3 + 2x_4 = 11$
$6x_1 + 14x_2 + 11x_3 + 6x_4 = 26$
$9x_1 + 11x_2 + 16x_3 + 5x_4 = 68$
$3x_1 + 13x_2 + 17x_3 + 12x_4 = 43$

PROBLEM 17 Solve system of equation by using

a) By using Gauss elimination
b) By using Gauss-Jordan elimination
c) LU decomposition (Dolittle)
d) LU Craut Decomposition
e) Gauss Seidel iteration
f) Conjugate-Gradian method

$$x_1 + 3x_2 + x_3 + 5x_4 = 4$$
$$2x_1 + x_2 + 3x_4 = 5$$
$$4x_1 + 2x_2 + 2x_3 + x_4 = 11$$
$$-3x_1 + x_2 + 3x_3 + 2x_4 = 3$$

PROBLEM 18 Solve system of equation by using

a) By using Gauss elimination
b) By using Gauss-Jordan elimination
c) LU decomposition (Dolittle)
d) LU Craut Decomposition

$x_1 + x_2 - x_3 = -3$
$6x_1 + 2x_2 + 2x_3 = 2$
$-3x_1 + 4x_2 + x_3 = 1$

PROBLEM 19 Find infinite norm two norm and m=3 norm of the matrix

$$\begin{bmatrix} 7 & 1 & 2 \\ -1 & 4 & -1 \\ 3 & 15 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 1 & 2 \\ -1 & 4 & -1 \\ 3 & 15 & 20 \end{bmatrix}$$

PROBLEM 20 Find the inverse matrix

$$\begin{bmatrix} 1.1348 & 3.8326 & 1.1651 & 3.4017 \\ 0.5301 & 1.7875 & 2.5330 & 1.5435 \\ 3.4129 & 4.9317 & 8.7643 & 1.3142 \\ 1.2371 & 4.9998 & 10.6721 & 0.0147 \end{bmatrix}$$

PROBLEM 21 Find the inverse matrix

$$\begin{bmatrix} 4 & 3 & -1 \\ 7 & -2 & 3 \\ 5 & -18 & 13 \end{bmatrix}$$

PROBLEM 22

Find two norm and inverse matrix

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/4 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

PROBLEM 23

Find two norm, infinite norm and and inverse matrix

$$\begin{bmatrix} 8 & 2 & 3 \\ 1 & -9 & 2 \\ 2 & 3 & 6 \end{bmatrix}$$

PROBLEM 24

Find two norm, infinite norm and and inverse matrix

$$\begin{bmatrix} 3 & -5 & 47 & 20 \\ 11 & 16 & 17 & 10 \\ 56 & 22 & 11 & -18 \\ 17 & 66 & -12 & 7 \end{bmatrix}$$

PROBLEM 25

Find two norm, infinite norm and and inverse matrix

$$\begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & & \ldots & & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \end{bmatrix}$$

4 Integration and derivatives

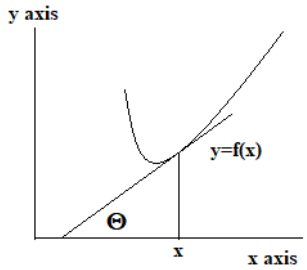
DERIVATIVES

DEFINITIONS

Definition of derivatives:

$$\frac{df(x)}{dx} = lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Other definitions



Derivative is the slope of tangent to the function at a given x value

$$\frac{df(x)}{dx} = \tan(\theta)$$

Let us apply basic definition to some functions:

$$f(x) = x^2$$

$$\frac{df(x)}{dx} = lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\frac{df(x)}{dx} = lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = lim_{\Delta x \to 0}(2x + \Delta x) = \boldsymbol{2x}$$

$$f(x) = x^3$$

$$\frac{df(x)}{dx} = lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

$$\frac{df(x)}{dx} = lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x} = lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}$$

$$\frac{df(x)}{dx} = lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + \Delta x^2)}{\Delta x} = lim_{\Delta x \to 0}(3x^2 + 3x\Delta x + \Delta x^2) = \mathbf{3x^2}$$

Some common analytic formulas for derivatives:

f(x)	$f'(x) = \frac{df(x)}{dx}$
x^n	nx^{n-1}
x^2	$2x$
x^3	$3x^2$
e^{ax}	ae^{ax}
$e^{u(x)}$	$u'(x)e^u$
e^{-x}	$-e^{-x}$
a^{bx}	$bln(a)a^{bx}$
a^x	$ln(a)a^x$
2^{3x}	$3ln(2)2^{3x}$
$ln(u(x))$	$\frac{u'(x)}{u(x)}$
cos(u(x))	-u'(x)sin(u(x))
sin(u(x))	u'(x)cos(u(x))
tan(u(x))	u'(x)$sec^2(u(x))$
arcsin $(u(x))$	$\frac{u'(x)}{\sqrt{1-u(x)^2}}$
arccos ($u(x)$)	$-\frac{u'(x)}{\sqrt{1-u(x)^2}}$
arctan $(u(x))$	$\frac{u'(x)}{1+u(x)^2}$
cosh(u(x))	u'(x)sinh(u(x))
sinh(u(x))	u'(x)cosh(u(x))
tanh(u(x))	u'(x)$sech^2(u(x))$
arcsinh $(u(x))$	$\frac{u'(x)}{\sqrt{1+u(x)^2}}$
arccosh ($u(x)$)	$-\frac{u'(x)}{\sqrt{u(x)^2-1}}$
arctanh $(u(x))$	$\frac{u'(x)}{1-u(x)^2}$
$u(x) \pm v(x)$	$u'(x) \pm v'(x)$
$u(x)v(x)$	$u'(x)v(x) + v'(x)u(x)$
$\frac{1}{u(x)}$	$-\frac{u'(x)}{u^2(x)}$
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - v'(x)u(x)}{v^2(x)}$

Examples of analytical derivations

EX1: take derivative of $y = f(x) = 2x^3 + 3x^2 + 5$

Answer: $f'(x) = 6x^2 + 6x$

EX2: take derivative of $y = f(x) = u(x)v(x) = (2x^3 + 3x^2 + 5)(x^2 - 2)$

$f'(x) = u'(x)v(x) + v'(x)u(x)$

$u'(x) = 6x^2 + 6x \qquad v'^{(x)} = 2x$

$f'(x) = (6x^2 + 6x)(x^2 - 2) + (2x^3 + 3x^2 + 5)(2x)$

EX3: take derivative of $y = f(x) = \frac{u(x)}{v(x)} = \frac{(2x^3+3x^2+5)}{(x^2-2)}$

$$f'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{v^2(x)}$$

$$f'(x) = \frac{(6x^2+6x)(x^2-2) - (2x^3+3x^2+5)(2x)}{(x^2-2)^2}$$

EX4: Use the chain rule to calculate $\frac{dy}{dx}$

$y(x) = u(x)^2 - 2u(x) + 1$ where $u(x) = (x^2+4)^2$

$\frac{dy(x)}{dx} = \frac{dy(x)}{du(x)}\frac{du(x)}{dx} = [2u(x)-2][2(x^2+4)2x] = [2(x^2+4)^2-2][2(x^2+4)2x]$

EX5: Use the chain rule to calculate $\frac{dy}{dx}$

$y(x) = 2u(x)^3 \quad u(v) = \sqrt{v^2+3} \quad v(x) = \frac{1}{x}$

$\frac{dy(x)}{du(x)} = 6u(x)^2 \quad \frac{du(v)}{dv} = (v^2+3)^{1/2} = \frac{1}{2}\frac{2v}{\sqrt{v^2+3}} \quad \frac{dv(x)}{dx} = -\frac{1}{x^2}$

$$\frac{dy(x)}{dx} = \frac{-6u(x)^2\, v(x)}{x^2\sqrt{v^2+3}}$$

EX6: Calculate the derivative of $y(x) = x^{(x^2+2x+1)}$

$$\ln(y(x)) = (x^2+2x+1)\ln(x)$$

$$\frac{\frac{dy(x)}{dx}}{y(x)} = (2x+2)\ln(x) + \frac{(x^2+2x+1)}{x}$$

$$\frac{dy(x)}{dx} = y(x)\left[(2x+2)\ln(x) + \frac{(x^2+2x+1)}{x}\right]$$

$$\frac{dy(x)}{dx} = x^{(x^2+2x+1)}\left[(2x+2)\ln(x) + \frac{(x^2+2x+1)}{x}\right]$$

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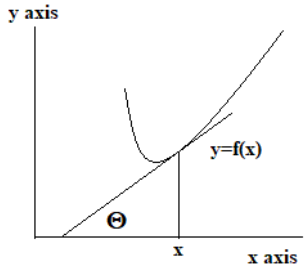

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$y(x) = 2u(x)^3 \quad u(v) = \sqrt{v^2 + 3} \qquad v(x) = \frac{1}{x}$

$\frac{dy(x)}{du(x)} = 6u(x)^2 \qquad \frac{du(v)}{dv} = (v^2 + 3)^{1/2} = \frac{1}{2}\frac{2v}{\sqrt{v^2+3}} \qquad \frac{dv(x)}{dx} = -\frac{1}{x^2}$

$$\frac{dy(x)}{dx} = \frac{-6u(x)^2\, v(x)}{x^2\sqrt{v^2 + 3}}$$

EX6: Calculate the derivative of $y(x) = x^{(x^2+2x+1)}$

$$\ln(y(x)) = (x^2 + 2x + 1)\ln(x)$$

$$\frac{\frac{dy(x)}{dx}}{y(x)} = (2x+2)\ln(x) + \frac{(x^2+2x+1)}{x}$$

$$\frac{dy(x)}{dx} = y(x)\left[(2x+2)\ln(x) + \frac{(x^2+2x+1)}{x}\right]$$

$$\frac{dy(x)}{dx} = x^{(x^2+2x+1)}\left[(2x+2)\ln(x) + \frac{(x^2+2x+1)}{x}\right]$$

In order to approximate derivation difference equations can be used.Difference equations can basically be derived from Taylor formulations

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \frac{f^{(4)}(x_i)}{4!}h^4 + \ldots$$

If the first derivative is solved from this equation it is found as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h - \frac{f^{(3)}(x_i)}{3!}h^2 - \frac{f^{(4)}(x_i)}{4!}h^3 + O(h^4)\ldots$$

or

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

Term O(h) is the order of error. Error is proportional to the difference parameter h. When h is getting smaller, error wil also shrinked. If the second derivative is written the same way:

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

In order to decrease the error, if the second derivative difference equation is substituted into thefirst derivative Taylor equation a first derivative equation with a second order error can be obtained.

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2h^2}h + O(h^2)$$

$$f'(x_i) = \frac{2f(x_{i+1}) - 2f(x_i) - f(x_{i+2}) + 2f(x_{i+1}) - f(x_i)}{h} + O(h^2)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

Difference equation can be written as forward or backward way. The backward difference equation: $f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$. By taken avarage of forward and bakward formulas can create a central formula

$$f'(x_i) = \frac{f'(x_i)ileri + f'(x_i)geri}{2} = \frac{\frac{f(x_i) - f(x_{i-1})}{h}}{2} + \frac{\frac{f(x_i) - f(x_{i-1})}{h}}{2} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

Difference formulas for calculating derivatives are given (as coefficients) in the following tables

Derivative (Difference) equations

Central finite difference

Derivativ	Accurac	−5	−4	−3	−2	−1	0	1	2	3	4	5
1	2					−1/2	0	1/2				
	4				1/12	−2/3	0	2/3	−1/12			

	6			−1/60	3/20	−3/4	0	3/4	−3/20	1/60		
	8		1/280	−4/10	1/5	−4/5	0	4/5	−1/5	4/105	−1/280	
2	2					1	−2	1				
	4				−1/12	4/3	−5/2	4/3	−1/12			
	6			1/90	−3/20	3/2	−49/18	3/2	−3/20	1/90		
	8		−1/560	8/315	−1/5	8/5	−205/7	8/5	−1/5	8/315	−1/560	
3	2				−1/2	1	0	−1	1/2			
	4			1/8	−1	13/8	0	−13/8	1	−1/8		
	6		−7/240	3/10	−169/120	61/30	0	−61/30	169/12	−3/10	7/240	
4	2				1	−4	6	−4	1			
	4			−1/6	2	−13/2	28/3	−13/2	2	−1/6		
	6		7/240	−2/5	169/60	−122/1	91/8	−122/1	169/60	−2/5	7/240	
5	2			−1/2	2	−5/2	0	5/2	−2	1/2		
	4		1/6	−3/2	13/3	−29/6	0	29/6	−13/3	3/2	−1/6	
	6	−13/28	19/36	−87/3	13/2	−323/4	0	323/48	−13/2	87/32	−19/36	13/28
6	2			1	−6	15	−20	15	−6	1		
	4		−1/4	3	−13	29	−75/2	29	−13	3	−1/4	
	6	13/240	−19/24	87/16	−39/2	323/8	−1023/	323/8	−39/2	87/16	−19/24	13/24

Forward finite difference

Derivative	Accuracy	0	1	2	3	4	5	6	7	8
1	1	−1	1							
	2	−3/2	2	−1/2						
	3	−11/6	3	−3/2	1/3					
	4	−25/12	4	−3	4/3	−1/4				
	5	−137/60	5	−5	10/3	−5/4	1/5			
	6	−49/20	6	−15/2	20/3	−15/4	6/5	−1/6		
2	1	1	−2	1						
	2	2	−5	4	−1					

	3	35/12	−26/3	19/2	−14/3	11/12				
	4	15/4	−77/6	107/6	−13	61/12	−5/6			
	5	203/45	−87/5	117/4	−254/9	33/2	−27/5	137/180		
	6	469/90	−223/10	879/20	−949/18	41	−201/10	1019/180	−7/10	
3	1	−1	3	−3	1					
	2	−5/2	9	−12	7	−3/2				
	3	−17/4	71/4	−59/2	49/2	−41/4	7/4			
	4	−49/8	29	−461/8	62	−307/8	13	−15/8		
	5	−967/120	638/15	−3929/40	389/3	−2545/24	268/5	−1849/120	29/15	
	6	−801/80	349/6	−18353/120	2391/10	−1457/6	4891/30	−561/8	527/30	−469/240
4	1	1	−4	6	−4	1				
	2	3	−14	26	−24	11	−2			
	3	35/6	−31	137/2	−242/3	107/2	−19	17/6		
	4	28/3	−111/2	142	−1219/6	176	−185/2	82/3	−7/2	
	5	1069/80	−1316/15	15289/60	−2144/5	10993/24	−4772/15	2803/20	−536/15	967/240

Backward finite difference

Derivative	Accuracy	0	-1	-2	-3	-4	-5	-6	-7	-8
1	1	1	-1							
	2	3/2	-2	1/2						
	3	11/6	-3	3/2	-1/3					
	4	25/12	-4	3	-4/3	1/4				
	5	137/60	-5	5	-10/3	5/4	-1/5			
	6	49/20	-6	15/2	-20/3	15/4	-6/5	1/6		
2	1	1	−2	1						
	2	2	−5	4	−1					
	3	35/12	−26/3	19/2	−14/3	11/12				
	4	15/4	−77/6	107/6	−13	61/12	−5/6			
	5	203/45	−87/5	117/4	−254/9	33/2	−27/5	137/180		

	6	469/90	−223/10	879/20	−949/18	41	−201/10	1019/180	−7/10	
3	1	1	-3	3	1					
	2	5/2	-9	12	-7	3/2				
	3	17/4	-71/4	59/2	-49/2	41/4	-7/4			
	4	49/8	-29	461/8	-62	307/8	-13	15/8		
	5	967/120	-638/15	3929/40	-389/3	2545/24	-268/5	1849/120	-29/15	
	6	801/80	-349/6	18353/120	-2391/10	1457/6	-4891/30	561/8	-527/30	469/240
4	1	1	−4	6	−4	1				
	2	3	−14	26	−24	11	−2			
	3	35/6	−31	137/2	−242/3	107/2	−19	17/6		
	4	28/3	−111/2	142	−1219/6	176	−185/2	82/3	−7/2	
	5	1069/80	−1316/15	15289/60	−2144/5	10993/24	−4772/15	2803/20	−536/15	967/240

In order to Show utilisation of the table, the derivative of f(x)=sin(x)

y=f(x)=sin(x)

Derivative	Accuracy	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	2	0.00000000	0.00000000	0.00000000	0.00000000	-0.50000000	0.00000000	0.50000000	0.00000000	0.00000000	0.00000000	0.00000000
	4	0.00000000	0.00000000	0.00000000	0.08333333	-0.66666667	0.00000000	0.66666667	-0.08333333	0.00000000	0.00000000	0.00000000
	6	0.00000000	0.00000000	-0.01666667	0.15000000	-0.75000000	0.00000000	0.75000000	-0.15000000	0.01666667	0.00000000	0.00000000
	8	0.00000000	0.00357143	-0.03809524	0.20000000	-0.80000000	0.00000000	0.80000000	-0.20000000	0.03809524	-0.00357143	0.00000000

Δx	0.01	-5	-4	-3	-2	-1	0	1	2	3	4	5		
x	0	-0.05	-0.04	-0.03	-0.02	-0.01	0	0.01	0.02	0.03	0.04	0.05		
y=f(x)	Accuracy	-0.049979	-0.039989	-0.029996	-0.019998667	-0.01	0	0.009999833	0.0199987	0.0299955	0.0399893	0.049979	Σ	f'(x)
	2	0	0	0	0	0.0049999	0	0.004999917	0	0	0	0	0.01	0.999983
	4	0	0	0	-0.001666556	0.0066666	0	0.006666556	-0.001667	0	0	0	0.01	1
	6	0	0	0.0004999	-0.0029998	0.0074999	0	0.007499875	-0.003	0.000499925	0	0	0.01	1
	8	0	-0.000143	0.0011427	-0.003999733	0.0079999	0	0.007999867	-0.004	0.001142686	-0.000143	0	0.01	1

Derivative	Accuracy	-5	-4	-3	-2	-1	0	1	2	3	4	5
2	2					1	-2	1				
	4				-0.083333333	1.3333333	-2.5	1.333333333	-0.083333			
	6			0.0111111	-0.15	1.5	-2.722222	1.5	-0.15	0.011111111		
	8		-0.001786	0.0253968	-0.2	1.6	-2.847222	1.6	-0.2	0.025396825	-0.001786	

Δx	0.01	-5	-4	-3	-2	-1	0	1	2	3	4	5		
x	0	-0.05	-0.04	-0.03	-0.02	-0.01	0	0.01	0.02	0.03	0.04	0.05		
y=f(x)	Accuracy	-0.049979	-0.039989	-0.029996	-0.019998667	-0.01	0	0.009999833	0.0199987	0.0299955	0.0399893	0.049979	Σ	f''(x)
	2	0	0	0	0	-0.01	0	0.009999833	0	0	0	0	0	0
	4	0	0	0	0.001666556	-0.013333	0	0.013333111	-0.001667	0	0	0	-2.2E-19	-2.2E-17
	6	0	0	-0.000333	0.0029998	-0.015	0	0.01499975	-0.003	0.000333283	0	0	2.71E-19	2.71E-17
	8	0	7.141E-05	-0.000762	0.003999733	-0.016	0	0.015999733	-0.004	0.00076179	-7.14E-05	0	2.17E-19	2.17E-17

Derivative	Accuracy	-5	-4	-3	-2	-1	0	1	2	3	4	5
3	2	0	0	0	-0.5	1	0	-1	0.5	0	0	0
	4	0	0	0.125	-1	1.625	0	-1.625	1	-0.125	0	0
	6	0	-0.029167	0.30000000	-1.40833333	2.03333333	0.00000000	-2.03333333	1.40833333	-0.30000000	0.02916667	0

Δx	0.0001	-5	-4	-3	-2	-1	0	1	2	3	4	5		Third
x	0	-0.05	-0.04	-0.03	-0.02	-0.01	0	0.01	0.02	0.03	0.04	0.05		Derivative
y=f(x)	Accuracy	-0.049979	-0.039989	-0.0299955	-0.019998667	-0.0099998	0	0.009999833	0.01999867	0.0299955	0.03998933	0.049979	Σ	f'''(x)
	2	0	0	0	0.009999333	-0.0099998	0	-0.009999833	0.00999933	0	0	0	-1E-06	-0.999975
	4	0	0	-0.0037494	0.019998667	-0.0162497	0	-0.016249729	0.01999867	-0.003749438	0	0	-1E-06	-1
	6	0	0.0011664	-0.0089987	0.028164789	-0.020333	0	-0.020332994	0.02816479	-0.00899865	0.00116636	0	-1E-06	-1

A derivative calculating program is given as an example

Program derivative(usage of difference equations)

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
//Derivative formulas
// Dr. Turhan Coban
// =====================================================
import java.io.*;

class f1 extends f_x
{ double func(double x)
  {return x*x*x*x*x;}
}

class NA81
{
public static double derivative(f_x f,double x)
{
double h0=0.001;
int i,m;
int n=7;
double T[][]=new double[n][n];
double h[];
h=new double[n];
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++) {h[i]=h0*Math.pow(r,i);}
for(i=0;i<n;i++)  {T[i][0]=dfdx(f,x,h[i]);}
for(m=1;m<n;m++)
{ for(i=0;i<n-m;i++)
  {T[i][m]=(h[i]*h[i]*T[i+1][m-1]-h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]- h[i+m]*h[i+m]);}
}
double xx=T[0][n-1];
return xx;
}

public static double dfdx_richardson(f_x f,double x)
{
double h0=0.01;
double h=h0;
int i,m;
int n=10;
double T[][]=new double[n][n];
for(i=0;i<n;i++)  {T[i][0]=dfdx(f,x,h);h/=2.0;}
for(m=1;m<n;m++)
{ for(i=0;i<n-m;i++)
  {T[i][m]=(4.0/3.0*T[i+1][m-1]-1.0/3.0*T[i][m-1]);}
}
double xx=T[0][n-1];
return xx;
}



public static double dfdx(f_x f,double x)
{ double h=0.00001;
  return dfdx(f,x,h);
}
```

```
public static double dfdx(f_x f,double x,double h)
{ double hh=1.0/h;
  return (-f.func(x+2.0*h)+8.0*f.func(x+h)-8.0*f.func(x-h)+f.func(x-2.0*h))/12.0*hh;
}

public static double d2fdx2(f_x f,double x)
{ double h=0.00001;
  return d2fdx2(f,x,h);
}

public static double d2fdx2(f_x f,double x,double h)
{
  double hh=1.0/h;
            return (-f.func(x+2.0*h)+16.0*f.func(x+h)-30.0*f.func(x)+16.0*f.func(x-h)-f.func(x-2.0*h))/(12.0)*hh*hh;}

public static double d3fdx3(f_x f,double x)
{ double h=0.00001;
  return d3fdx3(f,x,h);}

public static double d3fdx3(f_x f,double x,double h)
{  double hh=1.0/h;
            return (-f.func(x+3.0*h)+8.0*f.func(x+2.0*h)-13.0*f.func(x+h)+13.0*f.func(x-h)-8.0*f.func(x-2.0*h)+f.func(x-
3.0*h))/(8.0)*hh*hh*hh;}

  public static void main(String args[])
  {
  f1 b1=new f1();
  System.out.println("derivative dfdx    : "+dfdx(b1,1.0));
  System.out.println("derivative d2dfx2  : "+d2fdx2(b1,1.0));
  System.out.println("derivative d3dfx3  : "+d3fdx3(b1,1.0));

  System.out.println("derivative dfdx    : "+derivative(b1,1.0));
  System.out.println("derivative dfdx    : "+dfdx_richardson(b1,1.0));
  }
}
```

derivative(usage of difference equations)

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA81
derivative dfdx    : 4.999999999989272
derivative d2dfx2  : 20.00000674332961
derivative d3dfx3  : 60.14633235906783
derivative dfdx    : 5.000000000009831
derivative dfdx    : 4.99999999994733
> Terminated with exit code 0.
```

Derivative definitions in python

```
def f(x):
    y=x*x-2.0*x+5.0
    return y
def df(f,x):
    dx=0.001
    y=(f(x+dx)-f(x-dx))/(2.0*dx)
    return y
def d2f(f,x):
    dx=0.001
    y=(f(x+dx)-2.0*f(x)+f(x-dx))/(dx*dx)
    return y
x=float(input("x="))
y=f(x);
dy=df(f,x)
```

```
d2y=d2f(f,x)
print("x=",x,"y=",y,"dy=",dy,"d2y=",d2y)
```

```
runfile('E:/okul/SCO1/derivative1.py', wdir='E:/okul/SCO1')

x=1
x= 1.0 y= 4.0 dy= 0.0 d2y= 2.000000000279556
```

Derivative definitions in python

```
def f(x):
    y=x*x-2.0*x+5.0
    return y
def df(f,x):
    dx=0.001
    y=(-f(x+2.0*dx)+8.0*f(x+dx)-8.0*f(x-dx)+f(x-2.0*dx))/(12.0*dx)
    return y
def d2f(f,x):
    dx=0.001
    y=(-f(x+2.0*dx)+16.0*f(x+dx)-30.0*f(x)+16.0*f(x-dx)-f(x-2.0*dx))/(12.0*dx*dx)
    return y
x=float(input("x="))
y=f(x);
dy=df(f,x)
d2y=d2f(f,x)
print("x=",x,"y=",y,"dy=",dy,"d2y=",d2y)
```

```
runfile('E:/okul/SCO1/derivative2.py', wdir='E:/okul/SCO1')

x=1
x= 1.0 y= 4.0 dy= -7.401486830834377e-14 d2y= 2.00000000064963
```

Using if_x abstract class for derivatives in python

```
from math import *
from if_x import *;

class f1(if_x):
  def func(self,x):
      y=x*x-2;
      return y;

x=[1.2,2.3,3.4,4.5,5.6,6.0,5.7,5.1,4.6]
n=len(x)
f = f1()
for i in range(n):
    print("i = ",i," x = ",x[i],"y=f(x) = ",f.func(x[i]),"dy/dx=df(x)/dx = ",f.dfunc(x[i],1))
```

```
rom abc import ABC, abstractmethod
from math import *

class if_x(ABC):

    @abstractmethod
    def func(self,x):
        pass;

    def dfunc(self,x,n):
        dx=0.0001;
        if n==0: df=self.func(x);
        elif n==1: df=(3.0*self.func(x-4.0*dx)-32.0*self.func(x-3.0*dx)+168.0*self.func(x-2.0*dx)-672.0*self.func(x-
dx)+672.0*self.func(x+dx)-168.0*self.func(x+2.0*dx)+32.0*self.func(x+3.0*dx)-3.0*self.func(x+4.0*dx))/840.0/dx;
        elif n==2: df=(-14350.0*self.func(x)-9.0*self.func(x-4.0*dx)+128.0*self.func(x-3.0*dx)-1008.0*self.func(x-
2.0*dx)+8064.0*self.func(x-dx)+8064.0*self.func(x+dx)-1008.0*self.func(x+2.0*dx)+128.0*self.func(x+3.0*dx)-
9.0*self.func(x+4.0*dx))/5040.0/dx/dx;
        elif n==3: df=(-7.0*self.func(x-4.0*dx)+72.0*self.func(x-3.0*dx)-338.0*self.func(x-2.0*dx)+488.0*self.func(x-dx)-
488.0*self.func(x+dx)+338.0*self.func(x+2.0*dx)-72.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx;
        elif n==4: df=(2730.0*self.func(x)+7.0*self.func(x-4.0*dx)-96.0*self.func(x-3.0*dx)+676.0*self.func(x-2*dx)-1952.0*self.func(x-
dx)-1952.0*self.func(x+dx)+676.0*self.func(x+2.0*dx)-96.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx/dx;
        elif n==5: df=(self.func(x-4.0*dx)-9.0*self.func(x-3.0*dx)+26.0*self.func(x-2.0*dx)-29.0*self.func(x-dx)+29.0*self.func(x+dx)-
26.0*self.func(x+2.0*dx)+9.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/6.0/dx/dx/dx/dx/dx;
        elif n==6: df=(-150.0*self.func(x)-self.func(x-4.0*dx)+12.0*self.func(x-3.0*dx)-52.0*self.func(x-2.0*dx)+116.0*self.func(x-
dx)+116.0*self.func(x+dx)-52.0*self.func(x+2.0*dx)+12.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/4.0/dx/dx/dx/dx/dx/dx;
```

```
        elif n==7: df=(-self.func(x-4.0*dx)+6.0*self.func(x-3.0*dx)-14.0*self.func(x-2.0*dx)+14.0*self.func(x-dx)-
14.0*self.func(x+dx)+14.0*self.func(x+2.0*dx)-6.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/2.0/dx/dx/dx/dx/dx/dx/dx;
        elif n==8: df=(70.0*self.func(x)+self.func(x-4.0*dx)-8.0*self.func(x-3.0*dx)+28.0*self.func(x-2.0*dx)-56.0*self.func(x-dx)-
56.0*self.func(x+dx)+28.0*self.func(x+2.0*dx)-8.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/dx/dx/dx/dx/dx/dx/dx/dx;
        else: df=0;
        return df;

    def gauss_legendre_coefficients(self,x1,x2,n):
        #calculates legendre gauss-coefficients as coefficients of the integral
        #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

    def integral(self,x1,x2,n):
        #integral func(x)dx
        #integral of a function by using gauss-legendre quadrature
        #between x1 and x2
        a=self.gauss_legendre_coefficients(x1,x2,n)
        z=0
        for i in range(n):
            z=z+a[1][i]*self.func(a[0][i])
        return z
```

```
n [27]: runfile('E:/okul/SCO1/if_x_test.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
i =  0  x =  1.2 y=f(x) =  -0.56 dy/dx=df(x)/dx =  2.399999999999504
i =  1  x =  2.3 y=f(x) =  3.289999999999999 dy/dx=df(x)/dx =  4.599999999999715
i =  2  x =  3.4 y=f(x) =  9.559999999999999 dy/dx=df(x)/dx =  6.7999999999986755
i =  3  x =  4.5 y=f(x) =  18.25 dy/dx=df(x)/dx =  9.000000000005155
i =  4  x =  5.6 y=f(x) =  29.359999999999996 dy/dx=df(x)/dx =  11.200000000007698
i =  5  x =  6.0 y=f(x) =  34.0 dy/dx=df(x)/dx =  12.000000000013854
i =  6  x =  5.7 y=f(x) =  30.490000000000002 dy/dx=df(x)/dx =  11.400000000007774
i =  7  x =  5.1 y=f(x) =  24.009999999999998 dy/dx=df(x)/dx =  10.200000000005225
i =  8  x =  4.6 y=f(x) =  19.159999999999997 dy/dx=df(x)/dx =  9.200000000007734
```

```
 function f(x)
   y=x*x-2.0*x+5.0
   return y
   end
function df(f,x)
   dx=0.001
   y=(f(x+dx)-f(x-dx))/(2.0*dx)
   return y
   end
function d2f(f,x)
   dx=0.001
   y=(f(x+dx)-2.0*f(x)+f(x-dx))/(dx*dx)
   return y
```

```
    end
function read(s)
  print("enter x = ")
  x=readline()
  z=tryparse(Float64,x)
  print("z = $z")
  return z
  end

x=read("x=")
y=f(x);
dy=df(f,x)
d2y=d2f(f,x)
print("x=$x,y=$y,dy=,$dy,d2y= $d2y")
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" derivative1.jl
enter x = 1
z = 1.0x=1.0,y=4.0,dy=,0.0,d2y= 2.000000000279556
> Terminated with exit code 0.
```

Derivative definitions in python

```
 function f(x)
   y=x*x-2.0*x+5.0
   return y
   end

function f(x)
   y=x*x-2.0*x+5.0
   return y
   end
function df(f,x)
   dx=0.001
   y=(-f(x+2.0*dx)+8.0*f(x+dx)-8.0*f(x-dx)+f(x-2.0*dx))/(12.0*dx)
   return y
   end
function d2f(f,x)
   dx=0.001
   y=(-f(x+2.0*dx)+16.0*f(x+dx)-30.0*f(x)+16.0*f(x-dx)-f(x-2.0*dx))/(12.0*dx*dx)
   return y
   end
function read(s)
  print("enter x = ")
  x=readline()
  z=tryparse(Float64,x)
  print("z = $z")
  return z
  end


x=read("x=")
y=f(x);
dy=df(f,x)
d2y=d2f(f,x)
print("x=$x,y=$y,dy=,$dy,d2y= $d2y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" derivative2.jl
enter x = 1
z = 1.0x=1.0,y=4.0,dy=,-7.401486830834377e-14,d2y= 2.00000000064963
> Terminated with exit code 0.
```

Derivative in C++ using abstract class f_x

```
#include <stdio.h>
#include <iostream>
#include <math.h>
  #include <stdio.h>
#include <iostream>
#include <math.h>
#include <iomanip>
```

```
using namespace std;
class f_x{
public: virtual double func(double x)=0;
public: double dfunc(double x)
{ double h=0.0001;
  double hh=1.0/h;
  double df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-
h)+672.0*func(x+h)-168.0*func(x+2.0*h) +32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  return df;
}
public: double dfunc_n(double x,int order,double h)
{ //double h=0.0001;
  double hh=1.0/h;
  double df=0.0;
  if(order==0) df=func(x);
  else if(order==1) df=(3.0*func(x-4.0*h)-32.0*func(x-3.0*h)+168.0*func(x-2.0*h)-672.0*func(x-
h)+672.0*func(x+h)-168.0*func(x+2.0*h) +32.0*func(x+3.0*h) - 3.0*func(x+4.0*h))/840.0*hh;
  else if(order==2)
  {   double a[9]={-1.0/560.0,8.0/315.0,-1.0/5.0,8.0/5.0,-205.0/72.0,8.0/5.0,-1.0/5.0,8.0/315.0,-
1.0/560.0};
      df=(a[0]*func(x-4.0*h)+a[1]*func(x-3.0*h)+a[2]*func(x-2.0*h)+a[3]*func(x-
h)+a[4]*func(x)+a[5]*func(x+h)+a[6]*func(x+2.0*h)+a[7]*func(x+3.0*h)+a[8]*func(x+4.0*h))*h
h*hh;
  }
  else if(order==3)
  {   double a[9]={-7.0/240.0,3.0/10.0,-169.0/120.0,61.0/30.0,0.0,-61.0/30.0,169.0/120.0,-
3.0/10.0,7.0/240.0};
      df=(a[0]*func(x-4.0*h)+a[1]*func(x-3.0*h)+a[2]*func(x-2.0*h)+a[3]*func(x-
h)+a[4]*func(x)+a[5]*func(x+h)+a[6]*func(x+2.0*h)+a[7]*func(x+3.0*h)+a[8]*func(x+4.0*h))*h
h*hh*hh;
  }
  else if(order==4)
  {   double a[9]={7.0/240.0,-2.0/5.0,169.0/60.0,-122.0/15.0,91.0/8.0,-122.0/15.0,169.0/60.0,-
2.0/5.0,7.0/240.0};
      df=(a[0]*func(x-4.0*h)+a[1]*func(x-3.0*h)+a[2]*func(x-2.0*h)+a[3]*func(x-
h)+a[4]*func(x)+a[5]*func(x+h)+a[6]*func(x+2.0*h)+a[7]*func(x+3.0*h)+a[8]*func(x+4.0*h))*h
h*hh*hh*hh;
  }
  else if(order==5)
  {   double a[11]={-13.0/288.0,19.0/36.0,-87.0/32.0,13.0/2.0,-323.0/48.0,0,323.0/48.0,-
13.0/2.0,87.0/32.0,-19.0/36.0,13.0/288.0};
      df=(a[0]*func(x-5.0*h)+
          a[1]*func(x-4.0*h)+
          a[2]*func(x-3.0*h)+
          a[3]*func(x-2.0*h)+
          a[4]*func(x-h)+
          a[5]*func(x)+
          a[6]*func(x+h)+
          a[7]*func(x+2.0*h)+
          a[8]*func(x+3.0*h)+
          a[9]*func(x+4.0*h)+
          a[10]*func(x+5.0*h))*hh*hh*hh*hh*hh;
  }
    else if(order==6)
```

```
  {   double a[11]={13.0/240.0,-19.0/24.0,87.0/16.0,-39.0/2.0,323.0/8.0,-1023.0/20.0,323.0/8.0,-
39.0/2.0,87.0/16.0,-19.0/24.0,13.0/240.0};
     df=(a[0]*func(x-5.0*h)+
        a[1]*func(x-4.0*h)+
        a[2]*func(x-3.0*h)+
        a[3]*func(x-2.0*h)+
        a[4]*func(x-h)+
        a[5]*func(x)+
        a[6]*func(x+h)+
        a[7]*func(x+2.0*h)+
        a[8]*func(x+3.0*h)+
        a[9]*func(x+4.0*h)+
        a[10]*func(x+5.0*h))*hh*hh*hh*hh*hh*hh;
  }
  return df;
}
public: double dfunc_n(double x,int order)
{
double h=0.00001*pow(10,order);
return dfunc_n(x,order,h);
}
};

int main()
{
  class f1:public f_x{public:double func(double x){ return x*x*x*x; };};
  f1 f;
  double x=0.0;
  cout << " x : ";
  cin >>x;
  double y=f.dfunc_n(x,0);
  double dy=f.dfunc_n(x,1);
  double d2y=f.dfunc_n(x,2);
  double d3y=f.dfunc_n(x,3);
  double d4y=f.dfunc_n(x,4);
  double d5y=f.dfunc_n(x,5);
  double d6y=f.dfunc_n(x,6);
  cout<<"x = "<<x<<" y = "<<y<<" dy = "<<dy<<" d2y = "<<d2y<<" d3y = "<<d3y<<" d4y =
"<<d4y<<" d5y = "<<d5y<<" d6y = "<<d6y;
  return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" derivative

D:\okul\SCO1>derivative
 x : 1
x = 1 y = 1 dy = 4 d2y = 12 d3y = 24 d4y = 24 d5y = -5.68434e-014 d6y = 2.91038e-016
> Terminated with exit code 0.
```

```
int main()
{
  class f1:public f_x{public:double func(double x){ return sin(x); };};
  f1 f;
  double x=0.0;
```

```
   x=4.0*atan(1.0);
   double y=f.dfunc_n(x,0);
   double dy=f.dfunc_n(x,1);
   double d2y=f.dfunc_n(x,2);
   double d3y=f.dfunc_n(x,3);
   double d4y=f.dfunc_n(x,4);
   double d5y=f.dfunc_n(x,5);
   double d6y=f.dfunc_n(x,6);
   cout<<"x = "<<x<<" y = "<<y<<" dy = "<<dy<<" d2y = "<<d2y<<" d3y = "<<d3y<<" d4y =
"<<d4y<<" d5y = "<<d5y<<" d6y = "<<d6y;
   return 0;
}
```

```
---------- Capture Output ----------
> "D:\okul\SCO1\run.bat" derivative

D:\okul\SCO1>derivative
x = 3.14159 y = 1.22461e-016 dy = -1 d2y = 2.21923e-013 d3y = 1 d4y = 5.37764e-013 d5y = -
0.983548 d6y = -2.57988e-020
> Terminated with exit code 0.
```

The derivative formula coefficients can be calculated numerically by using Taylor expansion. The general derivation formula is applied into abstract class f_x in order to achieve a general derivation format of any given function. Algorithm used in here is given by Bength Fornberg[47] article "Generation of Finite Difference Formulas on Arbitrary Spaced Grids" an has the following form:

$$\left.\frac{d^m f(x)}{dx^m}\right|_{x=x_0} \approx \sum_{\nu=0}^{n} \delta_{n,\nu}^{m} f(x+\alpha_\nu h) \qquad m=0,1,\ldots,M;\; n=m,m+1,\ldots,N$$

Enter M,x_0,α_0 ,α_1 ,α_2 ,α_3 ,... ,α_N

$\delta_{0,0}^{0} := 1$

$c1 := 1$

for n:=1 to N do

  c2:=1

  for ν:=0 to n-1 do

    c3:=α_n - α_ν

    c2:=c2*c3

    if $n \le M$ *then* $\delta_{n-1,\nu}^{n} := 0$

    For m:=0 to min(n,M) do

      $\delta_{n,\nu}^{m} := ((\alpha_n - x_0)\delta_{n-1,\nu}^{n} - m\delta_{n-1,\nu}^{m-1})/c3$

      next m

    next ν

  For m:=0 to min(n,M) do

    $\delta_{n,n}^{m} := \frac{c1}{c2}\left[m\delta_{n-1,n-1}^{m-1} - (\alpha_{n-1} - x_0)\delta_{n-1,n-1}^{m}\right]$

    next m

  c1:=c2

 next n

Defining abstract class f_x by using the definition above in python

```
from abc import ABC, abstractmethod
from math import *

class f_x(ABC):
```

```
@abstractmethod
def func(self,x):
    pass;

def dfunc(self,x):
    h=1e-4
    n=1
    M=10
    return self.dnMfunc(x,n,M,h)

def d2func(self,x):
    h=1e-4
    n=2
    M=10
    return self.dnMfunc(x,n,M,h)

def d3func(self,x):
    h=1e-4
    n=3
    M=10
    return self.dnMfunc(x,n,M,h)

def dnfunc(self,x,n):
    h=1e-4
    M=10
    return self.dnMfunc(x,n,M,h)

def dnMfunc(self,x,N,Mi,hi):
    # order of the maximum derivative
    # N order of derivative
    # M degree of difference formula
    M=Mi;
    #double a[]=new double[0];
    h=hi;
    x0=0.0;
    alphai=[0,1,-1,2,-2,3,-3,4,-4,5,-5,6,-6,7,-7,8,-8,9,-9,10,-10,11,-11,12,-12,13,-13,14,-14,15,
    -15,16,-16,17,-17,18,-18,19,-19,20,-20,21,-21,22,-22,23,-23,24,-24,25,-25,26,-26,27,-27,28,-28,29,-29,30,-30,
    31,-31,32,-32,33,-33,34,-34,35,-35,36,-36,37,-37,38,-38,39,-39,40,-40,41,-41,42,-42,43,-43,44,-44,45,-45,46,
    -46,47,-47,48,-48,49,-49,50,-50,51,-51,52,-52,53,-53,54,-54,55,-55,56,-56,57,-57,58,-58,59,-59,60,-60,
    -61,61,-62,62,-63,63,-64,64,-65,65,-66,66,-67,67,-68,68,-70,70,-71,71,-72,72,-73,73,-74,74,-75,75,
    -76,76,-77,77,-78,78,-79,79,-80,80,-81,81,-82,82,-83,83,-84,84,-85,85,-86,86,-87,87,
    -88,88,-89,89,-90,90,-91,91,-92,92,-93,93,-94,94,-95,95,-96,96,-97,97,-98,98,-99,99,-100,100]
    alpha=alphai
    N1=len(alpha)-1;
    # M   degree of highest derivative
    #N+1 number of coefficients
    delta = [[[0 for x in range(M+1)] for y in range(N1+1)] for z in range(N1+1)];
    #double delta[][][]=new double[N1+1][N1+1][M+1];
    #double c1,c2,c3;
    delta[0][0][0]=1.0;
    c1=1.0
    for n in range(1,N1+1):
        c2=1
        for nu in range(0,n):
            c3=alpha[n]-alpha[nu]
            c2=c2*c3
            if n<=M: delta[n-1][nu][n]=0.0
            for m in range(0,(min(n,M)+1)):
                if m==0:
                    delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m])/c3
                else:
                    delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m]-m*delta[n-1][nu][m-1])/c3
        for m in range(0,(min(n,M)+1)):
            if m==0:delta[n][n][m]=c1/c2*(-(alpha[n-1]-x0)*delta[n-1][n-1][m])
            elif m!=0: delta[n][n][m]=c1/c2*(m*delta[n-1][n-1][m-1]-(alpha[n-1]-x0)*delta[n-1][n-1][m])
        c1=c2
    c=delta
    if Mi<N: M=N
    else: M=Mi
    h=hi;
    deriv=0
    h1=1.0/h
    h2=1.0;
    for j in range(N):
      h2=h2*h1
```

```
        for i in range(0,len(c[0])):
          deriv=deriv+c[M][i][N]*self.func(x+alpha[i]*h)
        return deriv*h2

    def gauss_legendre_coefficients(self,x1,x2,n):
        #calculates legendre gauss-coefficients as coefficients of the integral
        #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

    def integral(self,x1,x2,n):
        #integral func(x)dx
        #integral of a function by using gauss-legendre quadrature
        #between x1 and x2
        a=self.gauss_legendre_coefficients(x1,x2,n)
        z=0
        for i in range(n):
            z=z+a[1][i]*self.func(a[0][i])
        return z
```

```
from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
      y=x*x-2;
      return y;

x=[1.2,2.3,3.4,4.5,5.6,6.0,5.7,5.1,4.6]
n=len(x)
f = f1()
for i in range(n):
    print("i = ",i," x = ",x[i],"y=f(x) = ",f.func(x[i]),"dy/dx=df(x)/dx = ",f.dfunc(x[i]))
```

```
runfile('E:/okul/SCO1/f_x.py', wdir='E:/okul/SCO1')

runfile('E:/okul/SCO1/if_x_test.py', wdir='E:/okul/SCO1')
i =  0  x =  1.2 y=f(x) =  -0.56 dy/dx=df(x)/dx =  2.3999999999997
i =  1  x =  2.3 y=f(x) =  3.289999999999999 dy/dx=df(x)/dx =  4.6000000000321135
i =  2  x =  3.4 y=f(x) =  9.559999999999999 dy/dx=df(x)/dx =  6.800000000043178
i =  3  x =  4.5 y=f(x) =  18.25 dy/dx=df(x)/dx =  8.999999999982425
i =  4  x =  5.6 y=f(x) =  29.359999999999996 dy/dx=df(x)/dx =  11.199999999972274
i =  5  x =  6.0 y=f(x) =  34.0 dy/dx=df(x)/dx =  11.999999999983274
i =  6  x =  5.7 y=f(x) =  30.490000000000002 dy/dx=df(x)/dx =  11.400000000032911
i =  7  x =  5.1 y=f(x) =  24.009999999999998 dy/dx=df(x)/dx =  10.199999999996114
i =  8  x =  4.6 y=f(x) =  19.159999999999997 dy/dx=df(x)/dx =  9.199999999963438
```

The derivative formula coefficients can be calculated numerically by using Taylor expansion. The general derivation formula is applied into abstract class f_x in order to achieve a general derivation format of any given function. Algorithm used in here is given by Bength Fornberg[47] article “Generation of Finite Difference Formulas on Arbitrary Spaced Grids” an has the following form:

$$\left.\frac{d^m f(x)}{dx^m}\right|_{x=x_0} \approx \sum_{\nu=0}^{n} \delta_{n,\nu}^{m} f(x+\alpha_\nu h) \qquad m=0,1,\ldots,M;\ n=m,m+1,\ldots,N$$

Enter M,x_0,α_0 ,α_1 ,α_2 ,α_3 ,... ,α_N

$\delta_{0,0}^{0} := 1$

$c1 := 1$

for n:=1 to N do

  c2:=1

  for ν:=0 to n-1 do

    $c3 := \alpha_n - \alpha_\nu$

    c2:=c2*c3

    if $n \le M$ *then* $\delta_{n-1,\nu}^{n} := 0$

    For m:=0 to min(n,M) do

      $\delta_{n,\nu}^{m} := ((\alpha_n - x_0)\delta_{n-1,\nu}^{n} - m\delta_{n-1,\nu}^{m-1}) / c3$

      next m

    next ν

  For m:=0 to min(n,M) do

    $\delta_{n,n}^{m} := \frac{c1}{c2}\left[m\delta_{n-1,n-1}^{m-1} - (\alpha_{n-1} - x_0)\delta_{n-1,n-1}^{m}\right]$

    next m

  c1:=c2

 next n

```
 import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;
import static java.lang.Math.*;
  // single function single independent variable
  // example f=x*x
  // includes full set of derivatives
  // Reference :"Generation of Finite Difference Formulas on Arbitrary Spaced Grids",
  // Bength Fornberg, Mathematics of Computation, Volume 51, Number 184, October 1988
  // pages 699-706
  //interface (empty function) version
@FunctionalInterface
interface if_x
{  public double func(double x);
//first order derivative
default double dfunc(double x)
{double h=1.0e-4;
 int n=1;
 int M=10;
 return dfunc(x,n,M,h);
}
//second order derivative
default double dfunc2(double x)
{double h=1.0e-3;
 int n=2;
 int M=10;
 return dfunc(x,n,M,h);
}
default double dfunc(double x,int N,int Mi,double hi)
{// order of the maximum derivative
 // N order of derivative
 // M degree of difference formula
double c[][][];
double alpha[];
```

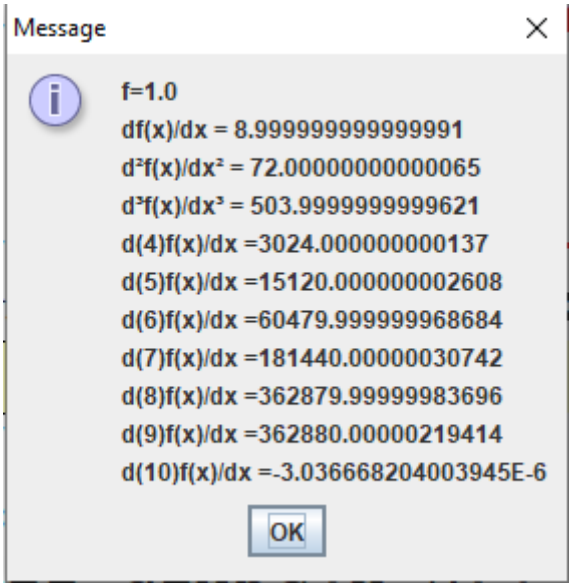
```
double h;
int M=Mi;
double a[]=new double[0];
h=0.01;
double x0=0;
double alphai[]={0,1,-1,2,-2,3,-3,4,-4,5,-5,6,-6,7,-7,8,-8,9,-9,10,-10,11,-11,12,-12,13,-13,14,-14,15,
-15,16,-16,17,-17,18,-18,19,-19,20,-20,21,-21,22,-22,23,-23,24,-24,25,-25,26,-26,27,-27,28,-28,29,-29,30,-30,
31,-31,32,-32,33,-33,34,-34,35,-35,36,-36,37,-37,38,-38,39,-39,40,-40,41,-41,42,-42,43,-43,44,-44,45,-45,46,
-46,47,-47,48,-48,49,-49,50,-50,51,-51,52,-52,53,-53,54,-54,55,-55,56,-56,57,-57,58,-58,59,-59,60,-60,
-61,61,-62,62,-63,63,-64,64,-65,65,-66,66,-67,67,-68,68,-70,70,-71,71,-72,72,-73,73,-74,74,-75,75,
-76,76,-77,77,-78,78,-79,79,-80,80,-81,81,-82,82,-83,83,-84,84,-85,85,-86,86,-87,87,
-88,88,-89,89,-90,90,-91,91,-92,92,-93,93,-94,94,-95,95,-96,96,-97,97,-98,98,-99,99,-100,100};
alpha=alphai;
int N1=alpha.length-1;
// M   degree of highest derivative
// N+1 number of coefficients
double delta[][][]=new double[N1+1][N1+1][M+1];
double c1,c2,c3;
delta[0][0][0]=1.0;
c1=1.0;
for(int n=1;n<=N1;n++)
{  c2=1;
   for(int nu=0;nu<=(n-1);nu++)
   {c3=alpha[n]-alpha[nu];
    c2=c2*c3;
    if(n<=M) delta[n-1][nu][n]=0.0;
    for(int m=0;m<=Math.min(n,M);m++)
    {
       if(m==0)
       { delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m])/c3;}
       else
       {delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m]-m*delta[n-1][nu][m-1])/c3;}
    }//next m
   }//next nu
   for(int m=0;m<=Math.min(n,M);m++)
   {   if(m==0)
                {delta[n][n][m]=c1/c2*(-(alpha[n-1]-x0)*delta[n-1][n-1][m]);}
                else
                {delta[n][n][m]=c1/c2*(m*delta[n-1][n-1][m-1]-(alpha[n-1]-x0)*delta[n-1][n-1][m]);}
   }//next m
   c1=c2;
}//next n
c=delta;
  if(Mi<N) M=N;
  else M=Mi;
  h=hi;
  double deriv=0;
  double h1=1/h;
  double h2=1;
  for(int j=0;j<N;j++)
  {h2*=h1;}
  for(int i=0;i<c[0].length;i++)
  {   deriv+=c[M][i][N]*func(x+alpha[i]*h);}
  return deriv*h2;}
default double dfunc(double x,int N)
{int M=30;double h=0.01*N*N;return dfunc(x,N,M,h);}
default double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{   //calculates legendre gauss-coefficients as coefficients of the integral
                //for n terms
                double EPS=3.0e-18;
                int m,j,i;
                double z1,z,xm,xl,pp,p3,p2,p1;
    //double x[]=new double[n];
    //double w[]=new double[n];
    double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
                m=(n+1)/2;
                xm=0.5*(x2+x1);
                xl=0.5*(x2-x1);
                for (i=1;i<=m;i++)
                { z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
```

```
                        do {
                                p1=1.0;
                                p2=0.0;
                                for (j=1;j<=n;j++) {
                                        p3=p2;
                                        p2=p1;
                                        p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                                }
                                pp=n*(z*p1-p2)/(z*z-1.0);
                                z1=z;
                                z=z1-p1/pp;
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
                }
return a;
}
default double integral(double x1,double x2)
{ int n=30;
  return integral(x1,x2,n);
}
default double integral(double x1,double x2,int n)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss legendre integral
double a[][]=new double[2][n];
a=gauss_legendre_coefficients(x1,x2,n);
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*func(a[0][i]);
}
return z;
}
default double bisection(double a,double b)
  {double b1=1.1*b;
   double r=(a+b)/2.0;
   double eps=1.0e-6;
   int nmax=100;
   int i=1;
   while(Math.abs(func(r))>eps && i<nmax)
   {if(func(a)*func(r)<0) b=r;
    else a=r;
    r=(a+b)/2.0;;
    i++;
   }
   if(i>=nmax) r=bisection(a,b1);
   return r;
  }
default double secant(double x)
{ double eps=1.0e-10;
  double y=1.0;
  double dy=0.0;
  int miter=100;
  int i=0;
  while(abs(y)>eps && i<miter)
  {y=func(x);
   dy=dfunc(x);
   x=x-y/dy;
   i++;
  }
  if(i>=miter)
    System.out.println("i="+i+"results may not be valid");
  return x;
}
default double illinois(double xa,double xu)
{
```

```
//Modified False position root finding: Illinois method
double test;
double p=0;
double es,ea;
double fa,fp,fu;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
double xold;
int ia=0,iu=0;
fa=func(xa);
fu=func(xu);
p=xu;
double xpold;
while(iter<maxit && ea>es)
{
xpold=p;
p=xu-fu*(xa-xu)/(fa-fu);
fp=func(p);
if(signum(fp)!=signum(fu))
{xa=xu;fa=fu;}
else
{fa=fa/2;}
xu=p;fu=fp;
iter++;
if(p!=0) ea=abs((p-xpold)/p)*100.0;
}
if(iter>=maxit) System.out.println("Maximum number of iteration is exceeded ");
return p;
}
default double newton_bisection(double x1, double x2)
{
double xacc=1.0e-10; //accuracy
double h=0.0001; //derivative function step
int MAXIT=100;  //Maximum number of iterations
int j;
double dfr,dx,dxold,fr,fh,fl;
double temp,b,a,r;
fl=func(x1);
fh=func(x2);
//Check if a root is existed in the given region
if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
System.out.println("There are no root in the given region or double root?");
if (fl == 0.0) return x1;
if (fh == 0.0) return x2;
if (fl < 0.0)   { a=x1;b=x2;}
else            { b=x1;a=x2;}
r=0.5*(x1+x2); //midpoint value
dxold=Math.abs(x2-x1);
dx=dxold;
fr=func(r);   //function value at midpoint
dfr=dfunc(r); //derivative of function value at midpoint
for (j=1;j<=MAXIT;j++)
{
if ((((r-b)*dfr-fr)*((r-a)*dfr-fr) > 0.0) || (Math.abs(2.0*fr) > Math.abs(dxold*dfr)))
   { //bisection step
                dxold=dx;
     dx=0.5*(b-a);
     r=a+dx;
     if (a == r) return r; //çözüm!!!!
   }
else{ //Newton-Raphson (Secant) step
                dxold=dx;
     dx=fr/dfr;
     temp=r;
     r -= dx;
     if (temp == r) return r;  //solution!!!!!
   }
if (Math.abs(dx) < xacc) return r; //solution!!!!!
fr=func(r);
```

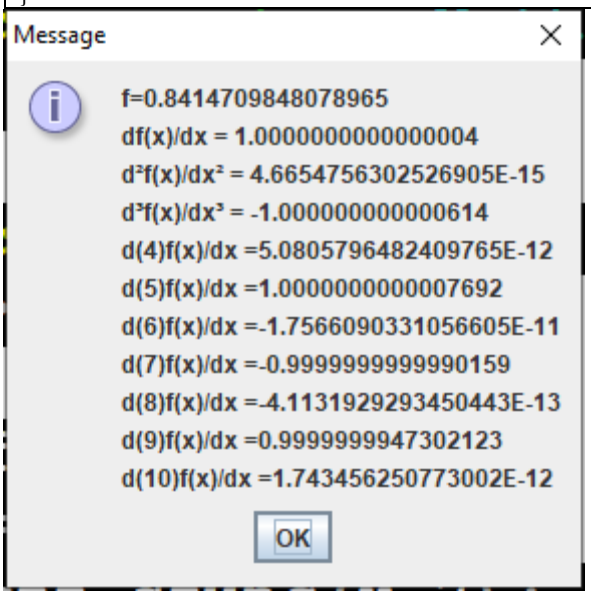
```
dfr=dfunc(r);
if (fr < 0.0)
a=r;
else
b=r;
}
System.out.println("Maximum number of iterations are exceeded");
return 0.0; //program should never reach here
          //dummy return
}
default double brent(double a,double b)
{
double test;
double p=0;
double es,ea;
double f1,f2,f3,fp;
int maxit=500,iter=0;
double tol=1.0e-15;
es=0.0000001;
double x1=a;f1=func(x1);
double x2=b;f2=func(x2);
double x3=(x1+x2)/2.0;f3=func(x3);
if(f1==0) return x1;
else if(f2==0) return x2;
else if(f3==0) return x3;
if(f1*f2>0) System.out.println("No root is existed in the given region");
p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
fp=func(p);
ea=Math.abs(x3-p);
while((ea>es)&&(iter<maxit))
{              if(Math.abs(f3)<tol) return x3;
   if((p<x1) && (p>x2))
   {p=(x1+x2)/2.0;
    if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
    else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   }
   else
   {
   if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
   else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=func(p);
   ea=Math.abs(x3-p);
   }
   iter++;
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return p;
}
default double enlarge(double x0,double dx)
{  //enlarge region untill a root is existed
              double x1=x0;
              double x2=x1+dx;
              int NTRY=200;
              double a[]=new double[2];
              double FACTOR=1.001;
              int j;
              double f1,f2;
              if (x1 == x2) System.out.println("Giriş değerleri yanlış verildi");
              f1=func(x1);
              f2=func(x2);
              for (j=1;j<=NTRY;j++)
              { if (f1*f2 < 0.0) {break;}
                else {x2+=dx;f2=func(x2);}
              }
              return x2;
}
}
```

```
 import javax.swing.*;
import static java.lang.Math.*;
public class if_xtest {

public static void main(String arg[])
{ if_x ff=x->x*x*x*x*x*x*x*x*x;
double x=1;
  double df=ff.dfunc(x,1);
  double d1f=ff.dfunc(x);
  double d2f=ff.dfunc(x,2);
  double d3f=ff.dfunc(x,3);
  double d4f=ff.dfunc(x,4);
  double d5f=ff.dfunc(x,5);
  double d6f=ff.dfunc(x,6);
  double d7f=ff.dfunc(x,7);
  double d8f=ff.dfunc(x,8);
  double d9f=ff.dfunc(x,9);
  double d10f=ff.dfunc(x,10);
 String s1="f="+ff.func(1.0)+"\ndf(x)/dx = "+df+"\nd"+\u00B2'+"f(x)/dx"+\u00B2'+" = "+d2f+
 "\nd"+\u00B3'+"f(x)/dx"+\u00B3'+" = "+d3f+"\nd(4)f(x)/dx ="+d4f+"\nd(5)f(x)/dx ="+d5f+"\nd(6)f(x)/dx ="+d6f+
 "\nd(7)f(x)/dx ="+d7f+"\nd(8)f(x)/dx ="+d8f+"\nd(9)f(x)/dx ="+d9f+"\nd(10)f(x)/dx ="+d10f;
 JOptionPane.showMessageDialog(null,s1);
}
}
```




```
import javax.swing.*;
import static java.lang.Math.*;
public class if_xtest {

public static void main(String arg[])
{ if_x ff=x->sin(x);
double x=0;
  double df=ff.dfunc(x,1);
  double d1f=ff.dfunc(x);
  double d2f=ff.dfunc(x,2);
  double d3f=ff.dfunc(x,3);
  double d4f=ff.dfunc(x,4);
  double d5f=ff.dfunc(x,5);
  double d6f=ff.dfunc(x,6);
  double d7f=ff.dfunc(x,7);
  double d8f=ff.dfunc(x,8);
  double d9f=ff.dfunc(x,9);
  double d10f=ff.dfunc(x,10);
 String s1="f="+ff.func(1.0)+"\ndf(x)/dx = "+df+"\nd"+\u00B2'+"f(x)/dx"+\u00B2'+" = "+d2f+
 "\nd"+\u00B3'+"f(x)/dx"+\u00B3'+" = "+d3f+"\nd(4)f(x)/dx ="+d4f+"\nd(5)f(x)/dx ="+d5f+"\nd(6)f(x)/dx ="+d6f+
 "\nd(7)f(x)/dx ="+d7f+"\nd(8)f(x)/dx ="+d8f+"\nd(9)f(x)/dx ="+d9f+"\nd(10)f(x)/dx ="+d10f;
 JOptionPane.showMessageDialog(null,s1);
}
```

```
}
```



Derivative coefficients for order of the highest derivative M = 4

Δx	df(x)/dx	d²f(x)/dx²	d³f(x)/dx³	d(4)f(x)/dx4
0.0	-0.0	-2.5	-0.0	6.0
1.0	0.6666666666666666	1.3333333333333333	-1.0	-4.0
-1.0	-0.6666666666666666	1.3333333333333335	1.0	-4.0
2.0	-0.08333333333333333	-0.08333333333333333	0.5	1.0
-2.0	0.08333333333333333	-0.08333333333333333	-0.5	1.0

Derivative coefficients for order of the highest derivative M = 6

Δx	df(x)/dx	d²f(x)/dx²	d³f(x)/dx³	d(4)f(x)/dx4	d(5)f(x)/dx5	d(6)f(x)/dx6
0.0	-0.0	-2.7222222222...	-0.0	9.3333333333...	-0.0	-20.0
1.0	0.75	1.5	-1.625	-6.5	2.5	15.0
-1.0	-0.75	1.5	1.625	-6.5	-2.5	15.0
2.0	-0.15	-0.15	1.0	2.0	-2.0	-6.0
-2.0	0.1500000000...	-0.15	-1.0	2.0	2.0	-6.0
3.0	0.0166666666...	0.0111111111...	-0.125	-0.1666666666...	0.5	1.0
-3.0	-0.0166666666...	0.0111111111...	0.125	-0.1666666666...	-0.5	1.0

Derivative coefficients for order of the highest derivative M = 8

Δx	df(x)/dx	d²f(x)/dx²	d³f(x)/dx³	d(4)f(x)/dx4	d(5)f(x)/dx5	d(6)f(x)/dx6	d(7)f(x)/dx7	d(8)f(x)/dx8
0.0	-0.0	-2.84722222222...	-0.0	11.375	-0.0	-37.5	-0.0	70.0
1.0	0.8	1.6	-2.03333333333...	-8.13333333333...	4.833333333333...	29.0	-7.0	-56.0
-1.0	-0.79999999999...	1.5999999999999...	2.033333333333...	-8.13333333333...	-4.83333333333...	29.0	7.0	-56.0
2.0	-0.19999999999...	-0.19999999999...	1.408333333333...	2.816666666666...	-4.33333333333...	-13.0	7.0	28.0
-2.0	0.200000000000...	-0.19999999999...	-1.40833333333...	2.816666666666...	4.333333333333...	-13.0	-7.0	28.0
3.0	0.038095238095...	0.025396825396...	-0.3	-0.39999999999...	1.499999999999...	3.0	-3.0	-8.0
-3.0	-0.03809523809...	0.025396825396...	0.3	-0.39999999999...	-1.5	3.0	3.0	-8.0
4.0	-0.00357142857...	-0.00178571428...	0.029166666666...	0.029166666666...	-0.16666666666...	-0.24999999999...	0.5	1.0
-4.0	0.003571428571...	-0.00178571428...	-0.02916666666...	0.029166666666...	0.166666666666...	-0.24999999999...	-0.5	1.0

Derivative coefficients for order of the highest derivative M = 10

Δx	df(x)/dx	d²f(x)/dx²	d³f(x)/dx³	d(4)f(x)/dx4	d(5)f(x)/dx5	d(6)f(x)/dx6	d(7)f(x)/dx7	d(8)f(x)/dx8	d(9)f(x)/dx9	d(10)f(x)/dx10
0.0	-0.0	-2.9272222222...	-0.0	12.741666666...	-0.0	-51.15	-0.0	154.0	-0.0	-252.0
1.0	0.8333333333...	1.6666666666...	-2.3180555555...	-9.2722222222...	6.7291666666...	40.375	-15.75	-126.0	21.0	210.0
-1.0	-0.8333333333...	1.6666666666...	2.3180555555...	-9.2722222222...	-6.7291666666...	40.375	15.75	-126.0	-21.0	210.0
2.0	-0.2380952380...	-0.2380952380...	1.7337301587...	3.4674603174...	-6.5	-19.5	17.0	68.0	-24.0	-120.0
-2.0	0.2380952380...	-0.2380952380...	-1.7337301587...	3.4674603174...	6.5	-19.5	-17.0	68.0	24.0	-120.0
3.0	0.0595238095...	0.0396825396...	-0.4830357142...	-0.6440476190...	2.7187499999...	5.4375	-8.625	-23.0	13.5	45.0
-3.0	-0.0595238095...	0.0396825396...	0.4830357142...	-0.6440476190...	-2.71875	5.4375	8.625	-23.0	-13.5	45.0
4.0	-0.0099206349...	-0.0049603174...	0.0833994708...	0.0833994708...	-0.5277777777...	-0.7916666666...	2.1666666666...	4.3333333333...	-4.0	-10.0
-4.0	0.0099206349...	-0.0049603174...	-0.0833994708...	0.0833994708...	0.5277777777...	-0.7916666666...	-2.1666666666...	4.3333333333...	4.0	-10.0
5.0	7.9365079365...	3.1746031746...	-0.0067791005...	-0.0054232804...	0.0451388888...	0.0541666666...	-0.2083333333...	-0.3333333333...	0.5	1.0
-5.0	-7.9365079365...	3.1746031746...	0.0067791005...	-0.0054232804...	-0.0451388888...	0.0541666666...	0.2083333333...	-0.3333333333...	-0.5	1.0

7.10 RICHARDSON EXTRAPOLATION IN DERIVATION

Richardson extrapolaton process that previously seen in integration can also be applied into derivation. Richardson extrapolation Formula was

$$D \cong D(h_2) + \frac{1}{(h_1/h_2)^2 - 1}[D(h_2) - D(h_1)]$$

If $h_2=h_1/2$ equation becomes

$$D \cong \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1)]$$

This equation can be written in the iterative two dimensional form as:

$$D_{j,k} \cong \frac{4^{k-1}D_{j+1,k-1} - D_{j,k-1}}{4^{k-1} - 1}$$

7.11 PAD'E APPROXIMATIONS

One of the most accurate derivative approximations can be derived by using Taylor series procedures. If the functional values at certain points are given Formula can be include the derivatives as well. For example, we can ask for the most accurate formula that includes f'_j, f'_{j+1} and f'_{j-1}
in addition to the functional values f_j, f_{j+1}, and f_{j-1} in addition to the functional values f_j, f_{j+1}, and f_{j-1}. we would write

$$f'_j + a_0 f_j + a_1 f_{j+1} + a_2 f_{j-1} + a_3 f'_{j+1} + a_4 f'_{j-1} = O(?)$$

And our task is then find the five coefficients $a_0, a_1, \ldots, a_4$ to maximize
the order of this approximation.
The Taylor table is

TAYLOR TABLE FOR A PAD'E SCHEME

	f_j	f_j'	f_j''	$f_j^{(3)}$	$f_j^{(4)}$	$f_j^{(5)}$
f_j'	0	1	0	0	0	0
a_0f_j	a_0	0	0	0	0	0
a_1f_{j+1}	a_1	a_1h	$a_1h^2/2$	$a_1h^3/6$	$a_1h^4/24$	$a_1h^5/120$
a_2f_{j-1}	a_2	$-a_2h$	$a_2h^2/2$	$-a_2h^3/6$	$a_2h^4/24$	$-a_2h^5/120$
a_3f_{j+1}	0	a_3	a_3h	$a_3h^2/2$	$a_3h^3/6$	$a_3h^4/24$
a_4f_{j-1}	0	a_4	$-a_4h$	$a_4h^2/2$	$-a_4h^3/6$	$a_4h^4/24$

we now sum all the rows and set as many of the lower order terms to zero as possible. We have five coefficients and can set the sum of the entries in columns 2 to 6 to zero. The linear equations for the coefficients are

$a_0 + a_1 + a_2 = 0$

$a_1h - a_2h + a_3 + a_4 = -1$

$$a_1\frac{h^2}{2} + a_2\frac{h^2}{2} + a_3h - a_4h = 0$$

$$a_1\frac{h}{3} - a_2\frac{h}{3} + a_3 + a_4 = 0$$

$$a_1\frac{h}{4} + a_2\frac{h}{4} + a_3 - a_4 = 0$$

The solution set is:

$a_0=0$ $a_1 = -\frac{3}{4h}$ $a_2 = \frac{3}{4h}$ $a_3 = a_4 = \frac{1}{4}$

This solution will lead to Padé Formula for numerical differential as

$$f'_{j+1} + f'_{j-1} + 4f'_j = \frac{3}{h}(f_{j+1} - f_{j-1}) + \frac{h^4}{30} f_j^{(5)}$$

This equation requires two more equation to solve complete set of differentiation. Additional equations can be taken as:

$$f'_0 + 2f'_1 = \frac{1}{h}(-\frac{5}{2} f_0 + 2f_1 + \frac{1}{2} f_2)$$

$$f'_n + 2f'_{n-1} = \frac{1}{h}(\frac{5}{2} f_n + 2f_{n-1} - \frac{1}{2} f_{n-2})$$

In matrix form equations can be written as

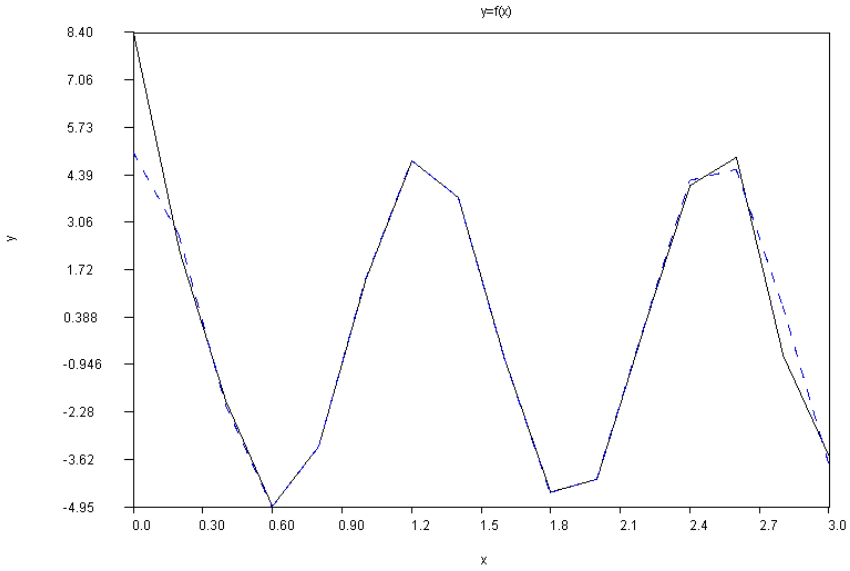
$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ & & & & 1 & 4 & 1 \\ & & & & & 2 & 1 \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ .. \\ .. \\ f'_{n-1} \\ f'_n \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} f_0 + 2f_1 + \frac{1}{2} f_2 \\ 3(f_2 - f_0) \\ 3(f_3 - f_1) \\ .. \\ .. \\ 3(f_n - f_{n-1}) \\ \frac{5}{2} f_n - 2f_{n-1} - \frac{1}{2} f_{n-2} \end{bmatrix}$$

This matrix is a tridiagonal matrix can be solved by using Thomas algorithm.

Program 7.11-1 derivative by Padé Formula

```
import java.io.*;
import javax.swing.*;

class f1 extends f_x
{ double func(double x)
  {return Math.sin(5.0*x);}
}
class f2 extends f_x
{//exact derivative of the function
double func(double x)
  {return 5.0*Math.cos(5.0*x);}
}

class NA81A
{
  public static double [] thomas(double a[][],double r[])
  {
  //Padé formula
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
```

```
return x;
}

public static double [] thomas(double f[],double e[],double g[],double r[])
{
int n=f.length;
double x[]=new double[n];
for(int k=1;k<n;k++)
   {e[k]=e[k]/f[k-1];
    f[k]=f[k]-e[k]*g[k-1];
   }
for(int k=1;k<n;k++)
   {r[k]=r[k]-e[k]*r[k-1];
   }
x[n-1]=r[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
   {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
return x;
}
public static double[][] df(f_x ff,double x0,double xn, int n)
{
int nn=n+1;
double h=(xn-x0)/n;
double y[]=new double[nn];
double x[]=new double[nn];
double f[]=new double[nn];
double g[]=new double[nn];
double e[]=new double[nn];
for(int i=0;i<=n;i++)
{x[i]=x0+h*i;}

for(int i=0;i<=n;i++)
{
 if(i==0) {f[i]=1;}
 else if(i==n) {f[i]=1;}
 else {f[i]=4;}
 if(i==1) {g[i]=2;}
 else if(i==n) {g[i]=0;}
 else {g[i]=1;}
 if(i==(n-1)) {e[i]=2;}
 else if(i==0) {e[i]=0;}
 else {e[i]=1;}
 if(i==0) {y[i]=1.0/h*(-2.5*ff.func(x[0])+2.0*ff.func(x[1])+0.5*ff.func(x[2]));}
 else if(i==n) {y[i]=1.0/h*(2.5*ff.func(x[n])-2.0*ff.func(x[n-1])-0.5*ff.func(x[n-1]));}
 else {y[i]=(1.0/h*3.0*(ff.func(x[i+1])-ff.func(x[i-1])));}
}
 double a[][]=new double[3][nn];
 a[0]=x;
 a[1]=y;
 a[2]=thomas(f,e,g,y);
 return a;
 }
public static void main(String args[]) throws IOException
{
 double a[][]=df(new f1(),0.0,3.0, 15);
 Plot pp=new Plot(a[0],a[2]);
 int n=a[0].length;
 double y[]=new double[n];
 f2 dy=new f2();
 for(int i=0;i<n;i++)
 {y[i]=dy.func(a[0][i]);
  System.out.println(a[0][i]+" "+a[2][i]+" "+y[i]);
 }
 pp.addData(a[0],y);
 pp.plot();
}
}
```



7.12 DIRECT DERIVATION OF THE NUMERICAL DATA

If data is given numerically, two different approach can be used to take the derivative of the function. The first approximation is to curve fit a function into the data and then take the derivative of the function. The second one is to fit difference equations directly to the given data
In order to define a function, interpolation equations can be used. For example if quadratic Lagrange Formula is used:

$$f_2(x)=\frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}f(x_{i-1})+\frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}f(x_i)+\frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}f(x_{i+1})$$

Derivative of this function will be:

$$f_2'(x)=\frac{(2x-x_i-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}f(x_{i-1})+\frac{(2x-x_{i-1}-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}f(x_i)+\frac{(2x-x_{i-1}-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}f(x_{i+1})$$

The same approach can be applied with differen degree of polynomials as well as differnet interpolation formulations

As another example to function definition (Curve fitting) cubic spline interpolation can be applied. If the cubic polynomial is in the form of:

$$s_k(x)=a_k(x-x_k)+b_k(x_{k+1}-x)+[(x-x_k)^3c_{k+1}+(x_{k+1}-x)^3c_k]/(6h_k) \quad 1\le k\le n$$

The derivative of this equation will be

$$s'_k(x)=a_k-b_k+[(x-x_k)^2c_{k+1}-(x_{k+1}-x)^2c_k]/(2h_k) \quad 1\le k\le n$$

Since cubic polynomial has continious first and second polynomials between each spliine function the derivative is also continious. An example program is given below. Data used is generated fron $f(x)=x^2$ fonction..

Program 7.12-1 data derivative with cubic spline interpolation

```
import java.io.*;
import javax.swing.*;
```

```
class NA81
{

  public static double [] thomas(double a[][],double r[])
  {
  //
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

  public static double [] thomas(double f[],double e[],double g[],double r[])
  {
  int n=f.length;
  double x[]=new double[n];
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }


  public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
  {
  // c0 second derivative of the first point in the first spline
  // cn second derivative of the last point in last spline
  int n=xi.length;
  double h[]=new double[n];
  double w[]=new double[n];
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double d[]=new double[n];
  double x[]=new double[n];
  double S[][]=new double[4][n];
  int k;
  for(k=0;k<(n-1);k++)
     {h[k]=xi[k+1]-xi[k];
      w[k]=(yi[k+1]-yi[k])/h[k];
```

```
            }
d[0]=c0;
d[n-1]=cn;
for(k=1;k<(n-1);k++)
   {d[k]=6.0*(w[k]-w[k-1]);}
f[0]=1.0;
f[n-1]=1.0;
g[0]=0.0;
g[n-1]=0.0;
e[0]=0.0;
e[n-1]=0.0;
for(k=1;k<(n-1);k++)
{f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
S[2]=thomas(f,e,g,d);
S[3]=xi;
for(k=0;k<(n-1);k++)
  {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
   S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
  }
return S;
}

public static double funcSpline(double S[][],double x)
{
int n=S[0].length;
double xx1=0;
double xx2=0;
double y=0;
double hk=0;
for(int k=0;k<(n-1);k++)
   {if(S[3][k]<=x && x<=S[3][k+1])
    {hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     break;
     }
   }
     if(y==0 && S[3][n-2]<=x )
     {
              int k=n-2;
              hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     }
return y;
}

public static double dfSpline(double xi[],double yi[],double c0,double cn,double x)
{ //cubic spline formulations
           double S[][]=cubic_spline(xi,yi,c0,cn);
  return dfSpline(S,x);
}

public static double dfSpline(double S[][],double x)
{
  //cubic spline derivative formulation
int n=S[0].length;
double xx1=0;
double xx2=0;
double y=0;
```

```
 double hk=0;
 for(int k=0;k<(n-1);k++)
    {if(S[3][k]<=x && x<=S[3][k+1])
     {hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]-S[1][k]+(xx1*xx1*S[2][k+1]-xx2*xx2*S[2][k])/(2.0*hk);
      break;
      }
    }
      if(y==0 && S[3][n-2]<=x )
      {
               int k=n-2;
               hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      }
 return y;
 }

 public static double  dfSpline(double xi[],double yi[],double x)
 {
   //naturaş cubic spline derivative formula
            double S[][]=cubic_spline(xi,yi,0,0);
   return dfSpline(S,x);
 }

  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1="a.txt";
  //JFileChooser fc=new JFileChooser();
  //if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  int n=x.length;
  double S[][]=cubic_spline(x,y,2.0,2.0);
  System.out.println(dfSpline(S,2.5));
  }
  }
```

Output 7.9-2 data derivative with cubic spline interpolation

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA81
5.005944788930356

> Terminated with exit code 0.
```

INTEGRATION

DEFINITIONS OF INTEGRATION

Integral defines the area under the curve. It can also be defined as antiderivative(inverse derivative) It can be defined as without limits, or with integration boundaries

$$I = \int f(x)dx$$

Or with limits

$$I = \int_a^b f(x)dx$$

Some analytical integration formulas are given below:

f(x)	$\int f(x)dx$
$\int x^n\,dx$	$\frac{x^{n+1}}{n+1}$
$\int x^3\,dx$	$\frac{x^4}{4}$
$\int e^x dx$	e^x
$\int e^{bx} dx$	$\frac{e^x}{b}$
$\int a^{bx}\,dx$	$\frac{a^{bx}}{\text{bln }(a)}$
$\int \cos(x)\,dx$	$\sin(x)$
$\int \sin(x)\,dx$	$-\cos(x)$
$\int sec^2(x)\,dx$	$\tan(x)$
$\int \cosh(x)\,dx$	$\sinh(x)$
$\int \sinh(x)\,dx$	$\cos\,\text{h}(x)$
$\int sech^2(x)\,dx$	$\tanh(x)$
$\int \frac{dx}{x}$	$\ln(\lvert x\rvert)$
$\int \frac{dx}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$
$\int -\frac{dx}{\sqrt{1-x^2}}$	arccos(x)
$\int \frac{dx}{1+x^2}$	arctan(x)+C
$\int \frac{dx}{a^2+x^2}$	$\frac{1}{a}arctan\left(\frac{x}{a}\right)$
$\int \frac{dx}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}arcsec\left(\left\lvert\frac{x}{a}\right\rvert\right)$
$\int \frac{dx}{\sqrt{a^2+x^2}}$	$ln\left(x+\sqrt{a^2+x^2}\right)$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$ln\left(\left\lvert x+\sqrt{a^2-x^2}\right\rvert\right)$
$\int \frac{dx}{a^2-x^2}$	$\frac{1}{2a}ln\left(\left\lvert\frac{x+a}{x-a}\right\rvert\right)$
$\int \frac{dx}{x\sqrt{a^2-x^2}}$	$-\frac{1}{a}ln\left(\left\lvert\frac{a+\sqrt{a^2-x^2}}{x}\right\rvert\right)$
$\int \frac{dx}{x\sqrt{a^2+x^2}}$	$-\frac{1}{a}ln\left(\left\lvert\frac{a+\sqrt{a^2+x^2}}{x}\right\rvert\right)$

$\int u(x)v'(x)\,dx$	$u(x)v(x) - \int u'(x)v(x)\,dx$
$\int ku(x)\,dx$	$k\int u(x)\,dx$
$\int ln(x)\,dx$	$xln(x) - x$

Examples of analytical integrations

EX1: take the integral of $y = f(x) = 2x^3 + 3x^2 + 5$

Answer: $I = \frac{2}{4}x^4 + \frac{3}{3}x^3 + 5x + C$

EX2: take integral of $y = f(x) = u'(x)v(x) = (2x^3 + 3x^2 + 5)(2x - 2)$

$v(x) = \frac{2}{4}x^4 + \frac{3}{3}x^3 + 5x \quad u'(x) = 2$

$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$

$$\int u(x)v'(x)\,dx = \left(\frac{2}{4}x^4 + \frac{3}{3}x^3 + 5x\right)(2x-2) - \int 2\left(\frac{2}{4}x^4 + \frac{3}{3}x^3 + 5x\right)dx$$

$$\int u(x)v'(x)\,dx = \left(\frac{2}{4}x^4 + \frac{3}{3}x^3 + 5x\right)(2x-2) - 2\left(\frac{2}{20}x^5 + \frac{3}{12}x^4 + \frac{5}{2}x^2\right) + C$$

EX3: take integral of

$$I = \int x^2 e^x dx$$

$$\frac{d}{dx}(uv) = udv + vdu$$

$$\int udv = uv - \int vdu$$

$u = x^2 \quad dv = e^x dx \quad du = 2xdx \quad v = e^x$

$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$

$u = x \quad dv = e^x dx \quad du = dx \quad v = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\left[xe^x - \int e^x dx\right] = x^2 e^x - 2xe^x + 2e^x + C$$

EX4

Calculate $\int \frac{3x^2+2x+3}{(x^2+1)^2}$

$$\frac{3x^2+2x+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Bx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)+Bx+D}{(x^2+1)^2}$$

$$\frac{3x^2+2x+3}{(x^2+1)^2} = \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}$$

So $A = 0 \;\; B = 3 \;\; C = 2 \;\; D = 0$

$$\int \frac{3x^2+2x+3}{(x^2+1)^2} = \int \frac{3dx}{(x^2+1)} + \int \frac{2xdx}{(x^2+1)^2}$$

$u = (x^2+1) \;\; du = 2xdx$

$$\int \frac{2xdx}{(x^2+1)^2} = \int \frac{du}{u^2} = \int u^{-2}du = -\frac{1}{u} = -\frac{1}{(x^2+1)}$$

$\int \frac{3x^2+2x+3}{(x^2+1)^2} = 3\text{atan}\,(x) - \frac{1}{(x^2+1)}$+c

EX5

Calculate $\int cos^2(x)dx$

$cos^2(x)+= 1$

$cos^2(x) - sin^2(x) = \cos\,(2x)$

$cos^2(x) = \frac{1+\cos\,(2x)}{2}$

$$\int cos^2(x)dx = \int \frac{1+\cos\,(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin\,(2x)}{4}$$

EX6

Calculate $\int xsin^2(x)dx$

$$\int xsin^2(x)dx = \int \frac{x(1-\cos\,(2x)}{2}dx = \int \frac{x}{2}dx - \int \frac{\text{xcos}(2x)}{2}dx = \frac{x^2}{4} - \int \frac{\text{xcos}(2x)}{2}dx$$

$$\int \frac{\text{xcos}(2x)}{2}dx$$

$$\int udv = uv - \int vdu$$

u=x dv=cos(2x)dx $du = dx$ $v = \int \cos(2x)\,dx = \frac{\sin\,(2x)}{2}$

$$\int \frac{\text{xcos}(2x)}{2}dx = x\frac{\sin\,(2x)}{2} - \int \frac{\sin\,(2x)}{2}dx = x\frac{\sin\,(2x)}{2} + \frac{\cos\,(2x)}{4}$$

$$\int xsin^2(x)dx = \frac{x^2}{4} - \left[x\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4}\right] + C$$

EX7 calculate

$$\int \sqrt{\boldsymbol{x^3+2x}}\,\boldsymbol{(3x^2+2)dx}$$

$$\boldsymbol{u = x^3+2x}$$

$$\boldsymbol{du = (x^3+2x)dx}$$

$$\int \sqrt{\boldsymbol{x^3+2x}}\,\boldsymbol{(3x^2+2)dx} = \int \boldsymbol{u^{1/2}du} = \frac{\boldsymbol{u^{3/2}}}{\boldsymbol{3/2}} + \boldsymbol{C} = \frac{\boldsymbol{2}}{\boldsymbol{3}}\left(\frac{\boldsymbol{u^{3/2}}}{\boldsymbol{3/2}} + \boldsymbol{C}\right)^{\boldsymbol{3/2}} + \boldsymbol{C}$$

EX8 calculate

$$I = \int_0^{\pi} \left[\frac{xsin(x)}{1 + cos^2(x)}\right] dx$$

$$I = \int_a^b f(x)dx$$

$$x = a + b - t$$

$$dx = -dt$$

$$I = \int_b^a f(a + b - t)(-dt)$$

Rewriting

$$I = \int_a^b f(a + b - x)dx$$

Replacing x with $(\pi + 0 - x)$

$$I = \int_0^{\pi} \left[\frac{(\pi - x)sin(\pi - x)}{1 + cos^2(\pi - x)}\right] dx$$

Adding our original integral to integral above

$$2I = \int_0^{\pi} \left[\frac{xsin(x)}{1 + cos^2(x)}\right] dx + \int_0^{\pi} \left[\frac{(\pi - x)sin(\pi - x)}{1 + cos^2(\pi - x)}\right] dx$$

$$I = \frac{1}{2}\int_0^{\pi} \left[\frac{xsin(x)}{1 + cos^2(x)} + \frac{(\pi - x)sin(\pi - x)}{1 + cos^2(\pi - x)}\right] dx$$

$$sin(\pi - x) = \sin(x)$$

$$cos^2(\pi - x) = cos^2(x)$$

$$I = \frac{1}{2}\int_0^{\pi} \left[\frac{\pi sin(x)}{1 + cos^2(x)}\right] dx$$

$$u = \cos(x)$$

$$du = -\sin(x)$$

$$I = \frac{\pi}{2}\int_1^{-1} \left[\frac{-du}{1 + u^2}\right] = \frac{\pi}{2}[\arctan(u)]_{-1}^{1} = \frac{\pi^2}{4}$$

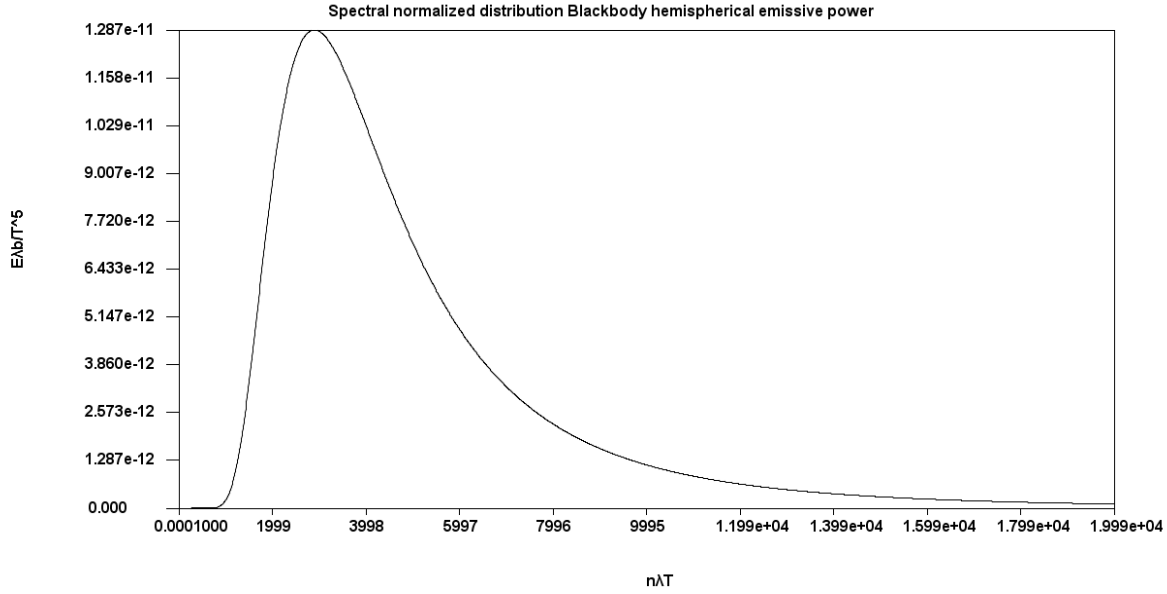
EX9 we would like to integrate Blackbody spectral diffuse power equation for all the wavelengths to find the total blackbody power for all spectrum.
Blackbody spectral diffuse emissive power (Plank's law)

$$\pi I_{\lambda,b}(\lambda,T)=E_{\lambda,b}(\lambda,T)=\frac{2\pi hc_0^2}{n^2\lambda^5[exp(hc_0/(n\lambda kT)-1]}$$

$$\pi I_{\lambda,b}(\lambda,T)=E_{\lambda,b}(\lambda,T)=\frac{C_1}{n^2\lambda^5[exp(C_2/(n\lambda T)-1]}$$

$$\frac{E_{\lambda,\beta}(\lambda,T)}{n^3T^5}=\frac{C_1}{(n\lambda T)^5[exp(C_2/(n\lambda T)-1]}$$

h		6.6260701500E-34	J s		
k		1.3806490000E-23	J/K		
co		2.9979245800E+08	m/s		
C1	$2\pi hc_0^2$	3.7417718522E-16	Wm^2	3.7417718522E+08	W $\mu m^4/m^2$
C2	hc_0	1.4387768775E-02	W/mK	1.4387768775E+04	W/μmK



$$\frac{E_b(T)}{n^3T^5}=\int_{\lambda=0}^{\infty}\frac{C_1d\lambda}{(n\lambda T)^5[exp(C_2/(n\lambda T)-1]}$$ Converting variables gives:

$$\xi=\frac{C_2}{n\lambda T}\quad \lambda=\frac{C_2}{n\xi T}\quad d\lambda=-\frac{C_2}{n\xi^2 T}d\xi$$

$$\frac{E_b(T)}{n^3T^5}==\int_{\xi=\infty}^{0}\frac{C_1\xi^5}{C_2^5}\left(-\frac{C_2}{\xi^2 T}d\xi\right)$$

$$E_b(T)=\left[\left(\frac{C_1}{C_2^4}\right)n^2T^4\int_{\xi=0}^{\infty}\frac{\xi^3}{[exp(\xi)-1]}d\xi\right]$$

$$\int_{\xi=0}^{\infty}\frac{\xi^3}{[\exp(\xi)-1]}d\xi=\int_{\xi=0}^{\infty}\frac{e^{-\xi}\xi^3}{e^{-\xi}[e^{\xi}-1]}d\xi=\int_{\xi=0}^{\infty}\frac{e^{-\xi}\xi^3}{[1-e^{-\xi}]}d\xi=\int_{\xi=0}^{\infty}\xi^3\sum_{n=1}^{\infty}e^{-n\xi}d\xi$$

In order to solve the integral a new variable change will be applied. Each summation in the integral can be evaluated seperately.

$$u=n\xi\quad du=nd\xi$$

$$\int_{\xi=0}^{\infty}\xi^3\sum_{n=1}^{\infty}e^{-n\xi}d\xi=\frac{1}{n^4}\int_{u=0}^{\infty}u^3e^{-u}du$$

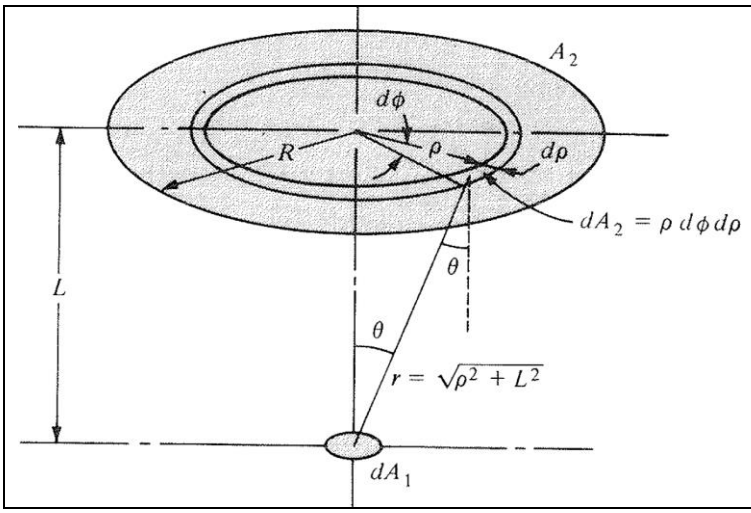
$\int_{u=0}^{\infty} u^3 e^{-u} du = -u^3 e^{-u}|_{u=0}^{\infty} + 3\int_{u=0}^{\infty} u^2 e^{-u} du$
$\int_{u=0}^{\infty} u^2 e^{-u} du = -u^2 e^{-u}|_{u=0}^{\infty} + 2\int_{u=0}^{\infty} u e^{-u} du$
$\int_{u=0}^{\infty} u e^{-u} du = -u e^{-u}|_{u=0}^{\infty} + 2\int_{u=0}^{\infty} e^{-u} du$
$\int_{u=0}^{\infty} e^{-u} du = -e^{-u}|_{u=0}^{\infty} = -(0-1) = 1$
$\int_{\xi=0}^{\infty} \xi^3 \sum_{n=1}^{\infty} e^{-n\xi}\, d\xi = \sum_{n=1}^{\infty} \frac{6}{n^4} = 6\frac{\pi^4}{90} = \frac{\pi^4}{15}$

Therefore

$$E_b(T) = \left[\left(\frac{C_1}{C_2^4}\right)\frac{\pi^4}{15}\right] n^2 T^4 = \sigma n^2 T^4$$

$$\sigma = \left[\left(\frac{C_1}{C_2^4}\right)\frac{\pi^4}{15}\right] = 5.6703744192 \times 10^{-8} \quad W/m^2K^4$$

EX10



A differential area dA_1 and a circle A_2 is given as the figure. L=1 m, R=1 m and dA_1=0.01 m^2. Calculate configuration factor

Analytical solution:

$$F_{dA_1-A_2} = \int_{A_2} \frac{\cos(\theta_1)\cos(\theta_2)}{\pi R^2} dA_2 = \int_{A_2} dF_{dA_1-dA_2}$$

$dA_2 = \rho d\phi d\rho$ where ϕ is azimuth angle. The distance r between the elemental surfaces dA_1 and dA_2 is given by

$$r^2 = \rho^2 + L^2$$

$$\cos(\theta_1) = \cos(\theta_2) = \cos(\theta) = \frac{L}{r} = \frac{L}{(\rho^2 + L^2)^{1/2}}$$

$$F_{dA_1-A_2} = \frac{1}{\pi}\int_{\rho=0}^{R}\int_{\phi=0}^{2\pi} \frac{L}{(\rho^2 + L^2)^{1/2}} \rho d\rho = \frac{R^2}{R^2 + L^2}$$

$$F_{dA_1-A_2} = 0.5$$

NEWTON-COTES FORMULATIONS : TRAPEZOIDAL, BOLE, SİMPSON 1/3 VE 3/8 METHODS

If function f(x) is given and ,it is desired to calculate integral of the function between integration limits a and b. Definition of the integral is the area inder the function between integration limits. The simplest approximation is to connect f(a) and f(b) points with a linear line and calculate the area of the trapezoidal

under the line. This approach is called trapezoidal rule of Newton-Cotes formulations (Area finding formulations)



Roger Cotes

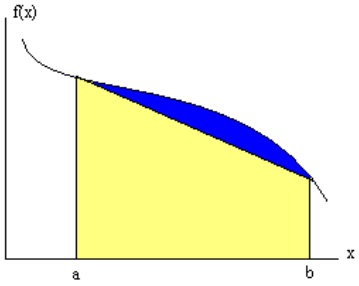


Figure 7.1-1 Finding integration with trapezoidal rule (yellow area) and error created by the method(blue area)

Integration formula :

$$I = \int_a^b f(x)dx = (b-a)\frac{f(b)+f(a)}{2}$$

$$I = \int_0^1 3x^2 dx = \frac{3x^3}{3}\Big|_0^1 = \frac{3}{3} = 1$$

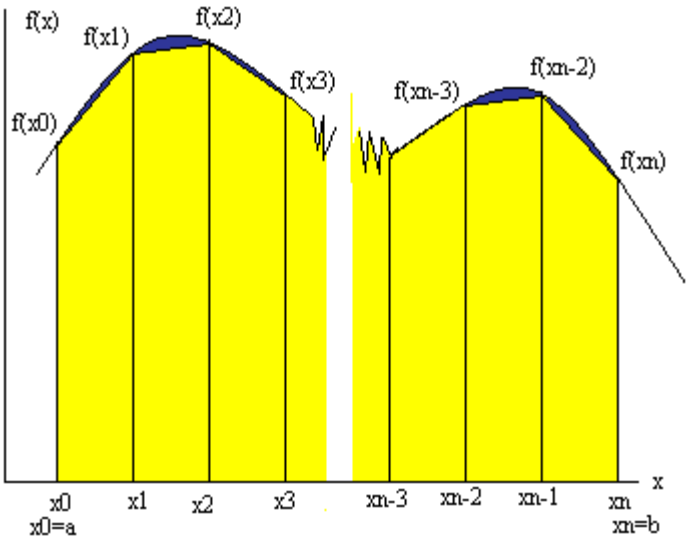


Figure 7.1-1 Finding integration with trapezoidal rule (yellow area) with multiple trapezoidal rule

If total area is divided into smaller areas amount of error will be reduced due to the fact that smaller piece of functions are closer to the line compare to a big piece of function. The same integration Formula for multipart equation will be:

$$I = \int_a^b f(x)dx = \frac{(b-a)}{2}\left[\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{n}\right]$$

Now assume an example integration function of

$$I = \int_0^1 \frac{4}{\pi}\sqrt{1-x^2}dx = 1.0$$

Since the value of the function is known exactly, it will be easy to evaluate the error of numerical function evaluation process.

Program 7.1-1 Trapezoidal intergral rule of Newton-Cotes approximation n=1 (java)

```
//=====================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula n points
// trapezoidal rule
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  double func(double x)
  {return 4.0/Math.PI*Math.sqrt(1.0-x*x);}
}

class NA51
{
```

```
public static double newton_cotes2(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 2 points
        //trapezoidal rule
        // ff integrating function
        // a,b integral limits
        // n : number of sub division h=(b-a)/n;
        double h=(b-a)/n;
        double h1=h;
        double sum=0;
        int p,i;
        double x[]=new double[3];
        double f[]=new double[3];
        for(p=0;p<n;p++)
        {  {for(i=0;i<2;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
           sum+=h*(f[0]+f[1])/2.0;
        }
        return sum;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  double i1= newton_cotes2(b1,0.0,1.0,1);
  double error=1-i1;
  System.out.println("integral of class         f1 : "+i1+"error = "+error);
}
}
```

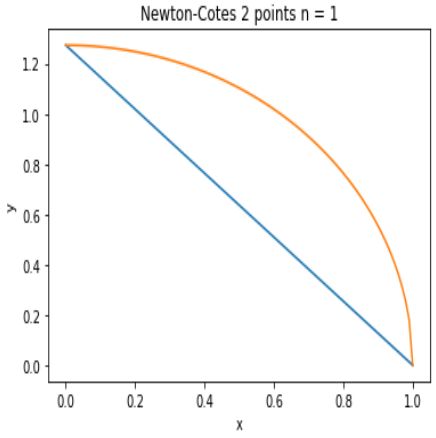
```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA51
integral of class         f1 : 0.6366197723675814error = 0.3633802276324186
> Terminated with exit code 0.
```

Trapezoidal intergral rule of Newton-Cotes approximation n=1 (python)

```
 from math import *
from if_x import *
from matplotlib import pyplot as plt
import numpy as np

class f1(if_x):
  def func(self,x):
     y=4.0/pi*sqrt(1.0-x*x)
     return y

def newton_cotes2(f,a,b,n):
    #Newton-cotes integral 2 points
    # trapezoidal rule
    # ff integrating function
    # a,b integral limits
    # n : number of sub division h=(b-a)/n;
    h=(b-a)/n;
    h1=h;
    sum=0.0
    nn=2*n
    x =[0 for w in range(2)]
    y =[0 for w in range(2)]
    xx =[0 for w in range(nn)]
    yy =[0 for w in range(nn)]
    j=0
    for p in range(n):
        for i in range(2):
            x[i]=a+i*h1+p*h
            xx[j]=x[i]
            y[i]=f.func(x[i])
```

```
            yy[j]=y[i]
            j=j+1
        sum=sum+h*(y[0]+y[1])/2.0
    n1=100
    xx1 =[0 for w in range(n1)]
    yy1 =[0 for w in range(n1)]
    dx=(b-a)/(n1-1)
    for i in range(n1):
       xx1[i]=dx*i
       yy1[i]=f.func(xx1[i])
    plt.plot(xx,yy,xx1,yy1)
    plt.xlabel("x")
    plt.ylabel("y")
    plt.title("Newton-Cotes 2 points n = "+str(n))
    plt.show()
    return sum

f=f1()
I= newton_cotes2(f,0.0,1.0,1)
print("I=",I)
```

```
runfile('E:/okul/SCO1/newton_cotes2.py', wdir='E:/okul/SCO1')

Reloaded modules: if_x

I= 0.6366197723675814
```



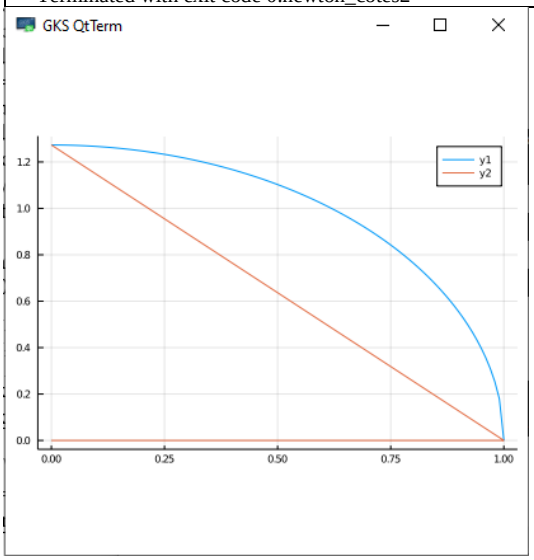
In order to see the error happened in the function visiually a graphical output form of the same program is created.

Trapezoidal intergral rule of Newton-Cotes approximation n=1 (Julia)

```
function  newton_cotes2(ff,a,b,n)
     #Newton-cotes integral 2 points
      #trapezoidal rule
      # ff integrating function
      # a,b integral limits
      # n : number of sub division h=(b-a)/n;
      h=(b-a)/n
      h1=h;
      sum=0.0;
      x=zeros(3)
      f=zeros(3)
      xi=zeros(n+1)
      fi=zeros(n+1)
      xi[1]=a
      fi[1]=ff(a)
      for p=1:n
        for i=1:2
          x[i]=a+(i-1)*h1+(p-1)*h
          f[i]=ff(x[i])
          xi[p+1]=x[i]
```

```
            fi[p+1]=f[i]
            end
          sum=sum+h*(f[1]+f[2])/2.0
        end
        xx=[sum,xi,fi]
        return xx
        end
using Plots
f(x)=4.0/pi*sqrt(1.0-x*x)
xx=newton_cotes2(f,0.0,1.0,1)
I=xx[1]
print("Integral = $I")
x1=0:0.01:1
y1=f.(x1)
x2=xx[2]
y2=xx[3]
p=plot(x1,y1)
plot!(p,x2,y2)
```

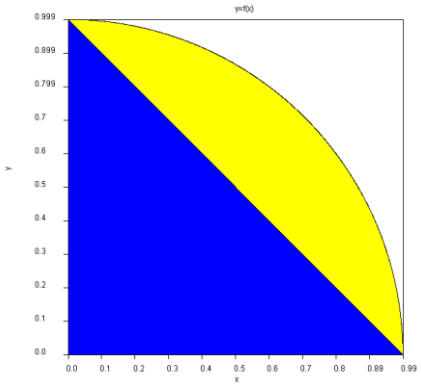
```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes2.jl
Integral = 0.6366197723675814
> Terminated with exit code 0.newton_cotes2
```



Program 7.1-2 Trapezoidal integral rule of Newton-Cotes approximation with plotting n=1 (java)

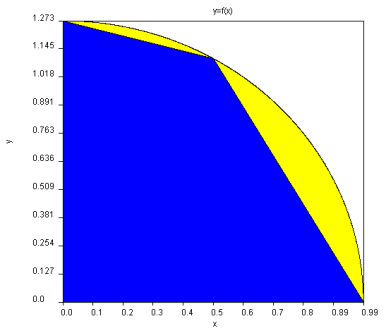
```
//=====================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula 4 points , n=3
// trapezoidal rule
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  double func(double x)
  {return 4.0/Math.PI*Math.sqrt(1.0-x*x);}
}


class NA51A
{

public static double newton_cotes2(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 2 points
        //trapezoidal rule
        double h=(b-a)/n;
```

```
        double h1=h;
        double sum=0;
        int p,i;
        int nn=2*n;
        double x[]=new double[2];
        double f[]=new double[2];
        double xi[]=new double[nn];
        double fi[]=new double[nn];
        int j=0;
        for(p=0;p<n;p++)
        {  {for(i=0;i<2;i++) {x[i]=a+i*h1+p*h;xi[j]=x[i];f[i]=ff.func(x[i]);fi[j]=f[i];j++;}}
           sum+=h*(f[0]+f[1])/2.0;
        }
        Plot pp=new Plot(xi,fi);
        pp.addFunction(ff,0.0,1.0,2000,2);
        pp.plot();
        return sum;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  double i1= newton_cotes2(b1,0.0,1.0,1);
  double error=1-i1;
  System.out.println("integral of class         f1 : "+i1+"error = "+error);

  }
}
```



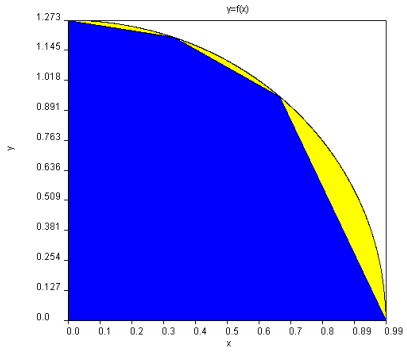
If the integral zone (0 to 1) is divided into two equal regions

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA51A
integral of class         f1 : 0.8696387816055828 error = 0.13036121839441717
```



If the integral zone (0 to 1) is divided into three equal regions

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA51A
integral of class       f1 : 0.9286860839333196error = 0.07131391606668036
```



If the integral zone (0 to 1) is divided into ten equal regions

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA51A
integral of class       f1 : 0.9881988750835945error = 0.011801124916405503
```

If the integral zone (0 to 1) is divided into hundred equal regions

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA51A
integral of class       f1 : 0.9996257879552262error = 3.742120447738362E-4
```

As it is seen from the example when trapezoidal steps are increased, approximation is getting better but even with 100 steps there are stil some error in the approximation. The next step in the approach is to use higher degrees of polinomials to connect points of integration. Quadratic (second degree) polinomial rule is called **Simpson 1/3 rule**. If step size of formulation is taken as h

$$h = \frac{b-a}{n}$$

Quadratic polynomial Newton-Cotes formula(Simpson 1/3 rule)

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{b-a}{6}[f(x_0) + 4f(x_1) + f(x_2)]}$$

or

$$h = \frac{b-a}{n} = \frac{b-a}{2}$$

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

(n=2 or three points are given in this Formula)

Quadratic polynomial Newton-Cotes formula(Simpson 1/3 rule) with multipoint approach

$$h = \frac{b-a}{n} = \frac{b-a}{2}$$

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + \sum_{i=2,4,6}^{n-2} f(x_i) + f(x_n)\right]}$$

Cubic polynomial Newton-Cotes formula(Simpson 3/8 rule)

$$h = \frac{b-a}{n} = \frac{b-a}{3}$$

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]}$$

(n=3 or four points are required for this formula)

Fourth order polynomial Newton-Cotes formula(Bode rule)

$$h = \frac{b-a}{n} = \frac{b-a}{4}$$

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]}$$

(n=4 or five points are used in this Formula)

Fifth order polynomial Newton-Cotes formula(Boole rule)

$$h = \frac{b-a}{n} = \frac{b-a}{5}$$

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{5h}{288}[10f(x_0) + 75f(x_1) + 50f(x_2) + 50f(x_3) + 75f(x_4) + 19f(x_5)]}$$

(n=5 or six points are used in this Formula)

Eight order polynomial Newton-Cotes formula

$$h = \frac{b-a}{n} = \frac{b-a}{8}$$

$$\boldsymbol{I = \int_a^b f(x)dx = \frac{4h}{14175}[989f(x_0) + 5888f(x_1) - 928f(x_2) + 10496f(x_3) - 4540f(x_4) + 10496f(x_3)f(x_5) - 928f(x_6) + 5888f(x_7) + 989f(x_8)]}$$

(n=8 or nine points are used in this Formula)

Programs for this approaches are given below:

Program quadratic polynomial Newton - Cotes Formula (Simpson 1/3 rule) Java language

```
//=====================================================
//  Numerical Analysis package in java
//  example to show utilisation of integration ( integral )
//  and differentiation (derivative) functions
//  Dr. Turhan Coban
// =====================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {return 4.0/Math.PI*Math.sqrt(1.0-x*x);}
}

class NA52
{
public static double newton_cotes3(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 3 points
```

```
        //Simpson 1/3 rule

        double h=(b-a)/n;
        double h1=h/2.0;
        double sum=0;
        int p,i;
        double x[]=new double[3];
        double f[]=new double[3];
        for(p=0;p<n;p++)
        {  {for(i=0;i<3;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
          sum+=h*(f[0]+4.0*f[1]+f[2])/6.0;
        }
        return sum;
}

  public static void main(String args[]) throws IOException
  {
   f1 b1=new f1();
  double i1=  newton_cotes3(b1,0.0,1.0,1);
  double error =1-i1;
  System.out.println("integral of class         f1 : "+i1+"error="+error);
}
}
```

If the quadratic Newton-Cotes Formula (Simpson 1/3 rule) applied as one time integral zone (0 to 1)

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA52
integral of class         f1 : 0.9473117846849166error=0.052688215315083387
> Terminated with exit code 0.
```

If the quadratic Newton-Cotes Formula (Simpson 1/3 rule) applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA52
integral of class         f1 : 0.9815388228080559error=0.018461177191944067

> Terminated with exit code 0.
```

quadratic polynomial Newton - Cotes Formula (Simpson 1/3 rule) python language

```
from math import *
from if_x import *


class f1(if_x):
  def func(self,x):
      y=4.0/pi*sqrt(1.0-x*x)
      return y

def newton_cotes3(f,a,b,n):
    #Newton-cotes integral 3 points
    # Simpson 1/3 rule
    # ff integrating function
    # a,b integral limits
    # n : number of sub division h=(b-a)/n;
    h=(b-a)/n;
    h1=h/2.0;
```

```
    sum=0.0
    nn=2*n
    n1=3
    x =[0 for w in range(n1)]
    y =[0 for w in range(n1)]
    for p in range(n):
        for i in range(n1):
            x[i]=a+i*h1+p*h
            y[i]=f.func(x[i])
        sum=sum+h*(y[0]+4.0*y[1]+y[2])/6.0
    return sum

f=f1()
n=1
I= newton_cotes3(f,0.0,1.0,n)
print("n = ",n,"I=",I)
```

If the quadratic Newton-Cotes Formula (Simpson 1/3 rule) applied as one time integral zone (0 to 1)

```
 runfile('E:/okul/SCO1/newton_cotes3.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  1 I= 0.9473117846849166
```

If the quadratic Newton-Cotes Formula (Simpson 1/3 rule) applied as two times integral zone (0 to 1)

```
 runfile('E:/okul/SCO1/newton_cotes3.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  2 I= 0.9815388228080559
```

If the quadratic Newton-Cotes Formula (Simpson 1/3 rule) applied as 10 times integral zone (0 to 1)

```
 runfile('E:/okul/SCO1/newton_cotes3.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  10 I= 0.9983621080451945
```

quadratic polynomial Newton - Cotes Formula (Simpson 1/3 rule) Julia language

```
function  newton_cotes3(ff,a,b,n)
      #Newton-cotes integral 2 points
       #trapezoidal rule
       # ff integrating function
       # a,b integral limits
       # n : number of sub division h=(b-a)/n;
       h=(b-a)/n
       h1=h/2.0;
       sum=0.0;
       x=zeros(3)
       f=zeros(3)
       for p=1:n
         for i=1:3
           x[i]=a+(i-1)*h1+(p-1)*h
           f[i]=ff(x[i])
           end
         sum=sum+h1*(f[1]+4.0*f[2]+f[3])/3.0
       end
       return sum
       end
using Plots
f(x)=4.0/pi*sqrt(1.0-x*x)
I=newton_cotes3(f,0.0,1.0,1)
print("Integral Newton-cotes 1/3 = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes3.jl
Integral Newton-cotes 1/3 = 0.9473117846849166
> Terminated with exit code 0.
```

Program cubic polynomial Newton - Cotes Formula (Simpson 3/8 rule) java language

```
//=======================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula 4 points , n=3
// Simpson 3/8 rule
// Dr. Turhan Coban
// =======================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {
  return 4.0/Math.PI*Math.sqrt(1.0-x*x);
  }
}

class NA53
{

public static double newton_cotes4(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 4 points
        //Simpson 3/8 rule
        // ff integrating function
        // a,b integral limits
        // n : number of sub division h=(b-a)/n;
                double h=(b-a)/n;
        double h1=h/3.0;
        double sum=0;
        int p,i;
        double x[]=new double[4];
        double f[]=new double[4];
        for(p=0;p<n;p++)
        {  {for(i=0;i<4;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
           sum+=h*(f[0]+3.0*f[1]+3.0*f[2]+f[3])/8.0;
        }
        return sum;
}

public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  double i1= newton_cotes4(b1,0.0,1.0,2);
  double error=1-i1;
  System.out.println("integral of class         f1 : "+i1+"error = "+error);
}
}
```

If the cubic Newton-Cotes Formula (Simpson 3/8 rule) applied as one time integral zone (0 to 1)

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA53
integral of class         f1 : 0.965194372879037error = 0.03480562712096302


> Terminated with exit code 0.
```

If the cubic Newton-Cotes Formula (Simpson 3/8 rule) applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA53
integral of class        f1 : 0.9877870530380952error = 0.01221294696190478

> Terminated with exit code 0.
```

cubic polynomial Newton - Cotes Formula (Simpson 3/8 rule) python language

```
 from math import *
from if_x import *


class f1(if_x):
  def func(self,x):
     y=4.0/pi*sqrt(1.0-x*x)
     return y

def newton_cotes4(f,a,b,n):
   #Newton-cotes integral 4 points
   # Simpson 3/8 rule
   # ff integrating function
   # a,b integral limits
   # n : number of sub division h=(b-a)/n;
   h=(b-a)/n;
   h1=h/3.0;
   sum=0.0
   n1=4
   x =[0 for w in range(n1)]
   y =[0 for w in range(n1)]
   for p in range(n):
      for i in range(n1):
         x[i]=a+i*h1+p*h
         y[i]=f.func(x[i])
      sum=sum+h*(y[0]+3.0*y[1]+3.0*y[2]+y[3])/8.0
   return sum

f=f1()
n=1
I= newton_cotes4(f,0.0,1.0,n)
print("n = ",n,"I=",I)
```

If the cubic Newton-Cotes Formula (Simpson 3/8 rule) applied as one time integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes4.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  1 I= 0.965194372879037
```

If the cubic Newton-Cotes Formula (Simpson 3/8 rule) applied as two times integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes4.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  2 I= 0.9877870530380952
```

If the cubic Newton-Cotes Formula (Simpson 3/8 rule) applied as 10 times integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes4.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  10 I= 0.998914915292993
```

cubic polynomial Newton - Cotes Formula (Simpson 3/8 rule) Julia language

```
function  newton_cotes4(ff,a,b,n)
      #Newton-cotes integral 2 points
       #trapezoidal rule
       # ff integrating function
```

```
        # a,b integral limits
        # n : number of sub division h=(b-a)/n;
        h=(b-a)/n
        h1=h/3.0;
        sum=0.0;
        x=zeros(4)
        f=zeros(4)
        for p=1:n
          for i=1:4
            x[i]=a+(i-1)*h1+(p-1)*h
            f[i]=ff(x[i])
            end
          sum=sum+h*(f[1]+3.0*f[2]+3.0*f[3]+f[4])/8.0
        end
        return sum
        end
using Plots
f(x)=4.0/pi*sqrt(1.0-x*x)
I=newton_cotes4(f,0.0,1.0,1)
print("Integral Newton-cotes 3/8 = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes4.jl
Integral Newton-cotes 3/8 = 0.965194372879037
> Terminated with exit code 0.
```

If the cubic Newton-Cotes Formula (Simpson 3/8 rule) applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes4.jl
Integral Newton-cotes 3/8 = 0.9877870530380952
> Terminated with exit code 0.
```

Program fourth order polynomial Newton - Cotes Formula (Boole rule) java language

```
//=====================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula 5 points , n=4
// Boole formula
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  double func(double x)
  {
  return 4.0/Math.PI*Math.sqrt(1.0-x*x);
  }
}


class NA54
{

public static double boole(f_x ff,double a,double b, int n)
{  return newton_cotes5(ff,a,b,n);}

public static double newton_cotes5(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 5 points
        //Boole rule
        // ff integrating function
        // a,b integral limits
        // n : number of sub division h=(b-a)/n;
```

```
                double h=(b-a)/n;
        double h1=h/4.0;
        double sum=0;
        int p,i;
        double x[]=new double[5];
        double f[]=new double[5];
        for(p=0;p<n;p++)
        {  {for(i=0;i<5;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
          sum+=h*(7.0*f[0]+32.0*f[1]+12.0*f[2]+32.0*f[3]+7.0*f[4])/90.0;
        }
        return sum;
}

   public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  double i1=  newton_cotes5(b1,0.0,1.0,1);
  double i2 = 1.0;
  double error =i2-i1;
  System.out.println("integral of class         f1 : "+i1+" error="+error);
 }
}
```

If the fourth order polynomial Newton-Cotes Formula (Boolee rule) applied as one time integral zone (0 to 1)

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA54
integral of class         f1 : 0.9838206253495985 error=0.016179374650401535


> Terminated with exit code 0.
```

If the fourth order polynomial Newton-Cotes Formula (Boole rule) applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA54
integral of class         f1 : 0.9943031391613758 error=0.005696860838624196

> Terminated with exit code 0.
```

fourth order polynomial Newton - Cotes Formula (Boole rule) python language

```
 from math import *
from if_x import *


class f1(if_x):
  def func(self,x):
     y=4.0/pi*sqrt(1.0-x*x)
     return y

def newton_cotes5(f,a,b,n):
    #Newton-cotes integral 5 points
    # Bole rule
    # ff integrating function
    # a,b integral limits
    # n : number of sub division h=(b-a)/n;
    h=(b-a)/n;
```

```
    h1=h/4.0;
    sum=0.0
    n1=5
    x =[0 for w in range(n1)]
    y =[0 for w in range(n1)]
    for p in range(n):
        for i in range(n1):
            x[i]=a+i*h1+p*h
            y[i]=f.func(x[i])
        sum=sum+h*(7.0*y[0]+32.0*y[1]+12.0*y[2]+32.0*y[3]+7.0*y[4])/90.0
    return sum

f=f1()
n=1
I= newton_cotes5(f,0.0,1.0,n)
print("n = ",n,"I=",I)
```

If the fourth order polynomial Newton-Cotes Formula (Boolee rule) applied as one time integral zone (0 to 1)

```
 runfile('E:/okul/SCO1/newton_cotes5.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  1 I= 0.9838206253495985
```

If the fourth order polynomial Newton-Cotes Formula (Boole rule) applied as two times integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes5.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  2 I= 0.9943031391613758
```

If the fourth order polynomial Newton-Cotes Formula (Boole rule) applied as 10 times integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes5.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  10 I= 0.9994921635232705
```

fourth order polynomial Newton - Cotes Formula (Boole rule) Julia language

```
 function  newton_cotes5(ff,a,b,n)
      #Newton-cotes integral 2 points
       #trapezoidal rule
       # ff integrating function
       # a,b integral limits
       # n : number of sub division h=(b-a)/n;
       h=(b-a)/n
       h1=h/4.0;
       sum=0.0;
       x=zeros(5)
       f=zeros(5)
       for p=1:n
         for i=1:5
           x[i]=a+(i-1)*h1+(p-1)*h
           f[i]=ff(x[i])
           end
        sum=sum+h*(7.0*f[1]+32.0*f[2]+12.0*f[3]+32.0*f[4]+7.0*f[5])/90.0
       end
       return sum
       end
using Plots
f(x)=4.0/pi*sqrt(1.0-x*x)
I=newton_cotes5(f,0.0,1.0,1)
print("Integral Newton-cotes (boole rule) = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes5.jl
Integral Newton-cotes (boole rule) = 0.9838206253495985
> Terminated with exit code 0.
```

If the fourth order polynomial Newton-Cotes Formula (Boole rule) applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes5.jl
Integral Newton-cotes (boole rule) = 0.9943031391613758
> Terminated with exit code 0.
```

Program fifth order polynomial Newton - Cotes Formula

```
//=======================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula 6 points, n=5
//
// Dr. Turhan Coban
// =======================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {
  return 4.0/Math.PI*Math.sqrt(1.0-x*x);
  }
}

class NA55
{

public static double newton_cotes6(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 6 points, n=5
        // ff integrating function
        // a,b integral limits
        // n : number of sub division h=(b-a)/n;
                double h=(b-a)/n;
        double h1=h/5.0;
        double sum=0;
        int p,i;
        double x[]=new double[6];
        double f[]=new double[6];
        for(p=0;p<n;p++)
        {  {for(i=0;i<6;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
           sum+=h*(19.0*f[0]+75.0*f[1]+50.0*f[2]+50.0*f[3]+75.0*f[4]+19.0*f[5])/288.0;
        }
        return sum;
}

        public static void main(String args[]) throws IOException
        { f1 b1=new f1();
          double i1=  newton_cotes6(b1,0.0,1.0,1);
          double i2 = 1.0;
          double error =i2-i1;
          System.out.println("integral of class         f1 : "+i1+" error="+error);
        }
}
```

If the fifth order polynomial Newton-Cotes Formula applied as one time integral zone (0 to 1)

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA55
integral of class         f1 : 0.9872489399864514 error=0.012751060013548643
> Terminated with exit code 0.
```

If the fifth order polynomial Newton-Cotes Formula applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA55
integral of class        f1 : 0.9955081956385212 error=0.0044918043614787795

> Terminated with exit code 0.
```

fifth order polynomial Newton - Cotes Formula, Julia language

```
function  newton_cotes6(ff,a,b,n)
      #Newton-cotes integral 2 points
       #trapezoidal rule
       # ff integrating function
       # a,b integral limits
       # n : number of sub division h=(b-a)/n;
       h=(b-a)/n
       h1=h/5.0;
       sum=0.0;
       x=zeros(6)
       f=zeros(6)
       for p=1:n
         for i=1:6
           x[i]=a+(i-1)*h1+(p-1)*h
           f[i]=ff(x[i])
           end
         sum=sum+h*(19.0*f[1]+75.0*f[2]+50.0*f[3]+50.0*f[4]+75.0*f[5]+19.0*f[6])/288.0
       end
       return sum
       end
using Plots
f(x)=4.0/pi*sqrt(1.0-x*x)
I=newton_cotes6(f,0.0,1.0,1)
print("Integral Newton-cotes (boole rule) = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes6.jl
Integral Newton-cotes (boole rule) = 0.9872489399864514
> Terminated with exit code 0.
```

If the fifth order polynomial Newton-Cotes Formula applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes6.jl
Integral Newton-cotes (boole rule) = 0.9955081956385212
> Terminated with exit code 0.
```

Program eight order polynomial Newton - Cotes Formula

```
//======================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula 9 points, n=8
//
// Dr. Turhan Coban
// ======================================================
import java.io.*;


class f1 extends f_x
{
  double func(double x)
   {
```

```
  return 4.0/Math.PI*Math.sqrt(1.0-x*x);
  }
}


class NA56
{

public static double newton_cotes9(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 9 points, n=8
       // ff integrating function
       // a,b integral limits
       // n : number of sub division h=(b-a)/n;
                double h=(b-a)/n;
       double h1=h/8.0;
       double sum=0;
       int p,i;
       double x[]=new double[9];
       double f[]=new double[9];
       for(p=0;p<n;p++)
       {  {for(i=0;i<9;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
          sum+=h/28350.0*(989.0*f[0]+5888.0*f[1]-928.0*f[2]+10496.0*f[3]-4540.0*f[4]+
          10496.0*f[5]-928.0*f[6]+5888.0*f[7]+989.0*f[8]);
       }
       return sum;
}

       public static void main(String args[]) throws IOException
       { f1 b1=new f1();
         double i1=  newton_cotes9(b1,0.0,1.0,1);
         double i2 = 1.0;
         double error =i2-i1;
         System.out.println("integral of class         f1 : "+i1+" error="+error);
       }
}
```

If the eigth order polynomial Newton-Cotes Formula applied as one time integral zone (0 to 1)

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA56
integral of class         f1 : 0.9952701383291108 error=0.004729861670889179

> Terminated with exit code 0.
```

If the eigth order polynomial Newton-Cotes Formula applied as two times integral zone (0 to 1) and divided up into 2

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA56
integral of class         f1 : 0.998330566008359 error=0.0016694339916409895

> Terminated with exit code 0.
```

8th order Newton-cotes formula:python language

```
from math import *

def newton_cotes9(f,a,b,n):
   #Newton-cotes integral 5 points
   # Bole rule
```

```
    # ff integrating function
    # a,b integral limits
    # n : number of sub division h=(b-a)/n;
    h=(b-a)/n;
    h1=h/8.0;
    sum=0.0
    n1=9
    x =[0 for w in range(n1)]
    y =[0 for w in range(n1)]
    for p in range(n):
        for i in range(n1):
            x[i]=a+i*h1+p*h
            y[i]=f(x[i])
        sum=sum+h/28350.0*(989.0*y[0]+5888.0*y[1]-928.0*y[2]+10496.0*y[3]-4540.0*y[4]+10496.0*y[5]-
928.0*y[6]+5888*y[7]+989.0*y[8])
    return sum

f = lambda x: 4.0/pi*sqrt(1.0-x*x)
n=1
I= newton_cotes9(f,0.0,1.0,n)
print("n = ",n,"I=",I)
```

If the eight order polynomial Newton-Cotes Formula applied as one time integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes9.py', wdir='E:/okul/SCO1')
n =  1 I= 0.9952701383291108
```

If the eight order polynomial Newton-Cotes Formula applied as two time integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes9.py', wdir='E:/okul/SCO1')
n =  2 I= 0.998330566008359
```

If the eight order polynomial Newton-Cotes Formula applied as 10 time integral zone (0 to 1)

```
runfile('E:/okul/SCO1/newton_cotes9.py', wdir='E:/okul/SCO1')
n =  10 I= 0.9998508846967017
```

(n=8 or nine points are used in this Formula), Ocatave-matlab version

```
## Author: M. Turhan ÇOBAN
## Created: 2021-06-13

function retval = newton_cotes_integral (f,a,b,n,n_formula)
  % f f(x) function
  % a lower integration limit
  % b upper integration limit
  % n number of division ao integration area
  % n_formula:
  % 2 : Trapezoidal rule
  % 3 : Simpson 1/3 rule
  % 4 : Simpson 3/8 rule
  % 5 : Boole rule
  retval=0.0;
if n_formula==2
  h=(b-a)/n;
  ff=f(a)+f(b);
  for i=1:n-1
    x=a+i*h;
    ff=ff+2.0*f(x);
  endfor
  ff=h/2.0*ff;
  retval =ff;
elseif n_formula==3
  h=(b-a)/n;
  h1=h/2.0;
  ff=0.0;
  fi=zeros(1,n_formula);
  for p=1:n
    for i=1:n_formula
      x=a+(i-1)*h1+(p-1)*h;
      fi(i)=f(x);
    endfor
  ff=ff+h*(fi(1)+4.0*fi(2)+fi(3))/6.0;
  endfor
```

```
   retval =ff;
elseif n_formula==4
  h=(b-a)/n;
  h1=h/(n_formula-1.0);
  ff=0.0;
  fi=zeros(1,n_formula);
  for p=1:n
    for i=1:n_formula
      x=a+(i-1)*h1+(p-1)*h;
      fi(i)=f(x);
    endfor
   ff=ff+h*(fi(1)+3.0*fi(2)+3.0*fi(3)+fi(4))/8.0;
   endfor
   retval =ff;
elseif n_formula==5
  h=(b-a)/n;
  h1=h/(n_formula-1.0);
  ff=0.0;
  fi=zeros(1,n_formula);
  for p=1:n
    for i=1:n_formula
      x=a+(i-1)*h1+(p-1)*h;
      fi(i)=f(x);
    endfor
   ff=ff+h*(7.0*fi(1)+32.0*fi(2)+12.0*fi(3)+32.0*fi(4)+7.0*fi(5))/90.0;
   endfor
   retval =ff;
endif
endfunction
```

```
f=@(x)4.0/pi*sqrt(1-x*x);
A=zeros(9,4);
for i=1:9
  for j=1:4
    A(i,j)=newton_cotes_integral (f,0.0,1.0,i+1,j+1);
  endfor
endfor
A
```

```
>> newton_cotes_integral_example
A =

   0.869638781605583   0.981538822808056   0.987787053038095   0.994303139161376
   0.928686083933320   0.989983928453389   0.993370155644743   0.996903328383621
   0.953563812507437   0.993505369389293   0.995699791714253   0.997990058327888
   0.966722667056538   0.995357611092613   0.996925630013113   0.998562407842033
   0.974659467323372   0.996470866415002   0.997662580760462   0.998906694188832
   0.979876212126265   0.997200817377992   0.998145876379058   0.999132570366295
   0.983519980168829   0.997709765269226   0.998482890692546   0.999290126798250
   0.986183036565696   0.998081226366241   0.998728888532670   0.999405158893415
   0.988198875083594   0.998362108045195   0.998914915292993   0.999492163523271

>>
```

The results will be summarised as one table

Table 7.1-1 comparison of Newton-Cotes results for different degree of integrating polnomials

$$I = \int_0^1 \frac{4}{\pi}\sqrt{1-x^2}dx = 1.0$$

N	Integral approximation	Error ($I_{approximation}$-I_{exact})
Region is not divided up calculated as one piece		
1 (Trapezoidal)	0.6366197723675814	0.3633802276324186
2 (Simpson 1/3)	0.9473117846849166	0.052688215315083387
3 (Simpson 3/8)	0.965194372879037	0.03480562712096302
4 (Bode)	0.9838206253495985	0.016179374650401535
5	0.9872489399864514	0.012751060013548643
8	0.9952701383291108	0.004729861670889179
Region is divided into 2 equal pieces (0-0.5 and 0.5-1)		

1 (Trapezoidal)	0.8696387816055828	0.13036121839441717
2 (Simpson 1/3)	0.9815388228080559	0.018461177191944067
3 (Simpson 3/8)	0.9877870530380952	0.01221294696190478
4 (Bode)	0.9943031391613758	0.005696860838624196
5	0.9955081956385212	0.0044918043614787795
8	0.998330566008359	0.0044918043614787795
Region is divided into 10 equal pieces (0-0.1,0.1-0.2,…,0.9-1)		
1 (Trapezoidal)	0.9881988750835945	0.011801124916405503
2 (Simpson 1/3)	0.9983621080451945	0.0016378919548054593
3 (Simpson 3/8)	0.998914915292993	0.001085084707006967
4 (Bode)	0.9994921635232705	5.07836476729473E-4
5	0.9995994322513025	4.005677486974557E-4
8	0.9998508846967017	1.4911530329830835E-4

The same integration calculated by using excel

$$I = \int_0^1 \frac{4}{\pi}\sqrt{1-x^2}\,dx = 1.0$$

Direct integration Newton-Cotes formulas

x	f(x)	Trapezoidal	Simpson 1/3
0	1.27324	0.158531	0.314961
0.125	1.263253	0.156004	
0.25	1.232809	0.150821	0.294032
0.375	1.180325	0.142686	
0.5	1.102658	0.131036	0.246688
0.625	0.993922	0.114756	
0.75	0.842169	0.091161	0.137824
0.875	0.616404	0.038525	
1	0		
Integral		0.98352	0.993505

Direct integration Newton-Cotes formulas

x	f(x)	Trapezoidal	Simpson 1/3
0	1.27324	0.0795	0.158739
0.0625	1.27075	0.079188	
0.125	1.263253	0.07856	0.156223
0.1875	1.250658	0.077608	
0.25	1.232809	0.076321	0.151063
0.3125	1.209473	0.074681	
0.375	1.180325	0.072664	0.142972
0.4375	1.14492	0.070237	
0.5	1.102658	0.067355	0.131405
0.5625	1.052711	0.063957	
0.625	0.993922	0.059954	0.115302
0.6875	0.924607	0.055212	
0.75	0.842169	0.049513	0.092241
0.8125	0.742249	0.042458	
0.875	0.616404	0.033109	0.049764
0.9375	0.443069	0.013846	
1	0		
Integral		0.994162	0.99771

Direct integration Newton-Cotes formulas

x	f(x)	Trapezoidal	Simpson 1/3
0	1.27324	0.039779	0.079526
0.03125	1.272618	0.03974	
0.0625	1.27075	0.039662	0.079214
0.09375	1.267632	0.039545	

0.125	1.263253	0.039388	0.078587
0.15625	1.257601	0.039192	
0.1875	1.250658	0.038954	0.077636
0.21875	1.242403	0.038675	
0.25	1.232809	0.038354	0.076351
0.28125	1.221845	0.037989	
0.3125	1.209473	0.03758	0.074712
0.34375	1.19565	0.037125	
0.375	1.180325	0.036621	0.072698
0.40625	1.163438	0.036068	
0.4375	1.14492	0.035463	0.070274
0.46875	1.124692	0.034802	
0.5	1.102658	0.034084	0.067398
0.53125	1.078708	0.033303	
0.5625	1.052711	0.032457	0.064007
0.59375	1.024512	0.031538	
0.625	0.993922	0.030541	0.060014
0.65625	0.960715	0.029458	
0.6875	0.924607	0.028279	0.055289
0.71875	0.885243	0.026991	
0.75	0.842169	0.025577	0.04962
0.78125	0.794779	0.024016	
0.8125	0.742249	0.022276	0.042627
0.84375	0.683394	0.020309	
0.875	0.616404	0.018042	0.033463
0.90625	0.538252	0.015333	
0.9375	0.443069	0.011858	0.017774
0.96875	0.315813	0.004935	
1	0		0
Integral		0.997934	0.999191

EXAMPLE :

$$I = \int_0^1 \sin(1 - x + x^2)\, dx = 1.0$$

solve with bole rule by hand. Use n=2

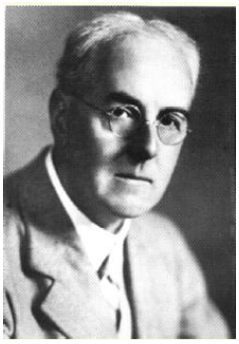
Note: Bole rule for n=1

$$I = \int_a^b f(x)dx = \frac{(b-a)}{90}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

Step size h=(b-a)/n

x0	0	0.841471	7	5.890297
x1	0.125	0.777465	32	24.87888
x2	0.25	0.726009	12	8.712104
x3	0.375	0.692988	32	22.17561
x4	0.5	0.681639	7	4.771471
				66.42836 0.005556 0.369046
x4	0.5	0.681639	7	4.771471
x5	0.625	0.692988	32	22.17561
x6	0.75	0.726009	12	8.712104
x7	0.875	0.777465	32	24.87888
x8	1	0.841471	7	5.890297
				66.42836 0.005556 0.369046
				I= 0.738093

RİCHARDSON AND AITKEN INTERPOLATION FORMULAS
RİCHARDSON EXTRAPOLATION (ROMBERG INTEGRATION) FORMULA



Lewis Fry Richardson

Richardson extrapolation formula can be applied to decrease integration error. Richardson extrapolation Formula has the folloving form :

$$I \cong I(h_2) + \frac{1}{\left(\frac{h_1}{h_2}\right)^2 - 1}[I(h_2) - I(h_1)]$$

In this Formula h_2 should be smaller than h_1 . If$h_2=h_1/2$ equation becomes:

$$I \cong \frac{4}{3}I(h_2) - \frac{1}{3}I(h_1)$$

As a starting interpolation any of the Newton-Cotes formulations can be taken. A General form of the above equation can be written as

$$I_{j,k} \cong \frac{4I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

If trapezoidal rule is used as starting point of this iterative formulation, it is called **Romberg Integration**



Werner Romberg

Consider the previous integral that solved by excel, if the results are integrated into the romberg integration formula:

$I = \int_0^1 \frac{4}{\pi}\sqrt{1-x^2}dx = 1.0$

Romberg: Trapezoidal

2*h	I0	I1	I2
0.125	0.98352	0.99771	0.999685
0.0625	0.994162	0.999191	
0.03125	0.997934		

Romberg: Simpson 1/3

2*h	I0	I1	I2
0.125	0.993505	0.999111	0.999877
0.0625	0.99771	0.999685	
0.03125	0.999191		

Program 7.2-1 Romberg Integration formula , java language

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
```

```
//Romberg Integration Formula
// Dr. Turhan Coban
// ======================================================
import java.io.*;
class f1 extends f_x
{
  double func(double x)
  {
  return 2.0/(1.0+2.0*x*x);
  }
}

class NA57
{
public static double integral_romberg(f_x ff,double a,double b)
{
//integral f(x)dx
//romberg integration
int n=8;//increase n to increase accuracy
double R[][];
R=new double[n+1][n+1];
int m=1;
double h=b-a;
double close=1;
double tol=1e-40;
int j=0,k=0;
double ret=0;
R[0][0]=h/2.0*(ff.func(a)+ff.func(b));
do
{
  j++;
  h/=2.0;
  trap(ff,a,h,j,m,R);
  m*=2;
  for(k=1;k<=j;k++)
  {
    R[j][k]=R[j][k-1]+(R[j][k-1]-R[j-1][k-1])/(Math.pow(4,k)-1.0);
    close=Math.abs(R[j-1][j-1]-R[j][j]);
    ret=R[j][k];
  }
}while(close>tol && j<n);
return ret;
}

public static void trap(f_x ff,double a,double h, int j,int m,double R[][])
{
    //Newton-cotes integral 2 points
              //  successive trapezoidal integration rule
    //  this program will be utilised in romberg integration
        double sum=0;
        int p;
        for(p=1;p<=m;p++)
          { sum+=ff.func(a+h*(2*p-1)); }
        R[j][0]=R[j-1][0]/2.0+h*sum;
}

  public static void main(String args[]) throws IOException
  {
  //Romberg integrali
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+
  integral_romberg(b1,-3.0,3.0));
```

```
}}
```

Romberg integration f(x)= 2.0/(1.0+2.0*x*x)

```
---------- Capture Output ----------
> "C:\co\java\bin\java.exe" integ7
integral of class        f1 : 3.7881660832213675

> Terminated with exit code 0.
```

If the iterative results of Romberg integration is given as a table

Table 7.2-1 Romberg integration of $I = \int_{-3}^{3} \frac{2}{1+2x^2} dx$

6.3157895	8.2105263							
4.2488038	3.5598086	3.2497608						
3.8058295	3.6581714	3.664729	3.6713158					
3.7866756	3.7802909	3.7884322	3.7903958	3.7908628				
3.7877768	3.7881439	3.7886674	3.7886711	3.7886644	3.7886622			
3.7880687	3.788166	3.7881675	3.7881596	3.7881576	3.7881571	3.7881569		
3.7881417	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661	
3.78816	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661	3.7881661

Solving of another integral $I = \int_{0}^{\pi} \sin(x) dx = 2$ by romberg Integration Formula

Table 8.2-2 Romberg integration $I = \int_{0}^{\pi} \sin(x) dx = 2$

1.5707963267948900	2.0943951023931900							
1.8961188979370300	2.0045597549844200	1.9985707318238300						
1.9742316019455500	2.0002691699483800	1.9999831309459800	2.0000055499796700					
1.9935703437723300	2.0000165910479300	1.9999997524545700	2.0000000162880400	1.9999999945872900				
1.9983933609701400	**2.0000010333694100**	**1.9999999961908400**	**2.0000000000596700**	**1.9999999999960300**	**2.0000000000013200**			
1.9995983886400300	2.0000000645300000	1.9999999999407000	2.0000000000002200	1.9999999999999900	1.9999999999999900	1.9999999999999900		
1.9998996001842000	2.0000000040322500	1.9999999999990700	2.0000000000000000	2.0000000000000000	2.0000000000000000	2.0000000000000000	2.0000000000000000	
1.9999749002350500	2.0000000002520000	1.9999999999999800	1.9999999999999900	1.9999999999999900	1.9999999999999900	1.9999999999999900	1.9999999999999900	1.9999999999999900

Program Romberg Integration formula , python language

```
 from math import *
def integral_romberg(f,a,b):
   #integral f(x)dx
   #romberg integration
   #increase n to increase accuracy
   n=8
   R=[[0 for z in range(n+1)] for w in range(n+1)]
   m=1
   h=b-a
   close=1
   tol=1e-40;
   j=0;k=0
   ret=0.0;
   R[0][0]=h/2.0*(f(a)+f(b))
   #for j in range(1,n):
   while(close>tol and j<n):
      j=j+1
```

```
        h=h/2.0
        trap(f,a,h,j,m,R)
        m=m*2
        for k in range(1,j+1):
            R[j][k]=R[j][k-1]+(R[j][k-1]-R[j-1][k-1])/(pow(4,k)-1.0)
            close=abs(R[j-1][j-1]-R[j][j])
            ret=R[j][k]
    return ret

def trap(f,a,h,j,m,R):
    sum=0
    for p in range(1,m+1):
        sum=sum+f(a+h*(2*p-1))
    R[j][0]=R[j-1][0]/2.0+h*sum

f = lambda x:2.0/(1.0+2.0*x*x)
I=integral_romberg(f,-3.0,3.0)
print("I=",I)
```

Romberg integration f(x)= 2.0/(1.0+2.0*x*x)

```
 runfile('E:/okul/SCO1/integral_romberg.py', wdir='E:/okul/SCO1')
I= 3.7881660832213675
```

Romberg integration with trapezoidal rule (Newton-Cotes 2)f(x)= 2.0/(1.0+2.0*x*x) Julia language

```
 function  newton_cotes2(ff,a,b,n)
      #Newton-cotes integral 2 points
       #trapezoidal rule
       # ff integrating function
       # a,b integral limits
       # n : number of sub division h=(b-a)/n;
       h=(b-a)/n
       h1=h;
       sum=0.0;
       x=zeros(2)
       f=zeros(2)
       for p=1:n
         for i=1:2
           x[i]=a+(i-1)*h1+(p-1)*h
           f[i]=ff(x[i])
           end
         sum=sum+h1*(f[1]+f[2])/2.0
       end
       return sum
       end
function romberg_integral(f, a,b,n)
    m = n
    R = zeros(Float64,n,m)
    # calculate Newton-cotes 2
    h=(a - b)/2.0
    for j=1:n
      m1=2*j
      R[j,1]=newton_cotes2(f,a,b,m1)
      y=R[j,1]
      println("R[$j,1] = $y")
    end
    for k=2:n
      for j=1:m-1
        n1=4^(k-1)
        n2=n1-1
         R[j,k]=(n1*R[j+1,k-1]-R[j,k-1])/n2
      end
    end
    return R
end

function output(R,nn)
   for i=1:nn
     for j=i:nn-1
```

```
        print(R[j,i])
        print(" ")
      end
      println(" ")
    end
    return nothing
  end

f(x)=2.0/(1.0+2*x*x)
a=-3.0
b=3.0
n=8
R=romberg_integral(f, a,b,n)
show(IOContext(stdout, :limit=>false), MIME"text/plain"(), R)
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" integral_romberg.jl
R[1,1] = 6.315789473684211
R[2,1] = 4.248803827751196
R[3,1] = 3.883040935672515
R[4,1] = 3.8058295411062657
R[5,1] = 3.789612338685158
R[6,1] = 3.786638312954102
R[7,1] = 3.7864174566839495
R[8,1] = 3.786675582381121
8X8 Matrix{Float64}:
 6.31579  3.55981   3.77454    3.78147     3.78454     3.78577     3.7864       3.7868
 4.2488   3.76112   3.78136    3.78453     3.78577     3.7864      3.7868      -0.337943
 3.88304  3.78009   3.78448    3.78576     3.7864      3.7868     -0.337691     0.00569917
 3.80583  3.78421   3.78574    3.7864      3.7868     -0.336684    0.00567821  -2.25949e-5
 3.78961  3.78565   3.78639    3.7868     -0.332657    0.00559463 -2.22469e-5   2.18218e-8
 3.78664  3.78634   3.78679   -0.316566    0.0052643  -2.08756e-5   2.04626e-8  -5.00041e-12
 3.78642  3.78676  -0.252451   0.00400716 -1.57143e-5   1.5361e-8   -3.75117e-12  2.28967e-16
 3.78668  0.0       0.0        0.0         0.0         0.0          0.0
> Terminated with exit code 0.
```

Romberg Integration, Ocatave-Matlab code

```
## Author: M. Turhan ÇOBAN
## Created: 2021-06-13
## Romberg integration with trapezoidal rule
function retval = romberg_integral (f,a,b)
  ##integral f(x)dx
  n=10;
  R=zeros(n,n);
  m=1;
  h=b-a;
  close=1;
  tol=1e-40;
  j=1;
  k=1;
  ret=0;
  R(1,1)=h/2.0*(f(a)+f(b));
 while close>tol && j<n
  j++;
  h=h/2.0;
  R(j,1)=newton_cotes_integral(f,a,b,2*j,2);
  m*=2;
  for k=2:j
   if k==n
    R(j,k)=R(j,k-1)+(R(j,k-1)-R(j-1,k-1))/(4^k-1.0)
   else
    R(j,k)=R(j,k-1)+(R(j,k-1)-R(j-1,k-1))/(4^k-1.0);
   endif
   close=abs(R(j-1,j-1)-R(j,j));
   ret=R(j,k);
  endfor
 endwhile
 retval = ret;
endfunction
```

```
>> romberg_integral(f,0.0,1.0)
R =

   0.6366        0        0        0        0        0        0        0        0        0
   0.9536   0.9747        0        0        0        0        0        0        0        0
```

```
    0.9747   0.9761   0.9761        0        0        0        0        0        0        0
    0.9835   0.9841   0.9842   0.9843        0        0        0        0        0        0
    0.9882   0.9885   0.9886   0.9886   0.9886        0        0        0        0        0
    0.9910   0.9912   0.9912   0.9913   0.9913   0.9913        0        0        0        0
    0.9929   0.9930   0.9930   0.9930   0.9930   0.9930   0.9930        0        0        0
    0.9942   0.9942   0.9943   0.9943   0.9943   0.9943   0.9943   0.9943        0        0
    0.9951   0.9952   0.9952   0.9952   0.9952   0.9952   0.9952   0.9952   0.9952        0
    0.9958   0.9959   0.9959   0.9959   0.9959   0.9959   0.9959   0.9959   0.9959   0.9959

ans = 0.9959
>>
```

AITKEN EXTRAPOLATION WITH NEWTON –COTES INTEGRATION FORMULAS

Another extrapolation Formula is te Aitken extrapolation. This equation is basically derived from Euler-McLaurin Formula. Aitken interpolation has the following form:

$$I = I_{4n} - \frac{(I_{4n} - I_{2n})^2}{(I_{4n} - I_{2n}) - (I_{2n} - I_n)}$$

By combining this equation with Simpson 1/3 formulation given above, more accurate results can be obtain

Program 7.2-2 SIMPSON 1/3 formula with aitken extrapolation , java language

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Simpson integration with Aitken extrapolation
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  double func(double x)
  {return x*Math.sqrt(x);}
}


class NA58
{

public static double newton_cotes3(f_x ff,double a,double b, int n)
{      //Newton-cotes integral 3 points
       //Simpson 1/3 rule

       double h=(b-a)/n;
       double h1=h/2.0;
       double sum=0;
       int p,i;
       double x[]=new double[3];
       double f[]=new double[3];
       for(p=0;p<n;p++)
       {  {for(i=0;i<3;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
          sum+=h*(f[0]+4.0*f[1]+f[2])/6.0;
       }
       return sum;
}

public static double Aitken_simpson1_3(f_x ff,double a,double b, int n)
{   //Aitken interpolation formula
```

```
   double I4n=newton_cotes3(ff,a,b,4*n);
   double I2n=newton_cotes3(ff,a,b,2*n);
   double In=newton_cotes3(ff,a,b,n);
   double I=I4n-(I4n-I2n)*(I4n-I2n)/((I4n-I2n)-(I2n-In));
   //System.out.println("I4n="+I4n+"I2n="+I2n+"In="+In+"I="+I);
   return I;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  for(int n=2;n<257;n*=2)
    System.out.println("n : "+n+"I = "+Aitken_simpson1_3(b1,0.0,1.0,n));
}
}
```

```
---------- Capture Output ---------- ---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA58
n : 2I = 0.39999984745800915
n : 4I = 0.3999999902487645
n : 8I = 0.3999999993877008
n : 16I = 0.3999999999617067
n : 32I = 0.3999999999976067
n : 64I = 0.3999999999998505
n : 128I = 0.3999999999999909
n : 256I = 0.4

> Terminated with exit code 0.
```

If Aitken intrepolation is combined with 8th degree polynomial Formula:

Program 7.2-3 Newton_Cotes 8th degree polynomial Formula with Aitken extrapolation, java language

```
//=====================================================
// Numerical Analysis package in java
// Newton-Cotes integral formula 9 points, n=8
// with Aitken extrapolation formula
// Dr. Turhan Coban
// =====================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {return x*Math.sqrt(x);}
}

class NA59
{
public static double newton_cotes9(f_x ff,double a,double b, int n)
{       //Newton-cotes integral 9 points, n=8
        // ff integrating function
        // a,b integral limits
        // n : number of sub division h=(b-a)/n;
                double h=(b-a)/n;
        double h1=h/8.0;
        double sum=0;
        int p,i;
        double x[]=new double[9];
        double f[]=new double[9];
        for(p=0;p<n;p++)
        {  {for(i=0;i<9;i++) {x[i]=a+i*h1+p*h;f[i]=ff.func(x[i]);}}
           sum+=h/28350.0*(989.0*f[0]+5888.0*f[1]-928.0*f[2]+10496.0*f[3]-4540.0*f[4]+
           10496.0*f[5]-928.0*f[6]+5888.0*f[7]+989.0*f[8]);
```

```
        }
        return sum;
}

public static double Aitken_newton_cotes9(f_x ff,double a,double b, int n)
{   //Aitken interpolation formula
                double I4n=newton_cotes9(ff,a,b,4*n);
    double I2n=newton_cotes9(ff,a,b,2*n);
    double In=newton_cotes9(ff,a,b,n);
    double I=I4n-(I4n-I2n)*(I4n-I2n)/((I4n-I2n)-(I2n-In));
    //System.out.println("I4n="+I4n+"I2n="+I2n+"In="+In+"I="+I);
    return I;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  for(int n=2;n<128;n*=2)
    System.out.println("n : "+n+"I = "+Aitken_newton_cotes9(b1,0.0,1.0,n));
}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA59
n : 2I = 0.39999999999998676
n : 4I = 0.39999999999999986
n : 8I = 0.39999999999999997
n : 16I = 0.39999999999999986
n : 32I = 0.4000000000000001
n : 64I = 0.4

> Terminated with exit code 0.
```

Simpson 1/3 formula with aitken extrapolation python code

```
from math import *
from if_x import *


class f1(if_x):
  def func(self,x):
     y=4.0/pi*sqrt(1.0-x*x)
     return y

def newton_cotes3(f,a,b,n):
    #Newton-cotes integral 4 points
    # Simpson 3/8 rule
    # ff integrating function
    # a,b integral limits
    # n : number of sub division h=(b-a)/n;
    h=(b-a)/n;
    h1=h/3.0;
    sum=0.0
    nn=2*n
    n1=4
    x =[0 for w in range(n1)]
    y =[0 for w in range(n1)]
    for p in range(n):
        for i in range(n1):
            x[i]=a+i*h1+p*h
            y[i]=f.func(x[i])
        sum=sum+h*(y[0]+3.0*y[1]+3.0*y[2]+y[3])/8.0
    return sum

def Aitken_simpson1_3(ff,a,b,n):
    #Aitken interpolation formula
    I4n=newton_cotes3(ff,a,b,4*n)
```

```
    I2n=newton_cotes3(ff,a,b,2*n)
    In=newton_cotes3(ff,a,b,n)
    I=I4n-(I4n-I2n)*(I4n-I2n)/((I4n-I2n)-(I2n-In))
    return I

f=f1()
n=2
while n<257:
    I= Aitken_simpson1_3(f,0.0,1.0,n)
    print("n = ",n,"I=",I)
    n=n*2
```

```
runfile('E:/okul/SCO1/newton_cotes3.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
n =  2 I= 0.9999928680748176
n =  4 I= 0.9999986596595479
n =  8 I= 0.99999975624287
n =  16 I= 0.9999999563192528
n =  32 I= 0.9999999922266964
n =  64 I= 0.9999999986213309
n =  128 I= 0.9999999997558837
n =  256 I= 0.9999999999568113
```

7.2 QUADRATURES: GAUSS-LEGENDRE, GAUSS - CHEBYCHEV, GAUSS -JACOBİ, GAUSS -HERMİT, GAUSS -LEQUERRE INTEGRATİON FORMULAS

A problem known from antique times, is to create a rectangle with the same area with a polynomial. IT is called a quadrature problem.

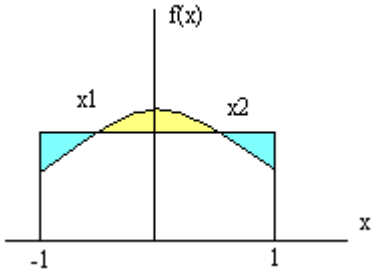


Figure 8.4-1a) a rectangle with the same area with a polynomial

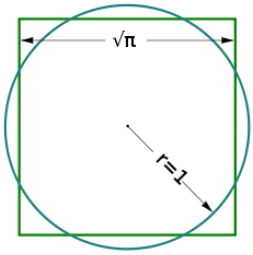


Figure 8.4-1b) A rectangle with the same area with a circle

If point x1 and x2 points intersecting the polynomial and rectangle is known, area calculation of the polynomial can be interchange with the area calculation of the rectangle. Or in more general means instead of integration process of summation can be substituted. As a general definition:

$$I_w(-1,1) = \int_{-1}^{1} w(x)f(x)dx \cong \sum_{k=1}^{n} c_k f(x_k)$$

Can be written. In this equation w(x) is the weight factor. In the first of a such formulation, Gauss-Legendre integral formulation, weight factor can be taken as 1.

$$I_w(-1,1) = \int_{-1}^{1} f(x)dx \cong \sum_{k=1}^{n} c_k f(x_k)$$

$$I_w(-1,1) = f(x)$$

If this integration is solved for a general polynamial equation.

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

Following relation is found:

$$\int_{-1}^{1} x^k dx = \frac{1-(-1)^{k+1}}{k+1}$$

$$\int_{-1}^{1} x^k dx \cong \frac{1-(-1)^{k+1}}{k+1}$$

This equation can be written in open form as:

$$c_1 x_1^k + c_2 x_2^k + \cdots + c_n x_n^k = \frac{1-(-1)^{k+1}}{k+1} \quad 0 \le k \le 2n$$

For a special case of n=2 equation becomes :

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$I_w(-1,1) = \int_{-1}^{1} f(x)dx \cong c_1 f(x_1) + c_2 f(x_2) = \int_{-1}^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx$$

$$c_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

$$= \int_{-1}^{1} (a_0)dx + \int_{-1}^{1} (a_1 x)dx + \int_{-1}^{1} (a_2 x^2)dx + \int_{-1}^{1} (a_3 x^3)dx$$

$$c_1 + c_2 = 2$$

$$c_1 x_1 + c_2 x_2 = 0$$

$$c_1 x_1^2 + c_2 x_2^2 = 2/3$$

$$c_1 x_1^3 + c_2 x_2^3 = 0$$

If this system of equation is solved roots and coefficients can be found as:

Roots: $x_{1,2} = \pm\frac{1}{\sqrt{3}}$ and Coefficients : $c_{1,2} = 1$

The general solution of this problem can be defined with the Legendre polynomials. Carl Friedrich Gauss was the first to derive the Gauss–Legendre quadrature rule. Carl Gustav Jacob Jacobi discovered the connection between the quadrature rule and the orthogonal family of Legendre polynomials.

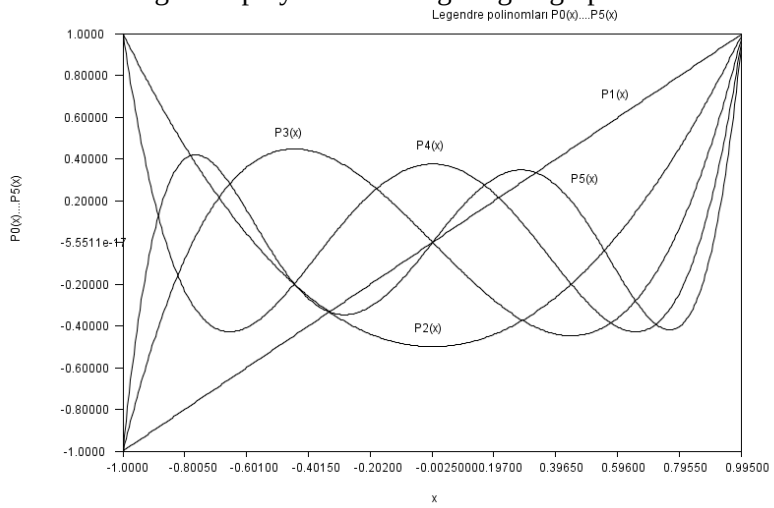
 Johann Carl Fredrich Gauss	 Carl Gustav Jacob Jacobi	 Adrien-Marie Legendre

Legendre polynomials has a general definition as:

$(k+1)P_{k+1}(x) = (2k+1)xP_k(x)$-k$P_{k-1}(x),\ \ k \geq 1$

and $P_0(x)=1$,$P_1(x)=x$

The first 6 Legendre polynomials are giving in graphic form.



The roots of Legendre Polynomials are the roots of the Gauss-Legendre integration Formula. For example if $P_2(x)$ value is calculated from the above general form:

$$P_2(x) = \frac{3x^2-1}{2}$$

The root of this is equal to$x_{1,2} = \pm\frac{1}{\sqrt{3}}$. In general roots can be calculated from Legendre polynomials as follows:

$$\theta_{n,k} = \frac{n-k+3/4}{n+1/2}\pi$$

$$x_{n,k} = \left[1 - \frac{1}{8n^2} + \frac{1}{8n^3} - \frac{1}{384n^4}\left(39 - \frac{29}{sin^2\theta_{n,k}}\right)\right] cos\theta_{n,k} + E(n^{-5})$$

This equation contains an error level of $\mathbf{n}^{-5}$. In order to decrease the error Newton-Raphson root finding method can be usefull.

$$x_{v+1} = x_v - \frac{1-x_v^2}{P_{n-1}(x_v) - x_v P_n(x_v)}$$

After finding the roots coeficients can be calculated as

$$c_1 x_1^{2j} + c_2 x_2^{2j} + \cdots + c_n x_n^{2j} = \frac{2}{2j+1} \quad 0 \leq j \leq n$$

Or

$$c_k = \frac{2(1-x_k^2)}{[nP_{n-1}(x_k)]^2}$$

As a last concept, consider that Gauss-Legendre integration limits are -1 and 1

Region x=[-1,1] can be converted to z=[a,b] by changing the variables

$$z = \left(\frac{b-a}{2}\right)x + \left(\frac{b-a}{2}\right) = \alpha x + \beta$$

In this case, integration Formula will became :

$$I_w(a,b) = \int_a^b f(x)dx \cong \alpha \sum_{k=1}^{n} c_k f(\alpha x_k + \beta)$$

First 10 Gauss-Legendre integration coefficients are given in the table

Table 8.4-1 Gauss – Legendre integration roots and coefficients

N	x_k	c_k

2	**-0.577350269189625**	**1.000000000000000**
	0.577350269189625	**1.000000000000000**
3	**-0.774596669241483**	**0.555555555555552**
	0.000000000000000	**0.888888888888888**
	0.774596669241483	**0.555555555555552**
4	**-0.861136311594052**	**0.347854845137447**
	-0.339981043584856	**0.652145154862546**
	0.339981043584856	**0.652145154862546**
	0.861136311594052	**0.347854845137447**
5	**-0.906179845938664**	**0.236926885056181**
	-0.538469310105683	**0.478628670499366**
	0.000000000000000	**0.568888888888888**
	0.538469310105683	**0.478628670499366**
	0.906179845938664	**0.236926885056181**
6	**-0.932469514203152**	**0.171324492379162**
	-0.661209386466264	**0.360761573048138**
	-0.238619186083196	**0.467913934572691**
	0.238619186083196	**0.467913934572691**
	0.661209386466264	**0.360761573048138**
	0.932469514203152	**0.171324492379162**
7	**-0.949107912342758**	**0.129484966168862**
	-0.741531185599394	**0.279705391489276**
	-0.405845151377397	**0.381830050505119**
	0.000000000000000	**0.417959183673469**
	0.405845151377397	**0.381830050505119**
	0.741531185599394	**0.279705391489276**
	0.949107912342758	**0.129484966168862**
8	**-0.960289856497536**	**0.101228536290369**
	-0.796666477413626	**0.222381034453374**
	-0.525532409916329	**0.313706645877887**
	-0.183434642495649	**0.362683783378362**
	0.183434642495649	**0.362683783378362**
	0.525532409916329	**0.313706645877887**
	0.796666477413626	**0.222381034453374**
	0.960289856497536	**0.101228536290369**
9	**-0.968160239507626**	**0.081274388361569**
	-0.836031107326635	**0.180648160694857**
	-0.613371432700590	**0.260610696402935**
	-0.324253423403808	**0.312347077040002**
	0.000000000000000	**0.330239355001259**
	0.324253423403808	**0.312347077040002**
	0.613371432700590	**0.260610696402935**
	0.836031107326635	**0.180648160694857**
	0.968160239507626	**0.081274388361569**
10	**-0.973906528517171**	**0.066671344308684**
	-0.865063366688984	**0.149451349150580**
	-0.679409568299024	**0.219086362515982**
	-0.433395394129247	**0.269266719309996**
	-0.148874338981631	**0.295524224714752**
	0.148874338981631	**0.295524224714752**
	0.433395394129247	**0.269266719309996**
	0.679409568299024	**0.219086362515982**
	0.865063366688984	**0.149451349150580**
	0.973906528517171	**0.066671344308684**

EXAMPLE : Calculate integral of $f(x) = \frac{\pi}{4}\sqrt{1-x^2}$ function between limits 0 and 1 with three points Gauss-Legendre integration formula by hand and by computer program

For n=3 α=(1-0)/2=0.5 β=(1+0)/2=0.5

I=0.5*(0.555555555555553 *f(0.5*-0.774596669241483+0.5)+ 0.888888888888889*f(0.5*0+0.5)+ 0.555555555555553 *f(0.5*0.774596669241483+0.5))

I=1.004609

excel versions of the same integral

For n=4

Gauss Integral Formula

a		0.0000000000

b		1.0000000000
a	(b-a)/2	0.5000000000
b	(b+a)/2	0.5000000000

rk	**Ck**	**xk=a+b*rk**	**yk=f(xk)**	**z=a*Ck*yk**
-0.861136312	**0.347854845**	0.0694318442	1.2701668325	0.2209168434
-0.339981044	**0.652145155**	0.3300094782	1.2019096333	0.3919097720
0.339981044	**0.652145155**	0.6699905218	0.9452143982	0.3082084950
0.861136312	**0.347854845**	0.9305681558	0.4661568155	0.0810774534
			Integral	1.002112564

For n=6

Gauss Integral Formula

a		0.0000000000
b		1.0000000000
a	(b-a)/2	0.5000000000
b	(b+a)/2	0.5000000000

rk	**Ck**	**xk=a+b*rk**	**yk=f(xk)**	**z=a*Ck*yk**
-0.932469514	**0.171324492**	0.0337652429	1.2725135329	0.1090063675
-0.661209386	**0.360761573**	0.1693953068	1.2548389257	0.2263488324
-0.238619186	**0.467913935**	0.3806904070	1.1773678593	0.2754534138
0.238619186	**0.467913935**	0.6193095930	0.9996800599	0.2338821151
0.661209386	**0.360761573**	0.8306046932	0.7090193556	0.1278934690
0.932469514	**0.171324492**	0.9662347571	0.3280671473	0.0281029687
			Integral	1.000687166

For n=10

Gauss Integral Formula

a		0.0000000000
b		1.0000000000
a	(b-a)/2	0.5000000000
b	(b+a)/2	0.5000000000

rk	**Ck**	**xk=a+b*rk**	**yk=f(xk)**	**z=a*Ck*yk**
-0.973906529	**0.066671344**	0.0130467357	1.2731311764	0.0424406835
-0.865063367	**0.149451349**	0.0674683167	1.2703383629	0.0949268911
-0.679409568	**0.219086363**	0.1602952159	1.2567754345	0.1376711792
-0.433395394	**0.269266719**	0.2833023029	1.2210757598	0.1643975319
-0.148874339	**0.295524225**	0.4255628305	1.1521912348	0.1702502107
0.148874339	**0.295524225**	0.5744371695	1.0422085462	0.1539989363
0.433395394	**0.269266719**	0.7166976971	0.8879368571	0.1195459222
0.679409568	**0.219086363**	0.8397047841	0.6914240280	0.0757407876
0.865063367	**0.149451349**	0.9325316833	0.4597517249	0.0343552578
0.973906529	**0.066671344**	0.9869532643	0.2050004799	0.0068338288
			Integral	1.000161229

For n=60

Gauss Integral Formula $f(x) = \frac{\pi}{4}\sqrt{1-x^2}$

a		0.0000000000
b		1.0000000000
a	(b-a)/2	0.5000000000
b	(b+a)/2	0.5000000000

rk	**Ck**	**xk=a+b*rk**	**yk=f(xk)**	**z=a*Ck*yk**
-0.999210123227436	0.002026811968873	0.0003949384	1.2732394454	0.0012903085
-0.995840525118838	0.004712729926953	0.0020797374	1.2732367912	0.0030002106
-0.989787895222221	0.007389931163346	0.0051060524	1.2732229468	0.0047045150
-0.981067201752598	0.010047557182288	0.0094663991	1.2731824942	0.0063961870
-0.969701788765052	0.012678166476816	0.0151491056	1.2730934350	0.0080702453
-0.955722255839996	0.015274618596784	0.0221388721	1.2729274803	0.0097217409
-0.939166276116423	0.017829901014207	0.0304168619	1.2726504171	0.0113456155
-0.920078476177627	0.020337120729457	0.0399607619	1.2722225443	0.0129366717
-0.898510310810046	0.022789516943998	0.0507448446	1.2715991671	0.0144895654

-0.874519922646898	0.025180477621521	0.0627400387	1.2707311395	0.0159988085
-0.848171984785929	0.027503556749925	0.0759140076	1.2695654443	0.0174587826
-0.819537526162145	0.029752491500789	0.0902312369	1.2680457997	0.0188637609
-0.788693739932264	0.031921219019296	0.1056531300	1.2661132804	0.0202079397
-0.755723775306585	0.034003892724946	0.1221381123	1.2637069454	0.0214854777
-0.720716513355730	0.035994898051084	0.1396417433	1.2607644611	0.0226905441
-0.683766327381355	0.037888867569243	0.1581168363	1.2572227133	0.0238173724
-0.644972828489477	0.039680695452381	0.1775135858	1.2530184000	0.0248603208
-0.604440597048510	0.041365551235585	0.1977797015	1.2480886002	0.0258139365
-0.562278900753944	0.042938892835936	0.2188605496	1.2423713133	0.0266730243
-0.518601400058569	0.044396478795787	0.2406993000	1.2358059656	0.0274327167
-0.473525841761707	0.045734379716114	0.2632370791	1.2283338822	0.0280885441
-0.427173741583078	0.046948988848912	0.2864131292	1.2198987211	0.0286365057
-0.379670056576798	0.048037031819971	0.3101649717	1.2104468712	0.0290731374
-0.331142848268448	0.048995575455757	0.3344285759	1.1999278109	0.0293955768
-0.281722937423261	0.049822035690550	0.3591385313	1.1882944300	0.0296016238
-0.231543551376029	0.050514184532509	0.3842282243	1.1755033149	0.0296897957
-0.180739964873425	0.051070156069856	0.4096300176	1.1615149982	0.0296593761
-0.129449135396945	0.051488451500981	0.4352754323	1.1462941736	0.0295104560
-0.077809333949537	0.051767943174910	0.4610953330	1.1298098787	0.0292439668
-0.025959772301248	0.051907877631221	0.4870201138	1.1120356457	0.0288617051
0.025959772301248	0.051907877631221	0.5129798862	1.0929496235	0.0283663477
0.077809333949537	0.051767943174910	0.5389046670	1.0725346712	0.0277614570
0.129449135396945	0.051488451500981	0.5647245677	1.0507784251	0.0270514770
0.180739964873425	0.051070156069856	0.5903699824	1.0276733412	0.0262417190
0.231543551376029	0.050514184532509	0.6157717757	1.0032167145	0.0253383371
0.281722937423261	0.049822035690550	0.6408614687	0.9774106755	0.0243482948
0.331142848268448	0.048995575455757	0.6655714241	0.9502621669	0.0232793209
0.379670056576798	0.048037031819971	0.6898350283	0.9217829005	0.0221398573
0.427173741583078	0.046948988848912	0.7135868708	0.8919892958	0.0209389978
0.473525841761707	0.045734379716114	0.7367629209	0.8609024020	0.0196864187
0.518601400058569	0.044396478795787	0.7593007000	0.8285478029	0.0183923025
0.562278900753944	0.042938892835936	0.7811394504	0.7949555081	0.0170672547
0.604440597048510	0.041365551235585	0.8022202985	0.7601598294	0.0157222152
0.644972828489477	0.039680695452381	0.8224864142	0.7241992435	0.0143683648
0.683766327381355	0.037888867569243	0.8418831637	0.6871162440	0.0130170282
0.720716513355730	0.035994898051084	0.8603582567	0.6489571814	0.0116795738
0.755723775306585	0.034003892724946	0.8778618877	0.6097720929	0.0103673124
0.788693739932264	0.031921219019296	0.8943468700	0.5696145247	0.0090913950
0.819537526162145	0.029752491500789	0.9097687631	0.5285413466	0.0078627110
0.848171984785929	0.027503556749925	0.9240859924	0.4866125631	0.0066917881
0.874519922646898	0.025180477621521	0.9372599613	0.4438911240	0.0055886953
0.898510310810046	0.022789516943998	0.9492551554	0.4004427432	0.0045629483
0.920078476177627	0.020337120729457	0.9600392381	0.3563357382	0.0036234215
0.939166276116423	0.017829901014207	0.9695831381	0.3116409221	0.0027782634
0.955722255839996	0.015274618596784	0.9778611279	0.2664316177	0.0020348207
0.969701788765052	0.012678166476816	0.9848508944	0.2207839805	0.0013995680
0.981067201752598	0.010047557182288	0.9905336009	0.1747782074	0.0008780470
0.989787895222221	0.007389931163346	0.9948939476	0.1285028199	0.0004748135
0.995840525118838	0.004712729926953	0.9979202626	0.0820735936	0.0001933953
0.999210123227436	0.002026811968873	0.9996050616	0.0357805413	0.0000362602
			Integral	1.000000837

A java version of Gauss-Legendre integration Formula will be written next to use n=10 formula given in the table. Our function is

$$I = \int_0^1 \frac{4}{\pi}\sqrt{1-x^2}dx = 1.0$$

Program 7.3-1 10 Point Gauss-Legendre integration

```
 //=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gauss-Legendre 10 points integration formula
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  public double func(double x)
  {
  return 4.0/Math.PI*Math.sqrt(1.0-x*x);
  }
}

class NA60
{
public static double integral(f_x f_xnt,double a,double b)
{
//integral f(x)dx
//integral of a function by using gauss-legendre quadrature
//coefficients are pre-calculated for 60 terms for [-1,1]
//band then utilises variable transform
 double r[],c[];
  r=new double[10];
 c=new double[10];
 r[0]=-0.973906528517171;
 r[1]=-0.865063366688984;
 r[2]=-0.679409568299024;
 r[3]=-0.433395394129247;
 r[4]=-0.148874338981631;
 r[5]=0.148874338981631;
 r[6]=0.433395394129247;
 r[7]=0.679409568299024;
 r[8]=0.865063366688984;
 r[9]=0.973906528517171;

 c[0]=0.066671344308684;
 c[1]=0.149451349150580;
 c[2]=0.219086362515982;
 c[3]=0.269266719309996;
 c[4]=0.295524224714752;
 c[5]=0.295524224714752;
 c[6]=0.269266719309996;
 c[7]=0.219086362515982;
 c[8]=0.149451349150580;
 c[9]=0.066671344308684;

int n=10;

double z=0;
double x,y;
double k1=(b-a)/2.0;
double k2=(b+a)/2.0;
double y1=0;
for(int i=0;i<n;i++)
{
x=k2+k1*r[i];
y=f_xnt.func(x);
```

```
y1=c[i]*y;
z+=y1;
}
return k1*z;
}

  public static void main(String args[]) throws IOException
  {
  //Gauss-Legendre integration formula with n=10
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+integral(b1,0.0,1.0));
}
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" NA60
integral of class         f1 : 1.0001612291825637

> Terminated with exit code 0.
```

Excel version of the program:

		n	xk	ck	$\alpha x_k + \beta$			
a	0	4	-0.8611363	0.3478548	0.069432	1.270167	0.441833687	
b	1		-0.339981	0.6521452	0.330009	1.20191	0.783819544	
α	0.5		0.339981	0.6521452	0.669991	0.945214	0.61641699	
β	0.5		0.8611363	0.3478548	0.930568	0.466157	0.162154907	
								I
							2.004225128	**1.002113**

The same program as a Python version

Program 10 Point Gauss-Legendre integration as Python code

```
 # Gauss - Legendre integral 10 coefficients
# python version

from math import *
def f1(x):
   return 4.0/pi*sqrt(1-x*x)

def integral(a,b):
  #integral f1(x)dx
  #integral of a function by using gauss-legendre quadrature
  #coefficients are pre-calculated for 10 terms for [-1,1]
  #band then utilises variable transform
  r=[-0.973906528517171,-0.865063366688984,-0.679409568299024,-0.433395394129247,-
0.148874338981631,0.148874338981631,0.433395394129247,0.679409568299024,0.865063366688984,0.973906528517171]
c=[0.066671344308684,0.149451349150580,0.219086362515982,0.269266719309996,0.295524224714752,0.295524224714752,0.2692
66719309996,0.219086362515982,0.149451349150580,0.066671344308684]
  n=10
  z=0
  k1=(b-a)/2.0;
  k2=(b+a)/2.0;
  for i in range(n):
    x=k2+k1*r[i]
    y=f1(x)
    z=z+k1*c[i]*y
  return z

a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
r=integral(a,b)
print("integral  : ",r)
```

```
 ---------- Capture Output ----------
> "D:\co\python\pythonw.exe" NA60.py
enter lower limit : 0
enter upper limit : 1
```

```
integral  :  1.0001612291825637

> Terminated with exit code 0.
```

The same program as a Octave(Matlab) version

Program Point Gauss-Legendre integration as Octave(Matlab) code

```
function z=integral(f,a,b)
% Gauss- Legendre integration n=10
  r=[-0.973906528517171,-0.865063366688984,-0.679409568299024,-0.433395394129247,-
0.148874338981631,0.148874338981631,0.433395394129247,0.679409568299024,0.865063366688984,0.973906528517171];
c=[0.066671344308684,0.149451349150580,0.219086362515982,0.269266719309996,0.295524224714752,0.295524224714752,0.2692
66719309996,0.219086362515982,0.149451349150580,0.066671344308684];
  n=10;
  z=0;
  k1=(b-a)/2.0;
  k2=(b+a)/2.0;
  for i =1:10
    x=k2+k1*r(i);
    y=f(x);
    z=z+k1*c(i)*y;
   endfor
endfunction
```

```
>> f=@(x)4/pi*sqrt(1-x*x);
>> integral(f,0,1)
ans =  1.00016122918256
>>
```

Program 10 Point Gauss-Legendre integration as Fortran 90 code

```
program GLintegral1
IMPLICIT NONE
REAL(KIND=8) a,b
a=0
b=1
print *,'integral(',a,',',b,') = ',integral(a,b)

CONTAINS
  FUNCTION func(x)
  REAL(KIND=8) func,x,PI
  PI=4.0*atan(1.0)
  func=4.0/PI*sqrt(1.0-x*x)
  END FUNCTION func

  FUNCTION integral(a,b)
  REAL(KIND=8) integral,a,b,z,x,y,y1,k1,k2
  INTEGER I,n
  REAL(KIND=8), DIMENSION(0:9) :: r,c
  r(0)=-0.973906528517171
  r(1)=-0.865063366688984
  r(2)=-0.679409568299024
  r(3)=-0.433395394129247
  r(4)=-0.148874338981631
  r(5)=0.148874338981631
  r(6)=0.433395394129247
  r(7)=0.679409568299024
  r(8)=0.865063366688984
  r(9)=0.973906528517171

  c(0)=0.066671344308684
  c(1)=0.149451349150580
  c(2)=0.219086362515982
  c(3)=0.269266719309996
  c(4)=0.295524224714752
  c(5)=0.295524224714752
  c(6)=0.269266719309996
  c(7)=0.219086362515982
  c(8)=0.149451349150580
  c(9)=0.066671344308684
  n=10
  z=0
  k1=(b-a)/2.0
  k2=(b+a)/2.0
```

```
  y1=0.0
  DO I = 0, n-1
    x=k2+k1*r(i)
    y=func(x)
    y1=c(i)*y
    z=z+y1
  END DO
  integral=k1*z
  END FUNCTION integral
end program GLintegral1
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" GLintegral1

E:\okul\SCO1>GLintegral1
 integral(   0.0000000000000000     ,   1.0000000000000000     ) =    1.0001612212606290

> Terminated with exit code 0.
```

10 Point Gauss-Legendre integration, Julia code

```
 function  integral(f_xnt, a, b)
#integral f(x)dx
#integral of a function by using gauss-legendre quadrature
#coefficients are pre-calculated for 60 terms for [-1,1]
#band then utilises variable transform
 r=zeros(Float64,10)
 c=zeros(Float64,10)
 r[1]=-0.973906528517171
 r[2]=-0.865063366688984
 r[3]=-0.679409568299024
 r[4]=-0.433395394129247
 r[5]=-0.148874338981631
 r[6]=0.148874338981631
 r[7]=0.433395394129247
 r[8]=0.679409568299024
 r[9]=0.865063366688984
 r[10]=0.973906528517171
 c[1]=0.066671344308684
 c[2]=0.149451349150580
 c[3]=0.219086362515982
 c[4]=0.269266719309996
 c[5]=0.295524224714752
 c[6]=0.295524224714752
 c[7]=0.269266719309996
 c[8]=0.219086362515982
 c[9]=0.149451349150580
 c[10]=0.066671344308684
 n=length(r)
 z=0.0
 k1=(b-a)/2.0
 k2=(b+a)/2.0
 y1=0.0
 for i=1:n
   x=k2+k1*r[i]
   y=f_xnt(x)
   y1=c[i]*y
   z+=y1
 end
return k1*z
end

f(x) = 4.0/pi*sqrt(1.0-x*x)
r=integral(f,0.0,1.0)
print("integral  : $r")
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_integral1.jl
integral  : 1.0001612291825637
> Terminated with exit code 0.
```

More terms can be used to achieve more accurate results. In the next program 60 terms of the Legendre coefficcients and roots are entered tho the program.

Program 60 Point Gauss-Legendre integration java code

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gauss-Legendre 10 points integration formula
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  public double func(double x)
  {
  return x*Math.sqrt(x);
  }
}

class NA60A
{
public static double integral(f_x f_xnt,double a,double b)
{
//integral f(x)dx
//integral of a function by using gauss-legendre quadrature
//coefficients are pre-calculated for 30 terms for [-1,1]
//band then utilises variable transform

double r[]=
{-0.99921012322743600,
-0.99584052511883800,
-0.98978789522222100,
-0.98106720175259800,
-0.96970178876505200,
-0.95572225583999600,
-0.93916627611642300,
-0.92007847617762700,
-0.89851031081004600,
-0.87451992264689800,
-0.84817198478592900,
-0.81953752616214500,
-0.78869373993226400,
-0.75572377530658500,
-0.72071651335573000,
-0.68376632738135500,
-0.64497282848947700,
-0.60444059704851000,
-0.56227890075394400,
-0.51860140005856900,
-0.47352584176170700,
-0.42717374158307800,
-0.37967005657679800,
-0.33114284826844800,
-0.28172293742326100,
-0.23154355137602900,
-0.18073996487342500,
-0.12944913539694500,
-0.07780933394953650,
-0.02595977230124780,
0.02595977230124780,
0.07780933394953650,
```

```
0.12944913539694500,
0.18073996487342500,
0.23154355137602900,
0.28172293742326100,
0.33114284826844800,
0.37967005657679800,
0.42717374158307800,
0.47352584176170700,
0.51860140005856900,
0.56227890075394400,
0.60444059704851000,
0.64497282848947700,
0.68376632738135500,
0.72071651335573000,
0.75572377530658500,
0.78869373993226400,
0.81953752616214500,
0.84817198478592900,
0.87451992264689800,
0.89851031081004600,
0.92007847617762700,
0.93916627611642300,
0.95572225583999600,
0.96970178876505200,
0.98106720175259800,
0.98978789522222100,
0.99584052511883800,
0.99921012322743600};
double c[]=
{0.00202681196887338,
0.00471272992695334,
0.00738993116334568,
0.01004755718228800,
0.01267816647681590,
0.01527461859678380,
0.01782990101420730,
0.02033712072945710,
0.02278951694399780,
0.02518047762152120,
0.02750355674992480,
0.02975249150078890,
0.03192121901929630,
0.03400389272494640,
0.03599489805108440,
0.03788886756924340,
0.03968069545238070,
0.04136555123558470,
0.04293889283593560,
0.04439647879578710,
0.04573437971611440,
0.04694898884891210,
0.04803703181997110,
0.04899557545575670,
0.04982203569055010,
0.05051418453250940,
0.05107015606985560,
0.05148845150098080,
0.05176794317491010,
0.05190787763122060,
0.05190787763122060,
0.05176794317491010,
0.05148845150098080,
0.05107015606985560,
0.05051418453250940,
0.04982203569055010,
0.04899557545575670,
0.04803703181997110,
0.04694898884891210,
0.04573437971611440,
0.04439647879578710,
0.04293889283593560,
0.04136555123558470,
0.03968069545238070,
0.03788886756924340,
0.03599489805108440,
0.03400389272494640,
```

```
0.03192121901929630,
0.02975249150078890,
0.02750355674992480,
0.02518047762152120,
0.02278951694399780,
0.02033712072945710,
0.01782990101420730,
0.01527461859678380,
0.01267816647681590,
0.01004755718228800,
0.00738993116334568,
0.00471272992695334,
0.00202681196887338};
int n=r.length;
double z=0;
double x,y;
double k1=(b-a)/2.0;
double k2=(b+a)/2.0;
for(int i=0;i<n;i++)
{
x=k2+k1*r[i];
y=f_xnt.func(x);
z+=k1*c[i]*y;
}
return z;
}

  public static void main(String args[]) throws IOException
  {
  //Gauss-Legendre integration formula with n=10
  f1 b1=new f1();
  System.out.println("integral of class        f1 : "+integral(b1,0.0,1.0));
}
}
```

```
---------- Capture Output ----------
> "C:\java\bin\javaw.exe" NA60A
integral of class        f1 : 0.39999999989833385

> Terminated with exit code 0.
```

Program 60 Point Gauss-Legendre integration, Julia code

```
 function  integral(f_xnt, a, b)
#integral f(x)dx
#integral of a function by using gauss-legendre quadrature
#coefficients are pre-calculated for 60 terms for [-1,1]
#band then utilises variable transform
r=[-0.99921012322743600,
-0.99584052511883800,
-0.98978789522222100,
-0.98106720175259800,
-0.96970178876505200,
-0.95572225583999600,
-0.93916627611642300,
-0.92007847617762700,
-0.89851031081004600,
-0.87451992264689800,
-0.84817198478592900,
-0.81953752616214500,
-0.78869373993226400,
-0.75572377530658500,
-0.72071651335573000,
-0.68376632738135500,
-0.64497282848947700,
-0.60444059704851000,
-0.56227890075394400,
-0.51860140005856900,
-0.47352584176170700,
-0.42717374158307800,
-0.37967005657679800,
-0.33114284826844800,
-0.28172293742326100,
```

-0.23154355137602900,
-0.18073996487342500,
-0.12944913539694500,
-0.07780933394953650,
-0.02595977230124780,
0.02595977230124780,
0.07780933394953650,
0.12944913539694500,
0.18073996487342500,
0.23154355137602900,
0.28172293742326100,
0.33114284826844800,
0.37967005657679800,
0.42717374158307800,
0.47352584176170700,
0.51860140005856900,
0.56227890075394400,
0.60444059704851000,
0.64497282848947700,
0.68376632738135500,
0.72071651335573000,
0.75572377530658500,
0.78869373993226400,
0.81953752616214500,
0.84817198478592900,
0.87451992264689800,
0.89851031081004600,
0.92007847617762700,
0.93916627611642300,
0.95572225583999600,
0.96970178876505200,
0.98106720175259800,
0.98978789522222100,
0.99584052511883800,
0.99921012322743600]
c=[0.00202681196887338,
0.00471272992695334,
0.00738993116334568,
0.01004755718228800,
0.01267816647681590,
0.01527461859678380,
0.01782990101420730,
0.02033712072945710,
0.02278951694399780,
0.02518047762152120,
0.02750355674992480,
0.02975249150078890,
0.03192121901929630,
0.03400389272494640,
0.03599489805108440,
0.03788886756924340,
0.03968069545238070,
0.04136555123558470,
0.04293889283593560,
0.04439647879578710,
0.04573437971611440,
0.04694898884891210,
0.04803703181997110,
0.04899557545575670,
0.04982203569055010,
0.05051418453250940,
0.05107015606985560,
0.05148845150098080,
0.05176794317491010,
0.05190787763122060,
0.05190787763122060,
0.05176794317491010,
0.05148845150098080,
0.05107015606985560,
0.05051418453250940,
0.04982203569055010,
0.04899557545575670,
0.04803703181997110,
0.04694898884891210,
0.04573437971611440,
0.04439647879578710,

```
 0.04293889283593560,
 0.04136555123558470,
 0.03968069545238070,
 0.03788886756924340,
 0.03599489805108440,
 0.03400389272494640,
 0.03192121901929630,
 0.02975249150078890,
 0.02750355674992480,
 0.02518047762152120,
 0.02278951694399780,
 0.02033712072945710,
 0.01782990101420730,
 0.01527461859678380,
 0.01267816647681590,
 0.01004755718228800,
 0.00738993116334568,
 0.00471272992695334,
 0.00202681196887338]
  n=length(r)
  z=0.0
  k1=(b-a)/2.0
  k2=(b+a)/2.0
  y1=0.0
  for i=1:n
    x=k2+k1*r[i]
    y=f_xnt(x)
    y1=c[i]*y
    z+=y1
  end
return k1*z
end

f(x) = 4.0/pi*sqrt(1.0-x*x)
r=integral(f,0.0,1.0)
print("integral  : $r")
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_integral2.jl
integral  : 1.000000837330801
> Terminated with exit code 0.
```

Matlab – Octave function for Gauss-Legendre integration, 60 points:

```
## Copyright (C) 2021 meral
## Author: M. Turhan ÇOBAN
## Created: 2021-06-13

function retval = Gauss_Legendre_integral(f,a,b)
r=[-0.999210123227436
-0.995840525118838
-0.989787895222221
-0.981067201752598
-0.969701788765052
-0.955722255839996
-0.939166276116423
-0.920078476177627
-0.898510310810046
-0.874519922646898
-0.848171984785929
-0.819537526162145
-0.788693739932264
-0.755723775306585
-0.720716513355730
-0.683766327381355
-0.644972828489477
-0.604440597048510
-0.562278900753944
-0.518601400058569
-0.473525841761707
-0.427173741583078
-0.379670056576798
```

-0.331142848268448
-0.281722937423261
-0.231543551376029
-0.180739964873425
-0.129449135396945
-0.077809333949537
-0.025959772301248
0.025959772301248
0.077809333949537
0.129449135396945
0.180739964873425
0.231543551376029
0.281722937423261
0.331142848268448
0.379670056576798
0.427173741583078
0.473525841761707
0.518601400058569
0.562278900753944
0.604440597048510
0.644972828489477
0.683766327381355
0.720716513355730
0.755723775306585
0.788693739932264
0.819537526162145
0.848171984785929
0.874519922646898
0.898510310810046
0.920078476177627
0.939166276116423
0.955722255839996
0.969701788765052
0.981067201752598
0.989787895222221
0.995840525118838
0.999210123227436
];
c=[0.002026811968873
0.004712729926953
0.007389931163346
0.010047557182288
0.012678166476816
0.015274618596784
0.017829901014207
0.020337120729457
0.022789516943998
0.025180477621521
0.027503556749925
0.029752491500789
0.031921219019296
0.034003892724946
0.035994898051084
0.037888867569243
0.039680695452381
0.041365551235585
0.042938892835936
0.044396478795787
0.045734379716114
0.046948988848912
0.048037031819971
0.048995575455757
0.049822035690550
0.050514184532509
0.051070156069856
0.051488451500981
0.051767943174910
0.051907877631221
0.051907877631221
0.051767943174910
0.051488451500981
0.051070156069856
0.050514184532509
0.049822035690550
0.048995575455757
0.048037031819971

```
0.046948988848912
0.045734379716114
0.044396478795787
0.042938892835936
0.041365551235585
0.039680695452381
0.037888867569243
0.035994898051084
0.034003892724946
0.031921219019296
0.029752491500789
0.027503556749925
0.025180477621521
0.022789516943998
0.020337120729457
0.017829901014207
0.015274618596784
0.012678166476816
0.010047557182288
0.007389931163346
0.004712729926953
0.002026811968873
];
n=length(r);
z=0;
k1=(b-a)/2.0;
k2=(b+a)/2.0;
y1=0;
for i=1:n
  x=k2+k1*r(i);
  y=f(x);
  y1=c(i)*y;
  z=z+y1;
endfor
retval =k1*z;
endfunction
```

```
>> Gauss_Legendre_integral(f,0.0,1.0)
ans = 1.000000837330801
>>
```

In the next example coeeficients are directly calculated from the Legendre polynomials, so storage of any coefficients is unnecessary. Of course it will have additional cost of calculating the coefficients.

Program variable Point Gauss-Legendre integration, Java code

```
 //=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// desired Legendre polynomials coefficients are internally calculated
// Gauss-Legendre n points integration formula
// Dr. Turhan Coban
// =====================================================
import java.io.*;
class f1 extends f_x
{
  public double func(double x)
  {
  return x*x;
  }
}
class NA61
{
public static double F1(double h,double r)
 {double H=h/r;
  double F=1.0/(H*H+1);
  return F;
 }
public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
```

```
            //calculates legendre gauss-coefficients as coefficients of the integral
            //for n terms
            double EPS=3.0e-15;
            int m,j,i;
            double z1,z,xm,xl,pp,p3,p2,p1;
   //double x[]=new double[n];
   //double w[]=new double[n];
   double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
            m=(n+1)/2;
            xm=0.5*(x2+x1);
            xl=0.5*(x2-x1);
            for (i=1;i<=m;i++)
            { z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                        do {
                                    p1=1.0;
                                    p2=0.0;
                                    for (j=1;j<=n;j++) {
                                                p3=p2;
                                                p2=p1;
                                                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                                    }
                                    pp=n*(z*p1-p2)/(z*z-1.0);
                                    z1=z;
                                    z=z1-p1/pp;
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
            }
return a;
}

public static double integral(f_x f_xnt,double x1,double x2,int n)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss legendre integral
double a[][]=new double[2][n];
a=gauss_legendre_coefficients(x1,x2,n);
System.out.println(Matrix.toString(a));
Text.print(Text.T(a));
//System.out.println(Matrix.toString(a));
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f_xnt.func(a[0][i]);
}
return z;
}

  public static void main(String args[])
  {
  //Gauss-Legendre integral of any desired point (general)
  f1 b1=new f1();

  double F12G=integral(b1,-1.0,1.0,3);
  System.out.println("F12G = "+F12G);
}
}
```

```
 ---------- Capture Output ----------
> "D:\co\java\bin\javaw.exe" NA61
      -0.774596669241483       0.000000000000000       0.774596669241483
       0.555555555555553       0.888888888888889       0.555555555555553

F12G = 0.6666666666666633
```

Program 7.3-2A variable Point Gauss-Legendre integration in Python code

```
 # Gauss - Legendre integral variable coefficients
# python version

from math import *
def f1(x):
  return x*x

def gauss_legendre_coefficients(x1,x2,n):
   #calculates legendre gauss-coefficients as coefficients of the integral
            #for n terms
            eps=3e-15
            m=int((n+1.0)/2.0)
            xm=0.5*(x2+x1)
            xl=0.5*(x2-x1)
            nmax=100
            a=[[0.0 for i in range(n)] for j in range(2)]
            for i in range(1,m+1):
                        z=cos(pi*((i-0.25)/(n+0.5)))
                        for ii in range(nmax):
                        #while abs(z-z1) > eps:
                                    p1=1.0;
                                    p2=0.0;
                                    for j in range(1,n+1):
                                       p3=p2
                                       p2=p1
                                       p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                                    pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                                    z1=z;
                                    z=z1-p1/pp
                                    if abs(z-z1) < eps: break
                        a[0][i-1]=xm-xl*z
                        a[0][n-i]=xm+xl*z
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
                        a[1][n-i]=a[1][i-1]
            return a

def integral(x1,x2,n):
  #integral f1(x)dx
  #integral of a function by using gauss-legendre quadrature
  #between x1 and x2
  a=gauss_legendre_coefficients(x1,x2,n)
  print(a)
  z=0
  for i in range(n):
    z=z+a[1][i]*f1(a[0][i])
  return z

a=-1.0#float(input("enter lower limit : "))
b=1.0#float(input("enter upper limit : "))
r=integral(a,b,3)
print("integral  : ",r)
```

```
 ---------- Capture Output ----------
> "D:\co\python\pythonw.exe" NA61.py
[[-0.7745966692414834, 0.0, 0.7745966692414834], [0.5555555555555527, 0.8888888888888888, 0.5555555555555527]]
integral  :  0.6666666666666633

> Terminated with exit code 0.
```

variable Point Gauss-Legendre integration, Julia code

```
    #calculates legendre gauss-coefficients as coefficients of the integral
   #for n terms
   function gauss_legendre_coefficients(x1,x2,n)
                A = zeros(Float64,2,n)
                eps=3.0e-15
                mx=(n+1)/2
                m=floor(Int,mx)
                xm=0.5*(x2+x1)
                xl=0.5*(x2-x1)
                nmax=20
                pp=0.0
                for i=1:m
                            z=cos(pi*((i-0.25)/(n+0.5)))
```

```
                        for ii=1:nmax
                                p1=1.0
                                p2=0.0
                                for j=1:n
                                  p3=p2
                                  p2=p1
                                  p1=(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                                end
                                pp=((z*p1-p2)*n/(z*z-1.0))
                                z1=z
                                z=z1-p1/pp
                                if abs(z-z1) < eps
                                  break
                                end
                        end
                        y1=xm-xl*z
                        y2=xm+xl*z
                        i1=n+1-i
                        A[1,i]=y1
                        A[1,i1]=y2
                        y3=2.0*xl/((1.0-z*z)*pp*pp)
                                A[2,i1]=y3
                         A[2,i]=y3
            end
            show(IOContext(stdout, :limit=>false), MIME"text/plain"(), A)
            println(" ")
            return A
  end
  #Gauss-Legendre integration
  function integral(f,x1,x2,n)
    #integral f1(x)dx
    #integral of a function by using gauss-legendre quadrature
    #between x1 and x2
    A=gauss_legendre_coefficients(x1,x2,n)
    z=0.0
    for i=1:n
      xx1=A[1,i]
      xx2=A[2,i]
      z=z+xx2*f(xx1)
    end
    return z
    end

f(x) = 4.0/pi*sqrt(1.0-x*x)
r=integral(f,0.0,1.0,10)
print("integral  : $r")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" integral_gauss.jl
2X10 Matrix{Float64}:
 0.0130467  0.0674683  0.160295  0.283302  0.425563  0.574437  0.716698  0.839705  0.932532   0.986953
 0.0333357  0.0747257  0.109543  0.134633  0.147762  0.147762  0.134633  0.109543  0.0747257  0.0333357
integral  : 1.0001612291825648
> Terminated with exit code 0.
```

```
---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" integral_gauss.jl
2Ã—50 Matrix{Float64}:
 0.000566798  0.00298402  0.00732296  0.0135678   0.0216945   0.0316717  0.0434607  0.057016   0.0722851  0.089209   0.107722
0.127753   0.149224   0.172052   0.196149   0.221421   0.247771   0.275097   0.303293  0.33225    0.361856   0.391996  0.422555
0.453413  0.484451   0.515549   0.546587  0.577445   0.608004  0.638144   0.66775    0.696707  0.724903   0.752229   0.778579
0.803851   0.827948   0.850776   0.872247   0.892278   0.910791   0.927715   0.942984   0.956539   0.968328   0.978305    0.986432
0.992677    0.997016   0.999433
 0.00145431   0.0033799   0.00529527  0.00719041  0.00905778  0.0108901  0.0126803  0.0144215  0.0161069  0.0177299  0.0192844
0.0207642  0.0221638  0.0234775  0.0247005  0.0258279  0.0268553  0.0277789  0.028595  0.0293004  0.0298925  0.030369
0.0307279  0.030968  0.0310883  0.0310883  0.030968  0.0307279  0.030369  0.0298925  0.0293004  0.028595  0.0277789  0.0268553
0.0258279  0.0247005  0.0234775  0.0221638  0.0207642  0.0192844  0.0177299  0.0161069  0.0144215  0.0126803  0.0108901
0.00905778  0.00719041  0.00529527  0.0033799  0.00145431
integral  : 1.0000014398835497
> Terminated with exit code 0.
```

Variable points Gauss-Legendre integration (coefficients calculated by using Gauss-Legendre polynomials)

```
## Copyright (C) 2017 Mustafa Turhan Çoban
## Created: 2017-08-25

function [a] = gauss_legendre_cof (x1,x2,n)
##calculates legendre gauss-coefficients as coefficients of the integral
##for n terms
eps=3e-15;
 nmax=100;
m=(n+1.0)/2.0;
xm=0.5*(x2+x1);
xl=0.5*(x2-x1);
for i=1:m
   z=cos(pi*((i-0.25)/(n+0.5)));
                          for ii=0:nmax
                             p1=1.0;
                             p2=0.0;
                             for j =1:n
                                       p3=p2;
                                       p2=p1;
                                       p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
      endfor
                            pp=n*(z*p1-p2)/(z*z-1.0);
                            z1=z;
                            z=z1-p1/pp;
     if abs(z-z1) < eps break; endif
   endfor
                          a(1,i)=xm-xl*z;
                          a(1,n+1-i)=xm+xl*z;
                          a(2,i)=2.0*xl/((1.0-z*z)*pp*pp);
                          a(2,n+1-i)=a(2,i);
   endfor
endfunction
```

```
function z=Gauss_Legendre_integral_n(f,a,b,n)
% Gauss- Legendre integration n
a=gauss_legendre_cof(a,b,n)
  z=0;
  for i =1:n
    z=z+a(2,i)*f(a(1,i));
   endfor
endfunction
```

```
>> f=@(x)4.0/pi*sqrt(1-x*x);
>> Gauss_Legendre_integral_n(f,0.0,1.0,100)
ans = 1.000000182645463
>>
```

Writing integral formulation into an abstract class and then using lambda variables are also practical in python language following code is such an example utilizing Gauss-Legendre code:
Abstract class if_x:

```
from abc import ABC, abstractmethod
from math import *

class if_x(ABC):

   @abstractmethod
   def func(self,x):
      pass;

   def dfunc(self,x,n):
      dx=0.001;
      if n==0: df=self.func(x);
      elif n==1: df=(3.0*self.func(x-4.0*dx)-32.0*self.func(x-3.0*dx)+168.0*self.func(x-2.0*dx)-672.0*self.func(x-
dx)+672.0*self.func(x+dx)-168.0*self.func(x+2.0*dx)+32.0*self.func(x+3.0*dx)-3.0*self.func(x+4.0*dx))/840.0/dx;
      elif n==2: df=(-14350.0*self.func(x)-9.0*self.func(x-4.0*dx)+128.0*self.func(x-3.0*dx)-1008.0*self.func(x-
2.0*dx)+8064.0*self.func(x-dx)+8064.0*self.func(x+dx)-1008.0*self.func(x+2.0*dx)+128.0*self.func(x+3.0*dx)-
9.0*self.func(x+4.0*dx))/5040.0/dx/dx;
      elif n==3: df=(-7.0*self.func(x-4.0*dx)+72.0*self.func(x-3.0*dx)-338.0*self.func(x-2.0*dx)+488.0*self.func(x-dx)-
488.0*self.func(x+dx)+338.0*self.func(x+2.0*dx)-72.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx;
```

```
        elif n==4: df=(2730.0*self.func(x)+7.0*self.func(x-4.0*dx)-96.0*self.func(x-3.0*dx)+676.0*self.func(x-2*dx)-1952.0*self.func(x-
dx)-1952.0*self.func(x+dx)+676.0*self.func(x+2.0*dx)-96.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx/dx;
        elif n==5: df=(self.func(x-4.0*dx)-9.0*self.func(x-3.0*dx)+26.0*self.func(x-2.0*dx)-29.0*self.func(x-dx)+29.0*self.func(x+dx)-
26.0*self.func(x+2.0*dx)+9.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/6.0/dx/dx/dx/dx/dx;
        elif n==6: df=(-150.0*self.func(x)-self.func(x-4.0*dx)+12.0*self.func(x-3.0*dx)-52.0*self.func(x-2.0*dx)+116.0*self.func(x-
dx)+116.0*self.func(x+dx)-52.0*self.func(x+2.0*dx)+12.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/4.0/dx/dx/dx/dx/dx/dx;
        elif n==7: df=(-self.func(x-4.0*dx)+6.0*self.func(x-3.0*dx)-14.0*self.func(x-2.0*dx)+14.0*self.func(x-dx)-
14.0*self.func(x+dx)+14.0*self.func(x+2.0*dx)-6.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/2.0/dx/dx/dx/dx/dx/dx/dx;
        elif n==8: df=(70.0*self.func(x)+self.func(x-4.0*dx)-8.0*self.func(x-3.0*dx)+28.0*self.func(x-2.0*dx)-56.0*self.func(x-dx)-
56.0*self.func(x+dx)+28.0*self.func(x+2.0*dx)-8.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/dx/dx/dx/dx/dx/dx/dx/dx;
        else: df=0;
        return df;

    def gauss_legendre_coefficients(self,x1,x2,n):
        #calculates legendre gauss-coefficients as coefficients of the integral
        #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

    def integral(self,x1,x2,n):
        #integral func(x)dx
        #integral of a function by using gauss-legendre quadrature
        #between x1 and x2
        a=self.gauss_legendre_coefficients(x1,x2,n)
        z=0
        for i in range(n):
            z=z+a[1][i]*self.func(a[0][i])
        return z
```

Gauss legendre example usung abstract class if_x and lambda variables, python code

```
# Gauss - Legendre integral variable coefficients
# python version integral methods are embedded to abstract class if_x

from math import *

#float(input("enter lower limit : "))
#float(input("enter upper limit : "))
class f1(if_x):func=lambda self,x:6.0*x*x-2;
f=f1();
r=f.integral(0.0,1.0,5)
print("integral  : ",r)
```

```
from abc import ABC, abstractmethod
from math import *

class if_x(ABC):

    @abstractmethod
    def func(self,x):
        pass;

    def dfunc(self,x,n):
```

```
        dx=0.0001;
        if n==0: df=self.func(x);
        elif n==1: df=(3.0*self.func(x-4.0*dx)-32.0*self.func(x-3.0*dx)+168.0*self.func(x-2.0*dx)-672.0*self.func(x-
dx)+672.0*self.func(x+dx)-168.0*self.func(x+2.0*dx)+32.0*self.func(x+3.0*dx)-3.0*self.func(x+4.0*dx))/840.0/dx;
        elif n==2: df=(-14350.0*self.func(x)-9.0*self.func(x-4.0*dx)+128.0*self.func(x-3.0*dx)-1008.0*self.func(x-
2.0*dx)+8064.0*self.func(x-dx)+8064.0*self.func(x+dx)-1008.0*self.func(x+2.0*dx)+128.0*self.func(x+3.0*dx)-
9.0*self.func(x+4.0*dx))/5040.0/dx/dx;
        elif n==3: df=(-7.0*self.func(x-4.0*dx)+72.0*self.func(x-3.0*dx)-338.0*self.func(x-2.0*dx)+488.0*self.func(x-dx)-
488.0*self.func(x+dx)+338.0*self.func(x+2.0*dx)-72.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx;
        elif n==4: df=(2730.0*self.func(x)+7.0*self.func(x-4.0*dx)-96.0*self.func(x-3.0*dx)+676.0*self.func(x-2*dx)-1952.0*self.func(x-
dx)-1952.0*self.func(x+dx)+676.0*self.func(x+2.0*dx)-96.0*self.func(x+3.0*dx)+7.0*self.func(x+4.0*dx))/240.0/dx/dx/dx/dx;
        elif n==5: df=(self.func(x-4.0*dx)-9.0*self.func(x-3.0*dx)+26.0*self.func(x-2.0*dx)-29.0*self.func(x-dx)+29.0*self.func(x+dx)-
26.0*self.func(x+2.0*dx)+9.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/6.0/dx/dx/dx/dx/dx;
        elif n==6: df=(-150.0*self.func(x)-self.func(x-4.0*dx)+12.0*self.func(x-3.0*dx)-52.0*self.func(x-2.0*dx)+116.0*self.func(x-
dx)+116.0*self.func(x+dx)-52.0*self.func(x+2.0*dx)+12.0*self.func(x+3.0*dx)-self.func(x+4.0*dx))/4.0/dx/dx/dx/dx/dx/dx;
        elif n==7: df=(-self.func(x-4.0*dx)+6.0*self.func(x-3.0*dx)-14.0*self.func(x-2.0*dx)+14.0*self.func(x-dx)-
14.0*self.func(x+dx)+14.0*self.func(x+2.0*dx)-6.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/2.0/dx/dx/dx/dx/dx/dx/dx;
        elif n==8: df=(70.0*self.func(x)+self.func(x-4.0*dx)-8.0*self.func(x-3.0*dx)+28.0*self.func(x-2.0*dx)-56.0*self.func(x-dx)-
56.0*self.func(x+dx)+28.0*self.func(x+2.0*dx)-8.0*self.func(x+3.0*dx)+self.func(x+4.0*dx))/dx/dx/dx/dx/dx/dx/dx/dx;
        else: df=0;
        return df;

    def gauss_legendre_coefficients(self,x1,x2,n):
        #calculates legendre gauss-coefficients as coefficients of the integral
        #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

    def integral(self,x1,x2,n):
        #integral func(x)dx
        #integral of a function by using gauss-legendre quadrature
        #between x1 and x2
        a=self.gauss_legendre_coefficients(x1,x2,n)
        z=0
        for i in range(n):
            z=z+a[1][i]*self.func(a[0][i])
        return z
```

Program 7.3-2B variable Point Gauss-Legendre integration in Octave(Matlab) code

```
## Copyright (C) 2017 Mustafa Turhan Çoban
## Created: 2017-08-25

function [a] = gauss_legendre_cof (x1,x2,n)
##calculates legendre gauss-coefficients as coefficients of the integral
##for n terms
eps=3e-15;
 nmax=100;
m=(n+1.0)/2.0;
xm=0.5*(x2+x1);
xl=0.5*(x2-x1);
for i=1:m
```

```
    z=cos(pi*((i-0.25)/(n+0.5)));
                        for ii=0:nmax
                          p1=1.0;
                          p2=0.0;
                          for j =1:n
                                    p3=p2;
                                  p2=p1;
                                    p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
      endfor
                          pp=n*(z*p1-p2)/(z*z-1.0);
                          z1=z;
                          z=z1-p1/pp;
     if abs(z-z1) < eps break; endif
   endfor
                        a(1,i)=xm-xl*z;
                        a(1,n+1-i)=xm+xl*z;
                        a(2,i)=2.0*xl/((1.0-z*z)*pp*pp);
                        a(2,n+1-i)=a(2,i);
   endfor
endfunction
```

```
function z=integGLn(f,a,b,n)
% Gauss- Legendre integration n
a=gauss_legendre_cof(a,b,n)
  z=0;
  for i =1:n
    z=z+a(2,i)*f(a(1,i));
   endfor
endfunction
```

```
Command Window
>>
>> f=@(x)x*x;
>> integGLn(f,-1,1,3)
a =

  -0.774596669241483   0.000000000000000   0.774596669241483
   0.555555555555553   0.888888888888889   0.555555555555553

ans =  0.666666666666663
>>
```

Program variable Point Gauss-Legendre integration in Fortran 90 code

```
program GLintegral2
IMPLICIT NONE
REAL(KIND=8) a,b
INTEGER n
a=-1
b=1
n=3
print *,'integral(',a,',',b,') = ',integral(a,b,n)

CONTAINS
  FUNCTION func(x)
  REAL(KIND=8) func,x,PI
  func=x*x
  END FUNCTION func

  FUNCTION integral(x1,x2,n)
  REAL(KIND=8) integral,EPS,PI,x1,x2,z1,z,xm,xl,pp,p3,p2,p1
  INTEGER n,m,i,j
  REAL(KIND=8), DIMENSION(0:n) :: a0
  REAL(KIND=8), DIMENSION(0:n) :: a1
  PI=4.0*atan(1.0)
  EPS=3.0e-15
  m=(n+1)/2.0
  xm=0.5*(x2+x1)
  xl=0.5*(x2-x1)
  DO i=1,m
    z=cos(PI*((i-0.25)/(n+0.5)))
    DO
      p1=1
```

```
      p2=0
      DO j=1,n
        p3=p2
        p2=p1
        p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j
      END DO
      pp=n*(z*p1-p2)/(z*z-1.0)
                z1=z
                z=z1-p1/pp
    IF(abs(z-z1)<EPS) EXIT
    END DO
    a0(i-1)=xm-xl*z
    a0(n-i)=xm+xl*z
    a1(i-1)=2.0*xl/((1.0-z*z)*pp*pp)
              a1(n-i)=a1(i-1);
  END DO
  DO i=0,n-1
    z=z+a1(i)*func(a0(i))
  END DO
  integral=z
  END FUNCTION integral
end program GLintegral2
```

```
---------- Capture Output ----------
> "E:\okul\SCO1\run.bat" GLintegral2

E:\okul\SCO1>GLintegral2
 integral(  -1.0000000000000000     ,   1.0000000000000000     ) =   0.66666666666666330

> Terminated with exit code 0.
```

In the Gauss-Legendre integral formulation weight function was taken as 1.

$$I_w(-1,1) = \int_{-1}^{1} w(x)f(x)dx \cong \sum_{k=1}^{n} c_k f(x_k)$$

If the weight function is taken as:

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

Gauss-Chebychev integration formula is created. This equation has the form of

$$I_w(-1,1) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx \cong \sum_{k=1}^{n} c_k f(x_k)$$

x_k in the equation is the roots of chebcychev polynomials. General definition of the Chbychev polynomials are given as:

$P_0(x)=1$

$P_1(x)=x$

$P_2(x)=2x^2-1$

$P_n(x)=2xP_{n-1}(x) – P_{n-2}(x)$, n=3,4,5..

It should be note that Chebychev polinomials are orthogonal polnomials. The roots of the polynomials are defined as:

$$x_k = cos\left(\frac{\left(k+\frac{1}{2}\right)\pi}{n}\right), \quad k = 0,1,2,..n-1$$

Coeeficients of the Formula can be calculated as:

$$c_k = \frac{\pi}{n}, \qquad k = 0,1,2,..n-1$$

x=[-1,1] integration range in Gauss-Chebychev Formula can be change to more general z=[a,b] range by using similar formulations used in Gauss-Legendre formulations.

As an example program if following integral is to be solved:

$$I = \int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}} dx$$

Program 7.3-3 Gauss-Chebychev integral, java code

```
import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;

class f1 extends f_x
{
  public double func(double x)
  {
  return Math.exp(x);
  }
}

public class integ_gauss_chebychev
{

public static double[][] gauss_chebychev_coefficients(int n)
{   //Gauss-Chebychev quadratürü katsayilarini hesaplar
    double sgngam=1;
    double a[][]=new double[2][n];
    for(int j = 1; j <=n; j++)
    {int k=j-1;
     a[0][k]=Math.cos((2.0*j-1.0)*Math.PI/(2.0*n));
     a[1][k]=Math.PI/(double)n;
    }
return a;
}

public static double gauss_chebychev_integral(f_x f_xnt,int n)
{
//n : number of integral coefficients
// this routine first generates gauss chebychev coefficients
// for [xa,xb] band
// then calculates gauss chebychev integral
double a[][]=new double[2][n];
a=gauss_chebychev_coefficients(n);
//System.out.println(Matrix.toString(a));

double toplam=0;
double ck;
double x,y;
double z1=0;
for(int k=0;k<n;k++)
{ toplam+=a[1][k]*f_xnt.func(a[0][k]); }
return toplam;
}

     public static void main(String[] args)
     {
               f1 b1=new f1();
     System.out.println("integral  : "+gauss_chebychev_integral(b1,50));
     }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA62
```

```
integral  : 3.9774632605064237

> Terminated with exit code 0.
```

Program Gauss-Chebychev integral, julia code

```
 function  gauss_chebychev_coefficients(n)
  #Gauss-Chebychev quadrature
   sgngam=1
   a=zeros(Float64,2,n)
   for j=1:n
     a[1,j]=cos((2.0*j-1.0)*pi/(2.0*n));
     a[2,j]=pi/n
   end
   return a
   end

function gauss_chebychev_integral(f_xnt,n)
#n : number of integral coefficients
# this routine first generates gauss chebychev coefficients
# for [xa,xb] band
# then calculates gauss chebychev integral
a=zeros(Float64,2,n)
a=gauss_chebychev_coefficients(n)
toplam=0.0
#ck
#x,y
z1=0.0
for k=1:n
  toplam+=a[2,k]*f_xnt(a[1,k])
  end
return toplam
end

f(x) = exp(x)
r=gauss_chebychev_integral(f,50)
print("gauss_chebychev_integral  : $r")
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" integral_gauss_chebychev.jl
gauss_chebychev_integral  : 3.9774632605064237
> Terminated with exit code 0.
```

Gauss-Jacobi (Mehler) integration Formula is in the form of

$$I_w(-1,1) = \int_{-1}^{1} w(x)f(x)dx = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}f(x)dx = \cong \sum_{k=0}^{n-1} c_k f(x_k)$$

In this equation x_k is the root of jacobi polynomials Jacobi polynomials are in the form of

$$J_n(x;\alpha,\beta) = \sum_{i=0}^{n} A_n x^n$$

Coefficients are defined as:

$$A_n = \frac{1}{2n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}$$

Gauss-Jacobi integration equation coefficients, c_k's are defined as :

$$c_k = -\frac{(2n+\alpha+\beta+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)!}\frac{2^{\alpha+\beta}}{J_n(x_k;\alpha,\beta)J_{n+1}(x_k;\alpha,\beta)}$$

In this equation Γ is the gamma function. One of the special form of equation is when α=0 and β=0. It is given Gauss_legendre equation. The second one is when α=-0.5 and β=-0.5. It is given the Gauss-Chebychev formulation. As it seen Gauss-Jacobi equation set is a kind of general solution with different possible solutions depends on α and β coefficients. In our example problem, the previous problem defined will be solved by using Gauss-Legendre polynomial.

$$I = \int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}}dx$$

Program variable point Gauss-Jacobi integration with α=0 and β=0 (Gauss-Legendre)

```
import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;

class f1 extends f_x
{
  public double func(double x)
  {
  return Math.exp(x)/Math.sqrt(1-x*x);
  }
}

public class integ_gauss_jacobi
{

   public static double [] lngamma(double x)
   {//logaritmik hata fonksiyonu
   //xx[0] hata fonksiyonunun logaritmasinin değerini
   //xx[1] hata fonksiyonunun kendisinin işaretini verir
            double xx[]=new double[2];
   double result;
   double a;
   double b;
   double c;
   double p;
   double q;
   double u;
   double w;
   double z;
   int i;
   double logpi;
   double ls2pi;
   double tmp;
   double sgngam = 1;
   logpi = 1.14472988584940017414;
   ls2pi = 0.91893853320467274178;
   if( x<-34.0 )
   {
      q = -x;
      double yy[]=lngamma(q);
      w = yy[0];
      tmp=yy[1];
      p = Math.floor(q);
      i = (int)Math.round(p);
      if( i%2==0 )
      {
         sgngam = -1;
```

```
    }
    else
    {
       sgngam = 1;
    }
    z = q-p;
    if( z>0.5 )
    {
       p = p+1;
       z = p-q;
    }
    z = q*Math.sin(Math.PI*z);
    result = logpi-Math.log(z)-w;
    xx[0]=result;
    xx[1]=sgngam;
    return xx;
 }
 if( x<13 )
 {
    z = 1;
    p = 0;
    u = x;
    while(u>=3)
    {
       p = p-1;
       u = x+p;
       z = z*u;
    }
    while(u<2)
    {
       z = z/u;
       p = p+1;
       u = x+p;
    }
    if( z<0 )
    {
       sgngam = -1;
       z = -z;
    }
    else
    {
       sgngam = 1;
    }
    if( u==2 )
    {
       result = Math.log(z);
       //return result;
       xx[0]=result;
       xx[1]=sgngam;
       return xx;
    }
    p = p-2;
    x = x+p;
    b = -1378.25152569120859100;
    b = -38801.6315134637840924+x*b;
    b = -331612.992738871184744+x*b;
    b = -1162370.97492762307383+x*b;
    b = -1721737.00820839662146+x*b;
    b = -853555.664245765465627+x*b;
    c = 1;
    c = -351.815701436523470549+x*c;
    c = -17064.2106651881159223+x*c;
```

```
        c = -220528.590553854454839+x*c;
        c = -1139334.44367982507207+x*c;
        c = -2532523.07177582951285+x*c;
        c = -2018891.41433532773231+x*c;
        p = x*b/c;
        result = Math.log(z)+p;
        //return result;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
    }
    q = (x-0.5)*Math.log(x)-x+ls2pi;
    if( x>100000000 )
    {
        result = q;
        //return result;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
    }
    p = 1/(x*x);
    if( x>=1000.0 )
    {
        q = q+((7.9365079365079365079365*0.0001*p-
2.777777777777777777778*0.001)*p+0.083333333333333333333)/x;
    }
    else
    {
        a = 8.11614167470508450300*0.0001;
        a = -5.95061904284301438324*0.0001+p*a;
        a = 7.9365034045771694394 5*0.0001+p*a;
        a = -2.77777777730099687205*0.001+p*a;
        a = 8.33333333333331927722*0.01+p*a;
        q = q+a/x;
    }
    xx[0]= q;
    xx[1]=sgngam;
    return xx;
    }
//====================

    public static double[][] gauss_jacobi_coefficients(int n,double alpha,double beta)
    {
        double eps=1.0e-30;
                int i = 0;
        int j = 0;
        double r = 0;
        double r1 = 0;
        double t1 = 0;
        double t2 = 0;
        double t3 = 0;
        double p1 = 0;
        double p2 = 0;
        double p3 = 0;
        double pp = 0;
        double an = 0;
        double bn = 0;
        double a = 0;
        double b = 0;
        double c = 0;
        double tmpsgn = 0;
        double alfbet = 0;
```

```
double temp = 0;
int its = 0;
double A[][]=new double[2][n];
for(i=0; i<=n-1; i++)
{   if( i==0 )
    {
       an = alpha/n;
       bn = beta/n;
       t1 = (1+alpha)*(2.78/(4+n*n)+0.768*an/n);
       t2 = 1+1.48*an+0.96*bn+0.452*an*an+0.83*an*bn;
       r = (t2-t1)/t2;
    }
    else
    {
       if( i==1 )
       {
          t1 = (4.1+alpha)/((1+alpha)*(1+0.156*alpha));
          t2 = 1+0.06*(n-8)*(1+0.12*alpha)/n;
          t3 = 1+0.012*beta*(1+0.25*Math.abs(alpha))/n;
          r = r-t1*t2*t3*(1-r);
       }
       else
       {
          if( i==2 )
          {
             t1 = (1.67+0.28*alpha)/(1+0.37*alpha);
             t2 = 1+0.22*(n-8)/n;
             t3 = 1+8*beta/((6.28+beta)*n*n);
             r = r-t1*t2*t3*(A[0][0]-r);
          }
          else
          {
             if( i<n-2 )
             {
                r = 3*A[0][i-1]-3*A[0][i-2]+A[0][i-3];
             }
             else
             {
                if( i==n-2 )
                {
                   t1 = (1+0.235*beta)/(0.766+0.119*beta);
                   t2 = 1/(1+0.639*(n-4)/(1+0.71*(n-4)));
                   t3 = 1/(1+20*alpha/((7.5+alpha)*n*n));
                   r = r+t1*t2*t3*(r-A[0][i-2]);
                }
                else
                {
                   if( i==n-1 )
                   {
                      t1 = (1+0.37*beta)/(1.67+0.28*beta);
                      t2 = 1/(1+0.22*(n-8)/n);
                      t3 = 1/(1+8*alpha/((6.28+alpha)*n*n));
                      r = r+t1*t2*t3*(r-A[0][i-2]);
                   }
                }
             }
          }
       }
    }
    alfbet = alpha+beta;
    do
    {
```

```
            temp = 2+alfbet;
            p1 = (alpha-beta+temp*r)*0.5;
            p2 = 1;
            for(j=2; j<=n; j++)
            {
               p3 = p2;
               p2 = p1;
               temp = 2*j+alfbet;
               a = 2*j*(j+alfbet)*(temp-2);
               b = (temp-1)*(alpha*alpha-beta*beta+temp*(temp-2)*r);
               c = 2*(j-1+alpha)*(j-1+beta)*temp;
               p1 = (b*p2-c*p3)/a;
            }
            pp = (n*(alpha-beta-temp*r)*p1+2*(n+alpha)*(n+beta)*p2)/(temp*(1-r*r));
            r1 = r;
            r = r1-p1/pp;
         }
         while( Math.abs(r-r1)>=eps*(1+Math.abs(r))*100 );
         A[0][i] = r;
         double yy[]=lngamma(alpha+n);
         double w1 = yy[0];
         tmpsgn=yy[1];
         yy=lngamma(beta+n);
         double w2 = yy[0];
         tmpsgn=yy[1];
         yy=lngamma(n+1);
         double w3 = yy[0];
         tmpsgn=yy[1];
         yy=lngamma(n+alfbet+1);
         double w4 = yy[0];
         tmpsgn=yy[1];
         A[1][i] = Math.exp(w1+w2-w3-w4)*temp*Math.pow(2, alfbet)/(pp*p2);
      }
   return A;
   }


public static double gauss_jacobi_integral(f_x f_xnt,int n,double alpha,double beta)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss hermite integral
double a[][]=new double[2][n];
a=gauss_jacobi_coefficients(n,alpha,beta);
System.out.println(Matrix.toString(a));
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f_xnt.func(a[0][i]);}
return z;
}

     public static void main(String[] args)
     {
               f1 b1=new f1();
     System.out.println(gauss_jacobi_integral(b1,5,0.0,0.0));
     }
}
```

Gauss - Jacobi integration with n=5 points

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA63
```

```
    0.906179845938664    0.538469310105683    0.000000000000000    -0.538469310105683    -0.906179845938664
    0.236926885056189    0.478628670499367    0.568888888888889    0.478628670499367     0.236926885056189

3.4865363609790476

> Terminated with exit code 0.
```

As it is seen Gauss-Legendre coefficients are obtained here correctly. Since n = 5 only accuracy is not very high. If it is solved with more coefficients :

Gauss - Jacobi integration with n=20 points α=0 and β=0

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA63

3.846342341165288

> Terminated with exit code 0.
```

Gauss - Jacobi integration with n=50 points α=0 and β=0

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA63
n=50 integral = 3.9242400049441444

> Terminated with exit code 0.
```

Gauss - Jacobi integration with n=100 points α=0 and β=0

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA63
n=100 integral = 3.9506992376105177

> Terminated with exit code 0.
```

If we change function and parameter to solve the same program as α=-0.5 and β=-0.5 (Gauss-Chebychev formula)

Program 7.3-5 variable point Gauss-Jacobi integration with α=-0.5 and β=-0.5 (Gauss-Legendre)

```
class f1 extends f_x
{
  public double func(double x)
  {  return Math.exp(x); }
}
……
System.out.println("n="+n+" integral = "+gauss_jacobi_integral(b1,n,-0.5,-0.5));
……
```

10 point variable point Gauss-Jacobi integration with α=-0.5 and β=-0.5 (Gauss-Legendre)

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA63
n=10 integral = 3.97746265002148

> Terminated with exit code 0.
```

Gauss-Hermit formula : this formulation uses the weight function and calculates integrals from minus infinity to plus infinity.

$$w(x) = e^{-x^2}$$

$$I_w(-1,1) = \int_{-\infty}^{\infty} w(x)f(x)dx = \int_{-\infty}^{\infty} e^{-x^2}f(x)dx = \cong \sum_{k=0}^{n-1} c_k f(x_k)$$

In this equation x_k is the roots of the Hermite polynomials. Hermit Polinomials defined as:
$P_{-1}(x)=0$
$P_0(x)=1$
$P_1(x)=2x$
$P_2(x)=4x^2-2$
$P_3(x)=8x^3-12x$
$P_n(x)=2xP_{n-1}(x) – 2(n-1)P_{n-2}(x)$, n=1,2,3,4,5..

Coefficients are given as:

$$c_k = \frac{2^{n-1}n!\sqrt{\pi}}{n^2[P_{n-1}(x_k)]^2}$$

Coefficients for the first 7 sets are given in the following table

Table Gauss –Hermit integration roots and coefficients

N	x_k	c_k
2	**+/-0.707167811**	**0.886226926**
3	**0**	**1.181635901**
	+/- 1.224744871391589	**0.295408975**
4	**+/- 0.524647623275290**	**0.80491409**
	+/- 1.650680123885784	**0.081312835**
5	**0**	**0.945308721**
	+/-0.9585724646	**0.393619323**
	+/-2.0201828705	**0.019953242**
6	**+/- 2.350604973674492**	**0.00453001**
	+/- 1.335849074013697	**0.15706732**
	+/- 0.436077411927616	**0.724629595**
7	**+/- 2.651961356835233**	**0.000971781**
	+/- 1.673551628767471	**0.054515583**
	+/- 0.816287882858965	**0.425607253**
	0	**0.810264618**

Gauss_Hermite integral equation example code :

Program Gauss-Hermite integration

```
import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;

class f1 extends f_x
{
  public double func(double x)
  {
  return 1/(1+x*x);
  }
}

public class NA64
{

public static double[][] gauss_hermite_coefficients(int n)
```

```
{
   double a[][]=new double[2][n];
   int i;
   int j;
   double r=0;
   double r1;
   double p1;
   double p2;
   double p3;
   double dp3;
   double pipm4;
   double eps=1e-10;
   pipm4 = Math.pow(Math.PI, -0.25);
   for(i = 0; i <= (n+1)/2-1; i++)
   {
      if( i==0 )
      { r =Math.sqrt((double)(2*n+1))-1.85575*Math.pow((double)(2*n+1), -1.0/6.0);}
      else if( i==1 )
         { r = r-1.14*Math.pow((double)(n), 0.426)/r; }
      else if( i==2 )
            { r = 1.86*r-0.86*a[0][0];}
      else if( i==3 )
            { r = 1.91*r-0.91*a[0][1];}
      else
            {r = 2*r-a[0][i-2];}
   do {
         p2 = 0;
         p3 = pipm4;
         for(j = 0; j <= n-1; j++)
         {   p1 = p2;
            p2 = p3;
            p3 = p2*r*Math.sqrt(2.0/(double)(j+1))-p1*Math.sqrt((double)(j)/(double)(j+1));
         }
         dp3 = Math.sqrt((double)(2.0*j))*p2;
         r1 = r;
         r = r-p3/dp3;
      }
      while(Math.abs(r-r1)>=eps*(1+Math.abs(r))*100.0);
   a[0][i] = r;
   a[1][i] = 2.0/(dp3*dp3);
   a[0][n-1-i] = -a[0][i];
   a[1][n-1-i] = a[1][i];
  }
  return a;
}

public static double gauss_hermite_integral(f_x f_xnt,int n)
{
//n : number of integral coefficients
// this routine first generates gauss hermite coefficients
// for [x1,x2] band
// then calculates gauss hermite integral
double a[][]=new double[2][n];
a=gauss_hermite_coefficients(n);
System.out.println(Matrix.toString(a));
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f_xnt.func(a[0][i])*Math.exp(a[0][i]*a[0][i]);}
return z;
}

     public static void main(String[] args)
     {
               f1 b1=new f1();
     System.out.println("integral  : "+gauss_hermite_integral(b1,20));
     }
}
```

Output Gauss-Hermite integration

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA64
      5.387480890011233      4.603682449550744      3.944764040115625      3.347854567383216      2.788806058428131
2.254974002089276      1.738537712116586      1.234076215395323      0.737473728545394      0.245340708300901      -
0.245340708300901      -0.737473728545394      -1.234076215395323      -1.738537712116586      -2.254974002089276      -
2.788806058428131      -3.347854567383216      -3.944764040115625      -4.603682449550744      -5.387480890011233
      0.000000000000223      0.000000000439934      0.000000108606937      0.000007802556261      0.000228338599065
0.003243773339370      0.024810520887341      0.109017206020022      0.286675503681034      0.462243669596553
0.462243669596553      0.286675503681034      0.109017206020022      0.024810520887341      0.003243773339370
0.000228338599065      0.000007802556261      0.000000108606937      0.000000000439934      0.000000000000223

integral  : 2.804683301721163

> Terminated with exit code 0.
```

Integration function

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Gauss-Hermite integration , python code

```
from math import *
from if_x import *


class f1(if_x):
  def func(self,x):
      y1=self.w(x)*1.0/(1+x*x)
      return y1;
  def w(self,x):
      w1=exp(x*x)
      return w1
def gauss_hermite_coefficients(n):
    a=[[0.0 for i in range(n)] for j in range(2)]
    r=0
    eps=1e-10;
    pipm4 = pow(pi, -0.25)
    nn=int((n+2)/2)
    for i in range(nn):
        if i==0:
            r =sqrt(float(2.0*n+1.0))-1.85575*pow(float(2.0*n+1.0), -1.0/6.0)
        elif i==1:
            r = r-1.14*pow(float(n), 0.426)/r
        elif i==2:
            r = 1.86*r-0.86*a[0][0]
        elif i==3:
            r = 1.91*r-0.91*a[0][1]
        else:
            r = 2*r-a[0][i-2]
        p2 = 0.0
        p3 = pipm4
        j1=0
        for j in range(n):
            p1 = p2
            p2 = p3
            p3 = p2*r*sqrt(2.0/float(j+1))-p1*sqrt(float(j)/float(j+1));
            j1=j
        j=j1+1
        dp3 = sqrt(float(2.0*j))*p2
        r1=r
        r = r-p3/dp3
        while abs(r-r1)>=eps*(1+abs(r))*100.0:
            p2 = 0.0
            p3 = pipm4
            for j in range(n):
                p1 = p2
                p2 = p3
                p3 = p2*r*sqrt(2.0/float(j+1))-p1*sqrt(float(j)/float(j+1));
                j1=j
            j=j1+1
            dp3 = sqrt(float(2.0*j))*p2
            r1=r
            r = r-p3/dp3
```

```
        a[0][i] = r
        a[1][i] = 2.0/(dp3*dp3);
        a[0][n-1-i] = -a[0][i];
        a[1][n-1-i] = a[1][i];
    return a

def gauss_hermite_integral(f_xnt,n):
    # n : number of integral coefficients
    # this routine first generates gauss hermite coefficients
    # for [x1,x2] band
    # then calculates gauss hermite integral
    a=[[0.0 for i in range(n)] for j in range(2)]
    a=gauss_hermite_coefficients(n)
    print("a=\n",a)
    z=0
    x1=0
    for i in range(n):
        x1=f_xnt.func(a[0][i])
        z=z+a[1][i]*x1
    return z


f=f1()
print("integral  : ",gauss_hermite_integral(f,7));
```

```
runfile('E:/okul/SCO1/gauss_hermite_integral.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
a=
 [[2.6519613568352334, 1.6735516287674717, 0.8162878828589646, -0.0, -0.8162878828589646, -1.6735516287674717, -
2.6519613568352334], [0.0009717812123401424, 0.05451557982996558, 0.42560725261012705, 0.8102646175568071,
0.42560725261012705, 0.05451557982996558, 0.0009717812123401424]]
integral  :  2.5512038801916415

In [81]:
```

```
f=f1()
print("integral  : ",gauss_hermite_integral(f,20));
```

```
runfile('E:/okul/SCO1/gauss_hermite_integral.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
a=
 [[5.387480890011233, 4.603682449550744, 3.944764040115625, 3.3478545673832163, 2.788806058428131, 2.2549740020892757,
1.7385377121165861, 1.234076215395323, 0.7374737285453943, 0.24534070830090127, -0.24534070830090127, -
0.7374737285453943, -1.234076215395323, -1.7385377121165861, -2.2549740020892757, -2.788806058428131, -
3.3478545673832163, -3.944764040115625, -4.603682449550744, -5.387480890011233], [2.2293936242484147e-13,
4.3993406240405176e-10, 1.0860693697609841e-07, 7.802556261260467e-06, 0.0002283859906453842, 0.0032437733393699776,
0.02481052088734086, 0.10901720602002156, 0.28667550368103384, 0.46224366960058466, 0.46224366960058466,
0.28667550368103384, 0.10901720602002156, 0.02481052088734086, 0.0032437733393699776, 0.0002283859906453842,
7.802556261260467e-06, 1.0860693697609841e-07, 4.3993406240405176e-10, 2.2293936242484147e-13]]
integral  :  2.80468330172924
```

```
f=f1()
print("integral  : ",gauss_hermite_integral(f,20));
```

```
runfile('E:/okul/SCO1/gauss_hermite_integral.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
a=
 [[13.40648733814491, 12.823799749487808, 12.342964222859674, 11.915061943114166, 11.521415400787031,
11.152404385585125, 10.802260753684715, 10.467185421342883, 10.144509941292847, 9.832269807777969, 9.528965823390115,
9.233420890219161, 8.944689217325475, 8.661996168134518, 8.384696940416266, 8.112247311162792, 7.844182384460821,
7.580100807857489, 7.319652822304577, 7.062531060248881, 6.808463352858802, 6.557207031921542, 6.308544361112136,
6.062278832614303, 5.818232135203517, 5.5762416493299245, 5.33615836013836, 5.097845105089136, 4.86117509179121,
4.6260306357871555, 4.392302078682684, 4.15988685513103, 3.928688683427671, 3.698616859318492, 3.4695856364185893,
3.241513679631013, 3.0143235803311557, 2.7879414239819895, 2.562296402372608, 2.3373204639068783, 2.112947996371188,
1.8891155374270083, 1.6657615087415094, 1.4428259702159327, 1.220250391218953, 0.9979774360981053, 0.7759507615401458,
0.554114823591617, 0.3324146923422318, 0.1107958724224395, -0.1107958724224395, -0.3324146923422318, -0.554114823591617,
-0.7759507615401458, -0.9979774360981053, -1.220250391218953, -1.4428259702159327, -1.6657615087415094, -
1.8891155374270083, -2.112947996371188, -2.3373204639068783, -2.562296402372608, -2.7879414239819895, -
3.0143235803311557, -3.241513679631013, -3.4695856364185893, -3.698616859318492, -3.928688683427671, -4.15988685513103, -
4.392302078682684, -4.6260306357871555, -4.86117509179121, -5.097845105089136, -5.33615836013836, -5.5762416493299245, -
5.818232135203517, -6.062278832614303, -6.308544361112136, -6.557207031921542, -6.808463352858802, -7.062531060248881, -
7.319652822304577, -7.580100807857489, -7.844182384460821, -8.112247311162792, -8.384696940416266, -8.661996168134518, -
8.944689217325475, -9.233420890219161, -9.528965823390115, -9.832269807777969, -10.144509941292847, -10.467185421342883,
-10.802260753684715, -11.152404385585125, -11.521415400787031, -11.915061943114166, -12.342964222859674, -
12.823799749487808, -13.40648733814491], [5.908067863103223e-79, 1.9728604274119031e-72, 3.083028963828961e-67,
```

9.019220432276765e-63, 8.518883076551092e-59, 3.459477733807211e-55, 7.191529461537315e-52, 8.597534439303944e-49,
6.42072208009955e-46, 3.185217604477932e-43, 1.100470663095527e-40, 2.748784872798951e-38, 5.1162325987743434e-36,
7.274572594631579e-34, 8.06743427795364e-32, 7.101812226172967e-30, 5.037791166163028e-28, 2.917350072619246e-26,
1.3948384373378411e-24, 5.561019523433332e-23, 1.8649961539456107e-21, 5.302313526665779e-20, 1.286832523068123e-18,
2.682491650786072e-17, 4.829834747689982e-16, 7.548896321650472e-15, 1.0288748904462765e-13, 1.2278784799174774e-12,
1.2879038035689307e-11, 1.191300622469033e-10, 9.74792119251736e-10, 7.075857257415599e-09, 4.568127498474821e-08,
2.629097480438831e-07, 1.3517971581501169e-06, 6.221524815349301e-06, 2.5676159379201832e-05, 9.517162777530007e-05,
0.00031729197102648603, 0.0009526921885249078, 0.002579273260030733 5, 0.006303000285579659, 0.013915665220208456,
0.027779127385918166, 0.05017581267742089, 0.08205182739122158, 0.12153798684410347, 0.16313003050278288,
0.19846284859264707, 0.21889262957968678, 0.21889262957968678, 0.19846284859264707, 0.16313003050278288,
0.12153798684410347, 0.08205182739122158, 0.05017581267742089, 0.027779127385918166, 0.013915665220208456,
0.006303000285579659, 0.002579273260030733 5, 0.0009526921885249078, 0.00031729197102648603, 9.517162777530007e-05,
2.5676159379201832e-05, 6.221524815349301e-06, 1.3517971581501169e-06, 2.629097480438831e-07, 4.568127498474821e-08,
7.075857257415599e-09, 9.74792119251736e-10, 1.191300622469033e-10, 1.2879038035689307e-11, 1.2278784799174774e-12,
1.0288748904462765e-13, 7.548896321650472e-15, 4.829834747689982e-16, 2.682491650786072e-17, 1.286832523068123e-18,
5.302313526665779e-20, 1.8649961539456107e-21, 5.561019523433332e-23, 1.3948384373378411e-24, 2.917350072619246e-26,
5.037791166163028e-28, 7.101812226172967e-30, 8.06743427795364e-32, 7.274572594631579e-34, 5.1162325987743434e-36,
2.748784872798951e-38, 1.100470663095527e-40, 3.185217604477932e-43, 6.42072208009955e-46, 8.597534439303944e-49,
7.191529461537315e-52, 3.459477733807211e-55, 8.518883076551092e-59, 9.019220432276765e-63, 3.083028963828961e-67,
1.9728604274119031e-72, 5.908067863103223e-79]]
integral : 2.9965945892152823

Gauss-Hermite integration, Julia code

```
 #Gauss- Hermit integral
function gauss_hermite_coefficients(n)
   A=OffsetArray{Float64}(undef,0:1,0:n-1)
   r=0.0
   r1=0.0
   p1=0.0
   p2=0.0
   p3=0.0
   dp3=0.0
   j=0
   i=Int64(0)
   eps=1e-15
   pipm4 =1.0/pi^0.25
   n1=floor(Int64,(n+1)/2-1)
   for i = 0:n1
      if i==0
        r =sqrt(Float64(2*n+1.0))-1.85575*(Float64(2*n+1))^(-1.0/6.0)
      elseif i==1
        r = r-1.14*Float64(n)^0.426/r
      elseif i==2
        r = 1.86*r-0.86*A[0,0]
      elseif i==3
        r = 1.91*r-0.91*A[0,1]
      else
        r = 2*r-A[0,i-2]
      end # if
      while(true)
         p2 = 0.0
         p3 = pipm4;
         for j = 0:n-1
            p1 = p2
            p2 = p3
            p3 = p2*r*sqrt(2.0/Float64(j+1))-p1*sqrt(Float64(j)/Float64(j+1))
            end # for j
         dp3 = sqrt(Float64(2.0*n))*p2
         r1 = r
         r = r-p3/dp3
         yy1=abs(r-r1)
         yy2=eps*(1.0+abs(r))*100.0
         if yy1<yy2
           break
         end
         end #while
   y1=r
   y2=2.0/(dp3*dp3)
   A[0,i] = y1
   A[1,i] = y2
   A[0,n-1-i] = -y1
```

```
   A[1,n-1-i] = y2
  end # for i
  return A
  end #function

function gauss_hermite_integral(f_xnt,n)
# n : number of integral coefficients
# this routine first generates gauss hermite coefficients
# for [x1,x2] band
# then calculates gauss hermite integral
# a=OffsetArray{Float64}(undef,0:1,0:n-1)
a=gauss_hermite_coefficients(n)
 show(IOContext(stdout, :limit=>false), MIME"text/plain"(), a)
 println(" ")

z=0
for i=0:n-1
 z+=a[1,i]*f_xnt(a[0,i])*exp(a[0,i]*a[0,i])
 end # for
return z
end # function



using OffsetArrays
f(x)=1.0/(1.0+x*x)
I=gauss_hermite_integral(f,20)
print("I = $I")
```

Gauss-Hermite integration, julia code

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_hermite_integral.jl
2X20 OffsetArray(::
Matrix{Float64}, 0:1, 0:19) with eltype Float64 with indices 0:1X0:19:
 5.38748     4.60368     3.94476     3.34785     2.78881     2.25497     1.73854     1.23408   0.737474  0.245341  -0.245341  -0.737474  -
1.23408   -1.73854   -2.25497   -2.78881   -3.34785   -3.94476   -4.60368   -5.38748
 2.22939e-13  4.39934e-10  1.08607e-7  7.80256e-6  0.000228339  0.00324377  0.0248105  0.109017  0.286676  0.462244  0.462244
0.286676  0.109017  0.0248105  0.00324377  0.000228339  7.80256e-6  1.08607e-7  4.39934e-10  2.22939e-13
I = 2.804683334304932
> Terminated with exit code 0.
```

Gauss-Laguerre formulation has the weight factor in the form of

$$w(x) = x^{\alpha}e^{-x}$$

The integral limits of this formulation is in between o and infinity. The equation is in the form of

$$I_w(0,\infty) = \int_0^{\infty} f(x)w(x)dx = \int_0^{\infty} x^{\alpha}e^{-x}f(x)dx \cong \sum_{k=0}^{n-1} c_k f(x_k)$$

In this equation x_k is the root of Laguerre polinomials. Leguerre polynomial has the form of:

$P_{-1}(x)=0$

$P_0(x)=1$

$(n+1)P^{\alpha}_{n+1}(x)=(-x+2*n+\alpha+1)P^{\alpha}_{n}(x) - (n+\alpha)P^{\alpha}_{n-1}(x)$, n=1,2,3,4,5...

Coefficients,

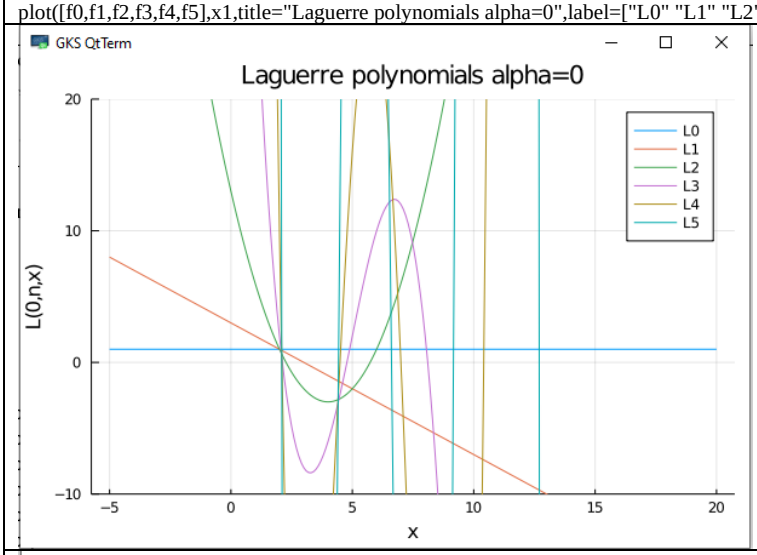
$$c_k = \frac{1}{x_k\left[\frac{dP_n(x_k)}{dx_k}\right]^2}$$

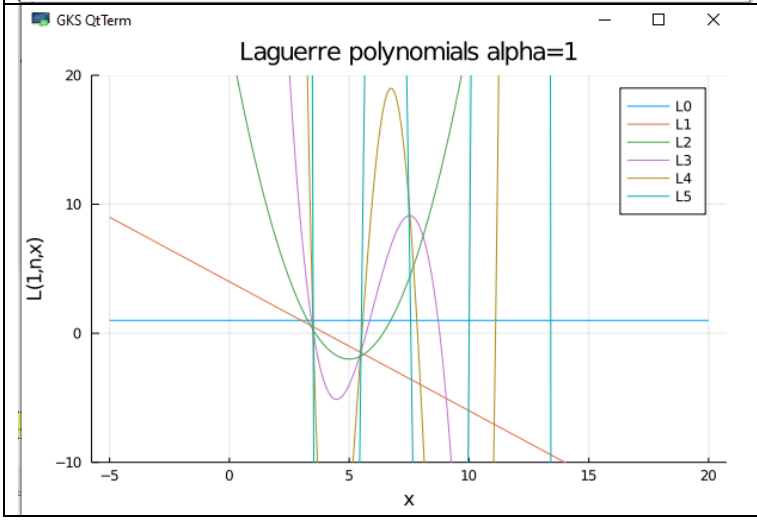
For the first 5 values, roots and coefficient of the equation is given in **Table 8.4-3** (α=0), **Table 8.4-4** (α=1) and **Table 8.4-5** (α=2) .

Gauss-Lequarre polynomials, julia code

```
#Laguerre polynomials
function L(alpha,n,x)
```

```
    P1=0.0
    if n==-1 P1=0.0
    elseif n==0 P1=1.0
    else
      P1=(-x+2.0*n+alpha+1.0)*L(alpha,n-1,x)-(n-alpha)*L(alpha,n-2,x)
    end
    return P1
    end

using Plots
x1=-5.0:0.1:20.0
alpha=0.0;
f0(x)=L(alpha,0,x)
f1(x)=L(alpha,1,x)
f2(x)=L(alpha,2,x)
f3(x)=L(alpha,3,x)
f4(x)=L(alpha,4,x)
f5(x)=L(alpha,5,x)
plot([f0,f1,f2,f3,f4,f5],x1,title="Laguerre polynomials alpha=0",label=["L0" "L1" "L2" "L3" "L4" "L5"],ylims = (-10.0,20.0),xaxis = ("x
```





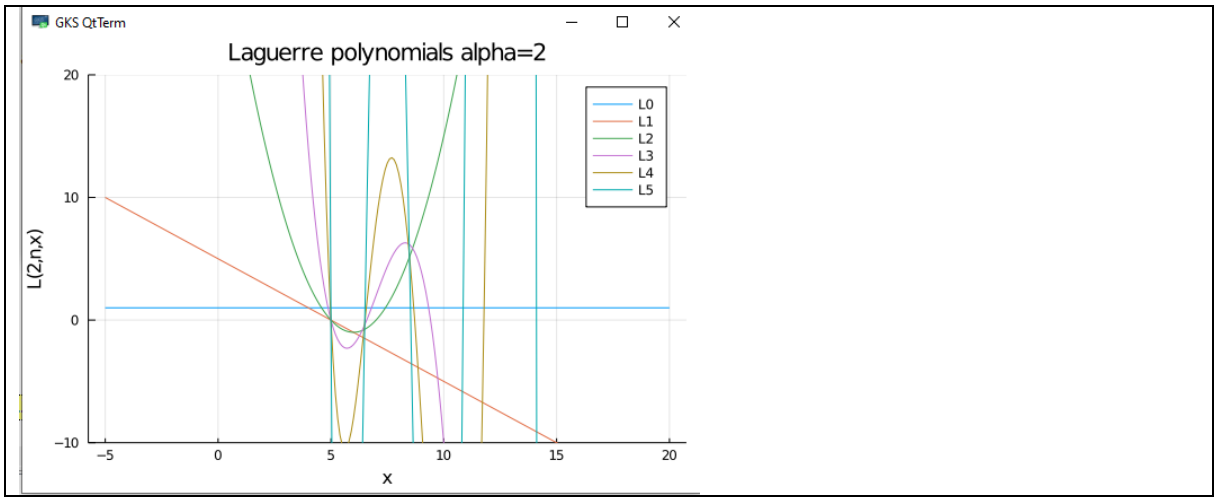


Table 7.3-9 Gauss – Laquerre integral roots and coefficients (α=0)

N	x_k	c_k
2	0.585786437626905	0.853553390592534
	3.414213562373090	0.146446609406723
3	0.415774556783479	0.711093009915089
	2.294280360279040	0.278517733568676
	6.289945082937470	0.010389256497354
4	0.322547689619392	0.603154104288259
	1.745761101158340	0.357418692437802
	4.536620296921120	0.038887908507759
	9.395070912301130	0.000539294705556
5	0.263560319718141	0.521755610467045
	1.413403059106510	0.398666821407258
	3.596425771040720	0.075942449681708
	7.085810005858830	0.003611758663878
	12.640800844275700	0.000023369972386

Table 7.3-10 Gauss – Laquerre integral roots and coefficients (α=1)

N	x_k	c_k
2	1.267949192431120	0.788675133123411
	4.732050807568870	0.211324865405185
3	0.935822227524088	0.588681472855445
	3.305407289332270	0.391216059222280
	7.758770483143630	0.020102459732679
4	0.743291927981432	0.446870579513482
	2.571635007646270	0.477635774492073
	5.731178751689090	0.074177784726293
	10.953894312683100	0.001315849683447
5	0.617030853278271	0.348014523429866
	2.112965958578520	0.502280674138866
	4.610833151017530	0.140915919102187
	8.399066971204840	0.008719893025972

Table 7.3-11 Gauss – Laquerre integral roots and coefficients (α=2)

N	x_k	c_k
2	2.000000000000000	1.500000000000000
	6.000000000000000	0.500000000000000
3	1.517387080677410	1.037494961490390
	4.311583133719520	0.905750004703039
	9.171029785603060	0.056755033772202
4	1.226763263500300	0.725524997698604
	3.412507358696940	1.063424292391060
	6.902692605851610	0.206696130999709
	12.458036771951100	0.004354579188558
5	1.031109144093380	0.520917396835042
	2.837212823953820	1.066705933159050
	5.620294272598700	0.383549720007093
	9.682909837664020	0.028564233510280

	15.828473921690000	**0.000262712802303**

If we investigate Gauss-Leguerre Formula with an example. We will calculate the integral

$$I = \int_0^{\infty} \frac{x}{e^x + 1} dx$$

Exact integral value I= 0.822467.

Program 7.3-8 Gauss-Laguerre integration, java version

```
import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;

class f1 extends f_x
{
  public double func(double x)
  {
  return x/(Math.exp(x)+1.0);
  }
}

public class NA65
{
   public static double [] lngamma(double x)
   {//logaritmik hata fonksiyonu
   //xx[0] hata fonksiyonunun logaritmasinin değerini
   //xx[1] hata fonksiyonunun kendisinin işaretini verir
            double xx[]=new double[2];
   double result;
   double a;
   double b;
   double c;
   double p;
   double q;
   double u;
   double w;
   double z;
   int i;
   double logpi;
   double ls2pi;
   double tmp;
   double sgngam = 1;
   logpi = 1.14472988584940017414;
   ls2pi = 0.91893853320467274178;
   if( x<-34.0 )
   {
      q = -x;
      double yy[]=lngamma(q);
      w = yy[0];
      tmp=yy[1];
      p = Math.floor(q);
      i = (int)Math.round(p);
      if( i%2==0 )
      {
         sgngam = -1;
      }
      else
      {
         sgngam = 1;
      }
      z = q-p;
```

```
    if( z>0.5 )
    {
       p = p+1;
       z = p-q;
    }
    z = q*Math.sin(Math.PI*z);
    result = logpi-Math.log(z)-w;
    xx[0]=result;
    xx[1]=sgngam;
    return xx;
 }
 if( x<13 )
 {
    z = 1;
    p = 0;
    u = x;
    while(u>=3)
    {
       p = p-1;
       u = x+p;
       z = z*u;
    }
    while(u<2)
    {
       z = z/u;
       p = p+1;
       u = x+p;
    }
    if( z<0 )
    {
       sgngam = -1;
       z = -z;
    }
    else
    {
       sgngam = 1;
    }
    if( u==2 )
    {
       result = Math.log(z);
       //return result;
       xx[0]=result;
       xx[1]=sgngam;
       return xx;
    }
    p = p-2;
    x = x+p;
    b = -1378.25152569120859100;
    b = -38801.6315134637840924+x*b;
    b = -331612.992738871184744+x*b;
    b = -1162370.97492762307383+x*b;
    b = -1721737.00820839662146+x*b;
    b = -853555.664245765465627+x*b;
    c = 1;
    c = -351.81570143652347O549+x*c;
    c = -17064.2106651881159223+x*c;
    c = -220528.590553854454839+x*c;
    c = -1139334.44367982507207+x*c;
    c = -2532523.07177582951285+x*c;
    c = -2018891.41433532773231+x*c;
    p = x*b/c;
    result = Math.log(z)+p;
```

```
        //return result;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
    }
    q = (x-0.5)*Math.log(x)-x+ls2pi;
    if( x>100000000 )
    {
        result = q;
        //return result;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
    }
    p = 1/(x*x);
    if( x>=1000.0 )
    {
        q = q+((7.9365079365079365079365*0.0001*p-
2.777777777777777777778*0.001)*p+0.083333333333333333333)/x;
    }
    else
    {
        a = 8.11614167470508450300*0.0001;
        a = -5.95061904284301438324*0.0001+p*a;
        a = 7.9365034045771694 3945*0.0001+p*a;
        a = -2.77777777730099687205*0.001+p*a;
        a = 8.33333333333331927722*0.01+p*a;
        q = q+a/x;
    }
    xx[0]= q;
    xx[1]=sgngam;
    return xx;
    }

public static double[][] gauss_laguerre_coefficients(int n,double alpha)
{   //Gauss-Laguerre quadratürü katsayilarini hesaplar
    double sgngam=1;
    double a[][]=new double[2][n];
    int i;
    int j;
    double r=0;
    double r1;
    double p1;
    double p2;
    double p3;
    double dp3;
    double tsg;
    double eps=1.0e-10;
    for(i = 0; i <= n-1; i++)
    {
        if( i==0 )
        {
            r = (1.0+alpha)*(3.0+0.92*alpha)/(1+2.4*n+1.8*alpha);
        }
        else
        {
            if( i==1 )
            {r = r+(15.0+6.25*alpha)/(1.0+0.9*alpha+2.5*n);}
            else
            {r = r+((1.0+2.55*(i-1))/(1.9*(i-1))+1.26*(i-1)*alpha/(1+3.5*(i-1)))/(1.0+0.3*alpha)*(r-a[0][i-2]);}
        }
        do
```

```
      {
         p2 = 0;
         p3 = 1;
         for(j = 0; j <= n-1; j++)
         {
            p1 = p2;
            p2 = p3;
            p3 = ((-r+2*j+alpha+1)*p2-(j+alpha)*p1)/(j+1);
         }
         dp3 = (n*p3-(n+alpha)*p2)/r;
         r1 = r;
         r = r-p3/dp3;
      }
      while(Math.abs(r-r1)>=eps*(1+Math.abs(r))*100);
      a[0][i] = r;
      double lng[]=lngamma((double)n);
      double lng1[]=lngamma((double)(alpha+n));
      a[1][i] = -Math.exp(lng1[0]-lng[0])/(dp3*n*p2);
      tsg=lng[1];
   }
return a;
}

public static double gauss_laquerre_integral(f_x f_xnt,double xa,int n,double alpha)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss hermite integral
double a[][]=new double[2][n];
a=gauss_laguerre_coefficients(n,alpha);
//System.out.println(Matrix.toString(a));
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f_xnt.func(a[0][i]+xa)*Math.exp(a[0][i])*Math.pow(a[0][i],-alpha);}
return z;
}

public static double gauss_laquerre_integral(f_x f_xnt,double xa,int n)
{
double alpha=1.0;
double a[][]=new double[2][n];
a=gauss_laguerre_coefficients(n,alpha);
//System.out.println(Matrix.toString(a));
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f_xnt.func(a[0][i]+xa)*Math.exp(a[0][i])/a[0][i];}
return z;
}

    public static void main(String[] args)
    {
            f1 b1=new f1();
    System.out.println("integral  : "+gauss_laquerre_integral(b1,0,30)+" exact= "+0.822467);
    }
}
```

Gauss-Laguerre integration

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA65
integral  : 0.8224670275951408 exact= 0.822467

> Terminated with exit code 0.
```

$$I = \int_0^{\infty} \frac{x}{e^x + 1} dx$$ Exact integral value I= 0.822467.

Program Gauss-Laguerre integration , Python version

```
 from math import *
from if_x import *
class f1(if_x):
    def __init__(self, alpha,xa):
      self.alpha=alpha
      self.xa=xa

    def func(self,x):
      y=x+self.xa
      y1=self.w(x)*x/(exp(x)+1)
      return y1;

    def w(self,x):
      if self.alpha==0:
         w1=exp(x)
      elif self.alpha==1:
         w1=exp(x)/x
      elif self.alpha==2:
         w1=exp(x)/(x*x)
      else:
         w1=exp(x)/pow(x,self.alpha)
      return w1

def gauss_laguerre_coefficients(n,alpha):
    #Gauss-Laguerre quadratürü katsayilarini hesaplar
    sngam=1;
    a=[[0.0 for i in range(n)] for j in range(2)]
    r=0;
    eps=1.0e-10
    for i in range(n):
       if i==0:
          r = (1.0+alpha)*(3.0+0.92*alpha)/(1+2.4*n+1.8*alpha)
       elif i==1:
          r = r+(15.0+6.25*alpha)/(1.0+0.9*alpha+2.5*n)
       else:
          r = r+((1.0+2.55*(i-1))/(1.9*(i-1))+1.26*(i-1)*alpha/(1+3.5*(i-1)))/(1.0+0.3*alpha)*(r-a[0][i-2])
       #  do blok
       p2 = 0;
       p3 = 1;
       for j in range(n):
          p1 = p2
          p2 = p3
          p3 = ((-r+2*j+alpha+1)*p2-(j+alpha)*p1)/(j+1)
       dp3 = (n*p3-(n+alpha)*p2)/r
       r1 = r;
       r = r-p3/dp3
       while abs(r-r1)>=eps*(1+abs(r))*100:
          p2 = 0;
          p3 = 1;
          for j in range(n):
             p1 = p2
             p2 = p3
```

```
            p3 = ((-r+2*j+alpha+1)*p2-(j+alpha)*p1)/(j+1)
          dp3 = (n*p3-(n+alpha)*p2)/r
          r1 = r;
          r = r-p3/dp3
       a[0][i] = r
       lng=lgamma(float(n))
       lng1=lgamma(float(alpha+n))
       a[1][i] = -exp(lng1-lng)/(dp3*n*p2)
    return a

def  gauss_laquerre_integral(f_xnt,alpha,n):
    #a=[[0.0 for i in range(n)] for j in range(2)]
    a=gauss_laguerre_coefficients(n,alpha)
    print(a)
    z=0.0
    for i in range(n):
       z=z+a[1][i]*f_xnt.func(a[0][i])
    return z



alpha=0.0
xa=0.0;
exact=0.822467
f=f1(alpha,xa)
print("integral  : ",gauss_laquerre_integral(f,alpha,30),"exact integral = ",exact)
```

```
runfile('E:/okul/SCO1/gauss_leguerre_integral.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
[[0.047407180540804436, 0.2499239167531601, 0.6148334543927677, 1.1431958256661, 1.8364545546225732, 2.696521874557215,
3.7258145077795097, 4.927293765849882, 6.3045155909650745, 7.8616932933702595, 9.603775985479261, 11.536546597956141,
13.666744693064238, 16.002221188981068, 18.55213484014315, 21.327204321783128, 24.340035764532693, 27.60555479678096,
31.141586701111237, 34.96965200824907, 39.11608494906789, 43.61365290848486, 48.5039861638042, 53.841385406507506,
59.6991218592355, 66.18061779443849, 73.44123859555988, 81.73681050672769, 91.55646652253684, 104.1575244310589],
[0.11604408459540358, 0.22085107571058027, 0.24139979501275974, 0.19463676844642563, 0.1237284159740859,
0.06367878042225962, 0.026860475337671705, 0.009338070904071624, 0.00268069689343822, 0.0006351291219520559,
0.00012390745991046017, 1.9828788490041096e-05, 2.5893509659220805e-06, 2.740942840536074e-07, 2.3328311650249643e-08,
1.5807455747673273e-09, 8.4274791224058e-11, 3.485161232810341e-12, 1.0990180557315944e-13, 2.588312617200288e-15,
4.4378377037318635e-17, 5.365916649660166e-19, 4.3939468922898974e-21, 2.3114097943795043e-23, 7.27458849806314e-26,
1.239149701197934e-28, 9.832375073911014e-32, 2.842323546988629e-35, 1.8786080302787062e-39, 8.745980212838154e-45]]
integral  :  0.822467013985489 exact integral= 0.822467
```

$$I = \int_0^\infty \frac{x}{e^x + 1} dx$$

Exact integral value I= 0.822467.

Program Gauss-Laguerre integration , Julia version

```
function gauss_laguerre_coefficients(n,alpha)
   #Gauss-Laguerre quadrature
   a=OffsetArray{Float64}(undef,0:1,0:n-1)
   i=0
   j=0
   r=0.0
   r1=0.0
   p1=0.0
   p2=0.0
   p3=0.0
   dp3=0.0
   eps=1.0e-10
   for i=0:n-1
      if i==0
         r = (1.0+alpha)*(3.0+0.92*alpha)/(1+2.4*n+1.8*alpha)
      else
         if i==1
           r = r+(15.0+6.25*alpha)/(1.0+0.9*alpha+2.5*n)
```

```
            else
              r = r+((1.0+2.55*(i-1))/(1.9*(i-1))+1.26*(i-1)*alpha/(1+3.5*(i-1)))/(1.0+0.3*alpha)*(r-a[0,i-2])
            end # if i==1
        end  # if i==0
        while(true)
            p2 = 0
            p3 = 1
            for j = 0:n-1
                p1 = p2
                p2 = p3
                p3 = ((-r+2*j+alpha+1)*p2-(j+alpha)*p1)/(j+1)
            end # for
            dp3 = (n*p3-(n+alpha)*p2)/r
            r1 = r
            r = r-p3/dp3
        if abs(r-r1)<eps*(1+abs(r))*100
          break
        end #if abs
        end # while
        a[0,i] = r
        lng=loggamma(Float64(n))
        lng1=loggamma(Float64(alpha+n))
        a[1,i] = -exp(lng1-lng)/(dp3*n*p2)
    end #for i
    show(IOContext(stdout, :limit=>false), MIME"text/plain"(), a)
    println(" ")
return a
end #function

function gauss_laquerre_integral(f_xnt,xa,n,alpha)
# n : number of integral coefficients
# this routine first generates gauss legendre coefficients
# for [x1,x2] band
# then calculates gauss hermite integral
# a=OffsetArray{Float64}(undef,0:1,0:n-1)
a=gauss_laguerre_coefficients(n,alpha)

z=0.0
for i=0:n-1
  x1=a[0,i]+xa
  z+=a[1,i]*f_xnt(x1)*exp(a[0,i])*a[0,i]^(-alpha)
end # for
return z
end # func

using OffsetArrays
using SpecialFunctions
f(x)=x/(exp(x)+1.0)
I=gauss_laquerre_integral(f,0.0,30,0.0)
print("integral  : $I  exact= 0.822467")
```

Gauss-Laguerre integration

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_laguerre_integral.jl
2X30 OffsetArray(::
Matrix{Float64}, 0:1, 0:29) with eltype Float64 with indices 0:1X0:29:
 0.0474072  0.249924  0.614833  1.1432    1.83645   2.69652    3.72581    4.92729     6.30452    7.86169     9.60378     11.5365
13.6667     16.0022     18.5521     21.3272     24.34       27.6056     31.1416     34.9697     39.1161     43.6137     48.504
53.8414     59.6991     66.1806     73.4412     81.7368     91.5565     104.158
 0.116044   0.220851  0.2414    0.194637  0.123728  0.0636788  0.0268605  0.00933807  0.0026807  0.000635129  0.000123907
1.98288e-5  2.58935e-6  2.74094e-7  2.33283e-8  1.58075e-9  8.42748e-11  3.48516e-12  1.09902e-13  2.58831e-15  4.43784e-17
5.36592e-19  4.39395e-21  2.31141e-23  7.27459e-26  1.23915e-28  9.83238e-32  2.84232e-35  1.87861e-39  8.74598e-45
integral  : 0.822467013985489  exact= 0.822467
> Terminated with exit code 0.
```

7.3 INTEGRATION FORMULATION WITH ADJUSTABLE ERROR

If integration in region **[a,b]** to be defined for a continious integrable function f(x), Integration can be written as:

$$I = \int_a^b f(x)dx$$

If a sub-region is defined for the integration. For example,

$$m = \frac{a-b}{2}$$

definition is given integral can be redefined as

$$I = \int_a^m f(x)dx + \int_m^b f(x)dx$$

For numerical integration it is generally assumed that the error of the second definition will be less than the first definition. It is possible to continue this process by dividing each integral to subregions again. The important concept in here is to stop dividing integral into smaller parts when the desired accuracy is achieved. In this section several integration methods with error control will be investigated.

7.4.1 Adjustable Simpson 1/3 integral

If simpson 1/3 integration equation is written in [a,b] region

$$I(a,b) = I(x_0,x_4) = \frac{h}{3}[f(x_0) + 4f(x_2) + f(x_4)]$$

$$\int_{x_0}^{x_4} f(x)dx = I(x_0,x_4) - h^5\frac{f^{(4)}(d_1)}{90}$$

Where h =(b-a)/2, x_0=a,x_2=(a+b)/2,x_4=b and d_1 is a number between a and b. and $h^5\frac{f^{(4)}(d_1)}{90}$ equation gives amount of error in the integral equation. $f^{(4)}$ term in the error equation is the fourth derivative of the function. If the integration region is divided into two subregion, equation becomes

$$I(x_0,x_2) = \frac{h}{6}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$I(x_2,x_4) = \frac{h}{6}[f(x_2) + 4f(x_3) + f(x_4)]$$

$$\int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx = I(x_0,x_2) + I(x_2,x_4) - \frac{h^5}{16}\frac{f^{(4)}(d_2)}{90}$$

In this equation

$$x_1 = \frac{a+b}{4}$$

$$x_3 = \frac{3(a+b)}{4}$$

d_2 is stil a value in between a and b.

If it is assumed that

$$f^{(4)}(d_1) = f^{(4)}(d_2)$$

Equation becomes

$$h^5\frac{f^{(4)}(d_1)}{90} \approx \frac{16}{15}[I(x_0,x_2) + I(x_2,x_4) - I(x_0,x_4)]$$

$$\frac{h^5}{16}\frac{f^{(4)}(d_1)}{90} \approx \frac{1}{15}[I(x_0,x_2) + I(x_2,x_4) - I(x_0,x_4)]$$

If we substitute this into integration equation

$$\left| \int_{x_0}^{x_4} f(x)dx - I(x_0, x_2) - I(x_2, x_4) \right| \approx \frac{1}{15} |I(x_0, x_2) + I(x_2, x_4) - I(x_0, x_4)|$$

In this case for error ε>0 , it can be written the equation

$$\frac{1}{15} |I(x_0, x_2) + I(x_2, x_4) - I(x_0, x_4)| < \varepsilon$$

If this condition is met, then approximately

$$\int_{x_0}^{x_4} f(x)dx \cong I(x_0, x_2) + I(x_2, x_4)$$

can be written. If the given condition can not be met, division process will be continue till the condition met. Described algorithm is used in the next program.

Program 7.4-1 Simson 1/3 integration with adjustable error, Java version

```
// ======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Adaptive simpson 1/3 integration
// Dr. Turhan Coban
// ======================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {return x*Math.sqrt(x);}
}

class NA66
{
public static double adaptive_simpson_integral(f_x ff,double a,double b)
{return adaptive_simpson_integral(ff,a,b,1.0e-20,30);}

public static double adaptive_simpson_integral(f_x ff,double a,double b,double eps)
{return adaptive_simpson_integral(ff,a,b,eps,30);}

public static double adaptive_simpson_integral(f_x ff,double a,double b,int MAX_ITERATION)
{return adaptive_simpson_integral(ff,a,b,1.0e-12,MAX_ITERATION);}

public static double adaptive_simpson_integral(f_x ff,double a,double b,double eps,int MAX_ITERATION)
{   int i;
    double h=(b-a);
    double h2=h/4.0;
            double x[]=new double[5];
    double f[]=new double[5];
    for(i=0;i<5;i+=2) {x[i]=a+i*h2;f[i]=ff.func(x[i]);}
            double sum=h*(f[0]+4.0*f[2]+f[4])/6.0;
            double Q=0;
    Q=simpson1_3(ff,x,f,sum,eps,0,MAX_ITERATION,Q);
    return Q;
}

public static double simpson1_3(f_x ff,double x[],double f[],double sum,double eps,int k,int MAX_ITERATION,double
Q)
{       //Newton-cotes integral 3 points
       //adaptive Simpson 1/3 rule
```

```
        double h=(x[4]-x[0]);
        double h1=h/2.0;
        double h2=h/4.0;
        int p,i;
        double x1[]=new double[5];
        double f1[]=new double[5];
        double x2[]=new double[5];
        double f2[]=new double[5];
        k++;
        for(i=1;i<5;i+=2) {x[i]=x[0]+i*h2;f[i]=ff.func(x[i]);}
        double sum1=h1*(f[0]+4.0*f[1]+f[2])/6.0;
        double sum2=h1*(f[2]+4.0*f[3]+f[4])/6.0;
        x1[0]=x[0];f1[0]=f[0];
        x1[2]=x[1];f1[2]=f[1];
        x1[4]=x[2];f1[4]=f[2];
        x2[0]=x[2];f2[0]=f[2];
        x2[2]=x[3];f2[2]=f[3];
        x2[4]=x[4];f2[4]=f[4];
        if(1.0/15.0*Math.abs(sum1+sum2-sum)<eps || k>MAX_ITERATION) Q+=sum1+sum2;
        else
Q+=simpson1_3(ff,x1,f1,sum1,eps,k,MAX_ITERATION,Q)+simpson1_3(ff,x2,f2,sum2,eps,k,MAX_ITERATION,Q);
        return Q;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+adaptive_simpson_integral(b1,0.0,1.0));
}
}
```

Simson 1/3 integration with the adaptive simpson integration

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA66
integral of class         f1 : 0.4

> Terminated with exit code 0.
```

If a version of the same program with actual plots of the calculating points are given, automatic adjustment process of the routine can be seen more clearly.

Program 7.4-2 Simson 1/3 integration with adjustable error (version with plot output)

```
// ======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Adaptive simpson 1/3 integration with plotting
// Dr. Turhan Coban
// ======================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {return 100.0/(x*x)*Math.sin(10.0/x); }
}

class NA67
{
```

```
public static double adaptive_simpson_integral(f_x ff,double a,double b)
{return adaptive_simpson_integral(ff,a,b,1.0e-20,30);}

public static double adaptive_simpson_integral(f_x ff,double a,double b,double eps)
{return adaptive_simpson_integral(ff,a,b,eps,30);}

public static double adaptive_simpson_integral(f_x ff,double a,double b,int MAX_ITERATION)
{return adaptive_simpson_integral(ff,a,b,1.0e-12,MAX_ITERATION);}

public static double adaptive_simpson_integral(f_x ff,double a,double b,double eps,int MAX_ITERATION)
{   int i;
   double h=(b-a);
   double h2=h/4.0;
            double x[]=new double[5];
   double f[]=new double[5];
   double w1[]=new double[3];
   double z1[]=new double[3];
   int j;
   for(i=0;i<5;i+=2) {j=i/2;x[i]=a+i*h2;f[i]=ff.func(x[i]);w1[j]=x[i];z1[j]=f[i];}
   double sum=h*(f[0]+4.0*f[2]+f[4])/6.0;
            double Q=0;
            Plot pp=new Plot(ff,a,b,100);
            pp.setPlotType(0,0);//
            pp.addData(w1,z1,28);
            int k=0;
   Q=simpson1_3(ff,x,f,sum,eps,k,MAX_ITERATION,Q,pp);
   pp.plot();
   return Q;
}

public static double simpson1_3(f_x ff,double x[],double f[],double sum,double eps,int k,int MAX_ITERATION,double
Q,Plot pp)
{
      //adaptive Simpson 1/3 rule

      double h=(x[4]-x[0]);
      double h1=h/2.0;
      double h2=h/4.0;
      int p,i;
      double x1[]=new double[5];
      double f1[]=new double[5];
      double x2[]=new double[5];
      double f2[]=new double[5];
      double w1[]=new double[2];
      double z1[]=new double[2];
      int j;
      for(i=1;i<5;i+=2) {j=i/2;x[i]=x[0]+i*h2;w1[j]=x[i];f[i]=ff.func(x[i]);z1[j]=f[i];}
      for(i=0;i<5;i++) {System.out.println(i+" "+x[i]+" "+f[i]);}
      {System.out.println("==================");}
      double sum1=h1*(f[0]+4.0*f[1]+f[2])/6.0;
      double sum2=h1*(f[2]+4.0*f[3]+f[4])/6.0;
      if(1.0/15.0*Math.abs(sum1+sum2-sum)<eps || k>MAX_ITERATION) Q+=sum1+sum2;
      else
      { x1[0]=x[0];f1[0]=f[0];
        x1[2]=x[1];f1[2]=f[1];
        x1[4]=x[2];f1[4]=f[2];
        x2[0]=x[2];f2[0]=f[2];
        x2[2]=x[3];f2[2]=f[3];
        x2[4]=x[4];f2[4]=f[4];
              k++;
              Q+=simpson1_3(ff,x1,f1,sum1,eps,k,MAX_ITERATION,Q,pp)+
          simpson1_3(ff,x2,f2,sum2,eps,k,MAX_ITERATION,Q,pp);
```

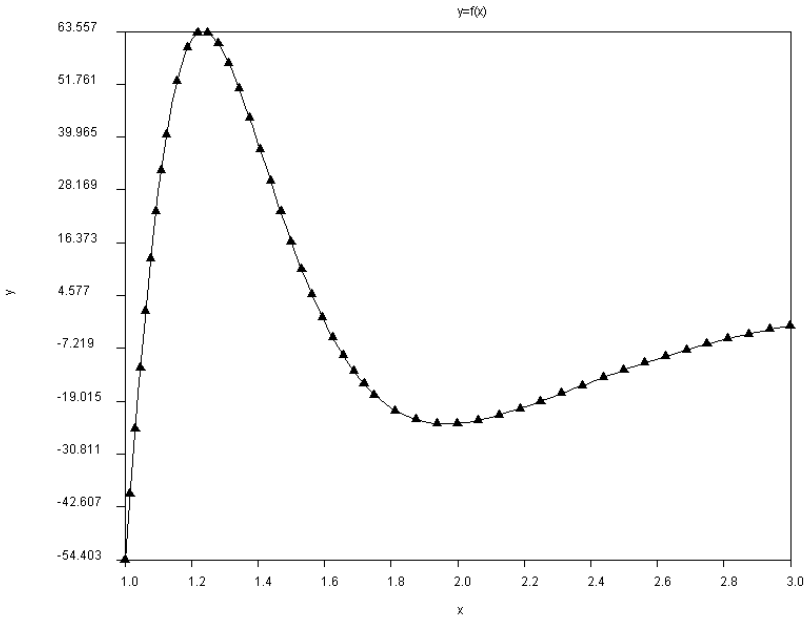
```
        }
        pp.addData(w1,z1,28);
        return Q;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+adaptive_simpson_integral(b1,1.0,3.0,1.0e-5,2));
  }
}
```

Adaptive Simson 1/3 integration (version with plot output)

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA67
0 1.0 -54.40211108893698
1 1.5 16.62894358094311
2 2.0 -23.973106866578462
3 2.5 -12.108839924926851
4 3.0 -2.1174218097276154
==================
0 1.0 -54.40211108893698
1 1.25 63.318927783896434
2 1.5 16.62894358094311
3 1.75 -17.590376763629592
4 2.0 -23.973106866578462
==================
0 1.0 -54.40211108893698
1 1.125 40.34415237327668
2 1.25 63.318927783896434
3 1.375 44.20625835648312
4 1.5 16.62894358094311
==================
0 1.0 -54.40211108893698
1 1.0625 1.1526986941820327
2 1.125 40.34415237327668
3 1.1875 59.814502478779694
4 1.25 63.318927783896434
==================
0 1.25 63.318927783896434
1 1.3125 56.45523901688928
2 1.375 44.20625835648312
3 1.4375 30.17788460030764
4 1.5 16.62894358094311
==================
0 1.5 16.62894358094311
1 1.625 -4.884405926189118
2 1.75 -17.590376763629592
3 1.875 -23.13470269347783
4 2.0 -23.973106866578462
==================
0 1.5 16.62894358094311
1 1.5625 4.773855430676219
2 1.625 -4.884405926189118
3 1.6875 -12.280554478542038
4 1.75 -17.590376763629592
==================
0 1.75 -17.590376763629592
1 1.8125 -21.101511468402688
2 1.875 -23.13470269347783
3 1.9375 -23.999652018301266
4 2.0 -23.973106866578462
```

```
==================
0 2.0 -23.973106866578462
1 2.25 -19.048239346741298
2 2.5 -12.108839924926851
3 2.75 -6.278745267070935
4 3.0 -2.1174218097276154
==================
0 2.0 -23.973106866578462
1 2.125 -22.144859946836526
2 2.25 -19.048239346741298
3 2.375 -15.542391685948601
4 2.5 -12.108839924926851
==================
0 2.0 -23.973106866578462
1 2.0625 -23.29043436266789
2 2.125 -22.144859946836526
3 2.1875 -20.69068321637011
4 2.25 -19.048239346741298
==================
0 2.25 -19.048239346741298
1 2.3125 -17.309323548876932
2 2.375 -15.542391685948601
3 2.4375 -13.79721050171754
4 2.5 -12.108839924926851
==================
0 2.5 -12.108839924926851
1 2.625 -8.988488771999702
2 2.75 -6.278745267070935
3 2.875 -3.9966027013802914
4 3.0 -2.1174218097276154
==================
0 2.5 -12.108839924926851
1 2.5625 -10.500944535981427
2 2.625 -8.988488771999702
3 2.6875 -7.579894531793721
4 2.75 -6.278745267070935
==================
0 2.75 -6.278745267070935
1 2.8125 -5.085116345062419
2 2.875 -3.9966027013802914
3 2.9375 -3.0091045529709124
4 3.0 -2.1174218097276154
==================
integral of class        f1 : -1.416364288717352
```

Graphic output of Adaptive simpson 1/3 integration



Simson 1/3 integration with adjustable error, Julia version

```
function  newton_cotes2(ff,a,b,n)
    #Newton-cotes integral 2 points
     #trapezoidal rule
     # ff integrating function
     # a,b integral limits
     # n : number of sub division h=(b-a)/n;
     h=(b-a)/n
     h1=h;
     sum=0.0;
     x=zeros(3)
     f=zeros(3)
     xi=zeros(n+1)
     fi=zeros(n+1)
     xi[1]=a
     fi[1]=ff(a)
     for p=1:n
       for i=1:2
         x[i]=a+(i-1)*h1+(p-1)*h
         f[i]=ff(x[i])
         xi[p+1]=x[i]
         fi[p+1]=f[i]
         end
       sum=sum+h*(f[1]+f[2])/2.0
     end
     xx=[sum,xi,fi]
     return xx
     end

function adaptive_simpson(ff,a,b,eps,MI)
 sum1=0.0
 sum2=0.0
 for i=1:MI
   sum1=newton_cotes2(ff,a,b,2*i)[1]
   m=(b-a)/2.0;
   sum2=newton_cotes2(ff,a,m,2*i)[1]+newton_cotes2(ff,m,b,2*i)[1]
   dsum=1.0/15.0*abs(sum2-sum1)
   if dsum<eps
     break
   end
```

```
   end
 return sum2
end

f(x)=x^1.5
I=adaptive_simpson(f,0.0,1.0,1.0e-8,500)
print("I = $I")
```

integration with the adaptive simpson integration

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" newton_cotes2.jl
I = 0.40000005006813727
> Terminated with exit code 0.
```

Program Simson 1/3 integration with adjustable error, Python version

```
 from math import *
from if_x import *
def simpson1_3(f,x,y,sum,eps,k,MI,Q):
    #Newton-cotes integral 3 points
    #adaptive Simpson 1/3 rule
    h=(x[4]-x[0]);
    h1=h/2.0;
    h2=h/4.0;
    x1=[0 for z in range(5)]
    y1=[0 for z in range(5)]
    x2=[0 for z in range(5)]
    y2=[0 for z in range(5)]
    k=k+1
    for i in range(5):
        x[i]=x[0]+i*h2
        y[i]=f.func(x[i])
    sum1=h1*(y[0]+4.0*y[1]+y[2])/6.0;
    sum2=h1*(y[2]+4.0*y[3]+y[4])/6.0;
    x1[0]=x[0];y1[0]=y[0];
    x1[2]=x[1];y1[2]=y[1];
    x1[4]=x[2];y1[4]=y[2];
    x2[0]=x[2];y2[0]=y[2];
    x2[2]=x[3];y2[2]=y[3];
    x2[4]=x[4];y2[4]=y[4];
    if 1.0/15.0*abs(sum1+sum2-sum)<eps or k>MI:
        Q=Q+sum1+sum2
    else:
        Q=Q+simpson1_3(f,x1,y1,sum1,eps,k,MI,Q)+simpson1_3(f,x2,y2,sum2,eps,k,MI,Q);
    return Q

def adaptive_simpson_integral(f,a,b,eps,MI):
    h=(b-a)
    h2=h/4.0
    x=[0 for z in range(5)]
    y=[0 for z in range(5)]
    for i in range(5):
        x[i]=a+i*h2
        y[i]=f.func(x[i])
    sum=h*(y[0]+4.0*y[2]+y[4])/6.0
    Q=0.0
    Q=simpson1_3(f,x,y,sum,eps,0,MI,Q)
    return Q

class f1(if_x):func=lambda self,x: x*sqrt(x);
f=f1()
eps=1.0e-10
MI=100
I=adaptive_simpson_integral(f,0.0,1.0,eps,MI)
print("I = ",I)
```

Simson 1/3 integration with the adaptive simpson integration

```
runfile('E:/okul/SCO1/ASimpsonIntegral.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
I =  0.4000000007335708
```

Adaptive Gauss-Legendre Integration

The error for the Gauss-Legendre formulas is specified generally by

$$E_t = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3} f^{(2n+2)}(\xi)$$

where n = the number of points minus one and $f(2n+2)(\xi)$ = the $(2n + 2)$th derivative of the function after the change of variable with ξ located somewhere on the interval from −1 to 1. It is clear that error level of Gauss-Legendre integration is far smaller than the Newton-Cotes formulas, therefore it will take less iteration to reach the desired error levels. Adjustable error version of the Gauss-Legendre integration is given below. The main idea here is to increase terms in the Gauss-Legendre integration untill the error level between two levels decrease below the desired error level.

Program Gauss-Legendre integration with the adjustable error, java version

```
// =======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Adaptive Gauss-legendre integration
// Dr. Turhan Coban
// =======================================================
import java.io.*;


class f1 extends f_x
{
  public double func(double x)
  {return x*Math.sqrt(x);}
}


class NA68
{
public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
        //calculates legendre gauss-coefficients as coefficients of the integral
        //for n terms
        double EPS=3.0e-15;
        int m,j,i;
        double z1,z,xm,xl,pp,p3,p2,p1;
         double a[][]=new double[2][n];
        m=(n+1)/2;
        xm=0.5*(x2+x1);
        xl=0.5*(x2-x1);
        for (i=1;i<=m;i++)  {
                z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                do {
                        p1=1.0;
                        p2=0.0;
                        for (j=1;j<=n;j++) {
                                p3=p2;
                                p2=p1;
                                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                        }
                        pp=n*(z*p1-p2)/(z*z-1.0);
                        z1=z;
                        z=z1-p1/pp;
```

```
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
                                }
return a;
}

public static double gauss_legendre_integral(f_x f,double x1,double x2,int n)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss legendre integral
double a[][]=new double[2][n];
a=gauss_legendre_coefficients(x1,x2,n);
//Text.print(Text.T(a));
//System.out.println(Matrix.toString(a));
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f.func(a[0][i]);}
return z;
}

public static double adaptive_gauss_legendre_integral(f_x f,double x1,double x2)
{return adaptive_gauss_legendre_integral(f,x1,x2,1.0e-12,100);}

public static double adaptive_gauss_legendre_integral(f_x f,double x1,double x2,double eps)
{return adaptive_gauss_legendre_integral(f,x1,x2,eps,100);}

public static double adaptive_gauss_legendre_integral(f_x f,double x1,double x2,double eps,double MAXITER)
{
int n=1;
double ans1=gauss_legendre_integral(f,x1,x2,n);
double ans2;
do
{ans2=ans1;n++;ans1=gauss_legendre_integral(f,x1,x2,n);}
while(Math.abs(ans2-ans1)>eps && n<MAXITER);
if(n==MAXITER) System.out.println("istenilen hata seviyesi tutturulamadi n="+n );
return ans2;
}

  public static void main(String args[]) throws IOException
  {
  //Gauss-Legendre integrali
  f1 b1=new f1();
  System.out.println("integral  : "+adaptive_gauss_legendre_integral(b1,0.0,1.0));
}
}
```

Adaptive Gauss-Legendre integration

```
---------- Capture Output ----------
> "C:\co\java\bin\java.exe" integ13
integral  : 0.3999999999829868
> Terminated with exit code 0.
```

Adjestable Gauss-Legendre integration, python version

```
from math import *
```

```
from if_x import *
def gauss_legendre_coefficients(x1,x2,n):
   #calculates legendre gauss-coefficients as coefficients of the integral
   #for n terms
      eps=3e-15
      m=int((n+1.0)/2.0)
      xm=0.5*(x2+x1)
      xl=0.5*(x2-x1)
      nmax=100
      a=[[0.0 for i in range(n)] for j in range(2)]
      for i in range(1,m+1):
         z=cos(pi*((i-0.25)/(n+0.5)))
         for ii in range(nmax):
            #while abs(z-z1) > eps:
            p1=1.0;
            p2=0.0;
            for j in range(1,n+1):
               p3=p2
               p2=p1
               p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
            pp=(float)(n*(z*p1-p2)/(z*z-1.0))
            z1=z;
            z=z1-p1/pp
            if abs(z-z1) < eps: break
         a[0][i-1]=xm-xl*z
         a[0][n-i]=xm+xl*z
         a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
         a[1][n-i]=a[1][i-1]
      return a

def integral(f,x1,x2,n):
      #integral func(x)dx
      #integral of a function by using gauss-legendre quadrature
      #between x1 and x2
      a=gauss_legendre_coefficients(x1,x2,n)
      z=0
      for i in range(n):
         z=z+a[1][i]*f.func(a[0][i])
      return z

def adaptive_integral(f,x1,x2):
   eps=1.0e-12
   MAXITER=100
   return adaptive_integral1(f,x1,x2,eps,MAXITER)

def adaptive_integral1(f,x1,x2,eps,MAXITER):
   n=1;
   ans1=integral(f,x1,x2,n)
   ans2=2.0*ans1
   while(abs(ans2-ans1)>eps and n<MAXITER):
      ans2=ans1
      n=n+1
      ans1=integral(f,x1,x2,n)
   if n==MAXITER: print("requird error level can not be achievd n="+n );
   return ans2


class f1(if_x):func=lambda self,x: x*sqrt(x)
f=f1()
a=0.0
b=1.0
I1=adaptive_integral(f,a,b)
print("I1  : ",I1);
```

```
runfile('E:/okul/SCO1/integral_GL3.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
I1  :  0.3999999999829868
```

Gauss-Kronrod integration formula

$$I_w(-1,1) \cong \int_{-1}^{1} f(x)dx \cong \sum_{k=1}^{n} c_k f(x_k^G) + \sum_{k=1}^{n+1} c_{n+k} f(x_k^K)$$

Gauss Kronrod integration formula uses two terms. First term is standard Gauss-Legendre integration term. The second term coefficients are taken as same as the Gauss-Legendre term, but roots are different. It is called Kronrod integration. In our example code constant integration of 15 terms will be used for each equation (total of 30 terms). Program utilises Gauss-Legendre terms as first, and then uses Gauss-Kronrod terms. To improve error reduction equation can be continued with half dividing technique. Gauss-Kronrod Formula is prefferred method for many package programs such as Mathlab.

Program Gauss-Kronrod integration formula, Java version

```
class f1 extends f_x
{
  double func(double x)
  {return x*Math.sqrt(x);}
}
public class NA69
{
    public static double gauss_kronrad_integral(f_x ff,double a,double b,double eps,int n)
    {
                //eps hata miktari
       int Aused=0;
                boolean sonuc;
       double integral=0;
       double A[][];
       int Aw = 0;
       double toplamhata = 0;
       double[] c;
       double[] xg;
       double[] xk;
       int nn = 0;
       int ng = 0;
       int i = 0;
       int j = 0;
       int h = 0;
       double v = 0;
       double k1 = 0;
       double k2 = 0;
       double intg = 0;
       double intk = 0;
       double ta = 0;
       double tb = 0;
       Aw = 4;
       A = new double[n-1+1][Aw-1+1];
       nn = 61;
       ng = 15;
       c = new double[nn];
       xk = new double[nn];
       xg = new double[nn];
       //Gauss katsayilari
       xg[0] = 0.00796819249616660561546588347467 4;
       xg[1] = 0.01846646831109095914230213191204 7;
       xg[2] = 0.02878470788332336934971917961129 2;
       xg[3] = 0.03879919256962704959680193644634 8;
       xg[4] = 0.04840267283059405290293814042280 8;
       xg[5] = 0.05749315621761906648172168940205 6;
       xg[6] = 0.06597422988218049512812851511596 2;
       xg[7] = 0.07375597473770520626824385002219 1;
       xg[8] = 0.08075589522942021535469493846053 0;
```

```
xg[9] = 0.08689978720108297980238753071512 6;
xg[10] = 0.09212252223778612871763270708761 9;
xg[11] = 0.09636873717464425963946862635181 0;
xg[12] = 0.09959342058679526706278028210356 9;
xg[13] = 0.10176238974840550459642895216855 4;
xg[14] = 0.10285265289355884034128563670541 5;
c[0] = 0.99948441005049063757132589570581 1;
c[1] = 0.99689348407464954027163005091869 5;
c[2] = 0.99163099687040459485862836610948 6;
c[3] = 0.98366812327974720997003258160566 3;
c[4] = 0.97311632250112626837469386842370 7;
c[5] = 0.96002186496830751221687102558179 8;
c[6] = 0.94437444474855997941583132403743 9;
c[7] = 0.92620004742927432587932427708047 4;
c[8] = 0.90557330769990779854652255892595 8;
c[9] = 0.88256053579205268154311646253022 6;
c[10] = 0.85720523354606109895865851065894 4;
c[11] = 0.82956576238276839744289811973250 2;
c[12] = 0.79972783582183908301366894232268 3;
c[13] = 0.76777743210482619491797734097450 3;
c[14] = 0.73379006245322680472617113136952 8;
c[15] = 0.69785049479331579693229238802664 0;
c[16] = 0.66006106412662696137005366814927 1;
c[17] = 0.62052618298924286114047755643118 9;
c[18] = 0.57934523582636169175602493217254 0;
c[19] = 0.53662414814201989926416979331107 3;
c[20] = 0.49248046786177857499369306120770 9;
c[21] = 0.44703376953808917678060990032285 4;
c[22] = 0.40040125483039439253547621154266 1;
c[23] = 0.35270472553087811347103720708937 4;
c[24] = 0.30407320227362507737267710719925 7;
c[25] = 0.25463692616788984643980512981780 5;
c[26] = 0.20452511668230989143895767100202 5;
c[27] = 0.15386991360858354696379467274325 6;
c[28] = 0.10280693796673703014709675131800 1;
c[29] = 0.05147184255531769583302521316672 3;
c[30] = 0.00000000000000000000000000000000 0;
//Kronrod katsayilari
xk[0] = 0.00138901369867700762455159122676 0;
xk[1] = 0.00389046112709988405126720184451 6;
xk[2] = 0.00663070391593129217331982636975 0;
xk[3] = 0.00927327965951776342844114689202 4;
xk[4] = 0.01182301525349634174223289885325 1;
xk[5] = 0.01436972950704580481245143244358 0;
xk[6] = 0.01692088918905327262757228942032 2;
xk[7] = 0.01941414119394238117340895105012 8;
xk[8] = 0.02182803582160919229716748573833 9;
xk[9] = 0.02419116207808060136568637072523 2;
xk[10] = 0.02650995488233310161060170933507 5;
xk[11] = 0.02875404876504129284397878535433 4;
xk[12] = 0.03090725756238776247288425294309 2;
xk[13] = 0.03298144705748372603181419101685 4;
xk[14] = 0.03497933802806002413749967073146 8;
xk[15] = 0.03688236465182122922391106561713 6;
xk[16] = 0.03867894562472759295034865153228 1;
xk[17] = 0.04037453895153595911199527975246 8;
xk[18] = 0.04196981021516424614714754128597 0;
xk[19] = 0.04345253970135606931683172811707 3;
xk[20] = 0.04481480013316266319235555161672 3;
xk[21] = 0.04605923827100698811627173555937 4;
xk[22] = 0.04718554656929915394526147818109 9;
xk[23] = 0.04818586175708712914077949229830 5;
```

```
xk[24] = 0.049055434555029778887528165367238;
xk[25] = 0.049795683427074206357811569379942;
xk[26] = 0.050405921402782346840893085653585;
xk[27] = 0.050881795898749606492297473049805;
xk[28] = 0.051221547849258772170656282604944;
xk[29] = 0.051426128537459025933862879215781;
xk[30] = 0.051494729429451567558340433647099;
for(i=nn-1; i>=nn/2; i--)
{c[i] = -c[nn-1-i];}
for(i=nn-1; i>=nn/2; i--)
{ xk[i] = xk[nn-1-i]; }
for(i=ng-1; i>=0; i--)
{ xg[nn-2-2*i] = xg[i];
  xg[1+2*i] = xg[i];
}
for(i=0; i<=nn/2; i++)
{ xg[2*i] = 0; }
k1 = 0.5*(b-a);
k2 = 0.5*(b+a);
intg = 0;
intk = 0;
for(i=0; i<=nn-1; i++)
{ v = ff.func(k1*c[i]+k2);
  intk = intk+v*xk[i];
  if( i%2==1 )
  {intg = intg+v*xg[i];}
}
intk = intk*(b-a)*0.5;
intg = intg*(b-a)*0.5;
A[0][0] = Math.abs(intg-intk);
A[0][1] = intk;
A[0][2] = a;
A[0][3] = b;
toplamhata = A[0][0];
if( toplamhata<eps )
{
   sonuc = true;
   integral = intk;
   Aused = 1;
   return integral;
}
Aused = 1;
for(h=1; h<=n-1; h++)
{
   Aused = h+1;
   gir(A, h, Aw);
   toplamhata = toplamhata-A[h-1][0];
   ta = A[h-1][2];
   tb = A[h-1][3];
   A[h-1][2] = ta;
   A[h-1][3] = 0.5*(ta+tb);
   A[h][2] = 0.5*(ta+tb);
   A[h][3] = tb;
   for(j=h-1; j<=h; j++)
   {
      k1 = 0.5*(A[j][3]-A[j][2]);
      k2 = 0.5*(A[j][3]+A[j][2]);
      intg = 0;
      intk = 0;
      for(i=0; i<=nn-1; i++)
      {   v = ff.func(k1*c[i]+k2);
         intk = intk+v*xk[i];
```

```
                if( i%2==1 )
                { intg = intg+v*xg[i];}
            }
            intk = intk*(A[j][3]-A[j][2])*0.5;
            intg = intg*(A[j][3]-A[j][2])*0.5;
            A[j][0] = Math.abs(intg-intk);
            A[j][1] = intk;
            toplamhata = toplamhata+A[j][0];
        }
        cik(A, h-1, Aw);
        cik(A, h, Aw);
        if( toplamhata<eps )
        {break;}
    }
    sonuc = toplamhata<eps;
    integral = 0;
    for(j=0; j<=Aused-1; j++)
    { integral = integral+A[j][1];}
    return integral;
}

public static void gir(double A[][],int n,int Awidth)
{
    int i = 0;
    int p = 0;
    double t = 0;
    int maxcp = 0;
    if( n==1 )
    { return; }
    for(i=0; i<=Awidth-1; i++)
    {   t = A[n-1][i];
        A[n-1][i] = A[0][i];
        A[0][i] = t;
    }//end of for
    p = 0;
    while( 2*p+1<n-1 )
    {   maxcp = 2*p+1;
        if( 2*p+2<n-1 )
        {   if( A[2*p+2][0]>A[2*p+1][0] )
            { maxcp = 2*p+2; }
        }
        if( A[p][0]<A[maxcp][0] )
        {   for(i=0; i<=Awidth-1; i++)
            {   t = A[p][i];
                A[p][i] = A[maxcp][i];
                A[maxcp][i] = t;
            }
            p = maxcp;
        }
        else
        { break; }
    } //end of while
}

public static void cik(double A[][],int n,int Awidth)
{
    int i = 0;
    int p = 0;
    double t = 0;
    int kk = 0;
    if( n==0 )
    {return;}
```

```
        p = n;
        while( p!=0 )
        {   kk = (p-1)/2;
            if( A[p][0]>A[kk][0] )
            { for(i=0; i<=Awidth-1; i++)
                { t = A[p][i];
                  A[p][i] = A[kk][i];
                  A[kk][i] = t;
                }
                p = kk;
            }
            else
            {break;}
        }
    }

  public static void main(String axgs[])
  {
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+gauss_konrad_integral(b1,0.0,1.0,1.0e-20,50));
}
}
```

```
---------- Capture Output ----------
> "C:\co\java\bin\javaw.exe" adaptive_Gauss_Kronrod_integral
integral of class         f1 : 0.4

> Terminated with exit code 0.
```

Program Gauss-Kronrod integration formula, Python version

```
from math import *
from if_x import *
def gauss_kronrod_integral(ff,a, b, eps,n):
    # eps hata miktarı
    Aused=0
    integral=0
    toplamhata = 0
    nn = 0
    ng = 0
    i = 0
    j = 0
    h = 0
    v = 0
    k1 = 0
    k2 = 0
    intg = 0
    intk = 0
    ta = 0
    tb = 0
    Aw = 4
    A = [[0.0 for i in range(Aw-1+1)] for j in range(n-1+1)]    #[n-1+1][Aw-1+1];
    nn = 61;
    ng = 15;
    x =[0.0 for i in range(nn)]
    ck =[0.0 for i in range(nn)]
    cg =[0.0 for i in range(nn)]
    #Gauss coefficients
    cg[0] = 0.00796819249616660561546588347467 4
    cg[1] = 0.01846646831109095914230213191204 7
    cg[2] = 0.02878470788332336934971917961129 2
    cg[3] = 0.03879919256962704959680193644634 8
    cg[4] = 0.04840267283059405290293814042280 8
    cg[5] = 0.05749315621761906648172168940205 6
    cg[6] = 0.06597422988218049512812851511596 2
    cg[7] = 0.07375597473770520626824385002219 1
```

```
    cg[8] = 0.080755895229420215354694938460530
    cg[9] = 0.086899787201082979802387530715126
    cg[10] = 0.092122522237786128717632707087619
    cg[11] = 0.096368737174644259639468626351810
    cg[12] = 0.099593420586795267062780282103569
    cg[13] = 0.101762389748405504596428952168554
    cg[14] = 0.102852652893558840341285636705415
    x[0] = 0.99948441005049063757132589570581l
    x[1] = 0.99689348407464954027163005091869 5
    x[2] = 0.99163099687040459485862836610948 6
    x[3] = 0.98366812327974720997003258160566 3
    x[4] = 0.97311632250112626837469386842370 7
    x[5] = 0.96002186496830751221687102558179 8
    x[6] = 0.94437444474855997941583132403743 9
    x[7] = 0.92620004742927432587932427708047 4
    x[8] = 0.90557330769990779854652255892595 8
    x[9] = 0.88256053579205268154311646253022 6
    x[10] = 0.85720523354606109895865851065894 4
    x[11] = 0.82956576238276839744289811973250 2
    x[12] = 0.79972783582183908301366894232268 3
    x[13] = 0.76777743210482619491797734097450 3
    x[14] = 0.73379006245322680472617113136952 8
    x[15] = 0.69785049479331579693229238802664 0
    x[16] = 0.66006106412662696137005366814927 1
    x[17] = 0.62052618298924286114047755643118 9
    x[18] = 0.57934523582636169175602493217254 0
    x[19] = 0.53662414814201989926416979331107 3
    x[20] = 0.49248046786177857499369306120770 9
    x[21] = 0.44703376953808917678060990032285 4
    x[22] = 0.40040125483039439253547621154266 1
    x[23] = 0.35270472553087811347103720708937 4
    x[24] = 0.30407320227362507737267710719925 7
    x[25] = 0.25463692616788984643980512981780 5
    x[26] = 0.20452511668230989143895767100202 5
    x[27] = 0.15386991360858354696379467274325 6
    x[28] = 0.10280693796673703014709675131800 1
    x[29] = 0.05147184255531769583302521316672 3
    x[30] = 0.00000000000000000000000000000000 0
    #Kronrod coefficients
    ck[0] = 0.00138901369867700762455159122676 0
    ck[1] = 0.00389046112709988405126720184451 6
    ck[2] = 0.00663070391593129217331982636975 0
    ck[3] = 0.00927327965951776342844114689202 4
    ck[4] = 0.01182301525349634174223289885325 1
    ck[5] = 0.01436972950704580481245143244358 0
    ck[6] = 0.01692088918905327262757228942032 2
    ck[7] = 0.01941414119394238117340895105012 8
    ck[8] = 0.02182803582160919229716748573833 9
    ck[9] = 0.02419116207808060136568637072523 2
    ck[10] = 0.02650995488233310161060170933507 5
    ck[11] = 0.02875404876504129284397878535433 4
    ck[12] = 0.03090725756238776247288425294309 2
    ck[13] = 0.03298144705748372603181419101685 4
    ck[14] = 0.03497933802806002413749967073146 8
    ck[15] = 0.03688236465182122922391106561713 6
    ck[16] = 0.03867894562472759295034865153228 1
    ck[17] = 0.04037453895153595911199527975246 8
    ck[18] = 0.04196981021516424614714754128597 0
    ck[19] = 0.04345253970135606931683172811707 3
    ck[20] = 0.04481480013316266319235555161672 3
    ck[21] = 0.04605923827100698811627173555937 4
    ck[22] = 0.04718554656929915394526147818109 9
    ck[23] = 0.04818586175708712914077949229830 5
    ck[24] = 0.04905543455502977888752816536723 8
    ck[25] = 0.04979568342707420635781156937994 2
    ck[26] = 0.05040592140278234684089308565358 5
    ck[27] = 0.05088179589874960649229747304980 5
    ck[28] = 0.05122154784925877217065628260494 4
    ck[29] = 0.05142612853745902593386287921578 1
    ck[30] = 0.05149472942945156755834043364709 9
    for i in range(nn-1,int(nn/2-1),-1):
        x[i] = -x[nn-1-i]
    for i in range(nn-1,int(nn/2-1),-1):
        ck[i] = ck[nn-1-i]
    for i in range(ng-1,-1,-1):
        cg[nn-2-2*i] = cg[i]
```

```
        cg[1+2*i] = cg[i]
    for i in range(0,int(nn/2+1)):
        cg[2*i] = 0
    k1 = 0.5*(b-a);
    k2 = 0.5*(b+a);
    intg = 0;
    intk = 0;
    for i in range(0,nn):
        v = ff.func(k1*x[i]+k2)
        intk = intk+v*ck[i];
        if  i%2==1:
            intg = intg+v*cg[i]
    intk = intk*(b-a)*0.5
    intg = intg*(b-a)*0.5
    A[0][0] = abs(intg-intk);
    A[0][1] = intk;
    A[0][2] = a;
    A[0][3] = b;
    toplamhata = A[0][0]
    if toplamhata<eps:
        sonuc = True
        integral = intk
        Aused = 1
        return integral
    Aused = 1
    for h in range(1,n):
        Aused = h+1
        gir(A, h, Aw)
        toplamhata = toplamhata-A[h-1][0]
        ta = A[h-1][2]
        tb = A[h-1][3]
        A[h-1][2] = ta
        A[h-1][3] = 0.5*(ta+tb)
        A[h][2] = 0.5*(ta+tb)
        A[h][3] = tb
        for j in range(h-1,h+1):
            k1 = 0.5*(A[j][3]-A[j][2])
            k2 = 0.5*(A[j][3]+A[j][2])
            intg = 0.0
            intk = 0.0
            for i in range(0,nn):
                v = ff.func(k1*x[i]+k2);
                intk = intk+v*ck[i];
                if  i%2==1:
                    intg = intg+v*cg[i]
            intk = intk*(A[j][3]-A[j][2])*0.5
            intg = intg*(A[j][3]-A[j][2])*0.5
            A[j][0] = abs(intg-intk)
            A[j][1] = intk
            toplamhata = toplamhata+A[j][0]

        cik(A, h-1, Aw);
        cik(A, h, Aw);
        if toplamhata<eps:
            break
    sonuc = toplamhata<eps
    integral = 0
    for j in range(0,Aused):
        integral = integral+A[j][1]
    return integral

def gir(A,n,Awidth):
    i = 0
    p = 0
    t = 0
    maxcp = 0
    if n==1:
        return
    for i in range(0,Awidth):
        t = A[n-1][i]
        A[n-1][i] = A[0][i]
        A[0][i] = t
    p = 0
    while 2*p+1<n-1:
        maxcp = 2*p+1;
        if 2*p+2<n-1:
```

```
            if A[2*p+2][0]>A[2*p+1][0]:
                maxcp = 2*p+2
        if  A[p][0]<A[maxcp][0]:
            for i in range(0,Awidth):
                t = A[p][i]
                A[p][i] = A[maxcp][i]
                A[maxcp][i] = t
            p = maxcp
        else:
            break



def cik(A,n,Awidth):
    i = 0
    p = 0
    t = 0
    kk = 0
    if n==0:
        return
    p = n
    while p!=0:
        kk =int( (p-1)/2)
        if A[p][0]>A[kk][0]:
            for i in range(0,Awidth):
                t = A[p][i];
                A[p][i] = A[kk][i]
                A[kk][i] = t
            p = kk
        else:
            break


class f1(if_x):func=lambda self,x: x*sqrt(x);
f=f1()
print("integral of class        f1 : ",gauss_kronrod_integral(f,0.0,1.0,1.0e-20,50))
```

```
runfile('E:/okul/SCO1/adaptive_Gauss_Kronrod_integral.py', wdir='E:/okul/SCO1')
Reloaded modules: if_x
integral of class        f1 :  0.4
```

Program Gauss-Kronrod integration formula , julia version

```
function  gauss_kronrad_integral(ff,a,b,eps,n)
    Aw = 4
    global A=OffsetArray{Float64}(undef,0:n-1,0:Aw-1)
    nn = 61
    ng = 15
    h=0
    i=0
    j=0
    v=0.0
    k1=0
    k2=0
    intg=0.0
    intk=0.0
    ta=0.0
    tb=0.0
    c = OffsetArray{Float64}(undef,0:nn-1)
    xk = OffsetArray{Float64}(undef,0:nn-1)
    xg =OffsetArray{Float64}(undef,0:nn-1)
    # Gauss-Legendre coefficients
    xg[0] = 0.00796819249616660561546588347467 4
    xg[1] = 0.01846646831109095914230213191204 7
    xg[2] = 0.02878470788332336934971917961129 2
    xg[3] = 0.03879919256962704959680193644634 8
    xg[4] = 0.04840267283059405290293814042280 8
    xg[5] = 0.05749315621761906648172168940205 6
    xg[6] = 0.06597422988218049512812851511596 2
    xg[7] = 0.07375597473770520626824385002219 1
```

xg[8] = 0.08075589522942021535469493846053 0
xg[9] = 0.08689978720108297980238753071512 6
xg[10] = 0.09212252223778612871763270708761 9
xg[11] = 0.09636873717464425963946862635181 0
xg[12] = 0.09959342058679526706278028210356 9
xg[13] = 0.10176238974840550459642895216855 4
xg[14] = 0.10285265289355884034128563670541 5
c[0] = 0.99948441005049063757132589570581 1
c[1] = 0.99689348407464954027163005091869 5
c[2] = 0.99163099687040459485862836610948 6
c[3] = 0.98366812327974720997003258160566 3
c[4] = 0.97311632250112626837469386842370 7
c[5] = 0.96002186496830751221687102558179 8
c[6] = 0.94437444474855997941583132403743 9
c[7] = 0.92620004742927432587932427708047 4
c[8] = 0.90557330769990779854652255892595 8
c[9] = 0.88256053579205268154311646253022 6
c[10] = 0.85720523354606109895865851065894 4
c[11] = 0.82956576238276839744289811973250 2
c[12] = 0.79972783582183908301366894232268 3
c[13] = 0.76777743210482619491797734097450 3
c[14] = 0.73379006245322680472617113136952 8
c[15] = 0.69785049479331579693229238802664 0
c[16] = 0.66006106412662696137005366814927 1
c[17] = 0.62052618298924286114047755643118 9
c[18] = 0.57934523582636169175602493217254 0
c[19] = 0.53662414814201989926416979331107 3
c[20] = 0.49248046786177857499369306120770 9
c[21] = 0.44703376953808917678060990032285 4
c[22] = 0.40040125483039439253547621154266 1
c[23] = 0.35270472553087811347103720708937 4
c[24] = 0.30407320227362507737267710719925 7
c[25] = 0.25463692616788984643980512981780 5
c[26] = 0.20452511668230989143895767100202 5
c[27] = 0.15386991360858354696379467274325 6
c[28] = 0.10280693796673703014709675131800 1
c[29] = 0.05147184255531769583302521316672 3
c[30] = 0.00000000000000000000000000000000 0
# Kronrod coefficients
xk[0] = 0.00138901369867700762455159122676 0
xk[1] = 0.00389046112709988405126720184451 6
xk[2] = 0.00663070391593129217331982636975 0
xk[3] = 0.00927327965951776342844114689202 4
xk[4] = 0.01182301525349634174223289885325 1
xk[5] = 0.01436972950704580481245143244358 0
xk[6] = 0.01692088918905327262757228942032 2
xk[7] = 0.01941414119394238117340895105012 8
xk[8] = 0.02182803582160919229716748573833 9
xk[9] = 0.02419116207808060136568637072523 2
xk[10] = 0.02650995488233310161060170933507 5
xk[11] = 0.02875404876504129284397878535433 4
xk[12] = 0.03090725756238776247288425294309 2
xk[13] = 0.03298144705748372603181419101685 4
xk[14] = 0.03497933802806002413749967073146 8
xk[15] = 0.03688236465182122922391106561713 6
xk[16] = 0.03867894562472759295034865153228 1
xk[17] = 0.04037453895153595911199527975246 8
xk[18] = 0.04196981021516424614714754128597 0
xk[19] = 0.04345253970135606931683172811707 3
xk[20] = 0.04481480013316266319235555161672 3
xk[21] = 0.04605923827100698811627173555937 4
xk[22] = 0.04718554656929915394526147818109 9

```
xk[23] = 0.048185861757087129140779492298305
xk[24] = 0.049055434555029778887528165367238
xk[25] = 0.049795683427074206357811569379942
xk[26] = 0.050405921402782346840893085653585
xk[27] = 0.050881795898749606492297473049805
xk[28] = 0.051221547849258772170656282604944
xk[29] = 0.051426128537459025933862879215781
xk[30] = 0.051494729429451567558340433647099
nn1=Int64(floor(nn/2))
for i=nn-1:-1:nn1
  c[i]=-c[nn-1-i]
end
for i=nn-1:-1:nn1
  xk[i]=xk[nn-1-i]
end
for i=ng-1:-1:0
  xg[nn-2-2*i]=xg[i]
  xg[1+2*i]=xg[i]
end
for i=0:nn1
  xg[2*i]=0
end
k1=0.5*(b-a)
k2=0.5*(b+a)
intg=0.0
intk=0.0
v=0.0
for i=0:nn-1
  v=ff(k1*c[i]+k2)
  intk=intk+v*xk[i]
  if i%2==1
    intg=intg+v*xg[i]
  end
end
intk=intk*(b-a)/2.0
intg=intg*(b-a)/2.0
A[0,0]=abs(intg-intk)
A[0,1]=intk
A[0,2]=a
A[0,3]=b
toplamhata=A[0,0]
integral=intk
if toplamhata<eps
   sonuc = true
   integral = intk
   Aused = 1
   return integral
end
Aused = 1
for h=1:n-1
    Aused = h+1
    gir(A, h, Aw)
    nh=h-1
    toplamhata = toplamhata-A[nh,0]
    ta = A[nh,2]
    tb = A[nh,3]
    A[nh,2] = ta
    A[nh,3] = 0.5*(ta+tb)
    A[h,2] = 0.5*(ta+tb)
    A[h,3] = tb
    for j=h-1:h
       k1 = 0.5*(A[j,3]-A[j,2])
```

```
            k2 = 0.5*(A[j,3]+A[j,2])
            intg = 0.0
            intk = 0.0
            for i=0:nn-1
              v = ff(k1*c[i]+k2)
               intk = intk+v*xk[i]
               if  i%2==1
                intg = intg+v*xg[i]
               end
            end # for i
            intk = intk*(A[j,3]-A[j,2])*0.5
            intg = intg*(A[j,3]-A[j,2])*0.5
            A[j,0] = abs(intg-intk)
            A[j,1] = intk
            toplamhata = toplamhata+A[j,0]
         end # for j
         cik(A,h-1,Aw)
         cik(A,h,Aw)
         if  toplamhata<eps
           break
         end
      end #for
      sonuc = toplamhata<eps
      integral = 0
      for j=0:Aused-1
        integral = integral+A[j,1]
      end # for j
return integral
end

function gir(A,n,Awidth)
    i = 0
    p = 0
    t = 0
    maxcp = 0
     if  n==1
       return
     end
     for i=0:Awidth-1
        t = A[n-1,i]
        A[n-1,i] = A[0,i]
        A[0,i] = t
     end # for
     p = 0
     while( 2*p+1<n-1 )
        maxcp = 2*p+1
        if (2*p+2)<(n-1)
           nx1= 2*p+2
           nx2=2*p+1
           if  A[nx1,0]>A[nx2,0]
             maxcp = 2*p+2
             end
        end
        if  A[p,0]<A[maxcp,0]
          for i=0:Awidth-1
             t = A[p,i]
             A[p,i] = A[maxcp,i]
             A[maxcp,i] = t
             end # for
          p = maxcp
        else
          break
```

```
          end # if
       end #of while
   end

function cik(A,n,Awidth)
       i = 0
       p = 0
       t = 0.0
       kk = 0
       if n==0
         return
       end
       p = n
       while( p!=0 )
          kk = Int64(floor((p-1)/2))
          if A[p,0]>A[kk,0]
          for i=0:Awidth-1
             t = A[p,i]
               A[p,i] = A[kk,i]
               A[kk,i] = t
             end # for
             p = kk
          else
            break
          end #if
       end # while
end

using OffsetArrays
f(x)=x^1.5
a=0.0
b=1.0
eps=1.0e-15
n=500
I=gauss_kronrad_integral(f,a,b,eps,n)
print("I = $I")
```

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" gauss_kronrad_integral.jl
I = 0.40000000000000024
> Terminated with exit code 0.
```

Julia Built-in Gauss-Kronroad formula integration:

```
using Pkg; Pkg.add("QuadGK")
```

```
using QuadGK
f(x)=x^1.5
I1=quadgk(f,0.0,1.0)[1]
print("integral  = $I1 I_exact= 0.4")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" integral_gk.jl
integral  = 0.39999999998326774 I_exact= 0.4
> Terminated with exit code 0.
```

7.4.4 Adaptive Newton-Cotes integration with Gander ve Gautschi[31] approach

In the previous adaptive integration approaches program follows a stopping criteria such as

if(Math.abs(I1-I2)< tolerance or

if(Math.abs((I1-I2)/I1) < tolerance

This type of stopping criteria might be unsuccessfull for some cases, and Maximum iteration error will be received. If we would like to further decrease the error and reach the machine error level a different type of error control can be used. If an error control such as

if(is+(I1-I2)==is)

is used, this control works only when I1 is equal to I2 with the machine accuracy. If this definition to be used with the halving approach

$m = \frac{(a-b)}{2}$ with an additional control definition can even give a better exit criteria

if((is+(I1-I2)==is) || (m<=a) || (b<=m))

Now this criteria will be applied to Newton-Cotes formulas. In this algorithm Newton Cotes Formula with 8th degree polynomial and quadratic polinomial (Simpson 1/3) will be used. 8th degree Newton cotes Formula :

$$is = \int_a^b f(x)dx = \frac{(b-a)}{28350}[989f(x_0) + 5888f(x_1) - 928f(x_2) + 10496f(x_3) - 4540f(x_4)$$
$$+10496f(x_5) - 928f(x_6) + 5888f(x_7) + 989f(x_8)]$$

Simpson 1/3integral in iterative form will be:

$$I1 = I = \int_a^b f(x)dx = I[a,b] = \frac{(b-a)}{6}[f(x_0) + 4f(x_2) + f(x_4)]$$

$$I2 = I[a,m] + I[m,b] = \int_a^m f(x)dx + \int_m^b f(x)dx =$$

$$\frac{(m-a)}{6}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{(b-m)}{6}[f(x_2) + 4f(x_3) + f(x_4)] =$$

$$\frac{(b-a)}{12}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

Program Gander and Gautschi adaptive Simpson 1/3 integral formulation, java version

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Adaptive Gander and Gautschi simpson 1/3 integration
// Dr. Turhan Coban
// =====================================================
import java.io.*;

class f1 extends f_x
{
  double func(double x)
  {return Math.sin(x);}
}

class NA70
{
public static double adaptive_simpson_integral2(f_x ff,double a,double b)
{   int i;
            Plot pp=new Plot(ff,a,b,100);
    double h=(b-a);
    double h2=h/4.0;
    double h3=h/8.0;
    double m=(a+b)/2.0;
    double x[]=new double[9];
    double y[]=new double[9];
```

```
    for(i=0;i<9;i++) {x[i]=a+i*h3;y[i]=ff.func(x[i]);}
    double is=h/28350.0*(989.0*y[0]+5888.0*y[1]-928.0*y[2]+10496.0*y[3]-4540.0*y[4]+
          10496.0*y[5]-928.0*y[6]+5888.0*y[7]+989.0*y[8]);
    double fa = y[0];
    double fm = y[4];
    double fb = y[8];
    if(is==0) is = b-a;
    double Q = adaptsimstp(ff,a,b,fa,fm,fb,is);
    return Q;
}

public static double adaptsimstp(f_x ff,double a,double b,double fa,double fm,double fb,double is)
{
  double m = (a + b)/2;
  double h = (b - a)/4;
  double x[] = {a + h, b - h};
  int n=x.length;
  double y[] = new double[n];
  for(int i=0;i<n;i++) {y[i]=ff.func(x[i]);}
  double fml = y[0];
  double fmr = y[1];
  double i1 = h/1.5 * (fa + 4*fm + fb);
  double i2 = h/3 * (fa + 4*(fml + fmr) + 2*fm + fb);
  double Q=0;
  i1 = (16.0*i2 - i1)/15.0; //romberg integrasyonu ile iyileştirme
  if (((is + (i1-i2)) == is) || (m <= a) || (b<=m))
  {
    if ((m <= a) || (b<=m))
       { System.out.println("istenilen toleransa ulaşilamadi");}
    Q = i1;
  }
  else
  {
    Q = adaptsimstp(ff,a,m,fa,fml,fm,is)+
        adaptsimstp(ff,m,b,fm,fmr,fb,is);
  }
  return Q;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+adaptive_simpson_integral2(b1,0.0,Math.PI));
  }
}
```

Gander and Gautschi adaptive Simpson 1/3 integral Formula with sample integral

$$I = \int_0^{\pi} \sin(x)dx = 2$$

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA70
integral of class         f1 : 2.0

> Terminated with exit code 0.
```

Program : Gander and Gautschi adaptive Simpson 1/3 integral formulation, Julia version

```
 # ======================================================
# Numerical Analysis package in Julia
# example to show utilisation of integration (integral)
# and differentiation (derivative) functions
```

```
# Adaptive Gander and Gautschi simpson 1/3 integration
# Dr. M. Turhan Coban
# ====================================================
function  adaptive_simpson_integral2(ff,a,b)
   i=0
   h=(b-a)
   h2=h/4.0
   h3=h/8.0
   m=(a+b)/2.0
   x=OffsetArray{Float64}(undef,0:8)
   y=OffsetArray{Float64}(undef,0:8)
   for i=0:8
     x[i]=a+i*h3
     y[i]=ff(x[i])
   end #for
   is=h/28350.0*(989.0*y[0]+5888.0*y[1]-928.0*y[2]+10496.0*y[3]-4540.0*y[4]+
        10496.0*y[5]-928.0*y[6]+5888.0*y[7]+989.0*y[8])
   fa = y[0]
   fm = y[4]
   fb = y[8]
   if is==0
    is = b-a
   end #if
   Q = adaptsimstp(ff,a,b,fa,fm,fb,is)
   return Q
end

function adaptsimstp(ff,a,b,fa,fm,fb,is)
  m = (a + b)/2
  h = (b - a)/4
  x=OffsetArray{Float64}(undef,0:1)
  x[0]=a + h
  x[1]=b - h
  n=size(x)[1]
  y=OffsetArray{Float64}(undef,0:n-1)
  for i=0:n-1
   y[i]=ff(x[i])
  end
  fml = y[0]
  fmr = y[1]
  i1 = h/1.5 * (fa + 4*fm + fb)
  i2 = h/3 * (fa + 4*(fml + fmr) + 2*fm + fb)
  global Q=0
  i1 = (16.0*i2 - i1)/15.0 #romberg integration
  nx1=is + (i1-i2)
  if nx1==is || m<= a || b<=m
    if m <= a || b<=m
      print("required tolerance can not be achieved")
    end #if m
    Q = i1
  else
    Q = adaptsimstp(ff,a,m,fa,fml,fm,is)+adaptsimstp(ff,m,b,fm,fmr,fb,is)
  end #if nx1
  return Q
  end #function


using OffsetArrays
f(x)=sin(x)
a=0.0
b=pi
I=adaptive_simpson_integral2(f,a,b)
```

```
print("I = $I")
```

Gander and Gautschi adaptive Simpson 1/3 integral Formula with sample integral

$$I = \int_0^{\pi} sin(x)dx$$

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" adaptive_simpson_integral2.jl
I = 2.0
> Terminated with exit code 0.
```

$$I = \int_0^{1} x^{1.5}dx$$

```
using OffsetArrays
f(x)=x^1.5
a=0.0
b=1.0
I=adaptive_simpson_integral2(f,a,b)
print("I = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" adaptive_simpson_integral2.jl
I = 0.4
> Terminated with exit code 0.
```

7.4-5 Adaptive Gauss-Lobatto integral with Gander ve Gautschi[31] approach

Gauss-Lobatto integration Formula is similar to Gauss integration Formula. The most important difference is that Lobatto Formula also covers end points

$$I_w(-1,1) \cong \int_{-1}^{1} f(x)dx \cong \sum_{k=0}^{n-1} c_k f(x_k)$$

For various n values Gauss-Lobatto coefficients and roots are given

Table 7.4-1 Gauss-Lobatto integration equation coefficients and roots

N	Xk	Ck
4	-1.000000000000000000000000	0.166666666666667000000000
	1.000000000000000000000000	0.166666666666667000000000
	-0.447213595499958000000000	0.833333333333333000000000
	0.447213595499958000000000	0.833333333333333000000000
5	-1.000000000000000000000000	0.100000000000000000000000
	1.000000000000000000000000	0.100000000000000000000000
	-0.654653670707977000000000	0.544444444444444000000000
	0.654653670707977000000000	0.544444444444444000000000
	0.000000000000000000000000	0.711111111111111000000000
6	-1.000000000000000000000000	0.066666666666666600000000
	1.000000000000000000000000	0.066666666666666600000000
	-0.765055323929465000000000	0.378474956297847000000000
	0.765055323929465000000000	0.378474956297847000000000
	-0.285231516480635000000000	0.554858377035486000000000
	0.285231516480635000000000	0.554858377035486000000000
7	-1.000000000000000000000000	0.052380952380952400000000
	1.000000000000000000000000	0.052380952380952400000000
	-0.816496580927726000000000	0.293877551020408000000000
	0.816496580927726000000000	0.293877551020408000000000
	-0.447213595499958000000000	0.425170068027211000000000
	-0.447213595499958000000000	0.425170068027211000000000
	0.000000000000000000000000	0.457142857142857000000000
12	-1.000000000000000000000000	0.015827191973480100000000
	1.000000000000000000000000	0.015827191973480100000000

	-0.9428824156954790000000	0.0942738402188500000000
	0.9428824156954790000000	0.0942738402188500000000
	-0.8164965809277260000000	0.1550719873365850000000
	0.8164965809277260000000	0.1550719873365850000000
	-0.6418533423457810000000	0.1888215739601820000000
	0.6418533423457810000000	0.1888215739601820000000
	-0.4472135954999580000000	0.1997734052268580000000
	0.4472135954999580000000	0.1997734052268580000000
	-0.2363831996621490000000	0.2249264653333390000000
	0.2363831996621490000000	0.2249264653333390000000
	0.0000000000000000000000	0.2426110719014070000000

As the previous simpson 1/3 approach **Gander ve Gautschi[31]** approach will be used as a stopping criteria in this program also. For the interval [a,b]

if((is+(I1 - I2) == is) || (m<=a) || (b<=m))

In this condition m is given as

$m = \frac{(a-b)}{2}$. For the is first integration value degree 12 Lobatto integration will be used.

$$is \cong \int_a^b f(x)dx \cong (b-a)*\{0.0158271919734801*[f(a)+f(b)]+$$
$$0.0942738402188500*[f(m-h*0.9428824156954790)+f(m+h*0.9428824156954790)]+$$
$$0.1550719873365850*[f(m-h*0.8164965809277260)+f(m+h*0.8164965809277260)]+$$
$$0.1888215739601820*[f(m-h*0.6418533423457810)+f(m+h*0.6418533423457810)]+$$
$$0.1997734052268580*[f(m-h*0.4472135954999580)+f(m+h*0.4472135954999580)]+$$
$$0.2249264653333390*[f(m-h*0.2363831996621490)+f(m+h*0.2363831996621490)]+$$
$$0.2426110719014070*[f(0)]\}$$

As iteration Formula 4th and 7th degree Gauss-Lobatto formulations will be used.

$$I1 \cong \int_a^b f(x)dx \cong (b-a)*\{0.1666666666666670*[f(a)+f(b)]+$$
$$0.8333333333333330*[f(m-h*0.4472135954999580)+f(m+h*0.4472135954999580)]\}$$

$$I2 \cong \int_a^b f(x)dx \cong (b-a)*\{0.0523809523809524*[f(a)+f(b)]+$$
$$0.2938775510204080*[f(m-h*0.8164965809277260)+f(m+h*0.8164965809277260)]+$$
$$0.4251700680272110*[f(m-h*0.4472135954999580)+f(m+h*0.4472135954999580)]+$$
$$0.4571428571428570*[f(0)]\}$$

Program code for this approach is given below.

Program 7.4-5 Gander and Gautschi adaptive Gauss-Lobatto integral formula, java version

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gander and Gautschi adaptive Gauss-Lobatto integral
// Dr. Turhan Coban
// =====================================================
import java.io.*;
class f1 extends f_x
{
  double func(double x)
  {//return 13.0*(x-x*x)*Math.exp(-1.5*x);
  return Math.sin(x);
  }
```

```
}

class NA71
{

public static double adaptive_Lobatto_integral(f_x ff,double a,double b)
{   int i;
            double m=(a+b)/2.0; double h=(b-a)/2.0;
   double alpha=Math.sqrt(2.0/3.0);
   double beta=1.0/Math.sqrt(5.0);
   double x1,x2,x3;
   x1=0.942882415695479719056351758431857202 32;
   x2=0.641853342345781305781235541329031883 54;
   x3=0.236383199662149880282223773492052925 99;
   double A,B,C,D,E,F,G;
   A=0.015827191973480183087169986733305510591;
   B=0.094273840218850045531282505077108171960;
   C=0.155071987336585396253635979802102986 80;
   D=0.188821573960182454420005339372971671 25;
   E=0.199773405226858526792068022066488402 46;
   F=0.224926465333339527016017687996395080 76;
   G=0.242611071901407733799640957903256352 33;
            double x[]={a,(m-x1*h),(m-alpha*h),(m-x2*h),(m-beta*h),(m-
x3*h),m,(m+x3*h),(m+beta*h),(m+x2*h),(m+alpha*h),(m+x1*h),b};
            int n=x.length;
            double y[]=new double[n];
            for(i=0;i<n;i++) y[i]=ff.func(x[i]);
            double fa=y[0];
            double fb=y[12];
   double i2=(h/6.0)*(y[0]+y[12]+5.0*(y[4]+y[8]));
   double i1=(h/1470.0)*(77.0*(y[0]+y[12])+432.0*(y[2]+y[10])+
    625.0*(y[4]+y[8])+672.0*y[6]);
   double
is=h*(A*(y[0]+y[12])+B*(y[1]+y[11])+C*(y[2]+y[10])+D*(y[3]+y[9])+E*(y[4]+y[8])+F*(y[5]+y[7])+G*y[6]);
   int s=(int)(is/Math.abs(is));
   if(s==0) s=1;
   double erri1=Math.abs(i1-is);
   double erri2=Math.abs(i2-is);
   double R;
   R=erri1/erri2;
   //if(R>0 && R<1) tol=tol/R;
   //is=s*Math.abs(is)*tol/eps;
   if(is==0) is=b-a;
  double Q=0;
  Q=Lobatto(ff,a,b,fa,fb,is);
  return Q;
}

public static double Lobatto(f_x ff,double a,double b,double fa,double fb,double is)
{
  double h=(b-a)/2;
  double m=(a+b)/2;
  double alpha=Math.sqrt(2.0/3.0);
  double beta=1.0/Math.sqrt(5.0);
  double mll=m-alpha*h;
  double ml=m-beta*h;
  double mr=m+beta*h;
  double mrr=m+alpha*h;
  double x[]={mll,ml,m,mr,mrr};
  int n=x.length;
  double y[]=new double[n];
  for(int i=0;i<n;i++) y[i]=ff.func(x[i]);
```

```
  double fmll=y[0];
  double fml=y[1];
  double fm=y[2];
  double fmr=y[3];
  double fmrr=y[4];
  double Q;
  double i2=(h/6)*(fa+fb+5*(fml+fmr));
  double i1=(h/1470)*(77*(fa+fb)+432*(fmll+fmrr)+625*(fml+fmr)+672*fm);
  boolean c1,c2,c3;
  double w1=i1-i2;;
  double w2=is;
  double w3=w2+w1;
  c1=(w3==w2);
  c2=(mll<=a);
  c3=(b<=mrr);
  if( c1|| c2 ||c3 )
  {
   if (((m <= a) || (b<=m)))
   {
      System.out.println("istenilen toleransa ulaşilamadi");
   }
   Q=i1;
  }
  else
  {
    Q=Lobatto(ff,a,mll,fa,fmll,is) + Lobatto(ff,mll,ml,fmll,fml,is)+
      Lobatto(ff,ml,m,fml,fm,is)   + Lobatto(ff,m,mr,fm,fmr,is)+
      Lobatto(ff,mr,mrr,fmr,fmrr,is)+Lobatto(ff,mrr,b,fmrr,fb,is);
  }
  return Q;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+adaptive_Lobatto_integral(b1,0.0,Math.PI));
  }
}
```

Gander and Gautschi adaptive Gauss-Lobatto integral formula

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA71
integral of class         f1 : 2.0

> Terminated with exit code 0.
```

Gander and Gautschi adaptive Gauss-Lobatto integral formula, Julia version

```
# ======================================================
# Numerical Analysis package in Julia
# example to show utilisation of integration (integral)
# and differentiation (derivative) functions
# Gander and Gautschi adaptive Gauss-Lobatto integral
# Dr. M. Turhan Coban
# ======================================================
function adaptive_Lobatto_integral(ff,a,b)
   i=0
          m=(a+b)/2.0
          h=(b-a)/2.0
   alpha=sqrt(2.0/3.0)
```

```
    beta=1.0/sqrt(5.0)
    x1=0.94288241569547971905635175843185720232
    x2=0.64185334234578130578123554132903188354
    x3=0.23638319966214988028222377349205292599
    A=0.01582719197348018308716998673330551059 1
    B=0.09427384021885004553128250507710817196 0
    C=0.15507198733658539625363597980210298680
    D=0.18882157396018245442000533937297167125
    E=0.19977340522685852679206802206648840246
    F=0.22492646533333952701601768799639508076
    G=0.24261107190140773379964095790325635233
    n=13
    x=OffsetArray{Float64}(undef,0:n-1)
    y=OffsetArray{Float64}(undef,0:n-1)
    x[0]=a;x[1]=m-x1*h;x[2]=m-alpha*h;x[3]=m-x2*h;x[4]=m-beta*h;x[5]=m-
x3*h;x[6]=m;x[7]=m+x3*h;x[8]=m+beta*h;x[9]=m+x2*h;x[10]=m+alpha*h;x[11]=m+x1*h;;x[12]=b
    #x=Float64[a,(m-x1*h),(m-alpha*h),(m-x2*h),(m-beta*h),(m-
x3*h),m,(m+x3*h),(m+beta*h),(m+x2*h),(m+alpha*h),(m+x1*h),b]
              for i=0:n-1
                y[i]=ff(x[i])
              end
              fa=y[0]
              fb=y[n-1]
    i2=(h/6.0)*(y[0]+y[12]+5.0*(y[4]+y[8]))
    i1=(h/1470.0)*(77.0*(y[0]+y[12])+432.0*(y[2]+y[10])+625.0*(y[4]+y[8])+672.0*y[6])
    is=h*(A*(y[0]+y[12])+B*(y[1]+y[11])+C*(y[2]+y[10])+D*(y[3]+y[9])+E*(y[4]+y[8])+F*(y[5]+y[7])+G*y[6])
    s=Int64(is/abs(is))
    if(s==0)
      s=1
    end #if
    erri1=abs(i1-is)
    erri2=abs(i2-is)
    R=erri1/erri2
    if(is==0)
      is=b-a
    end #if
  global Q=0
  Q=Lobatto(ff,a,b,fa,fb,is)
  return Q
end

function Lobatto(ff,a,b,fa,fb,is)
  h=(b-a)/2.0
  m=(a+b)/2.0
  alpha=sqrt(2.0/3.0)
  beta=1.0/sqrt(5.0)
  mll=m-alpha*h
  ml=m-beta*h
  mr=m+beta*h
  mrr=m+alpha*h
  n=5
  x=OffsetArray{Float64}(undef,0:n-1)
  x[0]=mll;x[1]=ml;x[2]=m;x[3]=mr;x[4]=mrr
  #x=[mll,ml,m,mr,mrr]
  #n=size(x)[1]
  y=OffsetArray{Float64}(undef,0:n-1)
  for i=0:n-1
   y[i]=ff(x[i])
  end #for
  fmll=y[0]
  fml=y[1]
  fm=y[2]
```

```
  fmr=y[3]
  fmrr=y[4]
  i2=(h/6)*(fa+fb+5*(fml+fmr))
  i1=(h/1470)*(77*(fa+fb)+432*(fmll+fmrr)+625*(fml+fmr)+672*fm)
  w1=i1-i2
  w2=is
  w3=w2+w1
  c1=(w3==w2)
  c2=(mll<=a)
  c3=(b<=mrr)
  if  c1|| c2 ||c3
   if (m <= a) || (b<=m)
      print("required accuracy can nor be achieved")
   end #if
   Q=i1
  else
    Q=Lobatto(ff,a,mll,fa,fmll,is) + Lobatto(ff,mll,ml,fmll,fml,is)+
      Lobatto(ff,ml,m,fml,fm,is)   + Lobatto(ff,m,mr,fm,fmr,is)+
      Lobatto(ff,mr,mrr,fmr,fmrr,is)+Lobatto(ff,mrr,b,fmrr,fb,is)
  end # if
  return Q
end #function

using OffsetArrays
f(x)=sin(x)
a=0.0
b=pi
I=adaptive_Lobatto_integral(f,a,b)
print("I = $I")
```

Gander and Gautschi adaptive Gauss-Lobatto integral formula

$$I = \int_0^{\pi} sin(x)dx$$

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" adaptive_Lobatto_integral.jl
I = 2.0
> Terminated with exit code 0.
```

$$I = \int_0^{1} x^{1.5}dx$$

```
using OffsetArrays
f(x)=x^1.5
a=0.0
b=1.0
I=adaptive_Lobatto_integral(f,a,b)
print("I = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" adaptive_Lobatto_integral.jl
I = 0.4
> Terminated with exit code 0.
```

In order to see how successfull the code will be a mathematical function test will be carried out as the next example. The definition of the error function :

$$erf(\eta) = \frac{1}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

Because of the difficulty of calculating this equation with integration Formula, it is generally calculated by serial solutions. In our example both serial solution and Gauss-Lobatto adaptive integration solution will be carried out and compared.

Program 7.4-6 Calculation of error function with Gander and Gautschi adaptive Gauss-Lobatto integral formula

```
//=====================================================
// Numerical Analysis package in java
// error function calculation by using
// Adaptive Gauss-Lobatto integral and serial solution
// Referance : Adapted Quadrature - Revisited
// Walter Gander - Walter Gautchi
// ftp.inf.ethz.ch
// Dr. Turhan Coban
// =====================================================

import java.io.*;
import javax.swing.*;

class f1 extends f_x
{ double func(double x)
  {return 2.0/Math.sqrt(Math.PI)*Math.exp(-x*x);}
}

class NA72
{

public static double gammp(double a,double x)
{
        //incomplete gamma function P(a,x)
        double gamser,gammcf;
        if (x < 0.0 || a <= 0.0) System.out.println("Invalid arguments in routine GAMMP");
        if (x < (a+1.0)) {
                gamser=gser(a,x);
                return gamser;
        } else {
                gammcf=gcf(a,x);
                return 1.0-gammcf;
        }
}
public static double gser(double a,double x)
{
        int ITMAX=100;
        double EPS=3.0e-7;
        int n;
        double sum,del,ap;
        double gln=gammln(a);
        double gamser=0;
        if (x <= 0.0) {
                if (x < 0.0) System.out.println("x less than 0 in routine GSER");
                gamser=0.0;
                return gamser;
        } else {
                ap=a;
                del=sum=1.0/a;
                for (n=1;n<=ITMAX;n++) {
                        ap += 1.0;
                        del *= x/ap;
                        sum += del;
                        if (Math.abs(del) < Math.abs(sum)*EPS) {
                                gamser=sum*Math.exp(-x+a*Math.log(x)-(gln));
                                return gamser;
                        }
                }
```

```
                System.out.println("a too large, ITMAX too small in routine GSER");
                return gamser;
        }
}

public static double gcf(double a,double x)
{
        int ITMAX=100;
        double EPS=3.0e-7;
        int n;
        double gln;
        double gammcf=0;
        double gold=0.0,g,fac=1.0,b1=1.0;
        double b0=0.0,anf,ana,an,a1,a0=1.0;
        gln=gammln(a);
        a1=x;
        for (n=1;n<=ITMAX;n++) {
                an=(double) n;
                ana=an-a;
                a0=(a1+a0*ana)*fac;
                b0=(b1+b0*ana)*fac;
                anf=an*fac;
                a1=x*a0+anf*a1;
                b1=x*b0+anf*b1;
                if (a1!=0) {
                        fac=1.0/a1;
                        g=b1*fac;
                        if (Math.abs((g-gold)/g) < EPS) {
                                gammcf=Math.exp(-x+a*Math.log(x)-gln)*g;
                                return gammcf;
                        }
                        gold=g;
                }
        }
        System.out.println("a too large, ITMAX too small in routine GCF");
   return gammcf;
}
public static double gammln(double xx)
{
        //gamma function
        double x,y,tmp,ser;
        double cof[]={76.18009172947146,-86.50532032941677,
                24.01409824083091,-1.231739572450155,
                0.1208650973866179e-2,-0.5395239384953e-5};
        int j;
        y=x=xx;
        tmp=x+5.5;
        tmp -= (x+0.5)*Math.log(tmp);
        ser=1.000000000190015;
        for (j=0;j<=5;j++) ser += cof[j]/++y;
        return -tmp+Math.log(2.5066282746310005*ser/x);
}
public static double  erf(double x)
{
  //error function from series solution
  return x < 0.0 ? -gammp(0.5,x*x) : gammp(0.5,x*x);
}

public static double  erf_int(double x)
{
  //error function from series solution
  f1 b1=new f1();
```

```
   return adaptive_Lobatto_integral(b1,0.0,x);
}


public static double adaptive_Lobatto_integral(f_x ff,double a,double b)
{double tol=Double.MIN_VALUE;
 return adaptive_Lobatto_integral(ff,a,b,tol);
}

public static double adaptive_Lobatto_integral(f_x ff,double a,double b,double tol)
{   int i;
            double m=(a+b)/2.0; double h=(b-a)/2.0;
   double alpha=Math.sqrt(2.0/3.0);
   double beta=1.0/Math.sqrt(5.0);
   double x1,x2,x3;
   double eps =Double.MIN_VALUE;
   //System.out.println("eps="+eps);
   x1=0.94288241569547971905635175843185720232;
   x2=0.64185334234578130578123554132903188354;
   x3=0.23638319966214988028222377349205292599;
   double A,B,C,D,E,F,G;
   A=0.01582719197348018308716998673330551059 1;
   B=0.09427384021885004553128250507710817196 0;
   C=0.15507198733658539625363597980210298680;
   D=0.18882157396018245442000533937297167125;
   E=0.19977340522685852679206802206648840246;
   F=0.22492646533333952701601768799639508076;
   G=0.24261107190140773379964095790325635233;
            double x[]={a,(m-x1*h),(m-alpha*h),(m-x2*h),(m-beta*h),(m-
x3*h),m,(m+x3*h),(m+beta*h),(m+x2*h),(m+alpha*h),(m+x1*h),b};
            int n=x.length;
            double y[]=new double[n];
            for(i=0;i<n;i++) y[i]=ff.func(x[i]);
            double fa=y[0];
            double fb=y[12];
   double i2=(h/6.0)*(y[0]+y[12]+5.0*(y[4]+y[8]));
   double i1=(h/1470.0)*(77.0*(y[0]+y[12])+432.0*(y[2]+y[10])+
    625.0*(y[4]+y[8])+672.0*y[6]);
   double
is=h*(A*(y[0]+y[12])+B*(y[1]+y[11])+C*(y[2]+y[10])+D*(y[3]+y[9])+E*(y[4]+y[8])+F*(y[5]+y[7])+G*y[6]);
   int s=(int)(is/Math.abs(is));
   if(s==0) s=1;
   double erri1=Math.abs(i1-is);
   double erri2=Math.abs(i2-is);
   double R;
   R=erri1/erri2;
   if(R>0 && R<1) tol=tol/R;
   //System.out.println("is="+is+"s="+s+"tol="+tol+"eps="+eps);
   if(is==0) is=b-a;
  double Q=0;
  Q=Lobatto(ff,a,b,fa,fb,is,tol);
  return Q;
}

public static double Lobatto(f_x ff,double a,double b,double fa,double fb,double is,double tol)
{
  double h=(b-a)/2;
  double m=(a+b)/2;
  double alpha=Math.sqrt(2.0/3.0);
  double beta=1.0/Math.sqrt(5.0);
  double mll=m-alpha*h;
  double ml=m-beta*h;
```

```
  double mr=m+beta*h;
  double mrr=m+alpha*h;
  double x[]={mll,ml,m,mr,mrr};
  int n=x.length;
  double y[]=new double[n];
  for(int i=0;i<n;i++) y[i]=ff.func(x[i]);
  double fmll=y[0];
  double fml=y[1];
  double fm=y[2];
  double fmr=y[3];
  double fmrr=y[4];
  double Q;
  double i2=(h/6)*(fa+fb+5*(fml+fmr));
  double i1=(h/1470)*(77*(fa+fb)+432*(fmll+fmrr)+625*(fml+fmr)+672*fm);
  double di=Math.abs(i1-i2);
  if (((is + di) == is) || (mll <= a) || (b<=mrr) || di<tol)
  {
   if (((m <= a) || (b<=m)))
   { System.out.println("istenilen toleransa ulaşilamadi delta i = "+di);}
   Q=i1;
  }
  else
  {
    Q=Lobatto(ff,a,mll,fa,fmll,is,tol) + Lobatto(ff,mll,ml,fmll,fml,is,tol)+
      Lobatto(ff,ml,m,fml,fm,is,tol)   + Lobatto(ff,m,mr,fm,fmr,is,tol)+
      Lobatto(ff,mr,mrr,fmr,fmrr,is,tol)+Lobatto(ff,mrr,b,fmrr,fb,is,tol);
  }
  return Q;
}

  public static void main(String args[]) throws IOException
  {
  double x=Double.parseDouble(JOptionPane.showInputDialog(" x = "));
  System.out.println("erf("+x+") = "+erf(x)+"integral erf="+erf_int(x));
  }
}
```

Calculation of error function with Gander and Gautschi adaptive Gauss-Lobatto integral Formula output

```
erf(0.1) = 0.1124629159993309  integral erf=0.1124629160182849
erf(0.4) = 0.4283923519241598  integral erf=0.42839235504666845
erf(1.0) = 0.8427007900291826  integral erf=0.8427007929497149
erf(2.0) = 0.995322264754383   integral erf=0.9953222650189527
```

CLENSHAW-CURTIS INTEGRATION

Clenshaw-Curtis formulation is based on $x = \cos(\theta)$ transformation. When this transformation applies, in the range of -1 to 1 integration equation becomes:

$$\int_{-1}^{1} f(x)dx = \int_{0}^{\pi} f(\cos(\theta))d(\cos(\theta)) = \int_{0}^{\pi} f(\cos(\theta))\sin(\theta)\,d\theta$$

Form. This integration can be opened as a series as:

$$f(\cos(\theta)) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\theta)$$

By applying this transformation integral can be carried out as:

$$\int_{0}^{\pi} f(\cos(\theta))\sin(\theta)\,d\theta = a_0 + \sum_{k=1}^{\infty} \frac{2a_{2k}}{1-(2k)^2}$$

Where

$$a_k = \frac{2}{k}\int_0^{\pi} f(\cos(\theta))\cos(k\theta)\,d\theta$$

This integration cam also open to series as:

$$a_k = \frac{2}{N}\left[\frac{f(1)}{2} + \frac{f(-1)}{2}(-1)^k + \sum_{n=1}^{N-1} f(cos\left(\frac{n\pi}{N}\right)cos\left(\frac{nk\pi}{N}\right)\right]$$

When the actual integral is in between limits of x_1 and x_2 values, coordinate conversion should be apply

$$k_2 = \frac{x_2 - x_1}{2}$$

$$k_1 = \frac{x_2 + x_1}{2}$$

$$\int_a^b f(x)dx = k_1\left[a_0 + \sum_{k=1}^{\infty}\frac{2a_{2k}}{1-(2k)^2}\right]$$

$$x = k_2 + k_1 cos\left(\frac{n\pi}{N}\right)$$

$$a_k = \frac{2}{N}\left[\frac{f(x_1)}{2} + \frac{f(x_2)}{2}(-1)^k + \sum_{n=1}^{N-1} f(x)cos\left(\frac{nk\pi}{N}\right)\right]$$

Program 7.5-1 Clenshaw-Curtis integrasyonu , java code

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Dr. Turhan Coban
// =====================================================
import java.io.*;


class f1 extends f_x
{
  public double func(double x)
  {return x*x;}
}

class integ12
{
public static double[] clenshaw_curtis_coefficients(f_x f,double x1,double x2,int N)
{   double a[]=new double[2*N+1];
                int cc=-1;
                int n,k;
                double x;
                double k1=(x2-x1)/2.0;
    double k2=(x2+x1)/2.0;
                for(k=0;k<2*N;k++)
                {
                a[k]=0.0;
                for (n=1;n<N;n++)
                {x=k2+k1*Math.cos(Math.PI*n/N);
                 a[k]+=f.func(x)*Math.cos(n*k*Math.PI/N);}
     cc*=-1;
                 a[k]+=f.func(x2)/2.0+f.func(x1)/2.0*cc;
                 a[k]*=2.0/n;
    }
    return a;
}
public static double Aitken_clenshaw_curtis(f_x ff,double a,double b, int n)
{   //Aitken interpolation formula
   double I8n=clenshaw_curtis_integral(ff,a,b,8*n);
    double I4n=clenshaw_curtis_integral(ff,a,b,4*n);
```

```
    double I2n=clenshaw_curtis_integral(ff,a,b,2*n);
    double In=clenshaw_curtis_integral(ff,a,b,n);
    double I=I4n-(I4n-I2n)*(I4n-I2n)/((I4n-I2n)-(I2n-In));
    return I;
}

public static double clenshaw_curtis_integral(f_x f,double x1,double x2,int N)
{
double a[]=clenshaw_curtis_coefficients(f,x1,x2,N);
double z=a[0];
double k1=(x2-x1)/2.0;
double k2=(x2+x1)/2.0;
for(int k=1;k<N;k+=2)
{ z+=2.0*a[2*k]/(1.0-4.0*k*k);}
return k1*z;
}
  public static void main(String args[]) throws IOException
  {
  //Gauss-Legendre integrali
  f1 b1=new f1();
  double I1=clenshaw_curtis_integral(b1,0.0,1.0,100);
  double I2=Aitken_clenshaw_curtis(b1,0.0,1.0,100);
  System.out.println("integral  : "+I1+"\nAitken interpolation corrrection = "+I2);

}}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" integ12
integral  : 0.3333301448018433
Aitken interpolation corrrection = 0.33333333158085837

> Terminated with exit code 0.
```

7.4 ONE AND MULTI-DIMENSIONAL MONTE-CARLO INTEGRAL

Monte Carlo integral is based on a very simple principle

$$\int_a^b f(x)dx = (b-a)f_{average}$$

$$x = a + (b-a)random\_number(0,1)$$

$$f_{average} = \frac{\sum_{i=0}^{n} f(x)}{n} \qquad n \gg 1$$

In the interval [a,b] a random x number is selected and function is evalueated in this point. When this sequaence is repeated for a very large number of times avarage value of the function can be calculated and directly used in the calculatetions of the integral. Integral is the calculateion of the avarage function value with the interval width. Due to fact that this method requires a huge amount of function evaluation, therefore it is not very practical for one variable integration, but in multivariable form it becomes a practical, sometimes only available approach.

Program One variable Monte Carlo integration, java code

```
//=======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Monte Carlo integration with one variable
// Dr. Turhan Coban
// =======================================================

import java.io.*;

class f1 extends f_x
{
```

```
  public double func(double x)
  {return 3.0*x*x;}
}

class NA73
{

public static double monte_carlo(f_x ff,double a,double b, long n)
{       //Monte-carlo integrali
        double h=(b-a);
        double sum=0.0;
        double x=0.0;
        double f;
        for(int i=0;i<n;i++)
        { x=a+(b-a)*Math.random();
          sum+=ff.func(x);
        }
        return (b-a)*sum/(double)n;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
  System.out.println("integral of class         f1 : "+monte_carlo(b1,0.0,1.0,1000000));
}}
```

One variable Monte-carlo integration

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA73
integral of class         f1 : 0.9990266020806676

> Terminated with exit code 0.
```

One variable Monte Carlo integration, julia code

```
 # ==================================================
# Numerical Analysis package in Julia
# example to show utilisation of integration (integral)
# and differentiation (derivative) functions
# Monte Carlo integration with one variable
# Dr. Turhan Coban
# ==================================================
function monte_carlo(ff,a,b,n)
      # Monte-carlo integral
       h=(b-a)
       sum=0.0
       x=0.0
       f
       for i=1:n
         x=a+(b-a)*rand()
         sum+=ff(x)
       end
       return (b-a)*sum/n
end


f(x)=x
a=0.0
b=1.0
n=50000
I=monte_carlo(f,a,b,n)
print("I = $I")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" monte_carlo_integral.jl
I = 0.49994511866715496
```

```
> Terminated with exit code 0.
```

If Monte Carlo equation is written for two independent variables

$$x = ax + (bx - ax) * random\_number(0,1)$$

$$y = ay + (by - ay) * random\_number(0,1)$$

$$f_{average} = \frac{\sum_{i=0}^{n} \overline{f}(x, y)}{n}, \qquad n >> 1$$

Integration subfield

$$\mu(D) = \int_{ay}^{by} \int_{ax=\alpha(x)}^{bx=\beta(x)} dy\, dx$$

integration

$$I = \mu(D)\, f_{average}$$

As it is seen from the equation process fro one variable integration and two variable integraton is almost the same. This is the main factor why it is frequently use as a high dimensional integration tool.

Program Monte Carlo integration with two independent variables

```
//=======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Monte Carlo integration with two variable
// Dr. Turhan Coban
// =======================================================
import java.io.*;

class f1 extends f_xj
{
public double func(double x[])
{
//çözümü istenen fonksiyon
double ff;
ff=2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1];
return -ff; //minimum değil maxsimum istendiğinden - ile çarptik
}
}

class NA74
{
public static double monte_carlo(f_xj ff,double a[],double b[], long n)
{       //Newton-cotes integral 6 points
                double x[]=new double[2];
                double h=1.0;
                double sum=0;
                for(int j=0;j<2;j++) {h*=(b[j]-a[j]);}
                for(int i=0;i<n;i++)
        { for(int j=0;j<2;j++) {x[j]=a[j]+(b[j]-a[j])*Math.random();}
          sum+=ff.func(x);
        }
        return h*sum/(double)n;
}

  public static void main(String args[]) throws IOException
  {
  f1 b1=new f1();
```

```
  double a[]=new double[2];
  double b[]=new double[2];
  a[0]=-1;
  a[1]=-1;
  b[0]=1;
  b[1]=1;
  System.out.println("integral of class        f1 : "+monte_carlo(b1,a,b,1000000));
}}
```

Output of Monte Carlo integration with two independent variables

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA74
integral of class        f1 : 4.009412425743507

> Terminated with exit code 0.
```

Program Monte Carlo integration with two independent variables , julia code

```
 # ==================================================
# Numerical Analysis package in Julia
# example to show utilisation of integration (integral)
# and differentiation (derivative) functions
# Monte Carlo integration with one variable
# Dr. M. Turhan Coban
# ==================================================
function monte_carlo(ff,a,b,n)
      # Monte-carlo integral
       h=(b-a)
       sum=0.0
       x=0.0
       f
       for i=1:n
         x=a+(b-a)*rand()
         sum+=ff(x)
       end
       return (b-a)*sum/n
end


f(x)=x
a=0.0
b=1.0
n=50000
I=monte_carlo(f,a,b,n)
print("I = $I")
```

Output of Monte Carlo integration with two independent variables

```
 ---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" monte_carlo_integral.jl
I = 0.49994511866715496
> Terminated with exit code 0.
```

7.7 MULTIDIMENSIONAL INTEGRATION BY PARAMETRIC METHODS

A two dimensional integration

$$I(D) = \int\int_D f(x,y)dydx$$

Where D is a region in xy plane. If this region is defined as $a \le x \le b$ and $c(x) \le y \le d(x)$, one way two calculate two dimensional integration is to repeat one dimensional integration twice. In the first step intehration function g(x) is calculated as an integration in y dimension and then function g(x) is solved to get integration result.

$$I = \int_{a}^{b} \int_{c(x)}^{d(x)} f(x,y)dydx$$

$$\mathrm{g(x)} = \int_{c(x)}^{d(x)} f(x,y)dy, \quad a \le x \le b$$

$$\mathrm{I(D)} = \int_{a}^{b} g(x)dx$$

Any integration formulation can be used for this purpose. All it has to be done is to evaluate integration Formula twice onece for each independent variable. As our first example Gauss-Legendre formulation will be used to evaluate double integration. Two code is defined. In the first one Gauss-Legendre coefficients are defined as constants, in the second one they are calculated through Legendre polynomials. In the two dimensional integration x dimesion limits are still given as constant values, but y dimension limits c(x) and d(x) should be given as functions for proper evaluation of the integral.

$$I = \int_{x=a}^{b} \int_{y=c(x)}^{d(x)} f(x,y)dydx = \int_{x=a}^{b} \frac{d(x)-c(x)}{2} \sum_{j=1}^{n} c_{n,j} f\left(x, \frac{(d(x)-c(x))r_{n,j} + d(x) + c(x)}{2}\right) dx$$

$$g(x) = \frac{d(x)-c(x)}{2} \sum_{j=1}^{n} c_{n,j} f\left(x, \frac{(d(x)-c(x))r_{n,j} + d(x) + c(x)}{2}\right)$$

$$I = \int_{x=a}^{b} \int_{y=c(x)}^{d(x)} f(x,y)dydx = \int_{x=a}^{b} g(x)dx = \frac{b-a}{2} \sum_{i=1}^{m} c_{m,i} f\left(\frac{b-a}{2} r_{m,i} + \frac{b+a}{2}\right)$$

In this equation $r_{n,j}$ root values and $c_{n,j}$ are equation coefficients for integration of y, and $r_{m,i}$ root values and $c_{m,i}$ are equation coefficients for integration of x. If n=m r and c coefficients will be same. These values can be calculated from Legendre polynomials or they can be directly given as an array in computer code.

Program 7.7-1 60 constant points two variable Gauss-Legendre integration formula , java code

```
//=======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gauss-Legendre integration with two variables
// constant number of coefficients (60 coefficients)
// and roots
// Dr. Turhan Coban
// =======================================================
import java.io.*;

abstract class fdi
{
public double a,b;
public fdi(double ai,double bi)
{a=ai;b=bi;}
public fdi()
```

```
{a=0;b=1;}
public void setab(double ai,double bi)
{a=ai;b=bi;}
abstract public  double c(double x);
abstract public  double d(double x);
abstract public  double f(double x[]);
}

class f1 extends fdi
{
  public f1(double ai,double bi)
  {super(ai,bi);}
  //lower integration limit function
  public double c(double x)
  {return 1.0;}
  //upper integration limit function
  public double d(double x)
  {return 1.5;}

  public double f(double x[])
  {
  //function required to integrate
  double ff;
  ff=Math.log(x[0]+2.0*x[1]);
  return ff;
  }
}


class NA75
{

public static double I(fdi f)
{
//integral f(x)dx
//integral of a function by using gauss-legendre quadrature
//coefficients are pre-calculated for 60 terms for [-1,1]
//band then utilises variable transform
 int i,j;
 int n=60;
 double r[],c[];
 r=new double[n];
 c=new double[n];
 r[ 0] = .15532579626752470000E-02;
 r[ 1] = .81659383601264120000E-02;
 r[ 2] = .19989067515846230000E-01;
 r[ 3] = .36899976285362850000E-01;
 r[ 4] = .58719732103973630000E-01;
 r[ 5] = .85217118808615820000E-01;
 r[ 6] = .11611128394758690000E+00;
 r[ 7] = .15107475260334210000E+00;
 r[ 8] = .18973690850537860000E+00;
 r[ 9] = .23168792592899010000E+00;
 r[10] = .27648311523095540000E+00;
 r[11] = .32364763723456090000E+00;
 r[12] = .37268153691605510000E+00;
 r[13] = .42306504319570830000E+00;
 r[14] = .47426407872234120000E+00;
 r[15] = .52573592127765890000E+00;
 r[16] = .57693495680429170000E+00;
 r[17] = .62731846308394490000E+00;
 r[18] = .67635236276543910000E+00;
```

```
 r[19] = .723516884769044500000E+00;
 r[20] = .768312074071009900000E+00;
 r[21] = .810263091494621400000E+00;
 r[22] = .848925247396658000000E+00;
 r[23] = .883888716052413100000E+00;
 r[24] = .914782881191384200000E+00;
 r[25] = .941280267896026400000E+00;
 r[26] = .963100023714637200000E+00;
 r[27] = .980010932484153700000E+00;
 r[28] = .991834061639873500000E+00;
 r[29] = .998446742037324800000E+00;
c[ 0] = .398409624808277900000E-02;
c[ 1] = .923323415554550000000E-02;
c[ 2] = .143923539416616700000E-01;
c[ 3] = .193995962848135300000E-01;
c[ 4] = .242013364152925900000E-01;
c[ 5] = .287465781088087200000E-01;
c[ 6] = .329871149410900800000E-01;
c[ 7] = .368779873688525700000E-01;
c[ 8] = .403779476147100900000E-01;
c[ 9] = .434498936005415000000E-01;
c[10] = .460612611188930500000E-01;
c[11] = .481843685873221200000E-01;
c[12] = .497967102933976400000E-01;
c[13] = .508811948742027500000E-01;
c[14] = .514263264467794200000E-01;
c[15] = .514263264467794200000E-01;
c[16] = .508811948742027500000E-01;
c[17] = .497967102933976400000E-01;
c[18] = .481843685873221200000E-01;
c[19] = .460612611188930500000E-01;
c[20] = .434498936005415000000E-01;
c[21] = .403779476147100900000E-01;
c[22] = .368779873688525700000E-01;
c[23] = .329871149410900800000E-01;
c[24] = .287465781088087200000E-01;
c[25] = .242013364152925900000E-01;
c[26] = .193995962848135300000E-01;
c[27] = .143923539416616700000E-01;
c[28] = .923323415554550000000E-02;
c[29] = .398409624808277900000E-02;
for(i=0;i<30;i++) {r[i+30]=-r[i];c[i+30]=c[i];}
int m=n;
double x[],h1,h2,J,a1,b1,c1,d1,k1,k2,JX,Q;
x=new double[2];
//1=========================
a1=f.a;
b1=f.b;
h1=(b1-a1)/2.0;
h2=(b1+a1)/2.0;
J=0;
//2=========================
for(i=0;i<m;i++)
{
//3=========================
JX=0;
x[0]=h1*r[i]+h2;
d1=f.d(x[0]);
c1=f.c(x[0]);
k1=(d1-c1)/2.0;
k2=(d1+c1)/2.0;
//4=========================
```

```
   for(j=0;j<n;j++)
   {
            x[1]=k1*r[j]+k2;
            Q=f.f(x);
            JX+=c[j]*Q;
            }
//System.out.println("JX="+JX);
//=========================
J=J+c[i]*k1*JX;
}
J=h1*J;
return J;
}

  public static void main(String args[]) throws IOException
  {
  f1 ff=new f1(1.4,2.0);
  System.out.println("integral of class         f1 : "+I(ff));
}
}
```

constant points two variable Gauss-Legendre integration formula output

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA75
integral of class         f1 : 0.429554527548266

> Terminated with exit code 0.
```

60 constant points two variable Gauss-Legendre integration formula , Julia code

```
# ======================================================
# Numerical Analysis package in Julia
# example to show utilisation of integration (egral)
# and differentiation (derivative) functions
# Gauss-Legendre integration with two variables
# constant number of coefficients (60 coefficients)
# and roots
# Dr. Turhan Coban
# ======================================================

function integral(f,a2,b2,c2,d2)
# integral f(x)dx
# integral of a function by using gauss-legendre quadrature
# coefficients are pre-calculated for 60 terms for [-1,1]
# band then utilises variable transform
  i=0
  j=0
  n=60
  global r=OffsetArray{Float64}(undef,0:n-1)
  global c=OffsetArray{Float64}(undef,0:n-1)
 r[0] = .155325796267524700000E-02
 r[1] = .816593836012641200000E-02
 r[2] = .199890675158462300000E-01
 r[3] = .368999762853628500000E-01
 r[4] = .587197321039736300000E-01
 r[5] = .852171188086158200000E-01
 r[6] = .116111283947586900000E+00
 r[7] = .151074752603342100000E+00
 r[8] = .189736908505378600000E+00
 r[9] = .231687925928990100000E+00
 r[10] = .276483115230955400000E+00
```

```
  r[11] = .323647637234560900000E+00
  r[12] = .372681536916055100000E+00
  r[13] = .423065043195708300000E+00
  r[14] = .474264078722341200000E+00
  r[15] = .525735921277658900000E+00
  r[16] = .576934956804291700000E+00
  r[17] = .627318463083944900000E+00
  r[18] = .676352362765439100000E+00
  r[19] = .723516884769044500000E+00
  r[20] = .768312074071009900000E+00
  r[21] = .810263091494621400000E+00
  r[22] = .848925247396658000000E+00
  r[23] = .883888716052413100000E+00
  r[24] = .914782881191384200000E+00
  r[25] = .941280267896026400000E+00
  r[26] = .963100023714637200000E+00
  r[27] = .980010932484153700000E+00
  r[28] = .991834061639873500000E+00
  r[29] = .998446742037324800000E+00
 c[0] = .398409624808277900000E-02
 c[1] = .923323415554550000000E-02
 c[2] = .143923539416616700000E-01
 c[3] = .193995962848135300000E-01
 c[4] = .242013364152925900000E-01
 c[5] = .287465781088087200000E-01
 c[6] = .329871149410900800000E-01
 c[7] = .368779873688525700000E-01
 c[8] = .403779476147100900000E-01
 c[9] = .434498936005415000000E-01
 c[10] = .460612611188930500000E-01
 c[11] = .481843685873221200000E-01
 c[12] = .497967102933976400000E-01
 c[13] = .508811948742027500000E-01
 c[14] = .514263264467794200000E-01
 c[15] = .514263264467794200000E-01
 c[16] = .508811948742027500000E-01
 c[17] = .497967102933976400000E-01
 c[18] = .481843685873221200000E-01
 c[19] = .460612611188930500000E-01
 c[20] = .434498936005415000000E-01
 c[21] = .403779476147100900000E-01
 c[22] = .368779873688525700000E-01
 c[23] = .329871149410900800000E-01
 c[24] = .287465781088087200000E-01
 c[25] = .242013364152925900000E-01
 c[26] = .193995962848135300000E-01
 c[27] = .143923539416616700000E-01
 c[28] = .923323415554550000000E-02
 c[29] = .398409624808277900000E-02
 for i=0:29
   r[i+30]=-r[i]
   c[i+30]=c[i]
 end
 m=n
 # 1=========================
 a1=a2
 b1=b2
 h1=(b1-a1)/2.0
 h2=(b1+a1)/2.0
 J=0
 # 2=========================
 for i=0:m-1
```

```
# 3=========================
JX=0
x=OffsetArray{Float64}(undef,0:1)
x[0]=h1*r[i]+h2
d1=d2(x[0])
c1=c2(x[0])
k1=(d1-c1)/2.0
k2=(d1+c1)/2.0
# 4=========================
   for j=0:n-1
            x[1]=k1*r[j]+k2
            Q=f(x)
            JX+=c[j]*Q
            end
# =========================
J=J+c[i]*k1*JX
end
J=h1*J
return J
end

using OffsetArrays
f(x)=log(x[0]+2.0*x[1])
a2=1.4
b2=2.0
c2(x)=1.0
d2(x)=1.5
I=integral(f,a2,b2,c2,d2)
print("integral = $I")
```

constant points two variable Gauss-Legendre integration formula output

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" integral_GL2.jl
integral = 0.429554527548266
> Terminated with exit code 0.
```

In the next example, instead of constant number of points Legendre transformation defined varible number of points are taken. C and r values are directly calculated from Legendre polynomials.

Program variable points two variable Gauss-Legendre integration formula , java code

```
//=======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gauss-Legendre integration with two variables
// variable number of coefficients and roots
// calculated by Legendre polynomials
// Dr. Turhan Coban
// =======================================================
import java.io.*;

abstract class fdi
{
public double a,b;
public fdi(double ai,double bi)
{a=ai;b=bi;}
public fdi()
```

```
{a=0;b=1;}
public void setab(double ai,double bi)
{a=ai;b=bi;}
abstract public  double c(double x);
abstract public  double d(double x);
abstract public  double f(double x[]);
}

class f1 extends fdi
{
  public f1(double ai,double bi)
  {super(ai,bi);}

  public double c(double x)
  {return 1.0;}

  public double d(double x)
  {return 1.5;}

  public double f(double x[])
  {
  //çözümü istenen fonksiyon
  double ff;
  ff=Math.log(x[0]+2.0*x[1]);
  return ff;
  }
}

class NA76
{
public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
            //calculates legendre gauss-coefficients as coefficients of the integral
            //for n terms
            double EPS=3.0e-15;
            int m,j,i;
            double z1,z,xm,xl,pp,p3,p2,p1;
             double a[][]=new double[2][n];
            m=(n+1)/2;
            xm=0.5*(x2+x1);
            xl=0.5*(x2-x1);
            for (i=1;i<=m;i++)  {
                        z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                        do {
                                    p1=1.0;
                                    p2=0.0;
                                    for (j=1;j<=n;j++) {
                                                p3=p2;
                                                p2=p1;
                                                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                                    }
                                    pp=n*(z*p1-p2)/(z*z-1.0);
                                    z1=z;
                                    z=z1-p1/pp;
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
                                    }
return a;
}
```

```
public static double I(fdi f,int n,int m)
{
//integral f(x,y)dxdy
//integral of a function by using gauss-legendre quadrature
//coefficients are pre-calculated for n,m terms for [-1,1]
//band then utilises variable transform
 int i,j;
 double rn[],cn[];
 double rm[],cm[];
 rn=new double[n];
 rm=new double[m];
 cn=new double[n];
 double a[][]=new double[2][n];
 a=gauss_legendre_coefficients(-1.0,1.0,n);
 rn=a[0];
 cn=a[1];
 a=gauss_legendre_coefficients(-1.0,1.0,m);
 rm=a[0];
 cm=a[1];
double x[],h1,h2,J,a1,b1,c1,d1,k1,k2,JX,Q;
x=new double[2];
a1=f.a;
b1=f.b;
h1=(b1-a1)/2.0;
h2=(b1+a1)/2.0;
J=0;
for(i=0;i<m;i++)
{
JX=0;
x[0]=h1*rm[i]+h2;
d1=f.d(x[0]);
c1=f.c(x[0]);
k1=(d1-c1)/2.0;
k2=(d1+c1)/2.0;
//4=========================
   for(j=0;j<n;j++)
   {
            x[1]=k1*rn[j]+k2;
            Q=f.f(x);
            JX+=cn[j]*Q;
            }
//System.out.println("JX="+JX);
//5=========================
J=J+cm[i]*k1*JX;
}
J=h1*J;
return J;
}


public static double I(fdi f,int n)
{return I(f,n,n);}

public static double I(fdi f)
{return I(f,60,60);}

  public static void main(String args[]) throws IOException
  {
  f1 ff=new f1(1.4,2.0);
  System.out.println("integral of class         f1 : "+I(ff));
}
```

```
}
```

variable points two variable Gauss-Legendre integration Formula output

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA76
integral of class        f1 : 0.4295545275482747

> Terminated with exit code 0.
```

Example function:

$$\int_{x=1.4}^{2}\int_{y=1}^{1.5} \ln(x+2y)\,dydx$$

Two dimensional Gauss-Legendre intgration, julia code

```
#integral_3D_GL.py
from math import *
#2 dimensional function
class f1:
  def __init__(self, a,b):
    self.a = a
    self.b = b
  def c(self,x):
      return 1.0
  def d(self,x):
      return 1.5
  def f(self,xx):
      ff=log(xx[0]+2.0*xx[1])
      return ff


def I(f):
    #integral f(x)dx
    #integral of a function by using gauss-legendre quadrature
    #coefficients are pre-calculated for 60 terms for [-1,1]
    #band then utilises variable transform
    n=60;
    r=[0 for z in range(n)]
    c=[0 for z in range(n)]
    r[ 0] = .15532579626752470000E-02;
    r[ 1] = .81659383601264120000E-02;
    r[ 2] = .19989067515846230000E-01;
    r[ 3] = .36899976285362850000E-01;
    r[ 4] = .58719732103973630000E-01;
    r[ 5] = .85217118808615820000E-01;
    r[ 6] = .11611128394758690000E+00;
    r[ 7] = .15107475260334210000E+00;
    r[ 8] = .18973690850537860000E+00;
    r[ 9] = .23168792592899010000E+00;
    r[10] = .27648311523095540000E+00;
    r[11] = .32364763723456090000E+00;
    r[12] = .37268153691605510000E+00;
    r[13] = .42306504319570830000E+00;
    r[14] = .47426407872234120000E+00;
    r[15] = .52573592127765890000E+00;
    r[16] = .57693495680429170000E+00;
    r[17] = .62731846308394490000E+00;
    r[18] = .67635236276543910000E+00;
    r[19] = .72351688476904450000E+00;
    r[20] = .76831207407100990000E+00;
    r[21] = .81026309149462140000E+00;
    r[22] = .84892524739665800000E+00;
    r[23] = .88388716052413100000E+00;
    r[24] = .91478288119138420000E+00;
    r[25] = .94128026789602640000E+00;
    r[26] = .96310002371463720000E+00;
    r[27] = .98001093248415370000E+00;
    r[28] = .99183406163987350000E+00;
```

```
    r[29] = .99844674203732480000E+00;
    c[ 0] = .39840962480827790000E-02;
    c[ 1] = .92332341555455000000E-02;
    c[ 2] = .14392353941661670000E-01;
    c[ 3] = .19399596284813530000E-01;
    c[ 4] = .24201336415292590000E-01;
    c[ 5] = .28746578108808720000E-01;
    c[ 6] = .32987114941090080000E-01;
    c[ 7] = .36877987368852570000E-01;
    c[ 8] = .40377947614710090000E-01;
    c[ 9] = .43449893600541500000E-01;
    c[10] = .46061261118893050000E-01;
    c[11] = .48184368587322120000E-01;
    c[12] = .49796710293397640000E-01;
    c[13] = .50881194874202750000E-01;
    c[14] = .51426326446779420000E-01;
    c[15] = .51426326446779420000E-01;
    c[16] = .50881194874202750000E-01;
    c[17] = .49796710293397640000E-01;
    c[18] = .48184368587322120000E-01;
    c[19] = .46061261118893050000E-01;
    c[20] = .43449893600541500000E-01;
    c[21] = .40377947614710090000E-01;
    c[22] = .36877987368852570000E-01;
    c[23] = .32987114941090080000E-01;
    c[24] = .28746578108808720000E-01;
    c[25] = .24201336415292590000E-01;
    c[26] = .19399596284813530000E-01;
    c[27] = .14392353941661670000E-01;
    c[28] = .92332341555455000000E-02;
    c[29] = .39840962480827790000E-02;
    for i in range(30):
        r[i+30]=-r[i]
        c[i+30]=c[i]
    m=n;
    x=[0 for z in range(2)]
    #1=========================
    a1=f.a
    b1=f.b;
    h1=(b1-a1)/2.0;
    h2=(b1+a1)/2.0;
    J=0;
    #2=========================
    for i in range(m):
        JX=0;
        x[0]=h1*r[i]+h2;
        d1=f.d(x[0]);
        c1=f.c(x[0]);
        k1=(d1-c1)/2.0;
        k2=(d1+c1)/2.0;
        #3=========================
        for j in range(n):
            x[1]=k1*r[j]+k2
            Q=f.f(x)
            JX+=c[j]*Q
        J=J+c[i]*k1*JX
    J=h1*J;
    return J


ff =f1(1.4,2.0)
I=I(ff)
print("integral = : ",I)
```

```
runfile('E:/okul/SCO1/integral_3D_GL.py', wdir='E:/okul/SCO1')
integral = :  0.429554527548266
```

In the next example Guass-Legendre polynomial coefficients will be generated by Legendre polynomial formulations. C and r values are directly calculated from Legendre polynomials.

Two dimensional Gauss-Legendre intgration, julia code

```
 #integral_3D_GL1.py
from math import *
#2 dimensional function
```

```
class f1:
  def __init__(self, a,b):
    self.a = a
    self.b = b
  def c(self,x):
      return 1.0
  def d(self,x):
      return 1.5
  def f(self,xx):
      ff=log(xx[0]+2.0*xx[1])
      return ff

def gauss_legendre_coefficients(x1,x2,n):
    #calculates legendre gauss-coefficients as coefficients of the integral
    #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

def I(f,n,m):
    #integral f(x)dx
    #integral of a function by using gauss-legendre quadrature
    #coefficients are pre-calculated for 60 terms for [-1,1]
    #band then utilises variable transform
    n=60;
    rn=[0 for z in range(n)]
    cn=[0 for z in range(n)]
    rm=[0 for z in range(n)]
    cm=[0 for z in range(n)]
    an=gauss_legendre_coefficients(-1.0,1.0,n)
    rn=an[0]
    cn=an[1]
    am=gauss_legendre_coefficients(-1.0,1.0,m)
    rm=am[0]
    cm=am[1]
    x=[0 for z in range(2)]
    #1=========================
    a1=f.a
    b1=f.b;
    h1=(b1-a1)/2.0;
    h2=(b1+a1)/2.0;
    J=0;
    #2=========================
    for i in range(m):
        JX=0;
        x[0]=h1*rm[i]+h2;
        d1=f.d(x[0]);
        c1=f.c(x[0]);
        k1=(d1-c1)/2.0;
        k2=(d1+c1)/2.0;
        #3=========================
        for j in range(n):
            x[1]=k1*rn[j]+k2
            Q=f.f(x)
```

```
        JX+=cn[j]*Q
      J=J+cm[i]*k1*JX
   J=h1*J;
   return J


ff =f1(1.4,2.0)
I=I(ff,60,60)
print("integral = : ",I)
```

```
runfile('E:/okul/SCO1/integral_3D_GL1.py', wdir='E:/okul/SCO1')
integral = :  0.4295545275482747
```

For the two variable integration Newton-Cotes formulas can also be used. As a example a Simpson 1/3 two dimensional integration code is prepared. Dor the quadratic polynomial Newton-Cotes equation (simpson 1/3 method) equations can be written as follows;
One dimensional orm was

$$h=\frac{(b-a)}{n}=\frac{(b-a)}{2}$$

$$I=\int_a^b f(x)dx=\frac{(b-a)}{6}[f(x_0)+4f(x_1)+f(x_2)]$$

If this Formula can be written in composite form.

$$I=\int_a^b f(x)dx=\frac{h}{3}\left[f(a)+2\sum_{j=1}^{(n/2)-1}f(x_{2j})+4\sum_{j=1}^{(n/2)}f(x_{2j-1})\right]$$

This Formula can be open to the second dimension similar to Gauss-Legendre integration
By defining $k=\frac{(d(x)-c(x))}{m}$ integration equation becomes

$$I=\int_{c(x)}^{d(x)} f(x,y)dy=\frac{k}{3}\left[f(x,y_0)+2\sum_{j=1}^{(m/2)-1}f(x,y_{2j})+4\sum_{j=1}^{(m/2)}f(x,y_{2j-1})+f(x,y_m)\right]$$

From here x integration can be carried out as:

$$I=\int_a^b\int_c^d f(x,y)dy=\int_a^b\frac{k}{3}\left[f(x,y_0)+2\sum_{j=1}^{(m/2)-1}f(x,y_{2j})+4\sum_{j=1}^{(m/2)}f(x,y_{2j-1})+f(x,y_m)\right]dx$$

The same integration equation is then applied to the outer integration. Complete code is given in the program 8.7.3 . In the example the same function as in previous Gauss-Legendre integration is taken to compare the results

Program two dimensional Simpson 1/3 integration formula , java code

```
//=====================================================
// Numerical Analysis package in java
// simpson integration of two dimensional function
// and differentiation (derivative) functions
// two dimensional Simpson 1/3 integration
// (quadratic Newton-Cotes formula)
// Dr. Turhan Coban
// =====================================================
import java.io.*;

abstract class fdi
{
public double a,b;
public fdi(double ai,double bi)
{a=ai;b=bi;}
public fdi()
```

```
{a=0;b=1;}
public void setab(double ai,double bi)
{a=ai;b=bi;}
abstract public  double c(double x);
abstract public  double d(double x);
abstract public  double f(double x[]);
}

class f1 extends fdi
{
  public f1(double ai,double bi)
  {super(ai,bi);}

  public double c(double x)
  {return 1.0;}

  public double d(double x)
  {return 1.5;}

  public double f(double x[])
  {
  //çözümü istenen fonksiyon
  double ff;
  ff=Math.log(x[0]+2.0*x[1]);
  return ff;
  }
}


class NA77
{

public static double I(fdi f,int m,int n)
{
//integral f(x)dx
//integral of a function by using simpson metod
double h=(f.b-f.a)/n;
double J1=0.0;
double J2=0.0;
double J3=0.0;
double x[]=new double[2];
double HX=0;
double K1,K2,K3;
double d,c;
double Q,L,J;
for(int i=0;i<=n;i++)
{
x[0]=f.a+i*h;
d=f.d(x[0]);
c=f.c(x[0]);
HX=(d-c)/m;
x[1]=c;K1=f.f(x);
x[1]=d;K1+=f.f(x);
K2=0;
K3=0;
  for(int j=1;j<=(m-1);j++)
  {x[1]=f.c(x[0])+j*HX;
  Q=f.f(x);
  if((j/2)*2==j) K2=K2+Q;
  else           K3=K3+Q;
  }
L=(K1+2.0*K2+4.0*K3)*HX/3.0;
if((i==0)||(i==n)) J1=J1+L;
else if ((i/2)*2==i) J2=J2+L;
else J3=J3+L;
}
J=h*(J1+2.0*J2+4.0*J3)/3.0;
return J;
}
```

```
  public static void main(String args[]) throws IOException
  {
  f1 ff=new f1(1.4,2.0);
  System.out.println("integral of class         f1 : "+I(ff,20,20));
}
}
```

two dimensional Simpson 1/3 integration formula

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA77
integral of class         f1 : 0.4295545273061155

> Terminated with exit code 0.
```

two dimensional Simpson 1/3 integration formula , python version

```
from math import *
#2 dimensional function
class f1:
  def __init__(self, a,b):
    self.a = a
    self.b = b
  def c(self,x):
      return 1.0
  def d(self,x):
      return 1.5
  def f(self,xx):
      xx1=xx[0]+2.0*xx[1]
      ff=log(xx1)
      return ff
def I(f,m,n):
    # integral f(x)dx
    # integral of a function by using simpson metod
    h=(f.b-f.a)/n
    J1=0.0
    J2=0.0
    J3=0.0
    x=[0.0 for i in range(2)]
    HX=0.0
    # double K1,K2,K3
    # double d,c
    # double Q,L,J
    for i in range(n+1):
        x[0]=f.a+h*i
        d=f.d(x[0])
        c=f.c(x[0])
        HX=(d-c)/float(m)
        x[1]=c
        K1=f.f(x)
        x[1]=d
        xk=f.f(x)
        K1=K1+xk
        K2=0.0
        K3=0.0
        for j in range(1,m):
            x[1]=f.c(x[0])+float(j)*HX
            Q=f.f(x)
            #if i==0:
            if(int(j/2)*2==j):
                K2=K2+Q
            else:
                K3=K3+Q
        L=(K1+2.0*K2+4.0*K3)*HX/3.0
        if (i==0) or (i==n):
            J1=J1+L
        elif int(i/2)*2==i:
            J2=J2+L
        else:
            J3=J3+L
    J=h*(J1+2.0*J2+4.0*J3)/3.0
    return J

ff =f1(1.4,2.0)
```

```
I=I(ff,20,20)
print("integral = : ",I)
```

```
runfile('E:/okul/SCO1/integral_3D_NC.py', wdir='E:/okul/SCO1')
integral = :  0.4295545273061155
```

Now the same principle can be used for three dimensional integration process. A three dimensional integral can be defined as:

$$I = \int_a^b \int_{c(x)}^{d(x)} \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)dzdydx$$

Gauss-Legendre integration will be used here as a solution method.

$$I = \int_0^1 \int_x^1 \int_0^y y^2 z dz dy dx$$

Program three dimensional Gauss-Legendre integration , java code

```
//=======================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gauss-Legendre integration with three independent
// variable number of coefficients and roots
// calculated by Legendre polynomials
// Dr. Turhan Coban
// =======================================================
import java.io.*;

abstract class fdi3
{
public double a,b;
public fdi3(double ai,double bi)
{setab(ai,bi);}
public void setab(double ai,double bi)
{a=ai;b=bi;}
abstract public  double c(double x);
abstract public  double d(double x);
abstract public  double alfa(double x[]);
abstract public  double beta(double x[]);
abstract public  double f(double x[]);
}

class f1 extends fdi3
{
  public f1(double ai,double bi)
  {super(ai,bi);}

  public double c(double x)
  {return x;}

  public double d(double x)
  {return 1.0;}

  public double alfa(double x[])
  {return 0.0;}

  public double beta(double x[])
  {return x[1];}
```

```
 public double f(double x[])
 {
 //sample function
 double ff;
 ff=x[1]*x[1]*x[2];
 return ff;
 }
}

class NA78
{
public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
        //calculates legendre gauss-coefficients as coefficients of the integral
        //for n terms
        double EPS=3.0e-15;
        int m,j,i;
        double z1,z,xm,xl,pp,p3,p2,p1;
   double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
        m=(n+1)/2;
        xm=0.5*(x2+x1);
        xl=0.5*(x2-x1);
        for (i=1;i<=m;i++)  {
                z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                do {
                        p1=1.0;
                        p2=0.0;
                        for (j=1;j<=n;j++) {
                                p3=p2;
                                p2=p1;
                                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                        }
                        pp=n*(z*p1-p2)/(z*z-1.0);
                        z1=z;
                        z=z1-p1/pp;
                } while (Math.abs(z-z1) > EPS);
                a[0][i-1]=xm-xl*z;
                a[0][n-i]=xm+xl*z;
                a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                a[1][n-i]=a[1][i-1];
                        }
return a;
}

public static double I(fdi3 f,int n,int m,int p)
{
//integral f(x)dx
//integral of a function by using gauss-legendre quadrature
//coefficients are pre-calculated for n,m terms for [-1,1]
//band then utilises variable transform
 int i,j;
 double rn[],cn[];
 double rm[],cm[];
 double rp[],cp[];
 rp=new double[p];
 cp=new double[p];
 int nn=Math.max(n,Math.max(m,p));
 int k;
 double x[]=new double[3];
 double a[][]=new double[2][nn];
 int mm=Math.max(Math.max(n,m),p);
```

```
 a=gauss_legendre_coefficients(-1.0,1.0,mm);
 rn=rm=rp=a[0];
 cn=cm=cp=a[1];
double h1,h2,J,a1,b1,c1,d1,k1,k2,JX,JY,Q,l1,l2,alfa1,beta1;
a1=f.a;
b1=f.b;
h1=(b1-a1)/2.0;
h2=(b1+a1)/2.0;
J=0;
for(i=0;i<m;i++)
{
JX=0;
x[0]=h1*rm[i]+h2;
d1=f.d(x[0]);
c1=f.c(x[0]);
k1=(d1-c1)/2.0;
k2=(d1+c1)/2.0;
//4=========================
for(j=0;j<n;j++)
{   JY=0;
   x[1]=k1*rn[j]+k2;
   beta1=f.beta(x);
   alfa1=f.alfa(x);
   l1=(beta1-alfa1)/2.0;
   l2=(beta1+alfa1)/2.0;
   for(k=0;k<p;k++)
   {
            x[2]=l1*rp[k]+l2;
            Q=f.f(x);
            JY+=cp[k]*Q;
   }
JX+=cn[j]*l1*JY;
}
//System.out.println("JX="+JX);
//5=========================
J=J+cm[i]*k1*JX;
}
J=h1*J;
return J;
}

public static double I(fdi3 f,int n)
{return I(f,n,n,n);}

public static double I(fdi3 f)
{return I(f,60,60,60);}

  public static void main(String args[]) throws IOException
  {
  f1 ff=new f1(0.0,1.0);          ;
  System.out.println("İntegral="+I(ff));
  }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" NA78
Integral=0.08333333333333286

> Terminated with exit code 0.
```

Example integral:

$$I = \int_0^1 \int_x^1 \int_0^y y^2 z dz dy dx$$

Three dimensional integration ,Python version

```
#integral_4D_GL1.py
from math import *
#2 dimensional function
class fdi3:
    def __init__(self, a,b):
      self.a = a
      self.b = b
    def c(self,x):
        return x
    def d(self,x):
        return 1.0
    def alfa(self,x):
        return 0
    def beta(self,x):
        return x[1]
    def f(self,x):
        ff=x[1]*x[1]*x[2]
        return ff

def gauss_legendre_coefficients(x1,x2,n):
    #calculates legendre gauss-coefficients as coefficients of the integral
    #for n terms
    eps=3e-15
    m=int((n+1.0)/2.0)
    xm=0.5*(x2+x1)
    xl=0.5*(x2-x1)
    nmax=100
    a=[[0.0 for i in range(n)] for j in range(2)]
    for i in range(1,m+1):
        z=cos(pi*((i-0.25)/(n+0.5)))
        for ii in range(nmax):
            #while abs(z-z1) > eps:
            p1=1.0;
            p2=0.0;
            for j in range(1,n+1):
                p3=p2
                p2=p1
                p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
            pp=(float)(n*(z*p1-p2)/(z*z-1.0))
            z1=z;
            z=z1-p1/pp
            if abs(z-z1) < eps: break
        a[0][i-1]=xm-xl*z
        a[0][n-i]=xm+xl*z
        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
        a[1][n-i]=a[1][i-1]
    return a

def I(f,n,m,p):
    # integral f(x)dx where x= x[0],x[1],x[2]
    #integral of a function by using gauss-legendre quadrature
    rp=[0.0 for i in range(p)]
    cp=[0.0 for i in range(p)]
    nn=max(n,max(m,p))
    x=[0.0 for i in range(3)]
    a=a=[[0.0 for i in range(nn)] for j in range(2)]
    mm=max(max(n,m),p);
    a=gauss_legendre_coefficients(-1.0,1.0,mm)
    rn=rm=rp=a[0]
    cn=cm=cp=a[1]
    # double h1,h2,J,a1,b1,c1,d1,k1,k2,JX,JY,Q,l1,l2,alfa1,beta1
    a1=f.a
    b1=f.b
    h1=(b1-a1)/2.0
    h2=(b1+a1)/2.0
```

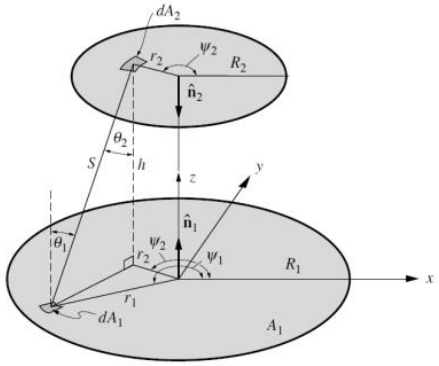
```
    J=0
    for i in range(m):
       JX=0.0
       x[0]=h1*rm[i]+h2
       d1=f.d(x[0])
       c1=f.c(x[0])
       k1=(d1-c1)/2.0
       k2=(d1+c1)/2.0
       #4=========================
       for j in range(n):
          JY=0;
          x[1]=k1*rn[j]+k2
          beta1=f.beta(x)
          alfa1=f.alfa(x)
          l1=(beta1-alfa1)/2.0
          l2=(beta1+alfa1)/2.0
          for k in range(p):
              x[2]=l1*rp[k]+l2
              Q=f.f(x)
              JY+=cp[k]*Q
          JX+=cn[j]*l1*JY;
       #5=========================
       J=J+cm[i]*k1*JX
    J=h1*J
    return J


ff =fdi3(0.0,1.0)
I=I(ff,60,60,60)
print("integral = : ",I)
```

```
runfile('E:/okul/SCO1/integral_4D_GL1.py', wdir='E:/okul/SCO1')
integral = :  0.08333333333333286
```

By using the same priciples, a fourth dimensional integral can also be solved.

$$I = \int_{x=a}^{b} \int_{y=c(x)}^{d(x)} \int_{z=\alpha(x,y)}^{\beta(x,y)} \int_{t=\gamma(x,y,z)}^{\theta(x,y,z)} f(x,y,z,t)dtdzdydx$$

As an example problem, let us look at radiation shape factor problem in radiative heat transfer. In this problem shape factor of two parallel round plate will be calculated.



Radiation shape factor is the percentage of radiative energy reached to surface A2, emitted from surface A1. If the Radius of plates are R1 and R2, and the distance in between is h, the radiation shape factor eqaution is given as:

$$F_{A1-A2} = \frac{1}{\pi^2 R_1^2} \int_{r1=0}^{R_1} \int_{r2=0}^{R_2} \int_{\psi_1=0}^{2\pi} \int_{\psi_2=0}^{2\pi} \frac{h^2 r_1 r_2 d\psi_1 d\psi_2 dr_1}{[h^2 + r_1^2 + r_2^2 - 2r_1 r_2 \cos(\psi_1 - \psi_2)]}$$

This equation has an analytical solution as:

$ds = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$

$ds_1 = R_1 d\,\psi_1(-sin\psi_1\vec{\imath} + cos\psi_1\vec{\jmath})$

$ds_2 = R_2 d\,\psi_2(-sin\psi_2\vec{i} + cos\psi_2\vec{j})$

$ds_1 ds_2 = R_1 R_2 d\psi_1 d\psi_2 (sin\psi_1 sin\psi_2 + cos\psi_1 cos\psi_2)$

$ds_1 ds_2 = R_1 R_2 cos(\psi_1 - \psi_2) d\psi_1 d\psi_2$

$R = [h^2 + R_1^2 + R_2^{-2} - 2R_1 R_2 cos(\psi_1 - \psi_2)]^{1/2}$

$F_{A_1-A_2} = \frac{R_1 R_2}{2\pi(\pi R_1^2)} \int_{\psi_1=0}^{2\pi} \int_{\psi_2=0}^{-2\pi} ln[h^2 + R_1^2 + R_2^2 - 2R_1 R_2 cos(\psi_1 - \psi_2)]^{1/2} cos(\psi_1 - \psi_2) d\psi_1 d\psi_2$

Integrating by parts

$F_{A_1-A_2} = -\frac{R_2}{\pi R_1}\left[sin\psi ln[h^2 + R_1^2 + R_2^2 - 2R_1 R_2 cos\psi]^{1/2}\Big|_{\psi=0}^{2\pi} - \right.$

$\left. R_1 R_2 \int_{\psi=0}^{2\pi} \frac{sin^2\psi}{h^2 + R_1^2 + R_2^{-2} - 2R_1 R_2 cos\psi} d\,\psi\right]$

$F_{A_1-A_2} = \frac{R_2}{\pi R_1} \int_{\psi=0}^{2\pi} \frac{sin^2\psi}{X - cos\psi} d\,\psi$ where $X = \frac{h^2 + R_1^2 + R_2^2}{2R_1 R_2}$

Analytical solution of the integral:

$$F_{A1-A2} = \frac{R_2}{R_1}\left(X - \sqrt{X^2 - 1}\right)$$

$$X = \frac{h^2 + R_1^2 + R}{2R_1 R_2}$$

In the following program numerical solution by Gauss-Legendre integration is obtained and compared to the analytical solution.

Program four dimensional Gauss-Legendre integration , java version

```
 //=======================================================
// Numerical Analysis package in java
// 4 dimensional Gauss-Legendre integration
// written by: Hamza TAŞ
// =======================================================
import java.io.*;
abstract class fdi4
{
public double a,b;
public fdi4(double ai,double bi)
{a=ai;b=bi;}
public fdi4()
{a=0;b=1;}
public void setab(double ai,double bi)
{a=ai;b=bi;}
abstract public  double c(double x);
abstract public  double d(double x);
abstract public  double alfa(double x[]);
abstract public  double beta(double x[]);
abstract public  double gama(double x[]);
abstract public  double teta(double x[]);
abstract public  double f(double x[]);
}

class f1 extends fdi4
{ double h,R1,R2;
  public f1(double hi,double R1i, double R2i)
  {h=hi;R1=R1i;R2=R2i;a=0;b=R1;}

  public double c(double x)
  {return 0;}

  public double d(double x)
  {return R2;}

   public double alfa(double x[])
  {return 0;}

  public double beta(double x[])
```

```
 {return 2.0*Math.PI;}

 public double gama(double x[])
 {return 0.0;}

 public double teta(double x[])
 {return 2.0*Math.PI;}

 public double f(double x[])
 {
 //çözümü istenen fonksiyon
 double ff;
 double z=(h*h+x[0]*x[0]+x[1]*x[1]-2.0*x[0]*x[1]*Math.cos(x[2]-x[3]));
 ff=1.0/(Math.PI*Math.PI*R1*R1)*h*h*x[0]*x[1]/(z*z);
 return ff;
 }
 public double  F12(double h,double R1,double R2)
 {double X=(h*h+R1*R1+R2*R2)/(2.0*R1*R2);
  double F=R2/R1*(X-Math.sqrt(X*X-1));
  return F;
 }
}

class NA78A
{
public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
        //calculates legendre gauss-coefficients as coefficients of the integral for n terms
        double EPS=3.0e-15;
        int m,j,i;
        double z1,z,xm,xl,pp,p3,p2,p1;
   double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
        m=(n+1)/2;
        xm=0.5*(x2+x1);
        xl=0.5*(x2-x1);
        for (i=1;i<=m;i++)  {
                z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                do {
                        p1=1.0;
                        p2=0.0;
                        for (j=1;j<=n;j++) {
                                p3=p2;
                                p2=p1;
                                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                        }
                        pp=n*(z*p1-p2)/(z*z-1.0);
                        z1=z;
                        z=z1-p1/pp;
                } while (Math.abs(z-z1) > EPS);
                a[0][i-1]=xm-xl*z;
                a[0][n-i]=xm+xl*z;
                a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                a[1][n-i]=a[1][i-1];
                        }
return a;
}
public static double I(fdi4 f,int n,int m,int p,int s)
{
 int i,j,k,h;
 double rn[],cn[];
 double rm[],cm[];
 double rp[],cp[];
 double rs[],cs[];
 rn=new double[n];
 rm=new double[m];
 rp=new double[p];
 rs=new double[s];
```

```
 cn=new double[n];
 cm=new double[m];
 cp=new double[p];
 cs=new double[s];
 double a[][]=new double[2][n];
 a=gauss_legendre_coefficients(-1.0,1.0,n);
 rn=a[0];
 cn=a[1];
 a=gauss_legendre_coefficients(-1.0,1.0,m);
 rm=a[0];
 cm=a[1];
 a=gauss_legendre_coefficients(-1.0,1.0,p);
 rp=a[0];
 cp=a[1];
 a=gauss_legendre_coefficients(-1.0,1.0,s);
 rs=a[0];
 cs=a[1];
double x[],h1,h2,J,a1,b1,c1,d1,k1,k2,JX,JY,JZ,Q,l1,l2,alfa1,beta1,gama1,teta1,m1,m2;
x=new double[4];
a1=f.a;
b1=f.b;
h1=(b1-a1)/2.0;
h2=(b1+a1)/2.0;
J=0;
for(i=0;i<m;i++)
{
JX=0;
x[0]=h1*rm[i]+h2;
d1=f.d(x[0]);
c1=f.c(x[0]);
k1=(d1-c1)/2.0;
k2=(d1+c1)/2.0;
     for(j=0;j<n;j++)
     {
               JY=0;
     x[1]=k1*rn[j]+k2;
     beta1=f.beta(x);
     alfa1=f.alfa(x);
     l1=(beta1-alfa1)/2.0;
     l2=(beta1+alfa1)/2.0;
          for(k=0;k<p;k++)
          {
                    JZ=0;
                    x[2]=l1*rp[k]+l2;
                    teta1=f.teta(x);
                    gama1=f.gama(x);
                    m1=(teta1-gama1)/2.0;
                    m2=(teta1+gama1)/2.0;
                         for (h=0;h<s;h++)
                         {
                                   x[3]=m1*rs[h]+m2;
                         Q=f.f(x);
                         JZ+=cs[h]*Q;
                }
         JY+=cp[k]*m1*JZ;
         }
    JX+=cn[j]*l1*JY;
     }
J=J+cm[i]*k1*JX;
}
J=h1*J;
return J;
}
public static double I(fdi4 f,int n)
{return I(f,n,n,n,n);}
public static double I(fdi4 f)
{return I(f,60,60,60,60);}
```

```
  public static void main(String args[]) throws IOException
  {
  double h=1.0;
  double R1=1.0;
  double R2=1.0;
  f1 ff=new f1(h,R1,R2);
  System.out.println("FA1-A2= "+I(ff)+" four dimensional auss Legendre integration ");
  System.out.println("FA1-A2= "+ff.F12(h,R1,R2)+ " analytical solution ");
  }
}
```

```
 ---------- Capture Output ----------
> "c:\java\bin\javaw.exe" NA78A
FA1-A2= 0.3819660112501026 four dimensional Gauss-Legendre integration
FA1-A2= 0.3819660112501051 analytical solution

> Terminated with exit code 0.
```

Program four dimensional Gauss-Legendre integration , Python version

```
#integral_5D_GL1.py
# Original program is written by Hamza Taş in Java Language
# as a part of PhD. level numerical heat radiation class
from math import *
#2 dimensional function
class fdi4:
   def __init__(self, hi,R1i,R2i):
     self.a = 0.0
     self.b = R1
     self.h=hi
     self.R1=R1i
     self.R2=R2i

   def c(self,x):
      return 0.0
   def d(self,x):
      return self.R2
   def alfa(self,x):
      return 0.0
   def beta(self,x):
      return 2.0*pi
   def gama(self,x):
      return 0.0
   def teta(self,x):
      return 2.0*pi
   def f(self,x):
      z=(h*h+x[0]*x[0]+x[1]*x[1]-2.0*x[0]*x[1]*cos(x[2]-x[3]))
      ff=1.0/(pi*pi*R1*R1)*h*h*x[0]*x[1]/(z*z)
      return ff
   # analytical solution of the integral function
   def F12(self,h,R1,R2):
      X=(h*h+R1*R1+R2*R2)/(2.0*R1*R2)
      F=R2/R1*(X-sqrt(X*X-1.0))
      return F


def gauss_legendre_coefficients(x1,x2,n):
   #calculates legendre gauss-coefficients as coefficients of the integral
   #for n terms
   eps=3e-15
   m=int((n+1.0)/2.0)
   xm=0.5*(x2+x1)
   xl=0.5*(x2-x1)
   nmax=100
   a=[[0.0 for i in range(n)] for j in range(2)]
   for i in range(1,m+1):
      z=cos(pi*((i-0.25)/(n+0.5)))
      for ii in range(nmax):
         #while abs(z-z1) > eps:
         p1=1.0;
         p2=0.0;
```

```
            for j in range(1,n+1):
                p3=p2
                p2=p1
                p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
            pp=(float)(n*(z*p1-p2)/(z*z-1.0))
            z1=z;
            z=z1-p1/pp
            if abs(z-z1) < eps: break
        a[0][i-1]=xm-xl*z
        a[0][n-i]=xm+xl*z
        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
        a[1][n-i]=a[1][i-1]
    return a

def I(f,n,m,p,s):
    rn=[0.0 for i in range(n)]
    rm=[0.0 for i in range(m)]
    rp=[0.0 for i in range(p)]
    rs=[0.0 for i in range(s)]
    cn=[0.0 for i in range(n)]
    cm=[0.0 for i in range(m)]
    cp=[0.0 for i in range(p)]
    cs=[0.0 for i in range(s)]
    #a=[[0.0 for i in range(n)] for j in range(2)]
    a=gauss_legendre_coefficients(-1.0,1.0,n)
    rn=a[0]
    cn=a[1]
    a=gauss_legendre_coefficients(-1.0,1.0,m)
    rm=a[0]
    cm=a[1]
    a=gauss_legendre_coefficients(-1.0,1.0,p)
    rp=a[0]
    cp=a[1]
    a=gauss_legendre_coefficients(-1.0,1.0,s)
    rs=a[0]
    cs=a[1]
    x=[0.0 for i in range(4)]
    a1=f.a
    b1=f.b
    h1=(b1-a1)/2.0
    h2=(b1+a1)/2.0
    J=0.0
    for i in range(m):
        JX=0.0
        x[0]=h1*rm[i]+h2
        d1=f.d(x[0])
        c1=f.c(x[0])
        k1=(d1-c1)/2.0
        k2=(d1+c1)/2.0
        for j in range(n):
            JY=0.0
            x[1]=k1*rn[j]+k2;
            beta1=f.beta(x);
            alfa1=f.alfa(x);
            l1=(beta1-alfa1)/2.0;
            l2=(beta1+alfa1)/2.0;
            for k in range(p):
                        JZ=0.0
                        x[2]=l1*rp[k]+l2
                        teta1=f.teta(x)
                        gama1=f.gama(x)
                        m1=(teta1-gama1)/2.0
                        m2=(teta1+gama1)/2.0
                for h in range(s):
                    x[3]=m1*rs[h]+m2
                    Q=f.f(x)
                    JZ=JZ+cs[h]*Q
                JY=JY+cp[k]*m1*JZ
            JX=JX+cn[j]*l1*JY
        J=J+cm[i]*k1*JX
    J=h1*J
    return J

h=1.0
R1=1.0
R2=1.0
```

```
ff =fdi4(h,R1,R2)
I=I(ff,60,60,60,60)
print("F_A1_A2 = ",I,"Numerical four directional Gauss-Legendre integration result")
print("F_A1_A2 = ",ff.F12(h,R1,R2),"Analytical solution")
```

```
runfile('E:/okul/SCO1/integral_5D_GL1.py', wdir='E:/okul/SCO1')
F_A1_A2 =  0.3819660112501026 Numerical four directional Gauss-Legendre integration result
F_A1_A2 =  0.3819660112501051 Analytical solution
```

7.8 DIRECT INTEGRATION OF DATA

If integration of a numerical data set is given two approches and be adapted for this process. The first one is to curve fit a function first and then integrate the fitted function. The second one is to apply an integration equation like Newton-Cotes equation directly to the data to achive integrated result without an intermediate process.

As an example to curve fitting integration a cubic spline integration will be described here. If a cubic polynomial of the following form is given as an interpolation formula:

$$s_k(x) = a_k(x - x_k) + b_k(x_{k+1} - x) + [(x - x_k)^3 c_{k+1} + (x_{k+1} - x)^3 c_k]/(6h_k) \quad 1 \le k \le n \quad (10.6-14)$$

Where x_k and y_k are data points. Derivatives of equation becomes

$$s'_k(x) = a_k - b_k + [(x - x_k)^2 c_{k+1} - (x_{k+1} - x)^2 c_k]/h_k \quad 1 \le k \le n \quad (10.6\text{-}15)$$

$$s''_k(x) = [(x - x_k)c_{k+1} - (x_{k+1} - x)c_k]/h_k \quad 1 \le k \le n$$

a_k ve b_k coefficients can be expressed as a function of c_k

$$b_k = \frac{[6y_k - h_k c_k]}{6h_k} \qquad 1 \le k \le n \qquad (10.6-16)$$

$$a_k = \frac{[6y_{k+1} - h_k^2 c_{k+1}]}{6h_k} \qquad 1 \le k \le n \qquad (10.6-17)$$

In this case only c_k terms left in the system of equations to be solved.

$$h_{k-1}c_{k-1} + 2(h_{k-1} - h_k)c_{k-1} + h_k c_{k+1}c_{k+1} = 6\left[\frac{y_{k+1} - y_k}{h_k} - \frac{y_k - y_{k-1}}{h_{k-1}}\right] \qquad 1 \le k \le n \qquad (10.6-18)$$

This system of equation has only n-2 terms to be solved. By making definition

$w_k = \frac{y_{k+1} - y_k}{h_k}$, $1 \le k \le n$ (10.6-19)

System of equation becomes

$$\begin{bmatrix} 1 & & & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & & \\ & h_2 & 2(h_2+h_3) & \cdots & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & 2(h_{n-3}+h_{n-2}) & h_{n-2} & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & & 1 & \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdots \\ c_{n-2} \\ c_{n-1} \\ c_n \end{Bmatrix} = \begin{Bmatrix} A \\ 6(w_2 - w_1) \\ 6(w_3 - w_2) \\ \cdots \\ 6(w_{n-2} - w_{n-3}) \\ 6(w_{n-1} - w_{n-2}) \\ B \end{Bmatrix} \quad (10.6-20)$$

Where A and B are the second derivative end conditions. A and B should be defined by user. Another important property of the above matrix is that it is a band matrix, therefore less amount of calculation is required to solve it (by using band matrix algorithms such as Thomas algorithm) compare to gauss elimination type of solution methods.

Integration of the equation could be described with the equation

$$\int_a^b S_k(x)dx = \sum_{x=a}^{b} a_k \frac{(x - x_k)^2}{2} - b_k \frac{(x_{k+1} - x)^2}{2} + c_k \frac{(x - x_k)^4}{24h_k} - d_k \frac{(x_{k+1} - x)^4}{24h_k}$$

For $a, x_0, x_1, x_2, \ldots x_n,$. An example program using this approach is given as:

Program 7.8-1 Direct integration of data by using cubic polynomial fitting

```
//=================================================
// Numerical Analysis package in java
```

```
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Direct data integration by using cubic spline
// curve fitting
// Dr. Turhan Coban
// ======================================================
import java.io.*;
import javax.swing.*;

class NA79
{

  public static double [] thomas(double a[][],double r[])
  {
  //
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

  public static double [] thomas(double f[],double e[],double g[],double r[])
  {
  int n=f.length;
  double x[]=new double[n];
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

  public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
  {
  // c0 second derivative of the first point
  // cn second derivative of the last point
  int n=xi.length;
  double h[]=new double[n];
  double w[]=new double[n];
  double f[]=new double[n];
  double e[]=new double[n];
```

```
  double g[]=new double[n];
  double d[]=new double[n];
  double x[]=new double[n];
  double S[][]=new double[4][n];
  int k;
  for(k=0;k<(n-1);k++)
     {h[k]=xi[k+1]-xi[k];
      w[k]=(yi[k+1]-yi[k])/h[k];
                }
  d[0]=c0;
  d[n-1]=cn;
  for(k=1;k<(n-1);k++)
     {d[k]=6.0*(w[k]-w[k-1]);}
  f[0]=1.0;
  f[n-1]=1.0;
  g[0]=0.0;
  g[n-1]=0.0;
  e[0]=0.0;
  e[n-1]=0.0;
  for(k=1;k<(n-1);k++)
  {f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
  S[2]=thomas(f,e,g,d);
  S[3]=xi;
  for(k=0;k<(n-1);k++)
    {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
     S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
    }
  return S;
  }

  public static double funcSpline(double S[][],double x)
  {
  int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y=0;
  double hk=0;
  for(int k=0;k<(n-1);k++)
     {if(S[3][k]<=x && x<=S[3][k+1])
      {hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       break;
       }
     }
       if(y==0 && S[3][n-2]<=x )
       {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       }
  return y;
  }

// Derivative formulations
// =================
  public static double dfSpline(double xi[],double yi[],double c0,double cn,double x)
  { //kübik şerit türev formülü
              double S[][]=cubic_spline(xi,yi,c0,cn);
```

```
   return dfSpline(S,x);
  }

  public static double dfSpline(double S[][],double x)
  {
    //derivative of the cubic spline formula
  int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y=0;
  double hk=0;
  for(int k=0;k<(n-1);k++)
     {if(S[3][k]<=x && x<=S[3][k+1])
      {hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]-S[1][k]+(xx1*xx1*S[2][k+1]-xx2*xx2*S[2][k])/(2.0*hk);
       break;
       }
     }
       if(y==0 && S[3][n-2]<=x )
       {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       }
  return y;
  }

 public static double  dfSpline(double xi[],double yi[],double x)
 {
    //natural cubic spline formula
              double S[][]=cubic_spline(xi,yi,0,0);
    return dfSpline(S,x);
 }
// Integration formulas
// ====================
  public static double intSpline(double S[][],double a,double b)
  {
    //cubic spline integration
  int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y1,y2;
  double hk=0;
  double toplam=0;
  y1=0;y2=0;
  for(int k=0;k<(n-1);k++)
     { hk=(S[3][k+1]-S[3][k]);
       //System.out.println("x1="+S[3][k]+"x2="+S[3][k+1]+"a="+a+"b="+b);
      if(a>S[3][k+1])
      {toplam=0;//System.out.println("şart 1 toplam = "+toplam);
      }//şart1
               else if(S[3][k]<=a && a<=S[3][k+1] && b>S[3][k+1])//şart 2
       {
       xx1=(a-S[3][k]);
       xx2=(S[3][k+1]-a);
       y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
       xx1=hk;
```

```
      xx2=0;
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      //System.out.println("şart 2toplam = "+toplam+"y1="+y1+"y2="+y2);
      }
      else if(S[3][k]<=a && a<=S[3][k+1] && S[3][k]<=b && b<=S[3][k+1])//şart 3
      {
      xx1=(a-S[3][k]);
      xx2=(S[3][k+1]-a);
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=(b-S[3][k]);
      xx2=(S[3][k+1]-b);
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      //System.out.println("şart 3 toplam = "+toplam+"y1="+y1+"y2="+y2);
      }
      else if(a<S[3][k] && b>=S[3][k+1]) //şart 4
      {
      xx1=0;
      xx2=hk;
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=hk;
      xx2=0;
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      //System.out.println("şart 4 toplam = "+toplam+"y1="+y1+"y2="+y2);
      }
      else if(S[3][k]<=b && b<=S[3][k+1])//şart 5
      {xx1=0;
       xx2=hk;
       y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
       xx1=(b-S[3][k]);
       xx2=(S[3][k+1]-b);
       y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-
xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
       toplam+=(y2-y1);
       //System.out.println("şart 5 toplam = "+toplam+"y1="+y1+"y2="+y2);
       }
       else break;
    }
  return toplam;
  }

  public static double intSpline(double xi[],double yi[],double c0,double cn,double a,double b)
  {double S[][]=cubic_spline(xi,yi,c0,cn);
   return intSpline(S,a,b);
  }

  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1="a.txt";
  //JFileChooser fc=new JFileChooser();
  //if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
```

```
double a[][]=Text.readDoubleT(s1);
x=a[0];
y=a[1];
int n=x.length;
double S[][]=cubic_spline(x,y,2.0,2.0);
System.out.println("s="+intSpline(S,0.0,10.0));
}
}
```

Following data is taken to integrate

```
0 0
1 1
2 4
5 25
6 36
7 49
8    64
9 81
10 100
```

Sample data is created from function $f(x)=x^2$. Therefore integration between 0 and 10 will give 333.3333333 as result.

Cubic spline direct data integration

```
---------- Capture Output ----------
> "C:\co\java\bin\java.exe" SCO11FB
s=333.3333333333334

> Terminated with exit code 0.
```

MULTIDIMENSIONAL INTEGRATION BY USING CONTOUR INTEGRATION

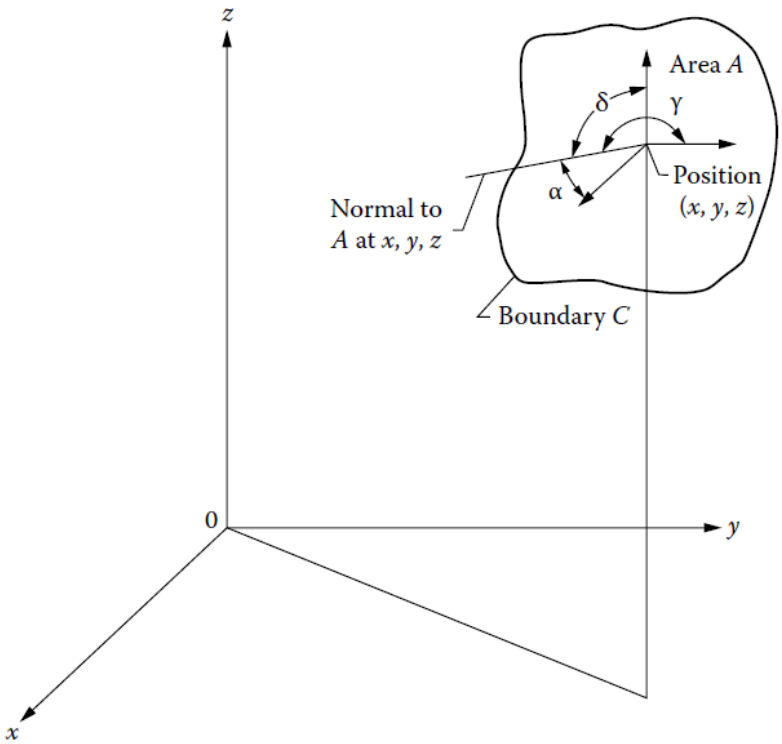
Stoke's theorem gives a way to reduce multiple integration to single integration. Consider the area A in the figure with countour boundary C. It is assumed that boundary is continuous or picewise-continuous. Let us also consider an arbitrary point on the surface as P(x,y,z). Normal to this point can be written as

Vector pointing from original to an arbitrary point $P(\mathrm{x},\mathrm{y},\mathrm{z}) = \left[x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right]$

Surface normal for area: $n = \left[l\vec{\imath} + m\vec{\jmath} + n\vec{k}\right] = \left[\cos(\alpha)\vec{\imath} + \cos(\beta)\vec{\jmath} + \cos(\gamma)\vec{k}\right]$

Where l,m,n are direction cosines of the surface normal vector, α, β and γ are angles of normal with x,y and z axeses respectively. And also let P,Q, and W be any twice differentiable functions of (x,y,z). Stokes' theorem in three dimensions states that

$$\oint_C (Pdx + Qdy + Wdz) = \int_A \left[\left(\frac{\partial W}{\partial y} - \frac{\partial Q}{\partial z}\right) l + \left(\frac{\partial P}{\partial z} - \frac{\partial W}{\partial x}\right) m + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) n\right] dA$$



Now this theory can be apply to solve configuration factor problems. Let us consider **configuration factors between a differential area and a finite area**

$$\cos(\theta_i) = \left[(x_j - x_i)l_i + (y_j - y_i)m_i + (z_j - z_i)n_i\right]/R$$
$$\cos(\theta_j) = \left[(x_i - x_j)l_j + (y_i - y_j)m_j + (z_i - z_j)n_j\right]/R$$

And

$$R^2 = (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2$$
$$R = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$
$$F_{dA_i - A_j} = \int_{A_2} \frac{\cos(\theta_i)\cos(\theta_j)}{\pi R^2} dA_j$$

This particular integral is called shape factor. It is an important part of radiation heat transfer calculations. For further details of radiation shape factor, please refer to radştşon heat transfer books. Integral formula converted to contour integration as:

$$F_{dA_i - A_j}$$
$$= \frac{1}{\pi} \int_{A_j} \frac{\left[(x_j - x_i)l_i + (y_j - y_i)m_i + (z_j - z_i)n_i\right]\left[(x_i - x_j)l_j + (y_i - y_j)m_j + (z_i - z_j)n_j\right]}{R^4} dA_j$$

Now let

$$f = \frac{\left[(x_j - x_i)l_i + (y_j - y_i)m_i + (z_j - z_i)n_i\right]}{\pi R^4}$$

then equation becomes

$$F_{dA_i - A_j} = \int_{A_j} \left[(x_i - x_j)fl_j + (y_i - y_j)fm_j + (z_i - z_j)fn_j\right] dA_j$$

Comparison of this equation with Stokes' equation gives

$$\left(\frac{\partial W}{\partial y} - \frac{\partial Q}{\partial z}\right) = (x_i - x_j)f$$

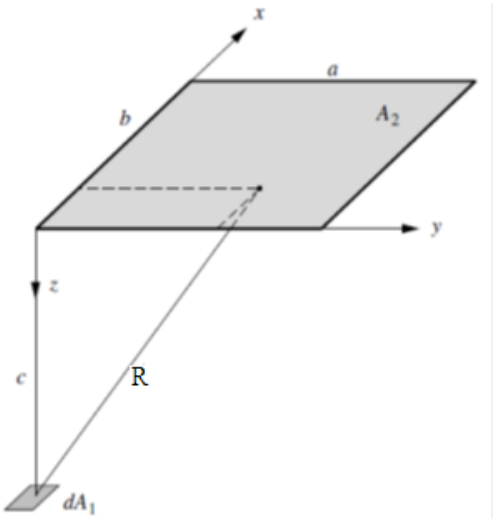
$$\left(\frac{\partial P}{\partial z}-\frac{\partial W}{\partial x}\right)=(y_i-y_j)f$$
$$\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=(z_i-z_j)f$$

Useful solution of these equations[98,99] are:

$$P=\frac{-m_i(z_j-z_i)+n_i(y_j-y_i)}{2\pi R^2}$$
$$Q=\frac{l_i(z_j-z_i)-n_i(x_j-x_i)}{2\pi R^2}$$
$$W=\frac{-l_i(y_j-y_i)+m_i(x_j-x_i)}{2\pi R^2}$$
$$F_{dA_i-A_j}=\oint_{C_j}(Pdx_j+Qdy_j+Wdz_j)$$
$$F_{dA_i-A_j}=\oint_{C_j}\left(\frac{-m_i(z_j-z_i)+n_i(y_j-y_i)}{2\pi R^2}dx_j+\frac{l_i(z_j-z_i)-n_i(x_j-x_i)}{2\pi R^2}dy_j\right.$$
$$\left.+\frac{-l_i(y_j-y_i)+m_i(x_j-x_i)}{2\pi R^2}dz_j\right)$$

$$F_{dA_i-A_j}=\frac{l_i}{2\pi}\oint_{C_j}\frac{(z_j-z_i)dy_j-(y_j-y_i)dz_j}{R^2}+\frac{m_i}{2\pi}\oint_{C_j}\frac{(x_j-x_i)dz_j-(z_j-z_i)dx_j}{R^2}+$$
$$\frac{n_i}{2\pi}\oint_{C_j}\frac{(y_j-y_i)dx_j-(x_j-x_i)dy_j}{R^2}$$

PROBLEM:



The shape in the left side is given. By using contour integration method find the shape factor analytically and numerically

$l_1=m_1=0\ \overrightarrow{n_1}=-\vec{k}\quad n_1=-1\quad F_{dA_1-A_2}=\frac{-1}{2\pi}\oint_{C_2}\frac{ydx-xdy}{R^2}\quad R^2=x^2+y^2+c^2$

$$F_{dA_1-A_2}=\frac{-1}{2\pi}\left[\int_{x=0}^{b}\frac{y}{R^2}dx\Big|_{y=0}+\int_{y=0}^{a}\frac{-x}{R^2}dy\Big|_{x=b}+\int_{x=b}^{0}\frac{y}{R^2}dx\Big|_{y=a}+\int_{x=a}^{0}\frac{y}{R^2}dy\Big|_{x=0}\right]$$

For the configuration $F_{dA_1-A_2}$ given as

$$F_{dA_1-A_2} = \frac{1}{2\pi}\left(\int_{y=0}^{a}\frac{bdy}{b^2+y^2+c^2} + \int_{x=0}^{b}\frac{adx}{x^2+a^2+c^2}\right)$$

Analytical solution gives

$$F_{dA_1-A_2} = \frac{1}{2\pi}\left(\frac{b}{\sqrt{b^2+c^2}}tan^{-1}\left(\frac{a}{\sqrt{b^2+c^2}}\right) + \frac{a}{\sqrt{a^2+c^2}}tan^{-1}\left(\frac{b}{\sqrt{a^2+c^2}}\right)\right)$$

Calculate first equation by using numerical integration and compare it with the analytical result
a= 1 m b= 1 m and c= 1 m

```
from math import *
from if_x import *;

class fx(if_x):
    def __init__(self, ai,bi,ci):
        self.a = ai
        self.b=bi
        self.c=ci
    def func(self,x):
        y=1.0/(2.0*pi)*a/(x*x+a*a+c*c)
        return y

class fy(if_x):
    def __init__(self, ai,bi,ci):
        self.a = ai
        self.b=bi
        self.c=ci
    def func(self,y):
        y=1.0/(2.0*pi)*b/(b*b+y*y+c*c);
        return y

def gauss_legendre_coefficients(x1,x2,n):
    #calculates legendre gauss-coefficients as coefficients of the integral
    #for n terms
        eps=3e-15
        m=int((n+1.0)/2.0)
        xm=0.5*(x2+x1)
        xl=0.5*(x2-x1)
        nmax=100
        a=[[0.0 for i in range(n)] for j in range(2)]
        for i in range(1,m+1):
            z=cos(pi*((i-0.25)/(n+0.5)))
            for ii in range(nmax):
                #while abs(z-z1) > eps:
                p1=1.0;
                p2=0.0;
                for j in range(1,n+1):
                    p3=p2
                    p2=p1
                    p1=(float)(((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j)
                pp=(float)(n*(z*p1-p2)/(z*z-1.0))
                z1=z;
                z=z1-p1/pp
                if abs(z-z1) < eps: break
```

```
            a[0][i-1]=xm-xl*z
            a[0][n-i]=xm+xl*z
            a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp)
            a[1][n-i]=a[1][i-1]
        return a

def F(a,b,c):
    A=b/sqrt(b*b+c*c)
    B=a/sqrt(b*b+c*c)
    C=a/sqrt(a*a+c*c)
    D=b/sqrt(a*a+c*c)
    f=1.0/(2.0*pi)*(A*atan(B)+C*atan(D))
    return f


def integral(f,x1,x2,n):
    #integral func(x)dx
    #integral of a function by using gauss-legendre quadrature
    #between x1 and x2
    a=gauss_legendre_coefficients(x1,x2,n)
    z=0
    for i in range(n):
        z=z+a[1][i]*f.func(a[0][i])
    return z

a=1.0
b=1.0
c=1.0
x1=fx(a,b,c)
y1=fy(a,b,c)
I1=integral(y1,0.0,a,20)+integral(x1,0.0,b,20)
print("I1 (contour integral) : ",I1)
print("F (analytical integral solutionl) : ",F(a,b,c));
```

```
runfile('D:/okul/SCO1/integral_contour_GL.py', wdir='D:/okul/SCO1')
Reloaded modules: if_x
I1 (contour integral) :  0.13853160599489273
F (analytical integral solution) :  0.13853160599489298
```

Contour integration for configuration factor between finite areas

In order to calculate contour integration for finite areas, the previously find contour integration between differential and finite area eqution integrated for area

$$A_i F_{A_i-A_j} = A_j F_{A_j-A_i} = \int_{A_i} F_{dA_i-A_j}\, dA_i$$

$$A_i F_{A_i-A_j} = A_j F_{A_j-A_i} = \frac{1}{2\pi} \oint_{C_j} \int_{A_i} \left[\frac{-m_i(z_j-z_i)+n_i(y_j-y_i)}{R^2} dx_j + \frac{l_i(z_j-z_i)-n_i(x_j-x_i)}{R^2} dy_j + \frac{-l_i(y_j-y_i)+m_i(x_j-x_i)}{R^2} dz_j) \right] dA_i$$

$$A_i F_{A_i-A_j} = A_j F_{A_j-A_i} = \frac{1}{2\pi} \oint_{C_j} \left[\left(\int_{A_i} \frac{-m_i(z_j-z_i)+n_i(y_j-y_i)}{R^2} dA_i \right) dx_j + \left(\int_{A_i} \frac{l_i(z_j-z_i)-n_i(x_j-x_i)}{R^2} dA_i \right) dy_j + \left(\int_{A_i} \frac{-l_i(y_j-y_i)+m_i(x_j-x_i)}{R^2} dA_i \right) dz_j) \right]$$

where the integrals have been rearranged, and dx_j, dy_j, and dz_j have been factored out since they are independent of the area integration over $A1$. Stokes' theorem is applied in turn to each of the three area integrals. Consider area integral

$\int_{A_i} \frac{-m_i(z_j-z_i)+n_i(y_j-y_i)}{R^2} dA_i$

Integral in stokes theorem gives

$$\left(\frac{\partial W}{\partial y} - \frac{\partial Q}{\partial z}\right) = 0$$

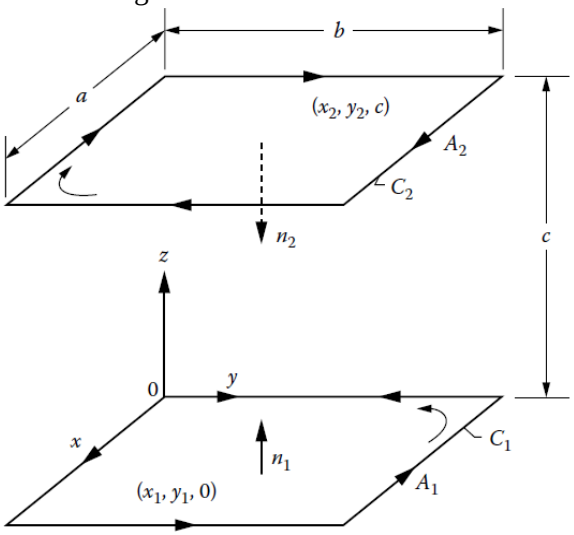
$$\left(\frac{\partial P}{\partial z} - \frac{\partial W}{\partial x}\right) = \frac{-(z_j - z_i)}{R^2}$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{(y_j - y_i)}{R^2}$$

Solution of this set of partial differential equations is P=ln(R) Q=0 and W=0 and

$$A_i F_{A_i-A_j} = A_j F_{A_j-A_i} = \frac{1}{2\pi} \oint_{C_1} \oint_{C_2} \left[\ln(R)\, dx_j dx_i + \ln(R)\, dy_j dy_i + \ln(R)\, dz_j dz_i\right]$$

PROBLEM: Find configuration factor by using contour integration for two parallel rectangular shown in the figure



$$F_{A_1-A_2} = F_{A_2-A_i} = \frac{1}{2\pi ab} \oint_{C_1} \oint_{C_2} \left[\begin{array}{l} \ln\left([(x_2 - x_1)^2 + (y_2 - y_1)^2 + c^2]^{\frac{1}{2}}\right) dx_2 dx_1 + \\ \ln\left([(x_2 - x_1)^2 + (y_2 - y_1)^2 + c^2]^{1/2}\right) dy_2 dy_1 \end{array}\right]$$

Opening contour integral for area 2

$F_{A_1-A_2} = F_{A_2-A_i}$

$$= \frac{1}{2\pi ab} \oint_{C_1} \left[\begin{array}{l} \left(\int_{x_2=0}^{a} \ln\left([(x_2 - x_1)^2 + (b - y_1)^2 + c^2]^{\frac{1}{2}}\right) dx_2\right) dx_1 + \left(\int_{x_2=a}^{0} \ln\left([(x_2 - x_1)^2 + (0 - y_1)^2 + c^2]^{\frac{1}{2}}\right) dx_2\right) dx_1 + \\ \left(\int_{y_2=0}^{b} \ln\left([(0 - x_1)^2 + (y_2 - y_1)^2 + c^2]^{1/2}\right) dy_2\right) dy_1 + \left(\int_{y_2=b}^{0} \ln\left([(a - x_1)^2 + (y_2 - y_1)^2 + c^2]^{1/2}\right) dy_2\right) dy_1 \end{array}\right]$$

Opening contour integral for area 2

$$F_{A_1-A_2} = F_{A_2-A_1}$$

$$= \frac{1}{2\pi ab}\begin{bmatrix} \int_{x_1=0}^{a}\left(\int_{x_2=0}^{a}\ln\left([(x_2-x_1)^2+(b-0)^2+c^2]^{\frac{1}{2}}\right)dx_2\right)dx_1 + \int_{x_1=0}^{a}\left(\int_{x_2=0}^{0}\ln\left([(x_2-x_1)^2+(0-0)^2+c^2]^{\frac{1}{2}}\right)dx_2\right)dx_1 + \\ \int_{x_1=a}^{0}\left(\int_{x_2=0}^{a}\ln\left([(x_2-x_1)^2+(b-b)^2+c^2]^{\frac{1}{2}}\right)dx_2\right)dx_1 + \int_{x_1=a}^{0}\left(\int_{x_2=a}^{0}\ln\left([(x_2-x_1)^2+(0-b)^2+c^2]^{\frac{1}{2}}\right)dx_2\right)dx_1 + \\ \int_{y_1=0}^{b}\left(\int_{y_2=0}^{b}\ln\left([(0-a)^2+(y_2-y_1)^2+c^2]^{\frac{1}{2}}\right)dy_2\right)dy_1 + \int_{y_1=0}^{b}\left(\int_{y_2=b}^{0}\ln\left([(a-a)^2+(y_2-y_1)^2+c^2]^{\frac{1}{2}}\right)dy_2\right)dy_1 + \\ \int_{y_1=b}^{0}\left(\int_{y_2=0}^{b}\ln\left([(0-0)^2+(y_2-y_1)^2+c^2]^{1/2}\right)dy_2\right)dy_1 + \int_{y_1=b}^{0}\left(\int_{y_2=b}^{0}\ln\left([(a-0)^2+(y_2-y_1)^2+c^2]^{1/2}\right)dy_2\right)dy_1 \end{bmatrix}$$

$$F_{A_1-A_2} = F_{A_2-A_1} = \frac{1}{\pi ab}\left[\int_{x_1=0}^{a}\left(\int_{x_2=0}^{a}\ln\left(\frac{[(x_2-x_1)^2+b^2+c^2]^{\frac{1}{2}}}{[(x_2-x_1)^2+c^2]^{\frac{1}{2}}}\right)dx_2\right)dx_1 + \int_{y_1=0}^{b}\left(\int_{y_2=0}^{b}\ln\left(\frac{[a^2+(y_2-y_1)^2+c^2]^{\frac{1}{2}}}{[(y_2-y_1)^2+c^2]^{1/2}}\right)dx_2\right)dx_1\right]$$

Analytical solution of the equation:

$X = \frac{a}{c} \quad Y = \frac{b}{c}$

$F_{A_1-A_2} = F_{A_2-A_1} = \frac{2}{\pi XY}\left\{ln\left[\frac{(1+X^2)(1+Y^2)}{(1+X^2+Y^2)}\right]^{1/2} + X\sqrt{1+Y^2}tan^{-1}\left(\frac{X}{\sqrt{1+Y^2}}\right) + Y\sqrt{1+X^2}tan^{-1}\left(\frac{Y}{\sqrt{1+X^2}}\right) - Xtan^{-1}(X) - Ytan^{-1}(Y)\right\}$

Gauss random distribution function

When n is a big number

Average: $\mu = \frac{\sum_{i=1}^{n} x_i}{n}$

Standard deviation: $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$

When n is relatively small number

Average: $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

Standard deviation: $s = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}}$

Laplace-Gauss distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Normal distribution $(\sigma = 1 \quad \mu = 0)$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Cumulative distribution function

The Laplace distribution is easy to integrate (if one distinguishes two symmetric cases) due to the use of the absolute value function. Its cumulative distribution function is as follows:

$$\phi(x) = \int_{x=-\infty}^{x} \varphi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x} e^{-\frac{1}{2}t^2} dt$$

$$\phi(\infty) = \int_{x=-\infty}^{\infty} \varphi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt = 1$$

Error function

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_{x=0}^{\infty} e^{t^2} dt$$

MacLaurin series opening of error function

$$\text{erf(x)} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

Error function- Cumulative distribution function relation

$$\phi(x) = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right]$$

Laplace-Gauss distribution function:

$$F(x) = \phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

```
# Gauss - Legendre integral 10 coefficients
# python version

from math import *

def fi(x):
   y= 1.0/(2.0*sqrt(pi))*exp(-0.5*x*x)
   return y

def erf1(x):
   plusminus=1
   power=x
   factorial=1
   total=0
   for n in range(100):
      k=2*n+1
      total+=plusminus*power/(factorial*k)
      plusminus*=-1
      power*=x*x
      factorial*=(n+1)
   return total*2.0/sqrt(pi)

def g(x):
   return 2.0/sqrt(pi) *exp(-x*x)

def erf2(x):
   return  integral(g,0,x)
  #gauss inetgral
def integral(f,a,b):
  #integral f1(x)dx
  #integral of a function by using gauss-legendre quadrature
  #coefficients are pre-calculated for 10 terms for [-1,1]
  #band then utilises variable transform
  r=[-0.973906528517171,-0.865063366688984,-0.679409568299024,-0.433395394129247,-
0.148874338981631,0.148874338981631,0.433395394129247,0.679409568299024,0.865063366688984,0.973906528517171]
```

```
c=[0.066671344308684,0.149451349150580,0.219086362515982,0.269266719309996,0.295524224714752,0.295524224714752,0.2692
66719309996,0.219086362515982,0.149451349150580,0.066671344308684]
  n=10
  z=0
  k1=(b-a)/2.0;
  k2=(b+a)/2.0;
  for i in range(n):
    x=k2+k1*r[i]
    y=f(x)
    z=z+k1*c[i]*y
  return z
# cumulative distribution functions
def fi1(x):
    y=0.5*(1.0+erf1(x/sqrt(2.0)))
    return y
def fi2(x):
    y=0.5*(1.0+erf2(x/sqrt(2.0)))
    return y
def fi3(x):
    y=0.5*(1.0+erf(x/sqrt(2.0)))
    return y

x=float(input("x="))
e1=erf1(x)
e2=erf2(x)
e3=erf(x)
print("erf1",e1,"erf2",e2,"erf3",e3)
fi1=fi1(x)
fi2=fi2(x)
fi3=fi3(x)
print("fi1",fi1,"fi2",fi2,"fi3",fi3)
```

```
runfile('E:/okul/SCO1/Laplace_gauss_didstribution.py', wdir='E:/okul/SCO1')

x=0.1
erf1 0.11246291601828491 erf2 0.11246291601828423 erf3 0.1124629160182849
fi1 0.539827837277029 fi2 0.5398278372770288 fi3 0.539827837277029
```

PROBLEMS (INTEGRATION & DERIVATION)

PROBLEM 1

of **$y=f(x)=1/(1+x^2)$** fonction between limits –1 ile 1 with 3 points Gauss-Legendre integration Formula

Note: Required roots and coefficients :

coefficients	Roots
c_0 0.555555555555553	x_0 0.774596669241483
c_1 0.888888888888889	x_1 0.000000000000000
c_2 0.555555555555553	x_2 0.774596669241483

PROBLEM 2

$$\int_{-3}^{3} \frac{2}{1+2x^2}\,dx$$

solve with bole rule. Use n=2

Note: Bole rule for n=1

$$I = (b-a)[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

Step size h=(b-a)/n

PROBLEM 3

$$f_{0-\lambda} = \frac{c_1}{\sigma} \int_{x=0}^{\lambda T} \frac{dx}{x^5 \left[exp\left(\frac{c_2}{x}\right) - 1\right]}$$

Function is known as black body radiation shape function. The values of coefficients in the equation are:

c_1 =3.743 x10^8 W.μm^4/m^2, c_2 =1.4387x10^4 μm.K, σ=5.67x10^{-8} W/(m^2.K^4).

Use Gauss-Legendre integration and for a giving λT calculate black body radiation shape factor. Create a table for the values of blackbody radiation shape factor from 5000 to infinity(?)

PROBLEM 4

$$I = \int_{x=0}^{1} \int_{y=0}^{1} \frac{dydx}{(1+y)^2}$$

Solve the above integration by using analytical methods and Gauss-Legendre integration

PROBLEM 5

Solve the integration

$$I = \int_{x=0}^{1} \frac{1}{1+x^2} dx$$

a) by analytical methods

b) by Trapezoidal rule with n=2(divide region into 2 parts) and calculate % error

c) Trapezoidal rule with n=2(divide region into 2 parts) and calculate % error

PROBLEM 6

Solve the integration

$$I = \int_{x=0}^{2} \frac{1}{\sqrt{1+x^2}} dx$$

By using Gauss-Legendre formulation

PROBLEM 7

For x=1

$$f(x) = ln\left(\frac{1}{1+x^2}\right)$$

a) Calculate analytical derivative
b) Calculate derivative with 3 points central difference formula with h=0.1
c) Calculate derivative with 3 points central difference formula with h=0.05
d) Calculate derivative with Richardson extrapolation formula with h1=0.5 and h2=0.25

PROBLEM 8

$$f(x) = \int_{-1}^{1} xexp(x)dx$$

Calculate the integration by

a) Trapezoidal rule integrali
b) Simpson 3/8 integration
c) Gauss-Legendre integration
d) Romberg integration

PROBLEM 9

$$\boldsymbol{f(x) = \int_{0}^{1} sin(1 - x + x^2)dx}$$

Calculate the integration by

e) Trapezoidal rule integrali
f) Simpson 3/8 integration
g) Gauss-Legendre integration
h) Romberg integration

PROBLEM 10

$$\boldsymbol{f(x) = \int_{-1}^{1} \frac{ln(1 + x^2)}{\sqrt{1 - x^2}} dx}$$

Calculate the integration by

i) Trapezoidal rule integrali
j) Simpson 3/8 integration
k) Gauss-Legendre integration
l) Romberg integration

PROBLEM 11

X	**0.99**	**2.10**	**3.22**	**4.40**	**5.70**	**7.12**	**8.01**	**8.37**	**9.32**	**9.98**
f(x)	**4.90**	**5.70**	**4.20**	**7.04**	**8.31**	**7.82**	**5.97**	**7.01**	**6.68**	**4.79**

Data given in the above table. Find the integration by using trapezoidal rule.

PROBLEM 12

Find the derivative of data given in problem 11 at point x=5

PROBLEM 13

Depth(H) and velocity(U) profile of a channel is given in below table

x, m	0	2	4	6	8	10	12	14	16	18	20
H, m	0	1.8	2	4	4	6	4	3.6	3.4	2.8	0
U m/s	0	0.03	0.045	0.055	0.065	0.12	0.07	0.06	0.05	0.04	0

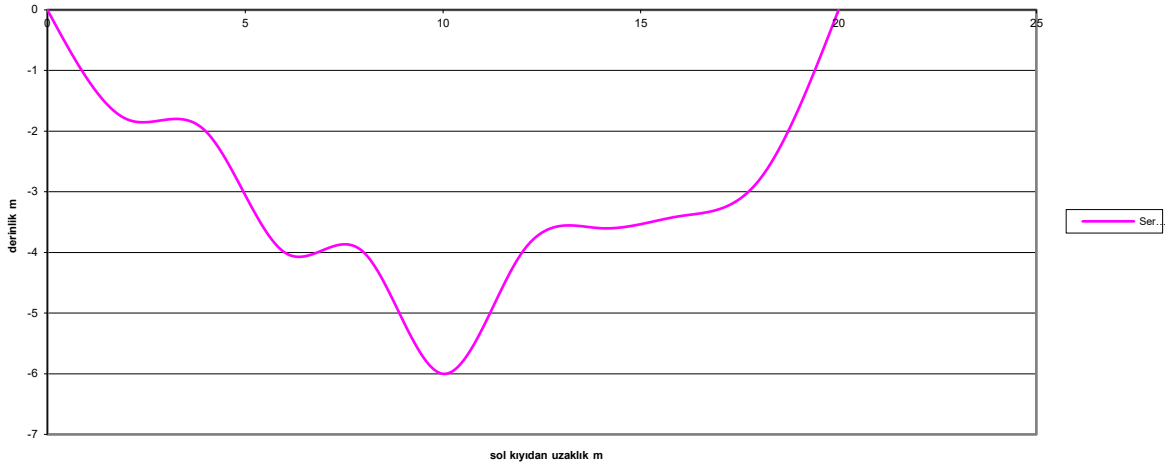
Cross sectional area of the channel can be calculated as :

$$A_c = \int_{0}^{B} H(x)dx$$

Flow rate of the channel will be

$$Q = \int_{0}^{B} U(x)H(x)dx$$

In this equations U is the velocity of the water measured at point x. Calculate cross sectional area and flow rate of the channel.



PROBLEM 14

Temperature profle of a solid body can be given as

$$\frac{dT}{dt} = -k(T - T_a)$$

In this equation T is the temperature, T_a is constant surface temperature.

A solid body with temperatures initially at T=90 C and, T_a =20 C put into a water container at temperature of T_a = 20 C. The temperature profile as a function of time is given at the table.

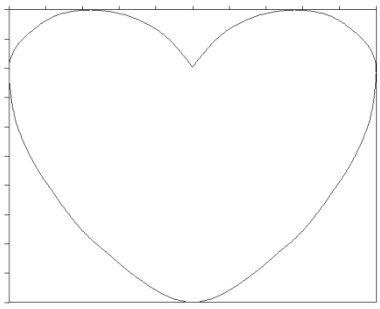
Zaman , t, s	0	5	10	15	20	25	30	35
Sicaklik T C	90	55	40	28	24	22	21.5	20.6

Calculate k thermal conductivity coefficient.

PROBLEM 15

Coordinate of a hard shape is given in the table. Find the area of the hard.

Xi	Yi	xi	yi	xi	yi	xi	Yi
241	127	331	157	215	246	149	120
258	107	323	178	199	232	160	106
283	96	312	197	184	217	175	97
307	97	298	217	170	197	199	96
322	106	283	232	159	178	224	107
333	120	267	246	151	157	241	127
334	138	241	258	148	138		



PROBLEM 16

$$\boldsymbol{I} = \int_{0.1}^{0.5} \int_{x^3}^{x^2} e^{y/x} dydx$$

Solve the integral by Gauss-Legendre method.

PROBLEM 17

$$\boldsymbol{I} = \int_{0.1}^{1} \int_{x}^{2x} (x^2 + y^2) dydx$$

Solve the equation with

a) 2 dimensional Gauss-Legendre integration
b) 2 dimensional simpson 1/3 integration
c) Develop an equation for 2 dimensional simpson. 3/8 integration and solve the integral

PROBLEM 18

$$\boldsymbol{I} = \int_{x=0}^{1} \int_{y=0}^{1} \frac{dydx}{(1+y)^2}$$

Solve the integral with Monte carlo method and check the results with Gauss-Legendre method.

PROBLEM 19

Show that equation below is correct

$$\pi = \int_{0}^{2} \sqrt{(4 - x^2)}\, dx$$

PROBLEM 20

$$I = \int_{0}^{1} e^x dx$$

Solve the integral for m=25,50,100,500,1000,10000,100000 with monte-carlo method and compare the result with the analytical solution.

PROBLEM 21

An angine of a vehicle with mass of 53400 kg and a velocity of v=30 m/is suddenly stopped at t=0. In this instant equation of the movement can be given as

$$5400v\frac{dv}{dx} = -8276v^2 - 2000$$

In the equation v(m/sn), is the velocity of the vehicle at time t. When velocity is 15 m/s calculate the total distance vehicle moved by using 4 point Gauss integration Formula .

PROBLEM 22

A vehicle is at complete stop. When it is started velocity of the vehicle is recorded for the first 5 minutes. Calculate the total distance vehicle moved by using trapezoidal rule as a direct data integration method.

t(minutes)	0	0,5	1,0	1,5	2,0	2,5	3,0	3,5	4,0	4,5	5,0
v (km/h)	0	4	7	11	15	21	25	30	32	35	40

Note : $\boldsymbol{v = \frac{dx}{dt}}$ $x - x_0 = \int_{t_0}^{t} v(t)dt$

PROBLEM 23

Upward velocity of a rocket can be calculated by using the formula

$$V = \frac{dz}{dt} = Uln\left(\frac{m_0}{m_0 - qt}\right) - gt$$

In this equation V is velocity, U is the exit velocity of burning gases from the rocket nozzle, m_o is the mass of the rocket at time t = 0 sn, q is fuel spending rate , g is garavitational acceleration (assume constant as, 9,8 m^2/s). If U = 2000 m/s, m_o = 150 000 kg, q = 2600 kg/s, Calculate the height rocket will climb in 30 seconds by using Romberg inntegration with k=4 [$O(h^8)$] order of error.

Note : $\boldsymbol{V = \frac{dH}{dt}}$ $H = \int_{t_{0=0}}^{t} V(t)dt$

PROBLEM 24

Calculate velocity and acceleration by using below data with

a) cental difference method [with $O(h^4)$ error level],
b) forward difference method [with $O(h^2)$ error level],
c) backward difference method [with $O(h^2)$ error level],

Time, t (s)	0	1	2	3	4	5	6	7	8
distance, x (m)	0	0.7	1.8	3.4	5.1	6.5	7.3	8.0	8.4

Note : Velocity $\boldsymbol{v = \frac{dx}{dt}}$Acceleration : $a = \frac{d^2x}{dt^2}$

PROBLEM 25

Integrate data by using trapezoidal rule

x	-3	-1	1	3	5	7	9	11
f(x)	1	-4	-9	2	4	2	6	-3

PROBLEM 26

In a river water height is measured in every 12.5 m distance. Water flow rate can be calculated by $Q = \int_{t_{0=0}}^{t} VdA$ integration. In this equation Q: water flow rate (m^3/s), V: water flow speed (m/s), A: cross-sectional area (m^2). If avarage water velocity is 0.4 m/s. Calculate water flow rate by using Simpson 1/3 rule

X(m)	0	12,5	25	37,5	50	62,5	75	87,5	100
f(X) m	11	7,5	5,5	0	1,5	4	5,5	8	11

PROBLEM 27

a)

$$\int_{0}^{10} \boldsymbol{f(x)dx}$$

calculate the integral value from the data given below

b) calculate the derivative of the function at x=3,4,5,6,7,8

X	0.99	2.10	3.22	4.40	5.70	7.12	8.01	8.37	9.32	9.98
f(x)	4.90	5.70	4.20	7.04	8.31	7.82	5.97	7.01	6.68	4.79

PROBLEM 28

By using Gauss-Legendre integral formulation calculate

$$I = \int_{-4}^{4} \frac{dx}{1+x^2}$$

integral for n=2,3,4,5 . Compare the result with analytical solution.

PROBLEM 29

By using **Adaptive** simpson 1/3 solve

$$I = \int_{-4}^{4} \frac{dx}{1+x^2}$$

integral with an error level of 1.0e-5

PROBLEM 30

By using Gauss-Chebychev integration equation solve the integral

$$I = \int_{-1}^{1} \frac{\cos(x)dx}{\sqrt{1-x^2}} = 2.40394$$

and compare the result with the exact value.

PROBLEM 31

By using trapezoidal rule calculate the integral

$$I = \int_{0}^{1} \frac{2dx}{2+\sin(10\pi x)} = 1.15470054$$

for n=2,4,8 ve 16

And then improve the result by using Richardson extrapolation Formula (Romberg integration).

PROBLEM 32

If velocity profile in a pipe is known, flow rate can be calculated as $Q = \int v\, dA$

In this equation v i the velocity of the fluid A is the cross-sectional area of the pipe. In a circular pipe A=(2 π r) and dA = (2 π r dr). If it is substituted into flow rate equation

Q =∫ v(2πr) dr is obtained. Where r is the radius. If the velocity distribution is given as a function of r as $v = 3[1-(r/r_0)]^{(1/7)}$ and if the radius of the pipe r_0= 0.1 m calculate mass flow rate. Select your own method.

PROBLEM 33

In the integrals below exact solutions are given. By using an adaptive integration method. Check out the results.

a)

$$I = \int_{0}^{1} \frac{4dx}{1+256(x-0.375)^2} = 0.7191938309210011$$

b)

$$I = \int_{0}^{1} x^{5/2}dx = \frac{2}{7}$$

c)

$$I = \int_0^1 x^{1/2} dx = \frac{2}{3}$$

d)

$$I = \int_0^1 \log(x) dx = -1$$

e)

$$I = \int_0^1 \log(|x - 0.7|) dx \,?\, 0.3\log(0.3) + 0.7\log(0.7) - 1.0$$

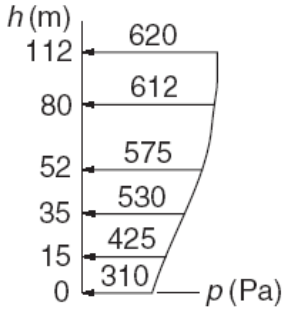
f)

$$I = \int_0^{12\pi} \mathrm{e}^{-x} \sin(50x) dx = \frac{50}{2501}(1 - e^{-2\pi})$$

g)

$$I = \int_{-9}^{10000} \frac{dx}{\sqrt{\lceil x \rceil}} = 206$$

PROBLEM 34



Wind pressure in a flat wall is measures as in the graphic. It is desired to find pressure effect center of the wall. It is calculated by using

$$\bar{h} = \frac{\int_0^{112} hp(h)dh}{\int_0^{112} p(h)dh}$$

Calculate the value.

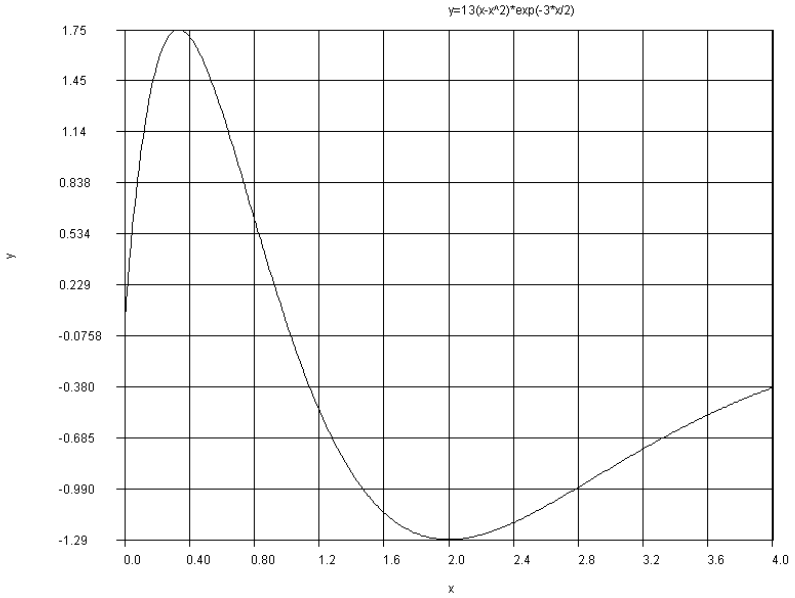
PROBLEM 35

For the analytically solved integration that result is given below:

Solve with with

a) Adaptive Simpson 1/3 method
b) Adaptive Gauss-Kronrod integral formula.

$$\int_0^4 13(x - x^2)e^{-3x/2} dx = \frac{4108e^{-6} - 52}{27} = -1.5487883725279481333$$



PROBLEM 36

v(m/s)	0	1	1.8	2.5	3.5	4.4	5.1	6
P(kW)	0	4.7	12.2	19.0	31.8	40.1	43.8	43.2

Power transmitted to the wheels of a vehicle is given in the table as a function of velocty. If the mass of the ar is m=2000 kg, Calculate the required time to accelarate the car from
0 m/s^2 to 6 m/s^2
Note :

$$\Delta t = m \int_{t=0\ s}^{6\ s} \frac{v}{P} dv$$

PROBLEM 37

$$I = \int_0^1 \int_1^2 \int_0^y e^{x+y+z} dz dy dx$$

Calculate the integral by using 3 dimensional Gauss-Legendre integration

PROBLEM 38

$$I = \int_0^{\pi} \int_1^x \int_0^{yx} \frac{1}{y} sin\left(\frac{z}{y}\right) dz dy dx$$

Calculate the integral by using 3 dimensional Gauss-Legendre integration

PROBLEM 39

Blackbody radiation function F(0- λT), λ(μm) T(degree Kelvin) can be calculated as

$$x = \frac{C_2}{\lambda T} = \frac{14387.69\ \mu mK}{\lambda T}$$

$$F(0 - \lambda T) = 1 - \frac{15}{\pi^4} \int_0^x \frac{x^3}{e^x - 1} dx$$

It can also be calculated from the series solution

$$F(0-\lambda T)=\frac{15}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{e^{-nx}}{n}\left(x^3+\frac{3x^2}{n}+\frac{6x}{n^2}+\frac{6}{n^3}\right)\right]$$

By creating a program that calculates the result with using series solution and integration. Compare the results. Resulting tables are given as a reference

λ T μ mK	F(0-λT)	λ T	F(0-λT)	λ T μ mK	F(0-λT)	λ T μ mK	F(0-λT)	λ T μ mK	F(0-λT)	λ T μ mK	F(0-λT)
1000	0.00032	2900	0.25056	4800	0.60754	6900	0.8022	10700	0.92709	18000	0.98081
1050	0.00056	2950	0.26191	4850	0.61428	7000	0.80808	10800	0.92872	18200	0.98137
1100	0.00091	3000	0.27323	4900	0.62089	7100	0.81374	10900	0.93031	18400	0.98191
1150	0.00142	3050	0.28453	4950	0.62737	7200	0.81918	11000	0.93185	18600	0.98243
1200	0.00213	3100	0.29578	5000	0.63373	7300	0.82443	11200	0.9348	18800	0.98293
1250	0.00308	3150	0.30697	5050	0.63996	7400	0.82949	11400	0.93758	19000	0.98341
1300	0.00432	3200	0.3181	5100	0.64608	7500	0.83437	11600	0.94021	19200	0.98387
1350	0.00587	3250	0.32915	5150	0.65207	7600	0.83907	11800	0.9427	19400	0.98431
1400	0.00779	3300	0.34011	5200	0.65795	7700	0.8436	12000	0.94505	19600	0.98474
1450	0.01011	3350	0.35097	5250	0.66371	7800	0.84797	12200	0.94728	19800	0.98516
1500	0.01285	3400	0.36173	5300	0.66937	7900	0.85219	12400	0.94939	20000	0.98555
1550	0.01605	3450	0.37238	5350	0.67491	8000	0.85625	12600	0.95139	20500	0.98649
1600	0.01972	3500	0.38291	5400	0.68034	8100	0.86018	12800	0.95329	21000	0.98735
1650	0.02388	3550	0.39332	5450	0.68566	8200	0.86397	13000	0.95509	21500	0.98814
1700	0.02854	3600	0.4036	5500	0.69089	8300	0.86763	13200	0.95681	22000	0.98886
1750	0.03369	3650	0.41375	5550	0.696	8400	0.87116	13400	0.95843	22500	0.98952
1800	0.03934	3700	0.42377	5600	0.70102	8500	0.87457	13600	0.95998	23000	0.99014
1850	0.04549	3750	0.43364	5650	0.70594	8600	0.87787	13800	0.96145	23500	0.9907
1900	0.05211	3800	0.44338	5700	0.71077	8700	0.88105	14000	0.96285	24000	0.99123
1950	0.0592	3850	0.45297	5750	0.7155	8800	0.88413	14200	0.96419	24500	0.99172
2000	0.06673	3900	0.46241	5800	0.72013	8900	0.88711	14400	0.96546	25000	0.99217
2050	0.07469	3950	0.47172	5850	0.72468	9000	0.88999	14600	0.96667	26000	0.99297
2100	0.08306	4000	0.48087	5900	0.72914	9100	0.89278	14800	0.96783	27000	0.99368
2150	0.0918	4050	0.48987	5950	0.73351	9200	0.89547	15000	0.96893	28000	0.99429
2200	0.10089	4100	0.49873	6000	0.73779	9300	0.89808	15200	0.96999	29000	0.99482
2250	0.11031	4150	0.50744	6050	0.74199	9400	0.9006	15400	0.971	30000	0.99529
2300	0.12003	4200	0.516	6100	0.74611	9500	0.90305	15600	0.97196	32000	0.99607
2350	0.13002	4250	0.52442	6150	0.75015	9600	0.90541	15800	0.97289	34000	0.99669
2400	0.14026	4300	0.53269	6200	0.75411	9700	0.9077	16000	0.97377	36000	0.99719
2450	0.15071	4350	0.54081	6250	0.758	9800	0.90992	16200	0.97461	38000	0.99759
2500	0.16136	4400	0.54878	6300	0.76181	9900	0.91207	16400	0.97542	40000	0.99792
2550	0.17217	4450	0.55662	6350	0.76554	10000	0.91416	16600	0.9762	45000	0.99851
2600	0.18312	4500	0.56431	6400	0.76921	10100	0.91618	16800	0.97694	50000	0.9989
2650	0.19419	4550	0.57186	6450	0.7728	10200	0.91814	17000	0.97765	55000	0.99917
2700	0.20536	4600	0.57927	6500	0.77632	10300	0.92004	17200	0.97834	60000	0.99935
2750	0.2166	4650	0.58654	6600	0.78317	10400	0.92188	17400	0.97899		
2800	0.22789	4700	0.59367	6700	0.78976	10500	0.92367	17600	0.97962		
2850	0.23922	4750	0.60067	6800	0.7961	10600	0.9254	17800	0.98023		

PROBLEM 40

Integral of blackbody spectral radiation equation for all wavelengths will give

$$E_b=\left[\frac{C_1}{C_2^4}\int_{x=0}^{\infty}\frac{x^3}{e^x-1}dx\right]T^4=\sigma T^4$$

In this equation C_1=3.74177489x10^8 Wmm^4/m^2 and C_2=1.438769x10^4 mm K . σ is called Stefan_Boltzman constant. Calculate this integral by using Gauss-Leguerre integral formula. The value of the Stephan Boltzman constant is given σ =5.670x10^{-8} W/m^{2}K^4

PROBLEM 41 Bessel function $J_n(x)$ can be defined as:

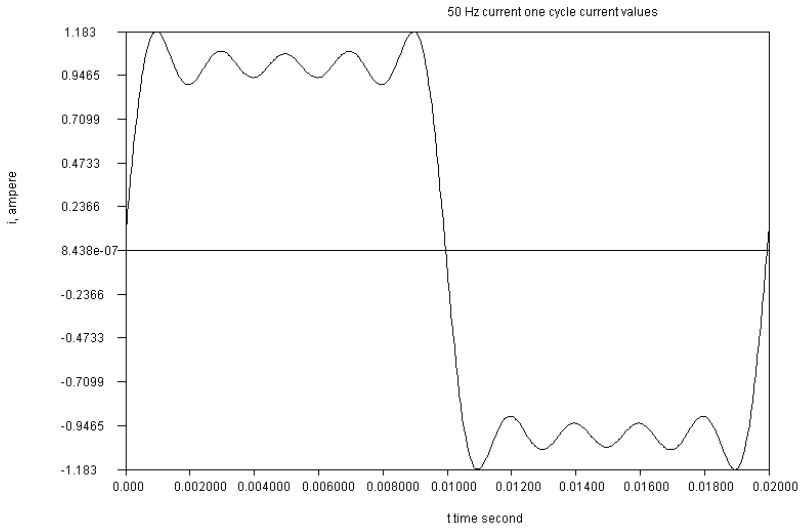
$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}cos(xsin(\theta)-n\theta)d\theta$$

Write a program to define such a function.

PROBLEM 42 In an AC-DC converter the signal defined with the following function is obtained. Signal frequency is 50 hertz (period T=1.0/50=0.02 second). A true root mean square current value for an alternative current signal is defined as:

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}$$

Calculate True RMS current for the given signal.



Note: In order to carry outintegral you should obtain $i(t)$ as a function y using data drom the plot above (digitizing and curve fitting)

PROBLEM 43 Work is defined as

Work =Force*Distance

In a real engineering problem force usually changes, Therefore Work for a variable force should be calculated as:

$$W = \int_{x_0}^{x_n} F(x)dx$$

In a system Force is measured as in the following table:

x(meter)	**0**	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**
F(Newton)	**0**	**9**	**12**	**15**	**18**	**23**	**16**	**13**	**11**

Calculate the work done by the system.

PROBLEM 44 Euler's constant is defined by the following equations:

$$\gamma = \int_{x=0}^{1} \left(\frac{1}{1+x^k} - e^{-x}\right)\frac{dx}{x} = \sum_{n=1}^{\infty}\left[\frac{1}{n} - ln\left(\frac{n+1}{n}\right)\right]$$

where $k > 0$

$$\gamma = 0.57721566490153286060651112 \ldots$$

Calculate **euler's** constant by integration and compare it with the series solution. You chose different integration methods to see the accuracy of the integration formula.

PROBLEM 45 Catalan's constant is defined by the following equations:

$$K = \int_{x=0}^{1} \left(\frac{\ln(x)}{1+x^2}\right)dx$$

Catalan's constant (K=0.915965594177) is defined by the following equations:

$$K = \frac{3}{64}\sum_{k=0}^{\infty}\frac{(-1)^k}{64^k}\left(\frac{32}{(12k+1)^2} - \frac{32}{(12k+2)^2} - \frac{32}{(12k+3)^2} - \frac{8}{(12k+5)^2} - \frac{16}{(12k+6)^2} - \frac{4}{(12k+7)^2} - \frac{4}{(12k+9)^2} - \frac{2}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right)$$

Calculate **Catalan's** constant by integration and compare it with the series solution. You chose different integration methods to see the accuracy

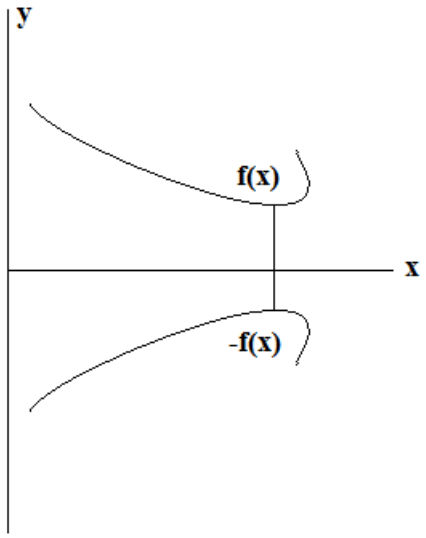
5 Optimisation

DEFINITION OF OPTIMIZATION AND GRAPHIC METHODS

Optimization is the name given to maximum or minimum finding process. When it is considered in mathematically, the root of the derivative of the function defines the optimum point.

$$\frac{df(x)}{dx} = 0$$

As the mathematical process, maximum and minimum is similar processes because
maximum (f(x)) = minimum(-f(x))

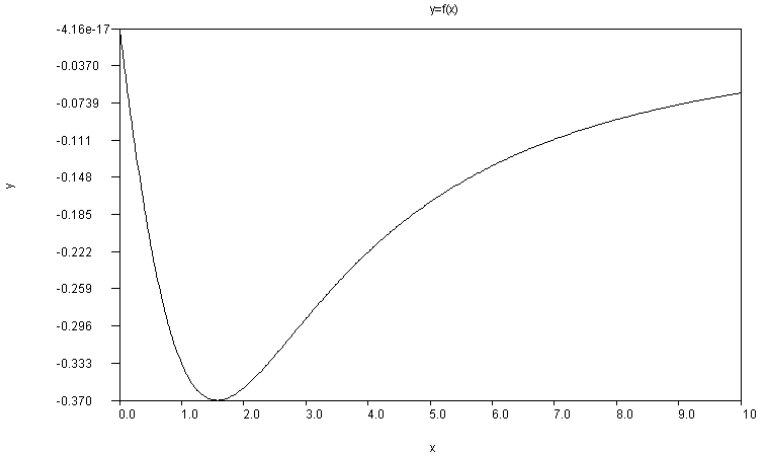


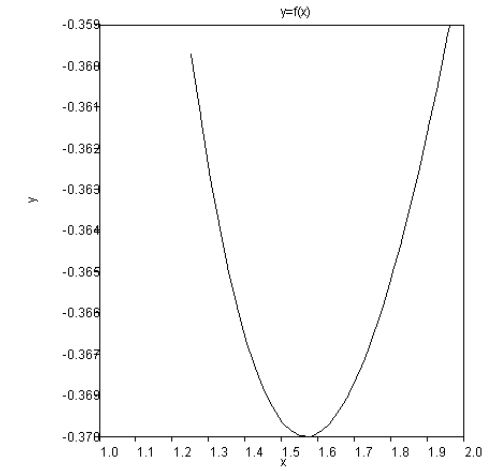
As a first example to one dimensional non-linear optimization, Plotting will be utilised.
Mathematically plotting means to subdivide the region into n equal parts and investigate the region of the minimum.

PROGRAM PlotT02 class as using plots in optimization example

```
import java.io.*;

class fa1 extends f_x
{
  double func (double c)
  {return -(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c));}
}

class PlotT02
{
    public static void main (String args[]) throws IOException
    {
    fa1 ff1=new fa1();
    Plot pp=new Plot(ff1,0.0,10.0,400,0);
    pp.plot();
    }
```

```
}
```





ove [1.5634218289085546 , -0.3696245733788395]

As it is seen from the plot windows, a guess of 1.5634 can be obtained for the optimum point.

ONE DIMENSIONAL NON-LINEAR FIBONNACHI SEARCH (GOLDEN SEARCH) METHOD

Series 0,1,1,2,3,5,8,13,21,34,… is called fibonnachi series. General definition is in the form of F(n)=F(n-1)+F(n-2) F(0)=0, F(1)=1

For the big fibonnaci terms the ration of two following fibonnachi number becomes constant. This constant is known as golden ratio,

$$R = \lim_{n\to\infty} \frac{F(n-1)}{F(n)} = \frac{\sqrt{5}-1}{2} = 0.618033989$$

If a minimum being searched in a given region (a,c). In order to determine if a minimum is existed, region should be divided up into three sub-regions. This requires avaluation of two intermediate data points. Neigbors of the minimum point remains and the third region is eliminated. In the new avaluation a new two points will be required. Golden ratio is given us a fine propery so that one of the selected points will always be one of the new search point. So that only one new point will be evaluated in each step of the evaluation.

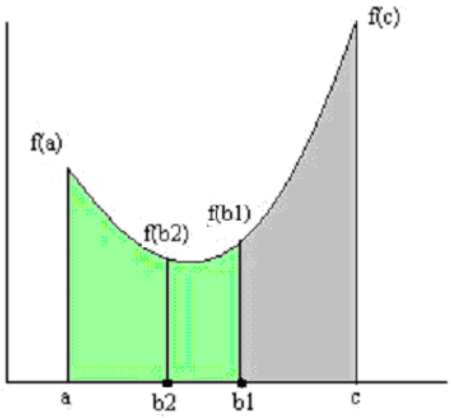


Figure 5.3-1 Golden ratio search

If minimumin in region [a,c] is required
The length of the region will be d0=(c – a) .
Then b1 is taken as b1=a+d0*R
And b2 is taken as b2 = c – d0*R = a + d0*R^2
And if it is assumed that minimum is in point b2, region (b1-c) will be thrown. Our new region will be d1=d0*R olacağindan
new b1 is taken as b1=a+d1*R=a+d0*R^2 which is the same as previous b2.
New b2 is taken n as b2 = c – d1*R = a + d1*R^2 = a + d0*R^3 .

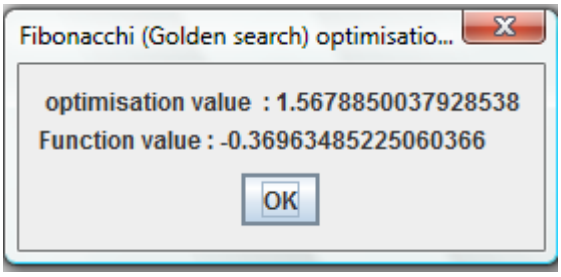
Since one of the point is the same, evaluation of only one new point is required for each iteration step. The Algorithm of Golden search is given in Program 5.3-1

PROGRAM Golden Ratio or Fibonnachi search, Java code

```
/ OPTIMIZATION Golden search or fibonnachi method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
class fb extends f_x
{
  public double func (double c)
  { return -(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c));}
}

public class NA31
{
public static double golden(f_x f,double a,double b,double epsilon)
{return golden(f,a,b,epsilon,1.0e-10,0);}
public static double golden(f_x f,double a,double b)
{return golden(f,a,b,1.0e-10,1.0e-10,0);}

public static double golden(f_x f,double a,double b,double epsilon,double delta,int print)
{
//  find the minimum of the function
// note  maximum f(x) = minimum (-f(x))
   double r1 = (Math.sqrt(5.0)-1.0)/2.0; // golden ratio
   double r2 = r1*r1;
   double h = b - a;
   double ya = f.func(a);
   double yb = f.func(b);
```

```
    double c = a + r2*h;
    double d = a + r1*h;
    double yc = f.func(c);
    double yd = f.func(d);
    int k = 1;
    double dp,dy,p,yp;
    while ((Math.abs(yb-ya)>epsilon) || (h>delta))
    {
      k++;
      if (yc<yd)
        {
        b = d;
        yb = yd;
        d = c;
        yd = yc;
        h = b - a;
        c = a + r2 * h;
        yc = f.func(c);
        }
      else
       {
         a = c;
         ya = yc;
         c = d;
         yc = yd;
         h = b - a;
         d = a + r1 * h;
         yd = f.func(d);
       }//end of if
    }//end of while
    dp = Math.abs(b-a);
    dy = Math.abs(yb-ya);
    p = a;
    yp = ya;
    if (yb<ya)
    {
      p = b;
      yp = yb;
    }
    if(print==1)
    {System.out.println("x min = "+p+"ymin = "+yp+"error of x ="+dp+"error of y"+dy); }
    return p;
}

    public static void main (String args[])
    {
            fb f=new fb();
    double a,b;
    a=Double.parseDouble(JOptionPane.showInputDialog(" enter lower limit of the function a : "));
    b=Double.parseDouble(JOptionPane.showInputDialog(" enter upper limit of the function b : "));
    double r;
    System.out.println("x=0.5"+f.func(0.5));
    System.out.println("x=1.5"+f.func(1.5));
    r= golden(f,a,b,1.0e-10,1.0e-5,0);
    JOptionPane.showMessageDialog(null," optimization value  : "+r+"\nFunction value : "+f.func(r),
    "Fibonacchi (Golden search) optimization : ",JOptionPane.PLAIN_MESSAGE);
```

```
    System.exit(0);
    }
}
```



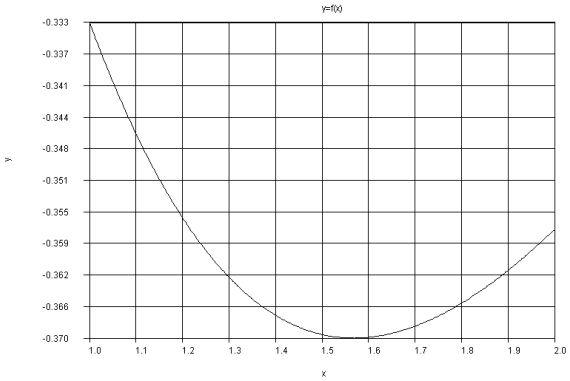
If search method is investigated step by step :

PROGRAM 5.2-1A Golden Ratio or Fibonnachi search by excel

Function f(c)=-(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c))

a	b	h	c	d	yc	yd
1	2	1	1.381966	1.618034	-0.366410476	-0.369427868
1.381966011	2	0.618034	1.618034	1.763932	-0.369427868	-0.366711114
1.381966011	1.763932	0.381966	1.527864	1.618034	-0.36949646	-0.369427868
1.381966011	1.618034	0.236068	1.472136	1.527864	-0.368819066	-0.36949646
1.472135955	1.618034	0.145898	1.527864	1.562306	-0.36949646	-0.369632208
1.527864045	1.618034	0.09017	1.562306	1.583592	-0.369632208	-0.369614176
1.527864045	1.583592	0.055728	1.54915	1.562306	-0.36960486	-0.369632208
1.549150281	1.583592	0.034442	1.562306	1.570437	-0.369632208	-0.369634304
1.562305899	1.583592	0.021286	1.570437	1.575462	-0.369634304	-0.369630023
1.562305899	1.575462	0.013156	1.567331	1.570437	-0.369634826	-0.369634304
1.562305899	1.570437	0.008131	1.565412	1.567331	-0.369634332	-0.369634826
1.565411519	1.570437	0.005025	1.567331	1.568517	-0.369634826	-0.369634819
1.565411519	1.568517	0.003106	1.566598	1.567331	-0.369634711	-0.369634826
1.56659776	1.568517	0.001919	1.567331	1.567784	-0.369634826	-0.369634851
1.567330897	1.568517	0.001186	1.567784	1.568064	-0.369634851	-0.36963485
1.567330897	1.568064	0.000733	1.567611	1.567784	-0.369634846	-0.369634851

Graphic exit of the function



Matlab (octave) version of the same program

PROGRAM Golden Ratio or Fibonnachi search by Octave(matlab)

```
function p=golden(f,a,b)
% golden search
r1=(sqrt(5)-1)/2;
r2=r1*r1;
h=b-a;
ya=f(a);
yb=f(b);
c=a+r2*h;
d=a+r1*h;
yc=f(c);
yd=f(d);
k=1;
```

```
epsilon=1.0e-6;
delta=1.0e-6;
while (abs(yb-ya) > epsilon & (h>delta))
   k++;
   if yc<yd
      b = d;
       yb = yd;
       d = c;
       yd = yc;
       h = b - a;
       c = a + r2 * h;
       yc = f(c);
    else
        a = c;
        ya = yc;
        c = d;
        yc = yd;
        h = b - a;
        d = a + r1 * h;
        yd = f(d);
    end
end
p=a;
yp=ya;
if yb<ya
   p=b;
   yp=yb;
end
endfunction
```

```
>> y=@(c)-(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c));
>> golden(y,1,4)
ans =  1.56541151876420
>>
```

Pyhthon version of the same program:

PROGRAM Golden Ratio or Fibonnachi search by Python language

```
 # Golden search optimisation method
# python version

from math import *
from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
     y=x*x-2*x+5;
     return y;

def golden(f,a,b):
  # golden search
  r1=(sqrt(5)-1)/2;
  r2=r1*r1;
  h=b-a;
  ya=f.func(a);
  yb=f.func(b);
  c=a+r2*h;
  d=a+r1*h;
  yc=f.func(c);
  yd=f.func(d);
  k=1;
  epsilon=1.0e-10;
  delta=1.0e-6;
  while abs(yb-ya) > epsilon:
    k=k+1;
    if yc<yd:
      b = d;
```

```
      yb = yd;
      d = c;
      yd = yc;
      h = b - a;
      c = a + r2 * h;
      yc = f.func(c);
    else:
      a = c;
      ya = yc;
      c = d;
      yc = yd;
      h = b - a;
      d = a + r1 * h;
      yd = f1(d);
  p=a;
  yp=ya;
  if yb<ya:
     p=b;
     yp=yb;
  return p;

a=float(input("enter lower limit : "));
b=float(input("enter upper limit : "));
f=f1();
r=golden(f,a,b)
print("golden search optimisation  : ",r)
```

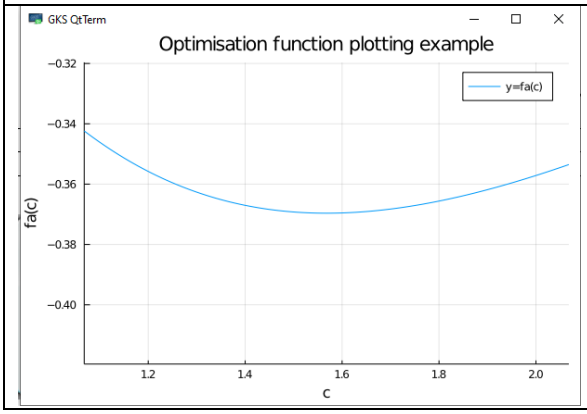
```
---------- Capture Output ----------
> "D:\co\python\pythonw.exe" golden.py
enter lower limit : 0
enter upper limit : 1
golden search optimisation  :  1.0

> Terminated with exit code 0.
```

PROGRAM : Golden Ratio or Fibonnachi search, Julia code

```
 # OPTIMIZATION Golden search or fibonnachi method
function golden(f,a,b,epsilon,delta,print)

#  find the minimum of the function
# note  maximum f(x) = minimum (-f(x))
    r1 = (sqrt(5.0)-1.0)/2.0 # golden ratio
    r2 = r1*r1
    h = b - a
    ya = f(a)
    yb = f(b)
    c = a + r2*h
    d = a + r1*h
    yc = f(c)
    yd = f(d)
    k = 1
    dp=0.0
    dy=0.0
    p=0.0
    yp=0.0
    while abs(yb-ya)>epsilon || h>delta
      k+=1
      if yc<yd
        b = d
        yb = yd
        d = c
        yd = yc
        h = b - a
```

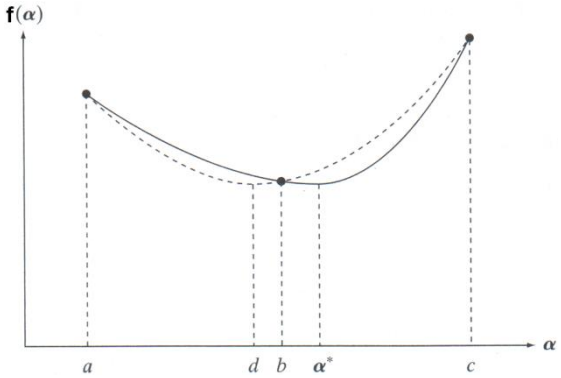
```
        c = a + r2 * h
        yc = f(c)
      else
        a = c
        ya = yc
        c = d
        yc = yd
        h = b - a
        d = a + r1 * h
        yd = f(d)
      end #end of if
    end #end of while
    dp = abs(b-a)
    dy = abs(yb-ya)
    p = a
    yp = ya
    if  yb<ya
      p = b
      yp = yb
    end # if
    if print==1
    print("x min = $p ymin = $yp error of x = $dp error of y $dy")
    end # if
    return p
end # function

using Plots
a=0.5
b=1.7
fa(c)=-2c/(4.0+0.8c+c^2+0.2*c^3)
r= golden(fa,a,b,1.0e-10,1.0e-5,0)
fr=fa(r)
print("optimisation value = $r function value = $fr")
x1=r-0.5:0.01:r+0.5
plot(fa,x1,title="Optimisation function plotting example",label="y=fa(c)",xlims=(r-0.5,r+0.5),ylims = (fr-
0.05,fr+0.05),xaxis = "c",yaxis = "fa(c)")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_fibonnachi.jl
optimisation value = 1.5678909109047903 function value = -0.3696348522520682
> Terminated with exit code 0.
```



ONE DIMENSIONAL NON-LINEAR QUADRATIC POLYNOMIAL METHOD

Golden search is a linear search method, it will always Works if a minimum(Maximum) is existed in the given region. But convergence speed is relatively slow. A quadratic search should theoretiacally be given values closer to the actual optimum value.



Minimum value of a quadratic equation is given with the Formula

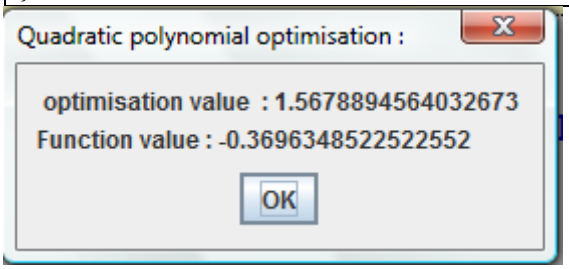
$$d = b + 0.5\frac{(c-b)^2[f(a)-f(b)]-(a-b)^2[f(c)-f(a)]}{(c-b)[f(a)-f(b)]-(a-b)[f(c)-f(a)]}$$

After finding point d, point b is replaced with this point and iteration continues.

Program Quadratic polynomial search, java code

```
// OPTIMIZATION
//  2nd degree polynomial (Quadratic polynomial) optimization
//  of function f(x)
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fb extends f_x
{
   public double func (double c)
   { return -(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c));}
}

public class NA32
{
public static double polinom2derece(f_x f,double x0,double x2)
{double x1=(x0+x2)/2.0;
 return polinom2derece(f,x0,x1,x2);
}
public static double polinom2derece(f_x f,double x0,double x1,double x2)
{
double epsilon=1.0e-10;
double delta=1.0e-5;
int print=0;
return polinom2derece(f,x0,x1,x2,epsilon,delta,print);
}
public static double polinom2derece(f_x f,double x0,double x1,double x2,double epsilon,double delta,int print)
{
int maxit=100;
double f0 = f.func(x0);
    double f1 = f.func(x1);
    double f2 = f.func(x2);
    double f3 = f2;
    double x3 = 0;
```

```
    double h = x1 - x0;
    double k=1;
    double dd=Math.abs(f1-f0);
    while ((dd >epsilon) || (h>delta))
    {
      k++;
      x3=(f0*(x1*x1-x2*x2)+f1*(x2*x2-x0*x0)+f2*(x0*x0-x1*x1))/
            (2*f0*(x1-x2)+2.0*f1*(x2-x0)+2.0*f2*(x0-x1));
      f3=f.func(x3);
      if(x3 >= x0 && x3<x1)
        {x2=x1;f2=f1;x1=x3;f1=f3;}
      else if(x3 >= x1 && x3 <x2)
        {x0=x1;f0=f1;x1=x3;f1=f3; }
      else if(x3 >x2)
        {x0=x1;f0=f1;x1=x2;f1=f2;x2=x3;f2=f3; }
      else if(x3 < x0)
        {x0=x3;f0=f3;x1=x0;f1=f0;x2=x1;f2=f1; }
      dd=Math.abs(f1-f0);
      h=Math.abs(x1-x0);
      if(k>maxit) break;
    }//end of while
    if(print==1)
    {System.out.println("x  = "+x3+"f = "+f3+"hata x ="+h+"hata f(x)=y "+dd); }
    return x3;
}

     public static void main (String args[])
     {
               fb f=new fb();
     double x0,x1,x2;
     x0=Double.parseDouble(JOptionPane.showInputDialog(" enter lower limit of the function  x0 : "));
     x1=Double.parseDouble(JOptionPane.showInputDialog(" enter middle value of the function x1 : "));
     x2=Double.parseDouble(JOptionPane.showInputDialog(" enter upper limit of the function  x2 : "));
     double r;
     r= polinom2derece(f,x0,x1,x2,1.0e-10,1.0e-5,1);
    JOptionPane.showMessageDialog(null," optimization value  : "+r+"\nFunction value : "+f.func(r),
     "Quadratic polynomial optimization : ",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);
     }
}
```



Quadratic polynomial search, Julia code

```
 # OPTIMIZATION quadratic polynomial
function  OPT_quadratic_poly(f,x0,x1,x2,epsilon,delta,print1)

maxit=100
    global f0 = f(x0)
    global f1 = f(x1)
    global f2 = f(x2)
    global f3 = f2
    global x3 = 0.0
```

```
    h = x1 - x0
    k=1
    dd=abs(f1-f0)
    while dd>epsilon || h>delta
      k+=1
      x3=(f0*(x1*x1-x2*x2)+f1*(x2*x2-x0*x0)+f2*(x0*x0-x1*x1))/(2*f0*(x1-x2)+2.0*f1*(x2-x0)+2.0*f2*(x0-x1))
      f3=f(x3)
      println("x3 = $x3 f3 = $f3")
      if x3 >=x0 && x3<x1
        x2=x1
        f2=f1
        x1=x3
        f1=f3
      elseif x3>=x1 && x3<x2
        x0=x1
        f0=f1
        x1=x3
        f1=f3
      elseif x3>x2
        x0=x1
        f0=f1
        x1=x2
        f1=f2
        x2=x3
        f2=f3
      elseif x3<x0
        x0=x3
        f0=f3
        x1=x0
        f1=f0
        x2=x1
        f2=f1
      end #if
      dd=abs(f1-f0)
      h=abs(x1-x0)
      if(k>maxit)
        break
      end # if
    end  #end of while
    if(print1==1)
      println("x  = $x3 f = $f3 ")
    end
    return x3
end # function

using Plots
x0=1.0
x1=1.5
x2=2.0
epsilon=1.e-10
delta=1.0e-10
print1=0
fa(c)=-2.0c/(4.0+0.8c+c^2+0.2*c^3)
r= OPT_quadratic_poly(fa,x0,x1,x2,epsilon,delta,print1)
fr=fa(r)
print("optimisation value = $r function value = $fr")
x1=r-0.5:0.01:r+0.5
plot(fa,x1,title="Optimisation function plotting example",label="y=fa(c)",xlims=(r-0.5,r+0.5),ylims = (fr-
0.05,fr+0.05),xaxis = "c",yaxis = "fa(c)")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" OPT_quadratic_poly.jl
x3 = 1.624045801526718 f3 = -0.3693761003047835
```

```
x3 = 1.5707114578848842 f3 = -0.3696341798401909
x3 = 1.5689145983133213 f3 = -0.36963476342950463
x3 = 1.5679570580017286 f3 = -0.36963485186545303
x3 = 1.5679088123360234 f3 = -0.36963485222046666
x3 = 1.5678909680728423 f3 = -0.3696348522520535
x3 = 1.5678897945555532 f3 = -0.36963485225224363
x3 = 1.5678894677852748 f3 = -0.3696348522522551
x3 = 1.5678894305561077 f3 = -0.36963485225225534
x3 = 1.5678894698582673 f3 = -0.36963485225225506
x3 = 2.0 f3 = -0.35714285714285715
x3 = 1.5678906524892626 f3 = -0.3696348522521276
x3 = 1.5678892563835765 f3 = -0.36963485225225295
optimisation value = 1.5678892563835765 function value = -0.36963485225225295
> Terminated with exit code 0.
```

Program Quadratic polynomial search, Ocatave-matlab code

```
## Author: M. Turhan ÇOBAN
## Created: 2021-06-05

function retval = quaratic_poly_opt (f,x0,x1,x2,epsilon,delta)
  maxit=100;
  f0 = f(x0);
  f1 = f(x1);
  f2 = f(x2);
  f3 = f2;
  x3 = 0.0;
  h = x1 - x0;
  k=1;
  dd=abs(f1-f0);
    while dd >epsilon || h>delta
      k=k+1;
      x3=(f0*(x1*x1-x2*x2)+f1*(x2*x2-x0*x0)+f2*(x0*x0-x1*x1))/(2*f0*(x1-x2)+2.0*f1*(x2-x0)+2.0*f2*(x0-x1));
      f3=f(x3);
      if x3 >= x0 && x3<x1
        x2=x1;f2=f1;x1=x3;f1=f3;
      elseif x3 >= x1 && x3 <x2
        x0=x1;f0=f1;x1=x3;f1=f3;
      elseif x3 >x2
        x0=x1;f0=f1;x1=x2;f1=f2;x2=x3;f2=f3;
      elseif x3 < x0
        x0=x3;f0=f3;x1=x0;f1=f0;x2=x1;f2=f1;
      end
      dd=abs(f1-f0);
      h=abs(x1-x0);
      if k>maxit
        break;
      end
    end
retval=x3;
endfunction
```

```
>> f=@(c)-(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c));
>> x0=0.0;
>> x1=1.0;
>> x2=3.0;
>> x=quaratic_poly_opt (f,x0,x1,x2,1e-10,1e-5)
x = 1.567889417006928
```

Program Quadratic polynomial search, Python code

```
from math import *
from if_x import *

def opt_quadratic_poly(f,x0,x1,x2):
    maxit=100;
    epsilon=1.0e-10;
    delta=1.0e-5;
    f0 = f.func(x0)
    f1 = f.func(x1)
    f2 = f.func(x2)
```

```
    f3 = f2;
    x3 = 0;
    h = x1 - x0
    k=1
    dd=abs(f1-f0)
    while dd >epsilon or h>delta:
      k=k+1
      x3=(f0*(x1*x1-x2*x2)+f1*(x2*x2-x0*x0)+f2*(x0*x0-x1*x1))/(2*f0*(x1-x2)+2.0*f1*(x2-x0)+2.0*f2*(x0-x1))
      f3=f.func(x3);
      if x3 >= x0 and x3<x1:
          x2=x1
          f2=f1
          x1=x3
          f1=f3;
      elif x3 >= x1 and x3 <x2:
          x0=x1
          f0=f1
          x1=x3
          f1=f3
      elif x3 >x2 :
          x0=x1
          f0=f1
          x1=x2
          f1=f2
          x2=x3
          f2=f3
      elif x3 < x0 :
         x0=x3
         f0=f3
         x1=x0
         f1=f0
         x2=x1
         f2=f1
      dd=abs(f1-f0)
      h=abs(x1-x0)
      if k>maxit :
         break
    return x3;

class f1(if_x):func=lambda self,x:x*x-x+5
f=f1()
a=float(input("enter lower limit : "))
b=float(input("enter upper limit : "))
c=(a+b)/2.0;
r=opt_quadratic_poly(f,a,c,b);
print(" optimum point  : ",r,"\nFUNCTION VALUE : ",f.func(r))
```

```
runfile('D:/okul/SCO1/opt_quadratic_poly.py', wdir='D:/okul/SCO1')
Reloaded modules: if_x

enter lower limit : 0.0

enter upper limit : 2.0
 optimum point  :  0.5
FUNCTION VALUE :  4.75
```

ONE DIMENSIONAL NON-LINEAR CUBIC POLYNOMIAL METHOD

If it is assumed that our polynomial is a cubic polynomial, a method similar to the quadratic polynomial method can be obtained. Hermite cubic polinomial is used for this purpose.

$P(x) = a(x-x_0)^3 + b(x-x_0)^2 + c(x-x_0) + d$
optimum(minimum , maximum or extremum) point of this polynomial in the $[x_0,x_1]$ region wil be calculated from formula

$$x - x_0 = \frac{-b \mp \sqrt{b^2 - 4ac}}{3a}$$

$$x^* = \mp \frac{-b + \sqrt{b^2 - 4ac}}{3a}$$

a b c d coefficients of this equation is unknown. This coefficients can be calculated from:

$p(x_0)=d=f(x_0)$ $p'(x_0)=c=f'(x_0)$

$a=(G-2H)/h$ $h=x_1-x_0$ $b=3H-G$

$F=[f(x_1)-f(x_0)]/h$

$G==[f'(x_1)-f'(x_0)]/h$

$H=[F-f'(x_0)]/h$

This type of formulation is not well suited for hand calculation, but is often highly efficient on the computer.

PROGRAM Cubic polynomial search, java code

```
//  ======================= //
//  Cubic polynomial optimization    //
//  ======================= //
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//============= Tanimi gereken fonksiyon ================

class fb extends f_x
{
   public double func (double x)
   { return x+25.0/(x*x);}
}
//=========================================================
public class NA33
{
public static double df(f_x f,double x,double h)
{ //numerical derivative
  double hh=1.0/h;
  return (-f.func(x+2.0*h)+8.0*f.func(x+h)-8.0*f.func(x-h)+f.func(x-2.0*h))/12.0*hh;
}
public static double cubic_search_opt(f_x f,double x0,double x1)
{
int nmax=500;
double tolerance=1.0e-15;
double f0,f1,df0,df1,fxs,dfxs;
double h=0.00001;
double xs;
int k=0;
double X,F,G,H,a,b,c;
  f0=f.func(x0);
  f1=f.func(x1);
  df0=df(f,x0,h);
  df1=df(f,x1,h);
while(Math.abs(x0-x1)>tolerance && k<nmax)
{ k++;
  X=1.0/(x1-x0);
  F=X*(f1-f0);
  G=X*(df1-df0);
  H=X*(F-df0);
  a=X*(G-2.0*H);
  b=3.0*H-G;
  c=df0;
  xs=x0+(Math.sqrt(b*b-3.0*a*c)-b)/(3.0*a);
  fxs=f.func(xs);
  dfxs=df(f,xs,h);
  if(dfxs > 0 ) {x1=xs;f1=fxs;df1=dfxs;}
  else         x0=xs;f0=fxs;df0=dfxs;
}
if(k>=nmax)  JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
```

```
   " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
 return (x0+x1)/2.0;
 }
     public static double[] inputdata(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }

     public static void main (String args[])
     {
             fb f=new fb();
     double x[]=inputdata("enter search area limits : xo and x1");
     double r= cubic_search_opt(f,x[0],x[1]);
     JOptionPane.showMessageDialog(null," minimum value  : "+r+"\nFunction value : "+f.func(r),
     "minimisation by using cubic search method : ",JOptionPane.PLAIN_MESSAGE);
     System.exit(0);
     }
 }
```

One dimensional cubic polynomial, octave-matlab code

```
## Author: M. Turhan ÇOBAN
## Created: 2021-06-05

function   retval=cubic_poly_opt(f,x0,x1)
  nmax=500;
  tolerance=1.0e-15;
  h=0.00001;
  k=0;
  f0=f(x0);
  f1=f(x1);
  df0=df(f,x0,h);
  df1=df(f,x1,h);
  while abs(x0-x1)>tolerance && k<nmax
    k=k+1;
    X=1.0/(x1-x0);
    F=X*(f1-f0);
    G=X*(df1-df0);
    H=X*(F-df0);
    a=X*(G-2.0*H);
    b=3.0*H-G;
    c=df0;
    xs=x0+(sqrt(b*b-3.0*a*c)-b)/(3.0*a);
    fxs=f(xs);
    dfxs=df(f,xs,h);
  if dfxs > 0
    x1=xs;f1=fxs;
    df1=dfxs;
  else
    x0=xs;f0=fxs;
    df0=dfxs;
  end
  endwhile
  retval=(x0+x1)/2.0;
endfunction
```

```
>> f=@(c)-(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c));
>> x=cubic_poly_opt(f,x0,x2)
```

```
x = 1.567889423366866
```

BISECTION METHOD

Bisection method is an important root finding method. By definition of optimisation which optimum point is the root of the derivative of the function, optimum van be obtained by finding root of derivative function

Bisection optimisation, Julia code

```
#first derivative
function df(f,x)
 dx=0.001
 c=[1.0/280.0 -4.0/105.0  1.0/5.0  -4.0/5.0  0  4.0/5.0  -1.0/5.0  4.0/105.0  -1.0/280.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end

function opt_bisection(f,a,b)
    b1=1.1*b
    r=(a+b)/2.0
    fr=f(r)
    fa=df(f,a)
    eps=1.0e-6
    nmax=100
    i=1
    while abs(fr)>eps && i<nmax
        if fa*fr<0
            b=r
        else
            a=r
            fa=fr
        end
        r=(a+b)/2.0
        fr=df(f,r)
        i=i+1
    end
    if i>=nmax
       r=bisection(f,a,b1)
    end
    return r
end


function enlarge(f,x0,dx)
    # enlarge region untill a root is existed
    x1=x0
    x2=x1+dx
    NTRY=200
    FACTOR=1.001
    j=0
    if x1 == x2
      print("input variables are wrong")
    end
    f1=f(x1)
    f2=f(x2)
    for j=1:NTRY+1
        if f1*f2 < 0.0
            break
        else
            x2=x2+dx
            f2=f(x2)
        end
    end
    return x2
end
a=0.0
b=2.0
fa(c)=-2.0c/(4.0+0.8c+c^2+0.2*c^3)
r= opt_bisection(fa,a,b)
```

```
fr=fa(r)
print("optimum value : $r f($r) = $fr")
fr=fa(r)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_bisection.jl
optimum value : 1.5678863525390625 f(1.5678863525390625) = -0.3696348522514579
> Terminated with exit code 0.
```

ONE DIMENSIONAL NON-LINEAR NEWTON-RAPHSON METHOD

Newton- Raphson is investigated as one of the root finding methods. Due to the fact that roots of the derivatives of a function is also given the optimum point makes Newton method a cendidate to find optimum through the roots of the derivative of the function.

Minimum of f(x)

Roots of $g(x) = \frac{df(x)}{dx} = 0$

Newton raphson formula

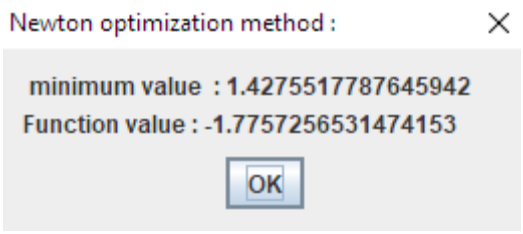
$$x_{n+1} = x_n - \frac{g(x)}{\frac{dg(x)}{dx}} = x_n - \frac{\frac{df(x)}{dx}}{\frac{d^2f(x)}{dx^2}}$$

Program Newton-Raphson method, java code

```
/// OPTIMIZATION
//  Newton-Raphson method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fc
{
  public double func(double x)
  { return -(2.0*Math.sin(x)-x*x/10.0);}
}

class fb extends f_x
{
  public double func(double x)
  { return -(2.0*Math.cos(x)-x/5.0);}
}
public class NA34
{
public static double derivative(f_x df,double x)
{
// derivative sub-program
// calculates derivative of function df
double h0=0.256;
int i,m;
int n=7;
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
 h[i]=0;
 for(int j=0;j<n;j++)
```

```
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
T[i][0]=( df.func(x + h[i]) - df.func( x - h[i]))/(2.0*h[i]);
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/
         (h[i]*h[i]- h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}

public static double newton(f_x df,double x)
{
// root finding newton-raphson program

int nmax=500;
double tolerance=1.0e-15;
double fx,dfx;
for(int i=0;i<nmax;i++)
{
fx=df.func(x);
dfx=derivative(df,x);
x-=fx/dfx;
if(Math.abs(fx)<tolerance)
{
return x;
}
}
return x;
}

public static double newton_2nd_derivative(f_x df,double x)
{
// Newton-Raphson with second derivative
int nmax=500;
double dx=1e-3;
double x1m;
double tolerance=1.0e-15;
double fx,fxm,dfx,dfxm,d2fx;
for(int i=0;i<nmax;i++)
{
fx=df.func(x);
fxm=df.func(x-dx);
dfx=derivative(df,x);
dfxm=derivative(df,(x-dx));
d2fx=-6.0/dx/dx*(fx-fxm)+2.0/dx*(2.0*dfx+dfxm);
x-=(fx/dfx+.5*fx*fx/(dfx*dfx*dfx)*d2fx);
if(Math.abs(fx)<tolerance)
```

```
{ return x; }
}
return x;
}
    public static void main (String args[])
    {
            fc  f=new fc();
            fb df=new fb();
    double x0;
    x0=Double.parseDouble(JOptionPane.showInputDialog(" enter guess value  x0 : "));
    double r;
    r= newton(df,x0);
    JOptionPane.showMessageDialog(null," minimum value  : "+r+"\nFunction value : "+f.func(r),
    "Newton optimization method : ",JOptionPane.PLAIN_MESSAGE);
    r= newton_2nd_derivative(df,x0);
    JOptionPane.showMessageDialog(null," minimum value  : "+r+"\nFunction value : "+f.func(r),
    "Newton with second derivative optimization method : ",JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Program Newton-Raphson method (Python code)

```
from math import *
from f_x import *;

class f1(f_x):
  def func(self,x):
     y=-(2.0*sin(x)-x*x/10.0);
     return y;

def newton(f,x):
   # Newton-Raphson optimisation method
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
     fx= f.dfunc(x,1);
     dfx=f.dfunc(x,2);
     x=x-fx/dfx
     if abs(fx)<tolerance: return x
  return x

f=f1();
x=float(input("enter initial guess : "))
r=newton(f,x);
print(" ROOT  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

```
---------- Capture Output ----------
> "D:\co\anaconda\pythonw.exe" newton_opt.py
enter initial guess : 1
 ROOT  :  1.4275517787644865
FUNCTION VALUE :  -1.775725653147415

> Terminated with exit code 0.
```

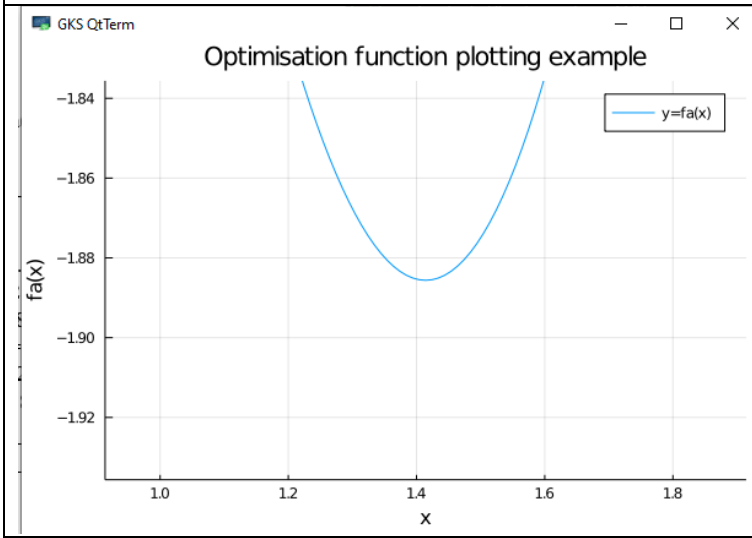
Program Newton-Raphson method, Julia code

```
 #first derivative
function df(f,x)
```

```
 dx=0.001
 c=[1.0/280.0 -4.0/105.0  1.0/5.0  -4.0/5.0  0  4.0/5.0  -1.0/5.0  4.0/105.0 -1.0/280.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end
#second derivative
function d2f(f,x)
 dx=0.001
 c=[-1.0/560.0  8.0/315.0 -1.0/5.0 8.0/5.0 -205.0/72.0 8.0/5.0 -1.0/5.0 8.0/315.0 -1.0/560.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx)
end
#third derivative
function d3f(f,x)
 dx=0.001
 c=[-7.0/240.0  3.0/10.0  -169.0/120.0  61.0/30.0 0.0  -61.0/30.0  169.0/120.0  -3.0/10.0
      7.0/240.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx*dx)
end

function newton(f,x)
nmax=100
tolerance=1.0e-10
for i=0:nmax
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x-=dfx/d2fx
  println("i = $i x = $x dfx = $dfx d2fx = $d2fx")
  if abs(dfx)<tolerance
    return x
  end #f
end #for
return x
end #function

using Plots
x0=1.001
epsilon=1.e-10
delta=1.0e-10
print1=0
fa(x)=1.0/3.0*x*x*x-2*x
x= newton(fa,x0)
fx=fa(x)
print("optimisation value = $x function value = $fx")
x1=x-0.5:0.01:x+0.5
plot(fa,x1,title="Optimisation function plotting example",label="y=fa(x)",xlims=(x-0.5,x+0.5),ylims = (fx-
0.05,fx+0.05),xaxis = "x",yaxis = "fa(x)")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_newton.jl
i = 0 x = 1.4995009989832748 dfx = -0.9979990000000115 d2fx = 2.0020000000712037
i = 1 x = 1.4166390181909343 dfx = 0.24850324595151324 d2fx = 2.9990019979643514
```

```
i = 2 x = 1.41421563870902 dfx = 0.006866107861130087 d2fx = 2.8332780368785055
i = 3 x = 1.4142135623746328 dfx = 5.8727691217008715e-6 d2fx = 2.8284312770122253
i = 4 x = 1.4142135623731422 dfx = 4.21624540836163e-12 d2fx = 2.8284271249763053
optimisation value = 1.4142135623731422 function value = -1.8856180831641267
> Terminated with exit code 0.
```



Newton-Raphson(secant) search by excel

Function: f(c)=-(2.0*c/(4.0+0.8*c+c*c+0.2*c*c*c))

secant							
$\Delta x =$	0.001					$\frac{df(x)}{dx}$	$\frac{d^2f(x)}{dx^2}$
x	$x+\Delta x$	$x-\Delta x$	$f(x)$	$f(x+\Delta x)$	$f(x+\Delta x)$		
1	1.001	0.999	-0.33333	-0.33348	-0.33319	-0.1444445	0.341481
1.422994	1.423994	1.421994	-0.36772	-0.36775	-0.36769	-0.0274624	0.210648
1.553364	1.554364	1.552364	-0.36962	-0.36962	-0.36961	-0.0024853	0.173085
1.567723	1.568723	1.566723	-0.36963	-0.36963	-0.36963	-2.82E-05	0.169166
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	-3.766E-09	0.16912
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	2.7756E-14	0.16912
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	0	0.16912
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	0	0.16912
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	0	0.16912
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	0	0.16912
1.56789	1.56889	1.56689	-0.36963	-0.36963	-0.36963	0	0.16912

5.6 ONE DIMENSIONAL NON-LINEAR BRENT METHOD

It is seen that quadratic formula can used to find with formula

$$x = b + 0.5\frac{(b-a)^2[f(b)-f(a)] - (b-c)^2[f(b)-f(a)]}{(b-a)[f(b)-f(a)] - (b-c)[f(b)-f(c)]}$$

But this Formula can be reached to maximum instead of minimum as well. One more secure approximation could be to use Golden search to approach to the minimum point and then utilise the quadratic polynomial method. This combined method is called Brent method.

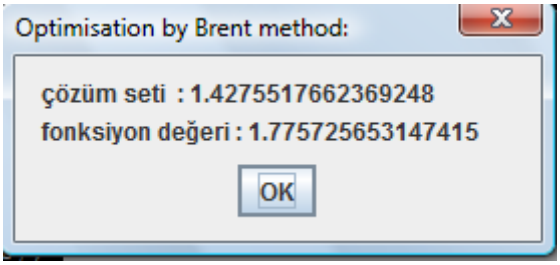
Program 5.6-1 Brent method, java code

```
/// OPTIMIZATION
//  one dimensional non-linear functions
//  Brent method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
```

```
import java.awt.event.*;
import javax.swing.*;

//============= Function to be defined ================
class fb extends f_x
{
  public double func (double x)
  { return -(2.0*Math.sin(x)-x*x/10.0);}
    //- is to obtain maximum value
}
//=========================================================

public class NA35
{
    public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}

public  static double brent(f_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}

public  static double[] brentf(f_x f,double ax,double bx,double cx,double tol)
{
// ax,bx,cx is three initial guesses (bx should be located between ax and cx)
// f : function (defined with f_x abstract class)
// tol tolerance
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //golden ratio
double ZEPS=1.0e-10;
double xmin;
double aa[]=new double[2];
            int iter;
            double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
            double e=0.0;
            d=0.0;
            a=((ax < cx) ? ax : cx);
            b=((ax > cx) ? ax : cx);
            x=w=v=bx;
            fw=fv=fx=f.func(x);
            for (iter=1;iter<=ITMAX;iter++) {
                        xm=0.5*(a+b);
                        tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                        if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                                    xmin=x;
```

```
                aa[0]=xmin;
                aa[1]=fx;
                return aa;
                }
                if (Math.abs(e) > tol1) {
                        r=(x-w)*(fx-fv);
                        q=(x-v)*(fx-fw);
                        p=(x-v)*q-(x-w)*r;
                        q=2.0*(q-r);
                        if (q > 0.0) p = -p;
                        q=Math.abs(q);
                        etemp=e;
                        e=d;
                        if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        else {
                                d=p/q;
                                u=x+d;
                                if (u-a < tol2 || b-u < tol2)
                                        d=SIGN(tol1,xm-x);
                        }
                } else {
                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                }
                u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                fu=f.func(u);
                if (fu <= fx) {
                        if (u >= x) a=x; else b=x;
                        {v=w;w=x;x=u;}
                        {fv=fw;fw=fx;fx=fu;}
                } else {
                        if (u < x) a=u; else b=u;
                        if (fu <= fw || w == x) {
                                v=w;
                                w=u;
                                fv=fw;
                                fw=fu;
                        } else if (fu <= fv || v == x || v == w) {
                                v=u;
                                fv=fu;
                        }
                }
        }
        System.out.println("maximum number of iterations are exceeded in BRENT optimization");
        xmin=x;
        aa[0]=xmin;
        aa[1]=fx;
        return aa;
}

    public static void main (String args[])
    {
             String s="enter three initial guess values with a space in between xo x1 x2: ";
    double [] x0=inputdata(s);
    fb f=new fb();
    double r1= brent(f,x0[0],x0[1],x0[2],1.0e-8);
    s=" çözüm seti  : "+r1+"\n fonksiyon değeri : "+(-f.func(r1));
    //fonksiyonun gerçek değerini elde etmek için -1 ile çarpilmiştir
    String s2="Optimization by Brent method:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    Plot pp=new Plot(f,0,2,100);
    pp.plot();
```

```
    }
}
```



A second version of Brent method can also consider derivatives of the function to determine movement of the function better is given as the next program.

Program dbrent method (brent method with derivative), java code

```
//==============================
//  OPTIMIZATION
//  One dimensional optimization
//  brent method with derivative
//==============================
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//============= Function to be defined ================
class fb extends f_x
{
  public double func (double x)
  { return -(2.0*Math.sin(x)-x*x/10.0);}
}
//============================================================

public class NA35A
{
            public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
   public static double MAX(double a,double b) {return (a > b ? a : b);}

   public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
    public static double df(f_x f,double x)
    {
// türev alt programi
// df fonksiyonunun türevini hesaplar
double h0=0.0256;
```

```
int i,m;
int n=7;
//turev of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
T[i][0]=( f.func(x + h[i]) - f.func( x - h[i]))/(2.0*h[i]);
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/
        (h[i]*h[i]- h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public  static double dbrent(f_x f,double ax,double bx,double cx,double tol)
{
// ax,bx,cx is three initial guesses (bx should be located between ax and cx)
// f : function (defined with f_x abstract class)
// tol tolerance
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altin oran
double ZEPS=1.0e-10;
            int iter;
            boolean ok1,ok2;
            double a,b,d,d1,d2,du,dv,dw,dx,e=0.0;
            d=0.0;
            double fu,fv,fw,fx,olde,tol1,tol2,u,u1,u2,v,w,x,xm;
            double xmin=bx;
            a=(ax < cx ? ax : cx);
            b=(ax > cx ? ax : cx);
            x=w=v=bx;
            fw=fv=fx=f.func(x);
            dw=dv=dx=df(f,x);
            for (iter=1;iter<=ITMAX;iter++) {
                        xm=0.5*(a+b);
                        tol1=tol*Math.abs(x)+ZEPS;
                        tol2=2.0*tol1;
                        if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                                    xmin=x;
                                    return xmin;
```

```
		}
		if (Math.abs(e) > tol1) {
			d1=2.0*(b-a);
			d2=d1;
			if (dw != dx)  d1=(w-x)*dx/(dx-dw);
			if (dv != dx)  d2=(v-x)*dx/(dx-dv);
			u1=x+d1;
			u2=x+d2;
			ok1 = (a-u1)*(u1-b) > 0.0 && dx*d1 <= 0.0;
			ok2 = (a-u2)*(u2-b) > 0.0 && dx*d2 <= 0.0;
			olde=e;
			e=d;
			if (ok1 || ok2) {
				if (ok1 && ok2)
					d=(Math.abs(d1) < Math.abs(d2) ? d1 : d2);
				else if (ok1)
					d=d1;
				else
					d=d2;
				if (Math.abs(d) <= Math.abs(0.5*olde)) {
					u=x+d;
					if (u-a < tol2 || b-u < tol2)
						d=SIGN(tol1,xm-x);
				} else {
					d=0.5*(e=(dx >= 0.0 ? a-x : b-x));
				}
			} else {
				d=0.5*(e=(dx >= 0.0 ? a-x : b-x));
			}
		} else {
			d=0.5*(e=(dx >= 0.0 ? a-x : b-x));
		}
		if (Math.abs(d) >= tol1) {
			u=x+d;
			fu=f.func(u);
		} else {
			u=x+SIGN(tol1,d);
			fu=f.func(u);
			if (fu > fx) {
				xmin=x;
				return xmin;
			}
		}
		du=df(f,u);
		if (fu <= fx) {
			if (u >= x) a=x; else b=x;
			//MOV3(v,fv,dv, w,fw,dw)
			{v=w;fv=fw;dv=dw;}
			//MOV3(w,fw,dw, x,fx,dx)
			{w=x;fw=fx;dw=dx;}
			//MOV3(x,fx,dx, u,fu,du)
			{x=u;fx=fu;dx=du;}
		} else {
			if (u < x) a=u; else b=u;
			if (fu <= fw || w == x) {
				//MOV3(v,fv,dv, w,fw,dw)
				{v=w;fv=fw;dv=dw;}
				//MOV3(w,fw,dw, u,fu,du)
				{w=u;fw=fu;dw=du;}
			} else if (fu < fv || v == x || v == w) {
				//MOV3(v,fv,dv, u,fu,du)
				{v=u;fv=fu;dv=du;}
```

```
                                }
                        }
                }
                System.out.println("maximum number of iterations are exceeded in DBRENT optimization");
                return xmin;
}
    public static void main (String args[])
    {
    String s="enter three initial guess values with a space in between xo x1 x2: ";
    double [] x0=inputdata(s);
    fb f=new fb();
    double r1= dbrent(f,x0[0],x0[1],x0[2],1.0e-8);
    s=" optimum value  : "+r1;
    String s2="Optimization by Brent method with derivatives:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    }
}
```

IMPROVING ONE DIMENSIONAL OPTIMIZATION BRACKET VALUES

Before starting the search for the optimal value search range can be improved, This also can ensure us to have an optimum point in the given range. Method mnbrak takes 3 initial gues points and enlarge the the given zone untill a minimum is existed. It uses golden search in this process.

Program mnbrak method (barketing the minimum value), java code

```
// OPTIMIZATION
//  one dimensional non-linear functions
//  mnbark bracket method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//============= Function to be defined ================
class fb extends f_x
{
   public double func (double x)
   { return -(2.0*Math.sin(x)-x*x/10.0);}
}
//============================================================

public class NA35B
{
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public  static double[] mnbrak(f_x f,double ax,double dx)
{
return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(f_x f,double ax)
{
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(f_x f,double ax,double bx,double cx)
{

int ITMAX=100;
double xmin;
```

```
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
        double ulim,u,r,q,fu,dum;
        double aa[]=new double[3];
        fa=f.func(ax);
        fb=f.func(bx);
        if (fb > fa) {
                //SHFT(dum,*ax,*bx,dum)
                {dum=ax;ax=bx;bx=dum;}
                //SHFT(dum,*fb,*fa,dum)
                {dum=fb;fb=fa;fa=dum;}
        }
        cx=(bx)+GOLD*(bx-ax);
        fc=f.func(cx);
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        while (fb > fc) {
                r=(bx-ax)*(fb-fc);
                q=(bx-cx)*(fb-fa);
                u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                        u=ulim;
                        fu=f.func(u);
                } else {
                        u=cx+GOLD*(cx-bx);
                        fu=f.func(u);
                }
                //SHFT(ax,bx,cx,u)
                {ax=bx;bx=cx;cx=u;}
                //SHFT(fa,fb,fc,fu)
                {fa=fb;fb=fc;fc=fu;}
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        }
                return aa;
```

```
}
//_____________________________________________________________
    public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
    public static void main (String args[])
    {
            String s="enter one or three initial guess value with a space in between: ";
    double [] x0=inputdata(s);
    fb f=new fb();
    double r1[];
    if(x0.length==3)      {r1= mnbrak(f,x0[0],x0[1],x0[2]);}
    else if(x0.length==1) {r1= mnbrak(f,x0[0]);}
    else{System.out.println("Wrong number of first initial guesses are given only the first value will be used");
            r1= mnbrak(f,x0[0]);}
    s=" new bracket values  : \n"+Matrix.toString(r1);
    //fonksiyonun gerçek değerini elde etmek için -1 ile çarpilmiştir
    String s2="bracket values with an ensured optimum value in the given range :";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    }
}
```

HALLEY ONE DIMENSIONAL OPTIMISATION METHOD

Consider the Taylor Formula, but this time assume that two term is remaining

$$f(x_{n+1}) = 0 = f(x_n) + \frac{f'(x_n)}{1!}(x_{n+1} - x_n) + \frac{f"(x_n)}{2!}(x_{n+1} - x_n)^2 + \frac{f^{(3)}(x_n)}{3!}(x_{n+1} - x_n)^3 + \cdots \qquad (2.11.1)$$

When second term of the Taylor Formula is also taken into acount the Newton-Raphson Formula takes the form:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)f'(x_n) - f(x_n)f"(x_n)}$$

This method is called **Halley's root finding method**.

In order to apply method to optimisation problems

Function will be replaced with its derivatives

$$x_{n+1} = x_n - \frac{2f'(x_n)f''(x_n)}{2f''(x_n)f''(x_n) - f'(x_n)f'''(x_n)}$$

Halley's root finding method., Python code

```
from math import *
from if_x import *

def halley_opt(f,x):
    # halley optimisation method
    nmax=100
    tolerance=1.0e-10
    for i in range(0,nmax):
      fx=f.dfunc(x,1);
      dfx=f.dfunc(x,2)
```

```
        d2fx=f.dfunc(x,3)
        x=x-2.0*fx*dfx/(2.0*dfx*dfx-fx*d2fx)
        if abs(fx)<tolerance: return x
      return x

def optimum_test(f,x):
   # optimum test through second derivatine
   # x is maximum-minimum-extremum point
   y=f.dfunc(x,3)
   s=""
   if y==0.0 :  s="extremum"
   elif y>0.0:  s="minimum"
   else :       s="maximum"
   return s

class f1(if_x):func=lambda self,x:x*x-x+5
f=f1()
x=1.0
r=halley_opt(f,x)
print(" optimum point  : ",r,"\nFUNCTION VALUE : ",f.func(r),optimum_test(f,r));
```

```
runfile('D:/okul/SCO1/halley_opt.py', wdir='D:/okul/SCO1')
Reloaded modules: if_x
 optimum point  :  0.499999999998834
FUNCTION VALUE :  4.75 maximum
```

ONE DIMENSIONAL THIELE METHOD

Thiele Method was investigated in Root finding section 7.9. Due to fact that root of the derivatives is given optimisation, Thiele Method can be utilised as optimisation method as follows:

$$g(x) = \frac{df(x)}{dx} = f'(x)$$

$$x_{n+1} = x_n - \frac{P_1(x_n)}{Q_1(x_n)} \quad (7.9.5)$$

$$P_1(x) = 4g(x)g'(x)[3g'(x)g'(x)g"(x)\text{-}3g(x)g"(x)g"(x) + g(x)g'(x)g'''(x)] \quad (7.9.6)$$

$$Q_1(x) = 12g'(x)g'(x)g'(x)g'(x)g(x)\text{-}18g(x)g'(x)g'(x)g"(x)g"(x) + 3g(x)g(x)\,g"(x)g"(x)g"(x) + 4g(x)g'(x)g'(x)g'(x)g'''(x) \quad (7.9.7)$$

$$x_{n+1} = x_n - \frac{P_2(x_n)}{Q_2(x_n)} \quad (7.9.8)$$

$$P_2(x) = g(x)[6g'(x)g'(x)\text{-}3g(x)g"(x)g"(x) + 2g(x)g'(x)g'''(x)] \quad (7.9.9)$$

$$Q_2(x) = 2g'(x)\left[3g'(x)g'(x)g"(x) - 3g(x)g"(x)g"(x) + g(x)g'(x)g'''(x)\right] \quad (7.9.10)$$

Thiele root finding method., Python code

```
# Thiele root finding method  for 1D optimisation

from math import *
from if_x import *

def Thiele_opt(f,x):
   #Thiele 1 method
   a=[0.0 for i in range(3)];
   nmax=500;
   tolerance=1.0e-10;
   gx=0;
   for i in range(0,nmax):
      gx=f.dfunc(x,1);
      dgx=f.dfunc(x,2);
      d2gx=f.dfunc(x,3);
      d3gx=f.dfunc(x,4);
      P1=4.0*gx*dgx*(3.0*dgx*dgx*d2gx-3.0*gx*d2gx*d2gx+gx*dgx*d3gx);
      Q1=12.0*dgx*dgx*dgx*dgx*d2gx-
18.0*gx*dgx*dgx*d2gx*d2gx+3.0*gx*gx*d2gx*d2gx*d2gx+4.0*gx*dgx*dgx*dgx*d3gx;
      x-=P1/Q1;
      if abs(gx)<tolerance:
         return x;
   print("Maximum number of iteration is exceeded in Thiele 1\n");
   return x;
```

```
class f1(if_x):func=lambda self,x:x*x-x+5
f=f1()
x=1.0
r=Thiele_opt(f,x)
print(" optimum point  : ",r,"Function = ",f.func(r));
```

```
n [81]: runfile('D:/okul/SCO1/Thiele_opt.py', wdir='D:/okul/SCO1')
Reloaded modules: if_x
 optimum point  :  0.49999999999816586 Function =  4.75
```

ONE DIMENSIONAL HASAN METHODS

Hasan[50] proposed several higher order methods for root finding with relatively less function calculations. Hasan Method can be utilised as optimisation method as follows:

$$g(x) = \frac{df(x)}{dx} = f'(x)$$

$$x_{n+1} = x_n - \frac{g(x_n)}{g'\left(x_n - \frac{g(x_n)}{2g'(x_n)}\right)} \qquad (8.7.1)$$

$$x_{n+1} = x_n - \frac{2g(x_n)}{g'(x_n) + g'\left(x_n - \frac{g(x_n)}{2g'(x_n)}\right)} \qquad (8.7.2)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'\left(x_n - \alpha\frac{f(x_n)}{f'(x_n)}\right) + f'\left(x_n - \beta\frac{f(x_n)}{f'(x_n)}\right)} \quad where\ \alpha + \beta = 1 \qquad (8.7.3)$$

$$x_{n+1} = x_n - \frac{2g(x_n)}{g'(x_n) + \frac{1}{r}g'(x_n)log\left(\frac{g'(x_n)g'(x_n) - g(x_n)g''(x_n)}{g'(x_n)g'(x_n)}\right)} \qquad (8.7.4)$$

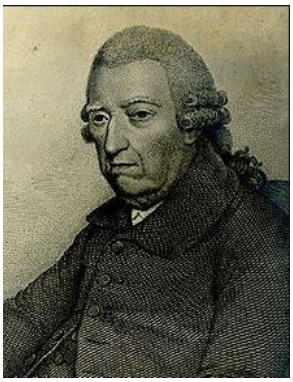
```
from math import *
from if_x import *

def hasan_opt(f,x):
    # finding real roots by using hasan method fourth order
    nmax=500;
    r=3;
    tolerance=1.0e-15;
    for iter in range(0,nmax):
       gx=f.dfunc(x,1);
       dgx=f.dfunc(x,2);
       d2gx=f.dfunc(x,3);
       x=x-gx/(dgx+1.0/r*dgx*log((dgx*dgx-gx*d2gx)/(dgx*dgx)));
       if abs(gx)<tolerance: return x;
    return x;


class f1(if_x):func=lambda self,x:x*x-x+5
f=f1()
x=1.0
r=hasan_opt(f,x)
print(" optimum point  : ",r,"\nFUNCTION VALUE : ",f.func(r));
```

```
runfile('D:/okul/SCO1/hasan_opt.py', wdir='D:/okul/SCO1')
Reloaded modules: if_x
 optimum point  :  0.5000000000009188
FUNCTION VALUE :  4.75
```

NEWTON-RAPHSON METHOD FOR NONLINEAR MULTIVARIABLE OPTIMIZATION



Sir Isaac Newton



Joseph Raphson

Root finding of multivariable functions by using Newton-Raphson method was prevously investigated. By remembering that optimum point is the root of the derivative of the function, the method can be utilise also to find optimum point.

If function $f(x_1,x_2,x_3,\ldots,x_n)$ First and second derivative and independent difference vector of the function can be written as:

$$\nabla f = \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \cdots\cdots \\ \frac{\partial f}{\partial x_n} \end{Bmatrix} \text{ and } \delta^{m+1} = \begin{Bmatrix} (x_1^{m+1} - x_1^m) \\ (x_2^{m+1} - x_2^m) \\ (x_3^{m+1} - x_3^m) \\ \cdots\cdots \\ (x_n^{m+1} - x_n^m) \end{Bmatrix} \text{ and } [H] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \frac{\partial^2 f}{\partial x_3 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Newton-Raphson iteration for (m+1) th iteration

$$[H^m]\{\delta^{m+1}\} = -\{\nabla f^m\}$$

As it is seen from the equation a linear system of equation is formed. The equation can be solved by using methods such as gauss elimination. An initial estimate for all the x values are required to start iterative solution. Second derivartive matrix is also called a Hessian matrix. First and second derivatives can also be calculated numerically, but that will increase the errors orthe number of iteration steps. The system of equation can also be written as:

$$\{\delta^{m+1}\} = [H^m]^{-1}[-\{\nabla f^m\}]$$

In this format$[H^m]^{-1}$ is the inverse Hessian matrix. If vector δ is written in open form, it becomes

$$\{x^{m+1}\} = \{x^m\} + [H^m]^{-1}[-\{\nabla f^m\}]$$

As a more general definition of this equation a step function α can be added into the term. It becomes

$$\{x^{m+1}\} = \{x^m\} + \alpha^m[H^m]^{-1}[-\{\nabla f^m\}]$$

In Newton-Raphson method step function α is equal to unity. In some other methods different utilisation of this general description will be invetigated.

Program Abstract class f_xj for function y=f(x0,x1,x2,..) (java version)

```
public abstract class f_xj extends Mathd
{  // single function  multi independent variable
   // a single value is returned indiced to equation_ref
abstract public double func(double x[]);
public double func(double x1,double y1)
{
double x[]=new double[2];
x[0]=x1;
x[1]=y1;
return func(x);
}
public double[] dfunc(double x[])
{int n=x.length;
```

```
 double c[]=new double[n];
 for(int i=0;i<n;i++)
 {c[i]=dfunc(x,i);}
 return c;
}

public double[][] d2func(double x[])
{int n=x.length;
 double c[][]=new double[n][n];

 for(int i=0;i<n;i++)
 {for(int j=0;j<n;j++)
   {   int xref[]={i,j};
                c[i][j]=d2func(x,xref);
   }
 }
 return c;
}
public double d2func(double x[],int x_ref[])
{// derivative of the function with respect to x_ref
double h0=0.0256;
int i,m;
int n=7;
double f1,f2;
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}
for(i=0;i<n;i++)
{
x1[x_ref[1]]+=h[i];
x2[x_ref[1]]-=h[i];
f1=dfunc(x1,x_ref[0]);
f2=dfunc(x2,x_ref[0]);
T[i][0]=( f1 - f2)/(2.0*h[i]);
x1[x_ref[1]]=x[x_ref[1]];
x2[x_ref[1]]=x[x_ref[1]];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public double dfunc(double x[],int x_ref)
{ // derivative of the function with respect to x_ref
double h0=0.015675863;
int i,m;
```

```
int n=7;
double f1,f2;
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
f1=func(x1);
f2=func(x2);
T[i][0]=( f1 - f2)/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}}
```

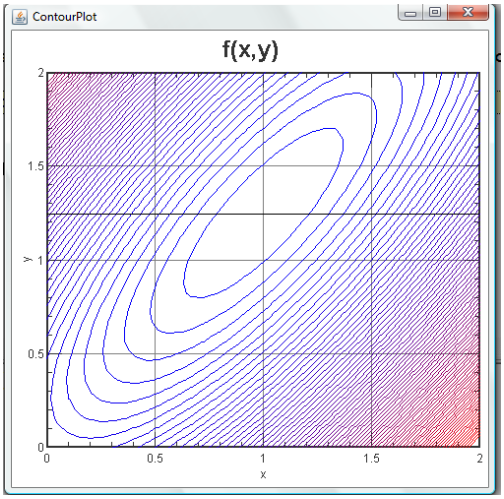
Program Multivariable non-linear Newton-Raphson method (java –abstact class-f-xj version)

```
// OPTIMIZATION
//  Newton-Raphson method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class f1 extends f_xj
{
public double func(double x[])
{//optimization function
double y=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
return y;
}}

public class NA34A
{
public static double[] gauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
```

```
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
              //Solving gauss elimination
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
              }
  }
  //back substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }
public static double[] newton_opt(f_xj f,double x[])
{ int k=x.length;
  int nmax=100;
  double tolerance=1.0e-10;
  double fx[];
  double dfx[][];
  double dx[];
  double total=0;
  for(int i=0;i<nmax;i++)
  { fx=f.dfunc(x);
    dfx=f.d2func(x);
    dx=gauss(dfx,fx);
    for(int j=0;j<k;j++)
      x[j]-=dx[j];
    total=0;
    for(int j=0;j<k;j++)
      total+=fx[j];
    if(Math.abs(total)<tolerance) return x;
  }
  return x;
}
public static double[] inputdata(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
```

```
    a[j++]=ax.doubleValue();
    }
    return a;
    }
    public static void main (String args[])
    {
                f1  f=new f1();
    double x[]=inputdata(" enter guess values  x0 : ");
    double r[];
    r=newton_opt(f,x);
    System.out.println(" minimum value  : "+Matrix.toString(r));
    }
}
```

```
---------- Capture Output ----------
> "D:\co\java\bin\java.exe" NA34A
 minimum value  :        0.999999999999083                    1.249999999998778
> Terminated with exit code 0.
```



Programif_xj interface (java language)

```
@FunctionalInterface
interface if_xj
{
  // single function  multi independent variable
  // a single value is returned indiced to equation_ref
public double func(double x[]);
default double func(double x1,double y1)
{
double x[]=new double[2];
x[0]=x1;
x[1]=y1;
return func(x);
}
default double[] dfunc(double x[])
{int n=x.length;
 double c[]=new double[n];
 for(int i=0;i<n;i++)
 {c[i]=dfunc(x,i);}
 return c;
}

default double[][] d2func(double x[])
{int n=x.length;
 double c[][]=new double[n][n];
 for(int i=0;i<n;i++)
 {for(int j=0;j<n;j++)
   {   int xref[]={i,j};
                c[i][j]=d2func(x,xref);
   }
 }
 return c;
}

default double d2func(double x[],int x_ref[])
```

```
{// derivative of the function with respect to x_ref
double h0=0.0256;
int i,m;
int n=7;
double f1,f2;
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref[1]]+=h[i];
x2[x_ref[1]]-=h[i];
f1=dfunc(x1,x_ref[0]);
f2=dfunc(x2,x_ref[0]);
T[i][0]=( f1 - f2)/(2.0*h[i]);
x1[x_ref[1]]=x[x_ref[1]];
x2[x_ref[1]]=x[x_ref[1]];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
default double dfunc(double x[],int x_ref)
{ // derivative of the function with respect to x_ref
double h0=0.0256;
int i,m;
int n=7;
double f1,f2;
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
```

```
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
f1=func(x1);
f2=func(x2);
T[i][0]=( f1 - f2)/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}}
```

Program Multivariable non-linear Newton-Raphson method (java –interface-if-xj – Lambda variable version)

```
/// OPTIMIZATION
//  Newton-Raphson method
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class NA34B
{
public static double[] gauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
              //Solving gauss elimination
    for(i=k+1;i<n;i++)
```

```
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //back substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }
public static double[] newton_opt(if_xj f,double x[])
{ int k=x.length;
  int nmax=100;
  double tolerance=1.0e-10;
  double fx[];
  double dfx[][];
  double dx[];
  double total=0;
  for(int i=0;i<nmax;i++)
  { fx=f.dfunc(x);
    dfx=f.d2func(x);
    dx=gauss(dfx,fx);
    for(int j=0;j<k;j++)
      x[j]-=dx[j];
    total=0;
    for(int j=0;j<k;j++)
      total+=Math.abs(fx[j]);
    if(total<tolerance) return x;
  }
  return x;
}
public static double[] inputdata(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }
     public static void main (String args[])
     {
     if_xj f=(double x[])-> 3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
     double x[]=inputdata(" enter guess values  x0 : ");
     double r[];
     r=newton_opt(f,x);
     System.out.println(" minimum value  : "+Matrix.toString(r));
     surfacePlot p=new surfacePlot(f,-2.0,2.0,-2.0,2.0,40);
     p.plot();
     }
}
```

Program Abstract class f_xj for function y=f(x0,x1,x2,..) (Python version)

```
from abc import ABC, abstractmethod
from math import *

class f_xj(ABC):
  # multifunction multi independent variable
  # vector of dependent variables are returned
  # example f[0]=x[0]+sin(x[1])
  #         f[1]=x[0]*x[1]-x[1]
```

```
# func(x) returns the value of f[0] and f[1]
# as a two dimensional vector

@abstractmethod
def func(self,x):
  pass;

def funcxy(self,x1,y1):
  n=2;
  x=[0 for i in range(n)];
  x[0]=x1;
  x[1]=y1;
  return self.func(x);

def dfunc_(self,x,x_ref):
   #derivative of the function with respect to x_ref
   h0=0.015675863;
   m=7;
   n=len(x);
   x1=[0 for i in range(n)];
   x2=[0 for i in range(n)];
   for i in range(n):
      x1[i]=x[i];
      x2[i]=x[i];
   #derivative of a simple function
   T=[[0 for i2 in range(m)] for j2 in range(m)];
   h=[0 for i in range(m)];
   for i in range(m):
      h[i]=0;
      for j in range(m):
         T[i][j]=0;
   h[0]=h0;
   r=0.5;
   for i in range(m):
      h[i]=h0*pow(r,i);
   for i in range(m):
      x1[x_ref]+=h[i];
      x2[x_ref]-=h[i];
      f1=self.func(x1);
      f2=self.func(x2);
      T[i][0]=( f1 - f2)/(2.0*h[i]);
      x1[x_ref]=x[x_ref];
      x2[x_ref]=x[x_ref];
   for k in range(1,m):
       for i in range(0,m-k):
          T[i][k]=(h[i]*h[i]*T[i+1][k-1] - h[i+k]*h[i+k]*T[i][k-1])/(h[i]*h[i]
          - h[i+k]*h[i+k]);
   xx=T[0][n-1];
   return xx;

def d2func_(self,x,x_ref):
   #derivative of the function with respect to x_ref
   h0=0.015675863;
   m=7;
   n=len(x);
   x1=[0 for i in range(n)];
   x2=[0 for i in range(n)];
   for i in range(n):
      x1[i]=x[i];
      x2[i]=x[i];
   #derivative of a simple function
   T=[[0 for i2 in range(m)] for j2 in range(m)];
   h=[0 for i in range(m)];
   for i in range(m):
      h[i]=0;
      for j in range(m):
         T[i][j]=0;
   h[0]=h0;
   r=0.5;
   for i in range(m):
      h[i]=h0*pow(r,i);
   for i in range(m):
      x1[x_ref[1]]+=h[i];
      x2[x_ref[1]]-=h[i];
      f1=self.dfunc_(x1,x_ref[0]);
      f2=self.dfunc_(x2,x_ref[0]);
```

```
        T[i][0]=( f1 - f2)/(2.0*h[i]);
        x1[x_ref[1]]=x[x_ref[1]];
        x2[x_ref[1]]=x[x_ref[1]];
    for k in range(1,m):
        for i in range(0,m-k):
            T[i][k]=(h[i]*h[i]*T[i+1][k-1] - h[i+k]*h[i+k]*T[i][k-1])/(h[i]*h[i]
            - h[i+k]*h[i+k]);
    xx=T[0][n-1];
    return xx;

 def dfunc(self,x):
    n=len(x);
    c=[0 for i in range(n)];
    for i in range(n):
        c[i]=self.dfunc_(x,i);
    return c;

 def d2func(self,x):
    n=len(x);
    x_ref=[0 for i2 in range(2)];
    c=[[0 for i2 in range(n)] for j2 in range(n)];
    for i in range(n):
        for j in range(n):
            x_ref[0]=i;
            x_ref[1]=j;
            c[i][j]=self.d2func_(x,x_ref);
    return c;
```

Program Multivariable non-linear Newton-Raphson method (Python version)

```
 from math import *
from f_xj import *;
from gauss import *;

class f1(f_xj):
  def func(self,x):
     y=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
     return y;

def newton(f,x):
   # Newton-Raphson root finding method
   k=len(x);
   nmax=100;
   tolerance=1.0e-10
   for i in range(nmax):
      fx= f.dfunc(x);
      dfx=f.d2func(x);
      dx=gauss(dfx,fx);
      for j in range(k):
         x[j]-=dx[j];
      total=0;
      for j in range(k):
         total+=fx[j];
      if abs(total)<tolerance: return x
   return x;

x=[1,1];
f=f1();
r=newton(f,x);
print("r=",r);
```

```
runfile('D:/okul/SCO1/newtond_opt.py', wdir='D:/okul/SCO1')
Reloaded modules: f_xj, gauss
r= [0.9999999999999799, 1.2499999999999776]
```

Newton optimisation method, Julia version

```
 #function set
function func(x)
ff=3.0*x[1]*x[1]-4.0*x[1]*x[2]+2.0*x[2]*x[2]-x[1]-x[2]
return ff
end

#single function with reference number i
```

```
function fi(f,x,i)
# system of non-linear equation
y=dfunc(f,x)
return y[i]
end

#first derivative
function dfunc(func,x)
 global  dx=0.001
 global c=[1.0/280.0 -4.0/105.0   1.0/5.0      -4.0/5.0  0 4.0/5.0      -1.0/5.0     4.0/105.0   -1.0/280.0]
 n=length(x)
 ff=zeros(Float64,n)
 global yy=ff
 x1=zeros(Float64,n)
   for ix=1:n
      for i=-4:4
        # create x1set for derivative input
        # ================================
        for k=1:n
          xx=x[k]
          if k!=ix
            x1[k]=xx
          else
            delta=Float64(i)*dx
            x1[k]=xx+delta
          end
        end # for k
        # ================================
        ff1=func(x1)
        ff[ix]+=c[i+5]*ff1
        yx=ff[ix]
      end # for i
      ff[ix]/=dx
      yx=ff[ix]
    end # for ix
 return ff
end # for function

#second derivative
function d2func(func,x)
 global  dx=0.001
 global c=[1.0/280.0 -4.0/105.0   1.0/5.0      -4.0/5.0  0 4.0/5.0      -1.0/5.0     4.0/105.0   -1.0/280.0]
 n=length(x)
 ff=zeros(Float64,n,n)
 global yy=ff[1]
 x1=zeros(Float64,n)
 for ieqn=1:n
   for ix=1:n
      for i=-4:4
        # create x1set for derivative input
        # ================================
        for k=1:n
          xx=x[k]
          if k!=ix
            x1[k]=xx
          else
            delta=Float64(i)*dx
            x1[k]=xx+delta
          end
        end # for k
        # ================================
        ff1=fi(func,x1,ieqn)
        ff[ieqn,ix]+=c[i+5]*ff1
        yx=ff[ieqn,ix]
      end # for i
      ff[ieqn,ix]/=dx
      yx=ff[ieqn,ix]
    end # for ix
 end # for ieqn
 return ff
end # for function

function gauss_pivot(A1,b)
              n=size(A1)[1]
              carpan=0.0
              toplam=0.0
```

```
            global x=Array{Float64}(undef,n)
            global a=Array{Float64}(undef,n,n)
            for i=1:n
              for j=1:n
                a[i,j]=A1[i,j]
              end
            end
            for k=1:n
            #pivoting
               p=k
               buyuk=abs(a[k,k])
               for ii=k+1:n
                 dummy=abs(a[ii,k])
                 if dummy > buyuk
                    buyuk=dummyp=ii
                 end #if dummy
               end #for ii
               if p!=k
                 for jj=k:n
                   dummy=a[p,jj]
                   a[p,jj]=a[k,jj]
                   a[k,jj]=dummy
                   end #for jj

                 dummy=b[p]
                 b[p]=b[k]
                 b[k]=dummy
               end #if p!
            # gauss elimination
   for i=k+1:n
               carpan=a[i,k]/a[k,k]
               a[i,k]=0.0
               for j=k+1:n
                 a[i,j]=a[i,j]-carpan*a[k,j]
               end #for j
               b[i]=b[i]-carpan*b[k]
            end #for i
            end #k
                        # back substituting
              x[n]=b[n]/a[n,n]
              n1=n-1
              for i=n1:-1:1
                toplam=0.0
                for j=i+1:n
                  toplam=toplam+a[i,j]*x[j]
                end #for j
                x[i]=(b[i]-toplam)/a[i,i]
              end #for i
            return x
end #function

function newton(func,x)
  # Newton-Raphson root finding method
  k=length(x)
  nmax=100
  tolerance=1.0e-10
  for i=1:nmax
    fx=dfunc(func,x)
    dfx=d2func(func,x)
    dx=gauss_pivot(dfx,fx)
    for j=1:k
       x[j]=x[j]-dx[j]
    end # for j
    total=0.0;
    for j=1:k
       total+=fx[j]
    end # for j
    if abs(total)<tolerance
      return x
    end
  end #for i
  return x
end # function


x=[1.0 1.0]
```

```
y=func(x)
dy=dfunc(func,x)
d2y=d2func(func,x)
println("x = $x \ny = $y dy = $dy d2y = $d2y")
r=newton(func,x)
println("r = $r")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_newton.jl
x = [1.0 1.0]
y = -1.0 dy = [1.0000000000000988, -1.0000000000000855]
d2y = [6.000000000161124 -4.000000000009855; -3.9999999999164455 3.9999999999791522]
r = [1.0000000000000777 1.2500000000001292]

> Terminated with exit code 0.
```

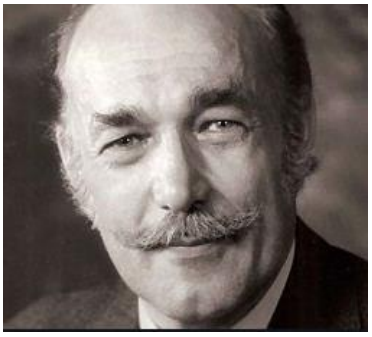
Julia has optimisation library as a package, This package can be use to carry out Newton optimisation

```
using Pkg; Pkg.add("Optim")
```

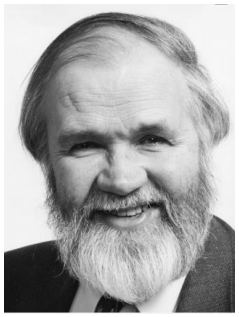
```
#Newton Optimisation by using Optim LBFGS

using Optim
func(x)=3.0*x[1]*x[1]-4.0*x[1]*x[2]+2.0*x[2]*x[2]-x[1]-x[2]
x=[1.0 1.0]
z=optimize(func,x, LBFGS())
print(z)
x=Optim.minimizer(z)
print("x = $x")
#y=func(x)
y=Optim.minimum(z)
print("\nminimum  = $y")
```

```
julia> include("opt_newton1.jl")
 * Status: success

 * Candidate solution
   Final objective value:     -1.125000e+00

 * Found with
   Algorithm:     L-BFGS

 * Convergence measures
   |x - x'|               = 1.39e-01 ≰ 0.0e+00
   |x - x'|/|x'|          = 1.11e-01 ≰ 0.0e+00
   |f(x) - f(x')|         = 1.39e-02 ≰ 0.0e+00
   |f(x) - f(x')|/|f(x')| = 1.23e-02 ≰ 0.0e+00
   |g(x)|                 = 1.47e-10 ≤ 1.0e-08

 * Work counters
   Seconds run:   1  (vs limit Inf)
   Iterations:    2
   f(x) calls:    7
   ∇f(x) calls:   7
x = [0.9999999999994034 1.2500000000357743]
minimum  = -1.1249999999999991
julia>
```

NELDER – MEAD SIMPLEX NONLINEAR MULTIVARIABLE OPTIMIZATION METHOD

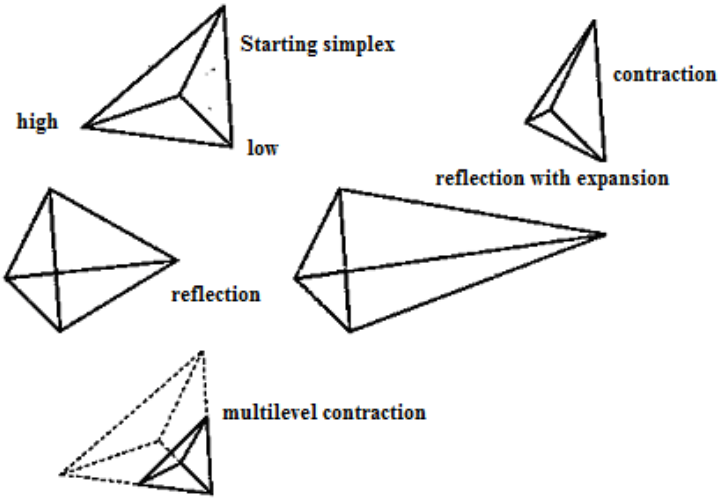


John Nelder



Roger Mead

The simplex method developed by Nelder and Mead method is a multidimensional search method . Simplex is an n dimensional geometric entity. It contains (n+1) points in n dimensional space. The method uses the concept of a simplex, which is a special polytop of $N + 1$ vertices in N dimensions. Examples of simplices include a line segment on a line, a triangle on a plane, a tetrahedron in three-dimensional space and so forth. Required processes for the simplex amobea's movements are shown in the figure



Process can be given as follows

- **1. Order** according to the values of the vertices:
  $\boldsymbol{f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{n+1})}$
- **2. Calculate** x_0, the center of gravity off all points except x_{n+1}
- **3. Reflection:** Compute reflected point $x_r = x_0 + \alpha(x_0 - x_{n+1})$

If the reflected point is better than the second worst, but not better than the best, i.e.:
$\boldsymbol{f(x_1) \leq f(x_2) \leq f(x_n)}$
then obtain a new simplex by replacing the worst point x_{n+1} with the reflected point x_r, and go to step1

- **4. Expansion:**

If the reflected point is the best point so far, $\boldsymbol{f(x_r) < f(x_1)}$ **Then** compute the expanded point
$x_e = x_0 + \gamma(x_0 - x_{n+1})$

If the expanded point is better than the reflected point, $\boldsymbol{f(x_e) < f(x_r)}$ **then** obtain a new simplex by replacing the worst point x_{n+1} with the expanded point x_e, and go to step 1.

Else obtain a new simplex by replacing the worst point x_{n+1} with the reflected point x_r, and go to step 1.

Else (i.e. reflected point is worse than second worst) continue at step 5.

- **5. Contraction:**

Here, it is certain that , $\boldsymbol{f(x_r) < f(x_n)}$
Compute contracted point $x_c = x_{n+1} + \rho(x_0 - x_{n+1})$

If the contracted point is better than the worst point, i.e. $\boldsymbol{f(x_c) < f(x_{n+1})}$

then obtain a new simplex by replacing the worst point x_{n+1} with the contracted point x_c, and go to step 1.

Else go to step 6.

- **6. Reduction:**

For all but the best point with

$x_i = x_1 + \sigma(x_i - x_1)$ for all $i \in \{2, \dots, n+1\}$ goto step 1

Note: $\boldsymbol{\alpha, \gamma, \rho}$ and σ are respectively the reflection, the expansion, the contraction and the shrink coefficient. Standard values are $\boldsymbol{\alpha = 1, \gamma = 2, \rho = 1/2}$ and $\boldsymbol{\sigma = 1/2}$

For the **reflection**, since x_{n+1} is the vertex with the higher associated value among the vertices, we can expect to find a lower value at the reflection of x_{n+1} in the opposite face formed by all vertices point x_i except x_{n+1}.

For the **expansion**, if the reflection point x_r is the new minimum along the vertices we can expect to find interesting values along the direction from x_o to x_r.

Concerning the **contraction**: If $f(x_r) > f(x_n)$ we can expect that a better value will be inside the simplex formed by all the vertices x_i.

The initial simplex is important, indeed, a too small initial simplex can lead to a local search, consequently the NM can get more easily stuck. So this simplex should depend on the nature of the problem. Instead of given n+1 point same process can be done by giving one point and a change vector by defining

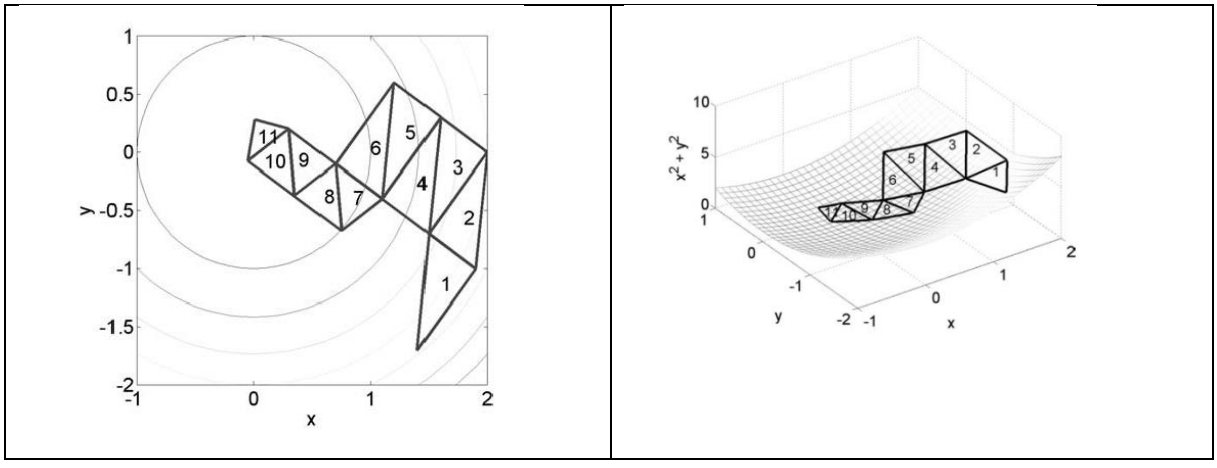
$x_{i-1} = P_i + \lambda\, dx_j$ where if i=j λ=1 else if $i \neq j$ λ=0 .

For example:

if $P = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$ and $dx = \begin{Bmatrix} da \\ db \\ dc \end{Bmatrix}$

$$x_0 = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \quad x_1 = \begin{Bmatrix} a + da \\ b \\ c \end{Bmatrix} \quad x_2 = \begin{Bmatrix} a \\ b + db \\ c \end{Bmatrix} \quad x_3 = \begin{Bmatrix} a \\ b \\ c + dc \end{Bmatrix}$$

Figure 6.2-1 Movements of the nelder-Mead simplex element to find the optimum point



Program Nelder and Mead simplex method, Java code

```
 // Nelder-Mead Simplex optimization method// Nelder-Mead Simplex optimization method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
```

```
import javax.swing.*;


class f1 extends f_xj
{
public double func(double x[])
{
//optimization function
double ff;
ff=(x[0]-2.0)*(x[0]-2.0)*(x[0]-2.0)*(x[0]-2.0)+(x[1]-1.0)*(x[1]-1.0)+
3.0*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0);
return ff;
}}

public class NA41
{
//6-2. Nelder and Mead Simplex multivariable nonlinear optimization method
//  Nelder & Mead 1965 Computer J, v.7, 308-313.
//___abstract class inputs_____________________________________________________________
public static double[] nelder(f_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
            int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
  {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
  }

  // Inlet variable definitions
  // fnelder : abstract multivariable function f(x)
  // x : independent variable set of n+1 simplex elements
  // maxiteration : maximum iteration number
  // tolerance :
      int NDIMS = x.length-1;
      int NPTS = x.length;
      int FUNC = NDIMS;
      int ncalls = 0;
        ////// construct the starting simplex ///////////////////
        //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
        double z[]=new double[NDIMS];
        double best = 1E99;
        ///////////////// calculate the first function values for the simplex ////////////////
        int iter=0;
        for (iter=1; iter<maxiteration; iter++)
        {
                  ///////////  define lo, nhi, hi (low high next_to_high //////////////
           int  ilo=0, ihi=0, inhi = -1; // -1 means missing
           double flo = p[0][FUNC];
           double fhi = flo;
           double pavg,sterr;
           for (i=1; i<NPTS; i++)
           {
             if (p[i][FUNC] < flo)
               {flo=p[i][FUNC]; ilo=i;}
             if (p[i][FUNC] > fhi)
               {fhi=p[i][FUNC]; ihi=i;}
           }
           double fnhi = flo;
```

```
inhi = ilo;
for (i=0; i<NPTS; i++)
  if ((i != ihi) && (p[i][FUNC] > fnhi))
    {fnhi=p[i][FUNC]; inhi=i;}
////////// exit criteria //////////////
if ((iter % 4*NDIMS) == 0)
{
   // calculate the avarage (including maximum value)
   pavg=0;
   for(i=0;i<NPTS;i++)
     pavg+=p[i][FUNC];
   pavg/=NPTS;
   double tot=0;
   if(printlist!=0)
   {  System.out.print(iter);
      for (j=0; j<=NDIMS; j++)
         { System.out.print(p[ilo][j]+" ");}
      System.out.println("");
   }
   for(i=0;i<NPTS;i++)
   { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
   sterr=Math.sqrt(tot/NPTS);
   //if(sterr < tolerance)
     { for (j=0; j<NDIMS; j++)
         { z[j]=p[ilo][j];}
       //break;
     }
   best = p[ilo][FUNC];
}

///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);
///////// reflect ////////////////
double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
   for (j=0; j<NDIMS; j++)
     p[ihi][j] = r[j];
   p[ihi][FUNC] = fr;
   continue;
}
if (fr < flo)  //// expand
{
   double e[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     e[j] = 3*ave[j] - 2*p[ihi][j];
   double fe = fnelder.func(e);
   if (fe < fr)
   {
```

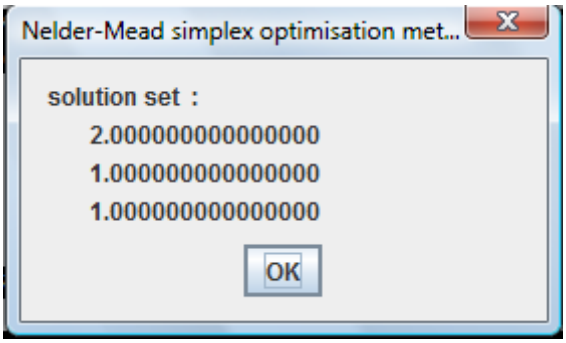
```
            for (j=0; j<NDIMS; j++)
              p[ihi][j] = e[j];
            p[ihi][FUNC] = fe;
            continue;
          }
          else
          {
            for (j=0; j<NDIMS; j++)
              p[ihi][j] = r[j];
            p[ihi][FUNC] = fr;
            continue;
          }
        }
        ////////////// shrink:
        if (fr < fhi)
        {
          double c[] = new double[NDIMS];
          for (j=0; j<NDIMS; j++)
            c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
          double fc = fnelder.func(c);
          if (fc <= fr)
          {
            for (j=0; j<NDIMS; j++)
              p[ihi][j] = c[j];
            p[ihi][FUNC] = fc;
            continue;
          }
          else   /////// shrink
          {
            for (i=0; i<NPTS; i++)
              if (i != ilo)
              {
                for (j=0; j<NDIMS; j++)
                  p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                p[i][FUNC] = fnelder.func(p[i]);
              }
            continue;
          }
        }

        if (fr >= fhi)   ///
        {
          double cc[] = new double[NDIMS];
          for (j=0; j<NDIMS; j++)
            cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
          double fcc = fnelder.func(cc);
          if (fcc < fhi)
          {
            for (j=0; j<NDIMS; j++)
              p[ihi][j] = cc[j];
            p[ihi][FUNC] = fcc;
            continue;
          }
          else    /////////
          {
            for (i=0; i<NPTS; i++)
              if (i != ilo)
              {
                for (j=0; j<NDIMS; j++)
                  p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                p[i][FUNC] = fnelder.func(p[i]);
              }
```

```
                }
            }
        }
     return z;
}
    public static double[] nelder(f_xj fnelder,double a[],double da[],double tolerance)
    {return nelder(fnelder,a,da,500,tolerance,0);}

    public static double[] nelder(f_xj fnelder,double a[],double da[])
    {return nelder(fnelder,a,da,500,1.0e-10,0);}

    public static double[] nelder(f_xj fnelder,double a[])
    {
                double [] da=new double[a.length];
        for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
        return nelder(fnelder,a,da);
    }
// if_xj interface methods
public static double[] nelder(if_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
            int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
  {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
  }

  // Inlet variable definitions
  // fnelder : abstract multivariable function f(x)
  // x : independent variable set of n+1 simplex elements
  // maxiteration : maximum iteration number
  // tolerance :
      int NDIMS = x.length-1;
      int NPTS = x.length;
      int FUNC = NDIMS;
      int ncalls = 0;
        ////// construct the starting simplex //////////////////
        //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
        double z[]=new double[NDIMS];
        double best = 1E99;
        ///////////////// calculate the first function values for the simplex ////////////////
        int iter=0;
        for (iter=1; iter<maxiteration; iter++)
        {
                  ///////////  define lo, nhi, hi (low high next_to_high //////////////
          int  ilo=0, ihi=0, inhi = -1; // -1 means missing
          double flo = p[0][FUNC];
          double fhi = flo;
          double pavg,sterr;
          for (i=1; i<NPTS; i++)
          {
            if (p[i][FUNC] < flo)
              {flo=p[i][FUNC]; ilo=i;}
            if (p[i][FUNC] > fhi)
              {fhi=p[i][FUNC]; ihi=i;}
          }
          double fnhi = flo;
          inhi = ilo;
```

```
for (i=0; i<NPTS; i++)
  if ((i != ihi) && (p[i][FUNC] > fnhi))
    {fnhi=p[i][FUNC]; inhi=i;}
////////// exit criteria //////////////
if ((iter % 4*NDIMS) == 0)
{
   // calculate the avarage (including maximum value)
   pavg=0;
   for(i=0;i<NPTS;i++)
     pavg+=p[i][FUNC];
   pavg/=NPTS;
   double tot=0;
   if(printlist!=0)
   {  System.out.print(iter);
      for (j=0; j<=NDIMS; j++)
        { System.out.print(p[ilo][j]+" ");}
      System.out.println("");
   }
   for(i=0;i<NPTS;i++)
   { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
   sterr=Math.sqrt(tot/NPTS);
   //if(sterr < tolerance)
     { for (j=0; j<NDIMS; j++)
         { z[j]=p[ilo][j];}
       //break;
     }
   best = p[ilo][FUNC];
}

///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);
/////////// reflect ////////////////
double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
   for (j=0; j<NDIMS; j++)
     p[ihi][j] = r[j];
   p[ihi][FUNC] = fr;
   continue;
}
if (fr < flo)  //// expand
{
   double e[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     e[j] = 3*ave[j] - 2*p[ihi][j];
   double fe = fnelder.func(e);
   if (fe < fr)
   {
     for (j=0; j<NDIMS; j++)
```

```
            p[ihi][j] = e[j];
          p[ihi][FUNC] = fe;
          continue;
        }
        else
        {
          for (j=0; j<NDIMS; j++)
            p[ihi][j] = r[j];
          p[ihi][FUNC] = fr;
          continue;
        }
      }
      ////////////// shrink:
      if (fr < fhi)
      {
        double c[] = new double[NDIMS];
        for (j=0; j<NDIMS; j++)
          c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
        double fc = fnelder.func(c);
        if (fc <= fr)
        {
          for (j=0; j<NDIMS; j++)
            p[ihi][j] = c[j];
          p[ihi][FUNC] = fc;
          continue;
        }
        else   /////// shrink
        {
          for (i=0; i<NPTS; i++)
            if (i != ilo)
            {
              for (j=0; j<NDIMS; j++)
                p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
              p[i][FUNC] = fnelder.func(p[i]);
            }
          continue;
        }
      }

      if (fr >= fhi)   ///
      {
        double cc[] = new double[NDIMS];
        for (j=0; j<NDIMS; j++)
          cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
        double fcc = fnelder.func(cc);
        if (fcc < fhi)
        {
          for (j=0; j<NDIMS; j++)
            p[ihi][j] = cc[j];
          p[ihi][FUNC] = fcc;
          continue;
        }
        else    /////////
        {
          for (i=0; i<NPTS; i++)
            if (i != ilo)
            {
              for (j=0; j<NDIMS; j++)
                p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
              p[i][FUNC] = fnelder.func(p[i]);
            }
        }
```

```
            }
          }
       return z;
}
     public static double[] nelder(if_xj fnelder,double a[],double da[],double tolerance)
     {return nelder(fnelder,a,da,500,tolerance,0);}

     public static double[] nelder(if_xj fnelder,double a[],double da[])
     {return nelder(fnelder,a,da,500,1.0e-10,0);}

     public static double[] nelder(if_xj fnelder,double a[])
     {
                    double [] da=new double[a.length];
          for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
          return  nelder(fnelder,a,da);
     }
  // input data
  //_____________________________________________________________________
    public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
            public static void main (String args[])
    {
    double [] x0=inputdata("enter optimum point first estimation values with one empty space in between: ");
    double [] dx0=new double[x0.length];
    for(int i=0;i<x0.length;i++) dx0[i]=0.2*x0[i];
    f1 f=new f1();
    double [] r1= nelder(f,x0,dx0);
    String s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



EXAMPLE PROBLEM

Solve the above function by using lambda variables, java code

```
import javax.swing.JOptionPane;
public class NA41A
{  public static void main (String args[])
    {if_xj f=(double x[])->(x[0]-2.0)*(x[0]-2.0)*(x[0]-2.0)*(x[0]-2.0)+(x[1]-1.0)*(x[1]-1.0)+
    3.0*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0)*(x[2]-1.0);
    double [] x0=NA41.inputdata("enter optimum point first estimation values with one empty space in between: ");
    double [] r1=NA41.nelder(f,x0);
    String s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

Python version of the above problem is given below:

```
from gauss import *;
from math import *;
from f_xj import *;
import numpy as np;
import matplotlib.pyplot as plt;
from mpl_toolkits.mplot3d import Axes3D
from random import *;

class f1(f_xj):
  def func(self,x):
     x1=x[0]-2.0
     x2=x[1]-1.0
     x3=x[2]-1.0
     y=x1*x1*x1*x1+x2*x2+3.0*x3*x3*x3*x3*x3*x3
     return y

class f2(f_xj):
  def func(self,x):
     y=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1]
     return y;

def  nelder(fnelder,a,da,maxiteration,tolerance,printlist):
   #print("a",a)
   #print("da",da)
   n=len(a)
   m=n+1
   x=[[0.0 for j in range(n)] for i in range(n+1)]
   p=[[0.0 for j in range(m)] for i in range(m)]
   s=""
   NDIMS = len(x)-1
   NPTS = len(x)
   FUNC = NDIMS
   ncalls = 0;
   for i in range(m):
      for j in range(n):
         if i==j:
            x[i][j]=a[j]+da[j]
            p[i][j]=x[i][j]
         else:
            x[i][j]=a[j]
            p[i][j]=x[i][j]
      p[i][FUNC] = fnelder.func(p[i])
      #print("x[",i,"]=",x[i])
      #print("p[",i,"]=",p[i])
   #print("p",p)
   #Inlet variable definitions
   #fnelder : abstract multivariable function f(x)
   #x : independent variable set of n+1 simplex elements
   #maxiteration : maximum iteration number
   #tolerance :

   #///// construct the starting simplex //////////////////
   z=[0.0 for i in range(NDIMS)]
   best = 1.0E99;
   #/////////////// calculate the first function values for the simplex /////////////////
   iter=0;
   for iter in range(1,maxiteration):
      #////////  define lo, nhi, hi (low high next_to_high //////////////
      ilo=0
      ihi=0
```

```
    inhi = -1
    # -1 means missing
    flo = p[0][FUNC];
    fhi = flo
    #print("flo=",flo,"fhi=",fhi)
    #double pavg,sterr;
    for i in range(1,NPTS):
       if p[i][FUNC] < flo:
          flo=p[i][FUNC]
          ilo=i
       if  p[i][FUNC] > fhi:
          fhi=p[i][FUNC]
          ihi=i
       #print("i=",i,"flo=",flo,"fhi=",fhi,"ilo=",ilo,"ihi=",ihi,"pifunc=",p[i][FUNC])
    fnhi = flo
    inhi = ilo
    #print("fnhi=",fnhi,"inhi=",inhi)
    for i in range(NPTS):
       if (i != ihi) and (p[i][FUNC] > fnhi):
          fnhi=p[i][FUNC]
          inhi=i
    #////////// exit criteria ////////////////
    if (iter % 4*NDIMS) == 0:
       #calculate the avarage (including maximum value)
       pavg=0
       for i in range(NPTS):
           pavg=pavg+p[i][FUNC]
       pavg=pavg/NPTS
       tot=0.0
       if printlist!=0:
          print(iter)
          for j in range(NDIMS):
             print(p[ilo][j]," ")
       for i in range(NPTS):
          tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg)
       sterr=sqrt(tot/NPTS)
       for j in range(NDIMS):
          z[j]=p[ilo][j]
       best = p[ilo][FUNC];
    # end of if iter%4*NDMS)==0
    #//// calculate avarage without maximum value //////
    ave=[0.0 for i in range(NDIMS)]
    for j in range(NDIMS):
       ave[j] = 0
    for i in range(NPTS):
       if i != ihi:
          for j in range(NDIMS):
             ave[j] = ave[j]+p[i][j];
    for j in range(NDIMS):
       ave[j]=ave[j]/(NPTS-1)
    #print("ave[",j,"]=",ave[j])
    #//////// reflect ////////////////
    #print("reflect")
    r=[0.0 for i in range(NDIMS)]
    for j in range(NDIMS):
       r[j] = 2.0*ave[j] - p[ihi][j]
    fr = fnelder.func(r)
    if (flo <= fr) and (fr < fnhi): #in zone: accept
       for j in range(NDIMS):
          p[ihi][j] = r[j]
       p[ihi][FUNC] = fr
       continue
    #///(////////////// expand
    if fr < flo:
       #print("expand")
       e=[0.0 for i in range(NDIMS)]
       for j in range(NDIMS):
          e[j] = 3.0*ave[j] - 2.0*p[ihi][j];
       fe = fnelder.func(e);
       if fe < fr:
          for j in range(NDIMS):
             p[ihi][j] = e[j]
          p[ihi][FUNC] = fe
          continue
       else:
          for j in range(NDIMS):
```

```
                p[ihi][j] = r[j]
            p[ihi][FUNC] = fr
            continue
        #////////// shrink:
        if fr < fhi:
            #print("srink")
            c=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                c[j] = 1.5*ave[j] - 0.5*p[ihi][j]
            fc = fnelder.func(c);
            if fc <= fr:
                for j in range(NDIMS):
                    p[ihi][j] = c[j];
                p[ihi][FUNC] = fc
                continue
            else:   #////// shrink
                for i in range(NPTS):
                    if i != ilo:
                        for j in range(NDIMS):
                            p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j]
                        p[i][FUNC] = fnelder.func(p[i])
                continue
        #/////////////// contract
        if fr >= fhi:
            #print("contract")
            cc=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
            fcc = fnelder.func(cc)
            if fcc < fhi:
                for j in range(NDIMS):
                    p[ihi][j] = cc[j];
                p[ihi][FUNC] = fcc
                continue
            else:
                for i in range(NPTS):
                    if i != ilo:
                        for j in range(NDIMS):
                            p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j]
                        p[i][FUNC] = fnelder.func(p[i]);
    return z

a=[1,1,1]
da=[0.5122,0.643,0.25]
f=f1()
r=nelder(f,a,da,100,1.0e-15,0)
print("r = ",r)
```

```
runfile('E:/okul/SCO1/nelder.py', wdir='E:/okul/SCO1')
Reloaded modules: gauss, f_xj
r =  [1.9999749583837203, 0.9999999853635089, 1.0023772156506 01]
```

Nelder and Mead simplex method, Julia code

```
 # 6-2. Nelder and Mead Simplex multivariable nonlinear optimization method
#  Nelder & Mead 1965 Computer J, v.7, 308-313.

function  nelder(fnelder,a,da)
            maxiteration=100
            tolerance=1e-6
            global pavg=0.0
    i=1
            j=1
            flo=0.0
            il0=1
            fhi=0.0
            ihi=1
            fnhi=0.0
            inhi=1
            fe=0.0
            fr=0.0
            fc=0.0
            fcc=0.0
            n=length(a)
```

```
            n1=n+1
x=zeros(Float64,n1,n)
x1=zeros(Float64,n)
global p=zeros(Float64,n1,n1)
f1=zeros(Float64,n)
for i=1:n1
  for j=1:n
     if i==j
       x[i,j]=a[j]+da[j]
       x1[j]=x[i,j]
       p[i,j]=x[i,j]
     else
       x[i,j]=a[j]
       x1[j]=x[i,j]
       p[i,j]=x[i,j]
     end # if i==j
  end # for j=1:n
  p[i,n1]=fnelder(x1)
end # for i=1:n1
   # Inlet variable definitions
   # fnelder : abstract multivariable function f(x)
   # x : independent variable set of n+1 simplex elements
   # maxiteration : maximum iteration number
   # tolerance :
            NDIMS = n
            NPTS = n1
            FUNC = n1
            ncalls = 0
            ave = zeros(Float64,NDIMS)
            # reflect
            r =zeros(Float64,NDIMS)
            # shrink:
    c=zeros(Float64,NDIMS)
    # shrink all dimensions
    cc=zeros(Float64,NDIMS)
    # expand
    e = zeros(Float64,NDIMS)
        # construct the starting simplex #########
        # p  [NPTS,NPTS] # [row,col] = [whichvx,coord,FUNC]
        z=zeros(Float64,NDIMS)
        best = 1e99
        global pavg=0.0
        # ####### calculate the first function values for the simplex ########
        iter=1
        for iter=1:maxiteration
                #  ####/  define lo, nhi, hi (low high next_to_high #######
                #println("Block 1 : entry to iter loop iter= $iter \n")
                #show(IOContext(stdout, :limit=>false), MIME"text/plain"(), p)
                #println("'\n flo= $flo fhi $fhi fnhi = $fnhi fr = $fr fe= $fe fc = $fc fcc=$fcc\n")
            ihi=1
            fnhi=0.0
            inhi=1
            fe=0.0
            fr=0.0
            fc=0.0
            fcc=0.0
           ilo=1
           ihi=1
           inhi = -1 # -1 means missing
           flo = p[1,FUNC]
           fhi = flo
           sterr=0.0
           for i=2:NPTS
             if p[i,FUNC] < flo
               flo=p[i,FUNC]
               ilo=i
             end  # if p[i,FUNC] < flo
             if p[i,FUNC] > fhi
               fhi=p[i,FUNC]
               ihi=i
```

```
    end # p[i,FUNC] > fhi
  end # for i=1:NPTS
  fnhi = flo
  inhi = ilo
  for i=1:NPTS
   if (i != ihi) && (p[i,FUNC] > fnhi)
     fnhi=p[i,FUNC]
     inhi=i
   end # if (i != ihi) && (p[i,FUNC] > fnhi)
  end # for i=1:NPTS
  ##### exit criteria #######
  ix=iter % (4*NDIMS);
  if ix == 1
     # calculate the average (including maximum value)
     pavg=0.0
     for i=1:NPTS
       pavg+=p[i,FUNC]
     end # for i=1:NPTS
     pavg/=NPTS
     tot=0.0
     for i=1:NPTS
       tot=(p[i,FUNC]-pavg)*(p[i,FUNC]-pavg)
     end #  for i=1:NPTS
     sterr=sqrt(tot/NPTS)
     for j=1:NDIMS
       z[j]=p[ilo,j]
     end # for j=1:NDIMS
     best = p[ilo,FUNC]
  end # if (iter % 4*NDIMS) == 0
  ##/ calculate avarage without maximum value ###
  ave = zeros(Float64,NDIMS)
  for i=1:NPTS
   if i != ihi
     for j=1:NDIMS
       ave[j] += p[i,j]
     end # for j=1:NDIMS
   end # if i != ihi
  end # for i=1:NPTS
  for j=1:NDIMS
     ave[j] /= (NPTS-1.0)
  end #  for j=1:NDIMS
  ####/ reflect ########
  for j=1:NDIMS
   r[j] = 2.0*ave[j] - p[ihi,j]
   r1=r[j]
   r2=ave[j]
   r3=p[ihi,j]
   #print("j = $j  r = $r1 ave = $r2 pihi,j = $r3\n")
  end # for j=1:NDIMS
  fr = fnelder(r)
  if (flo <= fr) && (fr < fnhi)  # in zone: accept
     for j=1:NDIMS
       p[ihi,j] = r[j]
     end # for j=1:NDIMS
     p[ihi,FUNC] = fr
     continue
  end # if (flo <= fr) && (fr < fnhi)
  if  fr < flo  ## expand
     for j=1:NDIMS
       e[j] = 3.0*ave[j] - 2.0*p[ihi,j]
     end # for j=1:NDIMS
     fe = fnelder(e)
     if fe < fr
       for j=1:NDIMS
         p[ihi,j] = e[j]
       end # for j=1:NDIMS
       p[ihi,FUNC] = fe
       continue
     else
       for j=1:NDIMS
```

```
                p[ihi,j] = r[j]
              end # for j=1:NDIMS
              p[ihi,FUNC] = fr
              continue
            end # if fe < fr
          end # if fr < flo

          ######/ shrink:
          if fr < fhi
            for  j=1:NDIMS
              c[j] = 1.5*ave[j] - 0.5*p[ihi,j]
            end # for  j=1:NDIMS
            fc = fnelder(c)
            if fc <= fr
              for  j=1:NDIMS
                p[ihi,j] = c[j]
              end # for  j=1:NDIMS
              p[ihi,FUNC] = fc
              continue
            else   ###/ shrink
              for i=1:NPTS
                if  i != ilo
                  for  j=1:NDIMS
                    p[i,j] = 0.5*p[ilo,j] + 0.5*p[i,j]
                    x1[j]=p[i,j]
                  end # for  j=1:NDIMS
                  p[i,FUNC] = fnelder(x1)
                end # if  i != ilo
              end # for i=1:NPTS
              continue
            end # if fc <= fr
          end # if fr < fhi
          if  fr >= fhi   #/
            for  j=1:NDIMS
              cc[j] = 0.5*ave[j] + 0.5*p[ihi,j]
            end # for  j=1:NDIMS
            fcc = fnelder(cc)
            if (fcc < fhi)
              for  j=1:NDIMS
                p[ihi,j] = cc[j]
              end # for  j=1:NDIMS
              p[ihi,FUNC] = fcc
              continue
            else    ####/
              for i=1:NPTS
                if (i != ilo)
                  for  j=1:NDIMS
                    p[i,j] = 0.5*p[ilo,j] + 0.5*p[i,j]
                    x1[j]=p[i,j]
                  end # for  j=1:NDIMS
                  p[i,FUNC] = fnelder(x1)
                end # if (i != ilo)
              end # for i=1:NPTS
            end # if (fcc < fhi)
          end #  if  fr >= fhi
          end  # for iter=1:maxiteration
      return z
 end

function  nelder(fnelder,a)
 n=length(a)
 da=zeros(Float64,n)
 for i=1:n
   dx=0.1*rand(Float64)
   da[i]=dx*a[i]
 end # for
return nelder(fnelder,a,da)
end # function

func(x)=(x[1]-2.0)*(x[1]-2.0)*(x[1]-2.0)*(x[1]-2.0)+(x[2]-1.0)*(x[2]-1.0)+
```

```
3.0*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)
a=[1.0 2 3]
z=nelder(func,a)
print("\nz = $z")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_nelder_mead.jl

z = [1.9998420725538413, 0.9999993740596262, 1.0116906510607162]
> Terminated with exit code 0.
```

Nelder & Mead simplex algorithm, Octave-matlab code

```
## Author: M. Turhan ÇOBAN
## Created: 2021-06-19
function retval = nelder_opt(f,a,da,maxiteration,tolerance)
   ## reflection
   alpha=1.0;
   ## contraction
   beta=0.5;
   ## expansion
   gamma=2.0;
   ## shrinkage
   delta=0.5;
   ## change basic multiplication parameters if needed
   n=length(a);
   m=n+1;
   NDIMS = n;
   NPTS = n+1;
   FUNC = n+1;
   ncalls = 0;
   p=zeros(m,m);
   x=zeros(m,n);
   for i=1:m
      for j=1:n
         if i==j
            x(i,j)=a(j)+da(j);
         else
            x(i,j)=a(j);
         endif
         p(i,j)=x(i,j);
      endfor
   endfor
   for i=1:m
     p(i,m)=f(x(i,:));
   endfor
   #Inlet variable definitions
   #fnelder  abstract multivariable function f(x)
   #x  independent variable set of n+1 simplex elements
   #maxiteration  maximum iteration number
   #tolerance

   #///// construct the starting simplex //////////////////
   best = 1.0E99;
   z=zeros(1,NDIMS);
   #/////////////// calculate the first function values for the simplex ////////////////
   iter=0;
   for iter=1:maxiteration
      ########  define lo, nhi, hi (low high next_to_high //////////////
      ilo=1;
      ihi=1;
      inhi = 0 ;
      # 0 means missing
      flo = p(1,FUNC);
      fhi = flo;
      for i=1:NPTS
         if p(i,FUNC) < flo
            flo=p(i,FUNC);
            ilo=i;
         endif
         if  p(i,FUNC) > fhi
            fhi=p(i,FUNC) ;
            ihi=i;
         endif
      endfor ## i=1:NPTS
```

```
##fprintf("fhi = %i  ihi = %i flo = %i ili= %i ",fhi,ihi,flo,ilo);
fnhi = flo;
inhi = ilo;
for i=1:NPTS
   if (i != ihi) && (p(i,FUNC) > fnhi)
      fnhi=p(i,FUNC);
      inhi=i;
   endif ##  (i != ihi) && (p(i,FUNC) > fnhi)
endfor ## i=1:NPTS
##fprintf("fnhi= %i inhi = %i\n",fnhi,inhi)
      #////////// exit criteria ////////////////
xx1=mod(iter,4);
xx2=xx1*NDIMS;
bx= (xx2==0);
if bx
  pavg=0.0;
  for i=1:NPTS
    pavg=pavg+p(i,FUNC);
  endfor
  pavg=pavg/NPTS;
  tot=0.0;
  for i=1:NPTS
    tot=(p(i,FUNC)-pavg)*(p(i,FUNC)-pavg);
  endfor
  sterr=sqrt(tot/NPTS);
  for j=1:NDIMS
      z(j)=p(ilo,j);
  endfor
  best = p(ilo,FUNC);
endif
# end of if iter%4*NDMS)==0
#//// calculate avarage without maximum value //////
for j=1:NDIMS
   ave(j) = 0;
endfor ## j=1:NDIMS
for i=1:NPTS
   if i != ihi
      for j=1:NDIMS
         ave(j) = ave(j)+p(i,j);
      endfor ## j=1:NDIMS
   endif # i != ihi
endfor ## i=1:NPTS
for j=1:NDIMS
   ave(j)=ave(j)/(NPTS-1);
end ##j=1:NDIMS
#//////// reflect ////////////////
xihi=ihi;
for j=1:NDIMS
   r(j) = ave(j)+alpha*(ave(j) - p(ihi,j));
endfor ## j=1:NDIMS
fr = f(r);
if (flo <= fr) && (fr < fnhi) # in zone accept reflection
   ##printf("reflect");
   for j=1:NDIMS
      p(ihi,j) = r(j);
   endfor ## j=1:NDIMS
   p(ihi,FUNC) = fr;
   continue;
endif ## flo <= fr) && (fr < fnhi)
#///(////////////// expand ///////////////
if fr < flo
   for j=1:NDIMS
      e(j) = ave(j)+gamma*(r(j)-ave(j));
   endfor ## j=1:NDIMS
   fe = f(e);
   if fe < fr
      ##fprintf("expand fe < fr");
      for j=1:NDIMS
         p(ihi,j) = e(j);
      endfor
      p(ihi,FUNC) = fe;
      continue;
   else
      ##fprintf("expand fe >= fr");
      for j=1:NDIMS
         p(ihi,j) = r(j);
```

```
            endfor ## j=1:NDIMS
            p(ihi,FUNC) = fr;
            continue;
          endif ## fe < fr
        endif ## fr < flo
        #/////////////// contract

        if fr >= fnhi
          if fr<fhi
            for j=1:NDIMS
              cc(j) = ave(j) + beta*(r(j)-ave(j));
            endfor ## j=1:NDIMS
            fcc = f(cc);
            if fcc <= fr ##outside
              ##fprintf("contract outside");
              for j=1:NDIMS
                 p(ihi,j) = cc(j);
              endfor ## j=1:NDIMS
              p(ihi,FUNC) = fcc;
              continue;
            endif ## fcc <= fr
          else ## fr>=fhi
            for j=1:NDIMS
              cc(j) = ave(j) + beta*(p(ihi,j)-ave(j));
            endfor ## j=1:NDIMS
            fcc = f(cc);
            if fcc<fhi
              ##fprintf("contract outside");
              for j=1:NDIMS
                 p(ihi,j) = cc(j);
              endfor ## j=1:NDIMS
              p(ihi,FUNC) = fcc;
              continue;
            endif ## fcc<fhi
          endif ## if fr<fhi else fr>=fhi
        endif ##  fr >= fnhi
        #////////// shrink
        ##fprintf("shrink");
              for i=1:NPTS
                 if i != ilo
                    for j=1:NDIMS
                       p(i,j) = p(i,j)+delta*(p(i,j)-p(ilo,j));
                    endfor
                    p(i,FUNC) = f(p(i));
                 endif
              continue;
              endfor
    endfor  ## iter=1:maxiteration
    retval =z;
endfunction
```

```
##  M. Turhan Coban
function y = f8(x)
  y=(x(1)-2.0)*(x(1)-2.0)*(x(1)-2.0)*(x(1)-2.0)+(x(2)-1.0)*(x(2)-1.0)+3.0*(x(3)-1.0)*(x(3)-1.0)*(x(3)-1.0)*(x(3)-1.0);;
endfunction
```

```
## Nelder optimisation test
y=@f8;
a=[1 1 1];
da=[0.123 .7654 0.1111];
maxiteration=200;
tolerance=1.0e-15;
nelder_opt(y,a,da,maxiteration,tolerance)
```

```
>> nelder_test
ans =

   2.000000031562442   0.999999999999997   1.000000012556213

>>
```

It should be note that Matlab-octave reference optimisation program fminsearch utilises Nelder & Mead simplex algorihm

Minimize a function of several variables

Syntax

- x = fminsearch(fun,x0)
- x = fminsearch(fun,x0,options)
- x = fminsearch(fun,x0,options,P1,P2,...)
- [x,fval] = fminsearch(...)
- [x,fval,exitflag] = fminsearch(...)
- [x,fval,exitflag,output] = fminsearch(...)

```
## Nelder optimisation
## fminsearch
y=@f8;
a=[1 1 1];
fminsearch(y,a)
```

```
>> nelder_test1
ans =

  1.999892735280956   1.000000049808283   0.999934132611535

>>
```

EXAMPLE PROBLEM

It is desired to find the minimum of function $\mathbf{f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1}$ with the initial value of {x}=[-2,2]

Function written in java :

Version 1: abstract class f_xj derived class f1, java code

```
 import javax.swing.JOptionPane;
class f1 extends f_xj
{
public double func(double x[])
{//optimization function
double ff;
ff=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
return ff;
}}

public class NA41B
{ public static void main (String args[])
    { f1 f=new f1();
    double [] x0=NA41.inputdata("enter optimum point first estimation values with one empty space in between: ");
    double [] r1=NA41.nelder(f,x0);
    String s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

Version 2: interface if_xj derived class f2, java code

```
import javax.swing.JOptionPane;
class f2 implements if_xj
{
public double func(double x[])
{
//optimization function
double ff;
ff=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
return ff;
}}
public class NA41C
{public static void main (String args[])
    {f2 f=new f2();
    double [] x0=NA41.inputdata("enter optimum point first estimation values with one empty space in between: ");
    double [] r1=NA41.nelder(f,x0);
    String s=" solution set  : \n"+Matrix.toStringT(r1);
```

```
    String s2="Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

Version 3: interface if_xj derived Lambda variable, java code

```
import javax.swing.JOptionPane;

public class NA41D
{public static void main (String args[])
    {if_xj f=(double x[])->3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
    double [] x0=NA41.inputdata("enter optimum point first estimation values with one empty space in between: ");
    double [] r1=NA41.nelder(f,x0);
    String s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

The results of first 50 iteration is recorded and results are plotted for first 17 iterations

Output 5.9-1 Sample Nelder-Mead output for the example program

```
iter=1p=
      -2.400000000000000       2.000000000000000       44.879999999999995
      -2.000000000000000       2.400000000000000       42.320000000000000
      -2.000000000000000       2.000000000000000       36.000000000000000
maximum y i=44.879999999999995 0 second maximum y i=42.32 1 minimum y i=36.0 2
iter=2p=
      -1.200000000000000       2.600000000000000       28.920000000000005
      -2.000000000000000       2.400000000000000       42.320000000000000
      -2.000000000000000       2.000000000000000       36.000000000000000
maximum y i=42.32 1 second maximum y i=36.0 2 minimum y i=28.920000000000005 0
iter=3p=
      -1.200000000000000       2.600000000000000       28.920000000000005
      -0.800000000000001       2.100000000000000       16.160000000000014
      -2.000000000000000       2.000000000000000       36.000000000000000
maximum y i=36.0 2 second maximum y i=28.920000000000005 0 minimum y i=16.160000000000014 1
iter=4p=
      -1.200000000000000       2.600000000000000       28.920000000000005
      -0.800000000000001       2.100000000000000       16.160000000000014
       0.999999999999999       3.050000000000002        5.355000000000019

maximum y i=28.920000000000005 0 second maximum y i=16.160000000000014 1 minimum y i=5.355000000000019 2
iter=5p=
       1.399999999999998       2.550000000000002        0.655000000000011
      -0.800000000000001       2.100000000000000       16.160000000000014
       0.999999999999999       3.050000000000002        5.355000000000019

maximum y i=16.160000000000014 1 second maximum y i=5.355000000000019 2 minimum y i=0.6550000000000114 0
iter=6p=
       1.399999999999998       2.550000000000002        0.655000000000011
       3.199999999999998       3.500000000000003        3.719999999999994
       0.999999999999999       3.050000000000002        5.355000000000019

maximum y i=5.355000000000019 2 second maximum y i=3.7199999999999944 1 minimum y i=0.6550000000000114 0
iter=7p=
       1.399999999999998       2.550000000000002        0.655000000000011
       3.199999999999998       3.500000000000003        3.719999999999994
       1.649999999999998       3.037500000000002        1.885312500000012

maximum y i=3.7199999999999944 1 second maximum y i=1.8853125000000115 2 minimum y i=0.6550000000000114 0
iter=8p=
       1.399999999999998       2.550000000000002        0.655000000000011
       2.362499999999998       3.146875000000002        1.302519531250004
       1.649999999999998       3.037500000000002        1.885312500000012

maximum y i=1.8853125000000115 2 second maximum y i=1.3025195312500042 1 minimum y i=0.6550000000000114 0
iter=9p=
       1.399999999999998       2.550000000000002        0.655000000000011
       2.362499999999998       3.146875000000002        1.302519531250004
```

```
    2.112499999999998    2.659375000000002    0.288925781250000

maximum y i=1.3025195312500042 1 second maximum y i=0.6550000000000114 0 minimum y i=0.2889257812500001 2
iter=10p=
    1.399999999999998    2.550000000000002    0.655000000000011
    1.149999999999998    2.062500000000001    -0.224687499999993
    2.112499999999998    2.659375000000002    0.288925781250000

maximum y i=0.6550000000000114 0 second maximum y i=0.2889257812500001 2 minimum y i=-0.22468749999999327 1
iter=11p=
    1.862499999999998    2.171875000000001    -0.374042968750002
    1.149999999999998    2.062500000000001    -0.224687499999993
    2.112499999999998    2.659375000000002    0.288925781250000

maximum y i=0.2889257812500001 2 second maximum y i=-0.22468749999999327 1 minimum y i=-0.3740429687500022 0
iter=12p=
    1.862499999999998    2.171875000000001    -0.374042968750002
    1.149999999999998    2.062500000000001    -0.224687499999993
    0.899999999999998    1.575000000000000    -0.753749999999998

maximum y i=-0.22468749999999327 1 second maximum y i=-0.3740429687500022 0 minimum y i=-0.7537499999999975 2
iter=13p=
    1.862499999999998    2.171875000000001    -0.374042968750002
    1.612499999999998    1.684375000000000    -0.686386718750004
    0.899999999999998    1.575000000000000    -0.753749999999998

maximum y i=-0.3740429687500022 0 second maximum y i=-0.6863867187500037 1 minimum y i=-0.7537499999999975 2
iter=14p=
    0.649999999999998    1.087499999999999    -0.932187499999998
    1.612499999999998    1.684375000000000    -0.686386718750004
    0.899999999999998    1.575000000000000    -0.753749999999998

maximum y i=-0.6863867187500037 1 second maximum y i=-0.7537499999999975 2 minimum y i=-0.9321874999999984 0
iter=15p=
    0.649999999999998    1.087499999999999    -0.932187499999998
    1.193749999999998    1.507812500000000    -1.079252929687500
    0.899999999999998    1.575000000000000    -0.753749999999998

maximum y i=-0.7537499999999975 2 second maximum y i=-0.9321874999999984 0 minimum y i=-1.0792529296874995 1
iter=16p=
    0.649999999999998    1.087499999999999    -0.932187499999998
    1.193749999999998    1.507812500000000    -1.079252929687500
    0.943749999999998    1.020312499999998    -1.061674804687500

maximum y i=-0.9321874999999984 0 second maximum y i=-1.0616748046875 2 minimum y i=-1.0792529296874995 1
iter=17p=
    0.859374999999998    1.175781249999999    -1.096405029296875
    1.193749999999998    1.507812500000000    -1.079252929687500
    0.943749999999998    1.020312499999998    -1.061674804687500
……..
iter=49p=
    1.000000969021338    1.250005597923464    -1.124999999956208
    0.999993292813279    1.249984113824493    -1.124999999786506
    1.000019269216091    1.250017992060081    -1.124999999625434

maximum y i=-1.1249999996254345 2 second maximum y i=-1.124999999786506 1 minimum y i=-1.1249999999562075 0
```

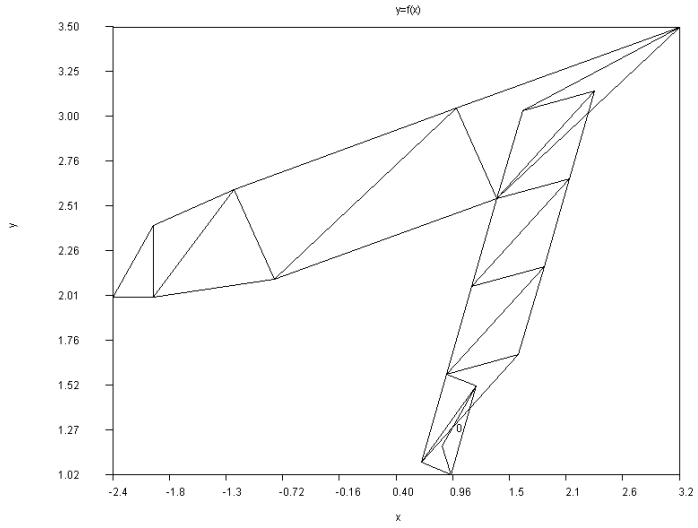
If we would like to see amobea movement graphically :

Program Graphic program to see amobe movement in the example, java code

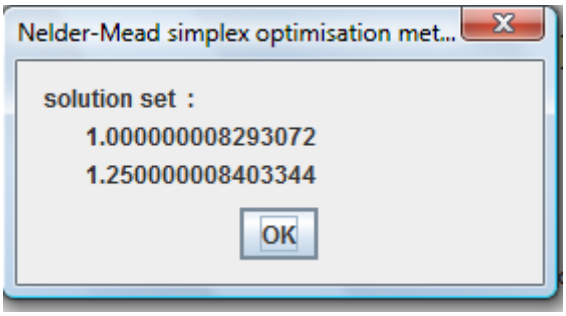
```
import java.io.*;

class PlotT01
{
   public static void main (String args[]) throws java.io.IOException
   {
            double x1[]={-2.4,-2,-2,-2.4};
            double y1[]={2,2.4,2,2};
            double x2[]={-1.2,-2,-2,-1.2};
            double y2[]={2.6,2.4,2,2.6};
```

```
            double x3[]={-1.2,-0.8,-2,-1.2};
            double y3[]={2.6,2.1,2,2.6};
            double x4[]={-1.2,-0.8,1,-1.2};
            double y4[]={2.6,2.1,3.05,2.6};
            double x5[]={1.4,-0.8,1,1.4};
            double y5[]={2.55,2.1,3.05,2.55};
            double x6[]={1.4,3.2,1,1.4};
            double y6[]={2.55,3.5,3.05,2.55};
            double x7[]={1.4,3.2,1.65,1.4};
            double y7[]={2.55,3.5,3.0375,2.55};
            double x8[]={1.4,2.3625,1.65,1.4};
            double y8[]={2.55,3.146875,3.0375,2.55};
            double x9[]={1.4,2.3625,2.1125,1.4};
            double y9[]={2.55,3.146875,2.659375,2.55};
            double x10[]={1.4,1.15,2.1125,1.4};
            double y10[]={2.55,2.0625,2.659375,2.55};
            double x11[]={1.8625,1.15,2.1125,1.8625};
            double y11[]={2.171875,2.0625,2.659375,2.171875};
            double x12[]={1.8625,1.15,0.9,1.8625};
            double y12[]={2.171875,2.0625,1.575,2.171875};
            double x13[]={1.8625,1.6125,0.9,1.8625};
            double y13[]={2.171875,1.684375,1.575,2.171875};
            double x14[]={0.65,1.6125,0.9,0.65};
            double y14[]={1.0874,1.684375,1.575,1.0874};
            double x15[]={0.65,1.19375,0.9,0.65};
            double y15[]={1.0874,1.5078125,1.575,1.0874};
            double x16[]={0.65,1.19375,0.94375,0.65};
            double y16[]={1.0874,1.5078125,1.0203125,1.0874};
            double x17[]={0.859375,1.19375,0.94375,0.859375};
            double y17[]={1.17578125,1.5078125,1.0203125,1.17578125};
            Plot pp=new Plot(x1,y1);
   pp.addData(x2,y2);
   pp.addData(x3,y3);
   pp.addData(x4,y4);
   pp.addData(x5,y5);
   pp.addData(x6,y6);
   pp.addData(x7,y7);
   pp.addData(x8,y8);
   pp.addData(x9,y9);
   pp.addData(x10,y10);
   pp.addData(x11,y11);
   pp.addData(x12,y12);
   pp.addData(x13,y13);
   pp.addData(x14,y14);
   pp.addData(x15,y15);
   pp.addData(x16,y16);
   pp.addData(x17,y17);
    pp.plot();
    }
}
```

Nelder-Mead actual amobea movement for the example function



result :



EXAMPLE PROBLEM

As a second example factory space selection problem is taken. In this problem, every city has a number of potential customer. In order to minimize transportation cost of the product new factory should be located in a place minimizing the following function

$$f = \sum_{i=1}^{n} N_i[(x - x_i)^2 + (y - y_i)^2]$$

Where N_i population of the city i and , x_i and y_i are location of the city. x,y are the location of the factory. Following city location function is given as a test function in file city.txt

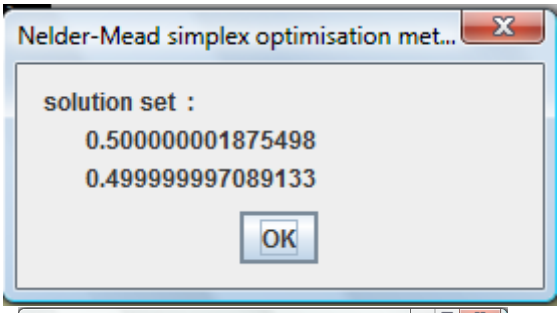
Population	X coordinate of the city	Y coordinate of the city
10	0	0
10	0	1
10	1	0
10	1	1

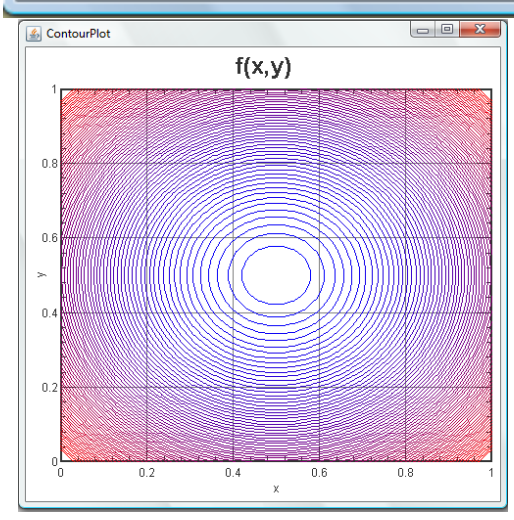
It is clear that optimum point for such a data will be located at [0.5,0.5]. In the example program nelder mead method is taken from class NA41

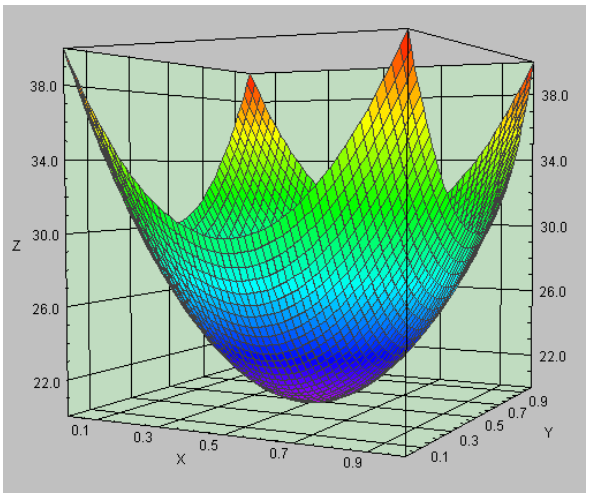
Program factory placement optimization, java code

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
class ff1 extends f_xj
{
double a[][];
public ff1(String s)
```

```
{a=Text.readDoubleT(s);}

public double func(double x[])
{
//optimization function
double ff=0;
int n=a[0].length;
for(int i=0;i<n;i++)
  ff+=a[0][i]*((x[0]-a[1][i])*(x[0]-a[1][i])+(x[1]-a[2][i])*(x[1]-a[2][i]));
return ff;
}
}

public class NA42
{   public static void main (String args[])
    {    String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=NA41.inputdata(s);
    double [] dx0=new double[x0.length];
    for(int i=0;i<x0.length;i++) dx0[i]=0.2*x0[i];
    ff1 f=new ff1("city.txt");
    double [] r1= NA41.nelder(f,x0);
    s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



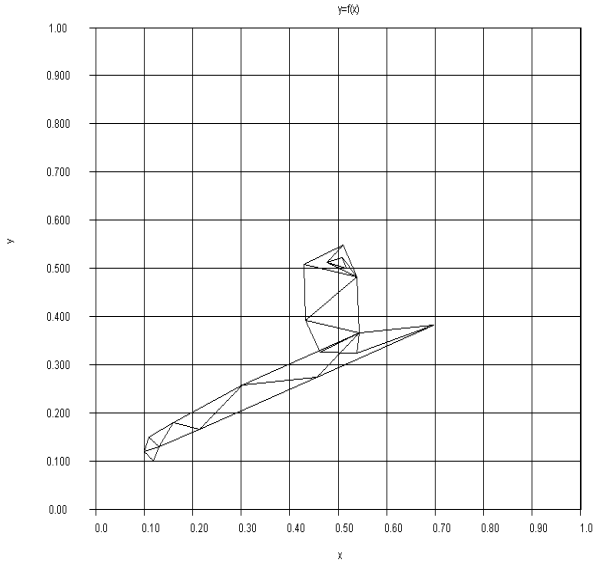





```
---------- Capture Output ----------
> "C:\co\java\bin\java.exe" NA41
iter=1p=
       0.120000000000000       0.100000000000000       32.176000000000000
       0.100000000000000       0.120000000000000       32.176000000000000
       0.100000000000000       0.100000000000000       32.800000000000004
maximum y i=32.800000000000004 2 second maximum y i=32.176 0 minimum y i=32.176 0
iter=2p=
```

```
    0.120000000000000    0.100000000000000    32.176000000000000
    0.100000000000000    0.120000000000000    32.176000000000000
    0.130000000000000    0.130000000000000    30.951999999999990
maximum y i=32.176 0 second maximum y i=32.176 1 minimum y i=30.95199999999999 2
iter=3p=
    0.110000000000000    0.150000000000000    30.983999999999995
    0.100000000000000    0.120000000000000    32.176000000000000
    0.130000000000000    0.130000000000000    30.951999999999990
maximum y i=32.176 1 second maximum y i=30.983999999999995 0 minimum y i=30.95199999999999 2
iter=4p=
    0.110000000000000    0.150000000000000    30.983999999999995
    0.160000000000000    0.180000000000000    28.719999999999995
    0.130000000000000    0.130000000000000    30.951999999999990
maximum y i=30.983999999999995 0 second maximum y i=30.95199999999999 2 minimum y i=28.719999999999995 1
iter=5p=
    0.215000000000000    0.165000000000000    27.737999999999985
    0.160000000000000    0.180000000000000    28.719999999999995
    0.130000000000000    0.130000000000000    30.951999999999990
maximum y i=30.95199999999999 2 second maximum y i=28.719999999999995 1 minimum y i=27.737999999999985 0
iter=6p=
    0.215000000000000    0.165000000000000    27.737999999999985
    0.160000000000000    0.180000000000000    28.719999999999995
    0.302500000000000    0.257500000000001    23.912499999999984
maximum y i=28.719999999999995 1 second maximum y i=27.737999999999985 0 minimum y i=23.912499999999984 2
iter=7p=
    0.215000000000000    0.165000000000000    27.737999999999985
    0.456250000000000    0.273750000000001    22.124124999999978
    0.302500000000000    0.257500000000001    23.912499999999984
maximum y i=27.737999999999985 0 second maximum y i=23.912499999999984 2 minimum y i=22.124124999999978 1
iter=8p=
    0.543750000000001    0.366250000000001    20.792124999999988
    0.456250000000000    0.273750000000001    22.124124999999978
    0.302500000000000    0.257500000000001    23.912499999999984
maximum y i=23.912499999999984 2 second maximum y i=22.124124999999978 1 minimum y i=20.792124999999988 0
iter=9p=
    0.543750000000001    0.366250000000001    20.792124999999988
    0.456250000000000    0.273750000000001    22.124124999999978
    0.697500000000001    0.382500000000002    22.112499999999997
maximum y i=22.124124999999978 1 second maximum y i=22.112499999999997 2 minimum y i=20.792124999999988 0
iter=10p=
    0.543750000000001    0.366250000000001    20.792124999999988
    0.538437500000001    0.324062500000001    21.297257812499986
    0.697500000000001    0.382500000000002    22.112499999999997
maximum y i=22.112499999999997 2 second maximum y i=21.297257812499986 1 minimum y i=20.792124999999988 0
iter=11p=
    0.543750000000001    0.366250000000001    20.792124999999988
    0.538437500000001    0.324062500000001    21.297257812499986
    0.462890625000000    0.326484375000001    21.259391113281236
maximum y i=21.297257812499986 1 second maximum y i=21.259391113281236 2 minimum y i=20.792124999999988 0
iter=12p=
    0.543750000000001    0.366250000000001    20.792124999999988
    0.433085937500000    0.390976562500001    20.654544067382805
    0.462890625000000    0.326484375000001    21.259391113281236
maximum y i=21.259391113281236 2 second maximum y i=20.792124999999988 0 minimum y i=20.654544067382805 1
iter=13p=
    0.543750000000001    0.366250000000001    20.792124999999988
    0.433085937500000    0.390976562500001    20.654544067382805
    0.539472656250001    0.482871093750001    20.074059600830080
maximum y i=20.792124999999988 0 second maximum y i=20.654544067382805 1 minimum y i=20.07405960083008 2
iter=14p=
    0.428808593750000    0.507597656250001    20.205037628173827
    0.433085937500000    0.390976562500001    20.654544067382805
    0.539472656250001    0.482871093750001    20.074059600830080
maximum y i=20.654544067382805 1 second maximum y i=20.205037628173827 0 minimum y i=20.07405960083008 2
iter=15p=
    0.428808593750000    0.507597656250001    20.205037628173827
    0.509667968750000    0.547363281250001    20.093470001220710
    0.539472656250001    0.482871093750001    20.074059600830080
maximum y i=20.205037628173827 0 second maximum y i=20.09347000122071 1 minimum y i=20.07405960083008 2
iter=16p=
```

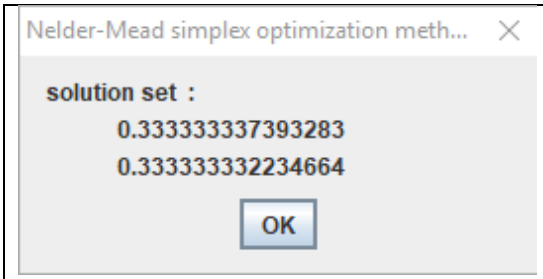
```
    0.476689453125000   0.511357421875001   20.026894905090334
    0.509667968750000   0.547363281250001   20.093470001220710
    0.539472656250001   0.482871093750001   20.074059600830080
maximum y i=20.09347000122071 1 second maximum y i=20.07405960083008 2 minimum y i=20.026894905090334 0
iter=17p=
    0.476689453125000   0.511357421875001   20.026894905090334
    0.508874511718750   0.522238769531251   20.022932793140413
    0.539472656250001   0.482871093750001   20.074059600830080
maximum y i=20.07405960083008 2 second maximum y i=20.026894905090334 0 minimum y i=20.022932793140413 1
iter=18p=
    0.476689453125000   0.511357421875001   20.026894905090334
    0.508874511718750   0.522238769531251   20.022932793140413
    0.516127319335938   0.499834594726564   20.010404711514713
maximum y i=20.07405960083008 2 second maximum y i=20.026894905090334 0 minimum y i=20.022932793140413 1
iter=52p=
    0.500002657994614   0.499999541116936   20.000000000291020
    0.499999313009312   0.500000148257349   20.000000000019753
    0.499999853685161   0.499998268210034   20.000000000120820
maximum y i=20.00000000029102 0 second maximum y i=20.00000000012082 2 minimum y i=20.000000000019753 1
> Terminated with exit code 0.
```

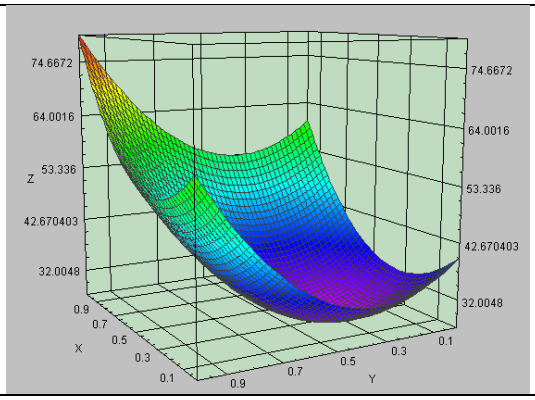


What will happen if the population of one of the city is different, for example

Population	X coordinate of the city	Y coordinate of the city
30	0	0
10	0	1
10	1	0
10	1	1

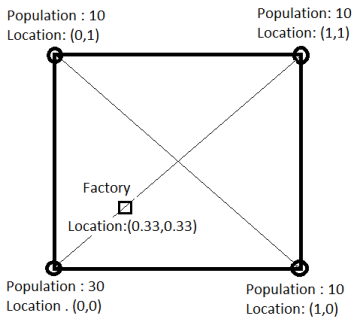
Then optimum point will move to:





What will happen if the population of one of the city is different, for example

Population	X coordinate of the city	Y coordinate of the city
30	0	0
10	0	1
10	1	0
10	1	1

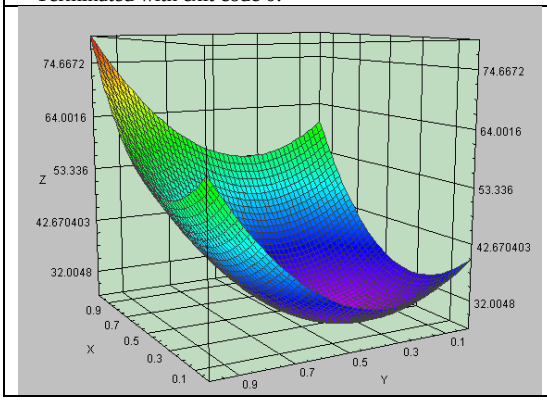


Then optimum point will move to:

Nelder-mead, julia

```
function func(x)
n=4
a=[30.0 10 10 10;0.0 0 1 1;0.0 1 0 1]
ff=0.0
for i=1:n
  ff+=a[1,i]*((x[1]-a[2,i])*(x[1]-a[2,i])+(x[2]-a[3,i])*(x[2]-a[3,i]))
end
return ff
end # function


a=[0.1 0.1]
z=nelder(func,a)
print("\nz = $z")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_nelder_mead.jl

z = [0.33333333193655074, 0.33333333362547546]
> Terminated with exit code 0.
```



Let us look at a realistic version of the example problem. Cities Latitude and longitudes of Turkish cities. What will be ideal place to locate the factory for minimising the transportation cost. It is assumed that each person has the same purchasing potential

The optimum factory location in Turkey:

Place Name	2021 Population	Latitude	Longitude
Adana, Turkey	1248900	37	35.321335

Afyonkarahisar, Turkey	146136	38.756886	30.538704
Amasya, Turkey	82896	40.652382	35.828819
Ankara, Turkey	3517182	39.925533	32.866287
Antalya, Turkey	758188	36.884804	30.704044
Artvin, Turkey	24266	41.183224	41.830982
Bolu, Turkey	96629	40.731647	31.589813
Bursa, Turkey	1412701	40.193298	29.074202
Denizli, Turkey	313238	37.783333	29.094715
Diyarbakir, Turkey	644763	37.91	40.240002
Edirne, Turkey	126470	41.674965	26.583481
Elazığ, Turkey	298004	38.680969	39.226398
Eskişehir, Turkey	514869	39.766193	30.526714
Gaziantep, Turkey	1065975	37.066666	37.383331
Giresun, Turkey	98864	40.917534	38.392654
Hatay, Turkey	210000	36.200001	36.166668
Isparta, Turkey	172334	37.768002	30.561905
Istanbul, Turkey	14804116	41.015137	28.97953
Izmir, Turkey	2500603	38.423733	27.142826
Izmit, Kocaeli, Turkey	196571	40.766666	29.916668
Kahramanmaras, Turkey	376045	37.575275	36.922821
Kayseri, Turkey	592840	38.734802	35.467987
Konya, Turkey	875530	37.874641	32.493156
Malatya, Turkey	441805	38.356869	38.309669
Manisa, Turkey	243971	38.630554	27.422222
Mersin, Turkey	537847	36.812103	34.641479
Nevşehir, Turkey	75527	38.626995	34.719975
Rize, Turkey	92772	41.025513	40.517666
Sakarya, Turkey	1042649	40.783333	30.4
Sanliurfa, Turkey	449549	37.158333	38.791668
Sivas, Turkey	264022	39.750359	37.015598
Tekirdag, Turkey	122287	40.977779	27.515278
Trabzon, Turkey	255083	41.002697	39.716763
Van, Turkey	371713	38.499817	43.378143
Yozgat, Turkey	87881	39.82206	34.808132
Zonguldak, Turkey	100229	41.451733	31.791344

Nelder-mead, julia

```
function func(x)
n=36
a=[1248900 146136 82896 3517182 758188 24266 96629 1412701 313238 644763 126470
298004 514869 1065975 98864 210000 172334 14804116 2500603 196571 376045 592840
875530 441805 243971 537847 75527 92772 1042649 449549 264022 122287 255083
371713 87881 100229;
37 38.756886 40.652382 39.925533 36.884804 41.183224 40.731647 40.193298 37.783333 37.91 41.674965 38.680969
39.766193 37.066666 40.917534 36.200001 37.768002 41.015137 38.423733 40.766666 37.575275 38.734802 37.874641
38.356869 38.630554 36.812103 38.626995 41.025513 40.783333 37.158333 39.750359 40.977779 41.002697 38.499817
39.82206 41.451733;
35.321335 30.538704 35.828819 32.866287 30.704044 41.830982 31.589813 29.074202 29.094715 40.240002 26.583481 39.226398
30.526714 37.383331 38.392654 36.166668 30.561905 28.97953 27.142826 29.916668 36.922821 35.467987 32.493156
38.309669 27.422222 34.641479 34.719975 40.517666 30.4 38.791668 37.015598 27.515278 39.716763 43.378143
34.808132 31.791344]
ff=0.0
for i=1:n
  ff+=a[1,i]*((x[1]-a[2,i])*(x[1]-a[2,i])+(x[2]-a[3,i])*(x[2]-a[3,i]))
end
return ff
end # function


a=[0.1 0.1]
z=nelder(func,a)
print("\n[Latitude,Longitude] = ",z)
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_nelder_mead.jl
[Latitude,Longitude] = [39.75600091923573, 31.245720445008125]
```

```
> Terminated with exit code 0.
```



Sakarya(40.70,30.4) could be the ideal place to locate the factory. Other alternatives will be Eskisehir(38.68,30.52) or Ankara(39.92,32.87)

Julia has optimisation library as a package, This package can be use to carry out Nelder-Mead optimisation Let us utilise the program now, to use much more cities anddifferent populations for each cities

Nelder-mead, julia

```
using Pkg; Pkg.add("Optim")
```

```
# 6-2. Nelder and Mead Simplex multivariable nonlinear optimization method
#  Nelder & Mead 1965 Computer J, v.7, 308-313.
using Optim
func(x)=(x[1]-2.0)*(x[1]-2.0)*(x[1]-2.0)*(x[1]-2.0)+(x[2]-1.0)*(x[2]-1.0)+
3.0*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)*(x[3]-1.0)
a=[1.0,2.0,3.0]
z=optimize(func,a, NelderMead())
print(z)
x=Optim.minimizer(z)
print("x = $x")
#y=func(x)
y=Optim.minimum(z)
print("\nminimum  = $y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_nelder_mead3.jl
x = [1.991433188031417, 0.9999753150134735, 0.9683777585648278]
minimum  = 8.9951752856918e-9
> Terminated with exit code 0.
```

```
include("opt_nelder_mead3.jl")
 * Status: success

 * Candidate solution
   Final objective value:     8.995175e-09

 * Found with
   Algorithm:     Nelder-Mead

 * Convergence measures
   √(Σ(yᵢ-ȳ)²)/n ≤ 1.0e-08

 * Work counters
   Seconds run:   0  (vs limit Inf)
   Iterations:    65
   f(x) calls:    122
x = [1.991433188031417, 0.9999753150134735, 0.9683777585648278]
minimum  = 8.9951752856918e-9
```

NON-LINEAR MULTIDIMENSIONAL STEEPEST DESCENT METHOD

For the n dimensional non lineear f(x_0,x_1,x_2,…,x_n) function , we would like to find minimum value of the function. The General definition of the Newton –Raphson approach can be written as:

$$\{x^{m+1}\} = \{x^m\} + \alpha^m [H^m]^{-1}[-\{\nabla f^m\}]$$

where

$$\nabla f(x_1, x_2, x_3, .., x_n) = \begin{Bmatrix} \frac{\partial f(x_1,x_2,x_3,..,x_n)}{\partial x_1} \\ \frac{\partial f(x_1,x_2,x_3,..,x_n)}{\partial x_2} \\ \frac{\partial f(x_1,x_2,x_3,..,x_n)}{\partial x_3} \\ \cdots \\ \frac{\partial f(x_1,x_2,x_3,..,x_n)}{\partial x_n} \end{Bmatrix} \text{ and } [H] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \frac{\partial^2 f}{\partial x_3 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

is Hessian Matrix, α is a step function. In steepest descent method the value of inverse hessian, $[H]^{-1}$ is taken as unit vector [I] Then the definition of the equation becomes:

$$\{x^{m+1}\} = \{x^m\} + \alpha^m [-\{\nabla f^m\}]$$

In this method, As a first action a starting point for the iteration will be selected. If the starting function is P_0

$P^0 = f(x_1^0, x_2^0, x_3^0, .., x_n^0)$

In each iteration step values of the function and derivatives are calculated.

$d^k = -\nabla f(x_1^k, x_2^k, x_3^k, .., x_n^k)$ k=0..n

Then for each x value is increased by d^k with an unknown multiplier α. In the new equation as only unkown remains is unknown multiplier α (so equation converted to a one dimensional equation as a function of α)

$f(\alpha^k) = f(x^k + \alpha^k d^k) = f\left(\left[x_1^k - \alpha^k \frac{\partial f(x_1^k,x_2^k,x_3^k,..,x_n^k)}{\partial x_1}\right], \left[x_2^k - \alpha^k \frac{\partial f(x_1^k,x_2^k,x_3^k,..,x_n^k)}{\partial x_2}\right], \ldots, \left[x_n^k - \alpha^k \frac{\partial f(x_1^k,x_2^k,x_3^k,..,x_n^k)}{\partial x_n}\right]\right)$

α^k value minimizing function $f(\alpha^k)$ is found by using one of the one dimensional minimisation methods. The new x^{k+1} value calculated by using

$$x^{k+1} = x^k + \alpha^k d^k$$

Process continues until the error is small.

EXAMPLE PROBLEM

Find minimum value of function $f(x_0, x_1) = 3x_0^2 - 4x_0x_1 + 2x_1^2 - x_0 - x_1$with the initial value of $\{x^0\} = \begin{Bmatrix} x_0^0 \\ x_1^0 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$ by using steepest descent method.

$$\nabla f(x_0, x_1) = \begin{Bmatrix} 6x_0^0 - 4x_1^0 - 1 \\ -4x_0^0 + 4x_1^0 - 1 \end{Bmatrix} = \begin{Bmatrix} -21 \\ 15 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} x_0^0 \\ x_1^0 \end{Bmatrix} + \alpha^0 \begin{Bmatrix} 6x_0^0 - 4x_1^0 - 1 \\ -4x_0^0 + 4x_1^0 - 1 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix} + \alpha^0 \begin{Bmatrix} -21 \\ 15 \end{Bmatrix} = \begin{Bmatrix} -2 - 21\alpha^0 \\ 2 + 15\alpha^0 \end{Bmatrix}$$

If these values are substituted into the function

$f(x_0, x_1) = 3x_0^2 - 4x_0x_1 + 2x_1^2 - x_0 - x_1$ it becomes

$f(\alpha^0) = 3(-2 - 21\alpha^0)^2 - 4(-2 - 21\alpha^0)(2 + 15\alpha^0) + 2(-2 - 21\alpha^0)^2 - (-2 - 21\alpha^0) - (2 + 15\alpha^0)$

$f(\alpha^0) = 36 + 666\alpha^0 + 3033\alpha_0^2$

In order to find minimum of this function, analitical solution will be used in this example

$f'^{(\alpha^0)} = 666 + 6066\alpha^0 = 0 \;\; \alpha^0 = -0.109792284$

If this value is substituted into x_0^1 and x_1^1

For α^0=-0.109792284 $\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} 0.3056379 \\ 0.35311572 \end{Bmatrix}$

For α^1=-1.121212 $\begin{Bmatrix} x_0^2 \\ x_1^2 \end{Bmatrix} = \begin{Bmatrix} 0.9544105 \\ 1.2613973 \end{Bmatrix}$

For α^2=-0.109792284 $\begin{Bmatrix} x_0^3 \\ x_1^3 \end{Bmatrix} = \begin{Bmatrix} 0.9894 \\ 1.2363 \end{Bmatrix}$

For α^3=-1.121212 $\begin{Bmatrix} x_0^4 \\ x_1^4 \end{Bmatrix} = \begin{Bmatrix} 0.9993 \\ 1.250173 \end{Bmatrix}$

The same problem is also solved in the algorithm

Program steepest descent algorithm , java code

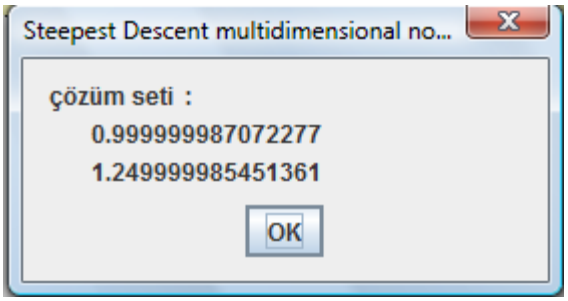
```
 // Multidimensional non-Linear optimization
// Steepest descent method
//
import javax.swing.*;
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import java.io.*;


class f1dim implements if_x
{
  // pseudofunction to convert multidimensional function to
  // one dimensional function
  if_xj ff1;
  double pc[];
  double xc[];
  double xt[];
  int n;
  public f1dim(if_xj ff,double pi[],double xi[])
  { n=pi.length;
    ff1=ff;
            pc=new double[n];
    xc=new double[n];
    xt=new double[n];
            for(int i=0;i<n;i++)
              {pc[i]=pi[i];xc[i]=xi[i];}
  }

  public double func(double x)
  {
           for (int j=0;j<n;j++) xt[j]=pc[j]+x*xc[j];
           return ff1.func(xt);
  }
}
public class NA43_i
{
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}

// 3. brent one dimensional optimization method

public  static double[] brentf(if_x f,double ax,double bx,double cx,double tol)
{
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altin oran
double ZEPS=1.0e-10;
```

```
double xmin;
double aa[]=new double[2];
        int iter;
        double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
        double e=0.0;
        d=0.0;
        a=((ax < cx) ? ax : cx);
        b=((ax > cx) ? ax : cx);
        x=w=v=bx;
        fw=fv=fx=f.func(x);
        for (iter=1;iter<=ITMAX;iter++) {
                xm=0.5*(a+b);
                tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                        xmin=x;
            aa[0]=xmin;
            aa[1]=fx;
            return aa;
                }
                if (Math.abs(e) > tol1) {
                        r=(x-w)*(fx-fv);
                        q=(x-v)*(fx-fw);
                        p=(x-v)*q-(x-w)*r;
                        q=2.0*(q-r);
                        if (q > 0.0) p = -p;
                        q=Math.abs(q);
                        etemp=e;
                        e=d;
                        if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        else {
                                d=p/q;
                                u=x+d;
                                if (u-a < tol2 || b-u < tol2)
                                        d=SIGN(tol1,xm-x);
                        }
                } else {
                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                }
                u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                fu=f.func(u);
                if (fu <= fx) {
                        if (u >= x) a=x; else b=x;
                        {v=w;w=x;x=u;}
                        {fv=fw;fw=fx;fx=fu;}
                } else {
                        if (u < x) a=u; else b=u;
                        if (fu <= fw || w == x) {
                                v=w;
                                w=u;
                                fv=fw;
                                fw=fu;
                        } else if (fu <= fv || v == x || v == w) {
                                v=u;
                                fv=fu;
                        }
                }
        }
        System.out.println("maximum iteration limit in BRENT method is exceeded");
        xmin=x;
        aa[0]=xmin;
        aa[1]=fx;
```

```
            return aa;
}

public  static double brent(if_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}

// 3. Steepest descent
//_____________________________________________________________________
public static double[] steepestdescent( if_xj f,double x[])
{
// steepestdescent method
// uses linmin and brent methods
double ti=1.0;
int i,ii,jj;
int nmax=500;
double tolerance=1.0e-15;
int n=x.length;
double b[];
b=new double[n];
double lambda;
double dy[]=new double[n];
i=0;
for(i=0;i<n;i++)
{
b[i]=1.0;
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{
   dy=f.dfunc(x);
   lambda=linmin(f,x,dy);
   b=Matrix.multiply(dy,lambda);
   x=Matrix.add(x,b);
   System.out.println("dy="+Matrix.toString(dy));
   System.out.println("i= "+i+"alfa="+lambda+"\n"+Matrix.toString(x));
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
  " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}
public  static double[] mnbrak(if_x f,double ax,double dx)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçilen uzaklik
return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(if_x f,double ax)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçiln uzaklik
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(if_x f,double ax,double bx,double cx)
{
// ax,bx,cx üç adet tahmin noktasi verildiğinde içinde minimum bulunan ax,bx,cx üç adetnoktayi bize arama
// sonucu bulur
// f : fonksiyon (f_x abstract sinifinda tanimlanmiş)
```

```
// tol tolerans
int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
        double ulim,u,r,q,fu,dum;
        double aa[]=new double[3];
        fa=f.func(ax);
        fb=f.func(bx);
        if (fb > fa) {
                //SHFT(dum,*ax,*bx,dum)
                {dum=ax;ax=bx;bx=dum;}
                //SHFT(dum,*fb,*fa,dum)
                {dum=fb;fb=fa;fa=dum;}
        }
        cx=(bx)+GOLD*(bx-ax);
        fc=f.func(cx);
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        while (fb > fc) {
                r=(bx-ax)*(fb-fc);
                q=(bx-cx)*(fb-fa);
                u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                        u=ulim;
                        fu=f.func(u);
                } else {
                        u=cx+GOLD*(cx-bx);
                        fu=f.func(u);
                }
                //SHFT(ax,bx,cx,u)
```

```
                        {ax=bx;bx=cx;cx=u;}
                        //SHFT(fa,fb,fc,fu)
                        {fa=fb;fb=fc;fc=fu;}
                        aa[0]=ax;
                        aa[1]=bx;
                        aa[2]=cx;
            }
                        return aa;
}
public static double linmin(if_xj f,double p[],double xi[])
{
            double tol=2.0e-4;
   int n=p.length;
   double pcom[],xicom[];
            int j;
            double xx,xmin,bx,ax;
            f1dim f1=new f1dim(f,p,xi);
            ax=0.0;
            xx=1.0;
            bx=2.0;
            double aa[]=mnbrak(f1,ax,xx,bx);
            ax=aa[0];
            xx=aa[1];
            bx=aa[2];
            xmin=brent(f1,ax,xx,bx,tol);
            return xmin;
}
//input data method
//_____________________________________________________________________
    public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
    public static void main (String args[])
    {
    String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=inputdata(s);
    if_xj f=(double x[])->3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
    double [] r1= steepestdescent(f,x0);
    s=" çözüm seti  : \n"+Matrix.toStringT(r1);
    String s2="Steepest Descent multidimensional non-linear optimization method:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    } }
```



Program steepest descent algorithm (Python version)

```
 from math import *
from f_xj import *;
from f_x import *;
from gauss import *;

class f1(f_xj):
  def func(self,x):
     y=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
     return y;

class f1dim(f_x):
   def __init__(self,ff,pi,xi):
     self.name = 'f1dim';
     self.ff1=ff;
     self.pc=pi;
     self.xc=xi;
   def func(self,x):
       n=len(self.pc);
      xt = [0.0 for ii in range(n)];
      for j in range(n):
          xt[j]=self.pc[j]+x*self.xc[j];
      return self.ff1.func(xt);

def linmin(f,p,xi):
   f1=f1dim(f,p,xi);
   xi=1.0;
   xmin=newton(f1,xi);
   return xmin;

def multiply_c_v(left,right):
   #multiplying a vector with a constant
   n=len(right);
   b = [0.0 for i in range(n)];
   for i in range(n):
      b[i]=left*right[i];
   return b;

def add(left,right):
   #addition of two vectors
   n1=len(left);
   n2=len(right);
   if n1>=n2: nMax=n1;
   else     : nMax=n2;
   b = [0.0 for i in range(nMax)];
   for i in range(n1):
      b[i]=b[i]+left[i];
   for i in range(n2):
      b[i]=b[i]+right[i];
   return b;

def steepestdescent(f,x):
   # steepestdescent method
   ti=1.0;
   nmax=100;
   tolerance=1.0e-10;
   n=len(x);
   b = [0.0 for ii in range(n)];
   dy = [0.0 for ii in range(n)];
   i=0;
   total=0;
   for i in range(n):
```

```
        b[i]=1.0;
        total+=abs(b[i]);
    i=0;
    while i<nmax and total>tolerance:
        dy=f.dfunc(x);
        lm=linmin(f,x,dy);
        b= multiply_c_v(lm,dy);
        x=add(x,b);
        total=0;
        for j in range(n):
            total+=abs(b[j]);
        i=i+1;
    return x;

def newton(f,x):
   # Newton-Raphson optimisation method
   # single variable
  nmax=100
  tolerance=1.0e-10
  for i in range(0,nmax):
      fx= f.dfunc(x,1);
      dfx=f.dfunc(x,2);
      x=x-fx/dfx
      if abs(fx)<tolerance: return x
  return x;

x=[1,1];
f=f1();
r=steepestdescent(f,x);
print("r=",r);
```

```
runfile('D:/okul/SCO1/steepest_descent_opt.py', wdir='D:/okul/SCO1')
Reloaded modules: f_xj, f_x, gauss
r= [1.0000095779092373, 1.2500087681731902]
```

Program steepest descent algorithm, Julia version

```
 # Multidimensional non-linear steepest descent algorithm

# one dimensional first derivative
#first derivative
function df(f,x)
 dx=0.001
 c=[1.0/280.0 -4.0/105.0        1.0/5.0     -4.0/5.0   0            4.0/5.0     -1.0/5.0   4.0/105.0 -1.0/280.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/dx
end

#one dimensional second derivative
function d2f(f,x)
 dx=0.001
 c=[-1.0/560.0         8.0/315.0 -1.0/5.0 8.0/5.0 -205.0/72.0 8.0/5.0 -1.0/5.0 8.0/315.0 -1.0/560.0]
 ff=0.0
 for i=-4:4
   ff+=c[i+5]*f(x+dx*i)
 end
 return ff/(dx*dx)
end

# multideimensional first derivative
# first derivative
function dfunc(func,x)
 global  dx=0.001
 global c=[1.0/280.0 -4.0/105.0            1.0/5.0     -4.0/5.0  0            4.0/5.0     -1.0/5.0   4.0/105.0 -1.0/280.0]
 n=length(x)
```

```
 ff=zeros(Float64,n)
 global yy=ff
 x1=zeros(Float64,n)
   for ix=1:n
     for i=-4:4
       # create x1set for derivative input
       # ================================
       for k=1:n
         xx=x[k]
         if k!=ix
           x1[k]=xx
         else
           delta=Float64(i)*dx
           x1[k]=xx+delta
         end
       end # for k
       # ================================
       ff1=func(x1)
       ff[ix]+=c[i+5]*ff1
       yx=ff[ix]
     end # for i
     ff[ix]/=dx
     yx=ff[ix]
   end # for ix
 return ff
end # for function

# one dimensional reduced function
# to be utilised in one dimensional optimisation routine
function f1dim(ff1,pc,xc,x)
   n=length(pc)
   xt =zeros(Float64,n)
   for j=1:n
      xt[j]=pc[j]+x*xc[j]
   end # for
   yy=ff1(xt)
   return yy
end # function

# minimisation function
function linmin(ff1,pc,xc)
   f1(x)=f1dim(f,pc,xc,x)
   xi=0.0
   xmin=newton(f1,xi)
   return xmin
end # function


# one dimensional newton-raphson (secant) optimisation
function newton(f,x)
# Newton-Raphson(secant) optimisation method
# single variable
nmax=100
tolerance=1.0e-10
for i=0:nmax
  dfx=df(f,x)
  d2fx=d2f(f,x)
  x-=dfx/d2fx
  if abs(dfx)<tolerance
    return x
  end #f
end #for
```

```
return x
end # function

# adding vectors
function add(left,right)
   #addition of two vectors
   n1=length(left)
   n2=length(right)
   if n1>=n2
      nMax=n1
   else
      nMax=n2
   end # if n1>=n2
   b =zeros(Float64,nMax)
   for i=1:n1
      b[i]=b[i]+left[i]
   end #  for i=1:n1
   for i=1:n2
      b[i]=b[i]+right[i]
   end # for i=1:n2
   return b
end # function add(left,right)

# multiplying vectors
function multiply_c_v(left,right)
   # multiplying a vector with a constant
   n=length(right)
   b = zeros(Float64,n)
   for i=1:n
      b[i]=left*right[i]
   end # for
   return b
end # function

# steepest descent optimisation
function steepestdescent(f,x)
   # steepestdescent method
   ti=1.0
   nmax=100
   tolerance=1.0e-10
   n=length(x)
   b =ones(Float64,n)
   xc =zeros(Float64,n)
   i=0
   total=0.0
   for i=1:n
     total+=b[i]
   end # for
   i=0
   while i<nmax && total>tolerance
      xc=dfunc(f,x)
      pc=x
      lm=linmin(f,pc,xc)
      b= multiply_c_v(lm,xc)
      x=add(x,b)
      total=0
      for j=1:n
         total+=abs(b[j])
      end
      i=i+1
   end # while
   return x
```

```
end # function

x=[-2,2]
n=length(x)
f(x)=3.0*x[1]*x[1]-4.0*x[1]*x[2]+2.0*x[2]*x[2]-x[1]-x[2]
r=steepestdescent(f,x)
print("r= $r")
```

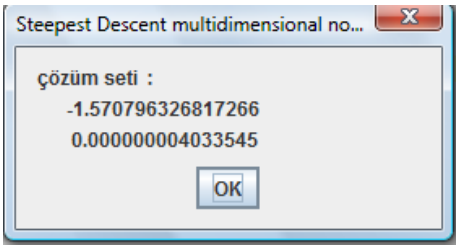
```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_steepest_descent.jl
r= [0.9999995947658935, 1.2499992827564628]
> Terminated with exit code 0.
```

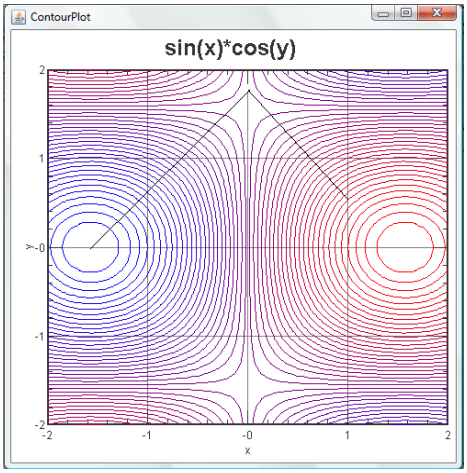
EXAMPLE PROBLEM

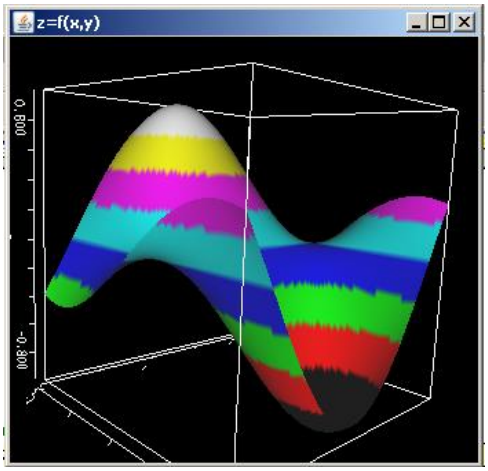
Find the minimum of the function f(x0,x1)=sin(x0)cos(x1) with the initial starting point {x}=[1,0.5]

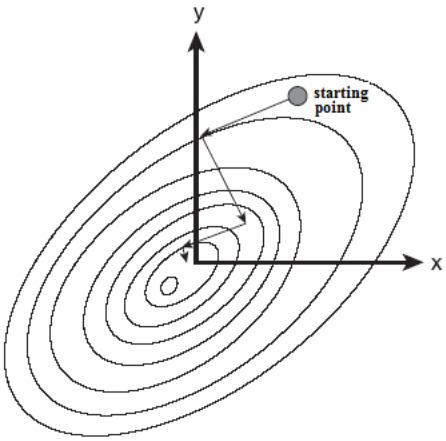
Sample function used in the problem

```
    public static void main (String args[])
    {
    String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=inputdata(s);
    if_xj f=(double x[])->(Math.sin(x[0])*Math.cos(x[1]));
    double [] r1= steepestdescent(f,x0);
    s=" çözüm seti  : \n"+Matrix.toStringT(r1);
    String s2="Steepest Descent multidimensional non-linear optimization method:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    } }
```









MODIFIED NEWTON METHOD(NEWTON RAPHSON STEEPEST DESCENT COMBINED METHOD)

If the initial guess is close to the actual value Newton-Raphson method performs well. But when it is far from the actual solution it is not behave so well. Modified Newton method combines Newton-Raphson method with steepest descent method to get a better convergence performance. Remembering general definition of optimizing equation

$$\{x^{m+1}\} = \{x^m\} + \alpha^m [H^m]^{-1}[-\{\nabla f^m\}]$$

Now both α and Hessian matrix can be taken into the calculation steps to get a combined method.

Program 5.11-1 Modified Newton Method(Combined Newton-Raphson and Steepest descent method), Java code

```
 // Modified Newton method
// uses Newton-Raphson together with Steepest descent
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class NA38A
{
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}
public static double[] dampednewton(if_xj f,double x[])
{
// Modified Newton method
// uses Newton-Raphson together with Steepest descent
// Steepest descent also uses linmin and brent methods of one dimensionaloptimisation
double ti=1.0;
int i,ii,jj;
int nmax=800;
double tolerance=1.0e-14;
int n=x.length;
double b[];
b=new double[n];
double lambda;
double g[]=new double[n];
double G[][]=f.d2func(x);//Hessian matrix
double g1,g2,g3,g4,g5;
//dy=new double[n][n];
i=0;
for(i=0;i<n;i++)
```

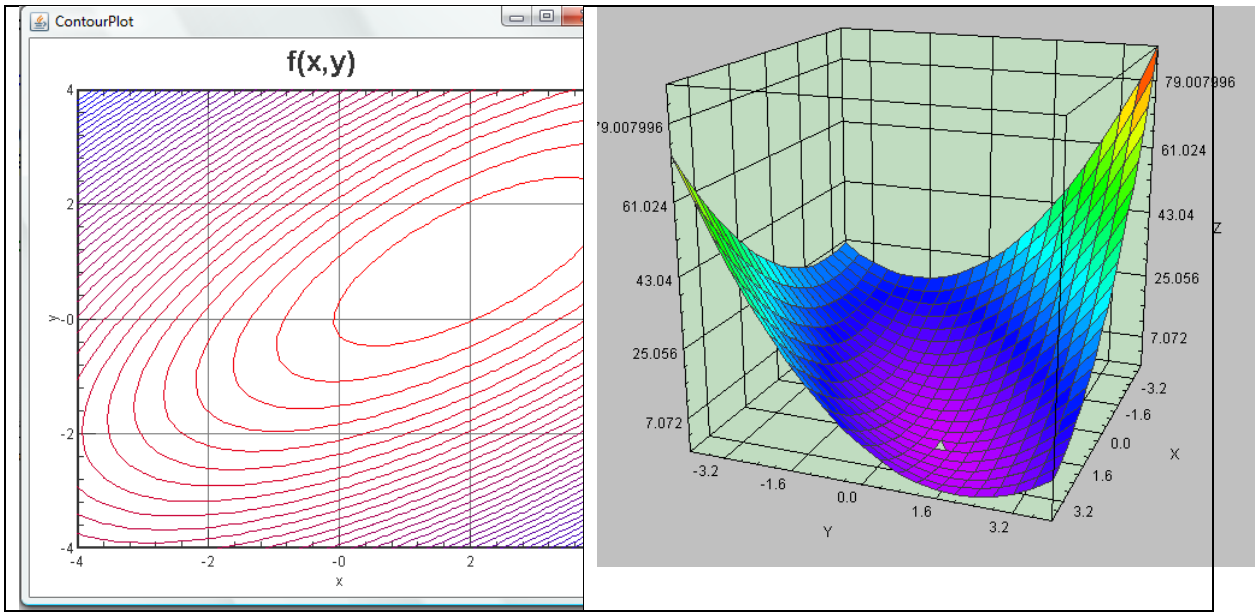
```
{
b[i]=1.0;
}
while( i++ < nmax && Matrix.abs(b) > tolerance )
{   g=f.dfunc(x);
   g1=Matrix.VT_V(g);
   g2=Matrix.VT_G_V(g,G);
   g1*=g1;
   g3=g1/g2;
   g4=Matrix.VT_G_V(g,Matrix.inverse(G));
   lambda=linmin(f,x,g);
   if(g4>Math.max(g3,0))
        b=Matrix.multiply(Matrix.divide(g,G),lambda);
   else
        b=Matrix.multiply(g,lambda);
   x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"WArn,ng : Maximum number of iteration exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}


public  static double[] mnbrak(if_x f,double ax,double dx)
{
return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(if_x f,double ax)
{
// dx ikinci derece polinom için gerekli iki ek noktayı oluşturmak için
// ana noktadan seçiln uzaklık
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(if_x f,double ax,double bx,double cx)
{
// ax,bx,cx üç adet tahmin noktası verildiğinde içinde minimum bulunan ax,bx,cx üç adetnoktayı bize arama
// sonucu bulur
// f : fonksiyon (if_x abstract sınıfında tanımlanmış)
// tol tolerans
int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
            double ulim,u,r,q,fu,dum;
            double aa[]=new double[3];
            fa=f.func(ax);
            fb=f.func(bx);
            if (fb > fa) {
                        //SHFT(dum,*ax,*bx,dum)
                        {dum=ax;ax=bx;bx=dum;}
                        //SHFT(dum,*fb,*fa,dum)
                        {dum=fb;fb=fa;fa=dum;}
            }
            cx=(bx)+GOLD*(bx-ax);
            fc=f.func(cx);
```

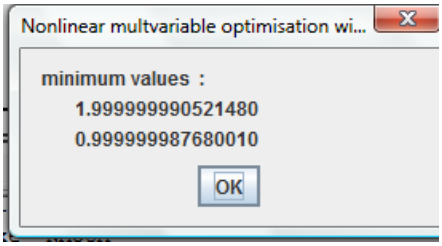
```
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        while (fb > fc) {
                r=(bx-ax)*(fb-fc);
                q=(bx-cx)*(fb-fa);
                u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                        u=ulim;
                        fu=f.func(u);
                } else {
                        u=cx+GOLD*(cx-bx);
                        fu=f.func(u);
                }
                //SHFT(ax,bx,cx,u)
                {ax=bx;bx=cx;cx=u;}
                //SHFT(fa,fb,fc,fu)
                {fa=fb;fb=fc;fc=fu;}
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        }
                return aa;
}
public  static double[] brentf(if_x f,double ax,double bx,double cx,double tol)
{
// ax,bx,cx üç adet tahmin noktası (bxax ile cx arasında yer almalıdır)
// f : fonksiyon (if_x abstract sınıfında tanımlanmış)
// tol tolerans
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altın oran
double ZEPS=1.0e-10;
double xmin;
double aa[]=new double[2];
// SHFT(double a,double b,double c,d) (a)=(b);(b)=(c);(c)=(d);
        int iter;
        double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
```

```
        double e=0.0;
        d=0.0;
        a=((ax < cx) ? ax : cx);
        b=((ax > cx) ? ax : cx);
        x=w=v=bx;
        fw=fv=fx=f.func(x);
        for (iter=1;iter<=ITMAX;iter++) {
                xm=0.5*(a+b);
                tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                        xmin=x;
            aa[0]=xmin;
            aa[1]=fx;
            return aa;
                }
                if (Math.abs(e) > tol1) {
                        r=(x-w)*(fx-fv);
                        q=(x-v)*(fx-fw);
                        p=(x-v)*q-(x-w)*r;
                        q=2.0*(q-r);
                        if (q > 0.0) p = -p;
                        q=Math.abs(q);
                        etemp=e;
                        e=d;
                        if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        else {
                                d=p/q;
                                u=x+d;
                                if (u-a < tol2 || b-u < tol2)
                                        d=SIGN(tol1,xm-x);
                        }
                } else {
                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                }
                u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                fu=f.func(u);
                if (fu <= fx) {
                        if (u >= x) a=x; else b=x;
                        {v=w;w=x;x=u;}
                        {fv=fw;fw=fx;fx=fu;}
                } else {
                        if (u < x) a=u; else b=u;
                        if (fu <= fw || w == x) {
                                v=w;
                                w=u;
                                fv=fw;
                                fw=fu;
                        } else if (fu <= fv || v == x || v == w) {
                                v=u;
                                fv=fu;
                        }
                }
        }
        System.out.println("Maximum iteration is exceeded in berent optimisation method");
        xmin=x;
        aa[0]=xmin;
        aa[1]=fx;
        return aa;
}

public  static double brent(if_x f,double ax,double bx,double cx,double tol)
```

```
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}
public static double linmin(if_xj f,double p[],double xi[])
{
        double tol=2.0e-4;
   int n=p.length;
   double pcom[],xicom[];
        int j;
        double xx,xmin,bx,ax;
        f1dim f1=new f1dim(f,p,xi);
        ax=0.0;
        xx=1.0;
        bx=2.0;
        double aa[]=mnbrak(f1,ax,xx,bx);
        ax=aa[0];
        xx=aa[1];
        bx=aa[2];
        xmin=brent(f1,ax,xx,bx,tol);
        return xmin;
}

public static double[][] linminiter(if_xj f,double p[],double xi[])
{   // gives new iteration point
   int n=p.length;
   double xmin=linmin(f,p,xi);
   double aa[][]=new double[2][n];
        for (int j=0;j<n;j++)
        {   xi[j] *= xmin;
                p[j] += xi[j];
                aa[0][j]=p[j];
                aa[1][j]=xi[j];
        }
   return aa;
}
// veri giriş çıkış metodu
//_____________________________________________________________________
    public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
    public static void main (String args[])
    {
         String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=inputdata(s);
    if_xj f=(double x[])->-(2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1]);
    double r1[]=dampednewton(f,x0);
    s=" minimum values  : \n"+Matrix.toStringT(r1);
    String s2="Nonlinear multvariable optimisation with Newton+Steepest descent:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
```

```
    surfacePlot sp=new surfacePlot(f,-4,4,-4,4);
    sp.plot();
    ContourPlot cp=new ContourPlot(f,-4,4,40,-4,4,40,40,false,"z=f(x,y)","x","y");
    cp.plot();
    }
}
```





NON-LINEAR MULTIDIMENSIONAL FLETCHER-REEVES CONJUGATE GRADIENT METHOD

Fletcher Reeves Method is a method between Modified Newton-Raphson and Conjugate-Gradient. Remembering the general equation Newton-Raphson type solution with a step function α

$\{x^{k+1}\} = \{x^k\} + \alpha^k [H^k]^{-1}[-\{\nabla f^k\}] = \{x^k\} + \alpha^k [H^k]^{-1}[-g^k] = \{x^k\} + \alpha^k \{d^k\}$

Where

$g^k = \nabla f\left(x_1^k, x_2^k, x_3^k, .., x_n^k\right), k = 0, .., n$ and

$d^k = [H^k]^{-1}[-\{g^k\}]$

$\{x^{k+1}\} = \{x^k\} + \alpha^k d^k$

Remembering that in this definition in pure Newton Raphson step function was taken as unitiy, In the other hand in the steepest descent inverse hessian was taken as unity. In the Fletcher-Reeves method neither is unity, but multiplied total term is calculating by assuming conjugate gradient vectors. Fletcher Reeves calculation starts from a Steepest descent step (assumed inverse hessian equals to unity in the first step)

$d^0 = g^0$

$g^k = \nabla f\left(x_1^k, x_2^k, x_3^k, .., x_n^k\right), k = 0, .., n$

For k=0 $\{x^{k+1}\} = \{x^k\} + \alpha^k \{d^k\}$

For the next steps $1 \leq k \leq n$

$$\beta^k = \frac{(g^k)^T(g^k)}{(g^{k-1})^T(g^{k-1})}$$
$$d^k = [-\{g^k\}] + \beta^k d^{k-1}$$
$$\{x^{k+1}\} = \{x^k\} + \alpha^k d^k$$

EXAMPLE PROBLEM

$$\{x^{k+1}\} = \{x^k\} + \alpha^k [H^k]^{-1}[-\{\nabla f^k\}] = \{x^k\} + \alpha^k [H^k]^{-1}[-g^k] = \{x^k\} + \alpha^k \{d^k\}$$

Our problem is finding minimum of function $f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1$ starting from {x}=[-2,2]

$$\nabla f = \begin{Bmatrix} 6x_0 - 4x_1 - 1 \\ 4x_1 - 4x_0 - 1 \end{Bmatrix} = \begin{Bmatrix} -21 \\ 15 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^0 \\ x_1^0 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} x_0^0 \\ x_1^0 \end{Bmatrix} + \alpha \begin{Bmatrix} 6x_0^0 - 4x_1^0 - 1 \\ 4x_1^0 - 4x_0^0 - 1 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix} + \alpha^0 \begin{Bmatrix} -21 \\ 15 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} -2 - 21\alpha^0 \\ 2 + 15\alpha^0 \end{Bmatrix}$$

Substituting this values into the equation

$f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1$

$y=3(-2-21\,\alpha)^2-4(-2-21\alpha)(2+15\alpha)+2(2+15\alpha)^2-(-2-21\alpha)-(2+15\alpha)$

$y = 36 +666\alpha+3033\,\alpha^2$

Minimum of this one dimensional function is needed. In the example programs brent method is used, in the example just analytical derivative method will be use to solve the minimum

$y' = 666+6066\,\alpha = 0 \quad \alpha=-0.109792284$

substituting α back into the x_0^1 and x_1^1 gives

$\alpha^0=-0.109792284$ için $\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} 0.305637982195846 \\ 0.353115727002967 \end{Bmatrix}$

So far, what is applied was steepest descent method. For the next iteration Flether-Reeves step can be taken

$$g = \nabla f = \begin{Bmatrix} 6x_0 - 4x_1 - 1 \\ 4x_1 - 4x_0 - 1 \end{Bmatrix} = \begin{Bmatrix} -0.578635014836797 \\ -0.810089020771512 \end{Bmatrix}$$

$$\beta^k = \frac{(g^k)^T(g^k)}{(g^{k-1})^T(g^{k-1})} = \frac{0.9910627019697276}{666} = 0.001488082135089681$$

$$d^k = -g^k + \beta^k d^{k-1} = \begin{Bmatrix} 0.60988473967368 \\ 0.787767788745167 \end{Bmatrix}$$

$$x^{k+1} = x^k + \alpha^k d^k = \begin{Bmatrix} 0.3056379 \\ .35311572 \end{Bmatrix} + \alpha \begin{Bmatrix} 0.60988473967368 \\ 0.787767788745167 \end{Bmatrix}$$

$$\alpha^1 = 1.138513513513$$

$$\begin{Bmatrix} x_0^2 \\ x_1^2 \end{Bmatrix} = \begin{Bmatrix} 1.000000000000001 \\ 1.24999999999998 \end{Bmatrix}$$

Since the function taken is almost linear the result is almost equal to solution values.

Fletcher-Reeves Conjugate Gradient method (NA43B.java), Java code

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
class f1dim implements if_x
```

```
{
  // pseudofunction to convert multidimensional function to
  // one dimensional function
  if_xj ff1;
  double pc[];
  double xc[];
  double xt[];
  int n;
  public f1dim(if_xj ff,double pi[],double xi[])
  { n=pi.length;
    ff1=ff;
          pc=new double[n];
    xc=new double[n];
    xt=new double[n];
          for(int i=0;i<n;i++)
            {pc[i]=pi[i];xc[i]=xi[i];}
  }

  public double func(double x)
  {
          for (int j=0;j<n;j++) xt[j]=pc[j]+x*xc[j];
          return ff1.func(xt);
  }
}
public class NA43B
{
public static double VT_V(double [] left)
{
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {tot+=left[i]*left[i];}
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {for(int j=0;j<n;j++) {aa[i][j]=left[i]*left[j];}}
return aa;
}

public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}
//===================

public static double[] df(if_xj f,double xxi[])
{
//G radian vector
// derivative of f(x0,x1,x2,...xn)
// [df/dx0, df/dx1,df/dx2,....,df/dxn]
// df/dxj  j=0...x.length
//
```

```
double a[]=new double[xxi.length];
for(int i=0;i<xxi.length;i++)
   {
   a[i]=dfdx(f,xxi,i);
   }
return a;
}


public static double dfdx(if_xj f,double xxi[],int xref)
{ int i;
              double h=1.0e-3;
  double hh=1.0/h;
  double xp2h[]=new double[xxi.length];
  double xph[]=new double[xxi.length];
  double xm2h[]=new double[xxi.length];
  double xmh[]=new double[xxi.length];
  double fp2h,fph,fm2h,fmh;
  for(i=0;i<xxi.length;i++)
  {
  if(i!=xref)
    {xp2h[i]=xxi[i];xph[i]=xxi[i];xm2h[i]=xxi[i];xmh[i]=xxi[i];}
  else {xp2h[i]=xxi[i]+2.0*h;xph[i]=xxi[i]+h;xm2h[i]=xxi[i]-2.0*h;xmh[i]=xxi[i]-h;}
  }
  fp2h=f.func(xp2h);
  fph=f.func(xph);
  fm2h=f.func(xm2h);
  fmh=f.func(xmh);
  return (-fp2h+8.0*fph-8.0*fmh+fm2h)/12.0*hh;
}

public static double d2fdx2(if_xj f,double xxi[],int xiref,int xjref)
{ //second derivative
  int i;
  double h=1.0e-4;
  double hh=1.0/h;
  int n=xxi.length;
  double ff;
  double xp2h[]=new double[xxi.length];
  double xph[]=new double[xxi.length];
  double xm2h[]=new double[xxi.length];
  double xmh[]=new double[xxi.length];
  double fp2h,fph,fm2h,fmh;
     for(int j=0;j<n;j++)
     {if(j!=xjref) {xp2h[j]=xxi[j];xph[j]=xxi[j];xm2h[j]=xxi[j];xmh[j]=xxi[j];}
      else {xp2h[j]=xxi[j]+2.0*h;xph[j]=xxi[j]+h;xm2h[j]=xxi[j]-2.0*h;xmh[j]=xxi[j]-h;}
     }
     fp2h=dfdx(f,xp2h,xiref);
     fph=dfdx(f,xph,xiref);
     fm2h=dfdx(f,xm2h,xiref);
     fmh=dfdx(f,xmh,xiref);
     ff=(-fp2h+8.0*fph-8.0*fmh+fm2h)/12.0*hh;
  return ff;
}
//===================
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public static double linmin(if_xj f,double p[],double xi[])
{
```

```
            double tol=2.0e-4;
   int n=p.length;
   double pcom[],xicom[];
            int j;
            double xx,xmin,bx,ax;
            f1dim f1=new f1dim(f,p,xi);
            ax=0.0;
            xx=1.0;
            bx=2.0;
            double aa[]=mnbrak(f1,ax,xx,bx);
            ax=aa[0];
            xx=aa[1];
            bx=aa[2];
            xmin=brent(f1,ax,xx,bx,tol);
            return xmin;
}

public static double[][] linminiter(if_xj f,double p[],double xi[])
{
   int n=p.length;
   double xmin=linmin(f,p,xi);
   double aa[][]=new double[2][n];
            for (int j=0;j<n;j++)
            {   xi[j] *= xmin;
                        p[j] += xi[j];
                        aa[0][j]=p[j];
                        aa[1][j]=xi[j];
            }
   return aa;
}
public  static double[] brentf(if_x f,double ax,double bx,double cx,double tol)
{

int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0;
double ZEPS=1.0e-10;
double xmin;
double aa[]=new double[2];
            int iter;
            double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
            double e=0.0;
            d=0.0;
            a=((ax < cx) ? ax : cx);
            b=((ax > cx) ? ax : cx);
            x=w=v=bx;
            fw=fv=fx=f.func(x);
            for (iter=1;iter<=ITMAX;iter++) {
                        xm=0.5*(a+b);
                        tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                        if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                                    xmin=x;
                  aa[0]=xmin;
                  aa[1]=fx;
                  return aa;
                        }
                        if (Math.abs(e) > tol1) {
                                    r=(x-w)*(fx-fv);
                                    q=(x-v)*(fx-fw);
                                    p=(x-v)*q-(x-w)*r;
                                    q=2.0*(q-r);
                                    if (q > 0.0) p = -p;
                                    q=Math.abs(q);
```

```
                                etemp=e;
                                e=d;
                                if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                                else {
                                        d=p/q;
                                        u=x+d;
                                        if (u-a < tol2 || b-u < tol2)
                                                d=SIGN(tol1,xm-x);
                                }
                        } else {
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        }
                        u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                        fu=f.func(u);
                        if (fu <= fx) {
                                if (u >= x) a=x; else b=x;
                                {v=w;w=x;x=u;}
                                {fv=fw;fw=fx;fx=fu;}
                        } else {
                                if (u < x) a=u; else b=u;
                                if (fu <= fw || w == x) {
                                        v=w;
                                        w=u;
                                        fv=fw;
                                        fw=fu;
                                } else if (fu <= fv || v == x || v == w) {
                                        v=u;
                                        fv=fu;
                                }
                        }
                }
                System.out.println("Maximum iteration limit is exceeded in brent method");
                xmin=x;
                aa[0]=xmin;
                aa[1]=fx;
                return aa;
}
public  static double[] mnbrak(if_x f,double ax,double dx)
{

return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(if_x f,double ax)
{
// dx ikinci derece polinom için gerekli iki ek noktayı oluşturmak için
// ana noktadan seçiln uzaklık
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(if_x f,double ax,double bx,double cx)
{

int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
```

```
fc=0;
		double ulim,u,r,q,fu,dum;
		double aa[]=new double[3];
		fa=f.func(ax);
		fb=f.func(bx);
		if (fb > fa) {
			//SHFT(dum,*ax,*bx,dum)
			{dum=ax;ax=bx;bx=dum;}
			//SHFT(dum,*fb,*fa,dum)
			{dum=fb;fb=fa;fa=dum;}
		}
		cx=(bx)+GOLD*(bx-ax);
		fc=f.func(cx);
			aa[0]=ax;
			aa[1]=bx;
			aa[2]=cx;
		while (fb > fc) {
			r=(bx-ax)*(fb-fc);
			q=(bx-cx)*(fb-fa);
			u=(bx)-((bx-cx)*q-(bx-ax)*r)/
				(2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
			ulim=(bx)+GLIMIT*(cx-bx);
			if ((bx-u)*(u-cx) > 0.0) {
				fu=f.func(u);
				if (fu < fc) {
					ax=bx;
					bx=u;
					fa=fb;
					fb=fu;
					return aa;
				} else if (fu > fb) {
					cx=u;
					fc=fu;
					return aa;
				}
				u=(cx)+GOLD*(cx-bx);
				fu=f.func(u);
			} else if ((cx-u)*(u-ulim) > 0.0) {
				fu=f.func(u);
				if (fu < fc) {
					//SHFT(bx,cx,u,cx+GOLD*(cx-bx))
					{bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
					//SHFT(fb,fc,fu,f.func(u))
					{fb=fc;fc=fu;fu=f.func(u);}
				}
			} else if ((u-ulim)*(ulim-cx) >= 0.0) {
				u=ulim;
				fu=f.func(u);
			} else {
				u=cx+GOLD*(cx-bx);
				fu=f.func(u);
			}
			//SHFT(ax,bx,cx,u)
			{ax=bx;bx=cx;cx=u;}
			//SHFT(fa,fb,fc,fu)
			{fa=fb;fb=fc;fc=fu;}
			aa[0]=ax;
			aa[1]=bx;
			aa[2]=cx;
		}
			return aa;
}
```

```
public  static double brent(if_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}
// 4. fletcher Reeves method
//_____________________________________________________________________
public static double[] fletcher_reeves( if_xj f,double x[])
{
int i,j,ii,jj,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-11;
int n=x.length;
double alpha;
double g[]=new double[n];
double ge[]=new double[n];
double d[]=new double[n];
double beta;
double alpha_d;
double Q[]=new double[n];
int nn=15;
i=0;
j=0;k=0;
while( j < nmax && gamma > tolerance )
{
   g=df(f,x);
   gamma=norm(g);
   if(k<1) {for(i=0;i<n;i++){d[i]=-g[i];};ge=g;}
   else
   {beta=VT_V(g)/VT_V(ge);
    for(i=0;i<n;i++){Q[i]=beta*d[i];};
   }
   for(i=0;i<n;i++){d[i]=-g[i]+Q[i];}
   alpha=linmin(f,x,d);
   for(i=0;i<n;i++){x[i]+=alpha*d[i];}
   ge=g;
   j++;k++;
   if(k>=nn) k=0; //tekrar bir en dik yaöaç metodu uygulaması
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
            public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

     public static void main (String args[])
    {
```

```
            String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=inputdata(s);
    if_xj f=(double x[])->3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
    double [] r1= fletcher_reeves(f,x0);
    s=" minimum values : \n"+Matrix.toStringT(r1);
    String s2="Fletcher-Reeves multivariable optimisation:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

NON-LINEAR MULTIDIMENSIONAL POLAK-RIBIERE CONJUGATE GRADIENT METHOD

A modified form of Fletcher Reeves method is called **Polak-Ribiere** method.

Fletcher Reeves definition $\beta^k = \frac{(g^k)^T(g^k)}{(g^{k-1})^T(g^{k-1})}$

With a small change of definition it becomes in Polack-Ribiere method

$\beta^k = \frac{(g^{k-1}-g^k)^T(g^k)}{(g^{k-1})^T(g^{k-1})}$

Polak-Ribiere Conjugate Gradient method with numerical derivatives (NA43C.java), Java code

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class NA43C
{

public static double VT_V(double [] left)
{
//multiplies a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
            aa[i][j]=left[i]*left[j];
            }
  }
return aa;
}

public static double  VT_X(double [] left,double [] right)
{
```

```
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}

public static double[] substract(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]-right[i];
  }
return b;
//end of vector substraction method
}

public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}
//====================

//====================
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public static double linmin(if_xj f,double p[],double xi[])
{
   double tol=2.0e-4;
   int n=p.length;
   double pcom[],xicom[];
   int j;
   double xx,xmin,bx,ax;
   f1dim f1=new f1dim(f,p,xi);
   ax=0.0;
   xx=1.0;
   bx=2.0;
   double aa[]=mnbrak(f1,ax,xx,bx);
   ax=aa[0];
   xx=aa[1];
```

```
    bx=aa[2];
    xmin=brent(f1,ax,xx,bx,tol);
            return xmin;
}

public static double[][] linminiter(if_xj f,double p[],double xi[])
{   // yeni iterasyon noktasını verir
    int n=p.length;
    double xmin=linmin(f,p,xi);
    double aa[][]=new double[2][n];
            for (int j=0;j<n;j++)
            {   xi[j] *= xmin;
                    p[j] += xi[j];
                    aa[0][j]=p[j];
                    aa[1][j]=xi[j];
            }
    return aa;
}
public  static double[] brentf(if_x f,double ax,double bx,double cx,double tol)
{
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altın oran
double ZEPS=1.0e-10;
double xmin;
double aa[]=new double[2];
// SHFT(double a,double b,double c,d) (a)=(b);(b)=(c);(c)=(d);
            int iter;
            double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
            double e=0.0;
            d=0.0;
            a=((ax < cx) ? ax : cx);
            b=((ax > cx) ? ax : cx);
            x=w=v=bx;
            fw=fv=fx=f.func(x);
            for (iter=1;iter<=ITMAX;iter++) {
                    xm=0.5*(a+b);
                    tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                    if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                            xmin=x;
                aa[0]=xmin;
                aa[1]=fx;
                return aa;
                    }
                    if (Math.abs(e) > tol1) {
                            r=(x-w)*(fx-fv);
                            q=(x-v)*(fx-fw);
                            p=(x-v)*q-(x-w)*r;
                            q=2.0*(q-r);
                            if (q > 0.0) p = -p;
                            q=Math.abs(q);
                            etemp=e;
                            e=d;
                            if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                    d=CGOLD*(e=(x >= xm ? a-x : b-x));
                            else {
                                    d=p/q;
                                    u=x+d;
                                    if (u-a < tol2 || b-u < tol2)
                                            d=SIGN(tol1,xm-x);
                            }
                    } else {
                            d=CGOLD*(e=(x >= xm ? a-x : b-x));
```

```
                        }
                        u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                        fu=f.func(u);
                        if (fu <= fx) {
                                if (u >= x) a=x; else b=x;
                                {v=w;w=x;x=u;}
                                {fv=fw;fw=fx;fx=fu;}
                        } else {
                                if (u < x) a=u; else b=u;
                                if (fu <= fw || w == x) {
                                        v=w;
                                        w=u;
                                        fv=fw;
                                        fw=fu;
                                } else if (fu <= fv || v == x || v == w) {
                                        v=u;
                                        fv=fu;
                                }
                        }
                }
                System.out.println("Maximum iteration in BRENT is exceeded");
                xmin=x;
                //minumum değer a[0] ile fonksiyon değeri a[1] ile geridöndürülmektedir
                aa[0]=xmin;
                aa[1]=fx;
                return aa;
}
public  static double[] mnbrak(if_x f,double ax,double dx)
{return mnbrak(f,ax,(ax-dx),(ax+dx));}

public  static double[] mnbrak(if_x f,double ax)
{return mnbrak(f,ax,(ax-0.1),(ax+0.1));}

public  static double[] mnbrak(if_x f,double ax,double bx,double cx)
{
int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
                double ulim,u,r,q,fu,dum;
                double aa[]=new double[3];
                fa=f.func(ax);
                fb=f.func(bx);
                if (fb > fa) {
                        //SHFT(dum,*ax,*bx,dum)
                        {dum=ax;ax=bx;bx=dum;}
                        //SHFT(dum,*fb,*fa,dum)
                        {dum=fb;fb=fa;fa=dum;}
                }
                cx=(bx)+GOLD*(bx-ax);
                fc=f.func(cx);
                        aa[0]=ax;
                        aa[1]=bx;
                        aa[2]=cx;
                while (fb > fc) {
                        r=(bx-ax)*(fb-fc);
                        q=(bx-cx)*(fb-fa);
```

```
                u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                        u=ulim;
                        fu=f.func(u);
                } else {
                        u=cx+GOLD*(cx-bx);
                        fu=f.func(u);
                }
                //SHFT(ax,bx,cx,u)
                {ax=bx;bx=cx;cx=u;}
                //SHFT(fa,fb,fc,fu)
                {fa=fb;fb=fc;fc=fu;}
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        }
                return aa;
}
public  static double brent(if_x f,double ax,double bx,double cx,double tol)
{ double aa[]=brentf(f,ax,bx,cx,tol);
  return aa[0];
}
// 4. Polak Ribiere Method
//_____________________________________________________________________
public static double[] polak_ribiere( if_xj f,double x[])
{
int i,j,ii,jj,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-7;
int n=x.length;
double alpha;
double g[]=new double[n];
double ge[]=new double[n];
double d[]=new double[n];
double beta;
double alpha_d;
```

```
double Q[]=new double[n];
int nn=15;
i=0;
j=0;k=0;
while( j < nmax && gamma > tolerance )
{   g=f.dfunc(x);
    gamma=norm(g);
    if(k<1) {for(i=0;i<n;i++){d[i]=-g[i];};ge=g;}
    else
    {   // Polack-Ribiere conjugate gradient definition
                beta=VT_X(substract(ge,g),g)/VT_V(ge);
     for(i=0;i<n;i++){Q[i]=beta*d[i];};
    }
    for(i=0;i<n;i++){d[i]=-g[i]+Q[i];}
    alpha=linmin(f,x,d);
    for(i=0;i<n;i++){x[i]+=alpha*d[i];}
    ge=g;
    j++;k++;
    if(k>=nn) k=0; //re-apply a step of steepest descent
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
                public static double[] verigir(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

                public static void main (String args[])
    {
                String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=verigir(s);
     if_xj f=(double x[])->3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
    double [] r1= polak_ribiere(f,x0);
    s=" minimum values : \n"+Matrix.toStringT(r1);
    String s2="Polak-Ribiere multivariable non linear optimisation:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

MULTIDIMENSIONAL NONLINEAR DAVIDON-FLETCHER-POWELL METHOD

The general equation was in the form of

$$\{x^{k+1}\}=\{x^k\}+\alpha^k\left[H^k\right]^{-1}[-\{\nabla f^k\}]=\{x^k\}+\alpha^k\left[H^k\right]^{-1}\{-g^k\}=\{x^k\}-\alpha^k\{d^k\}$$

where

$g^k=\nabla f(x_1^k,x_2^k,x_3^k,...,x_n^k)$, $k=0,...,n$ and

$d^k=\left[H^k\right]^{-1}g^k$

$$x^{k+1} = x^k - \alpha^k d^k$$

In this method, instead of Hessian an approximation to Hessian will be constructed. Let us define

$$Q_k = \left[H^k\right]^{-1} \quad , k \geq 0$$

$$d^k = Q_k\, g^k$$

$$x^{k+1} = x^k - \alpha_k\, Q_k g^k = x^k - \alpha_k\, d_k$$

To calculate Q in Davidon – Fletcher – Powell metod

$$\Delta g^k = g^{k+1} - g^k$$

$$r^k = Q_k \Delta g^k$$

$\Delta x^k = \{\delta^{k+1}\} = x^{k+1} - x^k$ olarak yazarsak

$$Q_{k+1} = Q_k - \frac{r^k (r^k)^T}{(r^k)^T \Delta g^k} + \frac{\Delta x^k (\Delta x^k)^T}{(\Delta x^k)^T \Delta g^k} \quad , \quad 0 \leq k < n$$

Formulas are used. Program definition of these is given in Program 5.14-1. A second version of the formulas are also existed as follows

$$Q_{k+1} = \beta_k [Q_k - \frac{r^k (r^k)^T}{(r^k)^T \Delta g^k}] + \frac{\Delta x^k (\Delta x^k)^T}{(\Delta x^k)^T \Delta g^k} \quad , \quad 0 \leq k < n$$

$$\beta_k = \frac{(\Delta x^k)^T \Delta x^k}{(r^k)^T \Delta g^k} \quad , \quad 0 \leq k < n$$

In this iteratif solution method, iteration starts as Q equals to unit matrix.

Q_0=I

$$x^{k+1} = x^k - \alpha_k\, Q_k g^k = x^k - \alpha_k\, d_k$$

α_k again is used by using one of the one variable optimization method. In our program Brent method is used. First step is always a steepest descent method also in this approach. After in every n step a steepest descent step is used

EXAMPLE PROBLEM

General equation : $\{x^{k+1}\} = \{x^k\} + \alpha^k \left[H^k\right]^{-1} [-\{\nabla f^k\}] = \{x^k\} + \alpha^k \left[H^k\right]^{-1} \{-g^k\} = \{x^k\} - \alpha^k \{d^k\}$

where

$g^k = \nabla f(x_1^k, x_2^k, x_3^k, ..., x_n^k)$, $k = 0, ..., n$ and

$$d^k = \left[H^k\right]^{-1} g^k$$

$$x^{k+1} = x^k - \alpha^k d^k$$

Problem is to find minimum of

$f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1$ with initial guess {x}=[-2,2]

$$\nabla f = \begin{Bmatrix} 6x_0 - 4x_1 - 1 \\ 4x_1 - 4x_0 - 1 \end{Bmatrix} = \begin{Bmatrix} -21 \\ 15 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^0 \\ x_1^0 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} x_0^0 \\ x_1^0 \end{Bmatrix} + \alpha \begin{Bmatrix} 6x_0^0 - 4x_1^0 - 1 \\ -4x_0^0 - 4x_1^0 - 1 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix} + \alpha^0 \begin{Bmatrix} -21 \\ 15 \end{Bmatrix}$$

x_0^1 = -2 -21α

x_1^1 = 2 +15α

If these values are substituted into the function

$f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1$

$y=3(-2-21\,\alpha)^2-4(-2-21\alpha)(2+15\alpha)+2(2+15\alpha)^2-(-2-21\alpha)-(2+15\alpha)$

The function becomes;

$y = 36 +666\alpha+3033\,\alpha^2$

Minimum of the function ca be find with analytical method as:

$y' = 666+6066\,\alpha = 0 \quad \alpha=-0.109792284$

If this value is substituted in x_0^1 and x_1^1

For α^0=-0.109792284 $\begin{Bmatrix} x_0^1 \\ x_1^1 \end{Bmatrix} = \begin{Bmatrix} 0.305637982195846 \\ 0.353115727002967 \end{Bmatrix}$

This first step was steepest descent, as a second step Davidon-Fletcher-Powell can be applied

$$g = \nabla f = \begin{Bmatrix} 6x_0 - 4x_1 - 1 \\ 4x_1 - 4x_0 - 1 \end{Bmatrix} = \begin{Bmatrix} -0.578635014836797 \\ -0.810089020771512 \end{Bmatrix}$$

$$\Delta g^k = g^{k+1} - g^k = = \begin{Bmatrix} 20.421364985163200 \\ -15.810089020771512 \end{Bmatrix}$$

$$Q = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

$$r^k = Q_k \Delta g^k = \begin{Bmatrix} 20.421364985163200 \\ -15.810089020771512 \end{Bmatrix}$$

$$\Delta x^k = \{\delta^{k+1}\} = x^{k+1} - x^k = \begin{Bmatrix} 2.305637982195846 \\ -1.646884272997033 \end{Bmatrix}$$

$$Q_{k+1} = Q_k - \frac{r^k (r^k)^T}{(r^k)^T \Delta g^k} + \frac{\Delta x^k (\Delta x^k)^T}{(\Delta x^k)^T \Delta g^k} = \begin{Bmatrix} 0.447456314953212 & 0.432131073481232 \\ 0.432131073481232 & 0.662335969913257 \end{Bmatrix}$$

$$d^k = Q_k\, g^k = \begin{Bmatrix} 0.608978529603123 \\ 0.786597267404034 \end{Bmatrix}$$

$$x^{k+1} = x^k + \alpha^k d^k = \begin{Bmatrix} 0.305637982195846 \\ 0.353115727002967 \end{Bmatrix} + \alpha \begin{Bmatrix} 0.608978529603123 \\ 0.786597267404034 \end{Bmatrix}$$

If these values are back substituted into the function

$f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1$ and minimum is solved again it is found that

α^1=1.1402077151335304:

$$\begin{Bmatrix} x_0^2 \\ x_1^2 \end{Bmatrix} = \begin{Bmatrix} 1.000000000000000 \\ 1.250000000000000 \end{Bmatrix}$$

Problem Davidon_Fletcher-Powell method (NA43D.java), Java code

```
// Davidon-Fletcher-Powell optimization method
// Metod Referansi : Numerical Analysis, 8th edition Richard L. Burden, J. Douglas Faires 3
// sayfa 617
```

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class f1 implements if_xj
{
public double func(double x[])
{
//çözümü istenen fonksiyon
double ff;
ff=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
return ff;
}}

public class NA43D
{
public static double VT_V(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}
public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
              aa[i][j]=left[i]*left[j];
              }
  }
return aa;
}
public static double  VT_X(double [] left,double [] right)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}
public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}
```

```
public static double turev(if_xj f,double x[],int denklem_ref,int x_ref)
{
double h0=0.256808;
int i,m;
int n=7;
double f1[];
f1=new double[x.length];
double f2[];
f2=new double[x.length];
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//turev of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}
for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i]*1.0094847;
f1=df(f,x1);
f2=df(f,x2);
T[i][0]=( f1[denklem_ref] - f2[denklem_ref])/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public static double[] df(if_xj f,double xxi[])
{
//Gredyen vektörü
// f(x0,x1,x2,...xn) founksiyonunun derivative vektörü
// [df/dx0, df/dx1,df/dx2,....,df/dxn]
```

```
// df/dxj  j=0...x.length
//
// bu fonksiyon bilgisayar tarafindan hesaplanmaktadir
// kullanici tarafindan tanimlanan f(x) fonksiyonunun
// sayisal olarak alinan derivativeidir.

//
double a[]=new double[xxi.length];
for(int i=0;i<xxi.length;i++)
   {
   a[i]=turev(f,xxi,i);
   }
return a;
}
public static double turev(if_xj f,double xxi[],int x_ref)
{
// df/dxj
double h0=0.256808;
int i,m;
int n=7;
double x1[];
x1=new double[xxi.length];
double x2[];
x2=new double[xxi.length];
for(i=0;i<xxi.length;i++)
{
x1[i]=xxi[i];
x2[i]=xxi[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
//h[i]=h0*Math.pow(r,i);
h0/=2.0;
h[i]=h0;
}
//first turev (difference formula)
for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
T[i][0]=(f.func(x1)-f.func(x2))/(2.0*h[i]);
x1[x_ref]=xxi[x_ref];
x2[x_ref]=xxi[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] -
```

```
  h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i] - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public static double[] turev(if_xj f_deriv,double x[])
{
// df/dxj  j=0...x.length
// This method calculates derivative of a function with more than one independent variable.
// Accuracy of method can be adjusted by changing variables h0 and n
// function input should be in the form given in abstract class
// if_xj,j=0...x.length = df/dx(x_ref)
double a[]=new double[x.length];
for(int x_ref=0;x_ref<x.length;x_ref++)
   {
   a[x_ref]=turev(f_deriv,x,x_ref);
   }
return a;
}
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}
public static double linmin(if_xj f,double p[],double xi[])
{
        double tol=2.0e-4;
    int n=p.length;
    double pcom[],xicom[];
        int j;
        double xx,xmin,bx,ax;
        f1dim f1=new f1dim(f,p,xi);
        ax=0.0;
        xx=1.0;
        bx=2.0;
        double aa[]=mnbrak(f1,ax,xx,bx);
        ax=aa[0];
        xx=aa[1];
        bx=aa[2];
        xmin=brent(f1,ax,xx,bx,tol);
        return xmin;
}

public static double[][] linminiter(if_xj f,double p[],double xi[])
{   // yeni iterasyon noktasini verir
    int n=p.length;
    double xmin=linmin(f,p,xi);
    double aa[][]=new double[2][n];
        for (int j=0;j<n;j++)
        {   xi[j] *= xmin;
                p[j] += xi[j];
                aa[0][j]=p[j];
                aa[1][j]=xi[j];
        }
    return aa;
}
public  static double[] brentf(if_x f,double ax,double bx,double cx,double tol)
{
// ax,bx,cx üç adet tahmin noktasi (bxax ile cx arasinda yer almalidir)
// f : fonksiyon (if_x abstract sinifinda tanimlanmiş)
// tol tolerans
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altin oran
double ZEPS=1.0e-10;
```

```
double xmin;
double aa[]=new double[2];
// SHFT(double a,double b,double c,d) (a)=(b);(b)=(c);(c)=(d);
        int iter;
        double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
        double e=0.0;
        d=0.0;
        a=((ax < cx) ? ax : cx);
        b=((ax > cx) ? ax : cx);
        x=w=v=bx;
        fw=fv=fx=f.func(x);
        for (iter=1;iter<=ITMAX;iter++) {
                xm=0.5*(a+b);
                tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                        xmin=x;
            aa[0]=xmin;
            aa[1]=fx;
            return aa;
                }
                if (Math.abs(e) > tol1) {
                        r=(x-w)*(fx-fv);
                        q=(x-v)*(fx-fw);
                        p=(x-v)*q-(x-w)*r;
                        q=2.0*(q-r);
                        if (q > 0.0) p = -p;
                        q=Math.abs(q);
                        etemp=e;
                        e=d;
                        if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        else {
                                d=p/q;
                                u=x+d;
                                if (u-a < tol2 || b-u < tol2)
                                        d=SIGN(tol1,xm-x);
                        }
                } else {
                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                }
                u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                fu=f.func(u);
                if (fu <= fx) {
                        if (u >= x) a=x; else b=x;
                        {v=w;w=x;x=u;}
                        {fv=fw;fw=fx;fx=fu;}
                } else {
                        if (u < x) a=u; else b=u;
                        if (fu <= fw || w == x) {
                                v=w;
                                w=u;
                                fv=fw;
                                fw=fu;
                        } else if (fu <= fv || v == x || v == w) {
                                v=u;
                                fv=fu;
                        }
                }
        }
        System.out.println("BRENT metodunda maksimum iterasyon sayisi aşildi");
        xmin=x;
        //minumum değer a[0] ile fonksiyon değeri a[1] ile geridöndürülmektedir
```

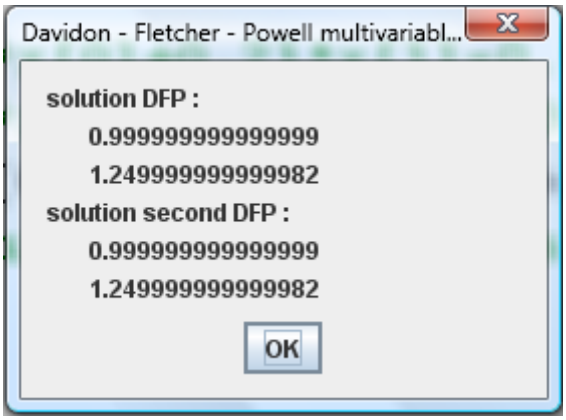
```
		aa[0]=xmin;
		aa[1]=fx;
		return aa;
}
public  static double[] mnbrak(if_x f,double ax,double dx)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçilen uzaklik
return mnbrak(f,ax,(ax-dx),(ax+dx));
}
public  static double[] mnbrak(if_x f,double ax)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçiln uzaklik
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(if_x f,double ax,double bx,double cx)
{
// ax,bx,cx üç adet tahmin noktasi verildiğinde içinde minimum bulunan ax,bx,cx üç adetnoktayi bize arama
// sonucu bulur
// f : fonksiyon (if_x abstract sinifinda tanimlanmiş)
// tol tolerans
int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
		double ulim,u,r,q,fu,dum;
		double aa[]=new double[3];
		fa=f.func(ax);
		fb=f.func(bx);
		if (fb > fa) {
			//SHFT(dum,*ax,*bx,dum)
			{dum=ax;ax=bx;bx=dum;}
			//SHFT(dum,*fb,*fa,dum)
			{dum=fb;fb=fa;fa=dum;}
		}
		cx=(bx)+GOLD*(bx-ax);
		fc=f.func(cx);
			aa[0]=ax;
			aa[1]=bx;
			aa[2]=cx;
		while (fb > fc) {
			r=(bx-ax)*(fb-fc);
			q=(bx-cx)*(fb-fa);
			u=(bx)-((bx-cx)*q-(bx-ax)*r)/
				(2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
			ulim=(bx)+GLIMIT*(cx-bx);
			if ((bx-u)*(u-cx) > 0.0) {
				fu=f.func(u);
				if (fu < fc) {
					ax=bx;
					bx=u;
					fa=fb;
					fb=fu;
					return aa;
				} else if (fu > fb) {
```

```
                                        cx=u;
                                        fc=fu;
                                        return aa;
                                }
                                u=(cx)+GOLD*(cx-bx);
                                fu=f.func(u);
                        } else if ((cx-u)*(u-ulim) > 0.0) {
                                fu=f.func(u);
                                if (fu < fc) {
                                        //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                        {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                        //SHFT(fb,fc,fu,f.func(u))
                                        {fb=fc;fc=fu;fu=f.func(u);}
                                }
                        } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                                u=ulim;
                                fu=f.func(u);
                        } else {
                                u=cx+GOLD*(cx-bx);
                                fu=f.func(u);
                        }
                        //SHFT(ax,bx,cx,u)
                        {ax=bx;bx=cx;cx=u;}
                        //SHFT(fa,fb,fc,fu)
                        {fa=fb;fb=fc;fc=fu;}
                        aa[0]=ax;
                        aa[1]=bx;
                        aa[2]=cx;
                }
                        return aa;
}
public  static double brent(if_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=1.0;
return b;
}
// Davidon - Fletcher-Powell metodu
//_____________________________________________________________________
public static double[] davidon_fletcher_powell( if_xj f,double x[])
{
int i,j,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-12 ;
int n=x.length;
double alpha;
double dx[]=new double[n];
double g[] =new double[n];
double ge[]=new double[n];
double dg[]=new double[n];
double d[] =new double[n];
double r[] =new double[n];
```

```
double rdg,dxdg;
double beta;
double alpha_d;
double Q[][]=I(n);
double Q1[][]=new double[n][n];
double Q2[][]=new double[n][n];
int nn=15;
i=0;
j=0;k=0;
g=turev(f,x);
while( j++ < nmax && gamma > tolerance )
{
   for(int ii=0;ii<n;ii++)
   {
    d[ii]=0.0;
    for(int kk=0;kk<n;kk++)
                {d[ii]=d[ii]-Q[ii][kk]*g[kk];}
   }
   System.out.println("j="+j+"d="+Matrix.toString(d));
   alpha=linmin(f,x,d);
   System.out.println("alfa="+alpha);
   if(alpha==0) break;
   for(i=0;i<n;i++){dx[i]=alpha*d[i];x[i]+=dx[i];ge[i]=g[i];}
   System.out.println("j="+j+"dx=\n"+Matrix.toString(dx));
   System.out.println("j="+j+"x=\n"+Matrix.toString(x));
   g=turev(f,x);
   System.out.println("j="+j+"g=\n"+Matrix.toString(g));
   for(i=0;i<n;i++){dg[i]=g[i]-ge[i];}
   gamma=norm(g);
   for(int ii=0;ii<n;ii++)
   {r[ii]=0.0;
               for(int kk=0;kk<n;kk++)
               {r[ii]+=Q[ii][kk]*dg[kk];}
   }
   rdg=VT_X(r,dg);
   dxdg=VT_X(dx,dg);
   Q1=V_VT(r);
   Q2=V_VT(dx);
   for(int ii=0;ii<n;ii++)
     for(int jj=0;jj<n;jj++)
       Q[ii][jj]=Q[ii][jj]-Q1[ii][jj]/rdg+Q2[ii][jj]/dxdg;
   gamma=norm(dx);
   System.out.println("j="+j+"Q=\n"+Matrix.toString(Q));
j++;k++;
if(k>=nn) {k=0;Q=I(n);}
}
if(j >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
  " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double[] davidon_fletcher_powell_V1( if_xj f,double x[])
{
int i,j,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-12 ;
int n=x.length;
double alpha;
double dx[]=new double[n];
double g[] =new double[n];
double ge[]=new double[n];
double dg[]=new double[n];
```

```
double d[] =new double[n];
double r[] =new double[n];
double rdg,dxdg;
double beta;
double alpha_d;
double Q[][]=I(n);
double Q1[][]=new double[n][n];
double Q2[][]=new double[n][n];
int nn=15;
i=0;
j=0;k=0;
g=turev(f,x);
while( j++ < nmax && gamma > tolerance )
{

   for(int ii=0;ii<n;ii++)
   {
    d[ii]=0.0;
    for(int kk=0;kk<n;kk++)
                {d[ii]=d[ii]-Q[ii][kk]*g[kk];}
   }
   alpha=linmin(f,x,d);
   if(alpha==0) break;
   for(i=0;i<n;i++){dx[i]=alpha*d[i];x[i]+=dx[i];ge[i]=g[i];}
   g=turev(f,x);
   for(i=0;i<n;i++){dg[i]=g[i]-ge[i];}
   gamma=norm(g);
   for(int ii=0;ii<n;ii++)
   {for(int kk=0;kk<n;kk++)
                {r[ii]=Q[ii][kk]*dg[kk];}
   }
   rdg=VT_X(r,dg);
   dxdg=VT_X(dx,dg);
   beta=dxdg/rdg;
   Q1=V_VT(r);
   Q2=V_VT(dx);
   for(int ii=0;ii<n;ii++)
     for(int jj=0;jj<n;jj++)
       Q[ii][jj]=beta*(Q[ii][jj]-Q1[ii][jj]/rdg)+Q2[ii][jj]/dxdg;
   gamma=norm(dx);
j++;k++;
if(k>=nn) {k=0;Q=I(n);}
}
if(j >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
  " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}
              public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
```

```
        public static void main (String args[])
    {
        String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=inputdata(s);
    f1 f=new f1();
    double [] r1= davidon_fletcher_powell(f,x0);
    double [] r2= davidon_fletcher_powell_V1(f,x0);
    s=" solution DFP : \n"+Matrix.toStringT(r1);
    s+=" solution second DFP : \n"+Matrix.toStringT(r2);
    String s2="Davidon - Fletcher - Powell multivariable nonlinear function optimization:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



MULTIVARIABLE NON-LINEAR BROYDEN-FLETCHER-GOLDBERG-SHANNO (BFGS) METHOD

Conjugate gradient methods, Fletcher-Reeves, Polak-Ribiere and Davidon-Fletcher-Powell were investigated inprevious subchapters. These methods were constructing a matrix to represent inverse Hessian matrix. Broyden-Fletcher-Goldberg-Shano methods are also similar, but instead of constructing an inverse hessian representation, it construct directly a hessian matrix representation. Then inverse is calculated by using some form of equation solving method. Algorithm of such a method can be summarised as:

1. Give an initial estimate vector, $x^{(0)}$. As the first guess of the Hessianunit matrix take unit matrix $H^{(0)} = I$. Define conversion factor ε and maximum number of iterations, nmax, start to iteration with the iteration counter k=0 Calculate the gradian vector $c^{(0)} = \nabla f(x^{(0)})$
2. Calculate the norm of gradian vector, $\|c^{(k)}\|$. If norm is smaller than conversion factor stop iteration $\|c^{(k)}\| < \varepsilon$ Result is obtained.
3. Solve vector d from linear system of equation $H^{(k)}d^{(k)} = -c^{(k)}$
4. Find $\alpha^{(k)}$ parameter in d direction from minimizing of one independent variable function $f(x^{(k)} + \alpha^{(k)}d^{(k)})$
5. It should be noted in here that the first iteration is a steepest descent iteration.
6. Calculate vector $x^{(k+1)} + \alpha^{(k)}d^{(k)}$
7. Calculate new Hessian Matrix as:
8. $H^{(k+1)} = H^{(k)} + D^{(k)} + E^{(k)}$

   $D^{(k)} = \frac{y^{(k)}[y^{(k)}]^T}{(y^{(k)}s^{(k)})}$ $\quad E^{(k)} = \frac{c^{(k)}[c^{(k)}]^T}{(c^{(k)}d^{(k)})}$

   $s^{(k)} = \alpha^{(k)}d^{(k)}$ $\quad y^{(k)} = c^{(k+1)} - c^{(k)}$

   $c^{(k+1)} = \nabla f(x^{(k+1)})$

9. Increase iteration parameter k and continiue to the next iteration.

In this algorithm derivatives can be calculated numericlly if so desired. As the example problem the same problem previously investigeted will be looked at: function $f(x_0,x_1)=3x_0^2-4x_0x_1+2x_1^2-x_0-x_1$ with initial conditions {x}=[-2,2]

Program Broyden - Fletcher-Goldfarb-Shanno (BFGS) method Java code

```
 // Broyden - Fletcher-Goldberg-Shanno (BFGS) method
// Method Reference : Intruduction to Optimum Design, Second Edition
// Jasbir S. Arora, Elsevier, ISBN: 0-12-064155-0
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class f1 implements if_xj
{
public double func(double x[])
{
//çözümü istenen fonksiyon
double ff;
ff=3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
//ff=5.0*x[0]*x[0]+2.0*x[0]*x[1]+x[1]*x[1]+7.0;
return ff;
}
}

public class NA43E
{
public static double VT_V(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
              aa[i][j]=left[i]*left[j];
              }
  }
return aa;
}

public static double[][] V_VT_y_s(double [] left,double [] y,double s[])
{
//multiplies a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
double ss=y_s(y,s);
double x1=y_s(y,s);
```

```
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {aa[i][j]=left[i]*left[j]/ss;}
  }
return aa;

}

public static double y_s(double [] y,double [] s)
{
//scalar multiply two vectors
int n=y.length;
double aa=0;
for(int i=0;i<n;i++)
  {aa+=y[i]*s[i];}
return aa;
}

public static double  VT_X(double [] left,double [] right)
{
//multiplies a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}

public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public static double linmin(if_xj f,double p[],double xi[])
{
        double tol=2.0e-10;
   int n=p.length;
   double pcom[],xicom[];
        int j;
        double xx,xmin,bx,ax;
        f1dim f1=new f1dim(f,p,xi);
        ax=0.0;
        xx=1.0;
        bx=2.0;
        double aa[]=mnbrak(f1,ax,xx,bx);
        ax=aa[0];
        xx=aa[1];
        bx=aa[2];
        xmin=brent(f1,ax,xx,bx,tol);
        return xmin;
}
```

```
public static double[][] linminiter(if_xj f,double p[],double xi[])
{
   int n=p.length;
   double xmin=linmin(f,p,xi);
   double aa[][]=new double[2][n];
            for (int j=0;j<n;j++)
            {   xi[j] *= xmin;
                        p[j] += xi[j];
                        aa[0][j]=p[j];
                        aa[1][j]=xi[j];
            }
   return aa;
}
public  static double[] brentf(if_x f,double ax,double bx,double cx,double tol)
{
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altın oran
double ZEPS=1.0e-12;
double xmin;
double aa[]=new double[2];
            int iter;
            double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
            double e=0.0;
            d=0.0;
            a=((ax < cx) ? ax : cx);
            b=((ax > cx) ? ax : cx);
            x=w=v=bx;
            fw=fv=fx=f.func(x);
            for (iter=1;iter<=ITMAX;iter++) {
                        xm=0.5*(a+b);
                        tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                        if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                                    xmin=x;
                  aa[0]=xmin;
                  aa[1]=fx;
                  return aa;
                        }
                        if (Math.abs(e) > tol1) {
                                    r=(x-w)*(fx-fv);
                                    q=(x-v)*(fx-fw);
                                    p=(x-v)*q-(x-w)*r;
                                    q=2.0*(q-r);
                                    if (q > 0.0) p = -p;
                                    q=Math.abs(q);
                                    etemp=e;
                                    e=d;
                                    if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                                    else {
                                                d=p/q;
                                                u=x+d;
                                                if (u-a < tol2 || b-u < tol2)
                                                            d=SIGN(tol1,xm-x);
                                    }
                        } else {
                                    d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        }
                        u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                        fu=f.func(u);
                        if (fu <= fx) {
                                    if (u >= x) a=x; else b=x;
```

```
                                {v=w;w=x;x=u;}
                                {fv=fw;fw=fx;fx=fu;}
                        } else {
                                if (u < x) a=u; else b=u;
                                if (fu <= fw || w == x) {
                                        v=w;
                                        w=u;
                                        fv=fw;
                                        fw=fu;
                                } else if (fu <= fv || v == x || v == w) {
                                        v=u;
                                        fv=fu;
                                }
                        }
                }
                System.out.println("maximum iteration is exceeded in BRENT");
                xmin=x;
                aa[0]=xmin;
                aa[1]=fx;
                return aa;
}
public  static double[] mnbrak(if_x f,double ax,double dx)
{
return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(if_x f,double ax)
{
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(if_x f,double ax,double bx,double cx)
{
int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
                double ulim,u,r,q,fu,dum;
                double aa[]=new double[3];
                fa=f.func(ax);
                fb=f.func(bx);
                if (fb > fa) {
                        //SHFT(dum,*ax,*bx,dum)
                        {dum=ax;ax=bx;bx=dum;}
                        //SHFT(dum,*fb,*fa,dum)
                        {dum=fb;fb=fa;fa=dum;}
                }
                cx=(bx)+GOLD*(bx-ax);
                fc=f.func(cx);
                        aa[0]=ax;
                        aa[1]=bx;
                        aa[2]=cx;
                while (fb > fc) {
                        r=(bx-ax)*(fb-fc);
                        q=(bx-cx)*(fb-fa);
                        u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
```

```
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                        u=ulim;
                        fu=f.func(u);
                } else {
                        u=cx+GOLD*(cx-bx);
                        fu=f.func(u);
                }
                //SHFT(ax,bx,cx,u)
                {ax=bx;bx=cx;cx=u;}
                //SHFT(fa,fb,fc,fu)
                {fa=fb;fb=fc;fc=fu;}
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        }
                return aa;
}
public  static double brent(if_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=1.0;
return b;
}

// Broyden - Fletcher-Goldfarb-Shanno (BFGS) method
//
// only function is given derivatives are calculated numerically
//_____________________________________________________________________
```

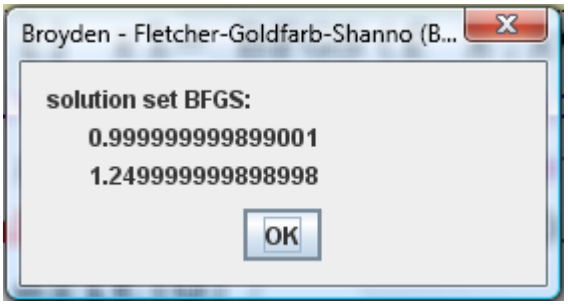
```
public static double[] BFGS( if_xj f,double x[])
{
int i,j,k;
int nmax=100;
double gamma=1.0;
double tolerance=1.0e-6 ;
int n=x.length;
double alpha;
double s[]=new double[n];
double c[] =new double[n];
double c1[] =new double[n];
double d[] =new double[n];
double y[] =new double[n];
double alpha_d;
double H[][]=I(n);
double H1[][]=I(n);
//double H[][]={{1,0},{0,0}};
double D[][]=new double[n][n];
double E[][]=new double[n][n];
int nn=15;
j=0;k=0;
c=f.dfunc(x);
for(i=0;i<c.length;i++)
{d[i]=-c[i];} //first iteration : steepest descent
gamma=norm(c);
while( k++ < nmax && gamma > tolerance )
{
   alpha=linmin(f,x,d);
   for(i=0;i<n;i++)
   {s[i]=alpha*d[i];
    x[i]+=s[i];
   }
   c1=f.dfunc(x);
   for(i=0;i<n;i++)
   {y[i]=c1[i]-c[i];}
   D=V_VT_y_s(y,y,s);
   E=V_VT_y_s(c,c,d);
   for(i=0;i<n;i++)
   {for(j=0;j<n;j++)
   {H[i][j]=H[i][j]+D[i][j]+E[i][j];
    H1[i][j]=H[i][j];
   }
   }
   for(i=0;i<n;i++)
   {c[i]=c1[i];
    c1[i]=-c1[i];
   }
   d=pivotlugauss(H1,c1);
   gamma=norm(c);
}
if(k >= nmax) JOptionPane.showMessageDialog(null,"Warning : Maximum number of iteration exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

// fonksiyon ve türev fonksiyonu verilmiş
//_____________________________________________________________

public static double[] BFGS( if_xj f,fi_xi df,double x[])
{
int i,j,k;
int nmax=100;
```

```
double gamma=1.0;
double tolerance=1.0e-6 ;
int n=x.length;
double alpha;
double s[]=new double[n];
double c[] =new double[n];
double c1[] =new double[n];
double d[] =new double[n];
double y[] =new double[n];
double alpha_d;
double H[][]=I(n);
double H1[][]=I(n);
//double H[][]={{1,0},{0,0}};
double D[][]=new double[n][n];
double E[][]=new double[n][n];
int nn=15;
j=0;k=0;
c=df.func(x);
for(i=0;i<c.length;i++)
{d[i]=-c[i];} //first iteration : steepest descent
gamma=norm(c);
while( k++ < nmax && gamma > tolerance )
{
   alpha=linmin(f,x,d);
   for(i=0;i<n;i++)
   {s[i]=alpha*d[i];
    x[i]+=s[i];
   }
   c1=df.func(x);
   for(i=0;i<n;i++)
   {y[i]=c1[i]-c[i];}
   D=V_VT_y_s(y,y,s);
   E=V_VT_y_s(c,c,d);
   for(i=0;i<n;i++)
   {for(j=0;j<n;j++)
   {H[i][j]=H[i][j]+D[i][j]+E[i][j];
    H1[i][j]=H[i][j];
   }
   }
   for(i=0;i<n;i++)
   {c[i]=c1[i];
    c1[i]=-c1[i];
   }
   d=pivotlugauss(H1,c1);
   gamma=norm(c);
}
if(k >= nmax) JOptionPane.showMessageDialog(null,"Uyarı maximum iterasyon sayısı aşıldı \n"+
  " çözüm geçerli olmıyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}


public static double[] multiply(double[] left,double right)
{
//multiplying a vector with a constant
int i;
int n=left.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {
  b[i]=right*left[i];
```

```
  }
return b;
}


public static double[] divide(double[] left,double[][] right)
{
return pivotlugauss(right,left);
}

  //en kucuk kareler metodu
  public static double[] pivotlugauss(double a[][],double b[])
  { //kısmi pivotlu gauss eleme yöntemi
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss eleme
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivotlama
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
              //gauss elemeyi çözme
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
           b[i]   =b[i]   -carpan*b[k];
                }
  }
  //geriye doğru yerine koyma
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] add(double[] left,double[] right)
{
```

```
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

public static double[][] add(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
  for(j=0;j<m1;j++)
    {
    b[i][j]=left[i][j]+right[i][j];
    }
  }
return b;
//end of matrix addition method
}
public static double[] inputdata(String s)
     {
     String s1=JOptionPane.showInputDialog(s);
     StringTokenizer token=new StringTokenizer(s1);
     int n=token.countTokens()-1;
     int m=n+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }
```

```
        public static void main (String args[])
    {
        String s="enter optimum point first estimation values with one empty space in between: ";
    double [] x0=inputdata(s);
    f1 f=new f1();
    double [] r1= BFGS(f,x0);
    s=" solution set BFGS: \n"+Matrix.toStringT(r1);
    String s2="Broyden - Fletcher-Goldberg-Shanno (BFGS) multivariable optimisation:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



HOMOTOPY OR CONTINUATION METHODS

Continuation method is defined at chapter 4.3 to solve non-linear system of equations. As we can remember from Newton-Raphson method optimization problem can be defined as solution of system of nonlinear equation for the derivative of the given function. System of equations were defined as:

$$\nabla f = \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{Bmatrix} \text{ and } \delta^{m+1} = \begin{Bmatrix} (x_1^{m+1} - x_1^m) \\ (x_2^{m+1} - x_2^m) \\ (x_3^{m+1} - x_3^m) \\ \cdots \\ (x_n^{m+1} - x_n^m) \end{Bmatrix} \text{and}$$

$$[H] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \frac{\partial^2 f}{\partial x_3 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

for the (k+1)th iteration

$[H^m]\{\delta^{m+1}\} = -\{\nabla f^m\}$

And for the inverse of the Hessian matrix, it can be written as

$\{\delta^{m+1}\} = [H^m]^{-1}[-\{\nabla f^m\}]$. In this equation the term $[H^m]^{-1}$ is inverse Hessian matrix .

In order to solve this equation Homotopy (Continuation) method given in chapter 4.3 will be used. Please refer to this chapter for the details of the method.

Continuation method for optimization (NA34F.java)

```
 // NA34F Continuation-homotopy optimization
// Method Referance : Numerical Analysis, 8th edition Richard L. Burden, J. Douglas Faires 3
// page 637

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class NA34F
{
// Continuity-homotopy method for optimisation
// (Solution of root of derivatives of function)
public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {b[i]=left*right[i];}
return b;
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
     b[ii]=0;
   }
   return b;
 }
 for(i=0;i<m1;i++)
   {
     b[i]=0;
     for(k=0;k<n1;k++)
              b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}

public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
```

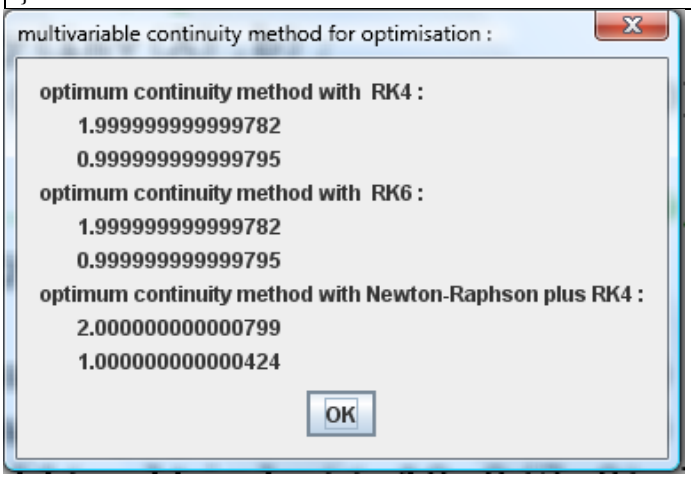
```
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

public static double[][] inv(double[][] a)
{
// INVERSION OF A MATRIX
// inversion by using gaussian elimination
// with full pivoting
int n=a.length;
int m=a[0].length;
double b[][];
b=new double[n][n];
int indxc[];
int indxr[];
double ipiv[];
indxc=new int[n];
indxr=new int[n];
ipiv=new double[n];
int i,j,k,l,ll,ii,jj;
int icol=0;
int irow=0;
double big,dum,pivinv,temp;
if(n!=m)
{
   System.out.println("Matrix must be square ");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                          b[ii][jj]=0.0;
                return b;
}
for(i=0;i<n;i++)
   for(j=0;j<n;j++)
       b[i][j]=a[i][j];
for(i=0;i<n;i++)
{
  big=0.0;
  for(j=0;j<n;j++)
  {
    if(ipiv[j] != 1)
      for(k=0;k<n;k++)
      {
              if(ipiv[k] == 0)
              {
                if(Math.abs(b[j][k]) >= big)
                {
                  big=Math.abs(b[j][k]);
                  irow=j;
                  icol=k;
```

```
                }
              }
              else if(ipiv[k] > 1 )
                {
                System.out.println("error : inverse of the matrix : singular matrix-1");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
                return b;
                }
     }
  }
  ++ ipiv[icol];
  if(irow != icol)
    for(l=0;l<n;l++)
    {
    temp=b[irow][l];
    b[irow][l]=b[icol][l];
    b[icol][l]=temp;
    }
  indxr[i]=irow;
  indxc[i]=icol;
  if(b[icol][icol] == 0.0)
  {
    System.out.println("error : inverse of the matrix : singular matrix-2");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
    return b;
  }
  pivinv=1.0/b[icol][icol];
  b[icol][icol]=1.0;
  for(l=0;l<n;l++) b[icol][l] *=pivinv;
  for(ll=0;ll<n;ll++)
  if(ll != icol)
  {
    dum=b[ll][icol];
    b[ll][icol]=0.0;
    for(l=0;l<n;l++) b[ll][l]-= b[icol][l]*dum;
  }
}
for(l=n-1;l>=0;l--)
{
  if(indxr[l] != indxc[l])
     for(k=0;k<n;k++)
     {
       temp=b[k][indxc[l]];
       b[k][indxc[l]]=b[k][indxr[l]];
       b[k][indxr[l]]=temp;
     }
}
return b;
}
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public static double[] continuationRK4(if_xj f,double x[],int N)
{
//=================================================
// Roots of nonlinear system of equation with Homotophy-Continuaton method
// 4.th degree  Runge-Kutta equations
```

```
// yi+1 = yi + (1/6)*( k1 + 2k2 +2k3+k4)h
// k1=f(xi,yi)
// k2=f(xi+0.5h,yi+0.5k1h)
// k3=f(xi+0.5h,yi+0.5k2h)
// k4=f(xi+h,yi +k3h)
//=================================================
// x  : independent variable vector
// y  : dependent variable vetor
// dy : derivativeof the dependent vector
// N  : number of points in Runge-Kutta differential equation solution

int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[4][n];
double A[][]=new double[n][n];
b=multiply(-h,f.dfunc(x));
for(i=0;i<N;i++)
{
x1=x;
A=f.d2func(x1);
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.5,k[0]));
A=f.d2func(x1);
k[1]=multiply(inv(A),b);
x1=add(x,multiply(0.5,k[1]));
A=f.d2func(x1);
k[2]=multiply(inv(A),b);
x1=add(x,k[2]);
A=f.d2func(x1);
k[3]=multiply(inv(A),b);
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/6.0*(k[0][j]+2.0*k[1][j]+2.0*k[2][j]+k[3][j]);}
}
return x;
}

public static double[] continuationRK6(if_xj f,double x[],int N)
{
//=================================================
// Roots of nonlinear system of equation with Homotophy-Continuaton method
// 6.th degree  Runge-Kutta equations
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
// k1=f(xi,yi)
// k2=f(xi+0.25h , yi+0.25k1h)
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
//=================================================
// x  : independent variable vector
// y  : dependent variable vetor
// dy : derivativeof the dependent vector
// N  : number of points in Runge-Kutta differential equation solution

int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
```

```
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[6][n];
double A[][]=new double[n][n];
b=multiply(-h,f.dfunc(x));
for(i=0;i<N;i++)
{
x1=x;
A=f.d2func(x1);
// k1=f(xi,yi)
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.25,k[0]));
A=f.d2func(x1);
// k2=f(xi+0.25h , yi+0.25k1h)
k[1]=multiply(inv(A),b);
x1=add(x,add(multiply(0.125,k[0]),multiply(0.125,k[1])));
A=f.d2func(x1);
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
k[2]=multiply(inv(A),b);
x1=add(x,add(multiply(-0.5,k[1]),k[2]));
A=f.d2func(x1);
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
k[3]=multiply(inv(A),b);
x1=add(x,add(multiply((3.0/16.0),k[0]),multiply((9.0/16.0),k[3])));
A=f.d2func(x1);
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
k[4]=multiply(inv(A),b);
x1=add(x,
      add(multiply((-3.0/7.0),k[0]),add(multiply((2.0/7.0),k[1]),
      add(multiply((12.0/7.0),k[2]),
         add(multiply((-12.0/7.0),k[3]),multiply((8.0/7.0),k[4])))));
A=f.d2func(x1);
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
k[5]=multiply(inv(A),b);
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/90.0*(7.0*k[0][j]+32.0*k[2][j]+12.0*k[3][j]+32.0*k[4][j]+7.0*k[5][j]);}
}
return x;
}

public static double[] newton_continuationRK4(if_xj f,double x[],int N)
{
// Newton-Raphson combined with Continuation (RK4) method

x=continuationRK4(f,x,4);
double ti=1.0;
int i;
int nmax=400;
double tolerance=1.0e-10;
int n=x.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
{
b[i]=1.0;
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{ b=Matrix.multiply(Matrix.divide(f.dfunc(x),f.d2func(x)),-ti);
  x=Matrix.add(x,b);
}
```

```
if(i >= nmax) JOptionPane.showMessageDialog(null,"Uyarı maximum iterasyon sayısı aşıldı \n"+
            " çözüm geçerli olmıyabilir","MAKSİMUM ITERASYON SAYISI
UYARISI",JOptionPane.WARNING_MESSAGE);
System.out.println();
return x;
}

public static double[] inputdata()
{
   String s=JOptionPane.showInputDialog("kök tahmin değerlerini arada bir boşluk bırakarak giriniz: ");
   StringTokenizer token=new StringTokenizer(s);
   int n=token.countTokens()-1;
   int m=n+1;
   double a[]=new double[m];
   int j=0;
   while(token.hasMoreTokens())
   {
   Double ax=new Double(token.nextToken());
   a[j++]=ax.doubleValue();
   }
   return a;
}

    public static void main (String args[])
    {
    double [] x0=inputdata();
    if_xj ff=(double x[])->2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1];
    double [] r= continuationRK4(ff,x0,10);
    double [] r1=continuationRK6(ff,x0,10);
    double [] r2=newton_continuationRK4(ff,x0,10);
    String s=" optimum continuity method with  RK4 : \n"+Matrix.toStringT(r);
    s+=" optimum continuity method with  RK6 : \n"+Matrix.toStringT(r1);
    s+=" optimum continuity method with Newton-Raphson plus RK4 : \n"+Matrix.toStringT(r2);
    String s1="multivariable continuity method for optimisation : ";
    JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



MULTIVARIABLE NONLINEAR BROYDEN SECANT METHOD WITH SHERMAN MORRISON FORMULA

Broyden secant method wasdefined in chapter 4.2. It is basically secant root finding method for multivariable functions.

As we can remember from Newton-Raphson method optimization problem can be defined as solution of system of nonlinear equation for the derivative of the given function. System of equations were defined as:

$$\nabla f = \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{Bmatrix} \text{ and } \delta^{m+1} = \begin{Bmatrix} (x_1^{m+1} - x_1^m) \\ (x_2^{m+1} - x_2^m) \\ (x_3^{m+1} - x_3^m) \\ \cdots \\ (x_n^{m+1} - x_n^m) \end{Bmatrix} \text{ and } [H] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \frac{\partial^2 f}{\partial x_3 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$[H^m]\{\delta^{m+1}\} = -\{\nabla f^m\}$

And for the inverse of the Hessian matrix, it can be written as

$\{\delta^{m+1}\} = [H^m]^{-1}[-\{\nabla f^m\}]$. In this equation the term $[H^m]^{-1}$ is inverse Hessian matrix .

Hesian matrix can be replaced by numerical differencedefinition of the essian $[H]$ as

$$\frac{\partial g_j}{\partial x_k}(x^{(i)}) \approx \frac{g_j(x^{(i)} + e_k h) - g_j(x^{(i)})}{h}$$

h in the equation is finite difference parameter (Δx) and ek is unit vektor (kth componet is 1 the other components are 0). i is the iteration number. Note that secant method is always approach to the roots less quickly compare to Newton-Raphson method. In order to improve number of iterations for the solutions, a modified version of secant method, Broyden method will be used. In Broyden method when x(0) is a given value, first iteration, x(1), will be calculated by using Newton-Raphson method. For the following steps

$[H(x^{(0)})]\ \{x^{(1)} - x^{(0)}\} = -\{g(x^{(0)}\}$ form of the equation can be changed to be

$[A_1]\ \{x^{(1)} - x^{(0)}\} = \{g(x^{(1)} - g(x^{(0)}\}$ In order to have this equation valid for every vector z, vector should satisfy following equations

$[A_1]\ \{z\} = [\nabla g(x^{(0)})]z \qquad \{x^{(1)} - x^{(0)}\}^t z = 0$ t

By using this defination Formula to calculate $[A_1]$ can be derived.

$$[A_1] = H(x^{(0)}) + \frac{[\{g(x^{(1)}) - g(x^{(0)})\} - H(x^{(0)})(x^{(1)} - x^{(0)})] * (x^{(1)} - x^{(0)})^t}{\left\| x^{(1)} - x^{(0)} \right\|_2^2}$$

Where the term $\left\| x^{(1)} - x^{(0)} \right\|_2^2$ is a two norm. Norms was defined in previous chapter. After finding $[A_1]$ new value for vector x can be constructed.

$$x^{(2)} = x^{(1)} - A_1^{-1} g(x^{(1)})$$

When $x^{(2)}$ is found, this value can be used to calculate $x^{(3)}$ .General Formula will become:

$$[A_i] = [A_{i-1}] + \frac{[\{g(x^{(1)}) - g(x^{(0)})\} - [A_{i-1}](x^{(1)} - x^{(0)})] * (x^{(1)} - x^{(0)})^t}{\left\| x^{(1)} - x^{(0)} \right\|_2^2}$$

$$x^{(i+1)} = x^{(i)} - A_i^{-1} g(x^{(i)})$$

One difficulty in this equation is the requirement of solving inverse of the matrix each time which requires solving of system of equation by a method such as Gauss elimination. But Gauss method is expensive in calculation time. A beter alternative could be using Sherman-Morrison approach. This approach calculates inverse directly as an iterative process. Matrix

A_i^{-1} can directly be calculated from value A_{i-1}^{-1}. Equations of Sherman-Morrison approach is as follows

$$s_i = \{x^{(1)} - x^{(i-1)}\}$$

$$y_i = \{f(x^{(i)}) - f(x^{(i-1)})\}$$

$$(A + xy^t)^{-1} = A^{-1} - \frac{A^{-1}xy^tA^{-1}}{1 + y^tA^{-1}x}$$

$$A_i^{-1} = \left(A^{-1} + \frac{y_i - A_{i-1}s_i}{\|s_i\|_2^2} s_i^t \right)^{-1}$$

$$A_i^{-1} = A^{-1} + \frac{A_{i-1}^{-1}\left(\frac{y_i - A_{i-1}s_i}{\|s_i\|_2^2} s_i^t \right) A_{i-1}^{-1}}{1 + s_i^t A_{i-1}^{-1}\left(\frac{y_i - A_{i-1}s_i}{\|s_i\|_2^2} s_i^t \right)} s_i^t$$

$$A_i^{-1} = A^{-1} + \frac{(s_i - A_{i-1}^{-1}y_i)s_i^t A_{i-1}^{-1}}{s_i^t A_{i-1}^{-1} y_i}$$

```
// NA34F Continuation-homotopy optimization
// Method Referance : Numerical Analysis, 8th edition Richard L. Burden, J. Douglas Faires 3
// page 637

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============ Tanimi gereken fonksiyon ================
abstract class fi_xi
{
  // multifunction multi independent variable
  // vector of dependent variables are returned
  // example f[0]=x[0]+sin(x[1])
  //         f[1]=x[0]*x[0]-x[1]
  // func(x) returns the value of f[0] and f[1]
  // as a two dimensional vector
abstract double[] func(double x[]);
}

abstract class fij_xi
{
  // multifunction multi independent variable
  // for n independent variable n*n matrix of
  // dependent variables are returned
  // example f[0][0]=x[0]+Math.sin(x[1])   f[0][1]=x[0]-x[1]
  //         f[1][0]=x[0]*x[0]-x[1]        f[1][1]=Math.exp(x[0]+x[1]*x[1]
  // func(x) returns the value of f[0][0], f[0][1]
  //                              f[1][0], f[1][1]
  // as a two dimensional matrix
  abstract double[][] func(double x[]);
}
```

```
//=========================
class f1 extends f_xj
{
      public double func(double x[])
     {//çözümü istenen fonksiyon
      double ff;
      ff=2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1];
     return -ff; //minimum değil maxsimum istendiğinden - ile çarptik
     }
}
//=========================

public class NA34F
{
// Continuity-homotopy method for optimization
// (Solution of root of derivatives of function)
public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {b[i]=left*right[i];}
return b;
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
    b[ii]=0;
   }
  return b;
 }
 for(i=0;i<m1;i++)
   {
    b[i]=0;
    for(k=0;k<n1;k++)
            b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}

public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
```

```
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

public static double[][] inv(double[][] a)
{
// INVERSION OF A MATRIX
// inversion by using gaussian elimination
// with full pivoting
int n=a.length;
int m=a[0].length;
double b[][];
b=new double[n][n];
int indxc[];
int indxr[];
double ipiv[];
indxc=new int[n];
indxr=new int[n];
ipiv=new double[n];
int i,j,k,l,ll,ii,jj;
int icol=0;
int irow=0;
double big,dum,pivinv,temp;
if(n!=m)
{
   System.out.println("Matrix must be square ");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                          b[ii][jj]=0.0;
                return b;
}
for(i=0;i<n;i++)
   for(j=0;j<n;j++)
       b[i][j]=a[i][j];
for(i=0;i<n;i++)
{
  big=0.0;
  for(j=0;j<n;j++)
  {
    if(ipiv[j] != 1)
      for(k=0;k<n;k++)
      {
              if(ipiv[k] == 0)
              {
                if(Math.abs(b[j][k]) >= big)
                {
                  big=Math.abs(b[j][k]);
                  irow=j;
                  icol=k;
```

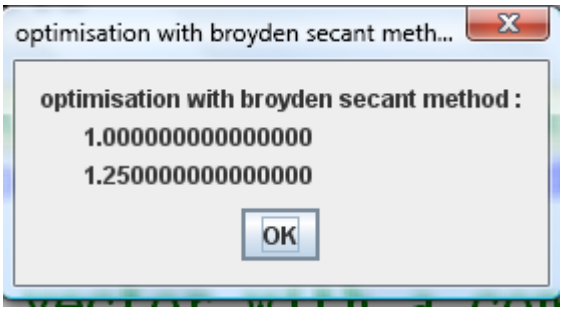
```
                }
              }
              else if(ipiv[k] > 1 )
                {
                System.out.println("error : inverse of the matrix : singular matrix-1");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
                return b;
                }
     }
  }
  ++ ipiv[icol];
  if(irow != icol)
    for(l=0;l<n;l++)
    {
    temp=b[irow][l];
    b[irow][l]=b[icol][l];
    b[icol][l]=temp;
    }
  indxr[i]=irow;
  indxc[i]=icol;
  if(b[icol][icol] == 0.0)
  {
    System.out.println("error : inverse of the matrix : singular matrix-2");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
    return b;
  }
  pivinv=1.0/b[icol][icol];
  b[icol][icol]=1.0;
  for(l=0;l<n;l++) b[icol][l] *=pivinv;
  for(ll=0;ll<n;ll++)
  if(ll != icol)
  {
    dum=b[ll][icol];
    b[ll][icol]=0.0;
    for(l=0;l<n;l++) b[ll][l]-= b[icol][l]*dum;
  }
}
for(l=n-1;l>=0;l--)
{
  if(indxr[l] != indxc[l])
     for(k=0;k<n;k++)
     {
       temp=b[k][indxc[l]];
       b[k][indxc[l]]=b[k][indxr[l]];
       b[k][indxr[l]]=temp;
     }
}
return b;
}
public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}

public static double[] df(f_xj f,double xxi[])
{
//Gredien vector
// f(x0,x1,x2,...xn) founksiyonunun türev vektörü
// [df/dx0, df/dx1,df/dx2,....,df/dxn]
// df/dxj  j=0...x.length
```

```
double a[]=new double[xxi.length];
for(int i=0;i<xxi.length;i++)
   {
   a[i]=dfdx(f,xxi,i);
   }
return a;
}
public static double dfdx(f_xj f,double xxi[],int xref)
{ int i;
              double h=1.0e-3;
  double hh=1.0/h;
  double xp2h[]=new double[xxi.length];
  double xph[]=new double[xxi.length];
  double xm2h[]=new double[xxi.length];
  double xmh[]=new double[xxi.length];
  double fp2h,fph,fm2h,fmh;
  for(i=0;i<xxi.length;i++)
  {
  if(i!=xref)
    {xp2h[i]=xxi[i];xph[i]=xxi[i];xm2h[i]=xxi[i];xmh[i]=xxi[i];}
  else {xp2h[i]=xxi[i]+2.0*h;xph[i]=xxi[i]+h;xm2h[i]=xxi[i]-2.0*h;xmh[i]=xxi[i]-h;}
  }
  fp2h=f.func(xp2h);
  fph=f.func(xph);
  fm2h=f.func(xm2h);
  fmh=f.func(xmh);
  return (-fp2h+8.0*fph-8.0*fmh+fm2h)/12.0*hh;
}
public static double d2fdx2(f_xj f,double xxi[],int xiref,int xjref)
{ //ikincitürev matrisi
  int i;
  double h=1.0e-4;
  double hh=1.0/h;
  int n=xxi.length;
  double ff;
  double xp2h[]=new double[xxi.length];
  double xph[]=new double[xxi.length];
  double xm2h[]=new double[xxi.length];
  double xmh[]=new double[xxi.length];
  double fp2h,fph,fm2h,fmh;
     for(int j=0;j<n;j++)
     {if(j!=xjref) {xp2h[j]=xxi[j];xph[j]=xxi[j];xm2h[j]=xxi[j];xmh[j]=xxi[j];}
      else {xp2h[j]=xxi[j]+2.0*h;xph[j]=xxi[j]+h;xm2h[j]=xxi[j]-2.0*h;xmh[j]=xxi[j]-h;}
     }
     fp2h=dfdx(f,xp2h,xiref);
     fph=dfdx(f,xph,xiref);
     fm2h=dfdx(f,xm2h,xiref);
     fmh=dfdx(f,xmh,xiref);
     ff=(-fp2h+8.0*fph-8.0*fmh+fm2h)/12.0*hh;
  return ff;
}

public static double[][]H(f_xj f,double x[])
{//Hessian Matrix
  int n=x.length;
  double dy[][]=new double[n][n];;
  for(int ii=0;ii<n;ii++)
  {
    for(int jj=0;jj<n;jj++)
    {   //Hessian matrx
        dy[ii][jj]=d2fdx2(f,x,ii,jj);
```

```
    }
  }
  return dy;
}
public static double[] continuationRK4(f_xj f,double x[],int N)
{
//=================================================
// Roots of nonlinear system of equation with Homotophy-Continuaton method
// 4.th degree  Runge-Kutta equations
// yi+1 = yi + (1/6)*( k1 + 2k2 +2k3+k4)h
// k1=f(xi,yi)
// k2=f(xi+0.5h,yi+0.5k1h)
// k3=f(xi+0.5h,yi+0.5k2h)
// k4=f(xi+h,yi +k3h)
//=================================================
// x  : independent variable vector
// y  : dependent variable vetor
// dy : derivativeof the dependent vector
// N  : number of points in Runge-Kutta differential equation solution
int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[4][n];
double A[][]=new double[n][n];
b=multiply(-h,df(f,x));
for(i=0;i<N;i++)
{
x1=x;
A=H(f,x1);
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.5,k[0]));
A=H(f,x1);
k[1]=multiply(inv(A),b);
x1=add(x,multiply(0.5,k[1]));
A=H(f,x1);
k[2]=multiply(inv(A),b);
x1=add(x,k[2]);
A=H(f,x1);
k[3]=multiply(inv(A),b);
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/6.0*(k[0][j]+2.0*k[1][j]+2.0*k[2][j]+k[3][j]);}
}
return x;
}

public static double[] continuationRK6(f_xj f,double x[],int N)
{
//=================================================
// Roots of nonlinear system of equation with Homotophy-Continuaton method
// 6.th degree  Runge-Kutta equations
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
// k1=f(xi,yi)
// k2=f(xi+0.25h , yi+0.25k1h)
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
//=================================================
// x  : independent variable vector
```

```
// y  : dependent variable vetor
// dy : derivativeof the dependent vector
// N  : number of points in Runge-Kutta differential equation solution

int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[6][n];
double A[][]=new double[n][n];
b=multiply(-h,df(f,x));
for(i=0;i<N;i++)
{
x1=x;
A=H(f,x1);
// k1=f(xi,yi)
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.25,k[0]));
A=H(f,x1);
// k2=f(xi+0.25h , yi+0.25k1h)
k[1]=multiply(inv(A),b);
x1=add(x,add(multiply(0.125,k[0]),multiply(0.125,k[1])));
A=H(f,x1);
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
k[2]=multiply(inv(A),b);
x1=add(x,add(multiply(-0.5,k[1]),k[2]));
A=H(f,x1);
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
k[3]=multiply(inv(A),b);
x1=add(x,add(multiply((3.0/16.0),k[0]),multiply((9.0/16.0),k[3])));
A=H(f,x1);
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
k[4]=multiply(inv(A),b);
x1=add(x,
      add(multiply((-3.0/7.0),k[0]),add(multiply((2.0/7.0),k[1]),
      add(multiply((12.0/7.0),k[2]),
         add(multiply((-12.0/7.0),k[3]),multiply((8.0/7.0),k[4])))));
A=H(f,x1);
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
k[5]=multiply(inv(A),b);
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/90.0*(7.0*k[0][j]+32.0*k[2][j]+12.0*k[3][j]+32.0*k[4][j]+7.0*k[5][j]);}
}
return x;
}
public static double[] newton_continuationRK4(f_xj f,double x[],int N)
{
// Newton-Raphson combined with Continuation (RK4) method
x=continuationRK4(f,x,4);
double ti=1.0;
int i;
int nmax=400;
double tolerance=1.0e-10;
int n=x.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
{
b[i]=1.0;
```

```
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{ b=Matrix.multiply(Matrix.divide(df(f,x),H(f,x)),-ti);
  x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
            " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI
UYARISI",JOptionPane.WARNING_MESSAGE);
System.out.println();
return x;
}
public static double[] inputdata()
{
   String s=JOptionPane.showInputDialog("kök tahmin değerlerini arada bir boşluk birakarak giriniz: ");
   StringTokenizer token=new StringTokenizer(s);
   int n=token.countTokens()-1;
   int m=n+1;
   double a[]=new double[m];
   int j=0;
   while(token.hasMoreTokens())
   {
   Double ax=new Double(token.nextToken());
   a[j++]=ax.doubleValue();
   }
   return a;
}
    public static void main (String args[])
    {f1 ff=new f1();
    double [] x0=inputdata();
    double [] r= continuationRK4(ff,x0,4);
    double [] r1= continuationRK6(ff,x0,4);
    double [] r2=newton_continuationRK4(ff,x0,4);
    String s=" optimum continuity method with  RK4 : \n"+Matrix.toStringT(r);
    s+=" optimum continuity method with  RK6 : \n"+Matrix.toStringT(r1);
    s+=" optimum continuity method with Newton-Raphson plus RK4 : \n"+Matrix.toStringT(r2);
    String s1="multivariable continuity method for optimization : ";
    JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



SCHOLASTIC METHODS : GENETIC ALGORITHMS

NATURAL SELECTION AND SYSTEM OF GENES IN NATURE

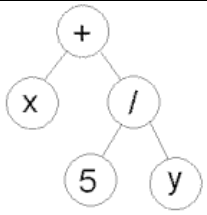
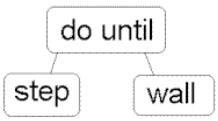
Genetic algorithms are a widely use modern programming method. This method based on natural selection mechanism in nature. Characteristics are carried in GENES, Father and Mother passes their genes to the child. Individuals carrying genes that given better characteristics to fit the nature has a

better chance of survival and to give offsprings, therefore more of their gene characteristics will be carried into the next generation. Sometimes gene structure passing throgh from parents can be changed by a mechanism called mutation. It is usuall a very small change. The result of this change could be deadly(less survival potential in environment) or sometimes make the individual fit the envronment better. If that is the case, his chance of passing his genes including this mutated one will be improved.

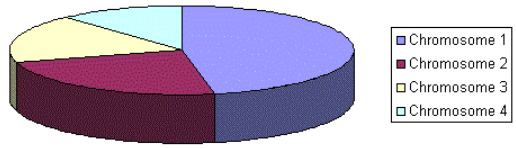
5.18.2 GENETIC ALGORITHMS, GENERAL DEFINITIONS

In general an algorithm of optimization by genetic algorithms can be summerized as:

[START] Set up a population of m member (one chromosome for each member, x gene-vector of independent variables)
[START] for each chromosome (vector of independent variables) calculate *f(x)* fitting value, repeat this for each individual in population
[NEW GENERATION] Generate new generations according to the following steps
[SELECTION] Select parents from population by considering their fitting values. Better fits should have a better chance to be selected but no one is eliminated completely from the selection process.
[PRODUCTION] Produce a new individual(s) from the parents. Replace parent generation with the new generation. Some of the successful parents can retain in the new generation
[MUTATION] mutate new generations with a ceratain (given) mutation rate
[ACCEPT] Add new individuals to the population.
[REPEAT] use the new generations to create new populations by using the same rule
[TEST] when the required conditions are occured or enough generations are generated stop the simulation
[LOOP] If the required conditions are not occured continue the process
Gens an choromosomes are the basic calculating units of the genetic calculations. Mathematically all the independent variales are an individual gene. . Chromsome is the vector of genes(independent varables). SAmple of some chromosome types utilised fort he computer simulation are given as the following table

chromosome A **chromosome B** **chromosome C**	**5.3243 0.4556 2.3293 2.4545** **ABDJEIFJDHDIERJFDLDFLFEGT** **(back), (foward), (right), (left), (turn)**
chromosome D	 (+ x (/ 5 y))
chromosome E	 (do_until step wall)

In the natural genes the fitting function is the survival in the nature, in mathmatical genes optimization (Maximisation) of the fitting function will be the base. Fitting functions usually added up and converted into the percantage form

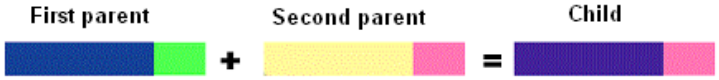


This percentage is called converted fitting function. Percentage in the total pipe is biger for the individuals with a beter fitting functions. Parents selected by using converted fitting functions and random number, so that individuals with a beter fitting has a beter chance to be selected. From the selected population of parent candidates actual parents are also selected by random selection. Then from parents gene the gene of th new population is created. Mathematically this process is called crossing. In crossing process genes of each parents are randomly selected to create the new child. For example if the gene is binary numbers and if single crossing is used, both parents gene is divided from a randomly selected point the first part of first parent is combined with the second part of second parent for the second part of the first parent combined with the first part of the second parent to crate a child.

For example for a binary group of 8 bit:

If 8*random(0-1) = 5 and if the first 5 bit is taken from the first parent and the remaining bits from the second parent : The new child:

11001011 + 11011111 = 11001111



Chromosome 1 (First Parent)	11011 | 00100110110
Chromosome 2 (Second parent)	11011 | 11000011110
1. Child	11011 | 11000011110
2. Child	11011 | 00100110110

If double crossing applied, two random numbers are generated and genes are divided into three parts. The first part is taken from the first parent, second part is taken from the second parent and the third part is taken from the first parent(or vice versa)

For example for a binary group of 8 bit i:

Random1=8*random(0-1) = 2

Random2=8*random(0-1) = 6

First 2 bit is taken from the first parent, from third to 6th bit is taken from the second parent and 7th and 8th bit is taken from the first parent.

11001011 + 11011111 = 11011111



As a third method every gene is selected by a random number decision process. If generated random number is smaller than 0.5 the first parent's binary digit is selected, if not second parents binary digit is selected.

11001011 + 11011111 = 11001011



5.18.3 GENETIC ALGORITHMS, OPTIMIZATION

In this section actual optimization process with genetic algorithms will be given in details. AS it is seen earlier in this chapter optimization process is minimisation or maximisation of a multivariable function

f(x1,x2,x3,…..xi)

It should again be rminded tt maximisation and minimisation is the same process with a sign change

max [f(x)] = min [- f(x)] = min [g(x)]

Another mathematical fact that will be used in genetic algorithms is that addition of a constant to the function will not change the optimum point

max [f(x)] = max [f(x) + C] = max [g()]

In genetic algorithm applications the nature of the algorithm is to find the maximum, in order to get correct percentages constant numbers are added into the function to ensure the resulting value is always positive.

In engineering optimization we usually face the real numbers. IF a real number is to be represented as a binary sequence, a conversion process between binary digits and real numbers are needed. If n+1 independent variable is existed as

(x0,x1,x2,..xi..xn)

In order to apply genetic algorithms lower and upper bounds of each variable should be known.

xo : xo_minimum xo_maximum

x1 : x1_minimum x1_maximum

….

xi : xi_minimum xi_maximum

…

xn : xn_minimum xn_maximum

definitins should be given. Ifbinary digit system is used number of representation digits also should be known. If the independent variable is defined as

xi_min <= xi <= xi_max N bit binary

Binary equivalent of N digits can be calculated as follows:

Binary group with n digit first converted into a 10 base integer number as

$$(b_n, b_{N-1}, b_{N-2}, \ldots b_1, b_0)_2 = (\sum_{i=0}^{N-1} b_i 2^i)_{10} = x'$$

Then the integer number is converted into a binary digit by using the following conversion process

$$x = x\_min + x'\left(\frac{x\_max - x\_min}{2^N - 1}\right)$$

Inverse process can also be apply to convert digits back to the binary digits. Each independent variables can be given with one gene, and the set of independent variables can be given b a a vector of genes. Evaluation function is assumd to be in the following form

```
class fa extends f_xj
{
 double func (double x[])
 { return (21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]));
 }}
```

The value sets of independent variables as whole together with the evaluated function value will be called as genotype (as borrowed from nature's genetic system). Number of chromosomes in each individual can be one or more. Number of individual will be set by the user of the program. For each individual starting variable values set by using random selection. Function values are calculated for each genotype. It is the fitting value of the genome. All fitting values are adde and then normalise to get percantage base fitting values. Percantage base fitting values are adde by starting from zero to 1and created cumulative fitting scale for all population. Cumulative fitting values are based on the scale of the fitting function. If a random number between 0 to 1 is selected, the chance of this number is fall in a region with a bigger fitting function is grater than that of a small one. After an individual in the population is selected as a parent cendidate, it can be removed from the selection list or it can remain in the list(a second selection can be allowed). From the selected parent candidates two parents are selected randomly to create a new child.In the new child population a selected percentage can be allowed (selected again randomly) to mutate. Mutation in a binary code can be achived to flip an

indivudual binary digit(from 0 to 1 or from 1 to 0). Process continues for a defined number of generations.

GENETIC ALGORITHMS, CLASS GENE

Genetic algorithms mimic the natural selection process of genes. The first structure used here to mimic natural genes are used binary numbers as defined in previous subchapter. Constructor of gene class can accept either double or binary numbers. Total bit length is a variable but it is limited to 64 bits (0—63). Gene class consist of gene crossing and mutation processes as well.

Program Gene class, java codeFew examples are given to explain gene processes. In the first example 5 63 bit gene is created and the results are listed

Program GeneTest class creating random genes 63 bits between 0 to 10 limit values, java code

```
public class geneTest
{
  public static void main(String arg[])
  {
  Gene x1=new Gene(0,10,63);
  for(int i=0;i<5;i++)
  { x1.setbits();System.out.println(x1.toString());}
  }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" geneTest
(1010000001011101010100011110011100000000000101101101010101101 01) 6.264239491429992
(1100100100000110111011000111000111110011010011111101110100001 10) 7.852618959616957
(0011100000011000010111010101011010100001001011110111010010100 11) 2.191217743394967
(1011010001110100011010100111101001111101101100001000010110100 10) 7.049013661432487
(1000010110011001101110011011111111100110010010100011010011100 00) 5.218769162659464
> Terminated with exit code 0.
```

Program 5.18.4-3 GeneTest1 class creating 63 bits between 0 to 10 limit values, value of 3.2, java code

```
public class geneTest1
{ public static void main(String arg[])
  { Gene x1=new Gene(3.2,0,10,63);System.out.println(x1.toString());}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" geneTest1
(0101000111101011100001010001111010111000010100011110110000000 00) 3.2
> Terminated with exit code 0.
```

Program GeneTest1 class creating 63 bits between 0 to 10 limit values, value of 3.2

```
public class geneTest2
{ public static void main(String arg[])
  {Gene x1=new Gene("0101000111101011100001010001111010111000010100011110110000000 00",0,10);
    System.out.println(x1.toString());}
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" geneTest2
(0101000111101011100001010001111010111000010100011110110000000 00) 3.2

> Terminated with exit code 0.
```

GENETIC ALGORITHMS, CLASSES GENOTYPE1 AND GENETIC1

Genotype1 can be considered as vector of Gene class variables. In means of Genetics, it represents a chromosome. In mathematically, it is a set of independent variables **(x0,x1,x2,..xi..xn)**
The main class of genetic algorithms is class Genetic1. Genetic optimization process with all the stages defined above calculated in this class. Some of the variables of Genetic1 should be defined by user. These variables are:

```
int    POPSIZE; //number of individuals in population
int    MAXGENS; //  maximum number of generations
int    NVARS;   // number of variables in chromosome (number of independent variables)
int    N;       //bit size of the binary genes
double PXOVER; //probability of cross-over;
double PMUTATION; //probability of mutation
```

method evaluate is the main method to do genetic algorithm calculations. It should be emhised again that all the function values in Genetic algorithms should be positive, if it is not the case, please add up a constant to convert the values into positive values. This proses does not change the optimum. Maximum of function $\mathbf{f(x_1,x_2,x_3,\ldots,x_n)}$ is same as the maximum of $\mathbf{g(x_1,x_2,x_3,\ldots,x_n) = f(x_1,x_2,x_3,\ldots,x_n) + C}$.
This process help to avoid errors during the calculations of cumulative percentage fitting function.

Program Genotype1 and Genetic1 classes, java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;
class genotype1i
{
// each genotype1 is member of the population
// Genotype is like a chromosone in real life genetic systems
// it can be thought as an evaluation point of a function with n variable
// f(xi)=f(x1,x2,x3,..,x_nvars)
public Gene G[]     ;  // string of variables makes a genotype1
public double Fitness;  // genotype1's fitness f(xi)
public double Upper[];  // gene's upper bound; xi_lower < xi < xi_upper
public double Lower[];  // Gene's lower bound; xi_lower < xi < xi_upper
public double RFitness; // the relative fitness
public double CFitness; // cumulative fitness
public int NVARS;       // Number of variables(Genes)
public int N;           // Number of bits defining each gene
public boolean Breeder;

 public genotype1i(int iNVARS,double low[],double high[],int iN,if_xj fi)
 {setgenotype1(iNVARS,low,high,iN,fi);}

 public void setgenotype1(int iNVARS,double low[],double high[],int iN,if_xj fi)
 {
 setN(iN);
 setNVARS(iNVARS);
 G=new Gene[NVARS];
 Upper=new double[NVARS];
 Lower=new double[NVARS];
 setLower(low);
 setUpper(high);
 setGene();
 setFitness(fi);
 }

 public genotype1i(genotype1i gi,if_xj fi)
 {setgenotype1(gi,fi);}
```

```
public void setgenotype1(genotype1i gi,if_xj fi)
{
setN(gi.N);
setNVARS(gi.NVARS);
G=new Gene[NVARS];
Upper=new double[NVARS];
Lower=new double[NVARS];
setLower(gi.Upper);
setUpper(gi.Lower);
for(int i=0;i<NVARS;i++)
  G[i]=gi.G[i].copyGene();
setFitness(fi);
}

public genotype1i copygenotype(if_xj fi)
{
genotype1i g1=new genotype1i(this,fi);
return g1;
}

public void setN(int iN)    {N=iN;}
public void setNVARS(int iNVARS)    {NVARS=iNVARS;}
public void setLower(double lbound[]) {for(int i=0;i<NVARS;i++) Lower[i]=lbound[i];}
public void setUpper(double ubound[]) {for(int i=0;i<NVARS;i++) Upper[i]=ubound[i];}
public void setFitness(if_xj fi)
{           Fitness=fi.func(getGene());}
public void setRFitness(double isum)  {RFitness=Fitness/isum;}
public void setCFitness(double ix)  {CFitness=ix;}
public double getLower(int j) {return Lower[j];}
public double getUpper(int j) {return Upper[j];}
public double getFitness() {return Fitness;}
public double getRFitness(){return RFitness;}
public double getCFitness(){return CFitness;}
public void breed() {Breeder=true;}
public void not_breed() {Breeder=false;}
public boolean isBreed() {return Breeder;}


public void setGene(int val)
{  G[val]=new Gene(Lower[val],Upper[val],N);}

public void setGene()
{  for(int j=0;j<NVARS;j++)
              setGene(j);
}

public double getGene(int val)
{ return G[val].getX();}

public double[] getGene()
{   double x[]=new double[NVARS];
            for(int j=0;j<NVARS;j++)
              x[j]=getGene(j);
            return x;
}

public String toString()
{
//x values, Fitness,relative fitness and cumulative fitness
String s="";
for(int j=0;j<NVARS;j++)
```

```
            s+=G[j].toString()+" F:";
  s+=getFitness();
  s+="RF:"+getRFitness();
  s+="CF:"+getCFitness();
  return s;
  }
}  //end of class genotype1

public class Genetic1i
{
            // Maximizing a function
            int    POPSIZE; //population size
            int    MAXGENS; //maximum number of generations;
            int    NVARS;   //number of independent variables of function
   int    N;       //number of genes in each variable
            double PXOVER; //probability of crossover;
   double PMUTATION; //proportion of mutated variables
   int Generation; //Current generation number
   int NBreeder; //Number of survivors in a generation
   int Best; //The best genotype1 in the population
   genotype1i Population[];
   //if_xj f;  // function to be avaluated by genetic algorithm

   public Genetic1i(int iPOPSIZE,int iMAXGENS,int iNVARS,int iN,
   double iPXOVER,double iPMUTATION)
   {
   POPSIZE=iPOPSIZE;
   MAXGENS=iMAXGENS;
   NVARS=iNVARS;
   N=iN;
   PXOVER=iPXOVER;
   PMUTATION=iPMUTATION;
   NBreeder=0;
   Population=new genotype1i[POPSIZE+1];
            }
            public void setPopulation(double low[],double up[],if_xj fi)
            {
            for(int i=0;i<POPSIZE+1;i++)
              { Population[i]=new genotype1i(NVARS,low,up,N,fi);}
            }

            public genotype1i[] copyPopulation(if_xj fi)
            {
            genotype1i Po[]=new genotype1i[POPSIZE+1];
            for(int i=0;i<POPSIZE+1;i++)
              { Po[i]=new genotype1i(Population[i],fi);
                Po[i].Fitness=Population[i].Fitness;
                        Po[i].RFitness=Population[i].RFitness;
                Po[i].CFitness=Population[i].CFitness;
              }
            return Po;
            }

            public void evaluate(if_xj fi)
            {
              int mem;
              int j;
              double x[]=new double[NVARS];
              for(mem=0;mem<POPSIZE;mem++)
              {
              //System.out.println(Population[mem].toString());
              Population[mem].setFitness(fi);
```

```
            if(Population[mem].getFitness() > Population[POPSIZE].getFitness())
              {
                    Best=mem;
                    Population[POPSIZE]=Population[mem];
              for(j=0;j<NVARS;j++)
                {
                Population[POPSIZE].G[j].copyGene(Population[mem].G[j]);
                      }//end of for(j=0..
                    }//end of if
            }//end of for(mem=0;..
          }


          public void setRFCF(genotype1i Pi[])
          {
          //calculates relative and cumulative fitness functions
          int mem;
          int PN=Pi.length;
          double sum=0.0;
          //total fitness of population
          for(mem=0;mem<PN;mem++)
            {sum+=Pi[mem].Fitness;}
          //System.out.println("sum="+sum);
          //calculate relative fitness of each genotype1
          for(mem=0;mem<PN;mem++)
            {Pi[mem].setRFitness(sum);}
          //calculate cumulative fitness
          Pi[0].setCFitness(Pi[0].getRFitness());
for(mem=1;mem<PN;mem++)
            {Pi[mem].setCFitness(Pi[mem-1].getCFitness()+Pi[mem].getRFitness());
            }
          }

          public void toString(genotype1i Pi[])
          {
          int mem;
          int PN=Pi.length;
          for(mem=0;mem<PN;mem++)
          {
          //list them
          System.out.println("Population["+mem+"]="+Pi[mem].toString());
          }
          }

          public void select(if_xj fi)
          {
          //select the new generation members of population
          double r;
          int mem;
          setRFCF(Population);
          //create a new population;
          genotype1i Po[]=copyPopulation(fi);
setRFCF(Po);
for(int i=0;i<POPSIZE;i++)
{   mem=0;
   r=Math.random();
   double gf1=Population[i].getCFitness();
            double gf2;
   if (gf1 > r) Po[i]=Population[mem].copygenotype(fi);
            for(mem=1;mem<POPSIZE;mem++)
     {
               gf2=Population[mem].getCFitness();
```

```
                if( gf2>=r && gf1< r ) {Po[i]=Population[mem].copygenotype(fi);break;}
        }
 }
 setRFCF(Po);
          Population=Po;
          evaluate(fi);
 //toString(Population);
 }

public void crossover(if_xj fi)
{
int i,count;
int mem;
int POne,a;
int PTwo,b;
int point;
//select members for breeding
int iselect[]=new int[POPSIZE];
int counter=0;
double r;
 for(mem=0;mem<POPSIZE;mem++)
          {
           r=Math.random();
           if(r < PXOVER)
             {
                    iselect[counter++]=mem;
             Population[mem].breed();
                    NBreeder++;
             }
           else
             Population[mem].not_breed();
          }
          //System.out.println("iselect="+Matrix.toString(iselect));
          //let also best of the population to breed
          Population[Best].breed();

//loop through the population select breeding pairs
 for(mem=0;mem<POPSIZE;mem++)
    {
      //select two in popolation in random
      a=(int)(Math.random()*NBreeder)+1;
      b=(int)(Math.random()*NBreeder)+1;
      count=0;
      POne=0;
      PTwo=0;
      //select two individuals for breeding
      for(i=0;i<POPSIZE;i++)
       {
       if(Population[i].isBreed())
         {
         count++;
         if(count==a) POne=count;
         if(count==b) PTwo=count;
                  }
                }
              //perform a crossover;
                genotype1i Po[]=new genotype1i[2];
                        Po[0]=Population[POne].copygenotype(fi);
                        Po[1]=Population[PTwo].copygenotype(fi);
                for(i=0;i<NVARS;i++)
                {
                        Population[POne].G[i].cross1(Math.random(),Po[0].G[i],Po[1].G[i]);
```

```
                        Population[PTwo].G[i].cross1(Math.random(),Po[0].G[i],Po[1].G[i]);
                        }
                        Population[POne].setFitness(fi);
                        Population[PTwo].setFitness(fi);
                }
              }

public void mutate(if_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*PMUTATION*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  Population[member].G[var].mutate();
          //after mutation recalculate the function value
  Population[member].setFitness(fi);
  }
}

public String report()
{
String s;
int i;
double best_value;
double avg;
double stdev;
double sum_square;
double square_sum;
double sum;
double xx;
sum=0;
sum_square=0.0;
for(i=0;i<POPSIZE;i++)
  {
          xx= Population[i].getFitness();
  sum+=xx;
  sum_square=xx*xx;
           }
avg=sum/(double)POPSIZE;
square_sum=sum*sum/(double)POPSIZE;
stdev=Math.sqrt(square_sum);
best_value=Population[POPSIZE].getFitness();
double aa[]=new double[NVARS+1];
for(i=0;i < NVARS;i++)
  {
  aa[i]=Population[POPSIZE].getGene(i);
          }
          aa[NVARS]=Population[POPSIZE].getFitness();

//s="Generation = "+Generation+"best value = "+Matrix.toString(aa)+"Average = "+avg+"Standart Deviation = "+stdev;
s="Generation = "+Generation+"best value = "+Matrix.toString(aa);
return s;
}

public double[] getBest()
```

```
  {
  double aa[]=new double[NVARS+1];
  for(int i=0;i < NVARS;i++)
   {
   aa[i]=Population[POPSIZE].getGene(i);
            }
            aa[NVARS]=Population[POPSIZE].getFitness();
  return aa;
  }

  public double[] calculate(if_xj fi,boolean report)
  {
  evaluate(fi);
  Generation=0;
  while(Generation<MAXGENS)
   {
   Generation++;
   NBreeder=(int)(0.8*POPSIZE);
   select(fi);
   crossover(fi);
   mutate(fi);
   evaluate(fi);
   //if(report)
     //System.out.println(report());
   }
  return getBest();
  }
}
```

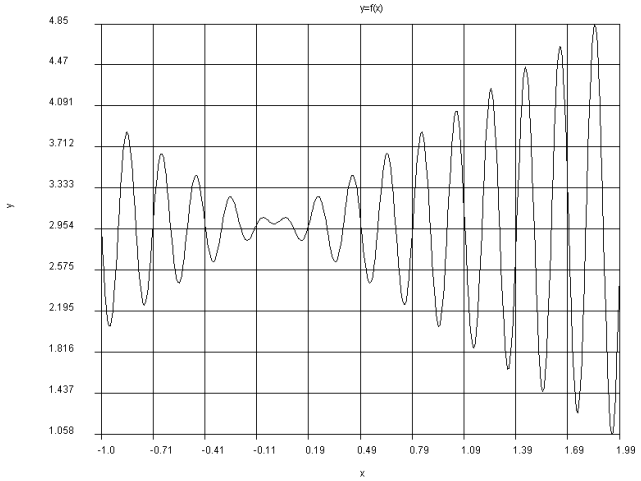
Now that a Genetic algorithm is available to us, optimization of actual functions can be carry out. The first example problem will be $f(x_0,x_1)=2x_0\ x_1+2x_0-x_0^2-2x_1^2+10$. This function was used in previous sub-sections of optimization, but an additional constant value of 10 is added to ensure positive cumulative numbers. Please remember that we do not have any dificulty solving this problem with previously given geometrical based optimization algorithms. Note that the result is probabilistic so that it will never be exactly the same like geometric methods.

Program Genetic1 class test program NA104i, java code

```
 import java.io.*;

class NA104i
{
   public static void main(String args[]) throws IOException
   {
            if_xj fx=(double x[])->2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1]+10.0;
   int N=30; //number of digits in binary genes
            int POPSIZE=200; //population size
   int MAXGENS=200; //maximum number of generations
   int NVARS=2; //number of variables (member of genes in genome
   double PXOVER=0.3; //probability of crossover
   double PMUTATION=0.02; //probability of mutation
   Genetic1i xx=new Genetic1i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   double low[]=new double[NVARS];
   double high[]=new double[NVARS];
   low[0]=-1.0;high[0]=5.0;
   low[1]=-1.0;high[1]=5.0;
   xx.setPopulation(low,high,fx);
   System.out.println("best=\n"+Matrix.toStringT(xx.calculate(fx,false)));
   }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA104i
best=
        1.998048807492525
        0.998046871272891
       11.999996185284427
> Terminated with exit code 0.
```

As a second example a one dimensional function $f(x_0)=x_0 \sin(10\pi x_0)+1$ function will be investigated . The plot of the function shows the difficulty of this function: it has many local maximum. It is very hard for geometric methods to get a solution of such function unless some jump mechanisms built into the solution.



Program Genetic1 class test program NA105, java code

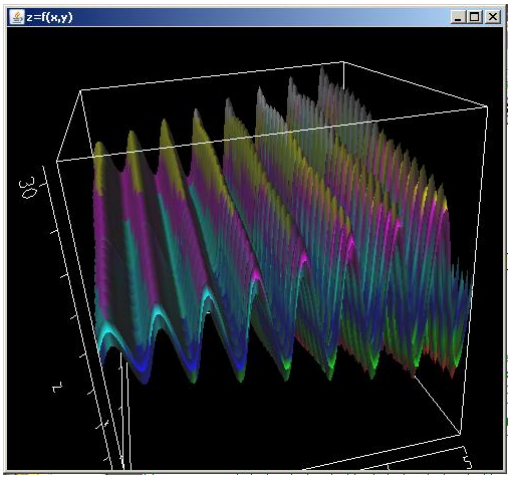
```
import java.io.*;

class NA105i
{
   public static void main(String args[]) throws IOException
   {
   int N=18;
          int POPSIZE=500;
   int MAXGENS=200;
   int NVARS=1;
   double PXOVER=0.3;
   double PMUTATION=0.02;
   Genetic1i xx=new Genetic1i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   double low[]={-1};
   double high[]={1};
   if_xj fx=(double x[])->(x[0]*Math.sin(10.0*Math.PI*x[0])+3.0);
   xx.setPopulation(low,high,fx);
   System.out.println("best=\n"+Matrix.toStringT(xx.calculate(fx,false)));
   }
}
```

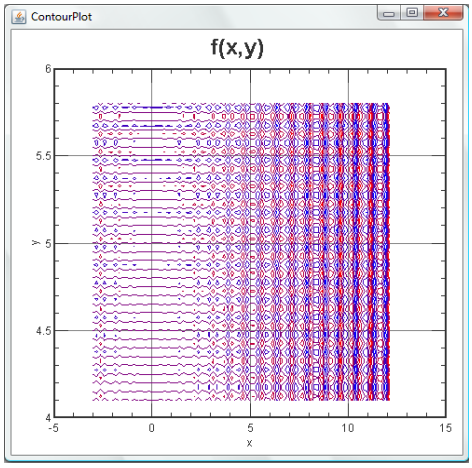
```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA105i
best=
        0.851073650641062
        3.850589565563568
> Terminated with exit code 0.
```

As a last example of class Genetic1 a two variable nonlimear function is defined as

f(x,y)=21.5+x*sin(4.0*π*y)+y*Math.sin(20.0* π *y)

The 3D plot of the function





It is clear from the plot that this problem also have the same kind of difficulty as the previous problem.

Program class test program NA106, java code

```
 import java.io.*;

class NA106i
{
   public static void main(String args[]) throws IOException
   {
   int N=18;
            int POPSIZE=100;
   int MAXGENS=300;
   int NVARS=2;
   double PXOVER=0.3;
   double PMUTATION=0.02;
   Genetic1i xx=new Genetic1i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   double low[]={-3,4.1};
   double high[]={12.1,5.8};
   if_xj fx=(double x[])->21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]);
   xx.setPopulation(low,high,fx);
   System.out.println("best=\n"+Matrix.toStringT(xx.calculate(fx,false)));
   }
}
```

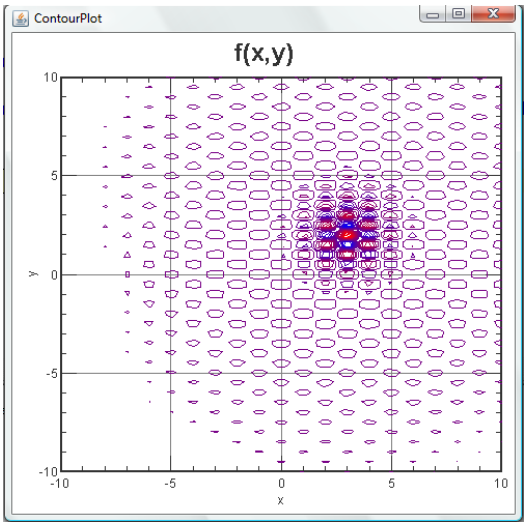
```
 ---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA106i
best=
          11.618906856181548
           5.025294209648932
          38.109299631193466


> Terminated with exit code 0.
```

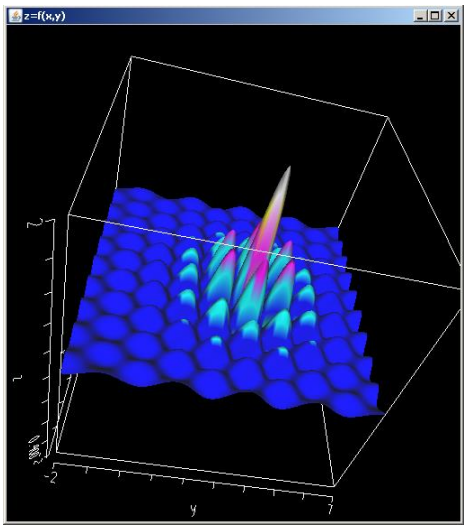
To see the statistical effect on the result if the same problem re-run with exactly the same parameters:

```
 ---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA106i
best=
          11.625358296807466
           4.923434156166672
          38.024865436893660


> Terminated with exit code 0.
```

Another interesting equation to optimise

z=f(x,y)= 1.0+cos(π*(x-3.0))*cos(2.0* π *(y-2.0))/(1+(x-3.0)*(x-3.0)+(y-2.0)*(y-2.0));





Program Genetic1 class test program NA107i, java code

```
 import java.io.*;

class NA107i
{
   public static void main(String args[]) throws IOException
   {
   int N=63;
   int POPSIZE=500;
   int MAXGENS=500;
   int NVARS=2;
   double PXOVER=0.3;
   double PMUTATION=0.02;
   Genetic1i xx=new Genetic1i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   double low[]={-10,-10};
   double high[]={10,10};
   if_xj fx=(double x[])->1.0+Math.cos(Math.PI*(x[0]-3.0))*Math.cos(2.0*Math.PI*(x[1]-2.0))/
   (1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0));
   xx.setPopulation(low,high,fx);
   System.out.println("best=\n"+Matrix.toStringT(xx.calculate(fx,false)));
   }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA107i
best=
        3.000030517579601
        1.999969482421839
        1.999999975157891

> Terminated with exit code 0.
```

We created similar program in Python programiing environment as well:

Gene class, Python code

```
 from abc import ABC, abstractmethod
from math import *
```

```
import random
from f_x import *


class Gene(ABC):

    def __init__(self, xming,xmaxg,N):
        self.xming=xming
        self.xmaxg=xmaxg
        self.N=N
        self.xN=2**N-1
        self.rand_float(N)


    #get a single bit of Gene
    def getBit(self,n):
        return self.xs[n]

    # creates a random string of p long 0 or 1 (binary)
    def rand_str(self,p):
        key1 = ""
        for i in range(p):
            key1=key1+str(random.randint(0, 1))
        return key1

    # set Gene from a string binary p long like "00101100.."
    def set_str(self,s):
        self.xs=s
        self.xp=int(s,2)
        self.x=self.xming+(self.xmaxg-self.xming)/self.xN*self.xp

    # create a random integer(10 base) xp based on p long binary
    def rand_int(self,p):
        key1 = ""
        for i in range(p):
            key1=key1+str(random.randint(0, 1))
        return int(key1,2)
    # create a random binary(2 base) xp based on p long binary
    def rand_bin(self,p):
        key1 = ""
        for i in range(p):
            key1=key1+str(random.randint(0, 1))
        x=int(key1,2)
        return bin(x)

    # create a random float xmin<=x<xmax
    # and set gene to this value
    def rand_float(self,p):
        key1 = ""
        for i in range(p):
            key1=key1+str(random.randint(0, 1))
        self.xs=key1
        self.xp=int(key1,2)
        self.x=self.xming+(self.xmaxg-self.xming)/self.xN*self.xp

    #print gene values
    def print(self):
        print("N=",self.N,"xs=",self.xs,"x=",self.x)

    #flip a single bit at position n if 0 to 1 if 1 to 0
    def flip(self,nn):
        if nn>self.N-1:
```

```
            nn=self.N-1
        else:
            nn=int(random.random()*self.N)
        if self.xs[nn]=='0': self.xs=self.xs[:nn]+'1'+self.xs[nn+1:]
        else: self.xs=self.xs[:nn]+'0'+self.xs[nn+1:]
        self.xp=int(self.xs,2)
        self.x=self.xming+(self.xmaxg-self.xming)/self.xN*self.xp

     #flip a single bit at a random position if 0 to 1 if 1 to 0
    def mutate(self):
        nn=int(random.random()*self.N)
        #print("nn=",nn)
        if self.xs[nn]=='0': self.xs=self.xs[:nn]+'1'+self.xs[nn+1:]
        else: self.xs=self.xs[:nn]+'0'+self.xs[nn+1:]
        self.xp=int(self.xs,2)
        self.x=self.xming+(self.xmaxg-self.xming)/self.xN*self.xp

    # split string binary gene into two pieces
    # seperated at point n
    def split(self,nn):
        #nn=int(random.random()*self.N)
        #print("nn=",nn)
        s1=self.xs[:nn]
        s2=self.xs[nn:]
        s=[s1,s2]
        return s

    # split string binary gene into three pieces
    # seperated at point n
    def split2(self,nn1,nn2):
        #nn=int(random.random()*self.N)
        #print("nn=",nn)
        if nn2>=nn1:
            s1=self.xs[:nn1]
            s2=self.xs[nn1:nn2]
            s3=self.xs[nn2:]
        else:
            s1=self.xs[:nn2]
            s2=self.xs[nn2:nn1]
            s3=self.xs[nn1:]
        s=[s1,s2,s3]
        return s

    # assign random value where xmin<=x<=xmax to gene
    def setbits(self):
        self.rand_float(self.N)

    def print(self):
        print("N=",self.N,"xs=",self.xs,"x=",self.x)

# assign s value where xmin<=x<=xmax to a gene
#return a new Gene
def setGene_str(s,xming,xmaxg):
    N=len(s)
    f=Gene(xming,xmaxg,N)
    f.xs=s
    f.xp=int(s,2)
    f.x=f.xming+(f.xmaxg-f.xming)/f.xN*f.xp
    return f

#creates a random int 0 or 1
def int_random():
```

```
    r=0
    if random.random()>=0.5:
        r=1
    return r

#creates a random int 0 or 1
def int3_random():
    r=0
    r1=random.random()
    if r1>=0.333 and r1<0.666:
        r=1
    else:
        r=2
    return r

#creates a random boolean True or False
def boolean_random():
    r=False
    if random.random()>=0.5:
        r=True
    return r

# assign x value where xmin<=x<=xmax to a gene
# return a new Gene
def setGene_float(x,xmin,xmax,N):
    f=Gene(xmin,xmax,N)
    f.x=x
    f.xp=int((f.x-f.xming)/(f.xmaxg-f.xming)*f.xN)
    ss=str(bin(f.xp))
    ss1=str(ss[2:])
    if len(ss1)!=N: ss1='0'+ss1
    f.xs=ss1
    return f

# cross correlation  1 of two Genes
def crossing1(f1,f2,N):
    s1=f1.split(N)
    s2=f2.split(N)
    ss1=s1[0]+s2[1]
    ss2=s2[0]+s1[1]
    ff1=setGene_str(ss1,f1.xming,f1.xmaxg)
    ff2=setGene_str(ss2,f2.xming,f2.xmaxg)
    if random.random()<=0.5:
        return ff1
    else:
        return ff2

# cross correlation2 of two Genes
def crossing2(f1,f2,N1,N2):
    s1=f1.split2(N1,N2)
    s2=f2.split2(N1,N2)
    ss1=s1[0]+s2[1]+s1[2]
    ss2=s2[0]+s1[1]+s2[2]
    ff1=setGene_str(ss1,f1.xming,f1.xmaxg)
    ff2=setGene_str(ss2,f2.xming,f2.xmaxg)
    if random.random()<=0.5:
        return ff1
    else:
        return ff2

# cross correlationN of two Genes
def crossingN(f1,f2):
```

```
    key1 = ""
    for i in range(f1.N):
        if boolean_random:
            key1=key1+f1.getBit(i)
        else:
            key1=key1+f2.getBit(i)
    fchild=setGene_str(key1,f1.xming,f1.xmaxg)
    return fchild

# cross correlationN of two Genes
# crossing1 crossing2 or crossing3 is selceted randomly
def crossingX(f1,f2):
    if int3_random()==0:
        N=int(f1.N*random.random())
        fchild=crossing1(f1,f2,N)
    elif int3_random()==1:
        N1=int(f1.N*random.random())
        N2=int(f1.N*random.random())
        if N1<=N2:
            fchild=crossing2(f1,f2,N1,N2)
        else:
            fchild=crossing2(f1,f2,N2,N1)
    else:
        fchild=crossingN(f1,f2)
    return fchild

# cross correlationN of two Genes
# crossing1 crossing2 or crossing3 is selceted randomly
def crossing(f1,f2):
    if int_random()==0:
        N=int(f1.N*random.random())
        fchild=crossing1(f1,f2,N)
    else:  # int_random()==1:
        N1=int(f1.N*random.random())
        N2=int(f1.N*random.random())
        if N1<=N2:
            fchild=crossing2(f1,f2,N1,N2)
        else:
            fchild=crossing2(f1,f2,N2,N1)
    return fchild
```

Few examples are given to explain gene processes. In the first example bit gene is created and cross correletion applied. The results are listed

GeneTest class creating random genes 32 bits between 0 to 3 limit values, Python

```
 N=32
xmin=0.0
xmax=3.0
f1=Gene(xmin,xmax,N)
f1.print()
f2=Gene(xmin,xmax,N)
f2.print()
f3=crossing(f1,f2)
f3.print()
```

```
runfile('E:/okul/SCO1/Gene_test.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, Gene, Genotype, Genetic
N= 32 xs= 11000001110101000110110100101110 x= 2.2714428632220818
N= 32 xs= 11110010101010111001100101111111 x= 2.8437927066916115
```

```
N= 32 xs= 11110010101010111001100101111110 x= 2.8437927059931196
```

Program 5.18.4-3 GeneTest1 class creating 63 bits between 0 to 10 limit values, value of 3.2

```
 from Gene import *
s="01010001111010111000010100011110101110000101000111101100000000"
f=setGene_str(s,0.0,10.0)
f.print()
print(f.split(5))
f1=setGene_float(3.2,0.0,10.0,f.N)
f1.print()
print(f1.split(5))
```

```
runfile('E:/okul/SCO1/Gene_test2.py', wdir='E:/okul/SCO1')
Reloaded modules: f_x
N= 63 xs= 0101000111101011100001010001111010111000010100011110110000000000 x= 3.2
['01010', '0011110101110000101000111101011100001010001111011000000000']
N= 63 xs= 0101000111101011100001010001111010111000010100011110110000000000 x= 3.2
['01010', '0011110101110000101000111101011100001010001111011000000000']

In [160]:
```

GENETIC ALGORITHMS, CLASSES GENOTYPE AND GENETIC IN PYTHON CODE

Class Genotype , Python (binary based)

```
 from abc import ABC, abstractmethod
from math import *
import random
from f_xj import *
from Gene import *


class Genotype(ABC):

    def __init__(self, f,NVARS,low1,high1,N):
        self.xmin=low1
        self.xmax=high1
        self.NVARS=NVARS
        self.N=N
        self.f=f
        self.G=[0.0 for j in range(self.NVARS)]
        self.x=[0.0 for j in range(self.NVARS)]
        for i in range(self.NVARS):
            self.G[i]=Gene(self.xmin[i],self.xmax[i],self.N)
            self.x[i]=self.G[i].x
        self.Fitness=self.f.func(self.x)
        self.RFitness=0.0
        self.CFitness=0.0
        self.Breeder=False

    def setGene(val):
        G[val]=Gene(self.xmin[val],self.xmax[val],self.N)
        return self.G[val]

    def getGene(val):
        return G[val]

    #print gene values
    def print(self):
        for i in range(self.NVARS):
            self.G[i].print()
        print("Fitness = ",self.Fitness)
        print("RFitness = ",self.RFitness)
```

```
        print("CFitness = ",self.CFitness)

    def setFitness(self):
        for i in range(self.NVARS):
            self.x[i]=self.G[i].x
        self.Fitness=self.f.func(self.x)

    def getFitness(self):
        return self.Fitness

    def setRFitness(self,sum):
        self.RFitness=self.Fitness/sum

    def setCFitness(self,x):
        self.CFitness=x

    def getCFitness(self):
        return self.CFitness

    def breed(self):
        self.Breeder=True

    def not_breed(self):
        self.Breeder=False

    def isBreed(self):
        return self.Breeder

    def copygenotype(self):
        g1=self
        return g1
    def mutate(self,PMUTATION):
        for i in range(self.NVARS):
            a=random.random()
            if a<PMUTATION:
                self.G[i].mutate()
def crossingGT(G1,G2):
    nv=G1.NVARS
    G3=Genotype(G1.f,G1.NVARS,G1.xmin,G1.xmax,G1.N)
    for i in range(nv):
        G3.G[i]=crossing(G1.G[i],G2.G[i])
```

Genotype examples:

Cross correlation and mutating processes, Python

```
from f_xj import *
from  Genotype import *

N=32
xmin=[0.0,0.0]
xmax=[5.0,5.0]
class f1(f_xj):func=lambda self,x: x[0]*x[0]+x[1]*x[1]
f = f1()
NVARS=2
N=32
G1=Genotype(f,NVARS,xmin,xmax,N)
G2=Genotype(f,NVARS,xmin,xmax,N)
print("G1 = ")
G1.print()
print("G2 = ")
G2.print()
print("breed")
G3=crossingGT(G1,G2)
G3.print()
G3.mutate(0.1)
G3.print()
```

```
runfile('E:/okul/SCO1/Genotype_test1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_x, Gene
G1 =
```

```
N= 32 xs= 01001111100000010011000011000011 x= 1.5528252014314814
N= 32 xs= 00000101110001011010001101101101 x= 0.11273486193100336
Fitness =  2.423975255295323
RFitness =  0.0
CFitness =  0.0
G2 =
N= 32 xs= 01110000101101101000001110011001 x= 2.2014247177172046
N= 32 xs= 00100110101001111111011001100010 x= 0.7550020168430642
Fitness =  5.416298833213369
RFitness =  0.0
CFitness =  0.0
breed
N= 32 xs= 01001111100000010011001111000011 x= 1.5528260955011532
N= 32 xs= 00100110101001111111011001100010 x= 0.7550020168430642
Fitness =  9.980521648925373
RFitness =  0.0
CFitness =  0.0
N= 32 xs= 01001111100000010011001111000011 x= 1.5528260955011532
N= 32 xs= 00100110101001111111011001100010 x= 0.7550020168430642
Fitness =  9.980521648925373
RFitness =  0.0
CFitness =  0.0
```

Class Genetic, Python

```
from abc import ABC, abstractmethod
from math import *
import random
from f_xj import *
from Genotype import *


class Genetic(ABC):
    def __init__(self,f,low,high,POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION):
        # function derived from f_xj class f(x0,x1,..xn)
        self.POPSIZE=POPSIZE   # population size
        self.MAXGENS=MAXGENS   # maksimum number of generations
        self.NVARS=NVARS    # number of variables
        #number of independent variable of function
        #number of Genes in genotype
        self.N=N
        #number of bits in a gene
        self.PXOVER=PXOVER
        #probability of cross-over
        self.PMUTATION=PMUTATION
        #proportion of mutated variables
        self.NBreeder=int(0.8*POPSIZE)
        #number of survivers in a generation
        self.Best=0
        self.f=f
        self.xmin=[0.0 for j in range(self.NVARS)]
        self.xmax=[0.0 for j in range(self.NVARS)]
        for i in range(self.NVARS):
            self.xmin[i]=low[i]
            self.xmax[i]=high[i]
        #Best genotype in the population
        self.Population=[0.0 for j in range(self.POPSIZE+1)]
        self.setPopulation()
        self.setRFCF()
        self.evaluate()

    def print(self):
        print("Population size = ",self.POPSIZE)
        print("Maximum number of Generations = ",self.MAXGENS)
        print("Number of Bits in Gene = ",self.N)
        print("Probability of crossover =",self.PXOVER)
        print("Propertion of mutated variables",self.PMUTATION)
        for i in range(self.POPSIZE+1):
            self.Population[i].print()


    def setPopulation(self):
        for i in range(self.POPSIZE+1):
            self.Population[i]=Genotype( self.f,self.NVARS,self.xmin,self.xmax,self.N)
```

```
            self.Population[i].setFitness()
        self.setRFCF()
        self.evaluate()


    def setRFCF(self):
        sum=0.0
        for i in range(self.POPSIZE):
            self.Population[i].setFitness()
            sum=sum+self.Population[i].Fitness

        for i in range(self.POPSIZE):
            self.Population[i].setRFitness(sum)
        sum=0.0
        for i in range(self.POPSIZE):
            sum=sum+self.Population[i].RFitness
            self.Population[i].setCFitness(sum)

    def evaluate(self):
        for i in range(self.POPSIZE):
            if self.Population[i].Fitness>self.Population[self.POPSIZE].Fitness:
                self.Population[self.POPSIZE]=self.Population[i]

    def breding_individual(self):
        #select members for breeding
        j=0
        for i in range(self.POPSIZE):
            self.Population[i].not_breed()
        for i in range(self.POPSIZE):
            r=random.random()
            if i==0 and r<self.Population[i].CFitness:
                j=i
            elif  r>=self.Population[i-1].CFitness and  r<=self.Population[i].CFitness:
                j=i
        return j


    def crossover(self):
        for mem in range(self.POPSIZE):
            POne= int(self.breding_individual())
            PTwo=int(self.breding_individual())
            Po=[0.0 for j in range(2)]
            Po[0]=self.Population[POne]
            Po[1]=self.Population[PTwo]
            self.Population[POne]=crossingGT(Po[0],Po[1])
            self.Population[PTwo]=crossingGT(Po[0],Po[1])
            self.Population[POne].mutate(self.PMUTATION)
            self.Population[PTwo].mutate(self.PMUTATION)
            self.setRFCF()
            self.evaluate()

    def getBest(self):
        aa=[0.0 for j in range(self.NVARS+1)]
        for i in range(self.NVARS):
            aa[i]=self.Population[self.POPSIZE].x[i]
        aa[self.NVARS]=self.Population[self.POPSIZE].Fitness
        return aa

    def select(self):
        Po=[0.0 for j in range(self.POPSIZE+1)]
        for i in range(self.POPSIZE+1):
            mem=0
            r=random.random()
            gf1=self.Population[i].getCFitness()
            gf2=0.0
            if gf1>r:
                Po[i]=self.Population[mem].copygenotype()
            for mm in range(1,self.POPSIZE):
                gf2=self.Population[mem].getCFitness()
                if gf2>r and gf1<r:
                    Po[i]=self.Population[mem].copygenotype()
                    break
        for i in range(self.POPSIZE):
            self.Population[i]=Po[i]
        self.setRFCF()
        self.evaluate()
```

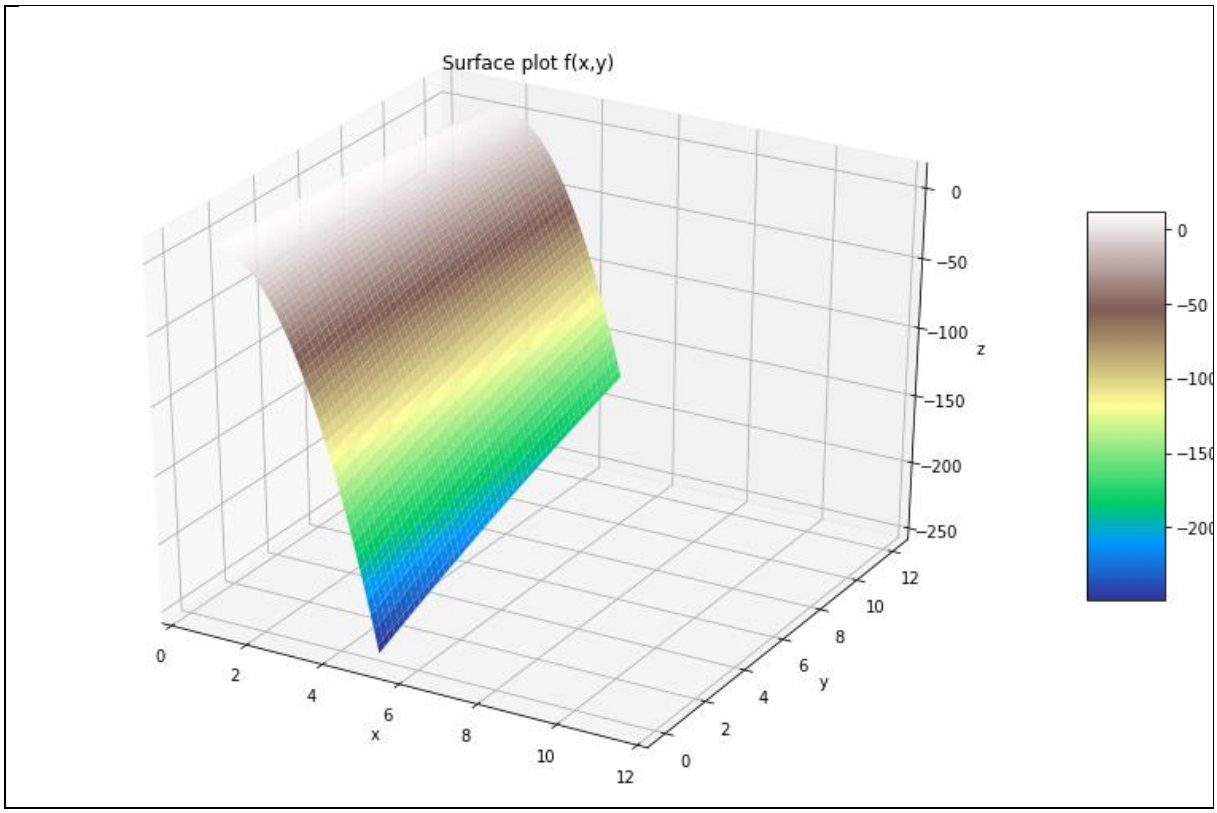
```
    def calculate(self):
        self.evaluate()
        Generation=0
        while Generation<self.MAXGENS:
            Generation=Generation+1
            self.crossover()
            self.evaluate()
        return self.getBest()

def setRFCF(Co):
    sum=0.0
    for i in range(Co.POPSIZE+1):
        Co.Population[i].setFitness()
        sum=sum+Co.Population[i].Fitness
    for i in range(Co.POPSIZE+1):
        Co.Population[i].setRFitness(sum)
        sum=0
        for i in range(Co.POPSIZE+1):
            sum=sum+Co.Population[i].RFitness
            Co.Population[i].setCFitness(sum)
```

Now that a Genetic algorithm is available to us, optimization of actual functions can be carry out. The first example problem will be $f(x_0,x_1)=2x_0\ x_1+2x_0-x_0^2-2x_1^2+10$. This function was used in previous sub-sections of optimization, but an additional constant value of 10 is added to ensure positive cumulative numbers. Please remember that we do not have any dificulty solving this problem with previously given geometrical based optimization algorithms. Note that the result is probabilistic so that it will never be exactly the same like geometric methods.

Genetic class test program , Python code

```
# -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
from f_xj import *
from Genetic import *
import  plot3D1 as P3D



class f1(f_xj):func=lambda self,x:2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1]+10.0
f=f1()

N=32 #number of digits in binary genes
POPSIZE=100 #population size
MAXGENS=50 # maximum number of generations
NVARS=2  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[1.0,1.0]
high1=[5.0,5.0]
g=Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION)
print(g.calculate())
xmin=low1[0]
xmax=high1[0]
Nx=100
ymin=0.0
ymax=12.0
Ny=100
PLabel="Optimisation: Genetic Algorithms"
XLabel="x"
YLabel="y"
ZLabel="z"
P3D.plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```

```
runfile('E:/okul/SCO1/Genetic_test2.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, Gene, Genotype, Genetic
[2.1468761491465562, 1.059217885150392, 11.988809270831494]
```



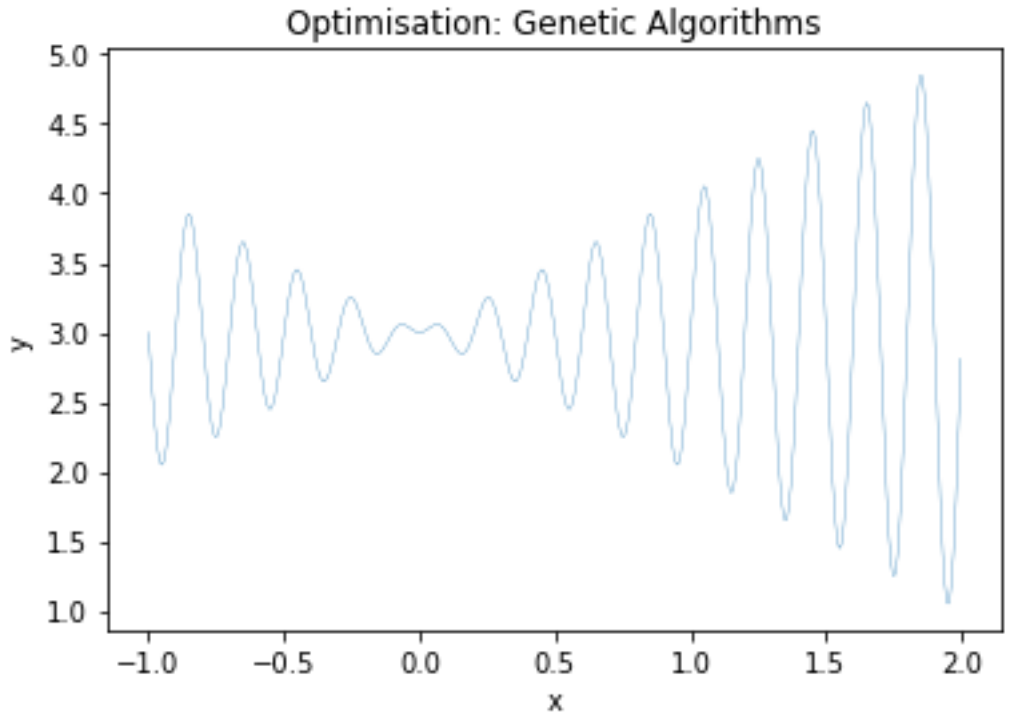
As a second example a one dimensional function $f(x_0)=x_0 \sin(10\pi x_0)+3$ function will be investigated . The plot of the function shows the difficulty of this function: it has many local maximum. It is very hard for geometric methods to get a solution of such function unless some jump mechanisms built into the solution.

Genetic algorithm test python code

```
# -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
from f_xj import *
from Genetic import *
import numpy as np;
import matplotlib.pyplot as plt;



class f1(f_xj):func=lambda self,x:x[0]*sin(10.0*pi*x[0])+3.0
f=f1()
class f2(f_x):func=lambda self,x:x*sin(10.0*pi*x)+3.0
ff=f2()
N=22 #number of digits in binary genes
POPSIZE=50 #population size
MAXGENS=50 # maximum number of generations
NVARS=1  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[-1.0]
high1=[2.0]
g=Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION)
print(g.calculate())

dx=3.0/1000.0
n=1000
y=[0 for z in range(n)];
x=[0 for z in range(n)];
for i in range(n):
    x[i]=-1.0+dx*i
```

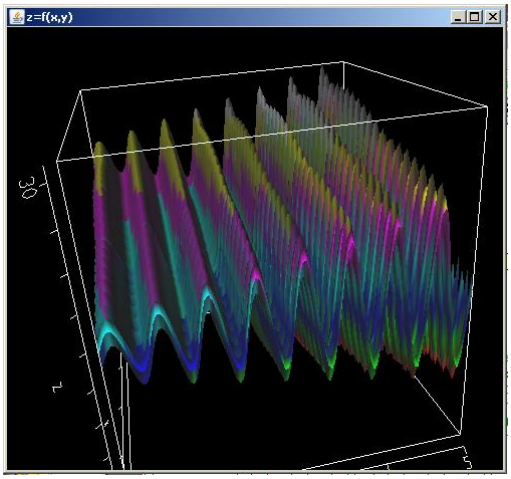
```
    y[i]=ff.func(x[i])
plt.plot(x,y,linewidth=0.3)
plt.xlabel("x")
plt.ylabel("y")
plt.title('Optimisation: Genetic Algorithms')
plt.show()
```

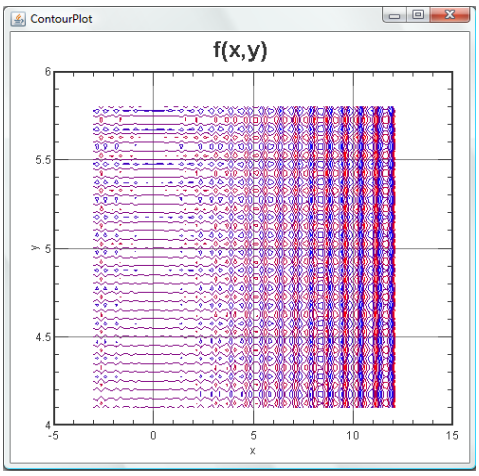
```
runfile('E:/okul/SCO1/Genetic_test.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, Gene, Genotype, Genetic
[1.8501109719541007, 4.850099728660397]
```



As a last example of class Genetic1 a two variable nonlimear function is defined as
f(x,y)=21.5+x*sin(4.0*π*y)+y*Math.sin(20.0* π *y)

The 3D plot of the function





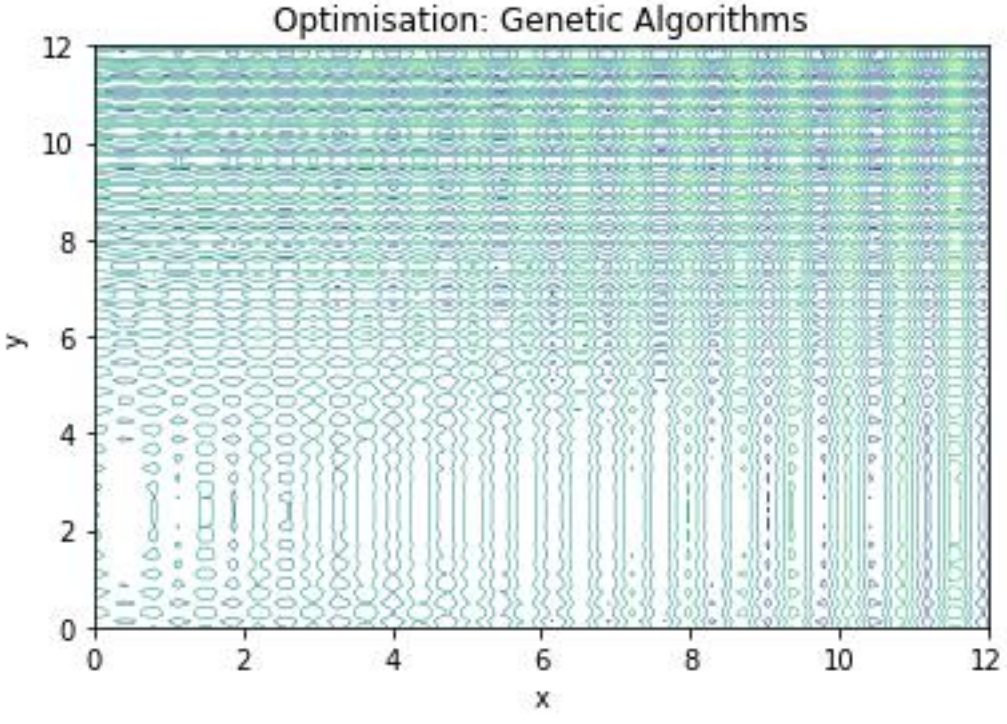
It is clear from the plot that this problem also have the same kind of difficulty as the previous problem.

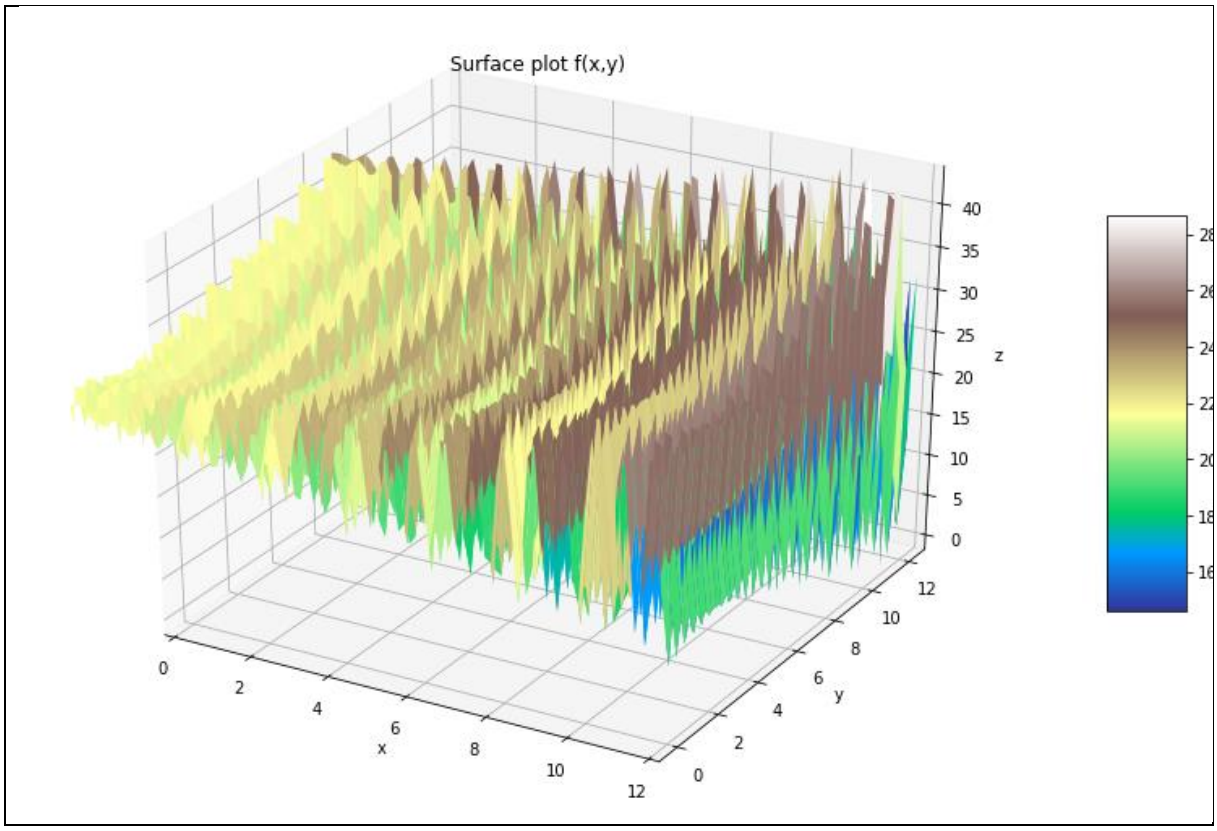
Program Genetic1 class test program genetictest1.py Python code

```
 # -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
```

```
from f_xj import *
from Genetic import *
import  plot3D1 as P3D



class f1(f_xj):func=lambda self,x:21.5+x[0]*sin(4.0*pi*x[0])+x[1]*sin(20.0*pi*x[1])
f=f1()

N=32 #number of digits in binary genes
POPSIZE=100 #population size
MAXGENS=50 # maximum number of generations
NVARS=2  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[-3.0,4.1]
high1=[12.1,5.8]
g=Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION)
print(g.calculate())
xmin=low1[0]
xmax=high1[0]
Nx=100
ymin=0.0
ymax=30.0
Ny=100
PLabel="Optimisation: Genetic Algorithms"
XLabel="x"
YLabel="y"
ZLabel="z"
P3D.plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```

```
runfile('E:/okul/SCO1/Genetic_test1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, Gene, Genotype, Genetic
[11.634943575466737, 5.224075439251976, 38.259492805578425]
```





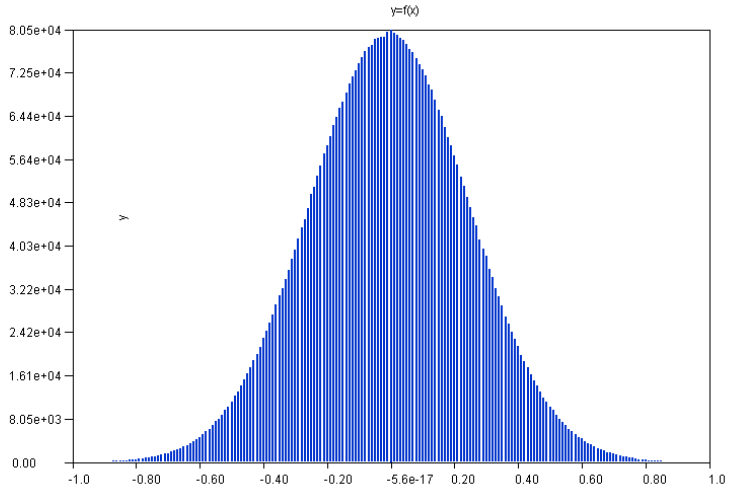
GENETİC ALGORITHMS BY USING GAUSS RANDOM NUMBERS

In the previous model, binary numbers are used as the base of constructing the genetic structure. In this second model Gauss random numbers will be used as the basis of the model. Gauss random numbers construct a bell shape distribution around the given point. We will start to create a Gauss random number algorithm.

Program 5.18.6-1Gausrandom method example , java code

```
import java.io.*; //java girdi cikti sinifini cagir
class randeneme
{
public static double  gaussrandom(double xort,double r)
{ int iset=0;
 double gset=0;
 double fac,rsq,v1,v2;
 double aret=0;
 if(iset==0)
 {
  do{
            v1=2.0*Math.random()-1.0;
            v2=2.0*Math.random()-1.0;
            rsq=v1*v1+v2*v2;
    }while(rsq>=1.0 || rsq==0.0);
  fac=Math.sqrt(-2.0*Math.log(rsq)/rsq);
  gset=v1*fac;
  iset=1;
  aret=v2*fac;
 }
 else
 {
 iset=0;
 aret=gset;
 }
 return aret/8.0*r+xort;
```

```
 }

public static void main(String args[])
 {
  double xort,r;
  double gr=0;
  double x[]=new double[201];
  double y[]=new double[201];
  xort=0.0;
  r=2.0;
  for(int i=0;i<=200;i++) {x[i]=xort-r/2+r*(double)i/200.0;}
  for(int i=0;i<=5000000;i++)
     {gr=gaussrandom(xort,r);for(int j=0;j<200;j++){if(gr>=x[j] && gr< x[j+1]){y[j]+=1.0;}}}
  Plot pp=new Plot(x,y);
  pp.setColor(0,0,51,204);
  pp.setPlotType(0,41);
  pp.plot();
}
 }
```



In the example program 5000000 Gauss random points are generated and plotted. Now genetic classes can be generated by using this Gauss Random numbers concept. This set of program is very similar except definition of the genes. With this type of gene definitions when cross-over of the genes are calculated border values should be considered carefully and not to exceed.

Program Gausrandom Gene class Gen.java , java code

```
 // Dr. M. Turhan ÇOBAN
// EGE University School of Engineering, Mechanical Engineering Department
// package : Numerical analysis package
// subject : optimization
// group   : Genetic Algorithms gauss random method
// Gen Class
// Gen class generates a double number represent a gene

import java.util.*;
import java.io.*;

public class Gen extends Object
{
public double xmin,xmax;//maximum and minimum limits of double number
public double x;

public  double  gaussrandom(double ixmin,double ixmax,double ix)
{
```

```
  double d1=ixmax-ix;
  double d2=ix-ixmin;
  double d=Math.abs(Math.min(d1,d2));
  ix=gaussrandom(ix,d);
  return ix;
}

public  void  gaussrandom()
{
  double d1=xmax-x;
  double d2=x-xmin;
  double d=Math.abs(Math.min(d1,d2));
  x=gaussrandom(x,d);
}

public  double  gaussrandom(double xort,double r)
{ //r yayılma bölgesi
   //xort ortalama değer
 int iset=0;
 double gset=0;
 double fac,rsq,v1,v2;
 double aret=0;
 if(iset==0)
 {
   do{
                v1=2.0*Math.random()-1.0;
                v2=2.0*Math.random()-1.0;
                rsq=v1*v1+v2*v2;
     }while(rsq>=1.0 || rsq==0.0);
   fac=Math.sqrt(-2.0*Math.log(rsq)/rsq);
   gset=v1*fac;
   iset=1;
   aret=v2*fac;
 }
 else
 {
 iset=0;
 aret=gset;
 }
 return aret/4.0*r+xort;
}

public Gen(double ixmin,double ixmax)
{//randomly generate the gene
 xmin=ixmin;
 xmax=ixmax;
 x=double_random();
}

public Gen(double s,double ixmin,double ixmax)
{
xmin=ixmin;
xmax=ixmax;
//generate gene equal to a given string
if((s>=xmin)&& (s<=xmax)) x=s;
else if(s<xmin) x=xmin;
else if(s>xmax) x=xmax;
}


public Gen()
{
xmin=0;
xmax=1.0;
x=xmin+Math.random();
}

public Gen(Gen g)
{
xmin=g.xmin;
```

```
xmax=g.xmax;
x=g.x;
}

public void copyGen(Gen g)
{
xmin=g.xmin;
xmax=g.xmax;
x=g.x;
}

public Gen copyGen()
{
 Gen g=new Gen(this);
 return g;
}

public Gen(Gen Father,Gen Mother)
{setGen(Father,Mother);}

public void  setGen(Gen Father,Gen Mother)
{
xmin=Father.xmin;
xmax=Father.xmax;
double r1=xmax-xmin;
double r=Math.random();

double d1=xmax-Father.x;
double d2=Father.x-xmin;
double d=Math.abs(Math.min(d1,d2));

double r2=gaussrandom(Father.x,d);
d1=xmax-Mother.x;
d2=Mother.x-xmin;
d=Math.abs(Math.min(d1,d2));
double r3=gaussrandom(Mother.x,d);
x=r2*r+r3*(1-r);
System.out.println("r2="+r2+"r3="+r3+"r="+r);
if(x<xmin) x=xmin;
else if(x>xmax) x=xmax;

}

public void cross1(double crossratio,Gen Father,Gen Mother)
{xmin=Father.xmin;
xmax=Father.xmax;
double r1=xmax-xmin;
double d1=xmax-Father.x;
double d2=Father.x-xmin;
double d=Math.abs(Math.min(d1,d2));
double gr1=gaussrandom(Father.x,d);
d1=xmax-Mother.x;
d2=Mother.x-xmin;
d=Math.abs(Math.min(d1,d2));
double gr2=gaussrandom(Mother.x,d);
x=gr1*crossratio+gr2*(1-crossratio);
if(x<xmin) x=xmin;
else if(x>xmax) x=xmax;
}

public void cross2(double crossratio,Gen Father,Gen Mother)
{xmin=Father.xmin;
xmax=Father.xmax;
double r1=xmax-xmin;
double d1=xmax-Father.x;
double d2=Father.x-xmin;
double d=Math.abs(Math.min(d1,d2));
double gr1=gaussrandom(Father.x,d);
d1=xmax-Mother.x;
d2=Mother.x-xmin;
```

```
d=Math.abs(Math.min(d1,d2));
double gr2=gaussrandom(Mother.x,d);
x=gr1*(1.0-crossratio)+gr2*crossratio;
if(x<xmin) x=xmin;
else if(x>xmax) x=xmax;
}
public double getX()
{//return double number equivalent of bit set number
 return x;
}
public boolean boolean_random()
{
if(Math.random()<=0.5) return false;
else return true;
}

public double double_random()
{ return (xmin+Math.random()*(xmax-xmin));}

public void mutate()
{
  double r=(xmax-xmin);
  double d1=xmax-x;
  double d2=x-xmin;
  double d=Math.abs(Math.min(d1,d2));
  x=gaussrandom(x,d);
}}
```

Program Gausrandom Genome (vector of gene) class class gentype.java , java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;


class gentypei
{
// each genotype1 is member of the population
// Genotype is like a chromosone in real life genetic systems
// it can be thought as an evaluation point of a function with n variable
// f(xi)=f(x1,x2,x3,..,x_nvars)
public Gen G[]      ;  // string of variables makes a genotype1
public double Fitness;  // genotype1's fitness f(xi)
public double Upper[];  // gene's upper bound; xi_lower < xi < xi_upper
public double Lower[];  // Gene's lower bound; xi_lower < xi < xi_upper
public double RFitness; // the relative fitness
public double CFitness; // cumulative fitness
public int NVARS;       // Number of variables(Genes)
public int N;           // Number of bits defining each gene
public boolean Breeder;

  public gentypei(int iNVARS,double low[],double high[],int iN,if_xj fi)
  {setgentypei(iNVARS,low,high,iN,fi);}

  public void setgentypei(int iNVARS,double low[],double high[],int iN,if_xj fi)
  {
  setN(iN);
  setNVARS(iNVARS);
  G=new Gen[NVARS];
  Upper=new double[NVARS];
  Lower=new double[NVARS];
  setLower(low);
  setUpper(high);
  setGen();
  setFitness(fi);
  }

  public gentypei(gentypei gi,if_xj fi)
  {setgentypei(gi,fi);}
```

```
 public void setgentypei(gentypei gi,if_xj fi)
 {
 setN(gi.N);
 setNVARS(gi.NVARS);
 G=new Gen[NVARS];
 Upper=new double[NVARS];
 Lower=new double[NVARS];
 setLower(gi.Upper);
 setUpper(gi.Lower);
 for(int i=0;i<NVARS;i++)
   G[i]=gi.G[i].copyGen();
 setFitness(fi);
 }

 public gentypei copygentypei(if_xj fi)
 {
 gentypei g1=new gentypei(this,fi);
 return g1;
 }

 public void setN(int iN)    {N=iN;}
 public void setNVARS(int iNVARS)    {NVARS=iNVARS;}
 public void setLower(double lbound[]) {for(int i=0;i<NVARS;i++) Lower[i]=lbound[i];}
 public void setUpper(double ubound[]) {for(int i=0;i<NVARS;i++) Upper[i]=ubound[i];}
 public void setFitness(if_xj fi)
 {              Fitness=fi.func(getGen());}
 public void setRFitness(double isum)  {RFitness=Fitness/isum;}
 public void setCFitness(double ix)  {CFitness=ix;}
 public double getLower(int j) {return Lower[j];}
 public double getUpper(int j) {return Upper[j];}
 public double getFitness() {return Fitness;}
 public double getRFitness(){return RFitness;}
 public double getCFitness(){return CFitness;}
 public void breed() {Breeder=true;}
 public void not_breed() {Breeder=false;}
 public boolean isBreed() {return Breeder;}


 public void setGen(int val)
 {  G[val]=new Gen(Lower[val],Upper[val]);}

 public void setGen()
 {  for(int j=0;j<NVARS;j++)
                 setGen(j);
 }

 public double getGen(int val)
 { return G[val].getX();}

 public double[] getGen()
 {   double x[]=new double[NVARS];
               for(int j=0;j<NVARS;j++)
                  x[j]=getGen(j);
               return x;
 }

 public String toString()
 {
 //x values, Fitness,relative fitness and cumulative fitness
 String s="";
 for(int j=0;j<NVARS;j++)
             s+=G[j].toString()+" F:";
 s+=getFitness();
 s+="RF:"+getRFitness();
 s+="CF:"+getCFitness();
 return s;
 }
}  //end of class gentypei
```

Program Gausrandom Genetic algorithm Genetic2i.java , java code

```
public class Genetic2
{
        // Maximizing a function
        int    POPSIZE; //population size
        int    MAXGENS; //maximum number of generations;
        int    NVARS;   //number of independent variables of function
  int    N;       //number of genes in each variable
        double PXOVER; //probability of crossover;
  double PMUTATION; //proportion of mutated variables
  int Generation; //Current generation number
  int NBreeder; //Number of survivors in a generation
  int Best; //The best gentype in the population
  gentype Population[];
  //f_xj f;  // function to be avaluated by genetic algorithm

  public Genetic2(int iPOPSIZE,int iMAXGENS,int iNVARS,int iN,
  double iPXOVER,double iPMUTATION)
  {
  POPSIZE=iPOPSIZE;
  MAXGENS=iMAXGENS;
  NVARS=iNVARS;
  N=iN;
  PXOVER=iPXOVER;
  PMUTATION=iPMUTATION;
  NBreeder=0;
  Population=new gentype[POPSIZE+1];
        }
        public void setPopulation(double low[],double up[],f_xj fi)
        {
        for(int i=0;i<POPSIZE+1;i++)
          { Population[i]=new gentype(NVARS,low,up,N,fi);}
        }

        public gentype[] copyPopulation(f_xj fi)
        {
        gentype Po[]=new gentype[POPSIZE+1];
        for(int i=0;i<POPSIZE+1;i++)
          { Po[i]=new gentype(Population[i],fi);
            Po[i].Fitness=Population[i].Fitness;
                    Po[i].RFitness=Population[i].RFitness;
            Po[i].CFitness=Population[i].CFitness;
          }
        return Po;
        }

        public void evaluate(f_xj fi)
        {
          int mem;
          int j;
          double x[]=new double[NVARS];
          for(mem=0;mem<POPSIZE;mem++)
          {
          //System.out.println(Population[mem].toString());
          Population[mem].setFitness(fi);
          if(Population[mem].getFitness() > Population[POPSIZE].getFitness())
            {
                    Best=mem;
                    Population[POPSIZE]=Population[mem];
            for(j=0;j<NVARS;j++)
              {
              Population[POPSIZE].G[j].copyGen(Population[mem].G[j]);
                      }//end of for(j=0..
                    }//end of if
          }//end of for(mem=0;..
        }


        public void setRFCF(gentype Pi[])
        {
```

```
            //calculates relative and cumulative fitness functions
            int mem;
            int PN=Pi.length;
            double sum=0.0;
            //total fitness of population
            for(mem=0;mem<PN;mem++)
              {sum+=Pi[mem].Fitness;}
            //System.out.println("sum="+sum);
            //calculate relative fitness of each gentype
            for(mem=0;mem<PN;mem++)
              {Pi[mem].setRFitness(sum);}
            //calculate cumulative fitness
            Pi[0].setCFitness(Pi[0].getRFitness());
 for(mem=1;mem<PN;mem++)
              {Pi[mem].setCFitness(Pi[mem-1].getCFitness()+Pi[mem].getRFitness());
              }
            }

            public void toString(gentype Pi[])
            {
            int mem;
            int PN=Pi.length;
            for(mem=0;mem<PN;mem++)
            {
            //list them
            System.out.println("Population["+mem+"]="+Pi[mem].toString());
            }
            }

            public void select(f_xj fi)
            {
            //select the new generation members of population
            double r;
            int mem;
            setRFCF(Population);
            //create a new population;
            gentype Po[]=copyPopulation(fi);
 setRFCF(Po);
 for(int i=0;i<POPSIZE;i++)
 {   mem=0;
    r=Math.random();
    double gf1=Population[i].getCFitness();
               double gf2;
    if (gf1 > r) Po[i]=Population[mem].copygentype(fi);
               for(mem=1;mem<POPSIZE;mem++)
      {
                 gf2=Population[mem].getCFitness();
                 if( gf2>=r && gf1< r ) {Po[i]=Population[mem].copygentype(fi);break;}
      }
 }
 setRFCF(Po);
            Population=Po;
            evaluate(fi);
 //toString(Population);
 }

public void crossover(f_xj fi)
{
int i,count;
int mem;
int POne,a;
int PTwo,b;
int point;
//select members for breeding
int iselect[]=new int[POPSIZE];
int counter=0;
double r;
 for(mem=0;mem<POPSIZE;mem++)
            {
             r=Math.random();
```

```
            if(r < PXOVER)
              {
                    iselect[counter++]=mem;
              Population[mem].breed();
                    NBreeder++;
              }
            else
              Population[mem].not_breed();
          }
          //System.out.println("iselect="+Matrix.toString(iselect));
          //let also best of the population to breed
          Population[Best].breed();

//loop through the population select breeding pairs
 for(mem=0;mem<POPSIZE;mem++)
    {
      //select two in popolation in random
      a=(int)(Math.random()*NBreeder)+1;
      b=(int)(Math.random()*NBreeder)+1;
      count=0;
      POne=0;
      PTwo=0;
      //select two individuals for breeding
      for(i=0;i<POPSIZE;i++)
        {
       if(Population[i].isBreed())
         {
         count++;
         if(count==a) POne=count;
         if(count==b) PTwo=count;
                  }
                }
              //perform a crossover;
                gentype Po[]=new gentype[2];
                      Po[0]=Population[POne].copygentype(fi);
                      Po[1]=Population[PTwo].copygentype(fi);
                for(i=0;i<NVARS;i++)
                {
                      Population[POne].G[i].cross1(Math.random(),Po[0].G[i],Po[1].G[i]);
                      Population[PTwo].G[i].cross1(Math.random(),Po[0].G[i],Po[1].G[i]);
                      }
                      Population[POne].setFitness(fi);
                      Population[PTwo].setFitness(fi);
              }
            }

public void mutate(f_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*PMUTATION*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  Population[member].G[var].mutate();
          //after mutation recalculate the function value
  Population[member].setFitness(fi);
  }
}

public String report()
{
String s;
int i;
double best_value;
```

```
  double avg;
  double stdev;
  double sum_square;
  double square_sum;
  double sum;
  double xx;
  sum=0;
  sum_square=0.0;
  for(i=0;i<POPSIZE;i++)
    {
                xx= Population[i].getFitness();
    sum+=xx;
    sum_square=xx*xx;
                }
  avg=sum/(double)POPSIZE;
  square_sum=sum*sum/(double)POPSIZE;
  stdev=Math.sqrt(square_sum);
  best_value=Population[POPSIZE].getFitness();
  double aa[]=new double[NVARS+1];
  for(i=0;i < NVARS;i++)
   {
   aa[i]=Population[POPSIZE].getGen(i);
                }
                aa[NVARS]=Population[POPSIZE].getFitness();

  //s="Generation = "+Generation+"best value = "+Matrix.toString(aa)+"Average = "+avg+"Standart Deviation = "+stdev;
  s="Generation = "+Generation+"best value = "+Matrix.toString(aa);
  return s;
  }

  public double[] getBest()
  {
  double aa[]=new double[NVARS+1];
  for(int i=0;i < NVARS;i++)
   {
   aa[i]=Population[POPSIZE].getGen(i);
                }
                aa[NVARS]=Population[POPSIZE].getFitness();
  return aa;
  }

  public double[] calculate(f_xj fi,boolean report)
  {
  evaluate(fi);
  Generation=0;
  while(Generation<MAXGENS)
    {
    Generation++;
    NBreeder=(int)(0.8*POPSIZE);
    select(fi);
    crossover(fi);
    mutate(fi);
    evaluate(fi);
    //if(report)
       //System.out.println(report());
    }
  return getBest();
  }
}
```

Program Gausrandom Genetic algorithm test program class NA108.java , java code

```
 import java.io.*;

class NA108

{
    public static void main(String args[]) throws IOException
    {
```

```
    int N=50;
            int POPSIZE=600;
    int MAXGENS=600;
    int NVARS=2;
    double PXOVER=0.3;
    double PMUTATION=0.02;
    if_xj fx=(double x[])->2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1]+10.0;
    Genetic2i xx=new Genetic2i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
    double low[]={-1,-1};
    double high[]={5,5};
    xx.setPopulation(low,high,fx);
    System.out.println("best=\n"+Matrix.toStringT(xx.calculate(fx,false)));
    }
}
```

The output of the test program:

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA108
best=
          2.009221343823353
          1.001161197373327
         11.999933685659865


> Terminated with exit code 0.
```

5.18.7 GENETİC ALGORITHMS BY USING A LONG TYPE VARIABLE AS A GENE

In a gene structure number of digits in the gene should be constant. It is similar in the natural system as well. In order to produce a child gene structure of both parents should be similar in structure . **A long integer with a setted number of digits** will be used as a gene structure in the next genetic algorithm code. In this two variable systems long type integer value bits can take all the possible values 0 to 9 in each digit, while double value can only corresponds the working space. As a maximum long number 18 digit x'_max=999999999999999999L is taken. In this case the relation between long number x' and double number x can be given as:

$$x = x_{min} + x'\left(\frac{x_{max} - x_{min}}{999999999999999999L}\right)$$

The double number can be converted back to the long integer gene number the same way.

$$x' = 999999999999999999L\left(\frac{x - x_{min}}{x_{max} - x_{min}}\right)$$

The remaining of genetic algorithm is the same as the previous algorithms. It should be note that even though the number of the gens are constant (18) for this algorithm, gen number input variable is kept in the program constructor, in order to make it same as the previous versions with the variable gene numbers.

Program 5.18.7-1 Gene constructed from long type integer numbers of 18 digits. Java code

```
// Dr. M. Turhan ÇOBAN
// EGE University School of Engineering, Mechanical Engineering Department
// package : Numerical analysis package
// subject : optimization
// group   : Genetic Algorithms
// Gene13 Class
// Gene3 class generates integer array to represent a double number
// actual binary defination changes depends on number of bits, xmin and xmax
// binary data stored in b BitSet array

import java.util.*;
import java.io.*;

public class Gene3 extends Object
{
//long imin=0;
int N=18;
```

```
public long imax=999999999999999999L;
public long ix;
public double x;//Limit value for integer(2^N-1)
public double xmin,xmax;//maximum and minimum limits of double number

public Gene3(double ixmin,double ixmax)
{//randomly generate the gene
 xmin=ixmin;
 xmax=ixmax;
 ix=(long)(imax*Math.random());
 x=xmin+(xmax-xmin)*(double)ix/(double)imax;
}


public Gene3(long ixi,double ixmin,double ixmax)
{//randomly generate the gene
 xmin=ixmin;
 xmax=ixmax;
 ix=ixi;
 x=xmin+(double)ix*(xmax-xmin)/(double)imax;
}

public Gene3(Gene3 g)
{
// copy a Gene from another one
copyGene(g);
}

public void copyGene(Gene3 g)
{
 ix=g.ix;
 x=g.x;
 xmin=g.xmin;
 xmax=g.xmax;
}

public Gene3 copyGene()
{Gene3 g=new Gene3(x,xmin,xmax);
 return g;
}

public Gene3(Gene3 Father,Gene3 Mother)
{cross(Math.random(),Father,Mother);}

public Gene3(double xi,double maxi,double mini)
{setGene(xi,maxi,mini);}


public Gene3(double crossratio,Gene3 Father,Gene3 Mother)
{   if(boolean_random()) cross(crossratio,Father,Mother);
   else                  cross(crossratio,Mother,Father);
}

public void  setGene(long iix,double ixmin,double ixmax)
{
 xmin=ixmin;
 xmax=ixmax;
 ix=iix;
 x=xmin+(xmax-xmin)*(double)ix/(double)imax;
}

public void  setGene(double idx,double ixmin,double ixmax)
{
 xmin=ixmin;
 xmax=ixmax;
 if(idx>=xmin && idx<=xmax)  x=idx;
 else if(idx<xmin) x=xmin;
 else x=xmax;
 ix=(long)((x-xmin)/(xmax-xmin)*imax+0.99);
}

public void  setGene(double idx)
{

 if(idx>=xmin && idx<=xmax)  x=idx;
 else if(idx<xmin) x=xmin;
```

```
 else x=xmax;
 ix=(long)((x-xmin)/(xmax-xmin)*imax);
}

public void cross(Gene3 Father,Gene3 Mother)
{double crossratio=Math.random();
cross(crossratio,Father,Mother);
}

public double getX()
{return x;}

public void cross(double crossratio,Gene3 Father,Gene3 Mother)
{
Long IntFx=new Long(Father.ix);
String SFx=IntFx.toString();
int nSFx=SFx.length();
Long IntMx=new Long(Mother.ix);
String SMx=IntMx.toString();
int nSMx=SMx.length();
if(nSFx<N)
{String ss="";for(int i=0;i<(N-nSFx);i++){ss+="0";};SFx=ss+SFx;}
if(nSMx<N)
{String ss="";for(int i=0;i<(N-nSMx);i++){ss+="0";};SMx=ss+SMx;}
int N2 = (int)(crossratio*N+0.02)/2;
int N1=2*N2;
int N3=N1+(N-N1)/2;
StringBuffer dest = new StringBuffer(N);
int i;
for (i = 0; i <N2; i++)
{ dest.append(SFx.charAt(i));
  dest.append(SMx.charAt(i));
}
for (i = N1; i <N3; i++)
{ dest.append(SFx.charAt(i));}
for (i = N3; i <N; i++)
{ dest.append(SMx.charAt(i));}
String s=dest.toString();
long iix=Long.parseLong(s);
ix=iix;
xmin=Father.xmin;
xmax=Father.xmax;
x=xmin+(xmax-xmin)*(double)ix/(double)imax;
}


public boolean boolean_random()
{
if(Math.random()<=0.5) return false;
else return true;
}

public void mutate()
{
int N1 = (int)(N*Math.random());
Long IntFx=new Long(ix);
String SFx=IntFx.toString();
int nSFx=SFx.length();
if(nSFx<N)
{String ss="";for(int i=0;i<(N-nSFx);i++){ss+="0";};SFx=ss+SFx;}
StringBuffer dest = new StringBuffer(N);
int i;
for (i = 0; i <N; i++)
{  if(i==N1)
   {  int i1=(int)(9.0*Math.random());
      String s2=Integer.toString(i1);
      dest.append(s2.charAt(0));
   }
  else {dest.append(SFx.charAt(i));}
}
String s=dest.toString();
long iix=Long.parseLong(s);
ix=iix;
x=xmin+(xmax-xmin)*(double)ix/(double)imax;
//==================
}
```

```
public String toString()
{String s;
 s="x="+x+"ix="+ix+"xmin="+xmin+"xmax="+xmax;
 return s;
}
}
```

Program Genetic algorithm constructed from long type integer numbers of 18 digits. , java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;


class genotype3i
{
// each genotype3i is member of the population
// Genotype is like a chromosone in real life genetic systems
// it can be thought as an evaluation point of a function with n variable
// f(xi)=f(x1,x2,x3,..,x_nvars)
public Gene3 G[]     ;  // string of variables makes a genotype3i
public double Fitness;  // genotype3i's fitness f(xi)
public double Upper[];  // gene's upper bound; xi_lower < xi < xi_upper
public double Lower[];  // Gene's lower bound; xi_lower < xi < xi_upper
public double RFitness; // the relative fitness
public double CFitness; // cumulative fitness
public int NVARS;       // Number of variables(Genes)
public int N;           // Number of bits defining each gene
public boolean Breeder;
  public genotype3i(int iNVARS,double low[],double high[],int iN,if_xj fi)
  {setgenotype(iNVARS,low,high,iN,fi);}
  public void setgenotype(int iNVARS,double low[],double high[],int iN,if_xj fi)
  {
  setN(iN);
  setNVARS(iNVARS);
  G=new Gene3[NVARS];
  Upper=new double[NVARS];
  Lower=new double[NVARS];
  setLower(low);
  setUpper(high);
  setGene();
  setFitness(fi);
  }

  public genotype3i(genotype3i gi,if_xj fi)
  {setgenotype(gi,fi);}

  public void setgenotype(genotype3i gi,if_xj fi)
  {
  setN(gi.N);
  setNVARS(gi.NVARS);
  G=new Gene3[NVARS];
  Upper=new double[NVARS];
  Lower=new double[NVARS];
  setLower(gi.Upper);
  setUpper(gi.Lower);
  for(int i=0;i<NVARS;i++)
    G[i]=gi.G[i].copyGene();
  setFitness(fi);
  }

  public genotype3i copygenotype(if_xj fi)
  {
  genotype3i g1=new genotype3i(this,fi);
  return g1;
  }

  public void setN(int iN)    {N=iN;}
  public void setNVARS(int iNVARS)    {NVARS=iNVARS;}
  public void setLower(double lbound[]) {for(int i=0;i<NVARS;i++) Lower[i]=lbound[i];}
```

```
  public void setUpper(double ubound[]) {for(int i=0;i<NVARS;i++) Upper[i]=ubound[i];}
  public void setFitness(if_xj fi)
  {             Fitness=fi.func(getGene());}
  public void setRFitness(double isum)  {RFitness=Fitness/isum;}
  public void setCFitness(double ix)  {CFitness=ix;}
  public double getLower(int j) {return Lower[j];}
  public double getUpper(int j) {return Upper[j];}
  public double getFitness() {return Fitness;}
  public double getRFitness(){return RFitness;}
  public double getCFitness(){return CFitness;}
  public void breed() {Breeder=true;}
  public void not_breed() {Breeder=false;}
  public boolean isBreed() {return Breeder;}


  public void setGene(int val)
  {  G[val]=new Gene3(Lower[val],Upper[val]);}

  public void setGene()
  {  for(int j=0;j<NVARS;j++)
                  setGene(j);
  }

  public double getGene(int val)
  { return G[val].getX();}

  public double[] getGene()
  {   double x[]=new double[NVARS];
               for(int j=0;j<NVARS;j++)
                 x[j]=getGene(j);
               return x;
  }

  public String toString()
  {
  //x values, Fitness,relative fitness and cumulative fitness
  String s="";
  for(int j=0;j<NVARS;j++)
               s+=G[j].toString()+" F:";
  s+=getFitness();
  s+="RF:"+getRFitness();
  s+="CF:"+getCFitness();
  return s;
  }
} //end of class genotype3i

public class Genetic3i
{
              // Maximizing a function
              int    POPSIZE; //population size
              int    MAXGENS; //maximum number of generations;
              int    NVARS;   //number of independent variables of function
    int    N;       //number of genes in each variable
              double PXOVER; //probability of crossover;
    double PMUTATION; //proportion of mutated variables
    int Generation; //Current generation number
    int NBreeder; //Number of survivors in a generation
    int Best; //The best genotype3i in the population
    genotype3i Population[];
    //if_xj f;  // function to be avaluated by genetic algorithm

    public Genetic3i(int iPOPSIZE,int iMAXGENS,int iNVARS,int iN,
    double iPXOVER,double iPMUTATION)
    {
    POPSIZE=iPOPSIZE;
    MAXGENS=iMAXGENS;
    NVARS=iNVARS;
    N=iN;
    PXOVER=iPXOVER;
    PMUTATION=iPMUTATION;
```

```
NBreeder=0;
Population=new genotype3i[POPSIZE+1];
        }
        public void setPopulation(double low[],double up[],if_xj fi)
        {
        for(int i=0;i<POPSIZE+1;i++)
          { Population[i]=new genotype3i(NVARS,low,up,N,fi);}
        }

        public genotype3i[] copyPopulation(if_xj fi)
        {
        genotype3i Po[]=new genotype3i[POPSIZE+1];
        for(int i=0;i<POPSIZE+1;i++)
          { Po[i]=new genotype3i(Population[i],fi);
            Po[i].Fitness=Population[i].Fitness;
                    Po[i].RFitness=Population[i].RFitness;
            Po[i].CFitness=Population[i].CFitness;
          }
        return Po;
        }

        public void evaluate(if_xj fi)
        {
          int mem;
          int j;
          double x[]=new double[NVARS];
          for(mem=0;mem<POPSIZE;mem++)
          {
          //System.out.println(Population[mem].toString());
          Population[mem].setFitness(fi);
          if(Population[mem].getFitness() > Population[POPSIZE].getFitness())
            {
                    Best=mem;
                    Population[POPSIZE]=Population[mem];
            for(j=0;j<NVARS;j++)
              {
              Population[POPSIZE].G[j].copyGene(Population[mem].G[j]);
                      }//end of for(j=0..
                    }//end of if
          }//end of for(mem=0;..
        }


        public void setRFCF(genotype3i Pi[])
        {
        //calculates relative and cumulative fitness functions
        int mem;
        int PN=Pi.length;
        double sum=0.0;
        //total fitness of population
        for(mem=0;mem<PN;mem++)
          {sum+=Pi[mem].Fitness;}
        //System.out.println("sum="+sum);
        //calculate relative fitness of each genotype3i
        for(mem=0;mem<PN;mem++)
          {Pi[mem].setRFitness(sum);}
        //calculate cumulative fitness
        Pi[0].setCFitness(Pi[0].getRFitness());
for(mem=1;mem<PN;mem++)
          {Pi[mem].setCFitness(Pi[mem-1].getCFitness()+Pi[mem].getRFitness());
          }
        }

        public void toString(genotype3i Pi[])
        {
        int mem;
        int PN=Pi.length;
        for(mem=0;mem<PN;mem++)
        {
        //list them
```

```
            System.out.println("Population["+mem+"]="+Pi[mem].toString());
            }
            }

            public void select(if_xj fi)
            {
            //select the new generation members of population
            double r;
            int mem;
            setRFCF(Population);
            //create a new population;
            genotype3i Po[]=copyPopulation(fi);
 setRFCF(Po);
 for(int i=0;i<POPSIZE;i++)
 {   mem=0;
    r=Math.random();
    double gf1=Population[i].getCFitness();
                double gf2;
    if (gf1 > r) Po[i]=Population[mem].copygenotype(fi);
                for(mem=1;mem<POPSIZE;mem++)
      {
                  gf2=Population[mem].getCFitness();
                  if( gf2>=r && gf1< r ) {Po[i]=Population[mem].copygenotype(fi);break;}
      }
 }
 setRFCF(Po);
            Population=Po;
            evaluate(fi);
 //toString(Population);
 }

public void crossover(if_xj fi)
{
int i,count;
int mem;
int POne,a;
int PTwo,b;
int point;
//select members for breeding
int iselect[]=new int[POPSIZE];
int counter=0;
double r;
 for(mem=0;mem<POPSIZE;mem++)
            {
              r=Math.random();
              if(r < PXOVER)
                {
                        iselect[counter++]=mem;
                Population[mem].breed();
                        NBreeder++;
                }
              else
                Population[mem].not_breed();
            }
            //System.out.println("iselect="+Matrix.toString(iselect));
            //let also best of the population to breed
            Population[Best].breed();

//loop through the population select breeding pairs
 for(mem=0;mem<POPSIZE;mem++)
    {
      //select two in popolation in random
      a=(int)(Math.random()*NBreeder)+1;
      b=(int)(Math.random()*NBreeder)+1;
      count=0;
      POne=0;
      PTwo=0;
      //select two individuals for breeding
      for(i=0;i<POPSIZE;i++)
        {
```

```
        if(Population[i].isBreed())
          {
          count++;
          if(count==a) POne=count;
          if(count==b) PTwo=count;
                    }
                  }
                 //perform a crossover;
                  genotype3i Po[]=new genotype3i[2];
                            Po[0]=Population[POne].copygenotype(fi);
                            Po[1]=Population[PTwo].copygenotype(fi);
                  for(i=0;i<NVARS;i++)
                  {
                            Population[POne].G[i].cross(Math.random(),Po[0].G[i],Po[1].G[i]);
                            Population[PTwo].G[i].cross(Math.random(),Po[0].G[i],Po[1].G[i]);
                            }
                            Population[POne].setFitness(fi);
                            Population[PTwo].setFitness(fi);
                 }
                }

public void mutate(if_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*PMUTATION*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  Population[member].G[var].mutate();
          //after mutation recalculate the function value
  Population[member].setFitness(fi);
  }
}

public String report()
{
String s;
int i;
double best_value;
double avg;
double stdev;
double sum_square;
double square_sum;
double sum;
double xx;
sum=0;
sum_square=0.0;
for(i=0;i<POPSIZE;i++)
  {
           xx= Population[i].getFitness();
  sum+=xx;
  sum_square=xx*xx;
           }
avg=sum/(double)POPSIZE;
square_sum=sum*sum/(double)POPSIZE;
stdev=Math.sqrt(square_sum);
best_value=Population[POPSIZE].getFitness();
double aa[]=new double[NVARS+1];
for(i=0;i < NVARS;i++)
 {
 aa[i]=Population[POPSIZE].getGene(i);
          }
          aa[NVARS]=Population[POPSIZE].getFitness();
```

```
  //s="Generation = "+Generation+"best value = "+Matrix.toString(aa)+"Average = "+avg+"Standart Deviation = "+stdev;
  s="Generation = "+Generation+"best value = "+Matrix.toString(aa);
  return s;
  }

  public double[] getBest()
  {
  double aa[]=new double[NVARS+1];
  for(int i=0;i < NVARS;i++)
    {
    aa[i]=Population[POPSIZE].getGene(i);
                }
                aa[NVARS]=Population[POPSIZE].getFitness();
  return aa;
  }

  public double[] calculate(if_xj fi,boolean report)
  {
  evaluate(fi);
  Generation=0;
  while(Generation<MAXGENS)
    {
    Generation++;
    NBreeder=(int)(0.8*POPSIZE);
    select(fi);
    crossover(fi);
    mutate(fi);
    evaluate(fi);
    //if(report)
       //System.out.println(report());
    }
  return getBest();
  }
  public double[] solution(if_xj fi)
  {
  evaluate(fi);
  Generation=0;
  while(Generation<MAXGENS)
    {
    Generation++;
    NBreeder=(int)(0.8*POPSIZE);
    select(fi);
    crossover(fi);
    mutate(fi);
    evaluate(fi);
    }
  double a[]=getBest();
  double b[]=new double[a.length-1];
  for(int i=0;i<b.length;i++)       {b[i]=a[i];}
  return b;
  }
}
```

Program Test program NA109 for Gene constructed from long type integer numbers of 9 digits., java code

```
 import java.io.*;

class NA109i
{
   public static void main(String args[]) throws IOException
   {
   int N=63;
    int POPSIZE=500;
   int MAXGENS=500;
   int NVARS=2;
   double PXOVER=0.3;
   double PMUTATION=0.02;
   Genetic3i xx=new Genetic3i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   double low[]={-10,-10};
```

```
    double high[]={10,10};
    if_xj fx=(double x[])->1.0+Math.cos(Math.PI*(x[0]-3.0))*Math.cos(2.0*Math.PI*(x[1]-2.0))/
    (1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0));
    xx.setPopulation(low,high,fx);
    System.out.println("best=\n"+Matrix.toStringT(xx.calculate(fx,false)));
    }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA109i
best=
        2.977777777777375
        2.000000020175298
        1.997071669188506


> Terminated with exit code 0.
```

GENETİC ALGORITHMS BY USING A LONG TYPE VARIABLE AS A GENE IN PYTHON CODE

int_gene constructed from integer numbers of 18 digits. Python code

```
 from abc import ABC, abstractmethod
from math import *
import random
from f_x import *


class int_gene(ABC):

    def __init__(self, xming,xmaxg):
        self.xming=xming
        self.xmaxg=xmaxg
        self.imax=999999999999999999
        self.xp=int(random.random()*self.imax)
        self.x=self.xming+self.xp*(self.xmaxg-self.xming)/self.imax
        self.xs=str(self.xp)
        self.N=18


    def rand_xp(self):
        self.xp=int(random.random()*self.imax)
    def rand_x(self):
        self.xp=int(random.random()*self.imax)
        self.x=self.xming+self.xp*(self.xmaxg-self.xming)/self.imax
    def mutate(self):
        n1=str(int(10.0*random.random()))
        nn=int(18*random.random())
        n2=str(self.xs[nn:nn+1])
        if(n1==n2 and n1==str(9)):
            n1=str(8)
        elif(n1==n2 and n1==str()):
            n1=str(1)
        elif(n1==n2):
            n1=str(int(n2)+1)
        self.xs=self.xs[:nn]+n1+self.xs[nn+1:]
        self.xp=int(self.xs)
        self.x=self.xming+self.xp*(self.xmaxg-self.xming)/self.imax

    def print(self):
        print("min=",self.xming,"max=",self.xmaxg,"xp=",self.xp,"x=",self.x)

    # seperated at point n
    def split(self,nn):
        #nn=int(random.random()*self.N)
        #print("nn=",nn)
        s1=self.xs[:nn]
        s2=self.xs[nn:]
        s=[s1,s2]
        return s
```

```
    # split string binary gene into three pieces
    # seperated at point n
    def split2(self,nn1,nn2):
        #nn=int(random.random()*self.N)
        #print("nn=",nn)
        if nn2>=nn1:
            s1=self.xs[:nn1]
            s2=self.xs[nn1:nn2]
            s3=self.xs[nn2:]
        else:
            s1=self.xs[:nn2]
            s2=self.xs[nn2:nn1]
            s3=self.xs[nn1:]
        s=[s1,s2,s3]
        return s
    # cross correlation  1 of two Genes
def crossing1(f1,f2,N):
    s1=f1.split(N)
    s2=f2.split(N)
    ss1=s1[0]+s2[1]
    ss2=s2[0]+s1[1]
    ff1=int(ss1)
    ff2=int(ss2)
    if random.random()<=0.5:
        ff=set_int_gene(f1.xming,f1.xmaxg,ff1)
    else:
        ff=set_int_gene(f1.xming,f1.xmaxg,ff2)
    return ff

# cross correlation2 of two Genes
def crossing2(f1,f2,N1,N2):
    s1=f1.split2(N1,N2)
    s2=f2.split2(N1,N2)
    ss1=s1[0]+s2[1]+s1[2]
    ss2=s2[0]+s1[1]+s2[2]
    ff1=int(ss1)
    ff2=int(ss2)
    if random.random()<=0.5:
        ff=set_int_gene(f1.xming,f1.xmaxg,ff1)
    else:
        ff=set_int_gene(f1.xming,f1.xmaxg,ff2)
    return ff

# crossing1 crossing2 or crossing3 is selceted randomly
def crossing(f1,f2):
        if int_random()==0:
            N=int(f1.N*random.random())
            fchild=crossing1(f1,f2,N)
        else:  # int_random()==1:
            N1=int(f1.N*random.random())
            N2=int(f1.N*random.random())
            if N1<=N2:
                fchild=crossing2(f1,f2,N1,N2)
            else:
                fchild=crossing2(f1,f2,N2,N1)
        return fchild

def set_int_gene(xming,xmaxg,xp):
    g=int_gene (xming,xmaxg)
    g.xp=xp
    g.x=g.xming+g.xp*(g.xmaxg-g.xming)/g.imax
    g.xs=str(xp)
    return g

#creates a random int 0 or 1
def int_random():
    r=0
    if random.random()>=0.5:
        r=1
    return r
```

A test for int_gene (mutation), Python code

```
from int_gene import *
```

```
xmin=0.0
xmax=3.0
f1=int_gene(xmin,xmax)
f1.print()
f2=int_gene(xmin,xmax)
f2.print()
print("f1 mutation")
f1.mutate()
f1.print()
print("f2 mutation")
f2.mutate()
f2.print()
```

```
runfile('E:/okul/SCO1/int_gene_test.py', wdir='E:/okul/SCO1')
Reloaded modules: f_x, int_gene
min= 0.0 max= 3.0 xp= 353017307410296384 x= 1.0590519222308892
min= 0.0 max= 3.0 xp= 381159414778967552 x= 1.1434782443369027
f1 mutation
min= 0.0 max= 3.0 xp= 353017307410496384 x= 1.0590519222314891
f2 mutation
min= 0.0 max= 3.0 xp= 381158414778967552 x= 1.1434752443369027
```

A test for int_gene (mutation), Python code

```
from int_gene import *

xmin=0.0
xmax=3.0
f1=int_gene(xmin,xmax)
f1.print()
f2=int_gene(xmin,xmax)
f2.print()
print("crossing of f1 and f2")
f3=crossing(f1,f2)
f3.print()
```

```
runfile('E:/okul/SCO1/int_gene_test1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_x, int_gene
min= 0.0 max= 3.0 xp= 3206613303700838840 x= 0.9619839911102516
min= 0.0 max= 3.0 xp= 946662408866570112 x= 2.8399872265997104
crossing of f1 and f2
min= 0.0 max= 3.0 xp= 9466624088665738840 x= 2.8399872265997215
```

Integer Genotype int_Genotype, Python code

```
from abc import ABC, abstractmethod
from math import *
import random
from f_xj import *
from int_gene import *


class int_Genotype(ABC):

    def __init__(self, f,NVARS,low1,high1):
        self.xmin=low1
        self.xmax=high1
        self.NVARS=NVARS
        self.N=18
        self.f=f
        self.G=[0.0 for j in range(self.NVARS)]
        self.x=[0.0 for j in range(self.NVARS)]
        for i in range(self.NVARS):
```

```
            self.G[i]=int_gene(self.xmin[i],self.xmax[i])
            self.x[i]=self.G[i].x
        self.Fitness=self.f.func(self.x)
        self.RFitness=0.0
        self.CFitness=0.0
        self.Breeder=False

    def setGene(val):
        G[val]=int_gene(self.xmin[val],self.xmax[val])
        return self.G[val]

    def getGene(val):
        return G[val]

    #print gene values
    def print(self):
        for i in range(self.NVARS):
            self.G[i].print()
        print("Fitness = ",self.Fitness)
        print("RFitness = ",self.RFitness)
        print("CFitness = ",self.CFitness)

    def setFitness(self):
        for i in range(self.NVARS):
            self.x[i]=self.G[i].x
        self.Fitness=self.f.func(self.x)

    def getFitness(self):
        return self.Fitness

    def setRFitness(self,sum):
        self.RFitness=self.Fitness/sum

    def setCFitness(self,x):
        self.CFitness=x

    def getCFitness(self):
        return self.CFitness

    def breed(self):
        self.Breeder=True

    def not_breed(self):
        self.Breeder=False

    def isBreed(self):
        return self.Breeder

    def copygenotype(self):
        g1=self
        return g1
    def mutate(self,PMUTATION):
        for i in range(self.NVARS):
            a=random.random()
            if a<PMUTATION:
                self.G[i].mutate()
def crossingGT(G1,G2):
    nv=G1.NVARS
    G3=int_Genotype(G1.f,G1.NVARS,G1.xmin,G1.xmax)
    for i in range(nv):
```

```
        G3.G[i]=crossing(G1.G[i],G2.G[i])
    return G3
```

int_Genotype example program:, Python code

```
from f_xj import *
from  int_Genotype import *

N=32
xmin=[0.0,0.0]
xmax=[5.0,5.0]
class f1(f_xj):func=lambda self,x: x[0]*x[0]+x[1]*x[1]
f = f1()
NVARS=2
N=32
G1=int_Genotype(f,NVARS,xmin,xmax)
G2=int_Genotype(f,NVARS,xmin,xmax)
print("G1 = ")
G1.print()
print("G2 = ")
G2.print()
print("G3 breed")
G3=crossingGT(G1,G2)
G3.print()
print("G3 mutate")
G3.mutate(1.0)
G3.print()
```

```
runfile('E:/okul/SCO1/int_Genotype_test1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_x, int_gene
G1 =
min= 0.0 max= 5.0 xp= 354311067151272704 x= 1.7715553357563636
min= 0.0 max= 5.0 xp= 296277299780326144 x= 1.4813864989016308
Fitness =  5.332914266774873
RFitness =  0.0
CFitness =  0.0
G2 =
min= 0.0 max= 5.0 xp= 222301085691244288 x= 1.1115054284562214
min= 0.0 max= 5.0 xp= 379478200327386432 x= 1.8973910016369322
Fitness =  4.835536930580449
RFitness =  0.0
CFitness =  0.0
G3 breed
min= 0.0 max= 5.0 xp= 354311067151272704 x= 1.7715553357563636
min= 0.0 max= 5.0 xp= 379477299327386432 x= 1.897386496636932
Fitness =  7.447702320452942
RFitness =  0.0
CFitness =  0.0
G3 mutate
min= 0.0 max= 5.0 xp= 354311067154272704 x= 1.7715553357713636
min= 0.0 max= 5.0 xp= 379477299377386432 x= 1.8973864968869323
Fitness =  7.447702320452942
RFitness =  0.0
CFitness =  0.0
```

Genetic algorithm constructed from long integer numbers of 18 digits. int_Genetic, Python code

```
 from abc import ABC, abstractmethod
from math import *
import random
from f_xj import *
from int_Genotype import *
```

```
class int_Genetic(ABC):
    def __init__(self,f,low,high,POPSIZE,MAXGENS,NVARS,PXOVER,PMUTATION):
        # function derived from f_xj class f(x0,x1,..xn)
        self.POPSIZE=POPSIZE   # population size
        self.MAXGENS=MAXGENS   # maksimum number of generations
        self.NVARS=NVARS    # number of variables
        #number of independent variable of function
        #number of Genes in genotype
        self.N=18
        #number of bits in a gene
        self.PXOVER=PXOVER
        #probability of cross-over
        self.PMUTATION=PMUTATION
        #proportion of mutated variables
        self.NBreeder=int(0.8*POPSIZE)
        #number of survivers in a generation
        self.Best=0
        self.f=f
        self.xmin=[0.0 for j in range(self.NVARS)]
        self.xmax=[0.0 for j in range(self.NVARS)]
        for i in range(self.NVARS):
            self.xmin[i]=low[i]
            self.xmax[i]=high[i]
        #Best genotype in the population
        self.Population=[0.0 for j in range(self.POPSIZE+1)]
        self.setPopulation()
        self.setRFCF()
        self.evaluate()

    def print(self):
        print("Population size = ",self.POPSIZE)
        print("Maximum number of Generations = ",self.MAXGENS)
        print("Number of Bits in Gene = ",self.N)
        print("Probability of crossover =",self.PXOVER)
        print("Propertion of mutated variables",self.PMUTATION)
        for i in range(self.POPSIZE+1):
            self.Population[i].print()


    def setPopulation(self):
        for i in range(self.POPSIZE+1):
            self.Population[i]=int_Genotype( self.f,self.NVARS,self.xmin,self.xmax)
            self.Population[i].setFitness()
        self.setRFCF()
        self.evaluate()


    def setRFCF(self):
        sum=0.0
        for i in range(self.POPSIZE):
            self.Population[i].setFitness()
            sum=sum+self.Population[i].Fitness

        for i in range(self.POPSIZE):
            self.Population[i].setRFitness(sum)
        sum=0.0
        for i in range(self.POPSIZE):
            sum=sum+self.Population[i].RFitness
            self.Population[i].setCFitness(sum)

    def evaluate(self):
        for i in range(self.POPSIZE):
            if self.Population[i].Fitness>self.Population[self.POPSIZE].Fitness:
                self.Population[self.POPSIZE]=self.Population[i]

    def breding_individual(self):
        #select members for breeding
        j=0
```

```
        for i in range(self.POPSIZE):
            self.Population[i].not_breed()
        for i in range(self.POPSIZE):
            r=random.random()
            if i==0 and r<self.Population[i].CFitness:
                j=i
            elif  r>=self.Population[i-1].CFitness and  r<=self.Population[i].CFitness:
                j=i
        return j


    def crossover(self):
        for mem in range(self.POPSIZE):
            POne= int(self.breding_individual())
            PTwo=int(self.breding_individual())
            Po=[0.0 for j in range(2)]
            Po[0]=self.Population[POne]
            Po[1]=self.Population[PTwo]
            self.Population[POne]=crossingGT(Po[0],Po[1])
            self.Population[PTwo]=crossingGT(Po[0],Po[1])
            self.Population[POne].mutate(self.PMUTATION)
            self.Population[PTwo].mutate(self.PMUTATION)
            self.setRFCF()
            self.evaluate()

    def getBest(self):
        aa=[0.0 for j in range(self.NVARS+1)]
        for i in range(self.NVARS):
            aa[i]=self.Population[self.POPSIZE].x[i]
        aa[self.NVARS]=self.Population[self.POPSIZE].Fitness
        return aa

    def select(self):
        Po=[0.0 for j in range(self.POPSIZE+1)]
        for i in range(self.POPSIZE+1):
            mem=0
            r=random.random()
            gf1=self.Population[i].getCFitness()
            gf2=0.0
            if gf1>r:
                Po[i]=self.Population[mem].copygenotype()
            for mm in range(1,self.POPSIZE):
                gf2=self.Population[mem].getCFitness()
                if gf2>r and gf1<r:
                    Po[i]=self.Population[mem].copygenotype()
                    break
        for i in range(self.POPSIZE):
            self.Population[i]=Po[i]
        self.setRFCF()
        self.evaluate()

    def calculate(self):
         self.evaluate()
         Generation=0
         while Generation<self.MAXGENS:
             Generation=Generation+1
             self.crossover()
             self.evaluate()
         return self.getBest()

def setRFCF(Co):
    sum=0.0
    for i in range(Co.POPSIZE+1):
        Co.Population[i].setFitness()
        sum=sum+Co.Population[i].Fitness
    for i in range(Co.POPSIZE+1):
        Co.Population[i].setRFitness(sum)
        sum=0
        for i in range(Co.POPSIZE+1):
            sum=sum+Co.Population[i].RFitness
```

```
        Co.Population[i].setCFitness(sum)
```

Test program for Gene constructed from integer numbers of 18 digits. Python code

```
 # -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
from f_xj import *
from int_Genetic import *
import numpy as np;
import matplotlib.pyplot as plt;



class f1(f_xj):func=lambda self,x:x[0]*sin(10.0*pi*x[0])+3.0
f=f1()
class f2(f_x):func=lambda self,x:x*sin(10.0*pi*x)+3.0
ff=f2()
N=22 #number of digits in binary genes
POPSIZE=50 #population size
MAXGENS=50 # maximum number of generations
NVARS=1  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[-1.0]
high1=[2.0]
g=int_Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,PXOVER,PMUTATION)
print(g.calculate())

dx=3.0/1000.0
n=1000
y=[0 for z in range(n)];
x=[0 for z in range(n)];
for i in range(n):
    x[i]=-1.0+dx*i
    y[i]=ff.func(x[i])
plt.plot(x,y,linewidth=0.3)
plt.xlabel("x")
plt.ylabel("y")
plt.title('Optimisation: Genetic Algorithms')
plt.show()
```

```
# -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
from f_xj import *
from int_Genetic import *
import numpy as np;
import matplotlib.pyplot as plt;



class f1(f_xj):func=lambda self,x:x[0]*sin(10.0*pi*x[0])+3.0
f=f1()
class f2(f_x):func=lambda self,x:x*sin(10.0*pi*x)+3.0
ff=f2()
N=22 #number of digits in binary genes
POPSIZE=100 #population size
MAXGENS=100 # maximum number of generations
NVARS=1  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[-1.0]
high1=[2.0]
g=int_Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,PXOVER,PMUTATION)
print(g.calculate())
```

```
dx=3.0/1000.0
n=1000
y=[0 for z in range(n)];
x=[0 for z in range(n)];
for i in range(n):
    x[i]=-1.0+dx*i
    y[i]=ff.func(x[i])
plt.plot(x,y,linewidth=0.3)
plt.xlabel("x")
plt.ylabel("y")
plt.title('Optimisation: Genetic Algorithms')
plt.show()
```

```
runfile('E:/okul/SCO1/int_Genetic_test.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, int_gene, int_Genotype, int_Genetic
[1.8502416190695525, 4.85018831522604]
```

```
# -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
from f_xj import *
from int_Genetic import *
import  plot3D1 as P3D



class f1(f_xj):func=lambda self,x:2.0*x[0]*x[1]+2.0*x[0]-x[0]*x[0]-2.0*x[1]*x[1]+10.0
f=f1()

N=18 #number of digits in integer genes
POPSIZE=100 #population size
MAXGENS=50 # maximum number of generations
NVARS=2  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[1.0,1.0]
high1=[5.0,5.0]
g=int_Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,PXOVER,PMUTATION)
print(g.calculate())
xmin=low1[0]
xmax=high1[0]
Nx=100
ymin=0.0
ymax=12.0
Ny=100
PLabel="Optimisation: Genetic Algorithms"
XLabel="x"
YLabel="y"
ZLabel="z"
P3D.plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```

```
runfile('E:/okul/SCO1/int_Genetic_test2.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, int_gene, int_Genotype, int_Genetic
[2.0299727032780286, 1.0563244328352681, 11.996133144615126]
```

```
# -*- coding: utf-8 -*-
"""
Created on Mon Nov  9 04:02:18 2020
@author: M. Turhan Çoban
"""
from f_xj import *
```

```
from int_Genetic import *
import  plot3D1 as P3D



class f1(f_xj):func=lambda self,x:21.5+x[0]*sin(4.0*pi*x[0])+x[1]*sin(20.0*pi*x[1])
f=f1()

N=18 #number of digits in binary genes
POPSIZE=100 #population size
MAXGENS=50 # maximum number of generations
NVARS=2  #number of variables (member of genes in genome
PXOVER=0.3 #probability of crossover
PMUTATION=0.02 #probability of mutation
low1=[-3.0,4.1]
high1=[12.1,5.8]
g=int_Genetic(f,low1,high1,POPSIZE,MAXGENS,NVARS,PXOVER,PMUTATION)
print(g.calculate())
xmin=low1[0]
xmax=high1[0]
Nx=100
ymin=0.0
ymax=12.0
Ny=100
PLabel="Optimisation: Genetic Algorithms"
XLabel="x"
YLabel="y"
ZLabel="z"
P3D.plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
P3D.contour_plot(f,xmin,xmax,Nx,ymin,ymax,Ny,PLabel,XLabel,YLabel,ZLabel)
```

```
runfile('E:/okul/SCO1/int_Genetic_test1.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj, f_x, int_gene, int_Genotype, int_Genetic, plot3D1
[12.08911500264852, 4.4257205915714986, 36.80182408286153]
```

5.17 STOCHASTIC METHODS: MONTE-CARLO METHOD

Monte carlo optimization as a method is very simple. Points between the given limits are selected randomly and evaluated. When number of points increase to very high numbers, statistically there will be a homogeneous coverege of all the search region. Record the best optimum found so far.What is achived is similar to plotting data, but in monte carlo method taken constant steps and walk through the steps are not needed because randomness takes care of uniformity of search when the total number is big enough.

Program 5.19-1 Monte carlo stochastic optimization method, Java code

```
 / Dr. M. Turhan ÇOBAN
//
// EGE University School of Engineering, Mechanical Engineering Department
// package : Numerical analysis package
// subject : optimization
// group   : Monte-Carlo optimization(maximum) Algorithms
// monte_carlo_opt Class
// class selects random points at the given area and selects the best
// results found
//
import java.util.*;
import java.io.*;


public class monte_carlo_opti extends Object
{
int N;
```

```
public double max[];
public double min[];
public double x;

public monte_carlo_opti(int Ni,double imin[],double imax[])
{
 max=new double[imin.length];
 min=new double[imin.length];
 N=Ni;
 int n=imin.length;
 for(int i=0;i<n;i++)
 {min[i]=imin[i];max[i]=imax[i];}
}

public double[] monte_carlo_max(if_xj fi)
{
 int n=min.length;
 double x[]=new double[n];
 double xmax[]=new double[n];
 double max_number=-1e99;
 double f;
 for(int k=0;k<N;k++)
 {
 for(int i=0;i<n;i++)
 {x[i]=min[i]+(max[i]-min[i])*Math.random();}
 f=fi.func(x);
 if(f>max_number)
   {  max_number=f;
      for(int i=0;i<n;i++)
         {xmax[i]=x[i];}
   }
 }
 return xmax;
}}
```

Program 5.19-2 Monte carlo stochastic optimization method example, Java code

```
 import java.io.*;

class NA110
{
   public static void main(String args[]) throws IOException
   {
   double low[]={-1};
   double high[]={2};
   monte_carlo_opti mco=new monte_carlo_opti(100000,low,high);
   if_xj fx=(double x[])->x[0]*Math.sin(10.0*Math.PI*x[0])+3.0;
   double xmax[]=mco.monte_carlo_max(fx);
   for(int i=0;i<xmax.length;i++)
   {System.out.println(xmax[i]);}
   }
}
```

```
 ---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA110
1.850556252949647

> Terminated with exit code 0.
```

Program 5.19-3 Monte carlo stochastic optimization method example, java code

```
 import java.io.*;

class NA111
{ public static void main(String args[]) throws IOException
   {
   double low[]={-10,-10};
   double high[]={10,10};
   monte_carlo_opti mco=new monte_carlo_opti(100000,low,high);
```

```
    if_xj fx=(double x[])->1.0+Math.cos(Math.PI*(x[0]-3.0))*Math.cos(2.0*Math.PI*(x[1]-2.0))/
    (1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0));
    double xmax[]=mco.monte_carlo_max(fx);
    for(int i=0;i<xmax.length;i++)
    {System.out.println(xmax[i]);}
    }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" NA111
2.9960487170622745
1.99874133664329

> Terminated with exit code 0.
```

Monte carlo stochastic optimization method example, Julia code

```
 #function set
function func(x)
ff=1.0+cos(pi*(x[1]-3.0))*cos(2.0*pi*(x[2]-2.0))/(1+(x[1]-3.0)*(x[1]-3.0)+(x[2]-2.0)*(x[2]-2.0))
return ff
end

function monte_carlo_opt(func,imin,imax,N)
  # monte-Carlo optimisation method
  min=imin
  max=imax
  n=length(min)
  global xmax=zeros(Float64,n)
  global x=zeros(Float64,n)
  max_number=-1.0e99
  for k=1:N
    for i=1:n
      x[i]=min[i]+(max[i]-min[i])*rand()
    end #for i=1:n
    f=func(x)
    if f>max_number
       max_number=f
       for i=1:n
         xmax[i]=x[i]
       end #for i=1:n
    end # if f>max_number
  end # for k
  return xmax
    end # function

imin=[-10.0 -10.0]
imax=[10.0 10.0]
N=10000
r=monte_carlo_opt(func,imin,imax,N)
y=func(r)
println("r = $r y = $y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_newton.jl
r = [3.112835283143717, 1.9680878153402759] y = 1.9065714544579773

> Terminated with exit code 0.
```

Sample test program: Julia code

```
 #function set
function func(x)
ff=x[1]*sin(10.0*pi*x[1])+3.0
return ff
end

function monte_carlo_opt(func,imin,imax,N)
  # monte-Carlo optimisation method
  min=imin
  max=imax
  n=length(min)
  global xmax=zeros(Float64,n)
```

```
   global x=zeros(Float64,n)
   max_number=-1.0e99
   for k=1:N
     for i=1:n
       x[i]=min[i]+(max[i]-min[i])*rand()
     end #for i=1:n
     f=func(x)
     if f>max_number
        max_number=f
        for i=1:n
          xmax[i]=x[i]
        end #for i=1:n
     end # if f>max_number
  end # for k
  return xmax
     end # function

imin=[-1.0]
imax=[2]
N=10000
r=monte_carlo_opt(func,imin,imax,N)
y=func(r)
println("r = $r y = $y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" opt_monte_carlo.jl
r = [1.8504975256833687] y = 4.8502714882191444

> Terminated with exit code 0.
```

5.20 STOCHASTIC METHODS: ITERATIVE HILL CLIMBING

Iterative Hill climbing method, a global maximum finding method similar to genetic algorithms and monte carlo. It is a simple method, yet it is more complicated than the Monte-Carlo method. As a basic strategy a point is selected and evaluated. Furthermore more point (but not excessively more) in the neigborhood of the point is evaluated then jumped to a new point and repeat the same process. If the method is given as an algorithm:

```
Iterative hill climbing algorithm
start
    t=0
    while
        local=false
        select a random point vc
        calculate the function for vc
        while
            select approximately 30 points in neighbourhood of vc and evaluate the function values
            In between these points select the maximum one as the solution vector
            if f(vc) < f(vn)  then  vc = vn
            else  local=true
        until local
        t=t+1
     until t=MAXIMUM
finish
```

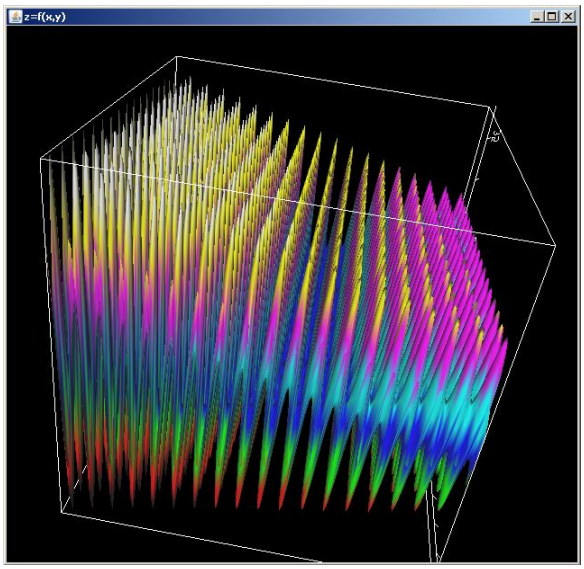
The basic problem in this code is the definition of neighbourhood region. In reality Genetic engineering Gene concept is just establishing this, a child gene is always in the neighbourhood of the parent genes. By using this property the method can be establish. A few example of describing the gene was given in genetic engineering method, the same methods can be utilised for iterative hill climbing as well. In our first class vector(point) vc is made by using binary gene

Program 5.20-1 Iterative hill climbing with binary gene , Java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;

public class iteratedhillclimberi
```

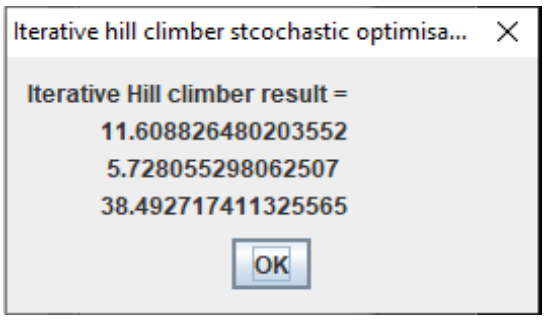
```
{
    // Maximizing a function
   //iterated hill climber
   // a version steepest ascent with statistical approach to variable
   genotype1i Vc;
   genotype1i Vn[];
   genotype1i best;
   int MAX;
   double t;
   boolean local;
   int POPSIZE;
   int NVARS;
   int N;//number of bit representation for each variable

   //if_xj f;  // function to be avaluated by genetic algorithm

   public iteratedhillclimberi(int iPOPSIZE,int iNVARS,int iN)
   {
   POPSIZE=iPOPSIZE;//Vn size number of neigbour variable f.e. 30
   NVARS=iNVARS; // number of variables equation to solve
   MAX=20;
   N=iN;           // number of bits represent each variable for example 32
   Vn=new genotype1i[POPSIZE+1];
   t=0;
   local=false;
               }

               public void setVc(double low[],double up[],if_xj fi)
               {
               //select originalgene andomly
               Vc=new genotype1i(NVARS,low,up,N,fi);
               }
               public void setBest(genotype1i gi,if_xj fi)
               {
               //select originalgene andomly
               best=new genotype1i(gi,fi);
               }
   public void setVc(genotype1i gi,if_xj fi)
               {
               //select original gene from other gene
               Vc=new genotype1i(gi,fi);
               }
               public void setVn(if_xj fi)
               {//select neighbouring genes similar to the original one
               for(int i=0;i<POPSIZE+1;i++)
                 { Vn[i]=new genotype1i(Vc,fi);}
                  mutate(fi);
               }


               public void evaluate(if_xj fi)
               {
                 int mem;
                 int j;
                 double x[]=new double[NVARS];
                 for(mem=0;mem<POPSIZE;mem++)
                 {
                 //System.out.println(Vn[mem].toString());
                 Vn[mem].setFitness(fi);
                 if(Vn[mem].getFitness() > Vn[POPSIZE].getFitness())
                    {
                              Vn[POPSIZE]=Vn[mem];
                    for(j=0;j<NVARS;j++)
                      {
                      Vn[POPSIZE].G[j].copyGene(Vn[mem].G[j]);
                                }//end of for(j=0..
                              }//end of if
                 }//end of for(mem=0;..
               }
```

```
            public void toString(genotype1i Pi[])
            {
            int mem;
            int PN=Pi.length;
            for(mem=0;mem<PN;mem++)
            {
            //list them
            System.out.println("Vn["+mem+"]="+Pi[mem].toString());
            }
            }


public void mutate(if_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  Vn[member].G[var].mutate();
  //after mutation recalculate the function value
  Vn[member].setFitness(fi);
  }
}


public double[] getBest(genotype1i b)
{
double aa[]=new double[NVARS+1];
for(int i=0;i < NVARS;i++)
 {
 aa[i]=b.getGene(i);
            }
            aa[NVARS]=b.getFitness();
return aa;
}

public double[] calculate(double low[],double up[],if_xj fi,boolean report)
{
 setVc(low,up,fi);// new set of random selection in each time
 setBest(Vc,fi);
            do
 {
            local=true;
 do
 {
 setVn(fi);
 evaluate(fi);
 if(Vn[POPSIZE].getFitness() > Vc.getFitness())
 {
              setVc(Vn[POPSIZE], fi);
              if(Vn[POPSIZE].getFitness() > best.getFitness()) setBest(Vn[POPSIZE],fi);
 }
 else local=false;
 } while(local);
 t++;
 setVc(low,up,fi); //new random value
 if(report)
   System.out.println(Matrix.toString(getBest(best)));
 } while(t<MAX);
return getBest(best);
}
```



Program Iterative hill climbing with binary gene example program, Java code

```
 //iterated hillclimber
//iterasyonlu en dik yamaç tırmanıcı
import java.io.*;
import javax.swing.JOptionPane;

class NA112
{
   public static void main(String args[]) throws IOException
   {
   int N=18;
              int POPSIZE=400;
   int MAXGENS=400;
   int NVARS=2;
   if_xj fax=(double x[])->21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]);
   iteratedhillclimberi xx=new iteratedhillclimberi(POPSIZE,NVARS,N);
   double low[]={-3,4.1};
   double high[]={12.1,5.8};
   double a[]=xx.calculate(low,high,fax,false);
   double x0[]=new double[NVARS];
   for(int i=0;i<NVARS;i++) x0[i]=a[i];
   String s="Iterative Hill climber result =\n"+Matrix.toStringT(a);
    String s2="Iterative hill climber stcochastic optimisation:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
   }
}
```



Program Iterative hill climbing with Gauss random gene , Java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;
public class IHCi
{
```

```
            // Maximizing a function
    // iterated Hill climber
    // a stochastic approach
    double T;
    double Vc[];
    double Vn[][];
    double best[];
    int MAX;//number
    int GLIMIT;//arka arkaya başarılı steplerin sayısı
    boolean local;
    int NVARS;//numberof variables
    int N;//number of bit representation for each variable
    double T0;
    double k,k0;//time index
    int POPSIZE;
    double low[];
    double high[];
    int t;
    //if_xj f;  // function to be avaluated by genetic algorithm

    public IHCi(int iPOPSIZE,int iNVARS,int iN,double Ti)
    {
            T=Ti;
            T0=T;
    POPSIZE=iPOPSIZE;
    NVARS=iNVARS; // number of variables equation to solve
    MAX=100;
    N=iN;           // number of bits represent each variable for example 32
    local=false;

    t=0;
    Vc=new double[NVARS];
    Vn=new double[POPSIZE+1][NVARS];
    best=new double[NVARS];
    low=new double[NVARS];
    high=new double[NVARS];
    local=true;
            }
public static double  gaussrandom(double xort,double r)
{ int iset=0;
 double gset=0;
 double fac,rsq,v1,v2;
 double aret=0;
 if(iset==0)
 {
  do{
            v1=2.0*Math.random()-1.0;
            v2=2.0*Math.random()-1.0;
            rsq=v1*v1+v2*v2;
    }while(rsq>=1.0 || rsq==0.0);
  fac=Math.sqrt(-2.0*Math.log(rsq)/rsq);
  gset=v1*fac;
  iset=1;
  aret=v2*fac;
 }
 else
 {
 iset=0;
 aret=gset;
 }
 return aret/4.0*r+xort;
 }

            public void setVc(double ilow[],double ihigh[],if_xj fi)
            {

            //select original gene randomly
            for(int i=0;i<NVARS;i++)
              {low[i]=ilow[i];high[i]=ihigh[i];Vc[i]=(high[i]-low[i])*Math.random();}
    }
```

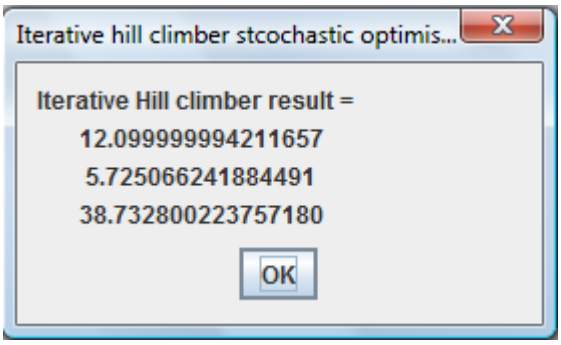
```
            public void setBest(double gi[],if_xj fi)
            {
            //select originalgene andomly
            for(int i=0;i<NVARS;i++)
              {best[i]=gi[i];}
            }

 public void setVc(double gi[],if_xj fi)
            {
            //select original gene from other gene
                        for(int i=0;i<NVARS;i++)
              {Vc[i]=gi[i];}
            }

            public void setVn(if_xj fi)
            {//select neighbouring genes similar to the original one
            for(int i=0;i<POPSIZE;i++)
              { for(int j=0;j<NVARS;j++)
                {Vn[i][j]=Vc[j];}}
                mutate(fi);
            }

public void mutate(if_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  //neigboring factor
  double d1=high[var]-Vn[member][var];
  double d2=Vn[member][var]-low[var];
  double d=Math.abs(Math.min(d1,d2));

  double R=gaussrandom(Vn[member][var],d);
  Vn[member][var] = R;
  //System.out.println("R="+R+"d1="+d1+"d2="+d2+"d="+d+"var = "+var+" "+Vn[member][var]);
  if(Vn[member][var]>high[var]) Vn[member][var]=high[var]-d*Math.random()*0.122561462;
  else if (Vn[member][var]<low[var]) Vn[member][var]=low[var]+d*Math.random()*0.122561462;;
            //after mutation recalculate the function value
  }
}


            public void evaluate(if_xj fi)
            {
              int mem;
              int j;
              double x[]=new double[NVARS];
              for(mem=0;mem<POPSIZE;mem++)
              {
              //System.out.println(Vn[mem].toString());
              if(getFitness(Vn[mem],fi) > getFitness(Vn[POPSIZE],fi))
                {
                        //Vn[POPSIZE]=Vn[mem];
                for(j=0;j<NVARS;j++)
                  {
                  Vn[POPSIZE][j]=Vn[mem][j];
                          }//end of for(j=0..
                        }//end of if
              }//end of for(mem=0;..
            }
```

```
  public double[] getBest(double[] b,if_xj fi)
  {
  double aa[]=new double[NVARS+1];
  for(int i=0;i < NVARS;i++)
    {
    aa[i]=b[i];
                }
                aa[NVARS]=getFitness(b,fi);
  return aa;
  }

  public double getFitness(double []xi,if_xj fi)
  {
  return fi.func(xi);
  }

  public double[] calculate(double low[],double up[],if_xj fi,boolean report)
  {
    local=true;
    setVc(low,up,fi); //new random value
                t=0;
    do
    {
    do
    {   setVn(fi);
        evaluate(fi);
        if(getFitness(Vn[POPSIZE],fi) > getFitness(Vc,fi))
        {  setVc(Vn[POPSIZE], fi);
                    if(getFitness(Vn[POPSIZE],fi) > getFitness(best,fi)) setBest(Vn[POPSIZE],fi);
        }
        else local=false;
    } while(local);
    if(report)
       System.out.println(Matrix.toString(getBest(best,fi)));
    t++;
    } while(t<MAX);
  return getBest(best,fi);
  }
}
```

Iterative hill climbing with Gauss random gene example problem, Java code

```
 // iterated hill climbing
//
import java.io.*;
import javax.swing.JOptionPane;

class NA113
{
   public static void main(String args[]) throws IOException
   {
   int N=18;
            int POPSIZE=30;
   int MAXGENS=400;
   int NVARS=2;
   if_xj fax=(double x[])->21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]);
   IHCi xx=new IHCi(POPSIZE,NVARS,N,20);
   double low[]={-3,4.1};
   double high[]={12.1,5.8};
   double a[]=xx.calculate(low,high,fax,false);
   double x0[]=new double[NVARS];
   for(int i=0;i<NVARS;i++) x0[i]=a[i];
   String s="Iterative Hill climber result =\n"+Matrix.toStringT(a);
   String s2="Iterative hill climber stcochastic optimisation";
   JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
```

```
    }
}
```



STOCHASTIC METHODS: SIMULATED ANNEALING

An alternative approach obtained from the thermodynamic analogy is the simulated annealing. If a metal is heated obove the melting point, atoms in the structure started randomly move. When the temperature drops movemets slow down, and slide to a lower energy level. At the end atoms self organise themselves in cristal structures with minimum energy levels. This process is known as annealing. The energy state of an annealed metal can be expressed by using Boltzman statistics.

$$p(e) = \alpha \, exp\left(\frac{-e}{kT}\right)$$

In this equation e>=0 is the energy level, T is the temperature, α=1/(kT), k is boltzman constant p(e) is the probability distribution function. Annealing can be taken as a natural optimization process. This phenemonen can be mimic in computer codes. Consider minimisation of function $\mathbf{f(x_1,x_2,x_3,\ldots,x_n)}$ fonksiyonunu minimize etme problemini göz önüne alalim. In order to establish the model an artificial Temperature value will be assumed. Optimization should be started with a big value of T. (In the starting of the annealing process the temperature of the metal is high) . As a starting point take an initial guess vector $x^0=\mathbf{(x_1,x_2,x_3,\ldots,x_n)^0}$ vektörünü alalim. If we change the initial vector a small amount Δx, distribution in Δx should follow the Gauss distribution. So in the neighbourhood of first point a second point can be obtain as $x^1=x^0+\Delta x$. The difference in the function value will be $\Delta f=f(x^1)-f(x^0)$. Gaussian distribution function is:

$$p(\Delta x_j) = \frac{1}{T\sqrt{2\pi}} \, exp\left(\frac{-\Delta x_j}{2T^2}\right) \quad 1 \le j \le n$$

If $\Delta f < 0$ we can accept the new point as a better approach to the minimum. But if $\Delta f \geq 0$ we do not automatically reject the second point. A Boltzmann distribution function is created and investigated.

$$p(\Delta f) = \frac{1}{kT} \, exp\left(\frac{-\Delta f}{kT}\right)$$

In order to decide accepting or rejecting the result a random number z is created in the range of [0,1/(kT)]. If p(Δf) > z accept x^1 as a new iteration value, If not reject it. Then a new Δx for a new vector is calculated. In time T temperature factor decreased as a some defined function of time. In the beginning of the iteration due to high temperatures the chance of accepting a new x value is high. In time when the T is getting smaller, the system became more conservative on accepting $\Delta f \geq 0$ cases. If this process is written as an algorithm

```
Simulated annealing algorithm
start
    t=0
    set a T valu.
     Select a v_c value as random
```

```
Calculate the function value for v_c
While
        while
            select a new point v_n in the neighbourhood of v_c
            if f(v_c) < f(v_n) then v_c = v_n (accepth the new point)
            else if tesadüfi sayi(0-1) < exp[(f(v_n)-f(v_c))/T] then v_c = v_n even if f(v_c) >= f(v_n) accept the new
point
        until (exit condition)
        t=t+1
    until (exit condition)
stop
```

Function T can be evaluated with different formulas, some examples are given below.

$T = \frac{T_0}{1+\log(1+t)}$

$T = T - \frac{\log(t_0)}{t\log(t)}$

$T = \frac{T_0}{1+t}$

As an actual program two version is developed. In the first version Gene1 and genotpe1 classes of genetic algorithms (binary genes) are used to define the neighbourhood area (only one bit is replaced in random-gene is mutated)

Program Simulated annealing with binary gene, java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;


public class simulatedannealingi
{
        // Maximizing a function
   //simulated annealing
   // a version of simulated annealing statistical approach
   double T; //Temperature
   genotype1i Vc;
   genotype1i Vn[];
   genotype1i best;
   genotype1i oldbest;
   int MAX;//number
   int GLIMIT;//arka arkaya başarılı steplerin sayısı
   boolean local;
   int NVARS;//numberof variables
   int N;//number of bit representation for each variable
   double T0;
   double k,k0;//time index
   int POPSIZE;
   int BESTLIMIT;
   int bestcounter;

   //if_xj f;  // function to be avaluated by genetic algorithm

   public simulatedannealingi(int iPOPSIZE,int iNVARS,int iN,double Ti)
   {
        T=Ti;
        T0=T;
   POPSIZE=iPOPSIZE;
   NVARS=iNVARS; // number of variables equation to solve
   MAX=20;
   N=iN;        // number of bits represent each variable for example 32
   Vn=new genotype1i[POPSIZE+1];
   local=false;
   GLIMIT=10;
```

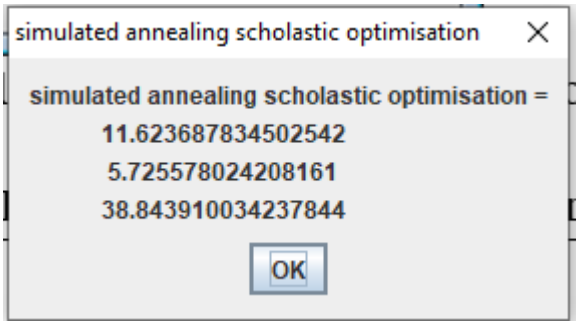
```
BESTLIMIT=3;
k=100.0;
k0=k;
bestcounter=0;
        }

        public void setVc(double low[],double up[],if_xj fi)
        {
        //select originalgene andomly
        Vc=new genotype1i(NVARS,low,up,N,fi);
        }
        public void setBest(genotype1i gi,if_xj fi)
        {
        //select originalgene andomly
        best=new genotype1i(gi,fi);
        }
        public void setoldBest(if_xj fi)
        {
        //select originalgene andomly
        oldbest=new genotype1i(best,fi);
        }

public void setVc(genotype1i gi,if_xj fi)
        {
        //select original gene from other gene
        Vc=new genotype1i(gi,fi);
        }
        public void setVn(if_xj fi)
        {//select neighbouring genes similar to the original one
        for(int i=0;i<POPSIZE+1;i++)
          { Vn[i]=new genotype1i(Vc,fi);}
           mutate(fi);
        }

        public void evaluate(if_xj fi)
        {
          int mem;
          int j;
          double x[]=new double[NVARS];
          for(mem=0;mem<POPSIZE;mem++)
          {
          //System.out.println(Vn[mem].toString());
          Vn[mem].setFitness(fi);
          if(Vn[mem].getFitness() > Vn[POPSIZE].getFitness())
            {
                    Vn[POPSIZE]=Vn[mem];
            for(j=0;j<NVARS;j++)
              {
              Vn[POPSIZE].G[j].copyGene(Vn[mem].G[j]);
                      }//end of for(j=0..
                    }//end of if
          }//end of for(mem=0;..
        }

        public void setRFCF(genotype1i Pi[])
        {
        //calculates relative and cumulative fitness functions
        int mem;
        int PN=Pi.length;
        double sum=0.0;
        //total fitness of Vn
        for(mem=0;mem<PN;mem++)
          {sum+=Pi[mem].Fitness;}
        //System.out.println("sum="+sum);
        //calculate relative fitness of each genotype1i
        for(mem=0;mem<PN;mem++)
          {Pi[mem].setRFitness(sum);}
        //calculate cumulative fitness
        Pi[0].setCFitness(Pi[0].getRFitness());
for(mem=1;mem<PN;mem++)
```

```
            {Pi[mem].setCFitness(Pi[mem-1].getCFitness()+Pi[mem].getRFitness());
             }
           }

           public void toString(genotype1i Pi[])
           {
           int mem;
           int PN=Pi.length;
           for(mem=0;mem<PN;mem++)
           {
           //list them
           System.out.println("Vn["+mem+"]="+Pi[mem].toString());
           }
           }


public void mutate(if_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  Vn[member].G[var].mutate();
           //after mutation recalculate the function value
  Vn[member].setFitness(fi);
  }
}

public double[] getBest(genotype1i b)
{
double aa[]=new double[NVARS+1];
for(int i=0;i < NVARS;i++)
 {
 aa[i]=b.getGene(i);
           }
           aa[NVARS]=b.getFitness();
return aa;
}
public double[] calculate(double low[],double up[],if_xj fi,boolean report)
{
 int good_swap=0;
           setVc(low,up,fi);// new set of random selection in each time
 setBest(Vc,fi);
           do
 {
           setoldBest(fi);
 do
 {  setVn(fi);
    evaluate(fi);
    if(Vn[POPSIZE].getFitness() > Vc.getFitness())
    {  setVc(Vn[POPSIZE], fi);
               if(Vn[POPSIZE].getFitness() > best.getFitness())
                  {setBest(Vn[POPSIZE],fi); }
                     if(oldbest.getFitness() == best.getFitness()) bestcounter++;
               if(bestcounter>BESTLIMIT) break;
               good_swap++;
    }
    else if(Math.random()<Math.exp((Vn[0].getFitness()-Vc.getFitness())/T))
    {  setVc(Vn[POPSIZE],fi);}
 if(report)
   System.out.println(Matrix.toString(getBest(best)));
 } while(good_swap<GLIMIT);
 k+=1.0;
```

```
     double lk=Math.log(k);
     T=T-Math.log(k0)/(k*lk*lk);
     setVc(low,up,fi); //new random value

     } while(k<MAX);
   return getBest(best);
   }
}
```

Program Simulated annealing with binary gene example program NA114.java, java code

```
 // simulated annealing
//
import java.io.*;
import javax.swing.JOptionPane;

class NA114
{   public static void main(String args[]) throws IOException
    {
    int N=18;
    int POPSIZE=1;
    int MAXGENS=400;
    int NVARS=2;
    if_xj fax=(double x[])->21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]);;
    simulatedannealingi xx=new simulatedannealingi(POPSIZE,NVARS,N,20);
    double low[]={-3,4.1};
    double high[]={12.1,5.8};
    double a[]=xx.calculate(low,high,fax,false);
    double x0[]=new double[NVARS];
    for(int i=0;i<NVARS;i++) x0[i]=a[i];
    String s="simulated annealing scholastic optimisation =\n"+Matrix.toStringT(a);
     String s2="simulated annealing scholastic optimisation ";
     JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    }
}
```



The second version will be using Gauss random gene to calculate neighbour point.

Program 5.21-3 Simulated annealing with Gauss random gene , java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;

public class SAi
{   // Maximizing a function
   // simulated annealing
   // a version of simulated annealing statistical approach

   double T; //Temperature
   double Vc[];
   double Vn[][];
   double best[];
   int MAX;//number
   int GLIMIT;//arka arkaya başarılı steplerin SAiyısı
   boolean local;
   int NVARS;//numberof variables
```

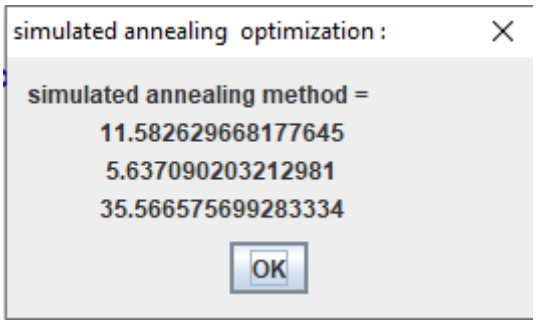
```
    int N;//number of bit representation for each variable
    double T0;
    double k,k0;//time index
    int POPSIZE;
    double low[];
    double high[];

    //if_xj f;  // function to be avaluated by genetic algorithm

    public SAi(int iPOPSIZE,int iNVARS,int iN,double Ti)
    {
                T=Ti;
                T0=T;
    POPSIZE=iPOPSIZE;
    NVARS=iNVARS; // number of variables equation to solve
    MAX=20;
    N=iN;           // number of bits represent each variable for example 32
    local=false;
    GLIMIT=30;
    k=100.0;
    k0=k;
    Vc=new double[NVARS];
    Vn=new double[POPSIZE+1][NVARS];
    best=new double[NVARS];
    low=new double[NVARS];
    high=new double[NVARS];
                }

public static double  gaussrandom(double xort,double r)
{ int iset=0;
 double gset=0;
 double fac,rsq,v1,v2;
 double aret=0;
 if(iset==0)
 {
   do{
                v1=2.0*Math.random()-1.0;
                v2=2.0*Math.random()-1.0;
                rsq=v1*v1+v2*v2;
     }while(rsq>=1.0 || rsq==0.0);
   fac=Math.sqrt(-2.0*Math.log(rsq)/rsq);
   gset=v1*fac;
   iset=1;
   aret=v2*fac;
 }
 else
 {
 iset=0;
 aret=gset;
 }
 return aret/4.0*r+xort;
 }
                public void setVc(double ilow[],double ihigh[],if_xj fi)
                {
                //select original gene randomly
                for(int i=0;i<NVARS;i++)
                   {low[i]=ilow[i];high[i]=ihigh[i];Vc[i]=(high[i]-low[i])*Math.random();}
    }
                public void setBest(double gi[],if_xj fi)
                {
                for(int i=0;i<NVARS;i++)
                   {best[i]=gi[i];}
                }

    public void setVc(double gi[],if_xj fi)
                {
                //select original gene from other gene
                                for(int i=0;i<NVARS;i++)
                   {Vc[i]=gi[i];}
                }
```

```
            public void setVn(if_xj fi)
            {//select neighbouring genes similar to the original one
            for(int i=0;i<POPSIZE+1;i++)
              { for(int j=0;j<NVARS;j++)
               {Vn[i][j]=Vc[i];}}
              mutate(fi);
            }
public void mutate(if_xj fi)
{
int i;
double lbound,hbound;
int nmutations;
int member;
int var;
nmutations=(int)((double)(POPSIZE*NVARS*N));
for(i=0;i<nmutations;i++)
  {
  member=(int)(Math.random()*POPSIZE);
  var=(int)(Math.random()*NVARS);
  //replace the old value with a new mutated one
  //neigboring factor
  double d1=high[var]-Vn[member][var];
  double d2=Vn[member][var]-low[var];
  double d=Math.min(d1,d2);
  double R=gaussrandom(Vn[member][var],d*Math.random());
  Vn[member][var]= R;
  if(Vn[member][var]>high[var]) Vn[member][var]=high[var]-Math.random()*0.22561462;
  else if (Vn[member][var]<low[var]) Vn[member][var]=low[var]+Math.random()*0.22561462;;
            //after mutation recalculate the function value
  }
}
            public void evaluate(if_xj fi)
            {
              int mem;
              int j;
              double x[]=new double[NVARS];
              for(mem=0;mem<POPSIZE;mem++)
              {
              //System.out.println(Vn[mem].toString());
              if(getFitness(Vn[mem],fi) > getFitness(Vn[POPSIZE],fi))
                {
                        //Vn[POPSIZE]=Vn[mem];
                for(j=0;j<NVARS;j++)
                  {
                  Vn[POPSIZE][j]=Vn[mem][j];
                          }//end of for(j=0..
                        }//end of if
              }//end of for(mem=0;..
            }

public double[] getBest(double[] b,if_xj fi)
{
double aa[]=new double[NVARS+1];
for(int i=0;i < NVARS;i++)
  {
  aa[i]=b[i];
            }
            aa[NVARS]=getFitness(b,fi);
return aa;
}

public double getFitness(double []xi,if_xj fi)
{
return fi.func(xi);
}

public double[] calculate(double low[],double up[],if_xj fi,boolean report)
{
  int good_swap=0;
```

```
            setVc(low,up,fi);// new set of random selection in each time
    setBest(Vc,fi);
            do
    {
    do
    {   setVn(fi);
       evaluate(fi);
       if(getFitness(Vn[POPSIZE],fi) > getFitness(Vc,fi))
       {  setVc(Vn[POPSIZE], fi);
                  if(getFitness(Vn[POPSIZE],fi) > getFitness(best,fi)) setBest(Vn[POPSIZE],fi);
                  good_swap++;
       }
       else if(Math.random()<Math.exp((getFitness(Vn[0],fi)-getFitness(Vc,fi))/(k*T)))
       {   setVc(Vn[POPSIZE],fi);}
    if(report)
      System.out.println(Matrix.toString(getBest(best,fi)));
    } while(good_swap<GLIMIT);
    k+=1.0;
    double lk=Math.log(k);
    T=T-Math.log(k0)/(k*lk*lk);
    setVc(low,up,fi); //new random value
    } while(k<MAX);
  return getBest(best,fi);
  }
}
```

Program Simulated annealing with Gauss random gene example program NA115.java, java code

```
 // simulated annealing
//
import java.io.*;
import javax.swing.JOptionPane;

class NA115
{
   public static void main(String args[]) throws IOException
   {
   int N=18;
            int POPSIZE=1;
   int MAXGENS=500;
   int NVARS=2;
   if_xj fax=(double x[])->21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]);
   SAi xx=new SAi(POPSIZE,NVARS,N,20);
   double low[]={-3,4.1};
   double high[]={12.1,5.8};
   double a[]=xx.calculate(low,high,fax,false);
   double x0[]=new double[NVARS];
   for(int i=0;i<NVARS;i++) x0[i]=a[i];
   String s="simulated annealing method =\n"+Matrix.toStringT(a);
   String s2="simulated annealing  optimization :";
   JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
   }
}
```



STOCHASTIC METHODS: FIREFLY ALGORITHM

You might notice amazing flashing lights of fireflys in the summer nights. The flashing light is produced by a process of bioluminescence. The true main purpose of such flashes are still on debate but one of the fundamental function is attracting mate. The rithmic flash, the rate of flashing and the amount of time form part of the signal system that brings both sexes together. We know that the light intensity at a particular distance from the light source obeys the inverse square law. That is to say, the light intensity decreases as the distance increases. Furthermore, the air absorbs light which becomes weaker and weaker as the distance increases. This physical factors are utilised by fireflies to find each other. The same principle can be utilised to create an artificial firefly for optimisation .The Firefly Algorithm was developed by Xin-She Yang [52], based on the idealization of the flashing characteristics of fireflies. In creating artificial firefly it is assumed that they are unisex, i.e. any firefly in the population can be attracted to any other firefly, Attractiveness is function of brightness. Brightness will decrease as a function of distance between. The less brighter one will move through the brighter one. If there are no brighter fire flies around, it will fly randomly. The brightness of the artificial firefly will directly be determined by objective function value. By using these principles, a pseudocode of the firefly algorithm can be written as:

Objective function $f(X)$, $X = (x_0, ...,x_n)$
Generate initial population of fireflies X^i $(i = 1,2, ...,m)$ *randomly*
Light intensity Ii at x≫ is determined by $f(X^i)$
Define light absorption coefficient γ
while $(t <$MaxGeneration)
for $i = 1 : n$ all n fireflies
    for $j = 1 : n$ all n fireflies (inner loop)
        if $(Ii < Ij)$, Move firefly i towards j; end i f
        Vary attractiveness with distance r as

$$\beta = \beta_0 e^{-\gamma d^2} \text{ or } \beta = \frac{\beta_0}{1+\gamma d^2}$$

$$d_{ij} = \sqrt{\sum_{k=1}^{n} (x_{i,k} - x_{i,k})^2}$$

        Evaluate new solutions and update light intensity

$$X^i = (1-\beta) X^i + \beta X^j + \alpha \, (gaussrandom() - \frac{1}{2})$$

    end for j
end for i
Rank the fireflies and find the current global best *gt*
end while
Postprocess results and visualization

A java program is developed for the above firefly algorithm . Program is extended also to use Nelder-Mead method so that the result can be tested by using Nelder_Mead

Program 5.22.1 FireFly optimisation firefly.java, java code

```
 import java.io.*;
import java.text.*;
import java.util.Locale;

public class fireflyi
{
double gamma,alpha,beta;
double alpha0,alphan;
int m;
int n;
// m number of fireflies
// n number of variables for each fireflyi
// I0 light intensity at source
// gamma absorbtion coefficient
// alpha size of the random step
// beta attractiveness
double x[][];
double I0[];
double ymin;
double xmin[];
```

```
int imin;
int ngeneration;
public fireflyi(if_xj ff,int mi,int ni, double xxmin[],double xxmax[],double gammai,double alpha0i,double alphani,int ngenerationi)
{
// initial population set
n=ni;
m=mi;
x=new double[m+1][n];
I0=new double[m+1];
xmin=new double[n];
gamma=gammai;
alpha0=alpha0i;
alphan=alphani;
ngeneration=ngenerationi;
for(int j=0;j<n;j++)
  {xmin[j]=xxmin[j]+Math.random()*(xxmax[j]-xxmin[j]);}
ymin=ff.func(xmin);
  //end of init
int l=0;
double d=0;
double t=0;
while(l<ngeneration)
{
// find minimum of population
for(int i=0;i<m;i++)
{for(int j=0;j<n;j++)
  {x[i][j]=xxmin[j]+Math.random()*(xxmax[j]-xxmin[j]);}
  I0[i]=ff.func(x[i]);
  if(I0[i]<ymin)
    {ymin=I0[i];imin=i;
    for(int k=0;k<n;k++)
    {xmin[k]=x[i][k];}
    };
}
for(int j=0;j<n;j++)
  {x[m][j]=xmin[j];}
  I0[m]=ymin;
l++;
double fi=0,fj=0;
for(int i=0;i<=m;i++)
{ //fi=ff.func(x[i]);
  for(int j=0;j<n;j++)
  {
    //fj=ff.func(x[j]);
    if(I0[i]<I0[j])
    {
    beta=attractiveness(i,j);
    for(int k=0;k<n;k++)
    {   //The best one will remain the rest will change
                if(i!=imin)
      { x[i][k]=(1.0-beta)*x[i][k]+beta*x[j][k]+alpha*(Math.random()-0.5);} }
    }
    else
    {
                for(int k=0;k<n;k++)
      {  if(i!=imin)
                  { x[i][k]=x[i][k]+alpha*(Math.random()-0.5); }
                }
    }
  }

}

    //alpha will change by time
    //(shrink to a smaller value)
    alpha=alphan+(alpha0-alphan)*Math.exp(-t);
    t=0.1*l;
}
}

public double distance(int i, int j)
{
double d=0;
for(int k=0;k<n;k++)
{d+=(x[i][k]-x[j][k])*(x[i][k]-x[j][k]);}
d=Math.sqrt(d);
```

```
return d;
}

public double attractiveness(int i,int j)
{   double d=distance(i,j);
                return I0[i]/(1+gamma*d*d);
                //return I0[i]*Math.exp(-gamma*d*d);
}

public String toString()
{ String s="ymin = "+ymin+"\nxmin = ";
  for(int k=0;k<n;k++)
  {s+=xmin[k]+" ";}
  return s;
}
public String toString(if_xj ff,double xx[])
{ String s="ymin = "+ff.func(xx)+"\nxmin = ";
  for(int k=0;k<n;k++)
  {s+=xx[k]+" ";}
  return s;
}
public double[] xmin()
{
  return xmin;
}

public static double[] nelder(if_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
                int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
   {for(j=0;j<x[0].length;j++)
     {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
      else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
     }
     p[i][j] = fnelder.func(p[i]);
   }


   // Inlet variable definitions
   // fnelder : abstract multivariable function f(x)
   // x : independent variable set of n+1 simplex elements
   // maxiteration : maximum iteration number
   // tolerance :
        int NDIMS = x.length-1;
        int NPTS = x.length;
        int FUNC = NDIMS;
        int ncalls = 0;
          ////// construct the starting simplex //////////////////
          //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
          double z[]=new double[NDIMS];
          double best = 1E99;
          /////////////// calculate the first function values for the simplex ////////////////
          int iter=0;
          for (iter=1; iter<maxiteration; iter++)
          {
                        ///////////  define lo, nhi, hi (low high next_to_high //////////////
             int  ilo=0, ihi=0, inhi = -1; // -1 means missing
             double flo = p[0][FUNC];
             double fhi = flo;
             double pavg,sterr;
             for (i=1; i<NPTS; i++)
             {
               if (p[i][FUNC] < flo)
                 {flo=p[i][FUNC]; ilo=i;}
               if (p[i][FUNC] > fhi)
                 {fhi=p[i][FUNC]; ihi=i;}
             }
             double fnhi = flo;
             inhi = ilo;
             for (i=0; i<NPTS; i++)
              if ((i != ihi) && (p[i][FUNC] > fnhi))
                 {fnhi=p[i][FUNC]; inhi=i;}
             ////////// exit criteria //////////////
             if ((iter % 4*NDIMS) == 0)
```

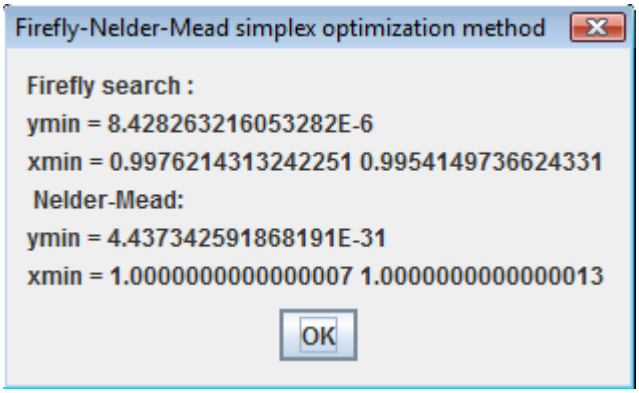
```
{
   // calculate the avarage (including maximum value)
   pavg=0;
   for(i=0;i<NPTS;i++)
     pavg+=p[i][FUNC];
   pavg/=NPTS;
   double tot=0;
   if(printlist!=0)
   {  System.out.print(iter);
      for (j=0; j<=NDIMS; j++)
         { System.out.print(p[ilo][j]+" ");}
      System.out.println("");
   }
   for(i=0;i<NPTS;i++)
   { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
   sterr=Math.sqrt(tot/NPTS);
   //if(sterr < tolerance)
     { for (j=0; j<NDIMS; j++)
          { z[j]=p[ilo][j];}
       //break;
     }
   best = p[ilo][FUNC];
}

///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);


///////// reflect ////////////////

double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
   for (j=0; j<NDIMS; j++)
     p[ihi][j] = r[j];
   p[ihi][FUNC] = fr;
   continue;
}

if (fr < flo)  //// expand
{
   double e[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     e[j] = 3*ave[j] - 2*p[ihi][j];
   double fe = fnelder.func(e);
   if (fe < fr)
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = e[j];
      p[ihi][FUNC] = fe;
      continue;
   }
   else
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = r[j];
      p[ihi][FUNC] = fr;
      continue;
   }
}

////////////// shrink:
```

```
            if (fr < fhi)
            {
               double c[] = new double[NDIMS];
               for (j=0; j<NDIMS; j++)
                 c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
               double fc = fnelder.func(c);
               if (fc <= fr)
               {
                  for (j=0; j<NDIMS; j++)
                    p[ihi][j] = c[j];
                  p[ihi][FUNC] = fc;
                  continue;
               }
               else   /////// daralt
               {
                  for (i=0; i<NPTS; i++)
                    if (i != ilo)
                    {
                       for (j=0; j<NDIMS; j++)
                         p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                       p[i][FUNC] = fnelder.func(p[i]);
                    }
                  continue;
               }
            }

            if (fr >= fhi)   ///
            {
               double cc[] = new double[NDIMS];
               for (j=0; j<NDIMS; j++)
                 cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
               double fcc = fnelder.func(cc);
               if (fcc < fhi)
               {
                  for (j=0; j<NDIMS; j++)
                    p[ihi][j] = cc[j];
                  p[ihi][FUNC] = fcc;
                  continue;
               }
               else    /////////
               {
                  for (i=0; i<NPTS; i++)
                    if (i != ilo)
                    {
                       for (j=0; j<NDIMS; j++)
                         p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                       p[i][FUNC] = fnelder.func(p[i]);
                    }
               }
            }
         }
      return z;
}
    public static double[] nelder(if_xj fnelder,double a[],double da[],double tolerance)
    {return nelder(fnelder,a,da,500,tolerance,0);}

    public static double[] nelder(if_xj fnelder,double a[],double da[])
    {return nelder(fnelder,a,da,500,1.0e-10,0);}

    public static double[] nelder(if_xj fnelder,double a[])
    {
                    double [] da=new double[a.length];
          for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
          return nelder(fnelder,a,da);
    }
}
```

Program 5.22.2 FireFly optimisation Example test case fireflytest.java

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class fireflytest
```

```
{
public static void main(String arg[])
{
if_xj ff=(double x[])->(1.0-x[0])*(1.0-x[0])+100*(x[1]-x[0]*x[0])*(x[1]-x[0]*x[0]);
double xmin[]={0,0};
double xmax[]={2.0,2.0};
double gamma=1.0;
double alpha0=0.5;
double alphan=0.01;
fireflyi  ffly=new fireflyi(ff,500,2,xmin,xmax,gamma,alpha0,alphan,1000);
double x0[]=ffly.xmin();
double r1[]= fireflyi.nelder(ff,x0);
String s="Firefly search : \n"+ffly.toString()+"\n Nelder-Mead: \n"+ffly.toString(ff,r1);
    String s2="Firefly-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
}
}
```



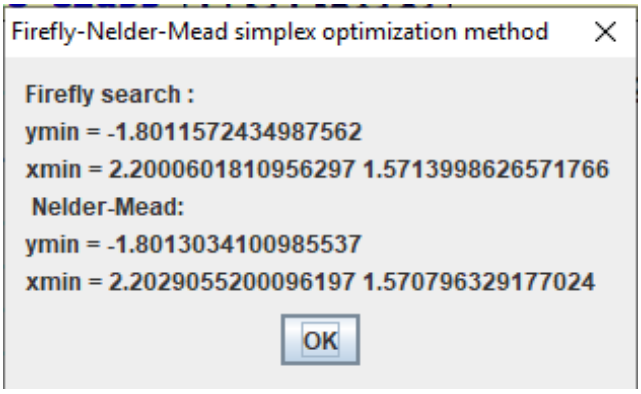
As a second example case Michaelewicz function with n=2 and m=10 is taken

$$f(x) = -\sum_{i=0}^{n}\left[\sin(x_i)\left[\sin\left(\frac{(i+1)x_i^2}{\pi}\right)\right]^{2m}\right]$$

Program 5.22.3 FireFly optimisation Example test case fireflytest.java

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fa implements if_xj
{
int n;
public double func(double x[])
{
//çözümü istenen fonksiyon
double ff=0;
int n=x.length;
for(int i=0;i<n;i++)
{ff+=-Math.sin(x[i])*Math.pow(Math.sin((i+1)*x[i]*x[i]/Math.PI),20);}
return ff; //maksimum testi
}
}

public class fireflytest1
{
public static void main(String arg[])
{
fa ff=new fa();
double xmin[]={0,0};
double xmax[]={2.5,2.5};
double gamma=1.0;
double alpha0=0.5;
double alphan=0.01;
```

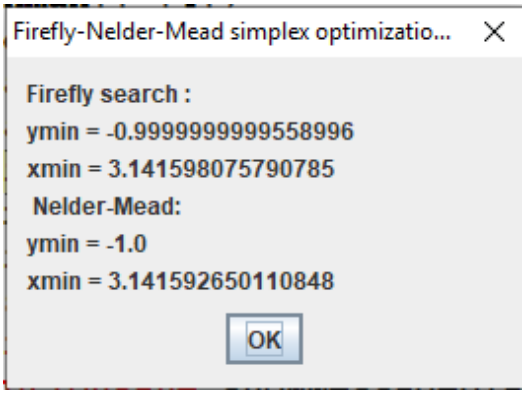
```
fireflyi  ffly=new fireflyi(ff,500,2,xmin,xmax,gamma,alpha0,alphan,1000);
double x0[]=ffly.xmin();
double r1[]= fireflyi.nelder(ff,x0);
String s="Firefly search : \n"+ffly.toString()+"\n Nelder-Mead: \n"+ffly.toString(ff,r1);
    String s2="Firefly-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
}
}
```

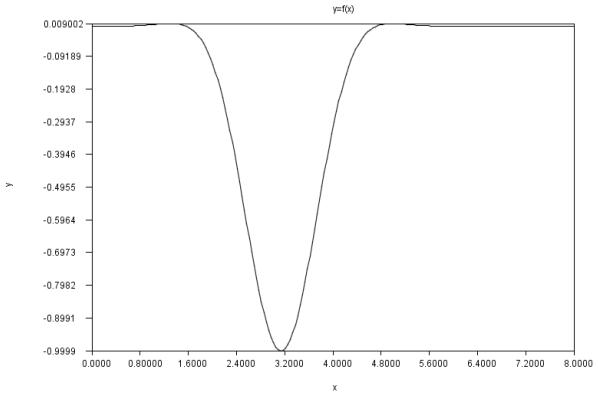


Another interesting test function is Easom's function: $f(x) = \cos(x)e^{-(x-\pi)^2}$ which has a solution x=π, y=f(x)=1

Program 5.22.4 FireFly optimisation Example test case fireflytest.java

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class fireflytest
{
public static void main(String arg[])
{
if_xj ff=(double x[])->Math.cos(x[0])*Math.exp(-(x[0]-Math.PI)*(x[0]-Math.PI));
double xmin[]={0};
double xmax[]={4};
double gamma=1.0;
double alpha0=0.5;
double alphan=0.01;
fireflyi  ffly=new fireflyi(ff,500,1,xmin,xmax,gamma,alpha0,alphan,1000);
double x0[]=ffly.xmin();
double r1[]= fireflyi.nelder(ff,x0);
String s="Firefly search : \n"+ffly.toString()+"\n Nelder-Mead: \n"+ffly.toString(ff,r1);
    String s2="Firefly-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
}
}
```






```
 from math import *
from f_xj import *
from matplotlib import pyplot as plt
import numpy as np
from random import *


class particle_swarm():

#class variables
#int m,n
#double alpha,beta,beta0,teta,gamma
#m number of population
#n number of variables for each individual
#double x[][];  location for each population member
#double y[];    function value of each population member
#double yxmin;  current best minimum
#double ygmin;  global best minimum
#double xmin[]; current best x
#double gmin[]; global best x
#double v[][];  velocity vector
#int ngeneration;  number of generations in iteration


    def __init__(self,ff,mi, xxmin,xxmax,alphai,betai,ngenerationi):
        #initial population set
        self.teta=0.5
        self.gamma=0.5
        #initial population set
        self.n=len(xxmin);
        self.m=mi;
        self.alpha=alphai
        self.beta0=betai
        self.beta=self.beta0
        self.ngeneration=ngenerationi
        self.x= [[0 for j in range(self.n)] for i in range(self.m)]
        self.v= [[0 for j in range(self.n)] for i in range(self.m)]
        self.y=[0 for i in range(self.m)]
        self.xmin=[0 for i in range(self.n)]
        self.gmin=[0 for i in range(self.n)]
        self.ngeneration=ngenerationi
        yxmin=1.0e50
        for i in range(self.m):
            for j in range(self.n):
                self.x[i][j]=xxmin[j]+random()*(xxmax[j]-xxmin[j])
                self.v[i][j]=random()
            self.y[i]=ff.func(self.x[i]);
            if self.y[i]<yxmin:
                yxmin=self.y[i]
                ygmin=self.y[i]
                for k in range(self.n):
                    self.xmin[k]=self.x[i][k]
                    self.gmin[k]=self.x[i][k]
        t=0
        while t<self.ngeneration:
            yxmin=1.0e50
            for i in range(self.m):
                for j in range(self.n):
```

```
                self.x[i][j]=self.x[i][j]+self.v[i][j]
                if self.x[i][j]<xxmin[j]:
                    self.x[i][j]=xxmin[j]
                if self.x[i][j]>xxmax[j]:
                    self.x[i][j]=xxmax[j]
                self.y[i]=ff.func(self.x[i])
                if self.y[i]<yxmin:
                    yxmin=self.y[i]
                    for k in range(self.n):
                        self.xmin[k]=self.x[i][k]
        for i in range(self.m):
            if self.y[i]<ygmin:
                ygmin=self.y[i]
                for k in range(self.n):
                    self.gmin[k]=self.x[i][k]
        for i in range(self.m):
            for j in range(self.n):
                self.v[i][j]=self.teta*self.v[i][j]+self.alpha*random()*(self.gmin[j]-self.xmin[j])+self.beta*random()*(self.xmin[j]-
self.x[i][j])
        t=t+1
        self.beta=self.beta0*pow(self.gamma,0.01*t)
```

5.21 STOCHASTIC METHODS: PARTICLE SWARM ALGORITHM

Particle swarm optimization (PSO) was developed by Kennedy and Eberhart in 1995, based on swarm behaviour such as fish and bird schooling in nature. PSO searches the space of an objective function by adjusting the trajectories of individual agents, called particles. The movement of the particles has both stochastic and deterministic elements. Each particle is attracted toward the best location of the given time and the best location of the near past (a current global). It has also tendency to move randomly. When a particle finds a location that is better than any previously found locations, then it updates it as the new current best for particle *i*.It also remembers all the currents best values to update a current global value. The aim is to find the global best among all the current best values available until no more improvements are available.

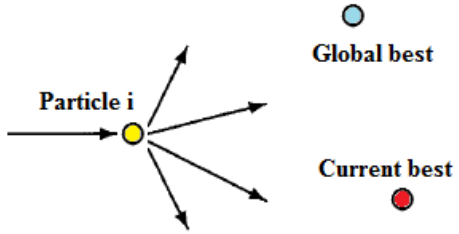


Figure 5.23.1 Particle will attract toward the global best, current best and also move randomly

The method is given in Table 5.23.1 as pseodocode. If x_i and v_i position vector and velocity for particle i, the new velocity vector is determined as:

$$v_i^{t+1} = v_i^t + \alpha^* \varepsilon_1^* (g^* - x_i^t) + \beta^* \varepsilon_2^* (x_i^* - x_i^t)$$

Where ε_1 and ε_1 are two random vectors between 0 and 1, g^* is the current global best x_i^* is local best. The parameters α and β are the learning parameters or acceleration constants. The initial velocity of the particles can be taken as zero, $v_i^{t=0} = 0$. The new position can then be updated by $x_i^{t+1} = x_i^t + v_i^{t+1}$. Although v_i can be any values, it is bounded in some range $[0, v_{max}]$.

There are many variants of the standart PSO algorithm. The most noticiable one is the version in which previous velocity function multiplied with an inertia function. This is called an accelerated PSO.

$$v_i^{t+1} = \theta v_i^t + \alpha^* \varepsilon_1^* (g^* - x_i^t) + \beta^* \varepsilon_2^* (x_i^* - x_i^t)$$

This is equivalent to introduce a virtual mass to stabilise the motion of the particles.

Table 5.23.1 Particle swarm algorithm

Objective function $f(P_i)$, $P_i = (x_0, ..., x_m)_i$ for particle i and m space dimension variable
Initialize locations P_i and velocity V_i of n particles by random selection Use as velocity vector:

$V_i^{t+1} = \theta V_i^t + \alpha^* random()[GPmin - x_i^t] + \beta^* random()[Pmin - x_i^t]$

Where θ is inertia function(an iterative variable between 0 and 1)

α and β are learning parameters or acceleration constants

$\alpha = \alpha_0 \gamma^t$ where $0 \le \gamma \le 1$ $\alpha_0 \cong 0.5..1$ for t is the virtual time

Find Global best GPmin from min{ $f(P_0)…f(P_n)$} (at t = 0)

while (criterion)

t = t + 1 (pseudo time or iteration counter)

for loop over all n particles and all m dimensions

Generate new velocity V_i^{t+1}

Calculate new locations $P_i^{t+1} = P_i^t + V_i^{t+1}$

Evaluate objective functions at new locations

Find the current best for each particle Pmin

Find the global best for each particle GPmin

end for

end while

Output the final results Pmin and GPmin

An example java code is developed. The java code and an example problem is given below:

Program 5.23.1 Particle swarm algorithm: particle_swarm1.java

```
 import java.io.*;
import java.text.*;
import java.util.Locale;

public class particle_swarm1i
{
int m;
int n;
double alpha,beta,beta0,teta,gamma;

// m number of population
// n number of variables for each individual
double x[][];  //location for each population member
double y[];    //function value of each population member
double yxmin;  //current best minimum
double ygmin;  //global best minimum
double xmin[]; //current best x
double gmin[]; //global best x
double v[][];  //velocity vector
int ngeneration;  //number of generations in iteration
if_xj ff;

public particle_swarm1i(if_xj ffi,int mi, double xxmin[],double xxmax[],double alphai,double betai,int ngenerationi)
{
teta=0.5;
gamma=0.5;
ff=ffi;
//initial population set
n=xxmin.length;
m=mi;
alpha=alphai;
beta0=betai;
beta=beta0;
ngeneration=ngenerationi;
x=new double[m][n];
v=new double[m][n];
y=new double[m];
xmin=new double[n];
gmin=new double[n];

ngeneration=ngenerationi;
yxmin=1.0e50;
for(int i=0;i<m;i++)
{for(int j=0;j<n;j++)
  {x[i][j]=xxmin[j]+Math.random()*(xxmax[j]-xxmin[j]);
   v[i][j]=Math.random();
```

```
  }

  y[i]=ff.func(x[i]);
  if(y[i]<yxmin)
    {yxmin=y[i];
     ygmin=y[i];
    for(int k=0;k<n;k++)
    {xmin[k]=x[i][k];
     gmin[k]=x[i][k];
    } //end for(int k=0;k<n;k++)
    }//end if(y[i]<yxmin)
}
//end of init
int t=0;

while(t<ngeneration)
{
yxmin=1.0e50;
for(int i=0;i<m;i++)
{  for(int j=0;j<n;j++)
   {x[i][j]=x[i][j]+v[i][j];
   if(x[i][j]<xxmin[j]) x[i][j]=xxmin[j];
   if(x[i][j]>xxmax[j]) x[i][j]=xxmax[j];
   }
   y[i]=ff.func(x[i]);
   if(y[i]<yxmin)
   {yxmin=y[i];
    for(int k=0;k<n;k++)
    {xmin[k]=x[i][k];}
   }
}
for(int i=0;i<m;i++)
{
   if(y[i]<ygmin)
   {ygmin=y[i];
   for(int k=0;k<n;k++)
   {gmin[k]=x[i][k];}
   }
}
for(int i=0;i<m;i++)
{
   for(int j=0;j<n;j++)
   {v[i][j]=teta*v[i][j]+alpha*Math.random()*(gmin[j]-xmin[j])+beta*Math.random()*(xmin[j]-x[i][j]);}
}//end for int i=..
t++;
beta=beta0*Math.pow(gamma,0.01*t);
}

}


public String toString()
{ String s="ymin = "+ff.func(xmin)+"\nxmin = ";
  for(int k=0;k<n;k++)
  {s+=xmin[k]+" ";}
  s+="\ny global min = "+ff.func(gmin)+"\nxmin = ";
  for(int k=0;k<n;k++)
  {s+=gmin[k]+" ";}
  return s;
}

public String toString(double x[])
{ String s="ymin = "+ff.func(x)+"\nxmin = ";
  for(int k=0;k<n;k++)
  {s+=x[k]+" ";}
  return s;
}

public double[] xmin()
{
  return xmin;
}
public double[] gmin()
{
  return gmin;
}
```

```
public static double[] nelder(if_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
                int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
  {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
  }


  // Inlet variable definitions
  // fnelder : abstract multivariable function f(x)
  // x : independent variable set of n+1 simplex elements
  // maxiteration : maximum iteration number
  // tolerance :
      int NDIMS = x.length-1;
      int NPTS = x.length;
      int FUNC = NDIMS;
      int ncalls = 0;
        ////// construct the starting simplex //////////////////
        //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
        double z[]=new double[NDIMS];
        double best = 1E99;
        /////////////// calculate the first function values for the simplex ////////////////

        int iter=0;

        for (iter=1; iter<maxiteration; iter++)
        {
                    ///////////  define lo, nhi, hi (low high next_to_high //////////////
          int  ilo=0, ihi=0, inhi = -1; // -1 means missing
          double flo = p[0][FUNC];
          double fhi = flo;
          double pavg,sterr;
          for (i=1; i<NPTS; i++)
          {
            if (p[i][FUNC] < flo)
              {flo=p[i][FUNC]; ilo=i;}
            if (p[i][FUNC] > fhi)
              {fhi=p[i][FUNC]; ihi=i;}
          }
          double fnhi = flo;
          inhi = ilo;
          for (i=0; i<NPTS; i++)
            if ((i != ihi) && (p[i][FUNC] > fnhi))
              {fnhi=p[i][FUNC]; inhi=i;}
          ////////// exit criteria //////////////
          if ((iter % 4*NDIMS) == 0)
          {
            // calculate the avarage (including maximum value)
            pavg=0;
            for(i=0;i<NPTS;i++)
              pavg+=p[i][FUNC];
            pavg/=NPTS;
            double tot=0;
            if(printlist!=0)
            {  System.out.print(iter);
              for (j=0; j<=NDIMS; j++)
                { System.out.print(p[ilo][j]+" ");}
              System.out.println("");
            }
            for(i=0;i<NPTS;i++)
            { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
            sterr=Math.sqrt(tot/NPTS);
            //if(sterr < tolerance)
              { for (j=0; j<NDIMS; j++)
                  { z[j]=p[ilo][j];}
                //break;
              }
            best = p[ilo][FUNC];
```

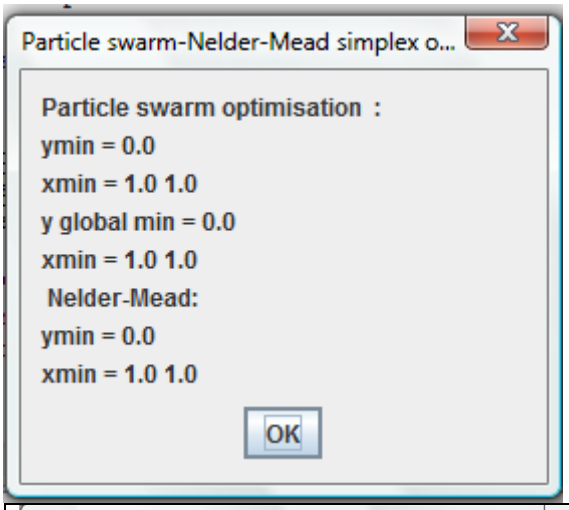
```
}

///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);


///////// reflect ////////////////

double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
   for (j=0; j<NDIMS; j++)
     p[ihi][j] = r[j];
   p[ihi][FUNC] = fr;
   continue;
}

if (fr < flo)  //// expand
{
   double e[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     e[j] = 3*ave[j] - 2*p[ihi][j];
   double fe = fnelder.func(e);
   if (fe < fr)
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = e[j];
      p[ihi][FUNC] = fe;
      continue;
   }
   else
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = r[j];
      p[ihi][FUNC] = fr;
      continue;
   }
}

////////////// shrink:

if (fr < fhi)
{
   double c[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
   double fc = fnelder.func(c);
   if (fc <= fr)
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = c[j];
      p[ihi][FUNC] = fc;
      continue;
   }
   else   /////// daralt
   {
      for (i=0; i<NPTS; i++)
        if (i != ilo)
        {
           for (j=0; j<NDIMS; j++)
             p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
           p[i][FUNC] = fnelder.func(p[i]);
        }
```

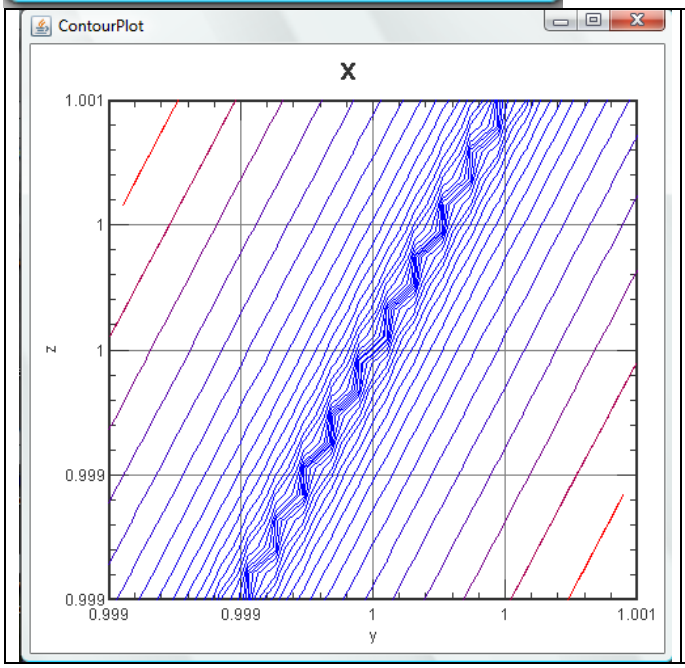
```
                    continue;
                }
            }

            if (fr >= fhi)   ///
            {
                double cc[] = new double[NDIMS];
                for (j=0; j<NDIMS; j++)
                  cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
                double fcc = fnelder.func(cc);
                if (fcc < fhi)
                {
                    for (j=0; j<NDIMS; j++)
                      p[ihi][j] = cc[j];
                    p[ihi][FUNC] = fcc;
                    continue;
                }
                else    /////////
                {
                    for (i=0; i<NPTS; i++)
                      if (i != ilo)
                      {
                          for (j=0; j<NDIMS; j++)
                            p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                          p[i][FUNC] = fnelder.func(p[i]);
                      }
                }
            }
          }
      return z;
}
     public static double[] nelder(if_xj fnelder,double a[],double da[],double tolerance)
     {return nelder(fnelder,a,da,500,tolerance,0);}

     public static double[] nelder(if_xj fnelder,double a[],double da[])
     {return nelder(fnelder,a,da,500,1.0e-10,0);}

     public static double[] nelder(if_xj fnelder,double a[])
     {
                        double [] da=new double[a.length];
            for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
            return nelder(fnelder,a,da);
     }
}
```

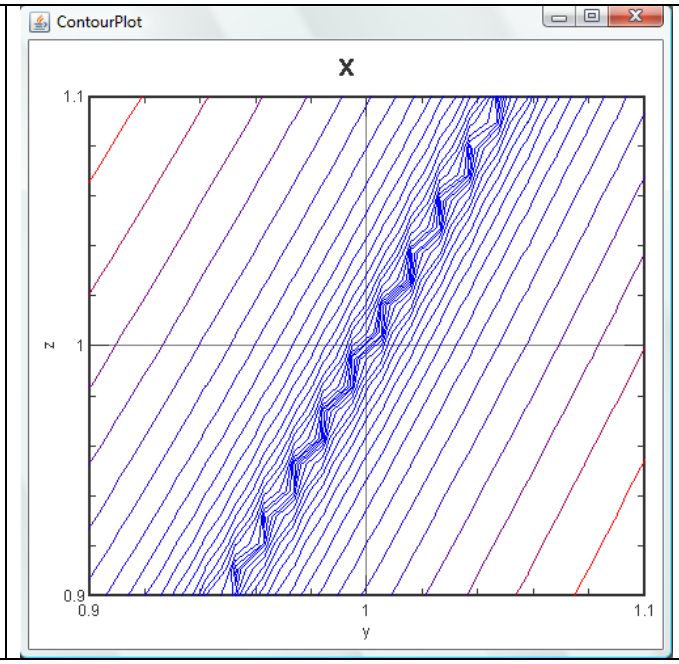
Program 5.23.2 Particle swarm test: particleswarmtest1.java

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fa implements if_xj
{
int n;
public double func(double x[])
{
double ff=(1.0-x[0])*(1.0-x[0])+100.0*(x[1]-x[0]*x[0])*(x[1]-x[0]*x[0]);
return ff; //maksimum testi
}}

public class particleswarmtest1
{
public static void main(String arg[])
{
fa ff=new fa();
double xmin[]={0,0};
double xmax[]={2.5,2.5};
particle_swarm1i ps=new particle_swarm1i(ff,500,xmin,xmax,2.0,0.7,500);
double r1[]= ps.nelder(ff,ps.gmin());
String s="Particle swarm optimisation  : \n"+ps.toString()+"\n Nelder-Mead: \n"+ps.toString(r1);
    String s2="Particle swarm-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);

}
```

```
}
```







Another example function:

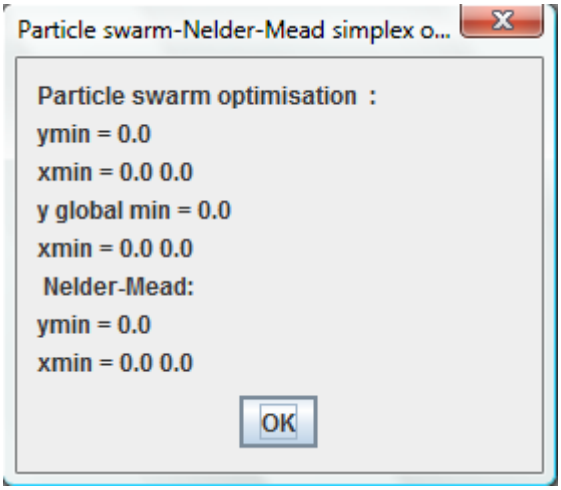
$$f(x) = \left(\sum_{i=0}^{d} |x_i|\right) \exp\left(-\sum_{i=0}^{d} \sin(x_i^2)\right) \qquad -2\pi \le x_i \le 2\pi$$

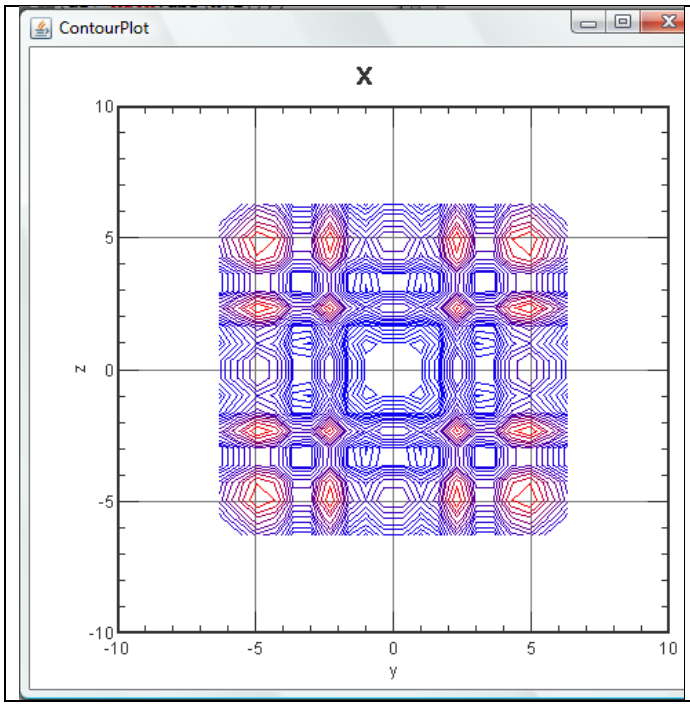
This function has a global minimum at (0,0) and the function value is also 0. It has a singularity at (0,0)

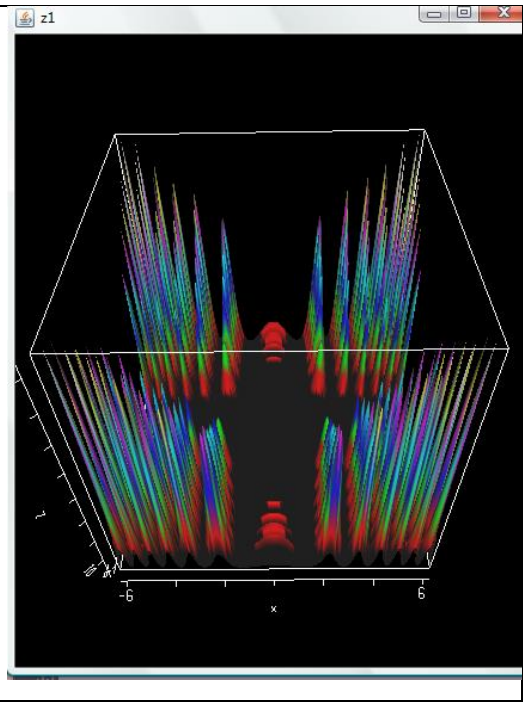
Program 5.23.3 Particle swarm test: particleswarmtest1.java

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fa implements if_xj
{
int n;
public double func(double x[])
```

```
{
//çözümü istenen fonksiyon
double ff=(1.0-x[0])*(1.0-x[0])+100.0*(x[1]-x[0]*x[0])*(x[1]-x[0]*x[0]);
return ff; //maksimum testi
}}


public class particleswarmtest1
{
public static void main(String arg[])
{
fa ff=new fa();
double xmin[]={0,0};
double xmax[]={2.5,2.5};
particle_swarm1i ps=new particle_swarm1i(ff,500,xmin,xmax,2.0,0.7,500);
double r1[]= ps.nelder(ff,ps.gmin());
String s="Particle swarm optimisation  : \n"+ps.toString()+"\n Nelder-Mead: \n"+ps.toString(r1);
    String s2="Particle swarm-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
}}
```







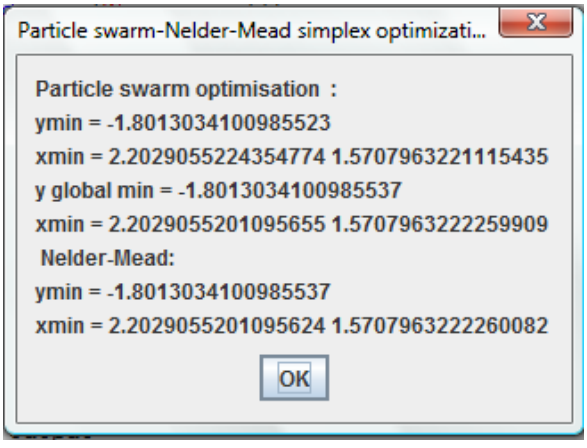
As another test function Michaelewicz function is given

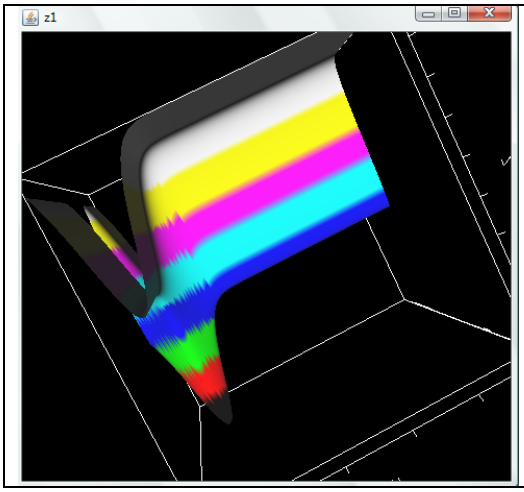
$$f(x) = -\sum_{i=0}^{n}\left[\sin(x_i)\left[\sin\left(\frac{(i+1)x_i^2}{\pi}\right)\right]^{2m}\right]$$

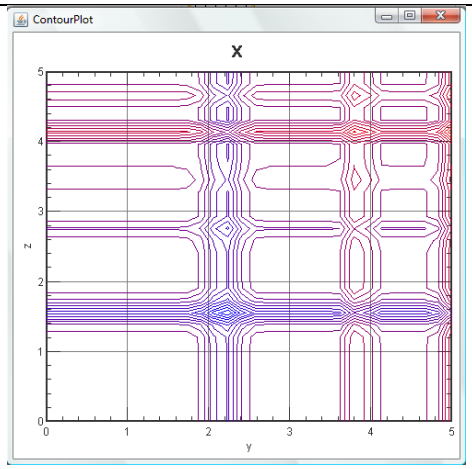
The minimum of the function f = -1.801 is located at (2.2029,1.5704796)

Program 5.23.4 Particle swarm test: particleswarmtest.java

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class fa extends f_xj
{int n;
public double func(double x[])
{double ff=0;
int n=x.length;
for(int i=0;i<n;i++)
{ff+=-Math.sin(x[i])*Math.pow(Math.sin((i+1)*x[i]*x[i]/Math.PI),20);}
return ff; //maksimum testi
}}

public class particleswarmtest1
{
public static void main(String arg[])
{
fa ff=new fa();
double xmin[]={0,0};
double xmax[]={2.5,2.5};
particle_swarm1 ps=new particle_swarm1(ff,500,xmin,xmax,2.0,0.7,500);
double r1[]= ps.nelder(ff,ps.gmin());
String s="Particle swarm optimisation  : \n"+ps.toString()+"\n Nelder-Mead: \n"+ps.toString(r1);
    String s2="Particle swarm-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
}
}
```







Python version of Particle swarm algorithm

Program Particle swarm algorithm python version

```
 # -*- coding: utf-8 -*-
"""
Created on Tue Sep  4 11:40:02 2018

@author: Mustafa Turhan Çoban
"""
from gauss import *;
from math import *;
from f_xj import *;
import numpy as np;
import matplotlib.pyplot as plt;
from mpl_toolkits.mplot3d import Axes3D
from random import *;

class f1(f_xj):
  def func(self,x):
      y=(1.0-x[0])*(1.0-x[0])+100.0*(x[1]-x[0]*x[0])*(x[1]-x[0]*x[0]);
      return y;

class particle_swarm():
    def __init__(self,ffi,mi,xxmin,xxmax,alphai,betai,ngenerationi):
        self.teta=0.5647645;
        self.gamma=0.6564537;
        self.ff=ffi;
        self.alpha=alphai;
        self.beta0=betai;
        self.beta=betai;
        self.n=len(xxmin);
        self.m=mi;  #number of species
        self.ngeneration=ngenerationi; #number of generations
        self.x= [[0 for i in range(self.n)] for j in range(self.m)];
        self.v= [[0 for i in range(self.n)] for j in range(self.m)];
        self.y= [0 for j in range(self.m)];
        self.xmin=[0 for j in range(self.n)];
        self.gmin=[0 for j in range(self.n)];
        yxmin=1.0e50;
        for i in range(self.m):
            for j in range(self.n):
                self.x[i][j]=xxmin[j]+random()*(xxmax[j]-xxmin[j]);
                self.v[i][j]=random();
                #print("vi",self.v[i][j],"   i",i,"j",j)
            self.y[i]=self.ff.func(self.x[i]);
            if self.y[i]<yxmin:
                yxmin=self.y[i];
                ygmin=self.y[i];
                for k in range(self.n):
                    self.xmin[k]=self.x[i][k];
                    self.gmin[k]=self.x[i][k];
        t=0;
        while t<self.ngeneration:
            yxmin=1.0e50;
            for i in range(self.m):
```

```
            for j in range(self.n):
                self.x[i][j]=self.x[i][j]+self.v[i][j];
                if self.x[i][j]<xxmin[j]: self.x[i][j]=xxmin[j];
                if self.x[i][j]>xxmax[j]: self.x[i][j]=xxmax[j];
            self.y[i]=self.ff.func(self.x[i]);
            if self.y[i]<yxmin:
               yxmin=self.y[i];
               for k in range(self.n):
                  self.xmin[k]=self.x[i][k];
        for i in range(self.m):
           if self.y[i]<ygmin: ygmin=self.y[i];
           for k in range(self.n):
              self.gmin[k]=self.x[i][k];
        for i in range(self.m):
            for j in range(self.n):
               self.v[i][j]=self.teta*self.v[i][j]+self.alpha*random()*(self.gmin[j]-self.x[i][j])+self.beta*random()*(self.xmin[j]-self.x[i][j]);
        t=t+1;
        self.beta=self.beta0*pow(self.gamma,0.1*t);

f=f1();
xmin=[0.0,0.0];
xmax=[2.0,2.0];
r1= particle_swarm(f,500,xmin,xmax,0.511545,0.771354,1000);
z=r1.xmin;
print("xmin = ",z,"fmin=",f.func(z));
```

```
runfile('D:/okul/SCO1/particle_swarm.py', wdir='D:/okul/SCO1')
Reloaded modules: gauss, f_xj
xmin =  [0.9999967950335541, 0.9999936653609952] fmin= 1.0838572191578976e-11
```

Program Particle swarm algorithm Julia version

```
#function set
function func(x)
ff=(1.0-x[1])*(1.0-x[1])+100.0*(x[2]-x[1]*x[1])*(x[2]-x[1]*x[1])
return ff
end

function particle_swarm_opt(ff,mi,  xxmin, xxmax, alphai, betai,ngenerationi)
teta=0.5
gamma=0.5
#initial population set
n=length(xxmin)
m=mi
alpha=alphai
beta0=betai
beta=beta0
ngeneration=ngenerationi
x=zeros(Float64,m,n)
v=zeros(Float64,m,n)
y=zeros(Float64,m)
xmin=zeros(Float64,n)
gmin=zeros(Float64,n)
x1=zeros(Float64,n)
ygmin=0
x2=0
ngeneration=ngenerationi
yxmin=1.0e50
# ======================
for i=1:m
 for j=1:n
   x2=xxmin[j]+rand()*(xxmax[j]-xxmin[j])
   x[i,j]=x2
   x1[j]=x2
   v[i,j]=rand()
  end # for j=1:n
  y[i]=ff(x1)
  y1=y[i]
  if y1<yxmin
     yxmin=y1
     ygmin=y1
    for k=1:n
     xmin[k]=x[i,k]
```

```
      gmin[k]=x[i,k]
    end #end k=1:n
  end #end if y[i]<yxmin
end #i=1:m
# end of init
# ======================
t=0
while(t<ngeneration)
yxmin=1.0e50
x2=0
for i=1:m
  for j=1:n
    x2=x[i,j]+v[i,j]
    x1[j]=x2
    x[i,j]=x2
    if x[i,j]<xxmin[j]
      x[i,j]=xxmin[j]
      x1[j]=xxmin[j]
    end # if
    if x[i,j]>xxmax[j]
      x[i,j]=xxmax[j]
      x1[j]=xxmax[j]
    end # if
  end # for j
  y[i]=ff(x1)
  if y[i]<yxmin
   yxmin=y[i]
   for k=1:n
     xmin[k]=x[i,k]
   end       # for k=1:n
  end # if y[i]<yxmin
end           # for i=1:m
for i=1:m
  if y[i]<ygmin
    ygmin=y[i]
    for k=1:n
      gmin[k]=x[i,k]
    end # for k=1:n
  end # if y[i]<ygmin
end # for i=1:m
for i=1:m
  for j=1:n
    v[i,j]=teta*v[i,j]+alpha*rand()*(gmin[j]-xmin[j])+beta*rand()*(xmin[j]-x[i,j])
  end # for j=1:n
end # end for i=1:m
f1=ff(xmin)
t=t+1
beta=beta0*gamma^(0.01*t)
end # while
return gmin
end # function

xmin=[0.0 0.0]
xmax=[2.0 2.0]
m=100
alpha=2.04865767
beta=0.78742864
ngeneration=500
r1=particle_swarm_opt(func,m, xmin,xmax,alpha,beta,ngeneration)
y=func(r1)
println("r = $r1 y = $y")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" particle_swarm_opt.jl
r = [1.0, 1.0] y = 0.0

> Terminated with exit code 0.
```

STOCHASTIC METHODS: BAT SEARCH ALGORITHM

Xin-She Yang (2010) [51] propose the Bat Algorithm (BA), BA is inspired by the research on the social behavior of bats. The BA is based on the echolocation behaviour of bats. Microbats use a type of sonar (echolocation) to detect prey, avoid obstacles, and locate their roosting crevices in the dark. These bats emit

a very loud sound pulse and listen for the echo that bounces back from the surrounding objects. Their pulses vary in properties and can be correlated with their hunting strategies, depending on the species. Based on the above description of bat process, Xin-She Yang proposes the Bat algorithm. The structure of the pseudo code of the Bat Algorithm is as follows :

Table 5.24.1 Bat Search algorithm

Objective function $f(x_1, x_2, \ldots, x_n)$
Initialize the bat population x_i (i = 1, 2, …, n) and v_i
Define pulse frequency f_i at x_i
Initialize pulse rates r_i and the loudness A_i
while (t < Max Number of iterations)
{
Generate new solutions by adjusting frequency
And updating velocities and locations/solutions:
$f_i = f_{min} + (f_{max} - f_{min})\beta$
$v_i^t = v_i^{t-1} + (x_i^t - x_*)f_i$
$x_i^t = x_i^{t-1} + v_i^t$
If (rand >*ri*)
{ Select a solution among the best solutions
Generate a local solution around the selected best solution
}
End if
Generate a new solution by flying randomly
If (rand < A_i and $f(x_i) < f(x_*)$)
{ Accept the new solutions
Increase r_i and reduce A_i
}
End if
Rank the bats and find the current best x^*
} **End while**
Postprocess results

Bats fly randomly with velocity *vi* at position *xi* with a fixed frequency f_{min}, varying wavelength λ and loudness *A0* to search the prey, they can automatically adjust the wavelength (or frequency) of their emitted pulses and adjust the rate of pulse emission $r \in [0,1]$, depending on the proximity of their target. Although the loudness can vary from a large positive *A*0 to a minimum constant value *Amin*. $\beta \in [0,1]$is a random vector drawn from a uniform random number generator. A program using above algorithm is developed. Code are listed below:

Program bat search algorithm

```
import java.io.*;
import java.text.*;
import java.util.Locale;

//Program to find minimum of the function
//for maximum finding change the sign of the function

public class bati
{
int m,n;
// n number of bat
// m number of variables (space dimension) for each bat
double x[][]; //bat population
double y[]; // function value;
double f[];  //pulse frequency
double v[][]; //velocity
double ri[];  //rate of pulse emmission
public double ymin;
public double xmin[];
double A[];
double beta;
double fmin,fmax;//minimum and maximum velocities
double A0,Amin,r0;
```

```
double alpha,gamma;
double t;
int ngeneration;
int imin;
double xxmin[];
double xxmax[];

public bati(if_xj ff,int ni, double xxmini[],double xxmaxi[],int ngenerationi)
{
// initial population set
xxmin=xxmini;
xxmax=xxmaxi;
fmin=0;
fmax=2;
A0=0.5;
Amin=0;
r0=0.5;
alpha=0.9;
gamma=0.9;
n=ni;
m=xxmin.length;
x=new double[n][m];
v=new double[n][m];
y=new double[n];
ri=new double[n];
f=new double[n];
xmin=new double[m];
A=new double[n];
double eps;
double AA=0;
double xold;
ngeneration=ngenerationi;
// initilize bat population
for(int i=0;i<n;i++)
{ ri[i]=Math.random();
for(int j=0;j<m;j++)
{
x[i][j]=xxmin[j]+Math.random()*(xxmax[j]-xxmin[j]);
}
y[i]=ff.func(x[i]);
f[i]=fmin+(fmax-fmin)*Math.random();
checkmin(i);
}
//initilize bat velocities
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
{
v[i][j]= v[i][j]+(x[i][j]-xmin[j])*(f[i]-fmin);
}
}
//initilize Ai and ri
for(int i=0;i<n;i++)
{ri[i]=r0*Math.random();
A[i]=A0;
}
int l;
t=0;l=0;
while(l<ngeneration)
{
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
{xold=x[i][j];
```

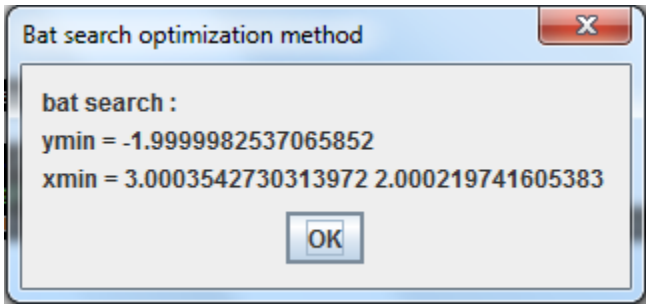
```
x[i][j]= x[i][j]+v[i][j];
//if out of range
int ic1=0;
int ic2=0;
while(x[i][j]<xxmin[j]&&ic1<5)  {x[i][j]= xold+Math.random()*v[i][j];ic1++;}
while(x[i][j]>xxmax[j]&& ic2<5) {x[i][j]= xold-Math.random()*v[i][j];ic2++;}
if(x[i][j]<xxmin[j])  x[i][j]=xxmin[j]+(xxmax[j]-xxmin[j])*Math.random();
if(x[i][j]>xxmax[j])  x[i][j]=xxmin[j]+(xxmax[j]-xxmin[j])*Math.random();
}
y[i]=ff.func(x[i]);
f[i]=fmin+(fmax-fmin)*Math.random();
checkmin(i);
}
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
{v[i][j]= v[i][j]+(x[i][j]-xmin[j])*f[i];}
}
for(int i=0;i<n;i++)
{ri[i]=r0*Math.random()*(1-Math.exp(-gamma*t));
A[i]=alpha*A[i];
}
//random walk
for(int i=0;i<n;i++)
{if(Math.random()>ri[i])
{x[i]=local_solution(xmin,xxmin,xxmax,0.05);
y[i]=ff.func(x[i]);
checkmin(i);
}
}

for(int i=0;i<n;i++)
{ if(Math.random()<A[i] && y[i]<ymin)
AA=avg(A);
for(int j=0;j<m;j++)
{eps=-1+2.0*Math.random();
x[i][j]=x[i][j]+eps*AA;}
y[i]=ff.func(x[i]);
checkmin(i);
ri[i]=r0*Math.random()*(1-Math.exp(-gamma*t));
A[i]=alpha*A[i];
}

for(int i=0;i<n;i++)
{  checkmin(i);}
l++;
t=l;
}//end of while

}

// if minimum is found replace ymin and xmin
// with the new founded value
public void checkmin(int i)
{
if(y[i]<ymin)
{ymin=y[i];imin=i;
for(int j=0;j<m;j++)
{xmin[j]=x[i][j];}
}
}

public static double avg(double xx[])
```

```
{ double toplam=0;
int nn=xx.length;
for(int i=0;i<nn;i++)
{toplam+=xx[i];}
toplam/=nn;
return toplam;
}

//Gauss random number distributed around xavg within a diameter of r
public static double  gaussrandom(double xavg,double r)
{ int iset=0;
double gset=0;
double fac,rsq,v1,v2;
double aret=0;
if(iset==0)
{
do{
        v1=2.0*Math.random()-1.0;
        v2=2.0*Math.random()-1.0;
        rsq=v1*v1+v2*v2;
}while(rsq>=1.0 || rsq==0.0);
fac=Math.sqrt(-2.0*Math.log(rsq)/rsq);
gset=v1*fac;
iset=1;
aret=v2*fac;
}
else
{
iset=0;
aret=gset;
}
return aret/8.0*r+xavg;
}

//Finds local solutions in the neighbourhood of the selected best soluton
// by using Gauss random number
public static double[] local_solution(double selected_best[],double xxmin[],double xxmax[],double ratio)
{
int m=xxmin.length;
double dx=1;
double xx[]=new double[m];
for(int j=0;j<m;j++)
{
dx=(xxmax[j]-xxmin[j])*ratio;
xx[j]=gaussrandom(selected_best[j],dx);
//we will not permit data out of range
int ic1=0;
int ic2=0;
while(xx[j]<xxmin[j]) {xx[j]=xxmin[j]+0.001*Math.random()*dx;ic1++;}
while(xx[j]>xxmax[j]) {xx[j]=xxmax[j]-0.001*Math.random()*dx;ic2++;}
}
return xx;
}

//output function
public double[] xmin()
{ return xmin;}

//output function
public String toString()
{ String s="ymin = "+ymin+"\nxmin = ";
for(int k=0;k<xmin.length;k++)
```

```
{s+=xmin[k]+" ";}
return s;
}

public void plot()
{
// m number of variables (space dimension) for each bat
if(m==2)
{   double z[][]=Matrix.T(x);
Plot pp=new Plot(z[0],z[1]);
pp.setPlotType(0,35);
pp.setMinMax(xxmin[0],xxmax[0],xxmin[1],xxmax[1]);
pp.plot();
}
}

}
```

Program 5.24.2 bat search algorithm test program

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class battest
{
public static void main(String arg[])
{
if_xj ff=(double x[])->(1.0-x[0])*(1.0-x[0])+100.0*(x[1]-x[0]*x[0])*(x[1]-x[0]*x[0]);
double xmin[]={0,0};
double xmax[]={4,4};
bati  b=new bati(ff,40,xmin,xmax,1000);
double x0[]=b.xmin();
String s="bat search : \n"+b.toString()  ;
String s2="Bat search optimization method ";
JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
b.plot();
}}
```



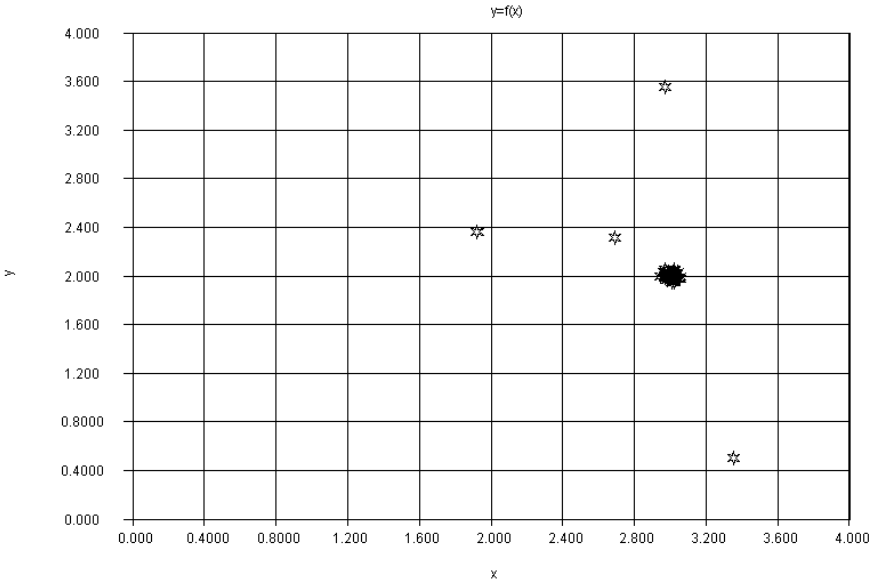


Figure 5.24.1 distribution of the bats at the solution space

STOCHASTIC METHODS: HARMONY AND MELODY SEARCH ALGORITHMS

Harmony Search (HS) is a relatively new heuristic optimization algorithm and it was first developed by Z. W. Geem *et cd.* in 2001[53]. Harmony search is a music-inspired metaheuristic optimization algorithm. It is inspired by the observation that the aim of music is to search for a perfect state of harmony. This harmony in music is analogous to find the optimum solution in an optimization process. The search process in optimization can be compared to a musician's improvisation process. This perfectly pleasing harmony is determined by the audio aesthetic standard. The aesthetic quality of a musical instrument is essentially determined by its pitch (or frequency), timbre (or sound quality), and amplitude (or loudness). Timbre is largely determined by the harmonic content which is in turn determined by the waveforms or modulations of the sound signal. However, the harmonics it can generate will largely depend on the pitch or frequency range of the particular instrument. In music theory, pitch p is often represented as a function of frequency, f, as

$$p = 69 + 12 * log_2 \left[\frac{f}{440Hz}\right]$$

or the inverse equation is

$$f = 440\left[2^{(p-69)/12}\right]$$

According to this a note A has a frequency of f_0=440 Hz or pitch p=69

Table 5.25.1 Harmony Search algorithm

Harmony Search
Objective function $f(x_1, x_2, \dots, x_n)$
Generate initial harmonics (real number arrays)
Define pitch adjusting rate (r_{pa}) and pitch limits
Define harmony memory accepting rate (T_{accept})
while (t <Max number of iterations)
Generate new harmonics by accepting best harmonics
Adjust pitch to get new harmonics (solutions)
if (rand> r_{accept}),
Choose an existing harmonic randomly
else i f (rand> r_{pa}),
Adjust the pitch randomly within a bandwidth
$x_{new} = x_{old} + b_p(2 * random - 1)$
else
Generate new harmonics via randomization
$p_a = p_{lower\ limit} + p_{range} random$
end if

Accept the new harmonics (solutions) if better
end while
Find the current best estimates

Program 5.25.1 harmony search algorithm

```
 import java.io.*;
import java.text.*;
import java.util.Locale;


public class HarmonySearch1i {
        public int m;       // number of variables
        public int HMS;     // Harmony memory size ( population size)
        public int maxIter;  // maximum number of iterations
        public double PAR;   // pitch adjusting rate
        public double BW;    // Bandwith
        public double HMCR;  // Harmony Memory consideration rate 0< HMCR< 1
        public double xmin[]; // minimum limits of independent variable
        public double xmax[]; //maximum limits of independent variable
        public double NCHV[];
        public double bestFitHistory[];  // best fit history
        public double bestHarmony[];     // Best Harmony
        public double worstFitHistory[]; // worst fit history
        public double HM[][];            //Harmony memory
        static int generation = 0;
        public boolean terminationCriteria = true;
        if_xj ff;

   public HarmonySearch1i(if_xj ffi,double xxmin[],double xxmax[],int maxIteration,double BWi,double
HMCRi,int HMSi,double PARi)
   {        ff=ffi;
            maxIter=maxIteration;
            m=xxmin.length;
            HMCR=HMCRi;
            HMS=HMSi;
            BW=BWi;
            xmin = xxmin;
                    xmax = xxmax;
                    NCHV = new double[m];
                    bestHarmony = new double[m + 1];
                    bestFitHistory = new double[maxIter + 1];
                    worstFitHistory = new double[maxIter + 1];
                    HM = new double[HMS][m + 1];
            xmin=xxmin;
            xmax=xxmax;
   }

   public double random(double low,double high)
        {return low+Math.random()*(high-low);}

        public int random(int low, int high)
        {return (int)(Math.floor(Math.random()*(high-low)+low+.5));}

        public void initiator()
        {
                int i;
                double curFit;
                for (i = 0; i < HMS; i++)
                {
                        for (int j = 0; j < m; j++) {HM[i][j] = random(xmin[j], xmax[j]);}
                        curFit = ff.func(HM[i]);  // fitness(HM[i]);
                        HM[i][m] = curFit;       // the fitness is stored in the last column of HM
                }
        }

        public boolean stopCondition() {
                if (generation > maxIter)
                        terminationCriteria = false;
```

```
		return terminationCriteria;
	}

	public void updateHarmonyMemory(double newFitness) {

		// find worst harmony
		double worst = HM[0][m];
		int worstIndex = 0;
		for (int i = 0; i < HMS; i++)
			if (HM[i][m] > worst) {
				worst = HM[i][m];
				worstIndex = i;
			}
		worstFitHistory[generation] = worst;
		// update harmony
		if (newFitness < worst) {
			for (int k = 0; k < m; k++)
				HM[worstIndex][k] = NCHV[k];
			HM[worstIndex][m] = newFitness;
		}
		// find best harmony
		double best = HM[0][m];
		int bestIndex = 0;
		for (int i = 0; i < HMS; i++)
			if (HM[i][m] < best) {
				best = HM[i][m];
				bestIndex = i;
			}
		bestFitHistory[generation] = best;
		if (generation > 0 && best != bestFitHistory[generation - 1]) {
			for (int k = 0; k < m; k++)
				bestHarmony[k] = HM[bestIndex][k];
			bestHarmony[m] = best;
		}
	}

	public void memoryConsideration(int varIndex) { NCHV[varIndex] = HM[random(0, HMS -
1)][varIndex];}

	public void pitchAdjustment(int varIndex) {

		double rand = Math.random();
		double temp = NCHV[varIndex];
		if (rand < 0.5) {
			temp += rand * BW;
			if (temp < xmax[varIndex])
				NCHV[varIndex] = temp;
		} else {
			temp -= rand * BW;
			if (temp > xmin[varIndex])
				NCHV[varIndex] = temp;
		}

	}

	public void randomSelection(int varIndex) {NCHV[varIndex] = random(xmin[varIndex],
xmax[varIndex]);}

	public double[]  xmin()
	{
		initiator();
		while (stopCondition()) {
			for (int i = 0; i < m; i++) {
				if (Math.random() < HMCR) {
					memoryConsideration(i);
					if (Math.random() < PAR)
						pitchAdjustment(i);
```

```
                                        } else
                                                randomSelection(i);
                                }
                                double currentFit;
                                currentFit = ff.func(NCHV);//fitness(NCHV);
                                updateHarmonyMemory(currentFit);
                                generation++;
                        }
                        return bestHarmony;
                }

public String toString()
{ double x[]=xmin();
  double y=x[x.length-1];
                String s="ymin = "+y+"\nxmin = ";
  for(int k=0;k<x.length-1;k++)
  {s+=x[k]+" ";}
  return s;
}

public void plot()
{
if(m==2)
{   double z[][]=Matrix.T(HM);
    for(int i=0;i<z[0].length;i++)
    {System.out.println(z[0][i]+"   "+z[1][i]);}
                Plot pp=new Plot(z[0],z[1]);
                pp.setPlotType(0,35);
                pp.setMinMax(xmin[0],xmax[0],xmin[1],xmax[1]);
                pp.plot();
}
}

}
```

Program 5.25.2 harmony search algorithm test program

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

public class Harmonytest {

                public static void main(String[] args) {

                                if_xj ff=(double x[])->-(1.0+Math.cos(Math.PI*(x[0]-3.0))*Math.cos(2.0*Math.PI*(x[1]-2.0))/
    (1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0)));
                                double xmin[]={0,0};
        double xmax[]={4,4};
        int maxIter=1000000;
        double BW=0.2;
        double HMCR=0.9;
        int HMS=10;
        double PAR=0.4;
                                HarmonySearch1i hs=new HarmonySearch1i(ff,xmin,xmax,maxIter,BW,HMCR,HMS,PAR);
                                String s="Harmony search : \n"+hs.toString()  ;
        String s2="Harmony search optimization method ";
        JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
        ContourPlot pp = new ContourPlot(ff,xmin[0],xmax[0],100,xmin[1],xmax[1],100,40, false,"Harmony search
optimisation", "x", "y");
                                pp.plot();
                }
}
```

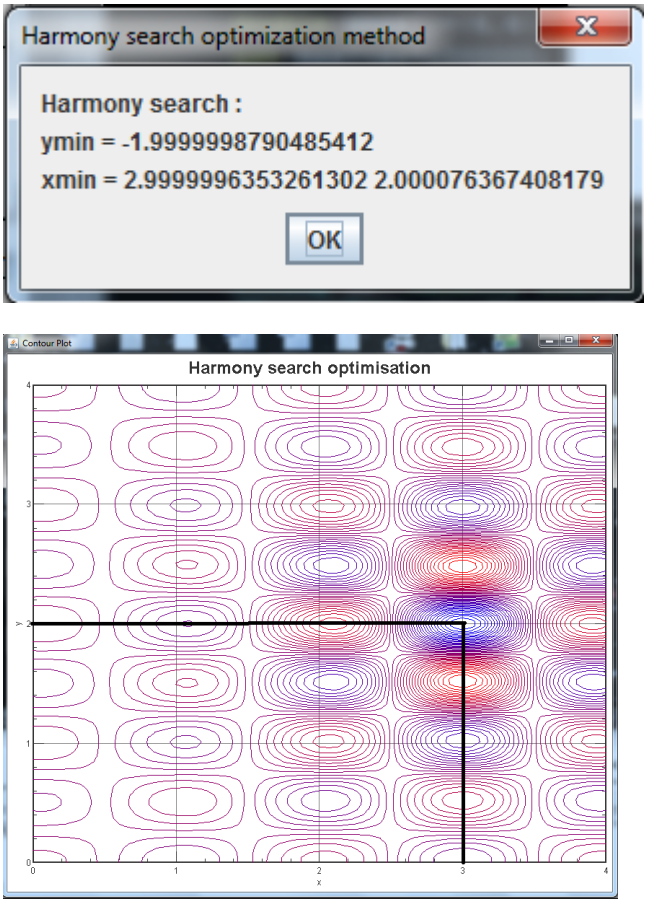


Figure 5.25.1 Contour plot of the harmony search example program

In order to enhance the efficiency of HS algorithm, a novel improved optimization technique, called Melody Search (MS) was proposed by Ashrafi and Dariane[54]. MS algorithm mimics the musical performance processes and interactive relations occurred between members of a musicians group; while, they are looking for the best series of pitches within a melodic line. In such a group, the existing several music players—with different tastes, ideas and experiences—can lead in achieving the best subsequence of pitches faster. Although the new algorithm adopts the basic concepts of HS, the structure is quite different. Unlike HS that uses a single harmony memory, MS algorithm employs several memories named Player Memory (PM). Memories interact to each other like musicians performance in a group. Furthermore, in the MS algorithm the possible range of each pitch for random selection can be changed going through different iterations.
MS algorithm is given as a pseudocode below:

Table 5.25.2 MELODY Search algorithm

Step 1. Initializing the optimization problem and adopting algorithm parameters Step 2. Initial Phase; 2.1. Initialize player memories (PM) 2.2. Improvise a new melody from each PM 2.3. Update each PM 2.4. Repeat sub-steps 2.2 and 2.3 until the criterion for stopping the initial phase is satisfied Step 3. Second phase; 3.1. Improvise a new melody from each PM according to the possible range of pitches 3.2. Update each PM 3.3. Determine the possible ranges of pitches for next improvisation (just for randomization) 3.4. Repeat sub-steps 3.1, 3.2 and 3.3 until the criterion for stopping the algorithm is satisfied (i.e. maximum number of iterations: NI)

Computer codes developed accoding to this algorithm.

Program 5.25.3 Melody search algorithm

```
 ////////////////////////////////////////////////////////////////////////////////////////////////
//Performance evaluation of an improved harmony search algorithm for numerical optimization : Melody Search (MS)
////////////////////////////////////////////////////////////////////////////////////////////////
//S.M. Ashrafi , A.B.Dariane////////////////////////////////////////////////////////////////////
// Programmed by : Oğuz Emrah Turgut
////////////////////////////////////////////////////////////////////////////////////////////////


public class melody_searchi
{
int D;// Dimension(Number of pitch)
int PMN;//number of player memory
int PMS;//player memory size
int Maxiter1;//maximum of iteration for  first phase
int Maxiter2;//maximum number of iteration for second phase
double PMCR;//player memory considering rate
double PARmax;// Maximum value of pitch adjustment rate
double PARmin;// Minimum value of pitch adjustment rate
double PAR;//Pitch adjustment rate(chenges within every iteration)
double Xnew[][];//Holds new pitches for stage 1
double Xnew2[];//Holds new pitches for stage 2
double[][] fitness;// Holds fitness values for stage 1
double[][] fitness2;// Holds fitness value for stage 2
double[] fitness3;
double[][][] MM;//Melody memory first stage
double[][] MM2;//Melody memory second stage
double[] Lower1;// Lower bound of first stage
double[] Upper1;// Upper bound of first stage
double[] Lower2;// Lower bound of second stage (determined by first stage)
double[] Upper2;// Upper bound of second stage (determined by first stage)
double[] maxval;
int[] maxindex;
double[] minval;
int[] minindex;
double[] maxval2;
int[] maxindex2;
double[] minval2;
int[] minindex2;
double[] bw;//bandwith
double[][] worst;
double[][] best;
double[][] best2;
double[] best3;
double[] worst3;
int L;//random integer between 1 and PMS
int H;//random integer between 1 and D
double min2;
int mindex2;
double max2;
int maxdex2;
if_xj ff;

	public melody_searchi(if_xj iff,int iPMN,int iPMS,int iMaxiter1,int iMaxiter2,double iPMCR,double[] iLower,double[] iUpper)
	{
		ff=iff;
		PMN=iPMN;
		PMS=iPMS;
		Maxiter1=iMaxiter1;
		Maxiter2=iMaxiter2;
		PMCR=iPMCR;
		Lower1=iLower;
		Upper1=iUpper;
		D=Upper1.length;
		Lower2=new double[D];
		Upper2=new double[D];
		maxval=new double[PMN];
		maxindex=new int[PMN];
		minval=new double[PMN];
		minindex=new int[PMN];
		maxval2=new double[PMN];
		maxindex2=new int[PMN];
		minval2=new double[PMN];
		minindex2=new int[PMN];
		fitness=new double[PMN][PMS];
		fitness2=new double[PMN][PMS];
		fitness3=new double[PMN];
```

```
worst=new double[PMN][D];
best=new double[PMN][D];
best2=new double[PMN][D];
best3=new double[D];
worst3=new double[D];
MM=new double[PMN][PMS][D];
MM2=new double[PMN][D];
Xnew=new double[PMN][D];
Xnew2=new double[D];
PARmax=2.0;
PARmin=0.0;

}

void initialize()
{
        // Improvisation a new melody
        for(int i=0;i<PMN;i++)
        {for(int j=0;j<PMS;j++)
         {for(int k=0;k<D;k++)
          {MM[i][j][k]=Lower1[k]+(Math.random()*(Upper1[k]-Lower1[k]));}}}

          //Calculate fitness of each player
          for(int i=0;i<PMN;i++)
          {for(int j=0;j<PMS;j++)
           {fitness[i][j]=ff.func(MM[i][j]);}}

          // Calculate best and worst melody for each player
           for(int i=0;i<PMN;i++)
           {
                   double dd[]=getmaxval_index(fitness[i]);
                   maxval[i]=dd[0];
                   maxindex[i]=(int)dd[1];
                  }

                  for(int i=0;i<PMN;i++)
           {
                   double dd2[]=getminval_index(fitness[i]);
                   minval[i]=dd2[0];
                   minindex[i]=(int)dd2[1];
                  }
                  for(int i=0;i<PMN;i++)
                  {
                    for(int k=0;k<D;k++)
                    {
                            best[i][k]=MM[i][minindex[i]][k];
                            }
                  }
                  for(int i=0;i<PMN;i++)
                  {
                    for(int k=0;k<D;k++)
                    {
                            worst[i][k]=MM[i][maxindex[i]][k];
                            }
                  }
        }


        double[][] solution()
        {
        initialize();

        /// FIRST STAGE /// Determine best melody for each music player

        int iter1=1;
                while(iter1<Maxiter1)
                {
                    for(int i=0;i<PMN;i++)
                    {
                            for(int k=0;k<D;k++)
                            {
                                    if(Math.random()<=PMCR) // memory consideration
                        {
            L=(int)(((double)PMS)*Math.random());
                                    H=(int)(((double)D)*Math.random());
            PAR=PARmax-((PARmax-PARmin)*((double)iter1)/((double)Maxiter1));
```

```
                                        if((iter1%2)==1)
                                        {
                                                Xnew[i][k]=MM[i][L][k]+(((Math.random()-0.5)*2.0)*((Upper1[k]-
Lower1[k])/200.0));
                                                }
                                        else
                                        {
                                                Xnew[i][k]=MM[i][L][H]+(((Math.random()-0.5)*2.0)*((Upper1[k]-
Lower1[k])/200.0));
                                                }
                                        if(Math.random()<=PAR) // pitch adjustment
                                        {
                                                Xnew[i][k]=best[i][k];
                                                }

                                    }
                                            else // randomization
                                            {
                                                    Xnew[i][k]=Lower1[k]+(Math.random()*(Upper1[k]-Lower1[k]));
                                                    }
                                }
                                }

                                for(int i=0;i<PMN;i++)   // Xnew is better than worst , exclude worst
                                {if(ff.func(Xnew[i])<ff.func(worst[i]))
                                   {for(int k=0;k<D;k++)
                                        {worst[i][k]=Xnew[i][k];}
                                    }
                                }

                                // Update  worst and best melody
                                for(int i=0;i<PMN;i++)
                          {for(int j=0;j<PMS;j++)
                            {fitness[i][j]=ff.func(MM[i][j]);}}


                        for(int i=0;i<PMN;i++)
                        {
                                double dd3[]=getmaxval_index(fitness[i]);
                                maxval[i]=dd3[0];
                                maxindex[i]=(int)dd3[1];
                                }

                                for(int i=0;i<PMN;i++)
                        {
                                double dd4[]=getminval_index(fitness[i]);
                                minval[i]=dd4[0];
                                minindex[i]=(int)dd4[1];
                                }

                                for(int i=0;i<PMN;i++)
                                {
                                  for(int k=0;k<D;k++)
                                  {
                                        best[i][k]=MM[i][minindex[i]][k];
                                       }
                                }

                                        for(int i=0;i<PMN;i++)
                                {
                                  for(int k=0;k<D;k++)
                                  {
                           MM[i][maxindex[i]][k]=worst[i][k]; // worst is now  new maxindexed MM
                                        }
                                }

                                iter1++;
                            }


                            /// SECOND STAGE /// Determine best harmony from selected melodies



                            // Determine new upper and lower bound
                            for(int i=0;i<PMN;i++)
```

```
                {
                  for(int j=0;j<PMS;j++)
                   {fitness2[i][j]=ff.func(MM[i][j]);}
}



                for(int i=0;i<PMN;i++)
{
                 double dd6[]=getminval_index(fitness2[i]);
                 minval2[i]=dd6[0];
                 minindex2[i]=(int)dd6[1];
                }
                for(int i=0;i<PMN;i++)
                {
                  for(int k=0;k<D;k++)
                  {
                            best2[i][k]=MM[i][minindex[i]][k];
                            }
                }

                for(int i=0;i<PMN;i++)
                {
                   for(int k=0;k<D;k++)
                   {
                               MM2[i][k]=best[i][k];
                            }
                }

                // MM2 now is new harmony matrix


                // Procedure for determining lower and upper bounds for every column
                double min=0.0;
                for(int k=0;k<D;k++)
                {
                            min=MM2[0][k];
                   for(int i=0;i<PMN;i++)
                   {
                            if(MM2[i][k]<min)
                            {min=MM2[i][k];}
                            }
                            Lower2[k]=min;
                }



                double max=0.0;
                for(int k=0;k<D;k++)
                {
                            max=MM2[0][k];
                   for(int i=0;i<PMN;i++)
                   {
                            if(MM2[i][k]>max)
                            {max=MM2[i][k];}
                            }
                            Upper2[k]=max;
                }


               // determine best and worst vector in MM2
               for(int i=0;i<PMS;i++)
               {fitness3[i]=ff.func(MM2[i]);}

               double dd7[]=getminval_index(fitness3);
               min2=dd7[0];
               mindex2=(int)dd7[1];

               double dd8[]=getmaxval_index(fitness3);
               max2=dd8[0];
               maxdex2=(int)dd8[1];

               for(int i=0;i<D;i++)
               {best3[i]=MM2[mindex2][i];
                worst3[i]=MM2[maxdex2][i];}
```

```
                int iter2=1;
                while(iter2<Maxiter2)
                {

                        for(int k=0;k<D;k++)
                        {
                                if(Math.random()<PMCR)//Memory consideration
                                {
                                        L=(int)(((double)PMS)*Math.random());
                            H=(int)(((double)D)*Math.random());
                            PAR=PARmax-((PARmax-PARmin)*((double)iter2)/((double)Maxiter2));
                                        if((iter2%2)==1)
                                        {
                                        Xnew2[k]=MM2[L][k]+(((Math.random()-0.5)*2.0)*((Upper2[k]-
Lower2[k])/200.0));
                                        }
                                        else
                                        {
                                        Xnew2[k]=MM2[L][H]+(((Math.random()-0.5)*2.0)*((Upper2[k]-
Lower2[k])/200.0));
                                        }
                                         if(Math.random()<PAR)// pitch adjustment
                                         {
                                                for(int kk=0;kk<D;kk++)
                                           {Xnew2[kk]=best3[kk];}

                                         }

                                }
                                else // Randomization
                                {
                                  Xnew2[k]=Lower2[k]+(Math.random()*(Upper2[k]-Lower2[k]));
                                }

                               }

                if(ff.func(Xnew2)<ff.func(worst3))
                {
                   for(int i=0;i<D;i++)
                   {worst3[i]=Xnew2[i];}

                }

                for(int i=0;i<PMN;i++)
            {fitness3[i]=ff.func(MM2[i]);}

            // get minimum index and best solution
            double[] dd9=getminval_index(fitness3);
            min2=dd9[0];
            mindex2=(int)dd9[1];
            for(int i=0;i<D;i++)
            {best3[i]=MM2[mindex2][i];}

            // get maximum index and replace new worst value
            double[] dd10=getmaxval_index(fitness3);
            max2=dd10[0];
            maxdex2=(int)dd10[1];
            for(int i=0;i<D;i++)
            {MM2[maxdex2][i]=worst3[i];}

                 // Procedure for determining lower and upper bounds for every column
                  min=0.0;
                 for(int k=0;k<D;k++)
                 {
                            min=MM2[0][k];
                    for(int i=0;i<PMN;i++)
                    {
                             if(MM2[i][k]<min)
                             {min=MM2[i][k];}
                            }
                            Lower2[k]=min;
                 }
```

```
                                max=0.0;
                                for(int k=0;k<D;k++)
                                {
                                        max=MM2[0][k];
                                   for(int i=0;i<PMN;i++)
                                   {
                                        if(MM2[i][k]>max)
                                        {max=MM2[i][k];}
                                        }
                                        Upper2[k]=max;
                                }

                                iter2++;

                                }
                                double[][] dp=new double[2][D];
                                dp[0][0]=min2;
                                for(int i=0;i<D;i++)
                                {dp[1][i]=best3[i];}

                                return dp;

                        }

                        double[] getmaxval_index(double a[])
                {
                double m=0.0;
                double b[]=new double[a.length];
                for(int i=0;i<a.length;i++)
                {b[i]=a[i];}
                double maxval=a[0];
                for(int j=0;j<a.length;j++)
                {if(a[j]>maxval){maxval=a[j];}}
                for(int i=0;i<b.length;i++)
                {if(b[i]==maxval){m=i;break;}}
                double dep2[]=new double[2];
                dep2[0]=maxval;
                dep2[1]=m;
                return dep2;
                }



                double[] getminval_index(double[] a)
                {
                double m=0.0;
                double b[]=new double[a.length];
                for(int i=0;i<a.length;i++)
                {b[i]=a[i];}
                double minval=a[0];
                for(int i=0;i<a.length;i++)
                {if(a[i]<minval){minval=a[i];}}
                for(int i=0;i<a.length;i++)
                {if(b[i]==minval){m=i;break;}};
                double[] dep=new double[2];
                dep[0]=minval;
                dep[1]=m;
                return dep;
                }

                void toStringnew()
                {
                        double out[][]=solution();
                        System.out.println("Optimized value = "+out[0][0]);
                        for(int i=0;i<D;i++)
                        {System.out.println("x["+i+"] = "+out[1][i]);}

                        }

}
```

Program Melody search algorithm test program

```
 public class melody_search_test1
{
```

```
public static void main(String args[])
        {   int PMN=100;// Number of player
    int PMS=100;// Player Memory
                int Maxiter1=5000;// Maximum iteration for first stage
                int Maxiter2=1000;// Maximum iteration for second stage
    double PMCR=0.99;// Player memory consideration
    double[] xmin={-100.0,-100.0};
    double[] xmax={100.0,100.0};
    if_xj ff=(double x[])->-Math.cos(x[0])*Math.cos(x[1])*Math.exp(-(x[0]-Math.PI)*(x[0]-Math.PI)-(x[1]-Math.PI)*(x[1]-
Math.PI));;
                melody_searchi ms=new melody_searchi(ff,PMN,PMS,Maxiter1,Maxiter2,PMCR,xmin,xmax);
                ms.toStringnew();
              }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" melody_search_test
Optimized value = -0.9999999997776523
x[0] = 3.1416013343921128
x[1] = 3.1416011903038434

> Terminated with exit code 0.
```

In recent years, quite a variation of HS is shown in the papers. Another good variant of the program is the Inteligent tuned Harmony Search Algorithm[55]. Pseudocode of this more complicated algorithm is given below. Please refer to the reference paper for further details.

Table Intelligent tuned Harmony Search algorithm

IHTS Procedure

Step 1: Initılise IHTS parameters

F(x) objective function

N: Number of variables

x_{i_lower} : Lower bound

x_{i_upper} : Upper bound

HMS: Harmony memory size=10

HMCR: Harmony Memory Consideration Rate=0.99

PAR_{min}=0

PAR_{max}=1

Maxiter=Maximum number of iterations

Step 2: Initilise Harmony Memory(HM)

Find best and worst harmony vectors for the HM

Step3: Improvise Harmony Memory (HM)

while (not Maxiter)

for i = 1 to number of decision variables N do

{

if rand [0, 1] <=HMCR

{ // (memory consideration) //

d = int(1 + (HMS _ 1) rand [0, 1])

x_i'=HM(d,i)

$$PAR(iter) = PAR_{max} - (PAR_{max} - PAR_{min})\left[\frac{iter}{\max iter}\right]$$

if rand [0, 1] $\leq$ PAR(iter)

{// (pitch adjustment) //

HM^{mean}=mean(HM(N+1));

if (HM(d, N + 1)) $\leq$ mean(HM(N + 1))

{

/ / (Group A, Both Intensification and Diversification) //

if rand [0, 1] $\leq$ 0.5

$$y_i = x_i^{Best} - (x_i^{Best} - x_i)rand[0,1]$$

else

$$y_i = x_i^{Best} - (x_i^{Worst} - x_i)rand[0,1]$$

Endif

}

Else

{

/ / (Group B, Only Diversification) //

m = int(1 + (N _ 1)/ rand [0, 1])

$x_m^{Best} = x_m^{Best}(x_{i\_upper} / x_{m\_upper})$

$y_i = x_i' - (x_m^{Best} - x_i')rand[0,1]$

}

Endif

$if\ (x_{i\_lower} \le y_i \le x_{i\_upper})$

$x_i' = y_i$

endif

}Endif // (pitch adjustment) //

Else

{// (memory consideration) //

$x_i' = x_{i\_lower} + (x_{i\_upper} - x_{i\_lower})rand[0,1]$

}Endif

}endfor

Step.4: Update Harmony Memory

Calculate the fitness of new harmony vector.

$if\ (f(x') \le f(x^{worst})$

$X^{worst}=x'$

Solution=f(x^{best})

End procedure

A program is written according to this algorithm is given below:

Program Intelligent Tune Harmony Search search algorithm

```
////////////////////////////////////////////////////////////////////////////////////////////////////
//Referance : An Intelligent Tuned Harmony Search algorithm for optimisation
//Parikshit Yadav, Rajesh Kumar, S.K. Panda, C.S. Chang
// Programmed by : Oğuz Emrah Turgut
////////////////////////////////////////////////////////////////////////////////////////////////////

public class intelligent_tuned_harmony_searchi
{
int N;
double Lower[];
double Upper[];
int HMS;
double HMCR;
double PARmin;
double PARmax;
double PAR;
int Maxiter;
double best[];
double worst[];
double y[];
int d;
int m;
double[][] HM;
double[] fitness;
double[] xdot;
int indexmin;
double minval;
int indexmax;
double maxval;
double fitnessmean;
if_xj ff;
        public intelligent_tuned_harmony_searchi(if_xj iff,double[] iLower,double[] iUpper,int iHMS,double iHMCR,int iMaxiter)
        {
                ff=iff;
                Maxiter=iMaxiter;
                Lower=iLower;
                Upper=iUpper;
                N=Upper.length;
                HMS=iHMS;
                HMCR=iHMCR;
                PARmin=0.0;
                PARmax=2.0;
                HM=new double[HMS][N];
                fitness=new double[HMS];
                best=new double[N];
                worst=new double[N];
```

```
y=new double[N];
xdot=new double[N];
}

void initialize()
{
        // introduce Harmony(improvisation)
 for(int i=0;i<HMS;i++)
   {for(int j=0;j<N;j++)
        {HM[i][j]=Lower[j]+(Upper[j]-Lower[j])*Math.random();}}

// calculate fitness
        for(int i=0;i<HMS;i++)
       {fitness[i]=ff.func(HM[i]);}

       // get minimum index
       double[] dd=getminval_index(fitness);
       minval=dd[0];
       indexmin=(int)dd[1];
       for(int i=0;i<N;i++)
       {best[i]=HM[indexmin][i];}

       // get maximum index
        double[] dd2=getmaxval_index(fitness);
        maxval=dd2[0];
        indexmax=(int)dd2[1];
        for(int i=0;i<N;i++)
        {worst[i]=HM[indexmax][i];}
    }


double[] getminval_index(double[] a)
{
double m=0.0;
double b[]=new double[a.length];
for(int i=0;i<a.length;i++)
{b[i]=a[i];}
double minval=a[0];
for(int i=0;i<a.length;i++)
{if(a[i]<minval){minval=a[i];}}
for(int i=0;i<a.length;i++)
{if(b[i]==minval){m=i;break;}};
double[] dep=new double[2];
dep[0]=minval;
dep[1]=m;
return dep;
}

double[] getmaxval_index(double a[])
{
double m=0.0;
double b[]=new double[a.length];
for(int i=0;i<a.length;i++)
{b[i]=a[i];}
double maxval=a[0];
for(int j=0;j<a.length;j++)
{if(a[j]>maxval){maxval=a[j];}}
for(int i=0;i<b.length;i++)
{if(b[i]==maxval){m=i;break;}}
double dep2[]=new double[2];
dep2[0]=maxval;
dep2[1]=m;
return dep2;
}
        double mean(double x[]) // mean of a vector
        {
                int n=x.length;double s1=0.0;
                for(int i=0;i<n;i++)
                {s1+=x[i];}
                return s1/(double)n;
                }
        double[][] solution()
        {
                 initialize() ;
                 int iter=0;
                 while(iter<Maxiter)
```

```
                                        {
                                                for(int i=0;i<N;i++)
                                                {
                                                        if(Math.random()<HMCR)
                                                        {
                                                                d=(int)(((double)HMS)*Math.random());
                                                xdot[i]=HM[d][i];
                                                                PAR=PARmax-((PARmax-
PARmin)*((double)iter)/((double)Maxiter));
                                                                if(Math.random()<PAR)
                                                                {
                                                                        fitnessmean=mean(fitness);
                                                                        if(fitness[d]<fitnessmean)
                                                                        {
                                                                                // Group A, Both intensification and
Diversification
                                                                                if(Math.random()<=0.5)
                                                                                {
                                                                                        y[i]=best[i]-(best[i]-HM[d][i])*
Math.random();
                                                                                }
                                                                                else
                                                                                {
                                                                                        y[i]=best[i]-(worst[i]-
HM[d][i])*Math.random();
                                                                                }
                                                                        }
                                                                        else
                                                                        { // Group B only diversification
                                                                                m=(int)(((double)N)*Math.random());
                                                                                y[i]=xdot[i]+(best[m]-
xdot[i])*Math.random();

                                                                        }

                                                                        if((Lower[i]<=y[i])&&(Upper[i]>y[i]))
                                                                        {xdot[i]=y[i];}

                                                                }
                                                        }
                                                                else
                                                                {
                                                                xdot[i]=Lower[i]+(Upper[i]-Lower[i])*Math.random();

                                                        }
                                                }
                                                if(ff.func(xdot)<=ff.func(worst))
                                                {
                                                        for(int i=0;i<N;i++)
                                                        {worst[i]=xdot[i];}
                                                }
                                                for(int i=0;i<HMS;i++)
                                        {fitness[i]=ff.func(HM[i]);}

                                        // get minimum index and best solution
                                        double[] dd=getminval_index(fitness);
                                        minval=dd[0];
                                        indexmin=(int)dd[1];
                                        for(int i=0;i<N;i++)
                                        {best[i]=HM[indexmin][i];}

                                        // get maximum index and replace new worst value
                                         double[] dd2=getmaxval_index(fitness);
                                         maxval=dd2[0];
                                         indexmax=(int)dd2[1];
                                         for(int i=0;i<N;i++)
                                         {HM[indexmax][i]=worst[i];}

                                         iter++;
                                }
                                double[][] dep=new double[2][N];
                                dep[0][0]=minval;
                                for(int i=0;i<N;i++)
                                {dep[1][i]=best[i];}
        return dep;
}
```

```
                                    void toStringnew()
                                    {
                                     double[][] out=solution();
                                     System.out.println("Optimized value = "+out[0][0]);
                                     for(int i=0;i<N;i++)
                                     {System.out.println("x["+i+"] = "+out[1][i]);}
                                    }
                                    }
```

Program Intelligent Tune Harmony Search search algorithm test program

```
 public class intelligent_tuned_harmony_search_test
{
        public static void main(String args[])
                {
                        double[] xmin={-10.0,-10.0};
                        double[] xmax={10.0,10.0};
                        int HMS=100;
                        double HMCR=0.99;
                        int Maxiter=50000;
                        if_xj ff=(double x[])->-Math.cos(x[0])*Math.cos(x[1])*Math.exp(-(x[0]-Math.PI)*(x[0]-
Math.PI)-(x[1]-Math.PI)*(x[1]-Math.PI));
                        intelligent_tuned_harmony_searchi iths=new
intelligent_tuned_harmony_searchi(ff,xmin,xmax,HMS,HMCR,Maxiter);
                        iths.toStringnew();
                      }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" intelligent_tuned_harmony_search_test
Optimized value = -1.0
x[0] = 3.1415926533665637
x[1] = 3.141592659303584

> Terminated with exit code 0.
```

5.22 STOCHASTIC METHODS: CUCKOO SEARCH ALGORITHM

This algorithm which is introduced by Yang and Deb[56] is constructed on brood parasitism of the cuckoos. Brood parasitism is a evoltionary strategy for cuckoos which is accomplished by laying their eggs in other kind of birds. If this alien egg is found by host bird, the nest(a place where eggs are hold) will be rejected or this egg will be thrown away. This behaviour is named as a evolutionary phenomena to mimic the egg apperance of local host birds. Regarding to this concept, following rules are employed .

- Cuckoos are allowed to lay one egg at a specific time, and drops it in a random nest
- Nest that contains high quality of eggs (nets with best fitness values) will carry over to next generation.
- The number of nest is fixed and a host bird can find an alien egg with a specified probability $p_a \in [0,1]$. If it happens, eggs which are dropped in nests will be thrown away or present nest will be built another place.

With this assumptions Cuckoo Search algorithm is developed . In order to carry out this algorithm effectively, Levy Flights are used to perform local and global searches in solution domain. Levy flights are kind of a random walk and simulated as foraging path of an animal and its next movement is based on its previous one. This pattern is also appliciable for natural phenomenas as biology, astronomy and physics. Behaviour of Levy Flight is proposed by

$$x_i^{t+1} = x_i^t + \alpha\theta Levy(\lambda)$$

Θ is donated by entery-wise multiplication, α is a step size and Lévy(λ) is Levy distribution. With the help of Levy distribution, displaced egg's position will be updated if this new position is better than the former one. According to literature survey, α value can be set to 1.0 . And also p_a parameter is suggested to be used as 0.25. Cuckoo Search algorithm pseudocode is given below

Program 5.26.1 Cuckoo Search algorithm

```
 import java.util.Arrays;
import java.util.Collections;
```

```
import java.util.List;
import java.util.Random;
import java.util.ArrayList;

/////////////////////////////////////////////////////////////////////////
// Enginering Optimisation by Cuckoo Search       //
//                                                                     //
//           Xin-She Yang          Suash Deb               //
/////////////////////////////////////////////////////////////////////////
// Programmed by : Oğuz Emrah Turgut

//REFERANCE
//Papers -- Citation Details:
// 1) X.-S. Yang, S. Deb, Cuckoo search via Levy flights,
// in: Proc. of World Congress on Nature & Biologically Inspired
// Computing (NaBIC 2009), December 2009, India,
// IEEE Publications, USA, pp. 210-214 (2009).
//http://arxiv.org/PS_cache/arxiv/pdf/1003/1003.1594v1.pdf
// 2) X.-S. Yang, S. Deb, Engineering optimization by cuckoo search,
//Int. J. Mathematical Modelling and Numerical Optimisation,
//Vol. 1, No. 4, 330-343 (2010).
//http://arxiv.org/PS_cache/arxiv/pdf/1005/1005.2908v2.pdf

public class cuckoo_search1i
{
int n;
int nd;
double pa;
double tol;
double nest[][];
double xmin[];
double xmax[];
if_xj ff;

    public cuckoo_search1i(if_xj iff,int in,double ipa,double itol,double ixmin[],double ixmax[])
    {
            n=in;
            nd=ixmin.length;
            pa=ipa;
            tol=itol;
            nest=new double[n][nd];
            xmin=new double[nd];
            xmax=new double[nd];
            ff=iff;
            double[] one=ones();
            for(int i=0;i<nd;i++)
            {xmin[i]=ixmin[i];xmax[i]=ixmax[i];}
            }

            double[] randn()
            {
            double a[]=new double[nd];
            for(int i=0;i<nd;i++)
            {a[i]=Math.random();}
            return a;
            }

            double[] ones()
    {
            double v[]=new double[nd];
            for(int i=0;i<nd;i++)
            {v[i]=1.0;}
            return v;
            }

            double [][][] initialize()
            {
            for(int i=0;i<n;i++)
            {for(int j=0;j<nd;j++)
               {nest[i][j]=xmin[j]+((xmax[j]-xmin[j])*Math.random());}
```

```
            }
            double[] fitness=new double[n];
            for(int i=0;i<n;i++)
            {fitness[i]=Math.pow(10,7);}
            double dp [][][]=new double [2][n][nd];
            for(int i=0;i<n;i++)
            {  for(int j=0;j<nd;j++)
              {dp[0][i][j]=nest[i][j];}
            }
            for(int i=0;i<n;i++)
            {dp[1][i][0]=fitness[i];}
            return dp;
            }

            double logGamma(double x)
            {
    double tmp =(x-0.5)*Math.log(x+4.5)-(x+4.5);
    double ser = 1.0 + 76.18009173    / (x + 0)   - 86.50532033    / (x + 1)+ 24.01409822    / (x + 2)   -  1.231739516   / (x + 3)+
0.00120858003 / (x + 4)   -  0.00000536382 / (x + 5);
    return tmp + Math.log(ser * Math.sqrt(2 * Math.PI));
    }
    double gamma(double x) { return Math.exp(logGamma(x)); }


   double[] getminval_index(double[] a)
            {
            double m=0.0;
            double b[]=new double[a.length];
            for(int i=0;i<a.length;i++)
            {b[i]=a[i];}
            double minval=a[0];
            for(int i=0;i<a.length;i++)
            {if(a[i]<minval){minval=a[i];}}
            for(int i=0;i<a.length;i++)
            {if(b[i]==minval){m=i;break;}};
            double[] dep=new double[2];
            dep[0]=minval;
            dep[1]=m;
            return dep;
            }

            double[][][] get_best_nest(double[][] nest,double[][] newnest,double[] fitness)
            {
            double fnew=0.0;
              for(int j=0;j<n;j++)
              {
                fnew=ff.func(newnest[j]);
                if(fnew<fitness[j])
                {fitness[j]=fnew;nest[j]=newnest[j];}
              }
              double[] dep=getminval_index(fitness);
              double fmin=dep[0];
              int index=(int)dep[1];
              double[] best=new double[nd];
              for(int i=0;i<nd;i++)
              {best[i]=nest[index][i];}
              double depp[][][]=new double[4][n][nd];
              depp[0][0][0]=fmin;
              for(int i=0;i<n;i++)
              {depp[1][i][0]=fitness[i];}
              for(int i=0;i<nd;i++)
              {depp[2][0][i]=best[i];}
              for(int i=0;i<n;i++)
              {for(int j=0;j<nd;j++)
                 {depp[3][i][j]=nest[i][j];}
              }
              return depp;
            }
            // Random permutation of matrix
            double[][] randperm(double a[][])
```

```
        {
        double y[][]=new double[a.length][a[0].length];
        double[] sw=new double[a[0].length];
        for(int i=0;i<a.length;i++)
        {y[i]=a[i];}
        int m=a.length;
        int rp=0;
        for(int i=0;i<m-1;i++)
                {rp=(int)((m-i)*Math.random()+i);sw=y[i];
                y[i]=y[rp];y[rp]=sw;
                }
        return y;
        }
        double[][] get_cuckoos(double[][] nest,double best[])
        {
        double[] s=new double[nd];
        double[] u=new double[nd];
        double[] v=new double[nd];
        double[] step=new double[nd];
        double[] stepsize=new double[nd];
        double beta=1.5;
        double sigma=Math.pow((gamma(1+beta)*Math.sin(3.1415*beta/2.0)/(gamma((1.0+beta)/2.0)*beta*Math.pow(2.0,(beta-
1.0)/2.0))),(1.0/beta));
        double[] sk=new double[nd];
        Random rndm=new Random();
        for(int j=0;j<n;j++)
        {
        s=nest[j];
          for(int i=0;i<nd;i++)
          {u[i]=(rndm.nextGaussian())*sigma;
                v[i]=(rndm.nextGaussian());
                step[i]=u[i]/Math.pow((Math.abs(v[i])),(1/beta) );
                stepsize[i]=0.01*step[i]*(s[i]-best[i]);
                }
                for(int k=0;k<nd;k++)
                {sk[k]=stepsize[k]*rndm.nextGaussian();
                 s[k]=sk[k]+s[k];
                }
                nest[j]=simplebounds(s);
        }
        return nest;
        }
        double[] simplebounds(double s[])
        {
          for(int i=0;i<nd;i++)
          {if(s[i]<xmin[i])
           {s[i]=xmin[i];}
           if(s[i]>xmax[i])
           {s[i]=xmax[i];}
          }
          return s;
        }
        double[][] emptynests(double[][] nest)
        {
                double[][] K=new double[n][nd];
           double[][] randmat=new double[n][nd];
                double[] perm1=new double[n];
                double[] perm2=new double[n];
                double[][] nest1=new double[n][nd];
                double[][] nest2=new double[n][nd];
                double[][] nest3=new double[n][nd];
                double[][] nest4=new double[n][nd];
                double[][] stepsize=new double[n][nd];
           double[][] randmat2=new double[n][nd];
                double[][] newnest=new double[n][nd];
           for(int i=0;i<n;i++)
           {for(int j=0;j<nd;j++)
                  {randmat[i][j]=Math.random();
                  if(randmat[i][j]<pa){K[i][j]=0.0;}
                  if(randmat[i][j]>pa){K[i][j]=1.0;}
```

```
                    }
                  }
                  for(int i=0;i<n;i++)
                  {for(int j=0;j<nd;j++)
                    {nest1[i][j]=nest[i][j];nest2[i][j]=nest[i][j];}}
                  nest3=Matrix.substract(randperm(nest1),randperm(nest2));
                  for(int i=0;i<n;i++)
                  {for(int j=0;j<nd;j++)
                   {stepsize[i][j]=nest3[i][j]*Math.random();}}
                   nest4=Matrix.dotproduct(stepsize,K);
                   newnest=Matrix.add(nest,nest4);
                   return newnest;
}
public void solution()
{
double nestt[][]=new double[n][nd];
double fitness[]=new double[n];
double bestnest[]=new double[nd];
double best[]=new double[nd];
double newnest[][]=new double[n][nd];
double d1[][][]=initialize();
for(int i=0;i<n;i++)
{for(int j=0;j<nd;j++)
  {nestt[i][j]=d1[0][i][j];}
}
for(int i=0;i<n;i++)
{fitness[i]=d1[1][i][0];}
double d2[][][]=get_best_nest(nestt,nestt,fitness);
double fmin=d2[0][0][0];
  for(int i=0;i<n;i++)
  {fitness[i]=d2[1][i][0];}
  for(int i=0;i<nd;i++)
  {bestnest[i]=d2[2][0][i];}
            for(int i=0;i<n;i++)
            {   for(int j=0;j<nd;j++)
              {nestt[i][j]=d2[3][i][j];}}
            int iter=0;
            double fnew=0.0;
            nestt=emptynests(nestt);
            //System.out.println(Matrix.toString(nestt));
            while(fmin>tol)
            {
                        newnest=get_cuckoos(nest,bestnest);
                        double d3[][][]=get_best_nest(nestt,newnest,fitness);
    fnew=d3[0][0][0];
    for(int i=0;i<n;i++)
    {fitness[i]=d3[1][i][0];}
    for(int i=0;i<nd;i++)
    {best[i]=d3[2][0][i];}
              for(int i=0;i<n;i++)
              {for(int j=0;j<nd;j++)
                {nestt[i][j]=d3[3][i][j];}
              }
              iter+=n;
                        newnest=emptynests(nestt);
                        //System.out.println(Matrix.toString(nestt));
                        double d4[][][]=get_best_nest(nestt,newnest,fitness);
     fnew=d4[0][0][0];
    for(int i=0;i<n;i++)
    {fitness[i]=d4[1][i][0];}
    for(int i=0;i<nd;i++)
    {best[i]=d4[2][0][i];}
              for(int i=0;i<n;i++)
              {for(int j=0;j<nd;j++)
                {nestt[i][j]=d4[3][i][j];}
              }
                        iter+=n;
                        if(fnew<fmin)
                        {fmin=fnew;bestnest=best;}
            }
```

```
                    System.out.println("Optimum value of a function = "+fmin);
                    for(int i=0;i<bestnest.length;i++)
                    {System.out.println("x["+i+"] = "+bestnest[i]);}
            }
}
```

Program 5.26.2 Cuckoo Search algorithm test program

```
public class cuckoosearchtest1
{
public static void main(String args[])
{
if_xj ff=(double x[])->3.0*x[0]*x[0]-4.0*x[0]*x[1]+2.0*x[1]*x[1]-x[0]-x[1];
int n=250;//Number of nests
double pa=0.25;// Discovery rate of alien eggs/solutions
double tol=1e-5;//minimum values of tolerance leads to better results
double xmin[]={0.0,0.0};//Lower bounds
double xmax[]={3.0,3.0};//Upper bounds
cuckoo_search1i cc=new cuckoo_search1i(ff,n,pa,tol,xmin,xmax);
cc.solution();
}
}
```

```
 ---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" cuckoosearchtest1
Optimum value of a function = -1.120827819699341
x[0] = 0.9863193554487094
x[1] = 1.2809568971894936

> Terminated with exit code 0.
```

STOCHASTIC METHODS: HUNTING SEARCH ALGORITHM

This method is a meta-heuristic algorithm which simulates the behavior of animals hunting in a group. Animals have different attributes to catch their preys. Wolfs and lions make a circle around its prey and try to hunt it. The proposed algorithm reveals that the hunters which are expressed before is a solution vector for a problem. These simulated hunters come together and catch the prey which, in essence, is a optimum point for this problem.

As it is said before , each “artificial hunter” corresponds to solution of the problem.But in actual hunting of animals there is a significant difference with this algorithm. In actual hunting, hunters see or smell their prey and set its position. But contrary, in optimization problems, solution is not indicated and optimum solution is found by several iterations. In group hunting of animals solution(prey) is dynamic and the hunters (design variables) must change its location. In optimization problems , solution is static and its position does not change during iteration process. To resolve this complexity, artificial hunters(design variables) move towards leader(design variables which has the best solution). Leader is assumed to be the best solution so far and other hunters move towards it. If artificial hunter’s new postion is better than leader , selected artificial hunter is became a new leader. After moving toward the leader, artificial hunters must correct its positions. This is established on “Hunting Group Consideration Rate(HGCR)”. And if they escape out of the ring , prey is surrounded by hunters with random distribition.Finally if the positions of hunters are too close to each other , they reorganize and try to find optimum point. Algorithm mainly consists of six main steps that follows
Step 1 : Initializing parameters of the algorithm
Step 2 : Determining and hunting group[HG]
Step 3 : Moving towards the leader
Step 4 : Changing current positions (if necessary)
Step 5 : Reorganizing the hunting group
Step 6 : Repeat Step 3,4,5 until maximum number of iteration is met
1.Step
First upper and lower limits of design elements are contructed to make a circle around the prey(to satisfy non out of bounds condition). Hunting Group Size (HGS), Maximum Movement toward the Leader (MML) and Hunting Group Consideration Rate(HGCR) are defined to start iterations in this step.
2.Step

Hunting group is set up with random numbers respect to submitted limiting conditions. Cost function (objective function=fitness) is calculated for each hunter and leader hunter is defined by its fitness in this step.

3.Step

With moving towards to leader , new solution vector with N design variables $x_i^{new} = (x_1, x_2, .., x_i, .. x_N)$ is constructed. Movements to leader is established by

$x_i^{new} = x_i + random * MML * (x_i^{Leader} - x_i)$

Where "rand" is a random number between 0 and 1, x_i^{Leader} is the *i*th position of the leader. For each hunting group element if new position of hunter is better than previous position, hunter will change its new position, but if not , hunter remains its position. MML parameter can have different values for various proplems. İts value is dependent of "epochs" which will be described in latter steps. For epochs with large number of iterations MML value is suggested as 0.05 while with small number of iterations its value is proposed as 0.4 .

4.Step

Position correction is built up to organize hunt process in this step.After moving towards the leader , hunters chooses their new solution in two ways: (1) real value correction (2) digital value correction. In this paper, real value correction is taken into account. New hunter position is selected from HG or real value correction by HGCR parameter which varies from 0.0 to 1.0(proposed value lies between 0.1 – 0.4). Value updating process is carried out as

$$x_i^j \leftarrow \begin{Bmatrix} x_i^j \in (x_i^1, x_i^2, x_i^3, \dots, x_i^{HGS}) \; with \; probability \; HGCR \; i = 1..N, j = 1..HGS \\ x_i^j = x_i^j \mp Ra \quad with \; probability \; (1 - HGCR) \end{Bmatrix}$$

Ra is a artificial distance between design variables and is updated during optimization process. Ra is an exponantial funtion and helps to find optimum value and served as

$$Ra = Ra_{min}[\max(x_i) - \min(x_i)] exp\left(\frac{ln\left[\frac{Ra_{min}}{Ra_{max}}\right] iter}{maxiter}\right)$$

Where "iter" is a number of iteration, max(x_i) and min(x_i) are maximum and minimum value of hunter vector (x_i). "Ra_{min}" and "Ra_{max}" are the artificial relative search radius of the hunter and "maxiter" is the maximum number of iteration.

Step 5.

During iteration process hunters may be trapped in a local extremums. In order to avoid this situation hunters are rearranged to find another optimum point. After several number of iterations, hunter reorganizes themselves. A new term is emerged as "epoch" which is certain number of search to prevent local extremum trapping. Rearragement process is proposed as

$x_i^{new} = x_i^{Leader} \mp rand[\max(x_i) - \min(x_i)] \alpha \exp(-\beta \, EN)$

Where α and β are real values to determine convergence of algorithm. Large values of α and small values of β are for the problems which has several local optimums. Contrary small values of α and large values of β are for small optimization problems with a small number of design varibles. EN is a counter which numerates epochs concluded so far. EN helps hunters to search more locally into optimum point.

Program 5.27.1 Hunting Search algorithm

```
 public class Hunting_searchi
{
int N;
int HGS;//Number of solution vectores in group
double MML;//Maximum movement toward the leader
double HGCR;//Hunting group consideration rate
double HG[][];
double Lower[];
double Upper[];
double dummyHG[][];
double fitness[];
double best[];
double worst[];
double alfa;
double beta;
double Ramax;
double Ramin;
int maxiter;
int epochiter;
double min;
double max;
int indexmin;
```

```
int indexmax;
int d;
double EN;
double[] ENcount;
if_xj ff;


    public Hunting_searchi(if_xj iff,int iHGS,double iMML,double iHGCR,double iLower[],double[] iUpper,double ialfa,double
ibeta,double iRamax,double iRamin,int imaxiter,int iepochiter)
    {
                ff=iff;
                HGS=iHGS;
                HGCR=iHGCR;
                MML=iMML;
                Ramax=iRamax;
                Ramin=iRamin;
                Lower=iLower;
                Upper=iUpper;
                epochiter=iepochiter;
                maxiter=imaxiter;
                N=Upper.length;
                HG=new double[HGS][N];
                dummyHG=new double[HGS][N];
                fitness=new double[HGS];
                best=new double[N];
                worst=new double[N];
                alfa=ialfa;
                beta=ibeta;
                EN=(double)maxiter/((double)HGS*(double)epochiter);
                EN=Math.ceil(EN);

                 //System.out.println(EN);
                 ENcount=new double[(int)EN];
                 for(int i=0;i<(int)EN;i++)
                 {ENcount[i]=i+1;}
                // System.out.println(Matrix.toString(ENcount));
               }



               double[] getminval_index(double[] a)
               {
               double m=0.0;
               double b[]=new double[a.length];
               for(int i=0;i<a.length;i++)
               {b[i]=a[i];}
               double minval=a[0];
               for(int i=0;i<a.length;i++)
               {if(a[i]<minval){minval=a[i];}}
               for(int i=0;i<a.length;i++)
               {if(b[i]==minval){m=i;break;}};
               double[] dep=new double[2];
               dep[0]=minval;
               dep[1]=m;
               return dep;
               }

               double[] getmaxval_index(double a[])
               {
               double m=0.0;
               double b[]=new double[a.length];
               for(int i=0;i<a.length;i++)
               {b[i]=a[i];}
               double maxval=a[0];
               for(int j=0;j<a.length;j++)
               {if(a[j]>maxval){maxval=a[j];}}
               for(int i=0;i<b.length;i++)
               {if(b[i]==maxval){m=i;break;}}
               double dep2[]=new double[2];
               dep2[0]=maxval;
               dep2[1]=m;
               return dep2;
               }

               void init()
```

```
{
   for(int i=0;i<HGS;i++)
   {
                for(int j=0;j<N;j++)
                {HG[i][j]=Lower[j]+(Math.random()*(Upper[j]-Lower[j]));}
               }

               //Calculate fitness
               for(int i=0;i<HGS;i++)
               {fitness[i]=ff.func(HG[i]);}

               // get minimum index and find best
               double[] dd=getminval_index(fitness);
               indexmin=(int)dd[1];
               for(int i=0;i<N;i++)
               {best[i]=HG[indexmin][i];}

               // get maximum index
               double[] dd2=getmaxval_index(fitness);
               indexmax=(int)dd2[1];
               for(int i=0;i<N;i++)
               {worst[i]=HG[indexmax][i];}
}

double max(double x[])
{
 int n=x.length;
 double maxv=x[0];
 for(int i=0;i<n;i++)
 {if(x[i]>maxv)
              {maxv=x[i];}}
 return maxv;
}

double min(double x[])
{
 int n=x.length;
 double minv=x[0];
 for(int i=0;i<n;i++)
 {if(x[i]<minv)
   {minv=x[i];}}
 return minv;

}


double[][] iterations()
{
               init();
               //System.out.println(Matrix.toString(HG));
               int iter=1;
               while(iter<maxiter)
               {

               for(int i=0;i<HGS;i++)
               {for(int j=0;j<N;j++)
                 {dummyHG[i][j]=HG[i][j]+(2.0*(Math.random()-0.5)*MML*(best[j]-HG[i][j]));}}

               for(int i=0;i<HGS;i++)
               {
                 if(ff.func(dummyHG[i])<ff.func(HG[i]))
                 {
                              for(int j=0;j<N;j++)
                              {HG[i][j]=dummyHG[i][j];}
                            }
               }

              //Calculate fitness
               for(int i=0;i<HGS;i++)
               {fitness[i]=ff.func(HG[i]);}

               // get minimum index and find best
               double[] dd=getminval_index(fitness);
               indexmin=(int)dd[1];
               for(int i=0;i<N;i++)
               {best[i]=HG[indexmin][i];}
```

```
                        for(int i=0;i<HGS;i++)
                        {
                          if(Math.random()<HGCR)
                          {
                            d=(int)(((double)HGS)*Math.random());
                            for(int j=0;j<N;j++)
                            {HG[i][j]=HG[d][j];}
                          }
                          else
                          {
                                    for(int j=0;j<N;j++)
                                    {
                                    HG[i][j]=HG[i][j]+(2.0*(Math.random()-0.5))*Ramin*(max(HG[i])-
min(HG[i]))*Math.exp(Math.log(Ramin/Ramax)*(double)iter/(double)maxiter);
                            }
                                  }
                        }


                        int tt=(int)((double)iter/((double)HGS*(double)epochiter));
                        for(int i=0;i<HGS;i++)
                        {
                           for(int j=0;j<N;j++)
                           {HG[i][j]=best[j]+(2.0*(Math.random()-0.5))*(max(HG[i])-min(HG[i]))*alfa*Math.exp(-beta*ENcount[tt]);
}
                        }




                        //Calculate fitness
                        for(int i=0;i<HGS;i++)
                        {fitness[i]=ff.func(HG[i]);}

                        // get minimum index and find best
                        double[] dd3=getminval_index(fitness);
                        min=dd3[0];
                        indexmin=(int)dd3[1];
                        for(int i=0;i<N;i++)
                        {best[i]=HG[indexmin][i];}

                        iter++;
                        }
                        double[][] out=new double[2][N];
                        for(int i=0;i<N;i++)
                        {out[0][i]=best[i];}
                        out[1][0]=min;
                        return out;

}

                void toStringnew()
                {
                          double[][] in=iterations();
                          System.out.println("Optimized value = "+in[1][0]);
                          for(int i=0;i<N;i++)
                          {System.out.println("x["+i+"] = "+in[0][i]);}
                        }

}
```

Program 5.27.2 Hunting Search algorithm test program

```
 public class Hunting_search_test
{
     public static void main(String args[])
              {
                        int maxiter=20000;
                        int epochiter=40;
                        int HGS=100;
                        double MML=0.3;
                        double HGCR=0.2;
                        double[] Lower={-5.0,-5.0};
                        double[] Upper={5.0,5.0};
```

```
                        double alfa=-0.1;
                        double beta=1.0;
                        double Ramax=1.0e-2;
                        double Ramin=1.0e-7;
                        if_xj ff=(double x[])->-Math.cos(x[0])*Math.cos(x[1])*Math.exp(-(x[0]-Math.PI)*(x[0]-Math.PI)-(x[1]-
Math.PI)*(x[1]-Math.PI));

                          Hunting_searchi hunt=new
Hunting_searchi(ff,HGS,MML,HGCR,Lower,Upper,alfa,beta,Ramax,Ramin,maxiter,epochiter);
                        hunt.toStringnew();
                }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" Hunting_search_test
Optimized value = -1.0
x[0] = 3.14159264888846
x[1] = 3.141592648868062

> Terminated with exit code 0.
```

STOCHASTIC METHODS: HARRIS' HAWKS ALGORITHM

Harris Hawks algorithm is a stachastic method based on hunting behavior of harris hawks[93].

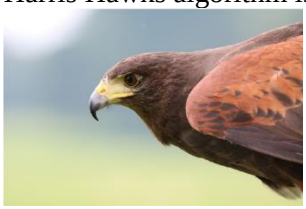


Figure Harris Hawk

Harris hawk, unlike other hawk species develop a group cooperative hunting behavior. Harris hawks shows evolved innovative team chasing capabilities in tracing, encircling, flushing out, and eventually attacking and catching their pray.

Their group behavior has Exploration and exploitation phases. In the exploration phase Haris' hawks track and detect their prey. But prey can not be seen directly all the time, in this case they fly randomly and perch to some position to detect the prays. Perching point will be selected on two different strategy, based on the position of the other members of the team and base on the possible position of the prey. As computer simulation of this can be given with the following formulations: Assume a random number q, and if q≥0.5 perch in a random position, if q<0.5 select a position based on position of the other members and possible prey location.

$$X(t+1) = \begin{cases} X_{rand}(t) - r_1[X_{rand}(t) - 2r_2X(t)] & if \quad q \geq 0.5 \\ \left(X_{prey}(t) - X_m(t)\right) - r_1[LB + r_4(UB - LB)] & if \quad q \geq 0.5 \end{cases}$$

Where $X(t+1)$ is the position vector of the hawks in the next iteration. $X_{prey}(t)$ is the position of the prey, $X(t)$ is the current position of hawks, r_1, r_2, r_3, r_4 and q are random numbers inside (0,1) range, which are updated in each iteration. LB and UB are search limits (upper and lower boundaries of variables). $X_{rand}(t)$ is a randomly selected hawk from the hawk group. $X_m(t)$ is the average position of the hawks. The average position of the hawks group is calculated by using mathematical averahe formula $X_m(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$ where $X_i(t)$ is the location of the each hawk fort he iteration t, N is the total number of hawks.

In order to translate from Exploration to exploitation phase prey energy is observed. Energy of the prey is modelled as:

$E = 2E_0\left(1 - \frac{t}{T}\right)$ where E indicates escaping energy of the prey, T is yhe maximum number of iterations, and E_0 is the initial state of its energy. When E is positive, prey is stenthening, when E is negative the prey is weakening. Exploration happens when $|E| \geq 1$ and explotation happens when $|E| < 1$. In exploitation phase, attacks the intented prey. Howewer prey often attemps to escape from

dangerous situation. Hence, different chasing styles occur in real situation. Suppose that r is the chance of a prey successfully escaping(r<0.5) or not successfully escaping(r≥0.5) before the surprise attack. The hawk behave two different mechanism to respond these, a hard or soft besiege.
When r≥ 0 and $|E| \geq 0.5$ soft besiege happens. In this case the prey has enough energy , and try to escape by some random movements. The Harris' hawks encircle it softly to make rabbit more exhausted the performs the surprise pounce. It is modelled as the following equations:

$$X(t+1) = \Delta X(t) - E\left[JX_{prey}(t) - X(t)\right]$$
$$\Delta X(t) = JX_{prey}(t) - X(t)$$
$$J = 2(1 - r_5)$$

Where $\Delta X(t)$ is the difference between the position vector of the rabbit and the current location in the iteration t, and r_5 is a random number. The hawks behave according to following pattern:

$$Y = X_{prey}(t) - E\left[JX_{prey}(t) - X(t)\right]$$
$$Z = Y + S * LF(D)$$

Where D is dimension of the problem aans S is a random vector of size 1xD and LF is the levy flight function.

$$LF(x) = 0.01 \frac{u\sigma}{|v|^{\frac{1}{\beta}}}$$
$$\sigma = \left(\frac{\Gamma(1+\beta) sin\left(\frac{\beta\pi}{2}\right)}{\Gamma(\frac{1+\beta}{2}) \beta \, 2^{\left(\frac{\beta-1}{2}\right)}} \right)^{\frac{1}{\beta}}$$

Where u,v are random numbers in (0,1) range, Γ(z) is gamma function which is defined as:

$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1} e^{-t} dt = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{t/n}$

where $\gamma = 0.57721566490153286..$ is the Euler-Mascheroni constant

Hence the final strategy becomes:

$$X(t+1) = \begin{cases} Y & if\ F(Y) < F(X(t)) \\ Z & if\ F(Z) < F(X(t)) \end{cases}$$

When r< 0.5 and $|E| < 0.5$ the prey does not have enough energy so hard besiege happens. Hawks directly attempts the surprise pounce to kill the prey.

$$Y = X_{prey}(t) - E\left[JX_{prey}(t) - X_m(t)\right]$$
$$Z = Y + S * LF(D)$$
$$X(t+1) = \begin{cases} Y & if\ F(Y) < F(X(t)) \\ Z & if\ F(Z) < F(X(t)) \end{cases}$$

Program 5.28-1 Harris-Hawk optimisation algorithm, function definition through if_xj interface

```
import java.util.*;
import java.io.*;
Written by: Oğuz Emrah Turgut
public class harris_hawks_optimization1
{

double[] Lower;
double[] Upper;
int N;
int D;
int Maxiter;
if_xj ff;
int iter;

double[] Rabbit_Location;
double Rabbit_Energy;
double[] CNVG;
double[][] X;
```

```
double[] X1;
double[] X2;
double[] fitnessX;
double[] mean2Dmat;
double[] X_rand;
double E1;
double E0;
double Escaping_Energy;

double q;
double r;
int rand_Hawk_index;

        public harris_hawks_optimization1(if_xj iff,int iN,double[] iLower,double[] iUpper, int iMaxiter)
        {
        Lower=iLower;
        Upper=iUpper;
        ff=iff;
        N=iN;
        D=Lower.length;
        Maxiter=iMaxiter;
        Rabbit_Location=new double[D];
        CNVG=new double[Maxiter];
        fitnessX=new double[N];
        X=new double[N][D];
        mean2Dmat=new double[D];
        X1=new double[D];
        X2=new double[D];
        X_rand=new double[D];
        Rabbit_Energy=1e80;
        }

         static double mean(double X[])
  {
           int m=X.length;
           double sum=0.0;
           for(int i=0;i<m;i++)
           {sum+=X[i];}
           return sum/(double)m;
        }

        static double[] mean2D(double XX[][])
        {
           int m=XX.length;
           int n=XX[0].length;
           double[] X=new double[m];
           double[] Y=new double[n];

            for(int j=0;j<n;j++)
            {
                                for(int i=0;i<m;i++)
                {X[i]=XX[i][j];}
                   Y[j]=mean(X);
            }
            return Y;
  }

   double logGamma(double x) {
   double tmp = (x - 0.5) * Math.log(x + 4.5) - (x + 4.5);
   double ser = 1.0 + 76.18009173    / (x + 0)   - 86.50532033    / (x + 1)
            + 24.01409822    / (x + 2)   -  1.231739516   / (x + 3)
            +  0.00120858003 / (x + 4)   -  0.00000536382 / (x + 5);
   return tmp + Math.log(ser * Math.sqrt(2 * Math.PI));
  }
   double gamma(double x) { return Math.exp(logGamma(x)); }

   double[] Levy()
   {
            double beta=1.5;
    double sigma=Math.pow((gamma(1+beta)*Math.sin(Math.PI*beta/2)/(gamma((1+beta)/2)*beta*Math.pow(2,((beta-
1)/2)))),(1/beta));
            Random rnd=new Random();
            double[] u=new double[D];
            double[] v=new double[D];
            double[] step=new double[D];
            for(int j=0;j<D;j++)
```

```
{u[j]=rnd.nextGaussian()*sigma;}
for(int j=0;j<D;j++)
{v[j]=rnd.nextGaussian();}
for(int j=0;j<D;j++)
{step[j]=u[j]/(Math.pow(Math.abs(v[j]),(1.0/beta)));}
return step;
}


double[][] boundary(double[][] XX)
{
        for(int i=0;i<N;i++)
                {for(int j=0;j<D;j++)
                {if((XX[i][j]<Lower[j])||(XX[i][j]>Upper[j]))
        {XX[i][j]=Lower[j]+((Upper[j]-Lower[j])*Math.random());}}}
        return XX;
}

void init()
{
   for(int i=0;i<N;i++)
   {for(int j=0;j<D;j++)
        {X[i][j]=(Lower[j]+((Upper[j]-Lower[j])*Math.random()));}}
}

double[][] solution()
{
 init();
 iter=0;

 while(iter<=Maxiter)
 {

        X=boundary(X);
        for(int i=0;i<N;i++)
        {fitnessX[i]=ff.func(X[i]);
          if(fitnessX[i]<Rabbit_Energy)
                {Rabbit_Energy=fitnessX[i];
                        for(int j=0;j<D;j++)
                        {Rabbit_Location[j]=X[i][j];}}}

                E1=2.0*(1.0-((double)iter/(double)Maxiter));

                for(int i=0;i<N;i++)
                {
                E0=2.0*Math.random()-1.0;
                Escaping_Energy=E1*E0;

                  if(Math.abs(Escaping_Energy)>=1.0)
                  {
                  q=Math.random();
                  rand_Hawk_index=(int)Math.floor((double)N*Math.random());
                    for(int j=0;j<D;j++)
                    {X_rand[j]=X[rand_Hawk_index][j];}
                        if(q<0.5)
                        {
                          for(int j=0;j<D;j++)
                          {X[i][j]=X_rand[j]-Math.random()*Math.abs(X_rand[j]-2.0*Math.random()*X[i][j]);}
                        }
                        else if (q>=0.5)
                        {
                                mean2Dmat=mean2D(X);
                                for(int j=0;j<D;j++)
                                {X[i][j]=(Rabbit_Location[j]-mean2Dmat[j])-
(Math.random()*(Lower[j]+((Upper[j]-Lower[j])*Math.random())));}

                        }

                }
                else if(Math.abs(Escaping_Energy)<1.0)
                {
                        r=Math.random();
                        if((r>=0.5)&&(Math.abs(Escaping_Energy)<0.5))
                        {
                          for(int j=0;j<D;j++)
```

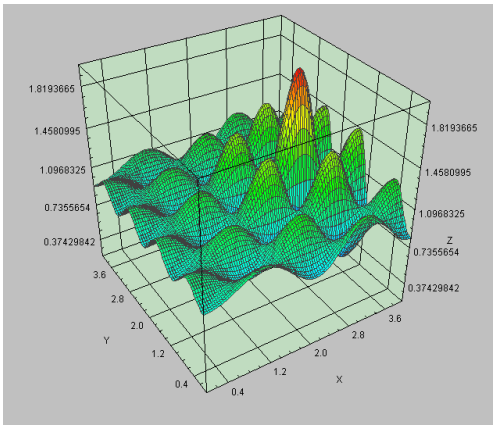
```
                                        {X[i][j]=Rabbit_Location[j]-Escaping_Energy*Math.abs(Rabbit_Location[j]-X[i][j]);}

                                      }
                                  if((r>=0.5)&&(Math.abs(Escaping_Energy)>=0.5))
                                      {
                                        for(int j=0;j<D;j++)
                                        {X[i][j]=(Rabbit_Location[j]-X[i][j])-Escaping_Energy*Math.abs(2.0*(1.0-
Math.random())*Rabbit_Location[j]-X[i][j]);}
                                      }
                                     if((r<0.5)&&(Math.abs(Escaping_Energy)>=0.5))
                                      {
                                        for(int j=0;j<D;j++)
                                        {X1[j]=Rabbit_Location[j]-Escaping_Energy*Math.abs(2.0*(1.0-
Math.random())*Rabbit_Location[j]-X[i][j]);}

                                        if( ff.func(X1)< ff.func(X[i]))
                                        {
                                                for(int j=0;j<D;j++)
                                                {X[i][j]=X1[j];}

                                              }
                                              else
                                              {
                                                double[] Levyout=Levy();
                                                for(int j=0;j<D;j++)
                                                {X2[j]=Rabbit_Location[j]-Escaping_Energy*Math.abs( 2.0*(1.0-
Math.random())*Rabbit_Location[j]-X[i][j]) + Math.random()*Levyout[j];  }

                                                 if( ff.func(X2)<ff.func(X[i]) )
                                                 {for(int j=0;j<D;j++)
                                                        {X[i][j]=X2[j];}}
                                              }


                                      }
                                       if((r<0.5)&&(Math.abs(Escaping_Energy)<0.5))
                                      {
                                                mean2Dmat=mean2D(X);
                                        for(int j=0;j<D;j++)
                                        {X1[j]=Rabbit_Location[j]-Escaping_Energy*Math.abs(2.0*(1.0-
Math.random())*Rabbit_Location[j]-mean2Dmat[j]);}

                                        if( ff.func(X1)< ff.func(X[i]))
                                        {
                                                for(int j=0;j<D;j++)
                                                {X[i][j]=X1[j];}

                                              }
                                              else
                                              {
                                                double[] Levyout=Levy();
                                                mean2Dmat=mean2D(X);
                                                for(int j=0;j<D;j++)
                                                {X2[j]=Rabbit_Location[j]-Escaping_Energy*Math.abs( 2.0*(1.0-
Math.random())*Rabbit_Location[j]-mean2Dmat[j]) + Math.random()*Levyout[j];  }

                                                 if( ff.func(X2)<ff.func(X[i]) )
                                                 {for(int j=0;j<D;j++)
                                                        {X[i][j]=X2[j];}}
                                              }


                                      }
                                  }

                        }
                        iter++;
                }

                double[][] out=new double[2][D];
                for(int j=0;j<D;j++)
                {out[0][j]=Rabbit_Location[j];}
                out[1][0]=Rabbit_Energy;
                return out;

              }
```

```
void toStringnew()
{
  double[][] out=solution();
  System.out.println("Optimized value = "+out[1][0]);
  for(int i=0;i<D;i++)
  {System.out.println("x["+i+"] = "+out[0][i]);}
}
public static void main(String arg[])
{//harris_hawks_optimization hho=new harris_hawks_optimization();
 //hho.toStringnew();
}

}
```

Program 5.28-2 Harris-Hawks optimisation algorithm rtest program , function definition through if_xj interface lambda variable

```
 import java.util.*;
import java.io.*;

public class harris_hawks_optimization_test3
{
public static void main(String arg[])
   {
   int iN=1000;
   if_xj f=(double x[])->1.0+Math.cos(Math.PI*(x[0]-3.0))*Math.cos(2.0*Math.PI*(x[1]-2.0))/
   (1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0));
   double[] iLower={0.0,0.0};
   double[] iUpper={4,4};
   int iMaxiter=200;
   //harris_hawks_optimization(f_xj iff,int iN,double[] iLower,double[] iUpper, int iMaxiter)
            harris_hawks_optimization1 hho=new harris_hawks_optimization1(f,iN,iLower,iUpper,iMaxiter);
   double[][] x0=hho.solution();
   String s="";
   s+="harris-hawks optimisation : \n"+Matrix.toStringT(x0[0]);
            double [] r2=NA41_i.nelder(f,x0[0]);
            s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
            System.out.println(s);
            surfacePlot p=new surfacePlot(f,iLower[0],iUpper[0],iLower[1],iUpper[1],100);
            p.plot();
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" harris_hawks_optimization_test3
harris-hawks optimisation :
        3.000000002599889
        2.480328761142911
Nelder-Mead solution :
        3.000000002599889
        2.480328761142911
```

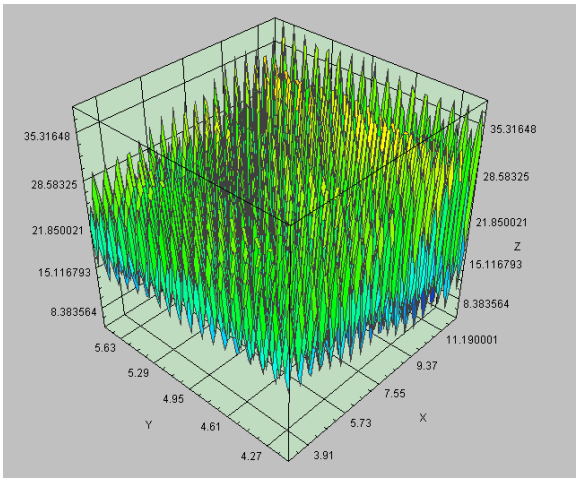


Program Harris-Hawks optimisation algorithm test program , function definition through if_xj interface lambda variable

```
import java.util.*;
import java.io.*;
public class harris_hawks_optimization_test4
{
```

```
public static void main(String arg[])
    {
    int iN=300;
    if_xj f=(double x[])->21.5+x[0]*Math.sin(4.0*Math.PI*x[0])+x[1]*Math.sin(20.0*Math.PI*x[1]);
    double[] iLower={3.0,4.1};
    double[] iUpper={12.1,5.8};
    int iMaxiter=200;
    //harris_hawks_optimization(f_xj iff,int iN,double[] iLower,double[] iUpper, int iMaxiter)
            harris_hawks_optimization1 hho=new harris_hawks_optimization1(f,iN,iLower,iUpper,iMaxiter);
    double[][] x0=hho.solution();
    String s="";
    s+="harris-hawks optimisation : \n"+Matrix.toStringT(x0[0]);
            double [] r2=NA41_i.nelder(f,x0[0]);
            s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
            System.out.println(s);
            surfacePlot p=new surfacePlot(f,iLower[0],iUpper[0],iLower[1],iUpper[1],70);
            p.plot();
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" harris_hawks_optimization_test4
harris-hawks optimisation :
        11.875533237082800
         5.775043861484454
Nelder-Mead solution :
        11.875533237082784
         5.775043861484401
```



STOCHASTIC METHODS: THERMAL EXCHANGE ALGORITHM

Thermal Exchange model[94] is based on cooling of the small (zero dimensional) object. In heat transfer this mechanism is called time dependent lumped solution. Consider a small body with volume V(m^3), surface area A(m^2), thermal convectivity at fluid-surface interaction h(W/m^2K), specific heat C_p (J/kgK), initial (time t=0 s) temperature T_0 (degree K) , environmental temperature T_b (degree K), and temperature of the body at time t is T(degree K), then time dependent variation of temperature of the object will be:

$$\frac{T-T_b}{T_0-T_b} = exp\left(-\frac{hA}{V\rho C_p}t\right)$$

Defining $\beta = \frac{hA}{V\rho C_p}$ equation becomes

$$\frac{T-T_b}{T_0-T_b} = exp(-\beta t) \text{ or } T = T_b + (T_0 - T_b)exp(-\beta t)$$

Based on these TEO is defined as follows:

Step 1: Initilization

The initial temperature of all the objects are determined in an *m* -dimensional search space.

$$T_i^0 = T_{min} + random(T_{max} - T_{min})$$

Where T_{min} and T_{max} are min and maximum bounds of the search region. Random is a ramdom number (0..1) region.

Step 2: Evaluation

Function values will be evaluated

Step 3: Saving

The group of best T vectors and their objective function value will be saved into memory.

Step 4: Group creating (for body and environment)

Agents will be divided into two equal groups. The first group will be treated as environment and the second group will be treated as cooling bodies. For example T_0 to $T_{n/2}$ will be environmental temperatures and $T_{n/2+1}$ to T_n will be cooling bodies.

Step 5: Defining β

Definition of β was given as $\beta = \frac{hA}{V\rho C_p}$ in this definition h is thermal conductivity coefficients and when h is bigger heat transfer will be higher so cooling effect will be higher, so the change of the β will determine cooling rate. In the algorithm, β will be defined as the cost of the object to cost of the wors object ratio.

$\beta = \frac{Cost(object)}{Cost(worst\ object)}$

Step 6: Defining t

$t = \frac{iteration}{Max\ iteration}$

Step 7: Escaping from local optima i

Environmental temperature is changed. Where c_1 and c_2 are controlling variables.

$T_i^{env} = (1 - \left(c_1 + c_2(1-t)\right) * random) T_i^{env}$

c_1 and c_2 are chosen either 0 or 1

Step 8: Updating the agent

The temperature of the each object is updated.

$T_i^{new} = T_i^{env} + (T_i^{old} - T_i^{new})\exp(-\beta t)$

Step 9: Escaping from local optima ii

The randomly selected parameter Pro (0,1) is introduced and it is specify whether a component of each cooling object must be changed. With each agent Pro is compared with a random number. If random(i)<Pro ith agent is selected randomly and its value is regenerated as

$T_{i,j} = T_{j,min} + random(T_{j,max} - T_{j,min})$

Where $T_{i,j}$ is the jth variable of agent i. $T_{j,max}$ and $T_{j,min}$ are lower and upper bounds of jth variable.

Step 10: Checking terminating conditions

The optimisation process will be terminated after a ficed number of iterations. If the termination does not satisfied it repeat fixed number of iterations.

Java code and example test cases are given below.

Program Thermal exhange optimisation algorithm, function definition through if_xj interface

```
import java.util.*;
import java.io.*;

public class Thermal_Exchange_Optimizationi
{
int N;
int D;
int Maxiter;
if_xj ff;
double[] Lower;
double[] Upper;
double[][] Therm_Object;
double[][] MEM_Therm_Object;
double[][] TRIAL_TO;
double[][] TRIAL;
double[]   TRIAL_fit;
double[] fitness_Therm_Object;
double[] sorted_fitness_Therm_Object;
double[] MEM_fitness_Therm_Object;
double[][] ENV_TO;
double[][] UPD_TO;
int MEM_TO;
double c1;
double c2;
double c;
```

```
double beta;
double p;
int index;
int NN;
int iter;
    public Thermal_Exchange_Optimizationi(if_xj iff,int iN,double[] iLower,double[] iUpper, int iMaxiter)
    {
            N=iN;
      ff=iff;
      Lower=iLower;
      Upper=iUpper;
      Maxiter=iMaxiter;
      D=Lower.length;
      MEM_TO=5;
      p=0.3;
      Therm_Object=new double[N][D];

      fitness_Therm_Object=new double[N];
      sorted_fitness_Therm_Object=new double[N];
      ENV_TO=new double[N][D];
      UPD_TO=new double[N][D];
      MEM_Therm_Object=new double[MEM_TO][D];
      MEM_fitness_Therm_Object=new double[MEM_TO];
      TRIAL=new double[N+MEM_TO][D];
      TRIAL_TO=new double[N+MEM_TO][D];
      TRIAL_fit=new double[N+MEM_TO];
      NN=N+MEM_TO;
    }

    double[][] sort_and_index(double[] A)
            {
                    ArrayList<Double> B=new ArrayList<Double>();
                    for(int i=0;i<A.length;i++)
                    {B.add(A[i]);}
                    ArrayList<Double> nstore=new ArrayList<Double>(B);
                    Collections.sort(B);
                    double[] ret=new double[B.size()];
                    Iterator<Double> iterator=B.iterator();
                    int ii=0;
                    while(iterator.hasNext())
                    {ret[ii]=iterator.next().doubleValue();ii++;}
                    int[] indexes=new int[B.size()];
                    for(int n=0;n<B.size();n++)
                    {indexes[n]=nstore.indexOf(B.get(n));}
                    double[][] outt=new double[2][B.size()];
                    for(int i=0;i<B.size();i++)
                    {outt[0][i]=ret[i];outt[1][i]=indexes[i];}
                    return outt;
            }

            double[] getminval_index(double[] a)
            {
            double m=0.0;
            double b[]=new double[a.length];
            for(int i=0;i<a.length;i++)
            {b[i]=a[i];}
            double minval=a[0];
            for(int i=0;i<a.length;i++)
            {if(a[i]<minval){minval=a[i];}}
            for(int i=0;i<a.length;i++)
            {if(b[i]==minval){m=i;break;}};
            double[] dep=new double[2];
            dep[0]=minval;
            dep[1]=m;
            return dep;
    }


            void init()
    {
               for(int i=0;i<N;i++)
               {for(int j=0;j<D;j++)
                        {Therm_Object[i][j]=Lower[j]+((Upper[j]-Lower[j])*Math.random());}}

                        for(int i=0;i<N;i++)
                        {fitness_Therm_Object[i]=ff.func(Therm_Object[i]);}
```

```
                    double[][] BB=sort_and_index(fitness_Therm_Object);
                    double[] I=new double[N];
                    for(int i=0;i<N;i++)
                     {I[i]=BB[1][i];}
                     for(int i=0;i<N;i++)
                     {sorted_fitness_Therm_Object[i]=BB[0][i];}

                    for(int i=0;i<N;i++)
                     {for(int j=0;j<D;j++)
                                {Therm_Object[i][j]=Therm_Object[(int)I[i]][j];}}

                                for(int i=0;i<MEM_TO;i++)
                                {MEM_fitness_Therm_Object[i]=ff.func(Therm_Object[i]);
                                        for(int j=0;j<D;j++)
                                        {MEM_Therm_Object[i][j]=Therm_Object[i][j];}}

        }

        double[][] environment(double[][] T, int iterr)
        {

                int mm=T.length;
                int nn=T[0].length;
                double[][] Env_TO=new double[mm][nn];
                double t=(double)iterr/(double)Maxiter;
                for(int i=0;i<N;i++)
                {
                 c1=Math.round(Math.random());
                 c2=Math.round(Math.random());
                        c=c1+c2*(1-t);
                 for(int j=0;j<D;j++)
                 {Env_TO[i][j]=(1.0-c*Math.random())*T[i][j];}
                }
               return Env_TO;
               }

               double[][] Newtonslaw(double[][] T,double[][] ENVTO,int iterr)
               {
                        double t=(double)iter/(double)Maxiter;
                        for(int i=0;i<N;i++)
                 {fitness_Therm_Object[i]=ff.func(T[i]);}
                 double[][] BB=sort_and_index(fitness_Therm_Object);
                 double[] I=new double[N];
                 for(int i=0;i<N;i++)
                  {I[i]=BB[1][i];}
                  for(int i=0;i<N;i++)
                  {sorted_fitness_Therm_Object[i]=BB[0][i];}
                  for(int i=0;i<N;i++)
                  {for(int j=0;j<D;j++)
                     {T[i][j]=T[(int)I[i]][j];}}
                        for(int i=0;i<N;i++)
                  {for(int j=0;j<D;j++)
                     {ENVTO[i][j]=ENVTO[(int)I[i]][j];}}

                        double[][] TO_NEW=new double[N][D];

                       for(int i=0;i<N;i++)
                       {
                         beta=sorted_fitness_Therm_Object[i]/sorted_fitness_Therm_Object[N-1];
                                if(i<N/2)
                                {
                                  for(int j=0;j<D;j++)
                                  {TO_NEW[i][j]=ENVTO[(N/2)-1+i][j]+(T[i][j]-ENVTO[(N/2)-1+i][j])*Math.exp(-
beta*t);}
                                                        if(Math.random()<p)
                                                        {index=(int)Math.floor(Math.random()*(double)D);
                                                         TO_NEW[i][index]=Lower[index]+((Upper[index]-
Lower[index])*Math.random());
                                                        }

                                }
                                else
                                {
                                  for(int j=0;j<D;j++)
```

```
                                        {TO_NEW[i][j]=ENVTO[i-(N/2)][j]+(T[i][j]-ENVTO[i-(N/2)][j])*Math.exp(-beta*t);}

                                            if(Math.random()<p)
                                                    {index=(int)Math.floor(Math.random()*(double)D);
                                                     TO_NEW[i][index]=Lower[index]+((Upper[index]-
Lower[index])*Math.random());
                                                    }

                                        }
        }

                return TO_NEW;
            }

            double[][] Update(double[][] T,double[][] T_M)
            {

                double[][] TT=new double[N][D];
                for(int i=0;i<N;i++)
                {for(int j=0;j<D;j++)
                        {TRIAL[i][j]=T[i][j];}}

               for(int i=N;i<NN;i++)
                {for(int j=0;j<D;j++)
                        {TRIAL[i][j]=T_M[i-N][j];}}

                for(int i=0;i<N;i++)
                {TRIAL_fit[i]=ff.func(TRIAL[i]);}


                double[][] AA=sort_and_index(TRIAL_fit);


                for(int i=0;i<NN;i++)
                {for(int j=0;j<D;j++)
                        {TRIAL_TO[i][j]=TRIAL[(int)AA[1][i]][j];}}

               for(int i=0;i<N;i++)
                {for(int j=0;j<D;j++)
                        {TT[i][j]=TRIAL_TO[i][j];}}

                return TT;
            }

            double[][] sorted(double[][] TT)
            {
                    int mm=TT.length;
                    int nn=TT[0].length;
                    double[][] TNEW=new double[mm][nn];
                    double[] fitnessTT=new double[mm];
                    for(int i=0;i<mm;i++)
                    {fitnessTT[i]=ff.func(TT[i]);}
                    double[][] FF=sort_and_index(fitnessTT);
                    for(int i=0;i<mm;i++)
                    {for(int j=0;j<nn;j++)
                      {TNEW[i][j]=TT[(int)FF[1][i]][j];}}

                    return TNEW;
            }



            double[][] Memory(double[][] TO, double[][] TO_M)
            {

                    double[] fit_TO_M=new double[MEM_TO];
                    double[] fit_TO=new double[N];
                    for(int i=0;i<MEM_TO;i++)
                    {fit_TO_M[i]=ff.func(TO_M[i]);}
                for(int i=0;i<N;i++)
                    {fit_TO[i]=ff.func(TO[i]);}

                    for(int i=0;i<N;i++)
                    {
                       for(int j=0;j<MEM_TO;j++)
                       {
```

```
                                        if(fit_TO[i]<fit_TO_M[j])
                                        {
                                                for(int jj=0;jj<D;jj++)
                                                {TO_M[j][jj]=TO[i][jj];}
                                                  fit_TO_M[j]=fit_TO[i];

                                                }
                                        }


                        }

                                return TO_M;
                }

                double[][] solution()
                {
                        init();
                        iter=0;

                        while(iter<Maxiter)
                        {
                                ENV_TO=environment(Therm_Object,iter);

                                UPD_TO=Newtonslaw(Therm_Object,ENV_TO,iter);

                                for(int i=0;i<N;i++)
                                {if( ff.func(UPD_TO[i])<ff.func(Therm_Object[i]))
                                 {for(int j=0;j<D;j++)
                                    {Therm_Object[i][j]=UPD_TO[i][j];}}}

                                MEM_Therm_Object=sorted(MEM_Therm_Object);
                                Therm_Object=sorted(Therm_Object);
                                Therm_Object=Update(Therm_Object,MEM_Therm_Object);

                                MEM_Therm_Object=Memory(Therm_Object,MEM_Therm_Object);
                                MEM_Therm_Object=sorted(MEM_Therm_Object);

                                iter++;

                        }

                 double[][] out=new double[2][D];
                for(int j=0;j<D;j++)
                {out[0][j]=MEM_Therm_Object[0][j];}
                 out[1][0]=ff.func(MEM_Therm_Object[0]);
                return out;



                      }


 void toStringnew()
{
  double[][] out=solution();
  System.out.println("Optimized value = "+out[1][0]);
  for(int i=0;i<D;i++)
  {System.out.println("x["+i+"] = "+out[0][i]);}
}


}
```

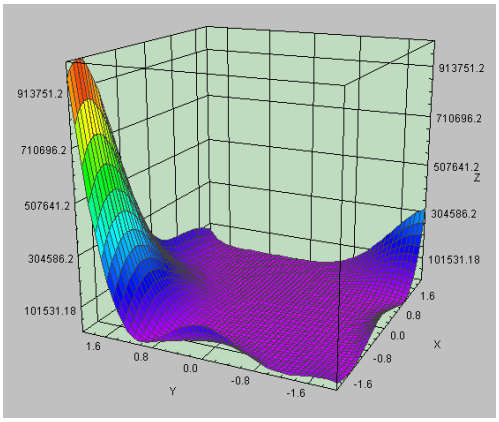
Test function: $\boldsymbol{f(x) = \left[1 + (x_0 + x_1 + 1)^2\left(19 - 14x_0 + 3x_0^2 - 14x_1 + 6x_0x_1 + 3x_1^2\right)\right] * \left[30 + (2x_0 - 3x_1)^2\left(18 - 32x_0 + 12x_0^2 + 48x_1 - 36x_0x_1 + 27x_1^2\right)\right]}$

Program 5.29-2 Thermal exhange optimisation test, function definition through if_xj interface

```
public class Thermal_Exchange_Optimization_test1
{
public static void main(String args[])
{   //Golstein & price function
        if_xj ff=(double x[])->(1.0+(x[0]+x[1]+1.0)*(x[0]+x[1]+1.0)*(19.0-14.0*x[0]+3.0*x[0]*x[0]-
14.0*x[1]+6.0*x[0]*x[1]+3.0*x[1]*x[1]))*
        (30.0+(2.0*x[0]-3.0*x[1])*(2.0*x[0]-3.0*x[1])*(18.0-32.0*x[0]+12.0*x[0]*x[0]+48.0*x[1]-36.0*x[0]*x[1]+27.0*x[1]*x[1]));
```

```
 double[] Lower={-2,-2};
 double[] Upper={2,2};
 int Maxiter=600;
 int N=48;
 Thermal_Exchange_Optimizationi wo=new   Thermal_Exchange_Optimizationi(ff,N,Lower,Upper,Maxiter);
 double[][] out=wo.solution();
 System.out.println("Optimized value = "+out[1][0]);
 for(int i=0;i<wo.D;i++)
 {System.out.println("x["+i+"] = "+out[0][i]);}
 surfacePlot pp=new surfacePlot(ff,Lower,Upper);
 pp.plot();
}
}
```

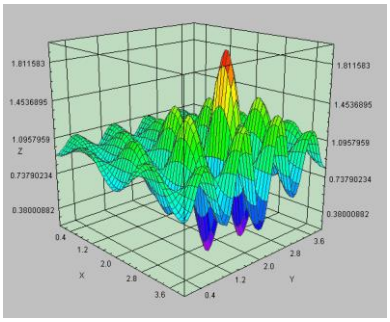
```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" Thermal_Exchange_Optimization_test1
Optimized value = 3.0014879299264376
x[0] = 0.002243055184328878
x[1] = -0.998527680225803
```



Program Thermal exhange optimisation test, function definition through if_xj interface

```
public class Thermal_Exchange_Optimization_test2
{
public static void main(String args[])
{
            if_xj ff=(double x[])->1.0+Math.cos(Math.PI*(x[0]-3.0))*Math.cos(2.0*Math.PI*(x[1]-2.0))/
   (1+(x[0]-3.0)*(x[0]-3.0)+(x[1]-2.0)*(x[1]-2.0));
 double[] Lower={0,0};
 double[] Upper={4,4};
 int Maxiter=600;
 int N=48;
 Thermal_Exchange_Optimizationi wo=new   Thermal_Exchange_Optimizationi(ff,N,Lower,Upper,Maxiter);
 double[][] out=wo.solution();
 System.out.println("Optimized value = "+out[1][0]);
 for(int i=0;i<wo.D;i++)
 {System.out.println("x["+i+"] = "+out[0][i]);}

 surfacePlot pp=new surfacePlot(ff,Lower,Upper);
 pp.plot();
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" Thermal_Exchange_Optimization_test2
Optimized value = 0.1936801694238951
x[0] = 2.998188195764863
x[1] = 2.4799952956638904
```



5.23 NON LINEAR OPTIMIZATION WITH BOUNDARY RESTRICTIONS

A non-linear optimization problem with boundary restrictions can be defined as

Minimum f(x)

Boundary restrictions : p(x)=0

q(x) * 0

In general these type of boundary restrictions are added up into the general equation as

For the first equation p(x)=0 define

$$P(x) = P(x_0, x_1, ..) = \sum_{k=0}^{r} p_k^2(x)$$

For the second equation q(x) $\geq$ 0 için

$$Q(x) = Q(x_0, x_1, ..) = \sum_{k=0}^{r} \min^2\{0, q_k(x)\}$$

Function min selects the minimum of 0 and $q_k(x)$.

The general function then becomes

$F(x) = f(x) + \mu[P(x) + Q(x)]$. μ in the equation is called punishment coefficient. For the second equation for q(x) $\geq$ 0 equation Q(x) can be also written as

$Q(x) = Q(x_0, x_1, ..) = \sum_{k=0}^{r} \min^2\{0, [q_k(x) - \varepsilon]\}$. In this equation ε >0 is a small number. With this small change refusal of the fiunction when the value is exactly equal to the restriction limit can be evoided. In the sample problem Fletcher – Reves Optimization method is used, but any unrestricted optimization methods can be used for this purpose.

Program Non-Restricted boundary fletcher reeves optimization

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

//OPTIMIZATION WITH BOUNDARY RESTRICTION
class f1 extends f_xj
{
public double func(double x[])
{
double ff;
double mu=0.0;
ff=(x[0]-1.0)*(x[0]-1.0)+(x[1]-2.0)*(x[1]-2.0)+(x[2]-3.0)*(x[2]-3.0)+mu*(P(x)+Q(x));
return ff;
}
public double P(double x[])
{// p(x)=0 boundary restriction
int n=x.length;
 return (x[0]+x[1]-2.0)*(x[0]+x[1]-2.0);
}
```

```
public double Q(double x[])
{// q(x)>=0 boundary restriction
double eps=1.0e-6;
int n=x.length;
double ax=x[1]-x[2]-3.0-eps;
double bx=0;
if(ax<0) bx=ax*ax;
 return bx;
}
}

public class NA116
{
public static double VT_V(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
              aa[i][j]=left[i]*left[j];
              }
  }
return aa;
}

public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

public static double turev(f_xj f,double x[],int denklem_ref,int x_ref)
{
double h0=0.256808;
int i,m;
int n=7;
double f1[];
f1=new double[x.length];
double f2[];
f2=new double[x.length];
double x1[];
x1=new double[x.length];
double x2[];
```

```
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i]*1.0094847;
f1=df(f,x1);
f2=df(f,x2);
T[i][0]=( f1[denklem_ref] - f2[denklem_ref])/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public static double[] df(f_xj f,double xxi[])
{
//Gredyen vektor
// f(x0,x1,x2,...xn) founksiyonunun türev vektörü
// [df/dx0, df/dx1,df/dx2,....,df/dxn]
// df/dxj  j=0...x.length
double a[]=new double[xxi.length];
for(int i=0;i<xxi.length;i++)
   {
   a[i]=turev(f,xxi,i);
   }
return a;
}
public static double turev(f_xj f,double xxi[],int x_ref)
{
// df/dxj
double h0=0.256808;
int i,m;
```

```
int n=7;
double x1[];
x1=new double[xxi.length];
double x2[];
x2=new double[xxi.length];
for(i=0;i<xxi.length;i++)
{
x1[i]=xxi[i];
x2[i]=xxi[i];
}
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
//h[i]=h0*Math.pow(r,i);
h0/=2.0;
h[i]=h0;
}
//first turev (difference formula)
for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
T[i][0]=(f.func(x1)-f.func(x2))/(2.0*h[i]);
x1[x_ref]=xxi[x_ref];
x2[x_ref]=xxi[x_ref];
}

for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] -
  h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i] - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public static double[] turev(f_xj f_deriv,double x[])
{
// df/dxj  j=0...x.length
// This method calculates turev of a function with more than one independent variable.
// Accuracy of method can be adjusted by changing variables h0 and n
// function input should be in the form given in abstract class
// f_xj,j=0...x.length = df/dx(x_ref)
double a[]=new double[x.length];
for(int x_ref=0;x_ref<x.length;x_ref++)
   {
   a[x_ref]=turev(f_deriv,x,x_ref);
   }
return a;
```

```
}

public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public static double linmin(f_xj f,double p[],double xi[])
{
        double tol=2.0e-4;
   int n=p.length;
   double pcom[],xicom[];
        int j;
        double xx,xmin,bx,ax;
        f1dim f1=new f1dim(f,p,xi);
        ax=0.0;
        xx=1.0;
        bx=2.0;
        double aa[]=mnbrak(f1,ax,xx,bx);
        ax=aa[0];
        xx=aa[1];
        bx=aa[2];
        xmin=brent(f1,ax,xx,bx,tol);
        return xmin;
}

public static double[][] linminiter(f_xj f,double p[],double xi[])
{   // yeni iterasyon noktasini verir
   int n=p.length;
   double xmin=linmin(f,p,xi);
   double aa[][]=new double[2][n];
        for (int j=0;j<n;j++)
        {   xi[j] *= xmin;
                p[j] += xi[j];
                aa[0][j]=p[j];
                aa[1][j]=xi[j];
        }
   return aa;
}
public  static double[] brentf(f_x f,double ax,double bx,double cx,double tol)
{
// ax,bx,cx üç adet tahmin noktasi (bxax ile cx arasinda yer almalidir)
// f : fonksiyon (f_x abstract sinifinda tanimlanmiş)
// tol tolerans
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altin oran
double ZEPS=1.0e-10;
double xmin;
double aa[]=new double[2];
// SHFT(double a,double b,double c,d) (a)=(b);(b)=(c);(c)=(d);
        int iter;
        double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
        double e=0.0;
        d=0.0;
        a=((ax < cx) ? ax : cx);
        b=((ax > cx) ? ax : cx);
        x=w=v=bx;
        fw=fv=fx=f.func(x);
        for (iter=1;iter<=ITMAX;iter++) {
                xm=0.5*(a+b);
                tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                        xmin=x;
```

```
                aa[0]=xmin;
                aa[1]=fx;
                return aa;
                }
                if (Math.abs(e) > tol1) {
                        r=(x-w)*(fx-fv);
                        q=(x-v)*(fx-fw);
                        p=(x-v)*q-(x-w)*r;
                        q=2.0*(q-r);
                        if (q > 0.0) p = -p;
                        q=Math.abs(q);
                        etemp=e;
                        e=d;
                        if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        else {
                                d=p/q;
                                u=x+d;
                                if (u-a < tol2 || b-u < tol2)
                                        d=SIGN(tol1,xm-x);
                        }
                } else {
                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                }
                u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                fu=f.func(u);
                if (fu <= fx) {
                        if (u >= x) a=x; else b=x;
                        {v=w;w=x;x=u;}
                        {fv=fw;fw=fx;fx=fu;}
                } else {
                        if (u < x) a=u; else b=u;
                        if (fu <= fw || w == x) {
                                v=w;
                                w=u;
                                fv=fw;
                                fw=fu;
                        } else if (fu <= fv || v == x || v == w) {
                                v=u;
                                fv=fu;
                        }
                }
        }
        System.out.println("BRENT metodunda maksimum iterasyon sayisi aşildi");
        xmin=x;
        //minumum değer a[0] ile fonksiyon değeri a[1] ile geridöndürülmektedir
        aa[0]=xmin;
        aa[1]=fx;
        return aa;
}
public  static double[] mnbrak(f_x f,double ax,double dx)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçilen uzaklik
return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(f_x f,double ax)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçiln uzaklik
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
```

```
}

public  static double[] mnbrak(f_x f,double ax,double bx,double cx)
{
// ax,bx,cx üç adet tahmin noktasi verildiğinde içinde minimum bulunan ax,bx,cx üç adetnoktayi bize arama
// sonucu bulur
// f : fonksiyon (f_x abstract sinifinda tanimlanmiş)
// tol tolerans
int ITMAX=100;
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
        double ulim,u,r,q,fu,dum;
        double aa[]=new double[3];
        fa=f.func(ax);
        fb=f.func(bx);
        if (fb > fa) {
                //SHFT(dum,*ax,*bx,dum)
                {dum=ax;ax=bx;bx=dum;}
                //SHFT(dum,*fb,*fa,dum)
                {dum=fb;fb=fa;fa=dum;}
        }
        cx=(bx)+GOLD*(bx-ax);
        fc=f.func(cx);
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        while (fb > fc) {
                r=(bx-ax)*(fb-fc);
                q=(bx-cx)*(fb-fa);
                u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
```

```
                                u=ulim;
                                fu=f.func(u);
                        } else {
                                u=cx+GOLD*(cx-bx);
                                fu=f.func(u);
                        }
                        //SHFT(ax,bx,cx,u)
                        {ax=bx;bx=cx;cx=u;}
                        //SHFT(fa,fb,fc,fu)
                        {fa=fb;fb=fc;fc=fu;}
                        aa[0]=ax;
                        aa[1]=bx;
                        aa[2]=cx;
                }
                        return aa;
}
public  static double brent(f_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}
// 4. fletcher Reeves metodu
//_____________________________________________________________________
public static double[] fletcher_reeves( f_xj f,double x[])
{
int i,j,ii,jj,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-4;
int n=x.length;
double alpha;
double g[]=new double[n];
double ge[]=new double[n];
double d[]=new double[n];
double beta;
double alpha_d;
double Q[]=new double[n];
int nn=15;
i=0;
j=0;k=0;
while( j++ < nmax && gamma > tolerance )
{
   g=turev(f,x);
   gamma=norm(g);
   if(k<1) {for(i=0;i<n;i++){d[i]=-g[i];};ge=g;}
   else
   {beta=VT_V(g)/VT_V(ge);
         for(i=0;i<n;i++){Q[i]=beta*d[i];};
   }
   for(i=0;i<n;i++){d[i]=-g[i]+Q[i];}
   alpha=linmin(f,x,d);
   for(i=0;i<n;i++){x[i]+=alpha*d[i];}
j++;k++;
if(k>=nn) k=0;
}
if(j >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
  " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}
              public static double[] verigir(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
```

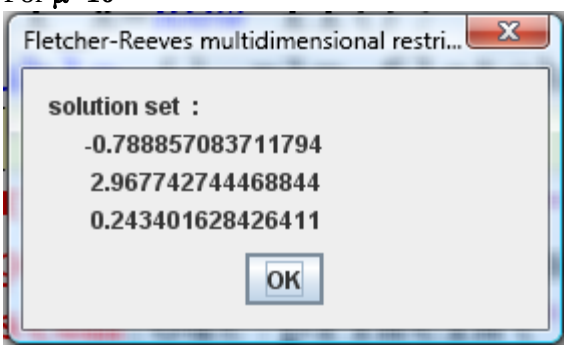
```
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }

    public static void main (String args[])
    {
   String s="enter initial root estimations with one space n between: ";
    double [] x0=verigir(s);
    f1 f_x=new f1();
    double [] r1= fletcher_reeves(f_x,x0);
    s=" çözüm seti  : \n"+Matrix.toStringT(r1);
    String s2="Fletcher-Reeves multidimensional restricted boundary optimization:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.out.println("y="+f_x.func(r1));
    System.exit(0);
    }
}
```

Our first example is without any restriction our equation is :
$f(x)=(x_0-1.0)^2+(x_1-2.0)^2+(x_2-3.0)^2$

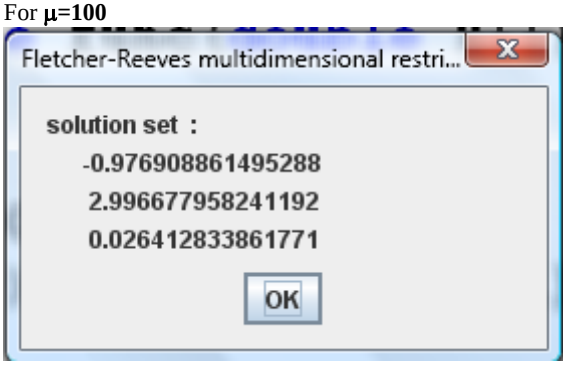


Now two boundary restriction will apply to the same function
$p(x) = x_0 + x_1 - 2.0=0$
$q(x) = x_1 - x_2 - 3.0 - \varepsilon \geq 0$

with this boundary restrictions equation becomes
$f(x)=(x_0-1.0)^2+(x_1-2.0)^2+(x_2-3.0)^2+\mu(p^2(x)+q^2(x))$
For **$\mu=10$**



For **μ=100**



As it is seen from the result for different values of μ the results are differs. μ is the weight function and the effects of restrictions increased when a bigger value of μ is used.

TRAVELLING SALESMAN PROBLEM (TSP)

Travelling salesman problem is a special kind of optimization problem. Problem satement is relatively simple. A travelling salesman will have to visit n cities in order to sell his products. The coordinates of the cities are set. How salesman can make the shortest travel? Problem can be considered as two basic form. In the first form starting and ending cities can be set. In the second form nothing is set.
The simplest way of the definitions of the function:

Distance= $\sum_{i=1}^{n}\left[(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\right]^{1/2}$ In this equation x_i and y_i is coordinates of nth city .

If the same Formula is written for a constant starting and ending cities

$$Distance = \left[(x_0 - x_{start})^2 + (y_0 - y_{start})^2\right]^{1/2} + \sum_{i=1}^{n}\left[(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\right]^{1/2} +$$

$$\left[(x_{final} - x_n)^2 + (y_{final} - y_n)^2\right]^{1/2}$$

In this equation (x_{start}, y_{start}) is the starting city coordinate and (x_{final}, y_{final}) is the destination city coordinate. In the normal version of the TSP problem every city must be travelled only once, but in some real problem this restriction does not have to be applied. The logical exact solution seems to be measured the distance from first city to the distance of all remaining cities travel there and continue on the process, but when the number of possibilities are concerned it is not practical. It should be also note that the distance between the cities does not have to fit the linear approach given in here (actual roads can be longer than the linear path). In this case a cross-reference table of the distances should be given to calculate it. Case equation will be:

Distance= $\sum_{i=1}^{n}\left[d_{i,i-1}\right]^{1/2}$ where $d_{i,(i-1)}$ given as a distance table (a matrix)

In here linear path will be assumed.

5.30.1 SOLUTION OF TSP BY GENETIC ALGORITHMS

Genetic algorithms can be applied to the TSP problem but, the definition of the gene structure and breeding procedures should be defined specifically to this problem, previous definition of Genetic algorithms was valid for a continious problem, but this of is a combination problem. The basic gene structure is formed by using ArrayList vector structure of Java. In order to do that a list of city corrdinates are needed as input. For example assume that the following city coordinates are given (it is stored in the file cities.txt)

```
100.0       0.0
100.0       0.0
95.10565162951535   30.901699437494738
80.90169943749474   58.778525229247315
```

```
58.77852522924731580.90169943749474
30.901699437494745 95.10565162951535
6.123233995736766E-15          100.0
-30.901699437494734            95.10565162951536
-58.7785252292473   80.90169943749474
-80.90169943749473 58.77852522924732
-95.10565162951535 30.901699437494752
-100.0      1.2246467991473532E-14
-95.10565162951538 -30.90169943749469
-80.90169943749474 -58.7785252292473
-58.77852522924732 -80.90169943749473
-30.901699437494756            -95.10565162951535
-1.8369701987210297E-14        -100.0
30.901699437494724 -95.10565162951536
58.77852522924729  -80.90169943749476
80.90169943749473  -58.778525229247336
95.10565162951535  -30.901699437494763
```

In this list the first line is the starting cities coordines, the second line is destination cities coordinates. The remaining is the cities to be visited in between starting and destination. The cities will be given a coordinate code according to the spacing in the given list. For example index of the third city (95.10565162951535, 30.901699437494738) is 0, the fourth city (80.90169943749474,58.778525229247315) is 1 etc. In the Array list visiting order of the index is tha basic gene combination. For example assume that ArrayList contains the following list

2	7	3	8	1	4	6	5

The actual travel is

Starting city-2-7-3-8-1-4-6-5-destination city. List can be arranged so that a repetation of the same city is restricted. Program mainly consist of two basic program TSP_gene and TSP_genetic. TSP_gene defines the gene structure as given above, breeding and mutation processes aealso included. Each individual has only one gene. The second program TSP_Genetic defines the defines genetic processes (select parents, breed, mutate new generations etc.) . The best solution is kept and copied to the new generation. A plot method is added to treck the path of the travel. List of numbers does not have much meaning in a case like this. It should be noted that due to statistical nature of GA the best result or the same result in each run should not be expected from the method.

Program solution of TSP by genetic algorithms, tsp_gene class

```
import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;
//çikiş ve variş noktalari belirlenmiş tur
public class TSP_gene
{
public int n;
public double xcikilan,ycikilan;
public double xvarilan,yvarilan;
public double x[],y[];
public double Fitness;    // uyum değeri
public double RFitness;   // relatif uyum
public double CFitness;   // kümülatif uyum
public boolean Breeder;
ArrayList<String> v;
String dizi;
public TSP_gene(double xi[],double yi[])
{n=xi.length-2;
v=new ArrayList<String>(n);
```

```
x=new double[n];
y=new double[n];
xcikilan=xi[0];
ycikilan=yi[0];
xvarilan=xi[1];
yvarilan=yi[1];
Breeder=false;
for(int i=0;i<n;i++) {x[i]=xi[i+2];y[i]=yi[i+2];v.add(""+i);}
Collections.shuffle(v);
//swap();
distance();
}

public void mutate()
{
int n1=(int)(n*Math.random());
int n2=(int)(n*Math.random());
//System.out.println("n="+n+"n1="+n1+"n2="+n2);
if(n1!=n2) Collections.swap(v, n1, n2);
}


public void swap()
{int pos[]=new int[n];
 int key[]=new int[n];
 int p,k;
 int n1=0;
for(int i=0;i<n;i++)
{ p=Collections.binarySearch(v,""+i);
  if(p==i) {pos[i]=p;n1++;}
}
for(int i=0;i<n1;i+=2)
{Collections.swap(v,pos[i],pos[i+1]);}
}

public TSP_gene(TSP_gene anne)
{n=anne.n;
 v=kopye(anne.v);
 xcikilan=anne.xcikilan;
 ycikilan=anne.ycikilan;
 xvarilan=anne.xvarilan;
 yvarilan=anne.yvarilan;
 x=new double[n];
 y=new double[n];
 for(int i=0;i<n;i++) {x[i]=anne.x[i];y[i]=anne.y[i];}
 distance();
}

public TSP_gene(int a[],double xi[],double yi[])
{
ArrayList<String> anne=add(a);
v=kopye(anne);
Collection torba=v;
n=xi.length-2;
xcikilan=xi[0];
ycikilan=yi[0];
xvarilan=xi[1];
yvarilan=yi[1];
x=new double[n];
y=new double[n];
for(int i=0;i<n;i++) {x[i]=xi[i+2];y[i]=yi[i+2];}
distance();
```

```
}

public TSP_gene(String dosya)
{double a[][]=Text.readDouble(dosya);
n=a[0].length-2;
v=new ArrayList<String>(n);
x=new double[n];
y=new double[n];
xcikilan=a[0][0];
ycikilan=a[1][0];
xvarilan=a[0][1];
yvarilan=a[1][1];
for(int i=0;i<n;i++) {x[i]=a[0][i+2];y[i]=a[1][i+2];v.add(""+i);}
Collections.shuffle(v);
distance();
}

   public TSP_gene(TSP_gene anne, TSP_gene baba)
   {
   n=anne.n;
   xcikilan=anne.xcikilan;
   ycikilan=anne.ycikilan;
   xvarilan=anne.xvarilan;
   yvarilan=anne.yvarilan;
   x=new double[n];
   y=new double[n];
   for(int i=0;i<n;i++)
   {x[i]=anne.x[i];y[i]=anne.y[i];}
   ArrayList<String> cocuk=s_cocuk(anne.v,baba.v);
            v=kopye(cocuk);
   distance();
            }

public TSP_gene(ArrayList<String> anne,double xi[],double yi[])
{
v=kopye(anne);
Collection torba=v;
n=xi.length-2;
xcikilan=xi[0];
ycikilan=yi[0];
xvarilan=xi[1];
yvarilan=yi[1];
x=new double[n];
y=new double[n];
for(int i=0;i<n;i++) {x[i]=xi[i+2];y[i]=yi[i+2];}
distance();
}

public void plot()
{
double xi[]=new double[n+2];
double yi[]=new double[n+2];
   int a=0;
   int b=0;
            String s1="";
            String s2="";
   Iterator i=v.iterator();
   int j=0;
   double distance;
   dizi="0";
   s1=(String)i.next();
   a=Integer.parseInt(s1);
```

```
    distance=Math.sqrt((x[a]-xcikilan)*(x[a]-xcikilan)+(y[a]-ycikilan)*(y[a]-ycikilan));
    xi[j]=xcikilan;
    yi[j]=ycikilan;
    j++;
    xi[j]=x[a];
    yi[j]=y[a];
    j++;
    while(i.hasNext())
    {
    s2=(String)i.next();
    b=Integer.parseInt(s2);

    distance+=Math.sqrt((x[a]-x[b])*(x[a]-x[b])+(y[a]-y[b])*(y[a]-y[b]));
    xi[j]=x[b];
    yi[j]=y[b];
    j++;
    a=b;
    }
    xi[j]=xvarilan;
    yi[j]=yvarilan;
    distance+=Math.sqrt((x[b]-xvarilan)*(x[b]-xvarilan)+(y[b]-yvarilan)*(y[b]-yvarilan));
    Fitness=1.0/distance;
    Plot pp=new Plot(xi,yi);
    double wx[]=new double[2];
    double wy[]=new double[2];
    wx[0]=xi[0];
    wy[0]=yi[0];
    wx[1]=xi[n+1];
    wy[1]=yi[n+1];
    pp.addData(wx,wy);
    pp.addData(xi,yi);
    pp.setPlotType(1,32);
    pp.setPlotType(2,22);
    pp.setPlabel("Travellin Salesman Problem distance = "+distance);
    pp.setXlabel("x, km");
    pp.setYlabel("y, km");
    pp.plot();
}

public double distance()
{
    int a=0;
    int b=0;
            String s1="";
            String s2="";
    Iterator i=v.iterator();
    int j=0;
    double distance;
    dizi="0";
    s1=(String)i.next();
    a=Integer.parseInt(s1);
    distance=Math.sqrt((x[a]-xcikilan)*(x[a]-xcikilan)+(y[a]-ycikilan)*(y[a]-ycikilan));
    while(i.hasNext())
    {
    s2=(String)i.next();
    b=Integer.parseInt(s2);
    distance+=Math.sqrt((x[a]-x[b])*(x[a]-x[b])+(y[a]-y[b])*(y[a]-y[b]));
    a=b;
    }
    distance+=Math.sqrt((x[b]-xvarilan)*(x[b]-xvarilan)+(y[b]-yvarilan)*(y[b]-yvarilan));
    Fitness=1.0/distance;
    return distance;
```

```
}

public void setFitness()
{Fitness=1.0/distance();}

public double getFitness()
{return Fitness;}

public void setRFitness(double RFitnessi)
{RFitness=RFitnessi;}

public double getRFitness()
{return RFitness;}

public void setCFitness(double CFitnessi)
{CFitness=CFitnessi;}

public double getCFitness()
{return CFitness;}
 public static String print(Collection c)
 {
  String s="";
  Iterator i=c.iterator();
  while(i.hasNext()) {s+=i.next()+"\n";}
  return s;
 }

 public String dizi()
 {return print(v);}

 public ArrayList<String>kopye(ArrayList<String>list1)
  {
     ArrayList<String> bilesimVectoru = new ArrayList<String>();
     String s1,s2;
     Iterator n1=list1.iterator();
     while(n1.hasNext())
     {
        s1=(String)n1.next();
        bilesimVectoru.add(s1);
     }
     return bilesimVectoru;
  }

  public static ArrayList<String> add(int a[])
  {
     ArrayList<String> bilesimVectoru = new ArrayList<String>();
     String s1;
     for(int i=0;i<a.length;i++)
     {
        s1=""+a[i];
        if(!bilesimVectoru.contains(s1))
           bilesimVectoru.add(s1);
     }
     return bilesimVectoru;
  }

  public ArrayList<String> bilesim(ArrayList<String>list1, int a)
  {
     ArrayList<String> bilesimVectoru = new ArrayList<String>();
     String s1,s2;
     Iterator n1=list1.iterator();
     String b=new String(""+a);
```

```
    while(n1.hasNext())
    {
       s1=(String)n1.next();
       bilesimVectoru.add(s1);
    }
    s2=(String)b;
       if(!bilesimVectoru.contains(s2))
          bilesimVectoru.add(s2);
    return bilesimVectoru;
}

public ArrayList<String> bilesim(ArrayList<String> list1, ArrayList<String> list2)
{
    ArrayList<String> bilesimVectoru = new ArrayList<String>();
    String s1,s2;
    Iterator n1=list1.iterator();
    Iterator n2=list2.iterator();
    while(n1.hasNext())
    {
       s1=(String)n1.next();
       bilesimVectoru.add(s1);
    }
    while(n2.hasNext())
    {
       s2=(String)n2.next();
       if(!bilesimVectoru.contains(s2))
          bilesimVectoru.add(s2);
    }
    return bilesimVectoru;
}

public  ArrayList<String> altküme(ArrayList<String> list1, int nn)
{
    ArrayList<String> bilesimVectoru = new ArrayList<String>();
    String s1,s2;
    Iterator n1=list1.iterator();
    for(int i=0;i<nn;i++)
    {
       s1=(String)n1.next();
       bilesimVectoru.add(s1);
    }
    return bilesimVectoru;
}

public  ArrayList<String> kalanküme(ArrayList<String> list1, int nn)
{
    ArrayList<String> bilesimVectoru = new ArrayList<String>();
    String s1,s2;
    Iterator n1=list1.iterator();

    for(int i=0;i<nn;i++)
    { s1=(String)n1.next();}

    while(n1.hasNext())
    {
       s1=(String)n1.next();
       bilesimVectoru.add(s1);
    }
    return bilesimVectoru;
}

public  ArrayList<String> s_cocuk(ArrayList<String> anne, ArrayList<String> baba)
```

```
    {int nn=(int)(n*Math.random())+1;
    //System.out.println("nn="+nn);
                ArrayList<String> anne1=altküme(anne,nn);
                ArrayList<String> anne2=kalanküme(anne,nn);
                ArrayList<String> baba1=altküme(baba,nn);
                ArrayList<String> baba2=kalanküme(baba,nn);
                ArrayList<String> cocuk=bilesim(anne1,baba1);
                ArrayList<String> cocuk1=bilesim(cocuk,anne2);
                ArrayList<String> cocuk2=bilesim(cocuk1,baba2);
                return cocuk2;
               }

               public String ciktiS()
    {
       Iterator enm = v.iterator();
       String s="  ("+xcikilan+","+ycikilan+")\n";
       String s1="";
       int a=0;
       while(enm.hasNext())
       {  s1=(String)enm.next();
          a=Integer.parseInt(s1);
                    s+=s1+" ("+x[a]+","+y[a]+") \n";
                 }
                 s+="  ("+xvarilan+","+yvarilan+") \n";
    return s;
    }

               public String[] ciktiSA()
    {
       Iterator enm = v.iterator();
       String s[]=new String[n];
       int i=0;
       while(enm.hasNext())
       {  s[i]= (String)enm.next()+" ";i++;}
    return s;
    }

    public String toString()
    {
               String s=ciktiS()+" distance = "+distance();
    return s;
    }
}
```

Program 5.30-2 solution of TSP by genetic algorithms, tsp_genetic class

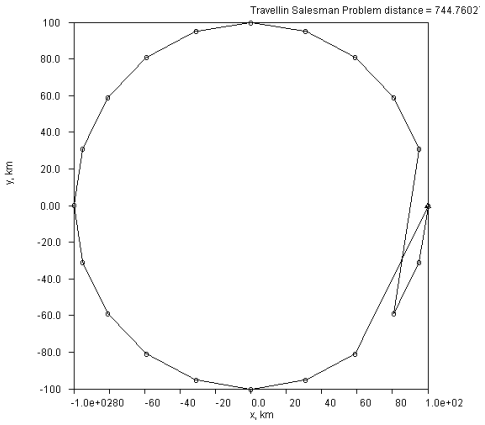
```
import java.io.*;
import java.text.*;
import java.util.Locale;

class TSP_genetic
{
public TSP_gene G[];    // string of variables makes a genotype1
public boolean Breeder;   //
public int    POPSIZE;          // population size
int    MAXGENS;          // maximum number of generations;
double PMUTATION;        // proportion of mutated variables
double PXOVER;           // probability of crossover;
int    Generation;       // Current generation number
int    NBreeder;   //   Number of survivors in a generation
int    Best; //The best genotype1 in the population
double toplameniyi=0;
```

```
double nesileniyi=0;
double xi[],yi[]; //City coordinates

 public TSP_genetic(String name,int POPSİZEi,int MAXGENSi,double PXOVERi,double PMUTATIONi )
 {double a[][]=Text.readDoubleT(name);
  double xi[]=a[0];
  double yi[]=a[1];
  int best;
  POPSIZE=POPSİZEi;
  PXOVER=PXOVERi;
  MAXGENS=MAXGENSi;
  PMUTATION=PMUTATIONi;
  G=new TSP_gene[POPSIZE+1];
  double totalfitness=0;
  for(int i=0;i<(POPSIZE+1);i++)
  {G[i]=new TSP_gene(xi,yi);G[i].setFitness();totalfitness+=G[i].getFitness();}
  double xR,xC=0;
  for(int i=0;i<(POPSIZE+1);i++)
  {xR=G[i].getRFitness()/totalfitness;xC+=xR;G[i].setRFitness(xR);G[i].setCFitness(xC);}
 }

 public void setPopulation()
 {
  double totalfitness=0;
  for(int i=0;i<POPSIZE;i++)
  {G[i]=new TSP_gene(xi,yi);}
   evaluate();
 }

 public void evaluate()
 {
  nesileniyi=0;
  double totalfitness=0;
  for(int i=0;i<POPSIZE+1;i++)
  {G[i].setFitness();
   double distance=1.0/G[i].getFitness();
   if(distance>nesileniyi) nesileniyi=distance;
   if(distance>toplameniyi) toplameniyi=distance;
   totalfitness+=G[i].getFitness();
  }
  double xR=0,xC=0,xRP=G[POPSIZE].getFitness()/totalfitness;
  for(int i=0;i<POPSIZE;i++)
  {xR=G[i].getFitness()/totalfitness;xC+=xR;G[i].setRFitness(xR);G[i].setCFitness(xC);
             if(xR > xRP)
                {
                        Best=i;
                        G[POPSIZE]=new TSP_gene(G[i]);
                        xRP=G[POPSIZE].getFitness()/totalfitness;
       }
  }
}
 public TSP_gene[] copyPopulation()
 {
            TSP_gene Po[]=new TSP_gene[POPSIZE+1];
            for(int i=0;i<(POPSIZE+1);i++)
              { Po[i]=new TSP_gene(G[i]);
                Po[i].Fitness=G[i].Fitness;
                        Po[i].RFitness=G[i].RFitness;
                Po[i].CFitness=G[i].CFitness;
              }
            return Po;
 }
```

```
            public void select()
            {
            double r,r1,r2,gf0,gf1;
            evaluate();
            TSP_gene Po[]=new TSP_gene[POPSIZE+1];
            NBreeder=0;
            int i,j;
            for(i=0;i<(POPSIZE+1);i++)
   {   r=Math.random();
       gf0=0;
       for(j=0;j<POPSIZE;j++)
       {
       gf1=G[j].getCFitness();
       if((r>=gf0) && (r<=gf1))
       {Po[NBreeder]=new TSP_gene(G[j]);
       NBreeder++;break;}
       gf0=gf1;
       }
   }
     int breed=(int)(POPSIZE*PXOVER);
     for(i=0;i<breed;i++)
                {
                            int a=(int)(Math.random()*NBreeder);
          int b=(int)(Math.random()*NBreeder);
          G[i]=new TSP_gene(Po[a],Po[b]);
          r2=Math.random();
          //mutation probability : PMUTATION
                 if(r2 < PMUTATION) G[i].mutate();
                }
                   for(i=breed;i<POPSIZE;i++)
                   {  int a=(int)(Math.random()*NBreeder);
            G[i]=new TSP_gene(Po[a]);
                   }
   }

   public void generations()
   {
            for(int i=0;i<MAXGENS;i++) {select();}
            }

   public TSP_gene selected_generation()
   {
   generations();
   G[POPSIZE].plot();
   return G[POPSIZE];
            }


  public static void main(String arg[])
  {
    TSP_genetic sg=new TSP_genetic("cities.txt",1000,1000,1.0,0.05 );
    System.out.println("selected generation : "+sg.selected_generation().toString());
  }

}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" TSP_genetic
selected generation :   (100.0,0.0)
18 (95.10565162951535,-30.901699437494763)
17 (80.90169943749473,-58.778525229247336)
```

```
0 (95.10565162951535,30.901699437494738)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
4 (6.123233995736766E-15,100.0)
5 (-30.901699437494734,95.10565162951536)
6 (-58.7785252292473,80.90169943749474)
7 (-80.90169943749473,58.77852522924732)
8 (-95.10565162951535,30.901699437494752)
9 (-100.0,1.2246467991473532E-14)
10 (-95.10565162951538,-30.90169943749469)
11 (-80.90169943749474,-58.7785252292473)
12 (-58.77852522924732,-80.90169943749473)
13 (-30.901699437494756,-95.10565162951535)
14 (-1.8369701987210297E-14,-100.0)
15 (30.901699437494724,-95.10565162951536)
16 (58.77852522924729,-80.90169943749476)
  (100.0,0.0)
 distance = 744.7602740406497
-142.35668789808918 122.48243559718966
```



Program 5.30-3 solution of TSP by genetic algorithms, tsp_gene1 class

```
    import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;
//çikiş ve variş noktalari belirlenmemiş tur
public class TSP_gene1
{
public int n;
public double x[],y[];
public double Fitness;    // uyum değeri
public double RFitness;   // relatif uyum
public double CFitness;   // kümülatif uyum
public boolean Breeder;

ArrayList<String> v;
String dizi;

public TSP_gene1(double xi[],double yi[])
{n=xi.length;
v=new ArrayList<String>(n);
x=new double[n];
y=new double[n];
```

```
Breeder=false;
for(int i=0;i<n;i++) {x[i]=xi[i];y[i]=yi[i];v.add(""+i);}
Collections.shuffle(v);
mesafe();
}

public void mutate()
{
int n1=(int)(n*Math.random());
int n2=(n1+1)%n;
//int n2=(int)(n*Math.random());
Collections.swap(v, n1, n2);
}


public void swap()
{int pos[]=new int[n];
 int key[]=new int[n];
 int p,k;
 int n1=0;
for(int i=0;i<n;i++)
{ p=Collections.binarySearch(v,""+i);
  if(p==i) {pos[i]=p;n1++;}
}
for(int i=0;i<n1;i+=2)
{Collections.swap(v,pos[i],pos[i+1]);}
}

public TSP_gene1(TSP_gene1 mother)
{n=mother.n;
 v=kopye(mother.v);
 x=new double[n];
 y=new double[n];
 for(int i=0;i<n;i++) {x[i]=mother.x[i];y[i]=mother.y[i];}
 mesafe();
}

public TSP_gene1(int a[],double xi[],double yi[])
{
ArrayList<String> mother=add(a);
v=kopye(mother);
Collection torba=v;
n=xi.length;
x=new double[n];
y=new double[n];
for(int i=0;i<n;i++) {x[i]=xi[i];y[i]=yi[i];}
mesafe();
}

public TSP_gene1(String dosya)
{double a[][]=Text.readDouble(dosya);
n=a[0].length;
v=new ArrayList<String>(n);
x=new double[n];
y=new double[n];
for(int i=0;i<n;i++) {x[i]=a[0][i];y[i]=a[1][i];v.add(""+i);}
Collections.shuffle(v);
mesafe();
}
        public TSP_gene1(TSP_gene1 mother, TSP_gene1 father)
   {
   n=mother.n;
```

```
    x=new double[n];
    y=new double[n];
    for(int i=0;i<n;i++)
    {x[i]=mother.x[i];y[i]=mother.y[i];}
    ArrayList<String> cocuk=s_cocuk(mother.v,father.v);
            v=kopye(cocuk);
    mesafe();
            }

public TSP_gene1(ArrayList<String> mother,double xi[],double yi[])
{
v=kopye(mother);
Collection torba=v;
n=xi.length;
x=new double[n];
y=new double[n];
for(int i=0;i<n;i++) {x[i]=xi[i];y[i]=yi[i];}
mesafe();
}

public void plot()
{
double xi[]=new double[n];
double yi[]=new double[n];
    int a=0;
    int b=0;
            String s1="";
            String s2="";
    Iterator i=v.iterator();
    int j=0;
    double mesafe=0;
    dizi="0";
    s1=(String)i.next();
    a=Integer.parseInt(s1);
    xi[j]=x[a];
    yi[j]=y[a];
    j++;
    while(i.hasNext())
    {
    s2=(String)i.next();
    b=Integer.parseInt(s2);
    mesafe+=Math.sqrt((x[a]-x[b])*(x[a]-x[b])+(y[a]-y[b])*(y[a]-y[b]));
    xi[j]=x[b];
    yi[j]=y[b];
    j++;
    a=b;
    }
    Plot pp=new Plot(xi,yi);
    double wx[]=new double[2];
    double wy[]=new double[2];
    wx[0]=xi[0];
    wy[0]=yi[0];
    wx[1]=xi[n-1];
    wy[1]=yi[n-1];
    pp.addData(wx,wy);
    pp.addData(xi,yi);
    pp.setPlotType(1,32);
    pp.setPlotType(2,22);
    pp.setPlabel("TSP problem distance = "+mesafe);
    pp.setXlabel("x, km");
    pp.setYlabel("y, km");
    pp.plot();
```

```
}

public double mesafe()
{
   int a=0;
   int b=0;
            String s1="";
            String s2="";
   Iterator i=v.iterator();
   int j=0;
   double mesafe;
   dizi="0";
   s1=(String)i.next();
   a=Integer.parseInt(s1);
   mesafe=0.0;
   while(i.hasNext())
   {
   s2=(String)i.next();
   b=Integer.parseInt(s2);
   mesafe+=Math.sqrt((x[a]-x[b])*(x[a]-x[b])+(y[a]-y[b])*(y[a]-y[b]));
   a=b;
   }
   Fitness=1.0/mesafe;
   return mesafe;
}

public void setFitness()
{Fitness=1.0/mesafe();}

public double getFitness()
{return Fitness;}

public void setRFitness(double RFitnessi)
{RFitness=RFitnessi;}

public double getRFitness()
{return RFitness;}

public void setCFitness(double CFitnessi)
{CFitness=CFitnessi;}

public double getCFitness()
{return CFitness;}
  public static String print(Collection c)
  {
    String s="";
    Iterator i=c.iterator();
    while(i.hasNext()) {s+=i.next()+"\n";}
    return s;
  }

  public String dizi()
  {return print(v);}

  public ArrayList<String>kopye(ArrayList<String>list1)
    {
       ArrayList<String> bilesimVectoru = new ArrayList<String>();
       String s1,s2;
       Iterator n1=list1.iterator();
       while(n1.hasNext())
       {
          s1=(String)n1.next();
```

```
            bilesimVectoru.add(s1);
        }
        return bilesimVectoru;
    }

    public static ArrayList<String> add(int a[])
    {
        ArrayList<String> bilesimVectoru = new ArrayList<String>();
        String s1;
        for(int i=0;i<a.length;i++)
        {
            s1=""+a[i];
            if(!bilesimVectoru.contains(s1))
                bilesimVectoru.add(s1);
        }
        return bilesimVectoru;
    }

    public ArrayList<String> bilesim(ArrayList<String>list1, int a)
    {
        ArrayList<String> bilesimVectoru = new ArrayList<String>();
        String s1,s2;
        Iterator n1=list1.iterator();
        String b=new String(""+a);
        while(n1.hasNext())
        {
            s1=(String)n1.next();
            bilesimVectoru.add(s1);
        }
        s2=(String)b;
            if(!bilesimVectoru.contains(s2))
                bilesimVectoru.add(s2);
        return bilesimVectoru;
    }

    public ArrayList<String> bilesim(ArrayList<String> list1, ArrayList<String> list2)
    {
        ArrayList<String> bilesimVectoru = new ArrayList<String>();
        String s1,s2;
        Iterator n1=list1.iterator();
        Iterator n2=list2.iterator();
        while(n1.hasNext())
        {
            s1=(String)n1.next();
            bilesimVectoru.add(s1);
        }
        while(n2.hasNext())
        {
            s2=(String)n2.next();
            if(!bilesimVectoru.contains(s2))
                bilesimVectoru.add(s2);
        }
        return bilesimVectoru;
    }

    public  ArrayList<String> altküme(ArrayList<String> list1, int nn)
    {
        ArrayList<String> bilesimVectoru = new ArrayList<String>();
        String s1,s2;
        Iterator n1=list1.iterator();
        for(int i=0;i<nn;i++)
        {
```

```
        s1=(String)n1.next();
        bilesimVectoru.add(s1);
    }
    return bilesimVectoru;
}

public  ArrayList<String> kalanküme(ArrayList<String> list1, int nn)
{
    ArrayList<String> bilesimVectoru = new ArrayList<String>();
    String s1,s2;
    Iterator n1=list1.iterator();

    for(int i=0;i<nn;i++)
    { s1=(String)n1.next();}

    while(n1.hasNext())
    {
        s1=(String)n1.next();
        bilesimVectoru.add(s1);
    }
    return bilesimVectoru;
}

public  ArrayList<String> s_cocuk(ArrayList<String> mother, ArrayList<String> father)
{int nn=(int)(n*Math.random())+1;
//System.out.println("nn="+nn);
            ArrayList<String> mother1=altküme(mother,nn);
            ArrayList<String> mother2=kalanküme(mother,nn);
            ArrayList<String> father1=altküme(father,nn);
            ArrayList<String> father2=kalanküme(father,nn);
            ArrayList<String> cocuk=bilesim(mother1,father1);
            ArrayList<String> cocuk1=bilesim(cocuk,mother2);
            ArrayList<String> cocuk2=bilesim(cocuk1,father2);
            return cocuk2;
           }

          public String ciktiS()
{
    Iterator enm = v.iterator();
    String s="";
    int a=0;
    String s1="";
    while(enm.hasNext())
    {  s1=(String)enm.next();
       a=Integer.parseInt(s1);
                s+=s1+" ("+x[a]+","+y[a]+") \n";
              }
return s;
}

          public String[] ciktiSA()
{
    Iterator enm = v.iterator();
    String s[]=new String[n];
    int i=0;
    while(enm.hasNext())
    {  s[i]= (String)enm.next()+" ";i++;}
return s;
}

public String toString()
{
```

```
            String s=ciktiS()+" mesafe = "+mesafe();
   return s;
   }
}
```

If starting and final destination is not important , TSP will change slightly

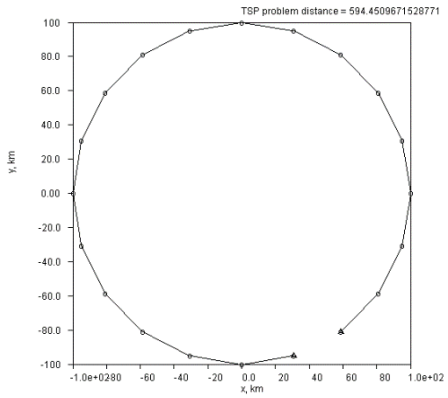
Program solution of TSP by genetic algorithms, tsp_gene1 class

```
import java.io.*;
import java.text.*;
import java.util.Locale;
//
class TSP_genetic1
{
public TSP_gene1 G[];    // string of variables makes a genotype1
public boolean Breeder;   //
public int    POPSIZE;          // population size
int    MAXGENS;          // maximum number of generations;
double PMUTATION;        // proportion of mutated variables
double PXOVER;           // probability of crossover;
int    Generation;       // Current generation number
int    NBreeder;   //   Number of survivors in a generation
int    Best; //The best genotype1 in the population
double toplameniyi=0;
double nesileniyi=0;
double xi[],yi[]; //şehir adresleri

 public TSP_genetic1(String name,int POPSİZEi,int MAXGENSi,double PXOVERi,double PMUTATIONi )
 {double a[][]=Text.readDoubleT(name);
  double xi[]=a[0];
  double yi[]=a[1];
  int best;
  POPSIZE=POPSİZEi;
  PXOVER=PXOVERi;
  MAXGENS=MAXGENSi;
  PMUTATION=PMUTATIONi;
  G=new TSP_gene1[POPSIZE+1];
  double totalfitness=0;
  for(int i=0;i<(POPSIZE+1);i++)
  {G[i]=new TSP_gene1(xi,yi);G[i].setFitness();totalfitness+=G[i].getFitness();}
  double xR,xC=0;
  for(int i=0;i<(POPSIZE+1);i++)
  {xR=G[i].getRFitness()/totalfitness;xC+=xR;G[i].setRFitness(xR);G[i].setCFitness(xC);}
 }

 public void setPopulation()
 {
  double totalfitness=0;
  for(int i=0;i<POPSIZE;i++)
  {G[i]=new TSP_gene1(xi,yi);}
   evaluate();
 }

 public void evaluate()
 {
  nesileniyi=0;
  double totalfitness=0;
  for(int i=0;i<POPSIZE+1;i++)
  {G[i].setFitness();
   double mesafe=1.0/G[i].getFitness();
   if(mesafe>nesileniyi) nesileniyi=mesafe;
```

```
   if(mesafe>toplameniyi) toplameniyi=mesafe;
   totalfitness+=G[i].getFitness();
  }
  double xR=0,xC=0,xRP=G[POPSIZE].getFitness()/totalfitness;
  for(int i=0;i<POPSIZE;i++)
  {xR=G[i].getFitness()/totalfitness;xC+=xR;G[i].setRFitness(xR);G[i].setCFitness(xC);
              if(xR > xRP)
                 {
                          Best=i;
                          G[POPSIZE]=new TSP_gene1(G[i]);
                          xRP=G[POPSIZE].getFitness()/totalfitness;
       }
  }
}
  public TSP_gene1[] copyPopulation()
  {
              TSP_gene1 Po[]=new TSP_gene1[POPSIZE+1];
              for(int i=0;i<(POPSIZE+1);i++)
                { Po[i]=new TSP_gene1(G[i]);
                  Po[i].Fitness=G[i].Fitness;
                          Po[i].RFitness=G[i].RFitness;
                  Po[i].CFitness=G[i].CFitness;
                 }
              return Po;
  }

              public void select()
              {
              double r,r1,r2,gf0,gf1;
              evaluate();
              //anne baba aday adaylarini seç
              TSP_gene1 Po[]=new TSP_gene1[POPSIZE+1];
              NBreeder=0;
              int i,j;
              for(i=0;i<(POPSIZE+1);i++)
   {   r=Math.random();
       gf0=0;
       for(j=0;j<POPSIZE;j++)
       {
       gf1=G[j].getCFitness();
       if((r>=gf0) && (r<=gf1))
       {Po[NBreeder]=new TSP_gene1(G[j]);
       //System.out.println("j= "+j+"gf0="+gf0+"gf1="+gf1+"r="+r+"NBreeder="+NBreeder);
       NBreeder++;break;}
       gf0=gf1;
       }
   }
   // seçilen anne, baba aday adaylarinin yeni çocuk yapmalarina olanak ver
     int breed=(int)(POPSIZE*PXOVER);
     for(i=0;i<breed;i++)
               {
                          int a=(int)(Math.random()*NBreeder);
          int b=(int)(Math.random()*NBreeder);
          G[i]=new TSP_gene1(Po[a],Po[b]);
          //System.out.println("a="+a+"b="+b+"cocuk="+G[i].toString());
          r2=Math.random();
          //mutasyon olasiliği : PMUTATION
                    if(r2 < PMUTATION) G[i].mutate();
                    //System.out.println(NBreeder+" "+G[POPSIZE].mesafe());
                }
               //listenin gerisini seçilen anne baba adaylariyla doldur
                  for(i=breed;i<POPSIZE;i++)
```

```
                {  int a=(int)(Math.random()*NBreeder);
          G[i]=new TSP_gene1(Po[a]);
                }
   }

   public void generations()
   {
            for(int i=0;i<MAXGENS;i++) {select();}
            }

   public TSP_gene1 selected_generation()
   {
   generations();
   G[POPSIZE].plot();
   return G[POPSIZE];
            }

  public static void main(String arg[])
  {
    TSP_genetic1 sg=new TSP_genetic1("cities1.txt",1000,1000,1.0,0.03 );
    System.out.println("selected generation : "+sg.selected_generation().toString());
  }

}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" TSP_genetic1
Selected heneration : 16 (30.901699437494724,-95.10565162951536)
15 (-1.8369701987210297E-14,-100.0)
14 (-30.901699437494756,-95.10565162951535)
13 (-58.77852522924732,-80.90169943749473)
12 (-80.90169943749474,-58.7785252292473)
11 (-95.10565162951538,-30.90169943749469)
10 (-100.0,1.2246467991473532E-14)
9 (-95.10565162951535,30.901699437494752)
8 (-80.90169943749473,58.77852522924732)
7 (-58.7785252292473,80.90169943749474)
6 (-30.901699437494734,95.10565162951536)
5 (6.123233995736766E-15,100.0)
4 (30.901699437494745,95.10565162951535)
3 (58.778525229247315,80.90169943749474)
2 (80.90169943749474,58.778525229247315)
1 (95.10565162951535,30.901699437494738)
0 (100.0,0.0)
19 (95.10565162951535,-30.90169943749476)
18 (80.90169943749473,-58.778525229247336)
17 (58.77852522924729,-80.90169943749476)
 mesafe = 594.4509671528771
```



SOLUTION OF TSP BY MONTE-CARLO ALGORITHM

Because TSP is a combinatory problem Monte-Carlo Method, like direct combinatoral calculations, can not give us a good solution. In order to show that a Monte-Carlo algoritm is developed and tried with very high number of iterations (100 million iteration). It is clear from the solution that, it is not enough to solve this problem only through randomness.

Program 5.30-4 solution of TSP by Monte-Carlo algorithms, tsp_gene1 class

```
import java.io.*;
import java.text.*;
import java.util.Locale;

class TSP_montecarlo
{
public TSP_gene G;    // string of variables makes a genotype1
public int    POPSIZE;          // population size
double xi[],yi[]; //şehir adresleri

  public TSP_montecarlo(String name,int POPSİZEi)
  {double a[][]=Text.readDoubleT(name);
   double xi[]=a[0];
   double yi[]=a[1];
   POPSIZE=POPSİZEi;
   TSP_gene G1=new TSP_gene(xi,yi);
   double toplameniyi=1e20;
   double distance;
   int counter;
   for(int i=0;i<POPSIZE;i++)
   {G1=new TSP_gene(xi,yi);
   distance=G1.distance();
   counter=i%1000000;
   if(counter==0) System.out.println("i="+i);
   if(distance < toplameniyi) {toplameniyi=distance;G=new TSP_gene(G1);System.out.println("i="+i+"\n"+G.toString());}
   }
   }

  public static void main(String arg[])
  {
    TSP_montecarlo sg=new TSP_montecarlo("cities.txt",100000000);
    //System.out.println("selected generations : "+sg.G.toString());
    sg.G.plot();
  }

}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" TSP_montecarlo
i=0
i=0
  (100.0,0.0)
17 (80.90169943749473,-58.778525229247336)
9 (-100.0,1.2246467991473532E-14)
10 (-95.10565162951538,-30.90169943749469)
7 (-80.90169943749473,58.77852522924732)
4 (6.123233995736766E-15,100.0)
15 (30.901699437494724,-95.10565162951536)
3 (30.901699437494745,95.10565162951535)
6 (-58.7785252292473,80.90169943749474)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
8 (-95.10565162951535,30.901699437494752)
12 (-58.77852522924732,-80.90169943749473)
13 (-30.901699437494756,-95.10565162951535)
14 (-1.8369701987210297E-14,-100.0)
5 (-30.901699437494734,95.10565162951536)
16 (58.77852522924729,-80.90169943749476)
0 (95.10565162951535,30.901699437494738)
18 (95.10565162951535,-30.901699437494763)
11 (-80.90169943749474,-58.7785252292473)
  (100.0,0.0)
 distance = 2400.9357446270287
i=3
  (100.0,0.0)
17 (80.90169943749473,-58.778525229247336)
14 (-1.8369701987210297E-14,-100.0)
12 (-58.77852522924732,-80.90169943749473)
11 (-80.90169943749474,-58.7785252292473)
13 (-30.901699437494756,-95.10565162951535)
10 (-95.10565162951538,-30.90169943749469)
16 (58.77852522924729,-80.90169943749476)
6 (-58.7785252292473,80.90169943749474)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
5 (-30.901699437494734,95.10565162951536)
9 (-100.0,1.2246467991473532E-14)
8 (-95.10565162951535,30.901699437494752)
0 (95.10565162951535,30.901699437494738)
18 (95.10565162951535,-30.901699437494763)
7 (-80.90169943749473,58.77852522924732)
4 (6.123233995736766E-15,100.0)
15 (30.901699437494724,-95.10565162951536)
  (100.0,0.0)
 distance = 2030.1843617772342
i=502
  (100.0,0.0)
0 (95.10565162951535,30.901699437494738)
2 (58.778525229247315,80.90169943749474)
12 (-58.77852522924732,-80.90169943749473)
13 (-30.901699437494756,-95.10565162951535)
10 (-95.10565162951538,-30.90169943749469)
7 (-80.90169943749473,58.77852522924732)
11 (-80.90169943749474,-58.7785252292473)
14 (-1.8369701987210297E-14,-100.0)
16 (58.77852522924729,-80.90169943749476)
15 (30.901699437494724,-95.10565162951536)
18 (95.10565162951535,-30.901699437494763)
```

4 (6.123233995736766E-15,100.0)
9 (-100.0,1.2246467991473532E-14)
6 (-58.7785252292473,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
17 (80.90169943749473,-58.778525229247336)
8 (-95.10565162951535,30.901699437494752)
5 (-30.901699437494734,95.10565162951536)
1 (80.90169943749474,58.778525229247315)
(100.0,0.0)
distance = 2012.5374983077775
i=575
(100.0,0.0)
2 (58.778525229247315,80.90169943749474)
0 (95.10565162951535,30.901699437494738)
1 (80.90169943749474,58.778525229247315)
3 (30.901699437494745,95.10565162951535)
5 (-30.901699437494734,95.10565162951536)
6 (-58.7785252292473,80.90169943749474)
4 (6.123233995736766E-15,100.0)
16 (58.77852522924729,-80.90169943749476)
7 (-80.90169943749473,58.77852522924732)
18 (95.10565162951535,-30.901699437494763)
8 (-95.10565162951535,30.901699437494752)
10 (-95.10565162951538,-30.90169943749469)
11 (-80.90169943749474,-58.7785252292473)
13 (-30.901699437494756,-95.10565162951535)
17 (80.90169943749473,-58.778525229247336)
9 (-100.0,1.2246467991473532E-14)
12 (-58.77852522924732,-80.90169943749473)
14 (-1.8369701987210297E-14,-100.0)
15 (30.901699437494724,-95.10565162951536)
(100.0,0.0)
distance = 1949.9796077260353
i=1148
(100.0,0.0)
16 (58.77852522924729,-80.90169943749476)
17 (80.90169943749473,-58.778525229247336)
18 (95.10565162951535,-30.901699437494763)
2 (58.778525229247315,80.90169943749474)
11 (-80.90169943749474,-58.7785252292473)
12 (-58.77852522924732,-80.90169943749473)
10 (-95.10565162951538,-30.90169943749469)
8 (-95.10565162951535,30.901699437494752)
9 (-100.0,1.2246467991473532E-14)
13 (-30.901699437494756,-95.10565162951535)
5 (-30.901699437494734,95.10565162951536)
7 (-80.90169943749473,58.77852522924732)
6 (-58.7785252292473,80.90169943749474)
14 (-1.8369701987210297E-14,-100.0)
0 (95.10565162951535,30.901699437494738)
15 (30.901699437494724,-95.10565162951536)
1 (80.90169943749474,58.778525229247315)
3 (30.901699437494745,95.10565162951535)
4 (6.123233995736766E-15,100.0)
(100.0,0.0)
distance = 1945.2569392748203
i=1473
(100.0,0.0)
18 (95.10565162951535,-30.901699437494763)
12 (-58.77852522924732,-80.90169943749473)
11 (-80.90169943749474,-58.7785252292473)
2 (58.778525229247315,80.90169943749474)

1 (80.90169943749474,58.778525229247315)
14 (-1.8369701987210297E-14,-100.0)
17 (80.90169943749473,-58.778525229247336)
16 (58.77852522924729,-80.90169943749476)
6 (-58.7785252292473,80.90169943749474)
9 (-100.0,1.2246467991473532E-14)
5 (-30.901699437494734,95.10565162951536)
7 (-80.90169943749473,58.77852522924732)
8 (-95.10565162951535,30.901699437494752)
3 (30.901699437494745,95.10565162951535)
4 (6.123233995736766E-15,100.0)
10 (-95.10565162951538,-30.90169943749469)
13 (-30.901699437494756,-95.10565162951535)
15 (30.901699437494724,-95.10565162951536)
0 (95.10565162951535,30.901699437494738)
(100.0,0.0)
distance = 1914.7548822898239
i=2518
(100.0,0.0)
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4 (6.123233995736766E-15,100.0)
3 (30.901699437494745,95.10565162951535)
5 (-30.901699437494734,95.10565162951536)
6 (-58.7785252292473,80.90169943749474)
7 (-80.90169943749473,58.77852522924732)
12 (-58.77852522924732,-80.90169943749473)
8 (-95.10565162951535,30.901699437494752)
9 (-100.0,1.2246467991473532E-14)
13 (-30.901699437494756,-95.10565162951535)
1 (80.90169943749474,58.778525229247315)
16 (58.77852522924729,-80.90169943749476)
15 (30.901699437494724,-95.10565162951536)
10 (-95.10565162951538,-30.90169943749469)
14 (-1.8369701987210297E-14,-100.0)
17 (80.90169943749473,-58.778525229247336)
18 (95.10565162951535,-30.901699437494763)
0 (95.10565162951535,30.901699437494738)
11 (-80.90169943749474,-58.7785252292473)
(100.0,0.0)
distance = 1909.6232492935656
i=4442
(100.0,0.0)
0 (95.10565162951535,30.901699437494738)
2 (58.778525229247315,80.90169943749474)
1 (80.90169943749474,58.778525229247315)
8 (-95.10565162951535,30.901699437494752)
12 (-58.77852522924732,-80.90169943749473)
13 (-30.901699437494756,-95.10565162951535)
14 (-1.8369701987210297E-14,-100.0)
11 (-80.90169943749474,-58.7785252292473)
16 (58.77852522924729,-80.90169943749476)
18 (95.10565162951535,-30.901699437494763)
17 (80.90169943749473,-58.778525229247336)
9 (-100.0,1.2246467991473532E-14)
10 (-95.10565162951538,-30.90169943749469)
7 (-80.90169943749473,58.77852522924732)
3 (30.901699437494745,95.10565162951535)
5 (-30.901699437494734,95.10565162951536)
6 (-58.7785252292473,80.90169943749474)
4 (6.123233995736766E-15,100.0)
15 (30.901699437494724,-95.10565162951536)
(100.0,0.0)

```
 distance = 1707.860830280625
i=9414
  (100.0,0.0)
2 (58.778525229247315,80.90169943749474)
1 (80.90169943749474,58.778525229247315)
8 (-95.10565162951535,30.901699437494752)
7 (-80.90169943749473,58.77852522924732)
6 (-58.7785252292473,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
4 (6.123233995736766E-15,100.0)
5 (-30.901699437494734,95.10565162951536)
15 (30.901699437494724,-95.10565162951536)
16 (58.77852522924729,-80.90169943749476)
18 (95.10565162951535,-30.901699437494763)
17 (80.90169943749473,-58.778525229247336)
9 (-100.0,1.2246467991473532E-14)
11 (-80.90169943749474,-58.7785252292473)
14 (-1.8369701987210297E-14,-100.0)
10 (-95.10565162951538,-30.90169943749469)
13 (-30.901699437494756,-95.10565162951535)
12 (-58.77852522924732,-80.90169943749473)
0 (95.10565162951535,30.901699437494738)
  (100.0,0.0)
 distance = 1644.5621964282616
i=14512
  (100.0,0.0)
9 (-100.0,1.2246467991473532E-14)
13 (-30.901699437494756,-95.10565162951535)
14 (-1.8369701987210297E-14,-100.0)
11 (-80.90169943749474,-58.7785252292473)
12 (-58.77852522924732,-80.90169943749473)
10 (-95.10565162951538,-30.90169943749469)
8 (-95.10565162951535,30.901699437494752)
7 (-80.90169943749473,58.77852522924732)
4 (6.123233995736766E-15,100.0)
5 (-30.901699437494734,95.10565162951536)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
0 (95.10565162951535,30.901699437494738)
18 (95.10565162951535,-30.901699437494763)
3 (30.901699437494745,95.10565162951535)
6 (-58.7785252292473,80.90169943749474)
17 (80.90169943749473,-58.778525229247336)
16 (58.77852522924729,-80.90169943749476)
15 (30.901699437494724,-95.10565162951536)
  (100.0,0.0)
 distance = 1630.24632213183
i=93035
  (100.0,0.0)
4 (6.123233995736766E-15,100.0)
6 (-58.7785252292473,80.90169943749474)
5 (-30.901699437494734,95.10565162951536)
7 (-80.90169943749473,58.77852522924732)
12 (-58.77852522924732,-80.90169943749473)
8 (-95.10565162951535,30.901699437494752)
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11 (-80.90169943749474,-58.7785252292473)
10 (-95.10565162951538,-30.90169943749469)
9 (-100.0,1.2246467991473532E-14)
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17 (80.90169943749473,-58.778525229247336)
15 (30.901699437494724,-95.10565162951536)
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14 (-1.8369701987210297E-14,-100.0)
18 (95.10565162951535,-30.901699437494763)
2 (58.778525229247315,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
1 (80.90169943749474,58.778525229247315)
0 (95.10565162951535,30.901699437494738)
 (100.0,0.0)
 distance = 1514.4486633644028
i=1000000
i=1399005
 (100.0,0.0)
16 (58.77852522924729,-80.90169943749476)
15 (30.901699437494724,-95.10565162951536)
14 (-1.8369701987210297E-14,-100.0)
13 (-30.901699437494756,-95.10565162951535)
11 (-80.90169943749474,-58.7785252292473)
12 (-58.77852522924732,-80.90169943749473)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
8 (-95.10565162951535,30.901699437494752)
9 (-100.0,1.2246467991473532E-14)
7 (-80.90169943749473,58.77852522924732)
10 (-95.10565162951538,-30.90169943749469)
4 (6.123233995736766E-15,100.0)
5 (-30.901699437494734,95.10565162951536)
6 (-58.7785252292473,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
17 (80.90169943749473,-58.778525229247336)
0 (95.10565162951535,30.901699437494738)
18 (95.10565162951535,-30.901699437494763)
 (100.0,0.0)
 distance = 1513.1324982330177
i=2000000
i=2171692
 (100.0,0.0)
18 (95.10565162951535,-30.901699437494763)
17 (80.90169943749473,-58.778525229247336)
13 (-30.901699437494756,-95.10565162951535)
9 (-100.0,1.2246467991473532E-14)
10 (-95.10565162951538,-30.90169943749469)
11 (-80.90169943749474,-58.7785252292473)
12 (-58.77852522924732,-80.90169943749473)
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16 (58.77852522924729,-80.90169943749476)
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6 (-58.7785252292473,80.90169943749474)
7 (-80.90169943749473,58.77852522924732)
8 (-95.10565162951535,30.901699437494752)
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2 (58.778525229247315,80.90169943749474)
4 (6.123233995736766E-15,100.0)
3 (30.901699437494745,95.10565162951535)
15 (30.901699437494724,-95.10565162951536)
0 (95.10565162951535,30.901699437494738)
 (100.0,0.0)
 distance = 1498.1779064811512
i=2419255
 (100.0,0.0)
0 (95.10565162951535,30.901699437494738)
1 (80.90169943749474,58.778525229247315)
3 (30.901699437494745,95.10565162951535)
6 (-58.7785252292473,80.90169943749474)

```
5 (-30.901699437494734,95.10565162951536)
2 (58.778525229247315,80.90169943749474)
4 (6.123233995736766E-15,100.0)
10 (-95.10565162951538,-30.90169943749469)
11 (-80.90169943749474,-58.7785252292473)
17 (80.90169943749473,-58.778525229247336)
18 (95.10565162951535,-30.901699437494763)
15 (30.901699437494724,-95.10565162951536)
14 (-1.8369701987210297E-14,-100.0)
13 (-30.901699437494756,-95.10565162951535)
12 (-58.77852522924732,-80.90169943749473)
8 (-95.10565162951535,30.901699437494752)
7 (-80.90169943749473,58.77852522924732)
9 (-100.0,1.2246467991473532E-14)
16 (58.77852522924729,-80.90169943749476)
  (100.0,0.0)
 distance = 1449.5497865351686
i=3000000
i=3436411
  (100.0,0.0)
0 (95.10565162951535,30.901699437494738)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
5 (-30.901699437494734,95.10565162951536)
6 (-58.7785252292473,80.90169943749474)
11 (-80.90169943749474,-58.7785252292473)
9 (-100.0,1.2246467991473532E-14)
10 (-95.10565162951538,-30.90169943749469)
8 (-95.10565162951535,30.901699437494752)
3 (30.901699437494745,95.10565162951535)
4 (6.123233995736766E-15,100.0)
7 (-80.90169943749473,58.77852522924732)
14 (-1.8369701987210297E-14,-100.0)
13 (-30.901699437494756,-95.10565162951535)
12 (-58.77852522924732,-80.90169943749473)
15 (30.901699437494724,-95.10565162951536)
18 (95.10565162951535,-30.901699437494763)
16 (58.77852522924729,-80.90169943749476)
17 (80.90169943749473,-58.778525229247336)
  (100.0,0.0)
 distance = 1353.0320496763034
i=4000000
i=5000000
i=6000000
i=6469430
  (100.0,0.0)
18 (95.10565162951535,-30.901699437494763)
0 (95.10565162951535,30.901699437494738)
11 (-80.90169943749474,-58.7785252292473)
10 (-95.10565162951538,-30.90169943749469)
8 (-95.10565162951535,30.901699437494752)
6 (-58.7785252292473,80.90169943749474)
5 (-30.901699437494734,95.10565162951536)
4 (6.123233995736766E-15,100.0)
1 (80.90169943749474,58.778525229247315)
2 (58.778525229247315,80.90169943749474)
3 (30.901699437494745,95.10565162951535)
7 (-80.90169943749473,58.77852522924732)
9 (-100.0,1.2246467991473532E-14)
14 (-1.8369701987210297E-14,-100.0)
15 (30.901699437494724,-95.10565162951536)
12 (-58.77852522924732,-80.90169943749473)
```

```
13 (-30.901699437494756,-95.10565162951535)
16 (58.77852522924729,-80.90169943749476)
17 (80.90169943749473,-58.778525229247336)
  (100.0,0.0)
 distance = 1319.5094061059226
i=7000000
i=7827971
  (100.0,0.0)
2 (58.778525229247315,80.90169943749474)
4 (6.123233995736766E-15,100.0)
6 (-58.7785252292473,80.90169943749474)
8 (-95.10565162951535,30.901699437494752)
7 (-80.90169943749473,58.77852522924732)
5 (-30.901699437494734,95.10565162951536)
1 (80.90169943749474,58.778525229247315)
3 (30.901699437494745,95.10565162951535)
9 (-100.0,1.2246467991473532E-14)
12 (-58.77852522924732,-80.90169943749473)
10 (-95.10565162951538,-30.90169943749469)
11 (-80.90169943749474,-58.7785252292473)
13 (-30.901699437494756,-95.10565162951535)
14 (-1.8369701987210297E-14,-100.0)
16 (58.77852522924729,-80.90169943749476)
15 (30.901699437494724,-95.10565162951536)
17 (80.90169943749473,-58.778525229247336)
18 (95.10565162951535,-30.901699437494763)
0 (95.10565162951535,30.901699437494738)
  (100.0,0.0)
 distance = 1266.711996027475
i=8000000
i=9000000
i=10000000
i=11000000
i=12000000
i=13000000
i=14000000
i=15000000
i=16000000
i=17000000
i=18000000
i=19000000
i=20000000
i=21000000
i=22000000
i=23000000
i=24000000
i=25000000
i=26000000
i=27000000
i=27211995
  (100.0,0.0)
0 (95.10565162951535,30.901699437494738)
18 (95.10565162951535,-30.901699437494763)
11 (-80.90169943749474,-58.7785252292473)
10 (-95.10565162951538,-30.90169943749469)
9 (-100.0,1.2246467991473532E-14)
3 (30.901699437494745,95.10565162951535)
2 (58.778525229247315,80.90169943749474)
1 (80.90169943749474,58.778525229247315)
4 (6.123233995736766E-15,100.0)
6 (-58.7785252292473,80.90169943749474)
5 (-30.901699437494734,95.10565162951536)
```

```
7 (-80.90169943749473,58.77852522924732)
8 (-95.10565162951535,30.901699437494752)
12 (-58.77852522924732,-80.90169943749473)
14 (-1.8369701987210297E-14,-100.0)
13 (-30.901699437494756,-95.10565162951535)
15 (30.901699437494724,-95.10565162951536)
16 (58.77852522924729,-80.90169943749476)
17 (80.90169943749473,-58.778525229247336)
  (100.0,0.0)
 distance = 1232.0491774494658
i=28000000
i=29000000
i=30000000
i=31000000
i=32000000
i=33000000
i=34000000
i=35000000
i=36000000
i=37000000
i=38000000
i=39000000
i=40000000
i=41000000
i=42000000
i=43000000
i=44000000
i=45000000
i=46000000
i=47000000
i=48000000
i=49000000
i=50000000
i=51000000
i=52000000
i=53000000
i=54000000
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i=59000000
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i=61000000
i=62000000
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i=64000000
i=65000000
i=66000000
i=67000000
i=68000000
i=69000000
i=70000000
i=71000000
i=72000000
i=73000000
i=74000000
i=75000000
i=76000000
i=77000000
i=78000000
i=79000000
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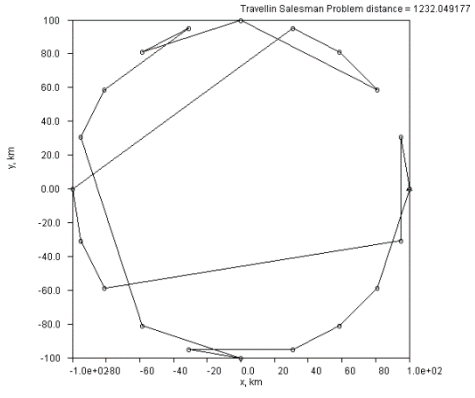
i=80000000
i=81000000
i=82000000
i=83000000
i=84000000
i=85000000
i=86000000
i=87000000
i=88000000
i=89000000
i=90000000
i=91000000
i=92000000
i=93000000
i=94000000
i=95000000
i=96000000
i=97000000
i=98000000
i=99000000



SOLUTION OF TSP BY ANT ALGORITHM

In the natural world, ants (initially) wander randomly, and upon finding food return to their colony while laying down pheromone trails. If other ants find such a path, they are likely not to keep travelling at random, but to instead follow the trail, returning and reinforcing it if they eventually find food . Over time, however, the pheromone trail starts to evaporate, thus reducing its attractive strength. The more time it takes for an ant to travel down the path and back again, the more time the pheromones have to evaporate. A short path, by comparison, gets marched over more frequently, and thus the pheromone density becomes higher on shorter paths than longer ones. Pheromone evaporation also has the advantage of avoiding the convergence to a locally optimal solution. If there were no evaporation at all, the paths chosen by the first ants would tend to be excessively attractive to the following ones. In that case, the exploration of the solution space would be constrained. Thus, when one ant finds a good (i.e., short) path from the colony to a food source, other ants are more likely to follow that path, and positive feedback eventually leads to all the ants' following a single path. The idea of the ant colony algorithm is to mimic this behavior with "simulated ants" walking around the representing the problem to solve. In figure 1 it is shown the behavior of ants when a barrier faced.

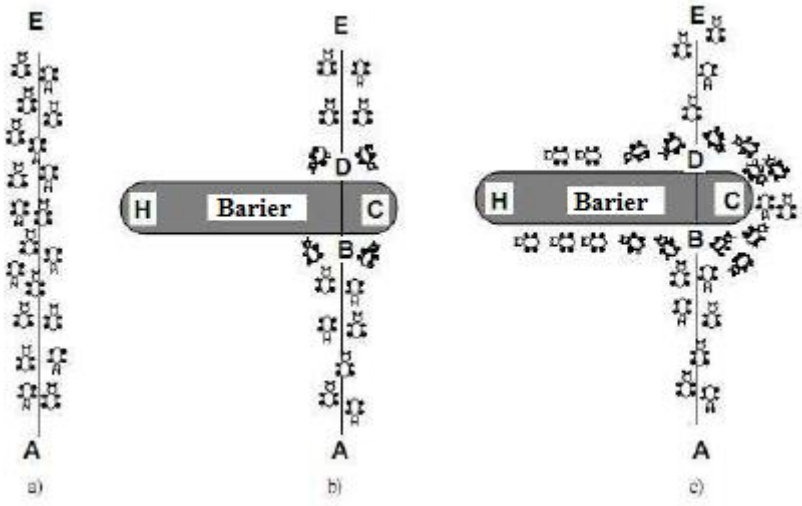


Figure 5.25-1 road selection by ants

Now let us investigate how to convert the basic principle into the computer code. Assume that ants can be reach to the food source by using two different paths. In this case probability of (m+1)th ant to select the first road can be written as:

$$p_{1(m+1)} = \frac{(m_1 + k)^h}{(m_1 + k)^h + (m_2 + k)^h}$$

In this equation m_1 and m_2 is the number of ant selected the first and second path. K and h in the equations are constants. The probability of (m+1)th ant to select the second road can be written as:

$P_{2(m+1)} = 1 - P_{1(m+1)}$

Simulated ants have some different properties that natural ants does not have

1. Simulated ants can have memory, for example in the TSP problems cities already visited can be memoried (keep in a variable)
2. They are npt completely blind, they can investigate the shortest road.
3. Their actual environment is much less time dependent.

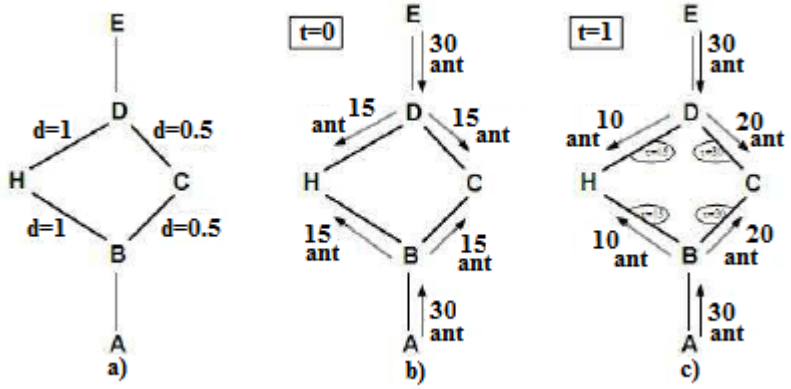


Figure 5.30-2 distribution of number of ants selected two different roads in time

When figüre 5.25-2 is investigated, ant behavior can be seen more closely.

- At the time t=0, 30 ant started to travel from A to E and another 30 ant started to travel from E to E
- As the first instance, ants travels in equal number to the cities H and C due to no previous pheromen trace on the road. Therefore each group will be consist of 15 ants
- Road B-C-D is shorter, therefore more pheromen will be collected on this path
- Art the time t=1 due to more pheromene in the B-C-D road more ants will prefer this road to the alternative one.

Now we have the basic idea to apply ant travel to the Travelling salesman problem. The following basic variables will be assumed

n : Number of the cities

d_{ij} : distance between cities i and j

$b_i(t)$: number of the ants in the city i, at the time t

m : total number of ants

$$m = \sum_{i=1}^{n} b_i(t)$$

$\tau_{ij}(t)$: pheromen density of the road ij at the time t

ρ : evaporation rate (0 to 1) In order to avoid infinite pheromen collection evaporation rate should ve selected smaller than 1. Note that 0 pheromen rate causes random movement also.

$\Delta\tau_{ij}^{k}$: the ration of pheromen density left by kth ant in the ij'th road to distance of this road.

$\Delta\tau_{ij}^{k} = \frac{Q}{L_k}$ If the kth ant used this particulare road

$\Delta\tau_{ij}^{k} = 0$ If the kth ant does not used this particulare road

L_k *;* total distance covered by the kth ant

Q; constant amount of pheromene left by each end per time

$Tabu_k$: The list of cities already travelled by kth ant

$$\eta_{ij} = \frac{1}{d_{ij}}$$

$p_{ij}^{k}(t)$: probability of going from city i to city j

If the city j is not in the list of already travelled cities;

$$p_{ij}^{k}(t) = \frac{\left[\tau_{ij}(t)\right]^{\alpha} \times \left[\eta_{ij}\right]^{\beta}}{\sum \left[\tau_{ik}(t)\right]^{\alpha} \times \left[\eta_{ik}\right]^{\beta}}$$

α ve β; are constant rate variables

If the city j is in the list of already travelled cities;

$$p_{ij}^{k}(t) = 0$$

Algorithm :

1. Start :
   a. t = 0 t : time counter
   b. NC = 0 NC :
   c. Starting pheromens for each road piece;
   d. Loacate m ants to the n cities.
2. s = 0 s : counter fort he list of the cities already travelled
   loop k = 1 to m
   for the kth ant place cities into the list of the cities already travelled
   repeat until list is full;
   a. s = s + 1
   b. k = 1 to m;
      i. select jth city to travel according to $p_{ij}^{k}(t)$ probability
      ii. move kth ant into the city j
      iii. add city j into the list of the cities already travelled
3. k = 1 to m
   a. let kth ant to travel from city $tabu_k(n)$ to $tabu_k(1)$

b. calculate distance travelled by the ant k; L_k
c. Record the shortest distance so far
d. For each ij path;
   i. k = 1 to m

If the kth ant used this road $\Delta\tau_{ij}^{k} = \frac{Q}{L_k}$

If the kth ant does not used this road $\Delta\tau_{ij}^{k} = 0$

$$\Delta\tau_{ij} = \Delta\tau_{ij} + \Delta\tau_{ij}^{k}$$

4. for each ij path calculate; $\tau_{ij}(t+n)$;
   a. $\tau_{ij}(t+n) = \rho\tau_{ij}(t) + \Delta\tau_{ij}$
   b. t = t + n
   c. NC = NC + 1
   d. for each ij path $\Delta\tau_{ij} = 0$
5.
   a. If NC < NC_{max} ; clear the lists of travelled cities and g oto item 2;
      If NC >=NC_{max} write shotest distance and stop*(stagnation behaviour)*

Several different versions of the ant colony TSP problems can be found in the literatüre.
For the TSP problems ant algorithm is given in the following program. As the example problem case, again cities around a circle is taken.

Program travelling salesman problem by using ants algorithm

```
public class antorj {
int id; //ant id number
int ssayi; //number of cities
int[] gyol; //gidilen güzergah
int asayi=0; //number of steps
double kmesafe; //total distance
double omesafe;
double feramon,guncel;

public antorj(int x,int y,double z) {
id=x;
ssayi=y;
feramon=z;
gyol=new int[ssayi];
kmesafe=0;
omesafe=400;
}

public void yuru(map w,double[][] x,int y,int z){
   double tolasilik;
   int isaretci;
   int a;
   int b;
   int yolsayisi=z;
   int bsehir,isaret,devam,devam2;
   int usehir;
   double of;

   a=0;
   b=0;
```

```
isaretci=0;
tolasilik=0;
asayi=y; //atilmis adim sayisi
usehir=gyol[asayi-1];
b=ssayi-asayi;
double[][] imkanliyol=new double[ssayi-asayi][2];
for(int i=0;i<yolsayisi;i++){
   if(a>=b){
      break;
   }

   isaretci=0;
   if(x[i][0]==usehir){
      for(int j=0;j<asayi;j++){
         if (x[i][1]==gyol[j]){
            isaretci=1;
              }
      }
      if (isaretci==0){
      imkanliyol[a][1]=x[i][1];
      imkanliyol[a][0]=x[i][2];
      a++;
      }
   } else
   {
   if(x[i][1]==usehir){
      for(int j=0;j<asayi;j++){
         if (x[i][0]==gyol[j]){
            isaretci=1;
              }
      }
       if(isaretci==0){
      imkanliyol[a][1]=x[i][0];
      imkanliyol[a][0]=x[i][2];
       a++;
       }

   }
   }


}

for(int i=0;i<ssayi-asayi;i++){
   tolasilik=imkanliyol[i][0]+tolasilik;
   //System.out.print(" imkanli yol : ");
   //System.out.print(gyol[0]);
   //System.out.print(" imkanli yollar : ");
   //System.out.print(imkanliyol[i][1]);
   //System.out.println(" ");
   //System.out.print(" imkanli yol : ");
   //System.out.println(imkanliyol[i][0]);

}

  of=Math.random();

   for(int i=0;i<ssayi-asayi;){
    if(of>imkanliyol[i][0]){
       i++;
       if(i>=(ssayi-asayi)){
          gyol[asayi]=(int) imkanliyol[i-1][1];
```

```
            }
          }
          else{
            //System.out.print(" imkanli yolda sorun var bence : ");
            //System.out.println(imkanliyol[i][1]);

            gyol[asayi]=(int) imkanliyol[i][1];
            break;
          }
         }
      kmesafe=kmesafe+w.cities[gyol[asayi-1]][(gyol[asayi])+3];
   }

   public double feramonsal(){
     return feramon;
   }
   public void ferguncel(double[][] x){
      guncel=feramon/kmesafe;
      for(int i=1;i<ssayi;i++){
         x[gyol[i-1]][gyol[i]]=x[gyol[i-1]][gyol[i]]+guncel;
      }
   }
    public void buharlastir(double[][] x,double y){
      for(int i=0;i<ssayi;i++){
         for(int j=0;j<ssayi;j++){
           x[i][j]=x[i][j]*y;
         }
              }
   }

}
```

```
import java.io.*;
import java.math.*;

class map { double[][] cities;
            public double mesafe(double x1,double x2,double y1,double y2)
            { double mesafe;
              mesafe=Math.pow((Math.pow((x2-x1),2)+Math.pow((y2-y1),2)),0.5);
              return mesafe;
            }
            public map(readcity x) {
            cities=new double[x.numberofcities][x.numberofcities+3];
            for(int i=0;i<x.numberofcities;i++) {
            cities[i][0]=i; //?ehrin ID si
            cities[i][1]=x.cities[i][0]; //?ehrin x koordinat?
            cities[i][2]=x.cities[i][1]; //?ehrin y koordinat?
            for(int k=3;k<(x.numberofcities+3);k++)
            {cities[i][k]=mesafe(x.cities[i][0],x.cities[k-3][0],x.cities[i][1],x.cities[k-3][1]);}}
                                  }
          }

class readcity
{   String dosyaismi=new String(" ");
   double[][] cities;
   int numberofcities;
   public readcity(String x)
   {   dosyaismi=x;
      cities=Text.readDouble(x);
      numberofcities=cities.length;
   }
}
```

```
public class ant{
                double mindistance;

      public static void quicksort(double[][] x,int y,int z){
      double[] mid=new double[3];
      double[] temp=new double[3];
      int k,j;
      k=y;
      j=z;
      mid[0]=x[(y+z)/2][0];
      mid[1]=x[(y+z)/2][1];
      mid[2]=x[(y+z)/2][2];
      do {while(x[k][2] < mid[2] & k < z) {k++;}
          while(x[j][2] > mid[2] & j > y) {j--;}
         if(k<=j){
            temp[2]=x[k][2];
            temp[1]=x[k][1];
            temp[0]=x[k][0];
            x[k][2]=x[j][2];
            x[k][1]=x[j][1];
            x[k][0]=x[j][0];
            x[j][2]=temp[2];
            x[j][1]=temp[1];
            x[j][0]=temp[0];
            k++;
            j--;
         }

      }while(k<=j);
      if(y<j) {
         quicksort(x,y,j);
      }
      if(k<z) {quicksort(x,k,z);}
      }

   public double readDistance()
   {return mindistance;}

            public  int[] ant_optimization(String dosya,double vaporizarion,double feramoni,int nant )
            {   double tolerans=1.0e-5;
      double dx=0.1;
      //vaporization 0-1
      double buharlasmaorani=vaporizarion; //feramonun buharlasma oranı
      //feramoni=5 (example)
      double feramon=feramoni; //feramon
      double beta=3; //
      double alfa=1; //
      double baslangicferamonu=1; //t=0 anindaki yoldaki feramon miktar?
      //int sakant=10; //iterasyon aras? korunacak maksimum kar?nca say?s?
      //int sakis; //saklanacak karınca işareti
      readcity sehir=new readcity(dosya); //şehirlerin okunması
      map bolge=new map(sehir); //mapnın hazırlanması
      double[][] feramondurumu=new double[sehir.numberofcities][sehir.numberofcities]; //yolların feramon durumunun
kaydedileceği
      double[][] deltaferamon=new double[sehir.numberofcities][sehir.numberofcities];
      //int antsayisi=1000; //koloni büyüklüğü
      int antsayisi=nant;
      int it=0;
      int maxit=10; //maksimum iterasyon sayisi
      int ilksehir;
      int numberofroads;
      double toplamprobability=0;
```

```
        double[][] probabilities=new double[sehir.numberofcities][sehir.numberofcities];
        double minprobability=0;
        double maxprobability=0;
        double tprobability=0;
        double mesafe;
        mindistance=0;
        double mindistanceeski=0.1;
        dx=Math.abs(mindistance-mindistanceeski);
        int minid=0;
        int[] minguzergah=new int[sehir.numberofcities];
        antorj[] antordusu=new antorj[antsayisi];
        //antorj[] sakkar=new antorj[sakant];
        numberofroads=0;
        //numberofroadsnin hesaplanması
        while (it<maxit  &&  dx>tolerans){
         mindistanceeski=mindistance;
         numberofroads=0;
        for(int i=1;i<sehir.numberofcities-1;i++){
           numberofroads=numberofroads+sehir.numberofcities-i;
        }
        numberofroads=numberofroads+1;
        //System.out.println(numberofroads);
        double[][] probabilitypasta=new double[numberofroads][3];

        //t=0 anındaki feramonun yollara konulması
       if(it==0){
        for(int i=0;i<(sehir.numberofcities-1);i++) {
           for(int j=i+1;j<(sehir.numberofcities);j++) {
              if(i==j){
                 feramondurumu[i][j]=0;
              }else{
                 feramondurumu[i][j]=baslangicferamonu;
              }
                    }
        }
       }
        //probability pastasinin hesaplanmasi
           //toplam olasiligin hesaplanmasi
        for(int i=0;i<sehir.numberofcities-1;i++){
           for(int j=i+1;j<(sehir.numberofcities);j++){
              if(i!=j){
                 toplamprobability=Math.pow((feramondurumu[i][j]),alfa)*Math.pow(bolge.cities[i][j+3],-beta);
                           }
           }
        }
        //toplamprobability=toplamprobability/2;
           //pastanin hesaplanması
        for(int i=0;i<sehir.numberofcities-1;i++){
           for(int j=i+1;j<(sehir.numberofcities);j++){
              if(i!=j){
                 probabilities[i][j]=(Math.pow((feramondurumu[i][j]),alfa)*Math.pow(bolge.cities[i][j+3],-
beta))/toplamprobability;
                 if(bolge.cities[i][j+3]==0){
                 }
                 if(i==0 & j==1){
                    maxprobability=probabilities[i][j];
                    minprobability=probabilities[i][j];
                    probabilitypasta[0][2]=minprobability;
                    probabilitypasta[0][0]=i;
                    probabilitypasta[0][1]=j;
                    probabilitypasta[numberofroads-1][2]=maxprobability;
                    probabilitypasta[numberofroads-1][0]=i;
```

```
                probabilitypasta[numberofroads-1][1]=j;
            } else {
                if(maxprobability<probabilities[i][j]) {
                    maxprobability=probabilities[i][j];
                    probabilitypasta[numberofroads-1][2]=maxprobability;
                    probabilitypasta[numberofroads-1][0]=i;
                    probabilitypasta[numberofroads-1][1]=j;
                } else {
                if(minprobability>probabilities[i][j]){
                    minprobability=probabilities[i][j];
                    probabilitypasta[0][0]=i;
                    probabilitypasta[0][1]=j;
                }
                }
            }
                        }
        else {
            probabilities[i][j]=0;
        }
    }
}
for(int i=1;i<numberofroads-1;i++){
    probabilitypasta[i][2]=minprobability;
}
    //pastanin siralanmasi
int l=0;
 for(int i=0;i<sehir.numberofcities-1;i++){
    for(int j=i+1;j<(sehir.numberofcities);j++){
      if(i!=j){
        probabilitypasta[l][2]=probabilities[i][j];
        probabilitypasta[l][0]=i;
        probabilitypasta[l][1]=j;
       l++;
        }
    }
 }
//probabilitiesi siralama
quicksort(probabilitypasta,0,numberofroads-1);
//t=0 anında ilk antların yerleştirilmesi
for(int i=0;i<antsayisi;i++){
    double x;
    antordusu[i]=new antorj(i,sehir.numberofcities,feramon);
    x=Math.random()*sehir.numberofcities;
    ilksehir=(int) x;
    antordusu[i].gyol[0]=ilksehir;
}
//antların yürümesi
for(int i=1;i<sehir.numberofcities;i++){
    for(int j=0;j<antsayisi;j++){
        antordusu[j].yuru(bolge,probabilitypasta,i,numberofroads);
    }
}
mindistance=antordusu[0].kmesafe;
minguzergah=antordusu[0].gyol;
for(int j=0;j<antsayisi;j++){
    if(antordusu[j].kmesafe<mindistance){
        mindistance=antordusu[j].kmesafe;
        minguzergah=antordusu[j].gyol;
    }
}

for(int j=0;j<antsayisi;j++){
```

```
            antordusu[j].ferguncel(deltaferamon);
        }
         for(int i=0;i<(sehir.numberofcities-1);i++) {
            for(int j=i+1;j<(sehir.numberofcities);j++) {
               if(i==j){
                  feramondurumu[i][j]=0;
               }else{
                  feramondurumu[i][j]=feramondurumu[i][j]*buharlasmaorani+deltaferamon[i][j];
               }
                     }
        }
            it++;
            dx=Math.abs(mindistance-mindistanceeski);
        }
        return minguzergah;
        }
}
```

```
import java.io.*;
import java.math.*;

class harita { double[][] sehirler;
              public double mesafe(double x1,double x2,double y1,double y2)
              { double mesafe;
                mesafe=Math.pow((Math.pow((x2-x1),2)+Math.pow((y2-y1),2)),0.5);
                return mesafe;
              }
              public harita(sehiroku x) {
              sehirler=new double[x.sehirsayisi][x.sehirsayisi+3];
              for(int i=0;i<x.sehirsayisi;i++) {
              sehirler[i][0]=i; //?ehrin ID si
              sehirler[i][1]=x.sehirler[i][0]; //?ehrin x koordinat?
              sehirler[i][2]=x.sehirler[i][1]; //?ehrin y koordinat?
              for(int k=3;k<(x.sehirsayisi+3);k++)
              {sehirler[i][k]=mesafe(x.sehirler[i][0],x.sehirler[k-3][0],x.sehirler[i][1],x.sehirler[k-3][1]);}}
                                  }
            }

class sehiroku
{   String dosyaismi=new String(" ");
    double[][] sehirler;
    int sehirsayisi;
    public sehiroku(String x)
    {   dosyaismi=x;
        sehirler=Text.readDouble(x);
        sehirsayisi=sehirler.length;
    }
}

public class ant{
                    double minmesafe;

        public static void quicksort(double[][] x,int y,int z){
        double[] mid=new double[3];
        double[] temp=new double[3];
        int k,j;
        k=y;
        j=z;
        mid[0]=x[(y+z)/2][0];
        mid[1]=x[(y+z)/2][1];
        mid[2]=x[(y+z)/2][2];
        do {while(x[k][2] < mid[2] & k < z) {k++;}
            while(x[j][2] > mid[2] & j > y) {j--;}
           if(k<=j){
              temp[2]=x[k][2];
              temp[1]=x[k][1];
```

```
            temp[0]=x[k][0];
            x[k][2]=x[j][2];
            x[k][1]=x[j][1];
            x[k][0]=x[j][0];
            x[j][2]=temp[2];
            x[j][1]=temp[1];
            x[j][0]=temp[0];
            k++;
            j--;
        }

    }while(k<=j);
    if(y<j) {
       quicksort(x,y,j);
    }
    if(k<z) {quicksort(x,k,z);}
    }

public double readMesafe()
{return minmesafe;}

            public  int[] ant_optimization(String dosya,double vaporizarion,double feramoni,int nkarinca )
            {   double tolerans=1.0e-5;
    double dx=0.1;
    //vaporization 0-1
    double buharlasmaorani=vaporizarion; //feramonun buharlasma oranı
    //feramoni=5 (örneğin)
    double feramon=feramoni; //feramon
    double beta=3; //
    double alfa=1; //
    double baslangicferamonu=1; //t=0 anindaki yoldaki feramon miktar?
    //int sakkarinca=10; //iterasyon aras? korunacak maksimum kar?nca say?s?
    //int sakis; //saklanacak karınca işareti
    sehiroku sehir=new sehiroku(dosya); //şehirlerin okunması
    harita bolge=new harita(sehir); //haritanın hazırlanması
    double[][] feramondurumu=new double[sehir.sehirsayisi][sehir.sehirsayisi]; //yolların feramon durumunun kaydedileceği
    double[][] deltaferamon=new double[sehir.sehirsayisi][sehir.sehirsayisi];
    //int karincasayisi=1000; //koloni büyüklüğü
    int karincasayisi=nkarinca;
    int it=0;
    int maxit=10; //maksimum iterasyon sayisi
    int ilksehir;
    int yolsayisi;
    double toplamolasilik=0;
    double[][] olasiliklar=new double[sehir.sehirsayisi][sehir.sehirsayisi];
    double minolasilik=0;
    double maxolasilik=0;
    double tolasilik=0;
    double mesafe;
    minmesafe=0;
    double minmesafeeski=0.1;
    dx=Math.abs(minmesafe-minmesafeeski);
    int minid=0;
    int[] minguzergah=new int[sehir.sehirsayisi];
    karincaorj[] karincaordusu=new karincaorj[karincasayisi];
    //karincaorj[] sakkar=new karincaorj[sakkarinca];
    yolsayisi=0;
    //yolsayisinin hesaplanması
    while (it<maxit  &&  dx>tolerans){
     minmesafeeski=minmesafe;
     yolsayisi=0;
    for(int i=1;i<sehir.sehirsayisi-1;i++){
       yolsayisi=yolsayisi+sehir.sehirsayisi-i;
    }
    yolsayisi=yolsayisi+1;
    //System.out.println(yolsayisi);
    double[][] olasilikpasta=new double[yolsayisi][3];

    //t=0 anındaki feramonun yollara konulması
   if(it==0){
```

```
for(int i=0;i<(sehir.sehirsayisi-1);i++) {
   for(int j=i+1;j<(sehir.sehirsayisi);j++) {
      if(i==j){
         feramondurumu[i][j]=0;
      }else{
         feramondurumu[i][j]=baslangicferamonu;
      }
            }
}
}
//olasilik pastasinin hesaplanmasi
   //toplam olasiligin hesaplanmasi
for(int i=0;i<sehir.sehirsayisi-1;i++){
   for(int j=i+1;j<(sehir.sehirsayisi);j++){
      if(i!=j){
         toplamolasilik=Math.pow((feramondurumu[i][j]),alfa)*Math.pow(bolge.sehirler[i][j+3],-beta);
                  }
   }
}
//toplamolasilik=toplamolasilik/2;
   //pastanin hesaplanması
for(int i=0;i<sehir.sehirsayisi-1;i++){
   for(int j=i+1;j<(sehir.sehirsayisi);j++){
      if(i!=j){
         olasiliklar[i][j]=(Math.pow((feramondurumu[i][j]),alfa)*Math.pow(bolge.sehirler[i][j+3],-beta))/toplamolasilik;
         if(bolge.sehirler[i][j+3]==0){
         }
         if(i==0 & j==1){
            maxolasilik=olasiliklar[i][j];
            minolasilik=olasiliklar[i][j];
            olasilikpasta[0][2]=minolasilik;
            olasilikpasta[0][0]=i;
            olasilikpasta[0][1]=j;
            olasilikpasta[yolsayisi-1][2]=maxolasilik;
            olasilikpasta[yolsayisi-1][0]=i;
            olasilikpasta[yolsayisi-1][1]=j;
         } else {
            if(maxolasilik<olasiliklar[i][j]) {
               maxolasilik=olasiliklar[i][j];
               olasilikpasta[yolsayisi-1][2]=maxolasilik;
               olasilikpasta[yolsayisi-1][0]=i;
               olasilikpasta[yolsayisi-1][1]=j;
            } else {
            if(minolasilik>olasiliklar[i][j]){
               minolasilik=olasiliklar[i][j];
               olasilikpasta[0][0]=i;
               olasilikpasta[0][1]=j;
            }
            }
         }
                  }
      else {
         olasiliklar[i][j]=0;
      }
   }
}
for(int i=1;i<yolsayisi-1;i++){
   olasilikpasta[i][2]=minolasilik;
}
   //pastanin siralanmasi
int l=0;
 for(int i=0;i<sehir.sehirsayisi-1;i++){
   for(int j=i+1;j<(sehir.sehirsayisi);j++){
     if(i!=j){
       olasilikpasta[l][2]=olasiliklar[i][j];
       olasilikpasta[l][0]=i;
       olasilikpasta[l][1]=j;
      l++;
       }
   }
```

```
        }
       //olasiliklari siralama
       quicksort(olasilikpasta,0,yolsayisi-1);
       //t=0 anında ilk karincaların yerleştirilmesi
       for(int i=0;i<karincasayisi;i++){
          double x;
          karincaordusu[i]=new karincaorj(i,sehir.sehirsayisi,feramon);
          x=Math.random()*sehir.sehirsayisi;
          ilksehir=(int) x;
          karincaordusu[i].gyol[0]=ilksehir;
       }
       //Karincaların yürümesi
       for(int i=1;i<sehir.sehirsayisi;i++){
          for(int j=0;j<karincasayisi;j++){
             karincaordusu[j].yuru(bolge,olasilikpasta,i,yolsayisi);
          }
       }
       minmesafe=karincaordusu[0].kmesafe;
       minguzergah=karincaordusu[0].gyol;
       for(int j=0;j<karincasayisi;j++){
          if(karincaordusu[j].kmesafe<minmesafe){
             minmesafe=karincaordusu[j].kmesafe;
             minguzergah=karincaordusu[j].gyol;
          }
       }

       for(int j=0;j<karincasayisi;j++){
          karincaordusu[j].ferguncel(deltaferamon);
       }
        for(int i=0;i<(sehir.sehirsayisi-1);i++) {
          for(int j=i+1;j<(sehir.sehirsayisi);j++) {
             if(i==j){
                feramondurumu[i][j]=0;
             }else{
                feramondurumu[i][j]=feramondurumu[i][j]*buharlasmaorani+deltaferamon[i][j];
             }
                    }
       }
          it++;
          dx=Math.abs(minmesafe-minmesafeeski);
       }
       return minguzergah;
       }
}
```

Program 5.256 ant algorithm saple testing program

```
public class anttest
{
        public static void main(String args[])
       {
       double evaporationrate=0.90; //pheromoneun buharlasma oran?
       double pheromone=5; //pheromone
       int totalant=1000;
       ant x=new ant();
       int[] tur=x.ant_optimization("cities.txt",evaporationrate,pheromone,totalant);
       for(int i=0;i<tur.length;i++){System.out.print(tur[i]+" ");}
       System.out.println("\nminimum mesafe = "+x.readMesafe());
       }
}
```

Sample test data cities.txt

```
100.0       0.0
100.0       0.0
95.10565162951535           30.901699437494738
80.90169943749474           58.778525229247315
58.778525229247315          80.90169943749474
```

30.901699437494745	95.10565162951535
6.123233995736766E-15	100.0
-30.901699437494734	95.10565162951536
-58.7785252292473	80.90169943749474
-80.90169943749473	58.77852522924732
-95.10565162951535	30.901699437494752
-100.0	1.2246467991473532E-14
-95.10565162951538	-30.90169943749469
-80.90169943749474	-58.7785252292473
-58.77852522924732	-80.90169943749473
-30.901699437494756	-95.10565162951535
-1.8369701987210297E-14	-100.0
30.901699437494724	-95.10565162951536
58.77852522924729	-80.90169943749476
80.90169943749473	-58.778525229247336
95.10565162951535	-30.901699437494763

Program output

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" anttest
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 0 1
minimum mesafe = 594.4509671528771

> Terminated with exit code 0.
```

LİNEAR MULTIVARIABLE OPTİMİZATION, SİMPLEX METHOD

As the last topic in optimization, Linear optimization will be investigated. Linear problems are existed only with the boundary restrictions. The general definition of the problem :

Maximum $Z=a_0x_0+ a_1x_1+ a_2x_2+ a_3x_3+\ldots\ldots+ a_nx_n$

Subject to:

$b_{00}x_0+ b_{01}x_0+ b_{02}x_0+ b_{03}x_0+\ldots\ldots+ b_{0n}x_0 <= A_0$

$b_{10}x_0+ b_{11}x_0+ b_{12}x_0+ b_{13}x_0+\ldots\ldots+ b_{1n}x_0 <= A_1$

$b_{20}x_0+ b_{21}x_0+ b_{22}x_0+ b_{23}x_0+\ldots\ldots+ b_{2n}x_0 <= A_2$

.....

$b_{n0}x_0+ b_{n,1}x_0+ b_{n2}x_0+ b_{n3}x_0+\ldots\ldots+ b_{nn}x_0 <= A_n$

If it is minimum instead of maximum we can simply change the sign of the function. An example case can be given as

Maximum Z=60x+50y

Subject to :

2x+4y<=80

3x+2y<=60

In order solve equation in Simplex method, form of the equation is changed as follows:

$Z-a_0x_0- a_1x_1- a_2x_2- a_3x_3-\ldots\ldots- a_nx_n = 0$

Subject to :

$b_{00}x_0+ b_{01}x_1+ b_{02}x_2+ b_{03}x_3+\ldots\ldots+ b_{0n}x_n + W_0 = A_0$

$b_{10}x_0+ b_{11}x_1+ b_{12}x_2+ b_{13}x_3+\ldots\ldots+ b_{1n}x_n + W_1 = A_1$

$b_{20}x_0+ b_{21}x_1+ b_{22}x_2+ b_{23}x_3+\ldots\ldots+ b_{2n}x_n + W_2 = A_2$

.....

$b_{n0}x_0+ b_{n,1}x_1+ b_{n2}x_2+ b_{n3}x_3+\ldots\ldots+ b_{nn}x_n + W_n = A_n$

where W's are additional slack variables. If the equation is written in matrix format

$$\begin{bmatrix} -a_0 & -a_1 & -a_2 & .... & -a_n & 0 & 0 & ... & 0 \\ b_{00} & b_{01} & b_{02} & .... & b_{0n} & 1 & 0 & ... & 0 \\ b_{10} & b_{11} & b_{12} & .... & b_{1n} & 0 & 1 & ... & 0 \\ ... & ... & ... & .... & ... & ... & ... & ... & ... \\ b_{n0} & b_{n1} & b_{n2} & .... & b_{nn} & 0 & 0 & ... & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ .... \\ x_n \\ W_0 \\ W_1 \\ W_2 \\ .... \\ W_n \end{bmatrix} = \begin{bmatrix} 0 \\ A_0 \\ A_1 \\ .... \\ A_n \end{bmatrix}$$

Due to inequality of the row and columns matrix can not be solved directly. Simplex method is used to tolve the system as follows:

1. In the main equation coefficient with the biggest negative number is selected. The column with this number is called critical column.
2. The value in the right hand side vector is divide by the value in the critical column. The row with the minimum value is selected as critical row.
3. Critical raw and column intersection point is called the critical value
4. Critical row is normalised by dividing to critical number
5. The following process applied to raws outside the critical row

   New value = old value – [value in the critical column and the same row]*[value in the critical raw in the same column] / critical number
6. Step 1 to 5 repeated until all values in the main equation becomes positive

We can look at the process through an example :
Maximum of Z=60x+50y
Subject to:
2x+4y<=80
3x+2y<=60
Simplex format :
$Z-60x-50y+0W_A+0W_B = 0$
$2x+4y + W_A+0W_B= 80$
$3x+2y +0W_A+ W_B=60$
matrix format:

$$\begin{bmatrix} -60 & -50 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 80 \\ 3 & 2 & 0 & 1 & 60 \end{bmatrix}$$

As the first iteration critical column is the column with number -60. In order to find the critical row t1=80/2=40 and t2=60/3=20 compared, since 20 is smaller raw 3 is selected as critical raw. Critical value is then selected as 3. Third row is diveded by 3 becomes

$$1 \quad \frac{2}{3} \quad 0 \quad \frac{1}{3} \quad 20$$

In the raws other than critical raw, a gauss elimination type process aplied to make column values 0
For example in order to make coefficient of x value in the first row equal to 0 following procedure is applied

3rd row(critical row)multiplied with 50 and Added to first row. In this case the values of the first row becomes:

$0 \quad -10 \quad 0 \quad 20 \quad 1200$ olacaktir.

At the end of these processes matrix becomes

$$\begin{bmatrix} 0 & -10 & 0 & 20 & 1200 \\ 0 & \frac{8}{3} & 1 & -\frac{2}{3} & 40 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 20 \end{bmatrix}$$

The same iterative process is repeated as a second time and result will be

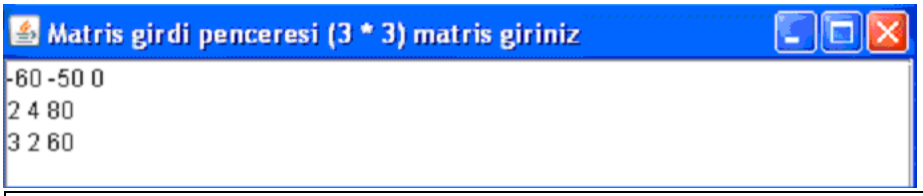
$$\begin{bmatrix} 0 & 0 & \frac{15}{4} & \frac{35}{2} & 1350 \\ 0 & 1 & \frac{3}{8} & -\frac{1}{4} & 15 \\ 1 & 0 & -\frac{1}{4} & \frac{1}{2} & 10 \end{bmatrix}$$

If Wa and Wb values are replaced with 0 value x=10 and y=15 is found. The function value becomes 1350 . Now the same problem will be solved by using the computer program

Program 5.26-1 Linear simplex optimization

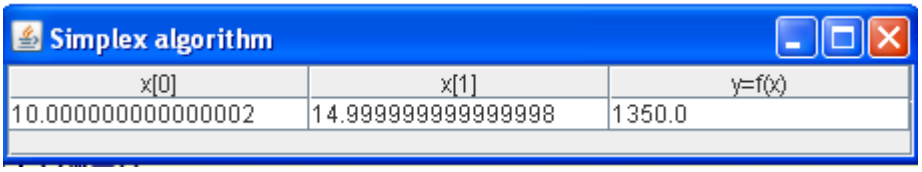
```
class simplex
{
    public static double[] simplex(double[][] a)
    {
    int n=a.length; //row
    int m=a[0].length; //column
    int m1=m-1;//
    int n1=n-1;
    int m2=m1+n1;
    int m3=m2+1;
    int keyrow=0;
    int keycolumn=0;
    double keynumber=0.0;
    double ratio=1.0;
    double b[][]=new double[n][m3];
    double d[]=new double[m];
    String s[]=new String[m];
    for(int i=0;i<n;i++)
    { b[i][m2]=a[i][m1]; }
    for(int i=0;i<n;i++)
    {for(int j=0;j<m1;j++) {b[i][j]=a[i][j];}}
    for(int i=0;i<n-1;i++)
      {
      for(int j=m1;j<m2;j++) {if(i==(j-m1)) b[i+1][j]=1.0;System.out.println("i="+i+"j="+j+"bij="+b[i+1][j]);}
      }
    double max;
    double min;
    System.out.println(Matrix.toString(b));
    for(int k=0;k<n1;k++)
    {
            max=-9.999e30;
    min=9.999e30;
    for(int j=0;j<m1;j++)
    {if( (b[0][j]<0) && Math.abs(b[0][j])>max) {max=Math.abs(b[0][j]);keycolumn=j;}}
    for(int i=1;i<n;i++)
    { ratio=b[i][m2]/b[i][keycolumn];
      if(ratio<min)  {min=ratio;keyrow=i;}
    }
```

```
    keynumber=b[keyrow][keycolumn];
    double c[][]=new double[n][m3];
    for(int i=0;i<n;i++)
    { for(int j=0;j<m3;j++) {c[i][j]=b[i][j];}}
    for(int i=0;i<n;i++)
    { if(i!=keyrow) {for(int j=0;j<m3;j++) {b[i][j]=c[i][j]-c[keyrow][j]*c[i][keycolumn]/keynumber;}}
      else {for(int j=0;j<m3;j++) {b[i][j]=c[i][j]/keynumber;}}
    }
    System.out.println("keynumber = "+keynumber);
    System.out.println("keyrow = "+keyrow+" keycolumn = "+keycolumn);
    System.out.println(Matrix.toString(b));
    }

    d[m1]=b[0][m2];
    s[m1]="y=f(x)";
    for(int j=0;j<m1;j++)
       { s[j]="x["+j+"]";}
    for(int i=1;i<n;i++)
    {
       for(int j=0;j<m1;j++)
       {
                    if((b[i][j]==1.0)&& (b[0][j]==0.0))
            { d[j]=b[i][m2];}
       }

    }
    Text.print(d,s,"Simplex algorithm");
    return d;
    }
            public static void main(String arg[])
            {
            double a[][]=Text.readDouble_TextArea();
            //double a[][]={{-60,-50,0},{2,4,80},{3,2,60}};
            double d[]=simplex(a);
            System.out.println(Matrix.toString(d));
            }
}
```




```
---------- Capture Output ----------
> "C:\co\java\bin\java.exe" OPO25
     -60.000000000000000      -50.000000000000000        0.000000000000000
0.000000000000000        0.000000000000000
       2.000000000000000        4.000000000000000          1.000000000000000
0.000000000000000          80.000000000000000
       3.000000000000000        2.000000000000000          0.000000000000000
1.000000000000000          60.000000000000000

keynumber = 3.0
keyrow = 2 keycolumn = 0
       0.000000000000000      -10.000000000000000        0.000000000000000
20.000000000000000     1200.000000000000000
```

```
      0.000000000000000      2.666666666666667      1.000000000000000      -
0.666666666666667     40.000000000000000
      1.000000000000000      0.666666666666667      0.000000000000000
0.333333333333333     20.000000000000000

keynumber = 2.666666666666667
keyrow = 1 keycolumn = 1
      0.000000000000000      0.000000000000000      3.750000000000000
17.500000000000000    1350.000000000000000
      0.000000000000000      1.000000000000000      0.375000000000000      -
0.250000000000000     14.999999999999998
      1.000000000000000      0.000000000000000     -0.250000000000000
0.500000000000000     10.000000000000002

     10.000000000000002            14.999999999999998            1350.000000000000000
```



6 Non-linear system of equations & solution methods

NEWTON-RAPHSON METHOD

One dimensional Newton-Raphson Formula was:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (4.1.1)$$

If the equation format changed a little, it can be written in the form of

$$\Delta x_n = x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} \qquad (4.1.2)$$

$$f'(x_n)\Delta x_n = -f(x_n) \qquad (4.1.3)$$

Now the same equation can be considered for a system of non-linear equation

$$[\nabla f] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (4.1.4) \qquad \{\Delta x\} = \begin{Bmatrix} \Delta x_1 \\ \Delta x_2 \\ \cdots \\ \Delta x_n \end{Bmatrix} \quad \{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{Bmatrix} \quad (4.1.5)$$

So multidimensional Newton-Raphson equation becomes

$$[\nabla f]\{\Delta x\} = -\{f\} \qquad (4.1.6)$$

The equation can be solved by using tecqniques such as gauss elimination. An initial estimate for all the x values are required to start iterative solution

First example function

$f_0(x_0,x_1)=x_0^2+ x_0\, x_1-10$

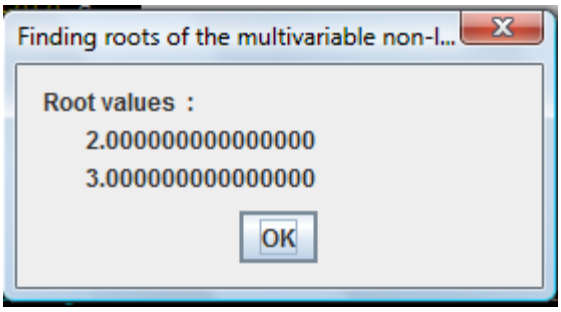
$f_1(x_0,x_1)=x_1+3\, x_0\, x_1^2-57$

$$[\nabla f] = \begin{bmatrix} 2x_0 + x_1 & x_0 \\ 3x_1^2 & 1 + 6x_0x_1 \end{bmatrix}$$

PROGRAM NA29 : Newton Raphson method both function and derivatives are defined , Java version

```
// NA29 Newton Raphson method
// to solve non-linear system of equation

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Tanimi gereken fonksiyon ================
abstract class fi_xi
{
  // multifunction multi independent variable
  // vector of dependent variables are returned
  // example f[0]=x[0]+sin(x[1])
  //         f[1]=x[0]*x[0]-x[1]
  // func(x) returns the value of f[0] and f[1]
  // as a two dimensional vector

  abstract double[] func(double x[]);
}

abstract class fij_xi
{
  // multifunction multi independent variable
  // for n independent variable n*n matrix of
  // dependent variables are returned
  // example f[0][0]=x[0]+Math.sin(x[1])   f[0][1]=x[0]-x[1]
  //         f[1][0]=x[0]*x[0]-x[1]        f[1][1]=Math.exp(x[0]+x[1]*x[1]
  // func(x) returns the value of f[0][0], f[0][1]
  //                              f[1][0], f[1][1]
  // as a two dimensional matrix
  abstract double[][] func(double x[]);
}

class fa extends fi_xi
{
public double[] func(double x[])
{
//system of non-linear equation
double ff[]=new double[2];
ff[0]=x[0]*x[0]+x[0]*x[1]-10.0;
ff[1]=x[1]+3.0*x[0]*x[1]*x[1]-57.0;
return ff;
}
}

class fb extends fij_xi
{
//derivative of system of non-linear equation
public double[][] func(double x[])
{
double a[][]=new double[2][2];
a[0][0]=2.0*x[0]+x[1];
a[0][1]=x[0];
a[1][0]=3.0*x[1]*x[1];
a[1][1]=1.0+6.0*x[0]*x[1];
return a;
} }

public class NA29
{
// Roots of system of non-linear equations
```

```
// By using Newton-Raphson Method
public static double[] newton(fi_xi f,fij_xi df,double x[])
{
//x vector of independent variables
//y vector of dependent variable
//dy derivative of vector of dependent variable
double ti=1.0;
int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
{
b[i]=1.0;
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{ b=Matrix.multiply(Matrix.divide(f.func(x),df.func(x)),-1.0);
  x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Maximum number of iterations are exceeded \n"+
            " Results might not be valid","MAXIMUM NUMBER OF ITERATIONS
ERROR",JOptionPane.WARNING_MESSAGE);
System.out.println();
return x;
}

public static double[] verigir()
{
   String s=JOptionPane.showInputDialog("ENTER FIRST ROOT ESTIMATES WİTH ONE EMPTY SPPACE IN
BETWEEN: ");
   StringTokenizer token=new StringTokenizer(s);
   int n=token.countTokens()-1;
   int m=n+1;
   double a[]=new double[m];
   int j=0;
   while(token.hasMoreTokens())
   {
   Double ax=new Double(token.nextToken());
   a[j++]=ax.doubleValue();
   }
   return a;
}

    public static void main (String args[])
    {
            fa f =new fa();
    fb df=new fb();
    double [] x0=verigir();
    double [] r= newton(f,df,x0);
    String s=" Root values  : \n"+Matrix.toStringT(r);
    String s1="Finding roots of the multivariable non-linear system of equations : ";
    JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



For initial guess x=[1 2], Octave-Matlab solution

```
function p=newton_raphson_n(f,x,tolerance,nmax)
% secant method for root finding
if nargin<5 nmax=100;end
if nargin<4 tolerance=1e-10;end
i=0;
fx=f(x);
while (abs(fx) > tolerance) && i<nmax
   fx=f(x);
   dfx=dfi(f,x);
   dx=dfx\fx';
   x=x-dx';
   i=i+1;
end
p=x;
if i>=nmax fprintf('Maximum number of iterations is exceeded the result may not be valid');end
end
```

```
##  M. Turhan Coban
function y = f1(x)
  y(1)=x(1)*x(1)+x(1)*x(2)-10.0;
  y(2)=x(2)+3.0*x(1)*x(2)*x(2)-57.0;
endfunction
```

```
>> y=@f1;
>> x=[1 2];
>> x=newton_raphson_n(y,x)
x =

   2   3

>>
```

Another example (Octave-Matlab code)

```
function y = f2(x)
  y(1)=3.0*x(1)-cos(x(2)*x(3))-0.5;
  y(2)=x(1)*x(1)-81.0*(x(2)+0.1)*(x(2)+0.1)+sin(x(3))+1.06;
  y(3)=exp(-x(1)*x(2))+20.0*x(3)+(10.0*pi-3.0)/3.0;
endfunction
```

```
>> format long
>> y=@f2;
>> x=[1 1 1];
>> x=newton_raphson_n(y,x)
x =

   4.999999999999999e-01  -1.062455701456471e-17  -5.235987755982988e-01
```

```
>>
```

PROGRAM Newton Raphson method both function and derivatives are defined , Julia version

```
 #function set
function func(x)
# system of non-linear equation
ff=zeros(Float64,2)
ff[1]=x[1]*x[1]+x[1]*x[2]-10.0
ff[2]=x[2]+3.0*x[1]*x[2]*x[2]-57.0
return ff
end

#single function with reference number i
function fi(f,x,i)
# system of non-linear equation
y=f(x)
return y[i]
end

#first derivative
function dfunc(func,x)
 global  dx=0.001
 global c=[1.0/280.0 -4.0/105.0          1.0/5.0    -4.0/5.0  0          4.0/5.0    -1.0/5.0   4.0/105.0 -1.0/280.0]
 n=length(x)
 ff=zeros(Float64,n,n)
 global yy=ff[1]
 x1=zeros(Float64,n)
 for ieqn=1:n
   for ix=1:n
     for i=-4:4
       # create x1set for derivative input
       # ================================
       for k=1:n
         xx=x[k]
         if k!=ix
           x1[k]=xx
         else
           delta=Float64(i)*dx
           x1[k]=xx+delta
         end
        end # for k
        # ================================
        ff1=fi(func,x1,ieqn)

        ff[ieqn,ix]+=c[i+5]* ff1
        yx=ff[ieqn,ix]
      end # for i
      ff[ieqn,ix]/=dx
      yx=ff[ieqn,ix]
    end # for ix
 end # for ieqn
 return ff
end # for function

function gauss_pivot(A1,b)
              n=size(A1)[1]
              carpan=0.0
              toplam=0.0
              global x=Array{Float64}(undef,n)
              global a=Array{Float64}(undef,n,n)
              for i=1:n
```

```
            for j=1:n
              a[i,j]=A1[i,j]
            end
          end
          for k=1:n
          #pivoting
            p=k
            buyuk=abs(a[k,k])
            for ii=k+1:n
              dummy=abs(a[ii,k])
              if dummy > buyuk
                buyuk=dummyp=ii
              end #if dummy
            end #for ii
            if p!=k
              for jj=k:n
                dummy=a[p,jj]
                a[p,jj]=a[k,jj]
                a[k,jj]=dummy
                end #for jj

              dummy=b[p]
              b[p]=b[k]
              b[k]=dummy
            end #if p!
          # gauss elimination
  for i=k+1:n
            carpan=a[i,k]/a[k,k]
            a[i,k]=0.0
            for j=k+1:n
              a[i,j]=a[i,j]-carpan*a[k,j]
            end #for j
            b[i]=b[i]-carpan*b[k]
          end #for i
          end #k
                    # back substituting
            x[n]=b[n]/a[n,n]
            n1=n-1
            for i=n1:-1:1
              toplam=0.0
              for j=i+1:n
                toplam=toplam+a[i,j]*x[j]
              end #for j
              x[i]=(b[i]-toplam)/a[i,i]
            end #for i
          return x
end #function

function newton(func,x)
  # Newton-Raphson root finding method
  k=length(x)
  nmax=100
  tolerance=1.0e-10
  for i=1:nmax
    fx=func(x)
    dfx=dfunc(func,x)
    dx=gauss_pivot(dfx,fx)
    for j=1:k
       x[j]=x[j]-dx[j]
    end # for j
    total=0.0;
    for j=1:k
```

```
        total+=fx[j]
      end # for j
      if abs(total)<tolerance
        return x
      end
    end #for i
    return x
end # function


x=[1.0 1.0]
y=func(x)
dy=dfunc(func,x)
println("x = $x \ny = $y dy = $dy")
r=newton(func,x)
println("r = $r")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_newton.jl
x = [1.0 1.0]
y = [-8.0, -53.0] dy = [2.999999999998608 1.0000000000000218; 2.9999999999981983 6.999999999992484]
r = [2.0 3.0]

> Terminated with exit code 0.
```

If the same problem is solved by hand starting from the initial guess x={1,2}

Newton-Raphson Non-linear equation solution

Initial guess

x0	1
x1	2

	4	1	dx0	7
	12	13	dx1	43

3	4	1	dx0	7
	0	10	dx1	22

dx0	1.2
dx1	2.2

First iteration result

x0	2.2
x1	4.2

	8.6	2.2	dx0	-4.08
	52.92	56.44	dx1	-63.624

6.153488	8.6	2.2	dx0	-4.08
	0	42.90233	dx1	-38.5178

dx0	-0.24475
dx1	-0.8978

Second iteration results

x0	1.955252
x1	3.302199

	7.212702	1.955252	dx0	-0.27964
	32.71355	39.73977	dx1	-10.2654

4.535547	7.212702	1.955252	dx0	-0.27964
	0	27.70816	dx1	-8.54467

dx0	0.044827
dx1	-0.30838

Third iteration results

x0	2.000079
x1	2.993818

	6.993975	2.000079	dx0	0.011814

	26.88883	36.92723	dx1	0.2264

3.844571	6.993975	2.000079	dx0	0.011814
	0	29.23778	dx1	0.180979

dx0	-8.1E-05
dx1	0.00619

x0	1.999998
x1	3.000008

	7.000003	1.999998	dx0	4.94E-07
	27.00014	37.00005	dx1	-0.00022

3.857161	7.000003	1.999998	dx0	4.94E-07
	0	29.28574	dx1	-0.00022

dx0	2.24E-06
dx1	-7.6E-06

Fourth iteration results

x0	2
x1	3

	7	2	dx0	1.2E-11
	27	37	dx1	-4E-11

3.857143	7	2	dx0	1.2E-11
	0	29.28571	dx1	-8.6E-11

dx0	2.56E-12
dx1	-2.9E-12

Fifth iteration results

x0	2
x1	3

PROGRAM Newton Raphson method only function are defined, derivatives are taken numerically

```
// NA30 Newton Raphson method
// to solve non-linear system of equation

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Tanimi gereken fonksiyon ================
abstract class fi_xi
{
  // multifunction multi independent variable
  // vector of dependent variables are returned
  // example f[0]=x[0]+sin(x[1])
  //         f[1]=x[0]*x[0]-x[1]
  // func(x) returns the value of f[0] and f[1]
  // as a two dimensional vector

  abstract double[] func(double x[]);
}

class fa extends fi_xi
{
public double[] func(double x[])
{
//system of non-linear equation
double ff[]=new double[2];
ff[0]=x[0]*x[0]+x[0]*x[1]-10.0;
ff[1]=x[1]+3.0*x[0]*x[1]*x[1]-57.0;
return ff;
}
}

public class NA30
```

```
{
// Roots of system of non-linear equations
// By using Newton-Raphson Method

public static double[] newtond( fi_xi f,double x[])
{
//x vector of independent variables
//y vector of dependent variable
//dy derivative of vector of dependent variable
// derivative vector calculated by numerical derivatives
double ti=1.0;
int i,ii,jj;
int nmax=500;
double tolerance=1.0e-15;
int n=x.length;
double b[];
b=new double[n];
double dy[][];
dy=new double[n][n];
i=0;
for(i=0;i<n;i++)
{
b[i]=1.0;
}
while( i++ < nmax && Matrix.abs(b) > tolerance )
{
  for(ii=0;ii<n;ii++)
  {
    for(jj=0;jj<n;jj++)
    {
       dy[ii][jj]=derivative(f,x,ii,jj);
    }
  }
    b=Matrix.multiply(Matrix.divide(f.func(x),dy),-ti);
    x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Maximum number of iterations are exceeded \n"+
            " Results might not be valid","MAXIMUM NUMBER OF ITERATIONS ERROR",JOptionPane.WARNING_MESSAGE);
return x;
}

public static double derivative(fi_xi f,double x[],int denklem_ref,int x_ref)
{
double h0=0.0256;
int i,m;
int n=7;
double f1[];
f1=new double[x.length];
double f2[];
f2=new double[x.length];
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
```

```
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
f1=f.func(x1);
f2=f.func(x2);
T[i][0]=( f1[denklem_ref] - f2[denklem_ref])/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}

public static double[] verigir()
{
    String s=JOptionPane.showInputDialog("ENTER FIRST ROOT ESTIMATES WİTH ONE EMPTY SPPACE IN BETWEEN: ");
    StringTokenizer token=new StringTokenizer(s);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
}

     public static void main (String args[])
     {
                fa f =new fa();
     double [] x0=verigir();
     String s2="Newton raphson method with the numerically calculated derivatives : ";
     double [] r1= newtond(f,x0);
     String s=" root vector  : \n"+Matrix.toStringT(r1);
     JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
     System.exit(0);
     }
}
```

The problem steps including numerical derivative is evaluated step by step as :

Newton-Raphson Non-linear equation solution

Initial guess

x0	1
x1	2

4.0000	1.0000	dx0	7.0000
12.0000	13.0000	dx1	43.0000

3	4	1	dx0	7

dx0	0.01				
x0+dx0	1.01	f1(x0+dx0,x1)	-6.9599	f2(x0+dx0,x1)	-42.88
x0+2dx0	1.02	f1(x0+2dx0,x1)	-6.9196	f2(x0+2dx0,x1)	-42.76
x0-dx0	0.99	f1(x0-dx0,x1)	-7.0399	f2(x0-dx0,x1)	-43.12
x0-2dx0	0.98	f1(x0-2dx0,x1)	-7.0796	f2(x0-2dx0,x1)	-43.24
x1+dx1	2.01	f1(x0,x1+dx1)	-6.99	f2(x0,x1+dx1)	-42.8697
x1+2dx1	2.02	f1(x0,x1+2dx1)	-6.98	f2(x0,x1+2dx1)	-42.7388
x1-dx1	1.99	f1(x0,x1-dx1)	-7.01	f2(x0,x1-dx1)	-43.1297

	0	10	dx1	22	x1-2dx1	1.98	f1(x0,x1-2dx1)	-7.02	f2(x0,x1-2dx1)	-43.2588

dx0	1.2
dx1	2.2

First iteration result

x0	2.2
x1	4.2

	8.6000	2.2000	dx0	-4.0800
	52.9200	56.4400	dx1	-63.6240
6.153488	8.6000	2.2000	dx0	-4.0800
	0	42.90233	dx1	-38.51777

					dx0	0.01				
					x0+dx0	2.21	f1(x0+dx0,x1)	4.1661	f2(x0+dx0,x1)	64.1532
					x0+2dx0	2.22	f1(x0+2dx0,x1)	4.2524	f2(x0+2dx0,x1)	64.6824
					x0-dx0	2.19	f1(x0-dx0,x1)	3.9941	f2(x0-dx0,x1)	63.0948
					x0-2dx0	2.18	f1(x0-2dx0,x1)	3.9084	f2(x0-2dx0,x1)	62.5656
					x1+dx1	4.21	f1(x0,x1+dx1)	4.102	f2(x0,x1+dx1)	64.18906
					x1+2dx1	4.22	f1(x0,x1+2dx1)	4.124	f2(x0,x1+2dx1)	64.75544
					x1-dx1	4.19	f1(x0,x1-dx1)	4.058	f2(x0,x1-dx1)	63.06026
					x1-2dx1	4.18	f1(x0,x1-2dx1)	4.036	f2(x0,x1-2dx1)	62.49784

dx0	-0.244748
dx1	-0.897801

Second iteration results

x0	1.955252
x1	3.302199

	7.2127	1.9553	dx0	-0.279637
	32.7135	39.7398	dx1	-10.26541
4.535547	7.212702	1.955252	dx0	-0.279637
	0	27.70816	dx1	-8.544666

					dx0	0.01				
					x0+dx0	1.9653	f1(x0+dx0,x1)	0.351864	f2(x0+dx0,x1)	10.59255
					x0+2dx0	1.9753	f1(x0+2dx0,x1)	0.424291	f2(x0+2dx0,x1)	10.91968
					x0-dx0	1.9453	f1(x0-dx0,x1)	0.20761	f2(x0-dx0,x1)	9.938276
					x0-2dx0	1.9353	f1(x0-2dx0,x1)	0.135783	f2(x0-2dx0,x1)	9.61114
					x1+dx1	3.3122	f1(x0,x1+dx1)	0.29919	f2(x0,x1+dx1)	10.6634
					x1+2dx1	3.3222	f1(x0,x1+2dx1)	0.318742	f2(x0,x1+2dx1)	11.06255
					x1-dx1	3.2922	f1(x0,x1-dx1)	0.260085	f2(x0,x1-dx1)	9.8686
					x1-2dx1	3.2822	f1(x0,x1-2dx1)	0.240532	f2(x0,x1-2dx1)	9.472962

dx0	0.044827
dx1	-0.308381

Third iteration results

x0	2.000079
x1	2.993818

	6.9940	2.0001	dx0	0.011814
	26.8888	36.9272	dx1	0.2264
3.844571	6.993975	2.000079	dx0	0.011814
	0	29.23778	dx1	0.180979

					dx0	0.01				
					x0+dx0	2.0101	f1(x0+dx0,x1)	0.058225	f2(x0+dx0,x1)	0.042488
					x0+2dx0	2.0201	f1(x0+2dx0,x1)	0.128465	f2(x0+2dx0,x1)	0.311377
					x0-dx0	1.9901	f1(x0-dx0,x1)	-0.081654	f2(x0-dx0,x1)	-0.495288
					x0-2dx0	1.9801	f1(x0-2dx0,x1)	-0.151294	f2(x0-2dx0,x1)	-0.764177
					x1+dx1	3.0038	f1(x0,x1+dx1)	0.008186	f2(x0,x1+dx1)	0.143472
					x1+2dx1	3.0138	f1(x0,x1+2dx1)	0.028187	f2(x0,x1+2dx1)	0.514545
					x1-dx1	2.9838	f1(x0,x1-dx1)	-0.031815	f2(x0,x1-dx1)	-0.595072
					x1-2dx1	2.9738	f1(x0,x1-2dx1)	-0.051816	f2(x0,x1-2dx1)	-0.962544

dx0	-8.09E-05
dx1	0.00619

x0	1.999998
x1	3.000008

Solution given in python language

PROGRAM: fi_xj abstract class to define function and derivatives of a vector function in python

```
from abc import ABC, abstractmethod
from math import *

class fi_xi(ABC):
  # multifunction multi independent variable
  # vector of dependent variables are returned
  # example f[0]=x[0]+sin(x[1])
  #         f[1]=x[0]*x[1]-x[1]
  # func(x) returns the value of f[0] and f[1]
  # as a two dimensional vector

  @abstractmethod
  def func(self,x):
    pass;

  def xp(self,x,j,m):
     k=len(x);
     h=0.001;
     xx=[0 for i in range(k)];
     for i in range(k):
        if i==m:
           xx[i]=x[i]+j*h;
        else: xx[i]=x[i];
```

```
    return xx;

 def dfunc(self,x,n):
    h=0.001;
    k=len(x);
    df=[[0 for i in range(k)] for j in range(k)];
    for i in range(k):
       for j in range(k):
          df[i][j]=self.dfun(x,n,i,j);
    return df;

 def dfun(self,x,n,j,m):
    k=len(x);
    ff=[[0 for i2 in range(k)] for j2 in range(9)];
    xp1=[[0 for i2 in range(k)] for j2 in range(9)];
    h=0.001;
    hh=1/h;
    x1=x[m];
    for jj in range(-4,5):
      j1=jj+4;
      xp1[j1]=self.xp(x,jj,m);
      ff[j1]=self.func(xp1[j1]);
    if n==0: df=ff[4][j];
    elif n==1:
       df=(3.0*ff[0][j]-32.0*ff[1][j]+168.0*ff[2][j]-672.0*ff[3][j]+672.0*ff[5][j]-168.0*ff[6][j]+32.0*ff[7][j]-3.0*ff[8][j])/840.0*hh;
    elif n==2:
       df=(-14350.0*ff[4][j]-9.0*ff[0][j]+128*ff[1][j]-1008*ff[2][j]+8064*ff[3][j]+8064.0*ff[5][j]-1008.0*ff[6][j]+128.0*ff[7][j]-
9.0*ff[8][j])/5040.0*hh*hh;
    elif n==3:
       df=(-7.0*ff[0][j]+72.0*ff[1][j]-338.0*ff[2][j]+488.0*ff[3][j]-488.0*ff[5][j]+338.0*ff[6][j]-72.0*ff[7][j]+7.0*ff[8][j])/240.0*hh*hh*hh;
    elif n==4:
       df=(2730.0*ff[4][j]+7.0*ff[0][j]-96.0*ff[1][j]+676.0*ff[2][j]-1952.0*ff[3][j]-1952.0*ff[5][j]+676.0*ff[6][j]-
96.0*ff[7][j]+7.0*ff[8][j])/240.0*hh*hh*hh*hh;
    elif n==5:
       df=(ff[0][j]-9.0*ff[1][j]+26.0*ff[2][j]-29.0*ff[3][j]+29.0*ff[5][j]-26.0*ff[6][j]+9.0*ff[7][j]-ff[8][j])/6.0*hh*hh*hh*hh*hh;
    elif n==6:
       df=(-150.0*ff[4][j]-ff[0][j]+12.0*ff[1][j]-52.0*ff[2][j]+116.0*ff[3][j]+116.0*ff[5][j]-52.0*ff[6][j]+12.0*ff[7][j]-
ff[8][j])/4.0*hh*hh*hh*hh*hh*hh;
    elif n==7:
       df=(-ff[0][j]+6.0*ff[1][j]-14.0*ff[2][j]+14.0*ff[3][j]-14.0*ff[5][j]+14.0*ff[6][j]-6.0*ff[7][j]+ff[8][j])/2.0*hh*hh*hh*hh*hh*hh*hh;
    elif n==8:
       df=(70.0*ff[4][j]+ff[0][j]-8.0*ff[1][j]+28.0*ff[2][j]-56.0*ff[3][j]-56.0*ff[5][j]+28.0*ff[6][j]-
8.0*ff[7][j]+ff[8][j])*hh*hh*hh*hh*hh*hh*hh*hh;
    else: df=0;
    return df;
```

PROGRAM 4.1-4 : newtond.py Newton-raphson meyhod to calculate non-linear roots in python

```
from math import *
from fi_xi import *;
from gauss import *;

class f1(fi_xi):
 def func(self,x):
    y=[0 for i in range(2)];
    y[0]=x[0]*x[0]+x[0]*x[1]-10.0;
    y[1]=x[1]+3.0*x[0]*x[1]*x[1]-57.0;
    return y;

def newton(f,x):
  # Newton-Raphson root finding method
  k=len(x);
  nmax=100;
  tolerance=1.0e-10
  for i in range(nmax):
    fx= f.func(x);
    dfx=f.dfunc(x,1);
    dx=gauss(dfx,fx);
    for j in range(k):
       x[j]=x[j]-dx[j];
    total=0;
    for j in range(k):
       total+=fx[j];
    if abs(total)<tolerance: return x
  return x;
```

```
f=f1();
x=[0 for i in range(2)];
x[0]=1.0;
x[1]=2.0;
r=newton(f,x)
print("r=",r);
```

```
runfile('D:/okul/SCO1/newtond.py', wdir='D:/okul/SCO1')
r= [2.0, 3.0]
```

4.2 BROYDEN SECANT METHOD WITH SHARMAN-MORRISON FORMULA

We would like to solve f(x)=0 linear system of equation. In these equations x={x1, x1,…, xn-1, xn} is a multidimensional independent variable set. As a second way of solving this system of equations secant method can be used instead of Newton-Raphson method. In Newton-Raphson approach was defined as :

$$[\nabla f] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (4.2.1) \qquad \{\Delta x\} = \begin{Bmatrix} \Delta x_1 \\ \Delta x_2 \\ \cdots \\ \Delta x_n \end{Bmatrix} \quad \{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{Bmatrix} \quad (4.2.2)$$

equation is used. This equation can be change to a new format:

$$[\nabla f]\{\Delta x\} = -\{f\} \qquad (4.2.3)$$

In this set derivative term $[\nabla f]$ be replaced by difference (numerical derivative)

$$\frac{\partial f_j(x^{(i)})}{\partial x_k} \approx \frac{f_j(x^{(i)}+e_k h)-f_j(x^{(i)})}{h} \quad (4.2.3)$$

h in the equation is finite difference parameter (Δx) and e_k is unit vektor (kth component is 1 the other components are 0). i is the iteration number. Note that secant method is always approach to the roots less quickly compare to Newton-Raphson method. In order to improve number of iterations for the solutions, a modified version of secant method, Broyden method will be used. In Broyden method when x(0) is a given value, first iteration, x(1), will be calculated by using Newton-Raphson method. For the following steps

$\left[\nabla f(x^{(0)})\right]\{x^{(1)} - x^{(0)}\} = -\{f(x^{(0)})\}$ (4.2.4) form of the equation can be changed to be

$[A_1]\{x^{(1)} - x^{(0)}\} = \{f(x^{(1)}) - f(x^{(0)})\}$ (4.2.5) In order to have this equation valid for every vector z, vector should satisfy following equations

$[A_1]\{z\} = \left[\nabla f(x^{(0)})\right]z$ (4.2.6) $\{x^{(1)} - x^{(0)}\}z = 0$ (4.2.7)

By using this defination Formula to calculate $[A_1]$ can be derived.

$$[A_1] = \left[\nabla f(x^{(0)})\right] + \frac{[\{f(x^{(1)})-f(x^{(0)})\}-\nabla f(x^{(0)})\{x^{(1)}-x^{(0)}\}]\{x^{(1)}-x^{(0)}\}^t}{\|x^{(1)}-x^{(0)}\|_2^2} \qquad (4.2.8)$$

Where the term $\|x^{(1)} - x^{(0)}\|_2$ (4.2.9) is a two norm. Norms was defined in previous chapter. After finding $[A_1]$ new value for vector x can be constructed.

$$x^{(2)} = x^{(1)} - [A_1]^{-1} f(x^{(1)}) \quad (4.2.10)$$

When $x^{(2)}$ is found, this value can be used to calculate $x^{(3)}$ .General Formula will become:

$$[A_i] = [A_{i-1}] + \frac{[\{f(x^{(i)})-f(x^{(i-1)})\}-\nabla f(x^{(i-1)})\{x^{(i)}-x^{(i-1)}\}]\{x^{(i)}-x^{(i-1)}\}^t}{\|x^{(i)}-x^{(i-1)}\|_2^2} \qquad (4.2.11)$$

$$x^{(i+1)} = x^{(i)} - [A_i]^{-1} f(x^{(i)}) \qquad (4.2.12)$$

One difficulty in this equation is the requirement of solving inverse of the matrix each time which requires solving of system of equation by a method such as Gauss elimination. But Gauss method is expensive in calculation time. A beter alternative could be using Sherman-Morrison approach. This approach calculates inverse directly as an iterative process. Matrix A_i^{-1} can directly be calculated from value A_{i-1}^{-1}. Equations of Sherman-Morrison approach is as follows

$$s_i = \left\{x^{(i)} - x^{(i-1)}\right\} \qquad (4.2.13)$$

$$y_i = \left\{f\left(x^{(i)}\right) - f\left(x^{(i-1)}\right)\right\} \qquad (4.2.14)$$

$$([\mathrm{A}] + xy^t)^t = [A]^{-1} - \frac{[\mathrm{A}]^{-1}xy^t[\mathrm{A}]^{-1}}{1+y^t[\mathrm{A}]^{-1}x} \qquad (4.2.15)$$

$$[A_i]^{-1} = \left([A_{i-1}]^{-1} + \frac{y_i - [A_{i-1}]s_i}{\|s_i\|_2^2} s_i^t\right)^{-1} \qquad (4.2.16)$$

$$[A_i]^{-1} = [A_i]^{-1} + \frac{[A_i]^{-1}\left(\frac{y_i - [A_{i-1}]s_i}{\|s_i\|_2^2}\right)[A_i]^{-1}}{1+s_i^t[A_i]^{-1}\left(\frac{y_i - [A_{i-1}]s_i}{\|s_i\|_2^2}\right)} s_i^t \qquad (4.2.17)$$

$$[A_i]^{-1} = [A_i]^{-1} + \frac{\left(s_i - [A_{i-1}]^{-1}y_i\right)[A_i]^{-1}}{s_i^t[A_{i-1}]^{-1}y_i} \qquad (4.2.18)$$

Due to the fact that this equation only involves multiplication processes, it is easier to use compare to Gauss elimination method. Algorithm of the method is given below.

Program Broyden secant method with sherman-Morrison Formula.

```
 // NA30A Broyden secant method with Sherman-Morrison formula to solve inverse matrix
// Solution of non-linear system of equations.
// Referans : Numerical Analysis, 8th edition Richard L. Burden, J. Douglas Faires 3
// page 617
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Tanimi gereken fonksiyon ================
abstract class fi_xi
{
  // multifunction multi independent variable
  // vector of dependent variables are returned
  // example f[0]=x[0]+sin(x[1])
  //         f[1]=x[0]*x[0]-x[1]
  // func(x) returns the value of f[0] and f[1]
  // as a two dimensional vector

  abstract double[] func(double x[]);
}

abstract class fij_xi
{
  // multifunction multi independent variable
  // for n independent variable n*n matrix of
  // dependent variables are returned
  // example f[0][0]=x[0]+Math.sin(x[1])   f[0][1]=x[0]-x[1]
  //         f[1][0]=x[0]*x[0]-x[1]        f[1][1]=Math.exp(x[0]+x[1]*x[1]
  // func(x) returns the value of f[0][0], f[0][1]
  //                              f[1][0], f[1][1]
```

```
  // as a two dimensional matrix
  abstract double[][] func(double x[]);
}

class fa extends fi_xi
{
public double[] func(double x[])
{
//function to be solved.
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
}
}

class fb extends fij_xi
{
fi_xi f;

public fb(fi_xi fi)
{f=fi;}

public double[][] func(double x[])
{
  int n=x.length;
  double a[][]=new double[n][n];
  for(int i=0;i<n;i++)
  {
    for(int j=0;j<n;j++)
    {
       a[i][j]=derivative(x,i,j);
    }
  }
return a;
}

public double derivative(double x[],int denklem_ref,int x_ref)
{
// df(x)/dx[x_ref]  x_ref=0...n
double h0=0.0156;
int i,m;
int n=7;
double f1[];
f1=new double[x.length];
double f2[];
f2=new double[x.length];
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//derivative of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
```

```
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
f1=f.func(x1);
f2=f.func(x2);
T[i][0]=( f1[denklem_ref] - f2[denklem_ref])/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}

}

public class NA30A
{
// SCO5E Sherman-Morrison formülü destekli broyden sekant metoduyla
//denklem sistemlerinin köklerini (çözümünü) bulmak

public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {b[i]=left*right[i];}
return b;
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
     b[ii]=0;
   }
   return b;
```

```
 }
 for(i=0;i<m1;i++)
   {
     b[i]=0;
     for(k=0;k<n1;k++)
              b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}

public static double[] multiply(double[] left,double[][] right)
{
//multiplication of one vector with one matrix
int ii,jj,i,j,k;
int m2=right[0].length;
int n2=right.length;
int m1=left.length;
double[] b;
b=new double[m1];
 if(n2 != m1)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n2;ii++)
   {
     b[ii]=0;
   }
   return b;
 }
 for(i=0;i<m2;i++)
   {
     b[i]=0;
     for(k=0;k<m1;k++)
              b[i]+=right[i][k]*left[k];
   }
return b;
//end of multiply of a vector and a matrix
}

public static double  VT_X(double [] left,double [] right)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}

public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
```

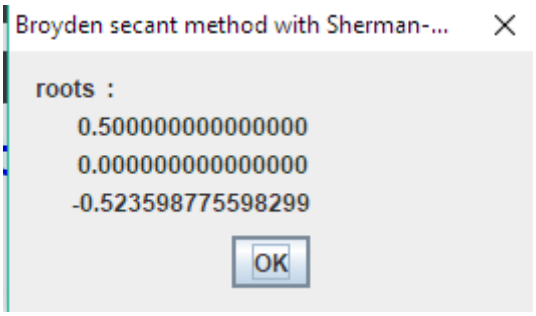
```
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

public static double[][] multiply(double left,double[][] right)
{
//multiplying a matrix with a constant
int i,j;
int n=right.length;
int m=right[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=right[i][j]*left;
  }
return b;
//end of multiplying a matrix with a constant double
}

public static double[][] add(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
  for(j=0;j<m1;j++)
    {
    b[i][j]=b[i][j]+left[i][j];
    }
  }
for(i=0;i<n2;i++)
  {
  for(j=0;j<m2;j++)
    {
    b[i][j]=b[i][j]+right[i][j];
    }
  }
return b;
//end of matrix addition method
}

public static double[] substract(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
```

```
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]-right[i];
  }
return b;
//end of vector addition method
}

public static double[][] inv(double[][] a)
{
// INVERSION OF A MATRIX
// inversion by using gaussian elimination
// with full pivoting
int n=a.length;
int m=a[0].length;
double b[][];
b=new double[n][n];
int indxc[];
int indxr[];
double ipiv[];
indxc=new int[n];
indxr=new int[n];
ipiv=new double[n];
int i,j,k,l,ll,ii,jj;
int icol=0;
int irow=0;
double big,dum,pivinv,temp;
if(n!=m)
{
   System.out.println("Matrix must be square ");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                          b[ii][jj]=0.0;
                return b;
}
for(i=0;i<n;i++)
   for(j=0;j<n;j++)
       b[i][j]=a[i][j];
for(i=0;i<n;i++)
{
  big=0.0;
  for(j=0;j<n;j++)
  {
    if(ipiv[j] != 1)
      for(k=0;k<n;k++)
      {
              if(ipiv[k] == 0)
              {
                if(Math.abs(b[j][k]) >= big)
                {
                  big=Math.abs(b[j][k]);
                  irow=j;
                  icol=k;
                }
              }
              else if(ipiv[k] > 1 )
                {
                System.out.println("error : inverse of the matrix : singular matrix-1");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                          b[ii][jj]=0.0;
                return b;
```

```
                }
      }
  }
  ++ ipiv[icol];
  if(irow != icol)
    for(l=0;l<n;l++)
    {
    temp=b[irow][l];
    b[irow][l]=b[icol][l];
    b[icol][l]=temp;
    }
  indxr[i]=irow;
  indxc[i]=icol;
  if(b[icol][icol] == 0.0)
  {
    System.out.println("error : inverse of the matrix : singular matrix-2");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                          b[ii][jj]=0.0;
    return b;
  }
  pivinv=1.0/b[icol][icol];
  b[icol][icol]=1.0;
  for(l=0;l<n;l++) b[icol][l] *=pivinv;
  for(ll=0;ll<n;ll++)
  if(ll != icol)
  {
    dum=b[ll][icol];
    b[ll][icol]=0.0;
    for(l=0;l<n;l++) b[ll][l]-= b[icol][l]*dum;
  }
}
for(l=n-1;l>=0;l--)
{
  if(indxr[l] != indxc[l])
     for(k=0;k<n;k++)
     {
       temp=b[k][indxc[l]];
       b[k][indxc[l]]=b[k][indxr[l]];
       b[k][indxr[l]]=temp;
     }
}
return b;
}
public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

public static double[][] multiply(double[] left,double[] right)
{
//multiplication of one vector with one matrix
int ii,jj,i,j,k;
int m2=left.length;
int n2=right.length;
int m1=left.length;
double[][] b;
b=new double[m2][n2];
 for(i=0;i<m2;i++)
   {
     for(k=0;k<n2;k++)
                b[i][k]=left[i]*right[k];
   }
```

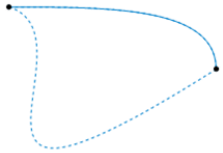
```
return b;
//end of multiply of a vector and a matrix
}

public static double[] broyden(fi_xi f,double x[])
{
//=====================================================
// Roots of non-linear system of equations
// Broyden secant method
fb J=new fb(f);
int i;
int nmax=400;
double tolerance=1.0e-15;
int n=x.length;
double b[]=new double[n];
double v[]=new double[n];
double s[]=new double[n];
double w[]=new double[n];
double y[]=new double[n];
double z[]=new double[n];
double p;
double ut[]=new double[n];
double x1[]=new double[n];
double A[][]=new double[n][n];
x1=x;
A=J.func(x1);
v=f.func(x1);
A=inv(A);
s=multiply(-1.0,multiply(A,v));
x=add(x,s);
int k=1;
while(k<nmax)
{
w=v;
v=f.func(x);
y=substract(v,w);
z=multiply(-1.0,multiply(A,y));
p=-VT_X(s,z);
ut=multiply(s,A);  //*********
A=add(A,multiply((1.0/p),multiply(add(s,z),ut)));
s=multiply(-1.0,multiply(A,v));
x=add(x,s);
if(norm(s)<tolerance) break;
k++;
}
return x;
}

public static double[] enterdata()
{
   String s=JOptionPane.showInputDialog("enter first gueses for roots with a sapce in between: ");
   StringTokenizer token=new StringTokenizer(s);
   int n=token.countTokens()-1;
   int m=n+1;
   double a[]=new double[m];
   int j=0;
   while(token.hasMoreTokens())
   {
   Double ax=new Double(token.nextToken());
   a[j++]=ax.doubleValue();
   }
   return a;
}

    public static void main (String args[])
    {
            fa f =new fa();
```

```
    double [] x0=enterdata();
    double [] r= broyden(f,x0);
    String s=" roots  : \n"+Matrix.toStringT(r);
    String s1="Broyden secant method with Sherman-Morrison iterative inverse matrix formula : ";
    JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



HOMOTOPY OR CONTINUATION METHODS

In topology, two continuous functions from one topological space to another are called **homotopic**(from Greek ὁμός homós "same, similar" and τόπος tópos "place") if one can be "continuously deformed" into the other, such a deformation being called a **homotopy** between the two functions.



When a problem of system of nonlinear equations of the form F(x)=0 desired to be solved, another interesting solution method is Homotopy or Continuation method. Assume that solution set to be found is x*. Consider a parametric function G(λ,x) in the form of

G(λ,x) = λF(x) + (1-λ)[F(x) - F(x(0))] (4.3.1)

Where λ=0 coresponds to initial guess of the solution , x(0) ,and where λ=1 value correspods the actual solution set x(1)= x* (4.3.2)

It is desired to be found G(λ,x) = 0 therefore for λ=0 equation becomes

G(λ,x) =G(0,x) = F(x) - F(x(0)) and for λ=1 (4.3.4)

0=G(1,x) = F(x) (4.3.4)

Therefore x(1)=x* solution set will be obtained. If a function G(λ,x) satisfies the above equation can be found, it will also give us the solution. Function G is called a homotopy between the function G(0,x) and G(1,x)=F(x) (4.3.5)

In order to find such a function, it is assumed to have a function

G(λ,x)=0 is existed and partial derivative of this function with respect to λ and x will also be zero

$$\frac{\partial G(\lambda,x)}{\partial \lambda} + \frac{\partial G(\lambda,x)}{\partial x} x'(\lambda) = 0 \qquad (4.3.6)$$

if x'(λ) is isolated form this equation, it becomes:

$$x'(\lambda) = -\left[\frac{\partial G(\lambda,x)}{\partial x}\right]^{-1}\left[\frac{\partial G(\lambda,x)}{\partial \lambda}\right] \quad (4.3.7)$$

If $G(\lambda,x)=\lambda F(x)+(1-\lambda)[F(x)-F(x(0))]$ the equation is substituted into the differential equation

$$\left[\frac{\partial G(\lambda,x)}{\partial \lambda}\right] = \begin{bmatrix} \frac{\partial f_1(x(\lambda))}{\partial x_1} & \frac{\partial f_1(x(\lambda))}{\partial x_2} & \frac{\partial f_1(x(\lambda))}{\partial x_{33}} \\ \frac{\partial f_2(x(\lambda))}{\partial x_1} & \frac{\partial f_2(x(\lambda))}{\partial x_2} & \frac{\partial f_2(x(\lambda))}{\partial x_3} \\ \frac{\partial f_3(x(\lambda))}{\partial x_1} & \frac{\partial f_3(x(\lambda))}{\partial x_2} & \frac{\partial f_3(x(\lambda))}{\partial x_3} \end{bmatrix} = J(x(\lambda)) \quad (4.3.8)$$

Forms a Jacobian matrix. and

$\left[\frac{\partial G(\lambda,x)}{\partial x}\right] = F(x(0))$ (4.3.9)

Differential equation becomes

$x'(\lambda) = \frac{dx(\lambda)}{d\lambda} = -[J(x(\lambda)]^{-1}F(x(0))$ $0 \leq \lambda \leq 1$ (4.3.10)

It is possible to solve such a differential equation by using initial value problem approaches, solution at x(1) will be given us the roots of the system of equation. Solutions of initial value problems will be given latter chapters in details, but A fourth order Runge-Kutta solution will be defined here to solve our homotopy problem. If equation

$\frac{dx(\lambda)}{d\lambda} = f(\lambda,x(\lambda))$ (4.3.11) is given the 4th order Runge-Kutta method to numerically solve this differential equation is defined as:

$x_{i+1} = x_i + (1/6)*(k_1 + 2k_2 +2k_3+k_4)h$ (4.3.12)

$k_1=f(\lambda_i,x_i)$

$k_2=f(\lambda_i+0.5h,x_i+0.5k_1h)$

$k_3=f(\lambda_i+0.5h,x_i+0.5k_2h)$

$k_4=f(\lambda_i+h,x_i +k_3h)$

In these equations h is finite difference step size. Solution starts by using the initial value λ=0 , x0(0) and adds h into λ in each iteration step. The example program uses 4th degree and 6th degree Runge-Kutta method to solve homotopy(Continuation problem). It should be note that Homotophy method is less dependent to initial value compare to methods such as Newton-Raphson therefore one possibility is to approachsolution with a relatively rough estimate with homotophy following with a Newton-Raphson type of method. In the following code methods of Homotophy with 4th and 6th order Runge-Kutta methods and combination of Homotophy with 4th order Runge-Kutta and Newton-Raphson method is given.

Program Homotophy (Contiuation) method (java version)

```
 / Continuation-homotopy
// solution of system of nonlinar equations (roots of equations)
// Referans : Numerical Analysis, 8th edition Richard L. Burden, J. Douglas Faires 3
// page 637
// All the required derivatives calculated numercally

import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
//============= Functions to be defined ================
class fa extends fi_xi
{
public double[] func(double x[])
{
//çözümü istenen fonksiyon
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
} }

public class NA30C
{
// (Continuity-homotopy) method to solve a system of nonlinear equations
//



public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
```

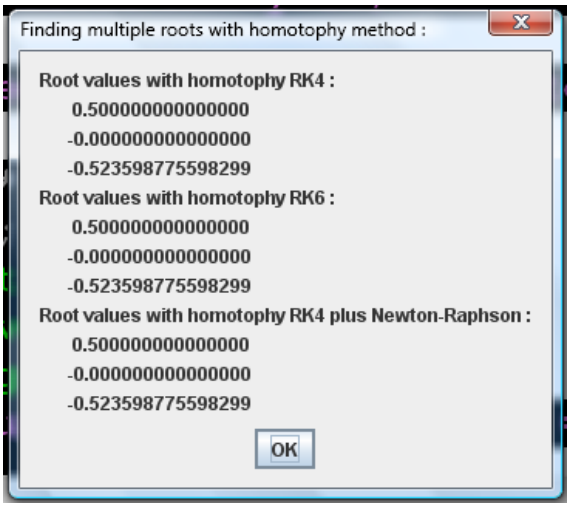
```
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {b[i]=left*right[i];}
return b;
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
    b[ii]=0;
   }
  return b;
 }
 for(i=0;i<m1;i++)
   {
    b[i]=0;
    for(k=0;k<n1;k++)
            b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}

public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
   b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
   b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}

public static double[][] inv(double[][] a)
{
// INVERSION OF A MATRIX
// inversion by using gaussian elimination
// with full pivoting
int n=a.length;
int m=a[0].length;
double b[][];
b=new double[n][n];
```

```
int indxc[];
int indxr[];
double ipiv[];
indxc=new int[n];
indxr=new int[n];
ipiv=new double[n];
int i,j,k,l,ll,ii,jj;
int icol=0;
int irow=0;
double big,dum,pivinv,temp;
if(n!=m)
{
  System.out.println("Matrix must be square ");
              for(ii=0;ii<n;ii++)
                for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
              return b;
}
for(i=0;i<n;i++)
  for(j=0;j<n;j++)
     b[i][j]=a[i][j];
for(i=0;i<n;i++)
{
 big=0.0;
 for(j=0;j<n;j++)
 {
  if(ipiv[j] != 1)
    for(k=0;k<n;k++)
    {
            if(ipiv[k] == 0)
            {
              if(Math.abs(b[j][k]) >= big)
              {
                big=Math.abs(b[j][k]);
                irow=j;
                icol=k;
              }
            }
            else if(ipiv[k] > 1 )
              {
              System.out.println("error : inverse of the matrix : singular matrix-1");
              for(ii=0;ii<n;ii++)
                for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
              return b;
              }
    }
 }
 ++ ipiv[icol];
 if(irow != icol)
  for(l=0;l<n;l++)
  {
  temp=b[irow][l];
  b[irow][l]=b[icol][l];
  b[icol][l]=temp;
  }
 indxr[i]=irow;
 indxc[i]=icol;
 if(b[icol][icol] == 0.0)
 {
  System.out.println("error : inverse of the matrix : singular matrix-2");
              for(ii=0;ii<n;ii++)
                for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
  return b;
 }
 pivinv=1.0/b[icol][icol];
```

```
  b[icol][icol]=1.0;
  for(l=0;l<n;l++) b[icol][l] *=pivinv;
  for(ll=0;ll<n;ll++)
  if(ll != icol)
  {
    dum=b[ll][icol];
    b[ll][icol]=0.0;
    for(l=0;l<n;l++) b[ll][l]-= b[icol][l]*dum;
  }
}
for(l=n-1;l>=0;l--)
{
  if(indxr[l] != indxc[l])
     for(k=0;k<n;k++)
     {
       temp=b[k][indxc[l]];
       b[k][indxc[l]]=b[k][indxr[l]];
       b[k][indxr[l]]=temp;
     }
}
return b;
}

public static double[] continuationRK4(fi_xi f,double x[],int N)
{
//=====================================================
// Roots of nonlinear system of equations Homotophy RK6
// 4.th order Runge-Kutta
// yi+1 = yi + (1/6)*( k1 + 2k2 +2k3+k4)h
// k1=f(xi,yi)
// k2=f(xi+0.5h,yi+0.5k1h)
// k3=f(xi+0.5h,yi+0.5k2h)
// k4=f(xi+h,yi +k3h)
//=====================================================
//x vector of independent variables
//y vector of dependent variables
//dy derivative vector of dependent variables
int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[4][n];
double A[][]=new double[n][n];
b=multiply(-h,f.func(x));
for(i=0;i<N;i++)
{
x1=x;
A=f.dfunc(x1);
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.5,k[0]));
A=f.dfunc(x1);
k[1]=multiply(inv(A),b);
x1=add(x,multiply(0.5,k[1]));
A=f.dfunc(x1);
k[2]=multiply(inv(A),b);
x1=add(x,k[2]);
A=f.dfunc(x1);
k[3]=multiply(inv(A),b);
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/6.0*(k[0][j]+2.0*k[1][j]+2.0*k[2][j]+k[3][j]);}
}
return x;
}

public static double[] continuationRK6(fi_xi f,double x[],int N)
```

```
{
//=====================================================
// Roots of nonlinear system of equations Homotophy RK6
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
// k1=f(xi,yi)
// k2=f(xi+0.25h , yi+0.25k1h)
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
//=====================================================
//x vector of independent variables
//y vector of dependent variables
//dy derivative vector of dependent variables
int i;
int nmax=400;
double tolerance=1.0e-20;
int n=x.length;
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[6][n];
double A[][]=new double[n][n];
b=multiply(-h,f.func(x));
for(i=0;i<N;i++)
{
A=f.dfunc(x);
// k1=f(xi,yi)
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.25,k[0]));
A=f.dfunc(x1);
// k2=f(xi+0.25h , yi+0.25k1h)
k[1]=multiply(inv(A),b);
x1=add(x,add(multiply(0.125,k[0]),multiply(0.125,k[1])));
A=f.dfunc(x1);
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
k[2]=multiply(inv(A),b);
x1=add(x,add(multiply(-0.5,k[1]),k[2]));
A=f.dfunc(x1);
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
k[3]=multiply(inv(A),b);
x1=add(x,add(multiply((3.0/16.0),k[0]),multiply((9.0/16.0),k[3])));
A=f.dfunc(x1);
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
k[4]=multiply(inv(A),b);
x1=add(x,
      add(multiply((-3.0/7.0),k[0]),add(multiply((2.0/7.0),k[1]),
      add(multiply((12.0/7.0),k[2]),
         add(multiply((-12.0/7.0),k[3]),multiply((8.0/7.0),k[4])))));
A=f.dfunc(x1);
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
k[5]=multiply(inv(A),b);
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/90.0*(7.0*k[0][j]+32.0*k[2][j]+12.0*k[3][j]+32.0*k[4][j]+7.0*k[5][j]);}
}
return x;
}
public static double[] newton_continuationRK4(fi_xi f,double x[])
{
// Roots of nonlinear system of equations Homotophy RK4 plus Newton-Raphson
//ti : weight function
//x vector of independent variables
//y vector of dependent variables
//dy derivative vector of dependent variables
x=continuationRK4(f,x,4);
double ti=1.0;
int i;
```

```
int nmax=400;
double tolerance=1.0e-10;
int n=x.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
{
b[i]=1.0;
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{ b=Matrix.multiply(Matrix.divide(f.func(x),f.dfunc(x)),-ti);
  x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Warning : Maximum number of iterations are exceeded \n"+
            " Results may not be valid","MAXIMUM ITERATIONS WARNING",JOptionPane.WARNING_MESSAGE);
System.out.println();
return x;
}
public static double[] enterdata()
{
   String s=JOptionPane.showInputDialog("enter initial root estimations with a space in between ");
   StringTokenizer token=new StringTokenizer(s);
   int n=token.countTokens()-1;
   int m=n+1;
   double a[]=new double[m];
   int j=0;
   while(token.hasMoreTokens())
   {
   Double ax=new Double(token.nextToken());
   a[j++]=ax.doubleValue();
   }
   return a;
}
    public static void main (String args[])
    { fa f =new fa();
    double [] x0=enterdata();
    double [] r= continuationRK4(f,x0,4);
    double [] r1= continuationRK6(f,x0,4);
    double [] r2= newton_continuationRK4(f,x0);
    String s=" Root values with homotophy RK4 : \n"+Matrix.toStringT(r);
    s+=" Root values with homotophy RK6 : \n"+Matrix.toStringT(r1);
    s+=" Root values with homotophy RK4 plus Newton-Raphson : \n"+Matrix.toStringT(r2);
    String s1="Finding multiple roots with homotophy method : ";
    JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Program Homotophy (Contiuation) method (python version)

```
# -*- coding: utf-8 -*-
"""
Created on Thu Aug 23 11:49:55 2018

@author: Mustafa Turhan Çoban
"""
from math import *
from fi_xi import *;
from gauss import *;
from newtond import *;

class fa(fi_xi):
  def func(self,x):

      ff=[0 for i in range(3)];
      ff[0]=3.0*x[0]-cos(x[1]*x[2])-0.5;
      ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+sin(x[2])+1.06;
      ff[2]=exp(-x[0]*x[1])+20.0*x[2]+(10.0*pi-3.0)/3.0;
      return ff;

def unit_matrix(n):
   #unit matrix of dimension n
   B = [[0 for x in range(n)] for y in range(n)];
   for i in range(0,n):
       for j in range(n):
           if i==j: B[i][j]=1.0;
   return B;

def invA(A):
    #inverse matrix (n=m)
    n=len(A);
    I= unit_matrix(n);
    B = [[0 for x in range(n)] for y in range(n)];
    for i in range(n):
        B[i]=gauss(A,I[i]);
    return B;

def continuationRK4(f,x,N):
    #=================================================
    # Roots of nonlinear system of equations Homotophy RK6
    # 4.th order Runge-Kutta
    # yi+1 = yi + (1/6)*( k1 + 2k2 +2k3+k4)h
    # k1=f(xi,yi)
    # k2=f(xi+0.5h,yi+0.5k1h)
    # k3=f(xi+0.5h,yi+0.5k2h)
    # k4=f(xi+h,yi +k3h)
    #=================================================
    #x vector of independent variables
    #y vector of dependent variables
    #dy derivative vector of dependent variables
    nmax=400;
    tolerance=1.0e-20;
    n=len(x);
    h=1.0/float(N);
    b=[0 for i in range(n)];
    x1=[0 for i in range(n)];
    k = [[0 for x in range(n)] for y in range(4)];
    A= [[0 for x in range(n)] for y in range(n)];
    b=multiply_c_v(-h,f.func(x));
    for i in range(N):
        x1=x;
        A=f.dfunc(x1,1);
        k[0]=multiply_m_v(invA(A),b);
        x1=add(x,multiply_c_v(0.5,k[0]));
        A=f.dfunc(x1,1);
        k[1]=multiply_m_v(invA(A),b);
        x1=add(x,multiply_c_v(0.5,k[1]));
        A=f.dfunc(x1,1);
        k[2]=multiply_m_v(invA(A),b);
        x1=add(x,k[2]);
        A=f.dfunc(x1,1);
        k[3]=multiply_m_v(invA(A),b);
        for j in range(n):
```

```
            x[j]=x[j]+1.0/6.0*(k[0][j]+2.0*k[1][j]+2.0*k[2][j]+k[3][j]);
    return x;

def continuationRK6(f,x,N):
    #=====================================================
    # Roots of nonlinear system of equations Homotophy RK6
    # yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
    # k1=f(xi,yi)
    # k2=f(xi+0.25h , yi+0.25k1h)
    # k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
    # k4=f(xi+0.5h , yi - 0.5k2h+k3h)
    # k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
    # k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
    #=====================================================
    #x vector of independent variables
    #y vector of dependent variables
    #dy derivative vector of dependent variables
    n=len(x);
    h=1.0/float(N);
    b=[0 for i in range(n)];
    x1=[0 for i in range(n)];
    k = [[0 for x in range(n)] for y in range(6)];
    A= [[0 for x in range(n)] for y in range(n)];
    b=multiply_c_v(-h,f.func(x));
    for i in range(N):
        A=f.dfunc(x,1);
        # k1=f(xi,yi)
        k[0]=multiply_m_v(invA(A),b);
        x1=add(x,multiply_c_v(0.25,k[0]));
        A=f.dfunc(x1,1);
        # k2=f(xi+0.25h , yi+0.25k1h)
        k[1]=multiply_m_v(invA(A),b);
        x1=add(x,add(multiply_c_v(0.125,k[0]),multiply_c_v(0.125,k[1])));
        A=f.dfunc(x1,1);
        # k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
        k[2]=multiply_m_v(invA(A),b);
        x1=add(x,add(multiply_c_v(-0.5,k[1]),k[2]));
        A=f.dfunc(x1,1);
        # k4=f(xi+0.5h , yi - 0.5k2h+k3h)
        k[3]=multiply_m_v(invA(A),b);
        x1=add(x,add(multiply_c_v((3.0/16.0),k[0]),multiply_c_v((9.0/16.0),k[3])));
        A=f.dfunc(x1,1);
        # k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
        k[4]=multiply_m_v(invA(A),b);
        x1=add(x,add(multiply_c_v((-3.0/7.0),k[0]),add(multiply_c_v((2.0/7.0),k[1]),
        add(multiply_c_v((12.0/7.0),k[2]),add(multiply_c_v((-12.0/7.0),k[3]),
        multiply_c_v((8.0/7.0),k[4])))))));
        A=f.dfunc(x1,1);
        # k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
        k[5]=multiply_m_v(invA(A),b);
        # yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
        for j in range(n):
            x[j]=x[j]+1.0/90.0*(7.0*k[0][j]+32.0*k[2][j]+12.0*k[3][j]+32.0*k[4][j]+7.0*k[5][j]);
    return x;

f=fa();
x=[1,1,1];
x= continuationRK6(f,x,4);
print("x : \n",x);
r=newton(f,x);
print("r : \n",r);
```

```
runfile('D:/okul/SCO1/homotopy.py', wdir='D:/okul/SCO1')
Reloaded modules: fi_xi, gauss, newtond
x :
 [0.6085068630782303, 0.0859198719076823, -0.44503655626229516]
r :
 [0.5, -4.1299852640664695e-18, -0.5235987755982989]
```

Program Homotophy (Contiuation) method (Julia version)

```
function func(x)
    ff=zeros(Float64,3)
    ff[1]=3.0*x[1]-cos(x[2]*x[3])-0.5
```

```
    ff[2]=x[1]*x[1]-81.0*(x[2]+0.1)*(x[2]+0.1)+sin(x[3])+1.06
    ff[3]=exp(-x[1]*x[2])+20.0*x[3]+(10.0*pi-3.0)/3.0
    return ff
end

#single function with reference number i
function fi(f,x,i)
# system of non-linear equation
y=f(x)
return y[i]
end

#first derivative
function dfunc(func,x)
 global  dx=0.001
 global c=[1.0/280.0 -4.0/105.0          1.0/5.0    -4.0/5.0  0          4.0/5.0    -1.0/5.0   4.0/105.0 -1.0/280.0]
 n=length(x)
 ff=zeros(Float64,n,n)
 global yy=ff[1]
 x1=zeros(Float64,n)
 for ieqn=1:n
  for ix=1:n
    for i=-4:4
      # create x1set for derivative input
      # ================================
      for k=1:n
        xx=x[k]
        if k!=ix
          x1[k]=xx
        else
          delta=Float64(i)*dx
          x1[k]=xx+delta
        end
      end # for k
      # ================================
      ff1=fi(func,x1,ieqn)

      ff[ieqn,ix]+=c[i+5]* ff1
      yx=ff[ieqn,ix]
    end # for i
    ff[ieqn,ix]/=dx
    yx=ff[ieqn,ix]
  end # for ix
 end # for ieqn
 return ff
end # for function


function continuationRK4(f,x,N)
   # ================================================
   # Roots of nonlinear system of equations Homotophy RK6
   # 4.th order Runge-Kutta
   # yi+1 = yi + (1/6)*( k1 + 2k2 +2k3+k4)h
   # k1=f(xi,yi)
   # k2=f(xi+0.5h,yi+0.5k1h)
   # k3=f(xi+0.5h,yi+0.5k2h)
   # k4=f(xi+h,yi +k3h)
   # ================================================
   # x vector of independent variables
   # y vector of dependent variables
   # dy derivative vector of dependent variables
   nmax=400
   tolerance=1.0e-20
   n=length(x)
   h=1.0/Float64(N)
   global b=zeros(Float64,n)
   x1=zeros(Float64,n)
```

```
    A= zeros(Float64,n,n)
    b=-h*f(x)
    for i=1:N
       x1=x
       A=dfunc(f,x1)
       k1=A\b
       kk=0.5*k1
       x1=x+kk'
       A=dfunc(f,x1)
       k2=A\b
       kk=0.5*k2
       x1=x+kk'
       A=dfunc(f,x1)
       k3=A\b
       kk=k3
       x1=x+kk'
       A=dfunc(f,x1)
       k4=A\b
       c=[1.0/6.0 2.0/6.0 2.0/6.0 1.0/6.0]
       kk=c[1]*k1
       kk+=c[2]*k2
       kk+=c[3]*k3
       kk+=c[4]*k4
       x+=kk'
    end
    return x
end

function newton(func,x)
  # Newton-Raphson root finding method
  k=length(x)
  nmax=100
  tolerance=1.0e-10
  for i=1:nmax
     fx=func(x)
     dfx=dfunc(func,x)
     dx=dfx\fx    #gauss_pivot(dfx,fx)
     for j=1:k
        x[j]=x[j]-dx[j]
     end # for j
     total=0.0;
     for j=1:k
        total+=fx[j]
     end # for j
     if abs(total)<tolerance
       return x
     end
  end #for i
  return x
end # function

x=[1 1 1]
x= continuationRK4(func,x,10)
print("x continuation : $x \n")
x=newton(func,x)
print("x newton : $x \n")
print("x : $x \n")
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" continuity.jl
x continuation : [0.4999682669519078 -3.9723398572444335e-5 -0.5235991652361665]
x newton : [0.5 -1.81723061024014e-17 -0.5235987755982989]
x : [0.5 -1.81723061024014e-17 -0.5235987755982989]

> Terminated with exit code 0.
```

In roots searching first estimate is given as a vector by the user. Another alternative way of searching non-linear roots can be achived by defining a search regoun and then finding a first estimate by random first estimate selection in the given region. This process can be repeated number of times and minimum value of summation of y=f(x) value can be considered as the best possible solution fort he given region. Note that there could be more than one solution set in the given region.

- Detemine search region $x_{i\_low}, x_{i\_high} \quad i = 0, N$
- Find a starting point randomly falling into search region $x_i = xx_{i\_low} + Random\ number(0,1) * (x_{i_{high}} - x_{i_{low}})$
- Find solution set
- Find $\sum_{i=0}^{N} f(x)$
- Compare them with the minimum value found so far
- Repeat the process for a given number of time and select the case with minimum total $\sum_{i=0}^{N} f(x)$ value

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.lang.Math;
class fa extends fi_xi
{
public double[] func(double x[])
{
//çözümü istenen fonksiyon
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
} }

public class iterative_continuity
{
public int Nlimit;  //search limit
public double low[];  //search area low limits
public double high[]; //search area high limits
public int N;  //Runge Kutta 6 differential eqn solving number of division
public int Neqn;  //number of independent set
// constructor
public iterative_continuity(int nlimiti, int Ni,double lowi[],double highi[])
{low=lowi;high=highi;Nlimit=nlimiti;N=Ni; Neqn=low.length;}
public iterative_continuity(int nlimiti,double lowi[],double highi[])
{low=lowi;high=highi;Nlimit=nlimiti;N=30; Neqn=low.length;}

// (Continuity-homotopy) method to solve a system of nonlinear equations
public  double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
int i;
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {b[i]=left*right[i];}
return b;
}
public  double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
```

```
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
     b[ii]=0;
   }
   return b;
 }
 for(i=0;i<m1;i++)
   {
     b[i]=0;
     for(k=0;k<n1;k++)
                b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}
public  double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}
public  double[][] inv(double[][] a)
{
// INVERSION OF A MATRIX
// inversion by using gaussian elimination
// with full pivoting
int n=a.length;
int m=a[0].length;
double b[][];
b=new double[n][n];
int indxc[];
int indxr[];
double ipiv[];
indxc=new int[n];
indxr=new int[n];
ipiv=new double[n];
int i,j,k,l,ll,ii,jj;
int icol=0;
int irow=0;
double big,dum,pivinv,temp;
if(n!=m)
{ System.out.println("Matrix must be square ");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                            b[ii][jj]=0.0;
                return b;
}
for(i=0;i<n;i++)
   for(j=0;j<n;j++)
       b[i][j]=a[i][j];
for(i=0;i<n;i++)
{
  big=0.0;
  for(j=0;j<n;j++)
  {
    if(ipiv[j] != 1)
      for(k=0;k<n;k++)
```

```
    {
            if(ipiv[k] == 0)
            {
              if(Math.abs(b[j][k]) >= big)
              {
                big=Math.abs(b[j][k]);
                irow=j;
                icol=k;
              }
            }
            else if(ipiv[k] > 1 )
              {
              System.out.println("error : inverse of the matrix : singular matrix-1");
              for(ii=0;ii<n;ii++)
                for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
              return b;
              }
    }
  }
  ++ ipiv[icol];
  if(irow != icol)
    for(l=0;l<n;l++)
    {
    temp=b[irow][l];
    b[irow][l]=b[icol][l];
    b[icol][l]=temp;
    }
  indxr[i]=irow;
  indxc[i]=icol;
  if(b[icol][icol] == 0.0)
  {
    System.out.println("error : inverse of the matrix : singular matrix-2");
                for(ii=0;ii<n;ii++)
                  for(jj=0;jj<n;jj++)
                        b[ii][jj]=0.0;
    return b;
  }
  pivinv=1.0/b[icol][icol];
  b[icol][icol]=1.0;
  for(l=0;l<n;l++) b[icol][l] *=pivinv;
  for(ll=0;ll<n;ll++)
  if(ll != icol)
  {
    dum=b[ll][icol];
    b[ll][icol]=0.0;
    for(l=0;l<n;l++) b[ll][l]-= b[icol][l]*dum;
  }
}
for(l=n-1;l>=0;l--)
{
  if(indxr[l] != indxc[l])
     for(k=0;k<n;k++)
     {
       temp=b[k][indxc[l]];
       b[k][indxc[l]]=b[k][indxr[l]];
       b[k][indxr[l]]=temp;
     }
}
return b;
}
public  double[] continuationRK6(fi_xi f,double x[])
{
//=====================================================
// Roots of nonlinear system of equations Homotophy RK6
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
// k1=f(xi,yi)
// k2=f(xi+0.25h , yi+0.25k1h)
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
//=====================================================
//x vector of independent variables
//y vector of dependent variables
//dy derivative vector of dependent variables
```

```
int i;
int nmax=1000;
double tolerance=1.0e-15;
int n=x.length;
double h=1.0/(double)N;
double b[]=new double[n];
double x1[]=new double[n];
double k[][]=new double[6][n];
double A[][]=new double[n][n];
b=multiply(-h,f.func(x));
for(i=0;i<N;i++)
{
A=f.dfunc(x);
// k1=f(xi,yi)
k[0]=multiply(inv(A),b);
x1=add(x,multiply(0.25,k[0]));
A=f.dfunc(x1);
// k2=f(xi+0.25h , yi+0.25k1h)
k[1]=multiply(inv(A),b);
x1=add(x,add(multiply(0.125,k[0]),multiply(0.125,k[1])));
A=f.dfunc(x1);
// k3=f(xi+0.25h , yi+0.125k1h+0.125k2h)
k[2]=multiply(inv(A),b);
x1=add(x,add(multiply(-0.5,k[1]),k[2]));
A=f.dfunc(x1);
// k4=f(xi+0.5h , yi - 0.5k2h+k3h)
k[3]=multiply(inv(A),b);
x1=add(x,add(multiply((3.0/16.0),k[0]),multiply((9.0/16.0),k[3])));
A=f.dfunc(x1);
// k5=f(xi+0.75h , yi + (3/16)k1h+(9/16)k4h)
k[4]=multiply(inv(A),b);
x1=add(x,
      add(multiply((-3.0/7.0),k[0]),add(multiply((2.0/7.0),k[1]),
      add(multiply((12.0/7.0),k[2]),
         add(multiply((-12.0/7.0),k[3]),multiply((8.0/7.0),k[4])))));
A=f.dfunc(x1);
// k6=f(xi+h , yi - (3/7)k1h+(2/7)k2h+(12/7)k3h - (12/7)k4h+(8/7)k5h)
k[5]=multiply(inv(A),b);
// yi+1 = yi + (1/90)*( 7k1 + 32k3 +12k4+32k5+7k6)h
for(int j=0;j<n;j++) {x[j]=x[j]+1.0/90.0*(7.0*k[0][j]+32.0*k[2][j]+12.0*k[3][j]+32.0*k[4][j]+7.0*k[5][j]);}
}
return x;
}

public static double[] newton(fi_xi f,double x[])
{
// Roots of nonlinear system of equations Homotophy RK4 plus Newton-Raphson
//ti : weight function
//x vector of independent variables
//y vector of dependent variables
//dy derivative vector of dependent variables
double ti=1.0;
int i;
int nmax=1000;
double tolerance=1.0e-15;
int n=x.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
{
b[i]=1.0;
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{ b=Matrix.multiply(Matrix.divide(f.func(x),f.dfunc(x)),-ti);
  x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Warning : Maximum number of iterations are exceeded \n"+
            " Results may not be valid","MAXIMUM ITERATIONS WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}
public  double[] newton_continuationRK6(fi_xi f,double x[])
{
// Roots of nonlinear system of equations Homotophy RK4 plus Newton-Raphson
//ti : weight function
//x vector of independent variables
```

```
//y vector of dependent variables
//dy derivative vector of dependent variables
x=continuationRK6(f,x);
double ti=1.0;
int i;
int nmax=1000;
double tolerance=1.0e-15;
int n=x.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
{
b[i]=1.0;
}
i=0;
while( i++ < nmax && Matrix.abs(b) > tolerance )
{ b=Matrix.multiply(Matrix.divide(f.func(x),f.dfunc(x)),-ti);
  x=Matrix.add(x,b);
}
if(i >= nmax) JOptionPane.showMessageDialog(null,"Warning : Maximum number of iterations are exceeded \n"+
            " Results may not be valid","MAXIMUM ITERATIONS WARNING",JOptionPane.WARNING_MESSAGE);
return x;
}

public static double[] enterdata()
{
   String s=JOptionPane.showInputDialog("enter initial root estimations with a space in between ");
   StringTokenizer token=new StringTokenizer(s);
   int n=token.countTokens()-1;
   int m=n+1;
   double a[]=new double[m];
   int j=0;
   while(token.hasMoreTokens())
   {a[j++]=Double.parseDouble(token.nextToken());}
   return a;
}
public double[][] findContinuity(fi_xi f)
{    double ymin=3e40;
     int nbest=0;
     double x1[]=new double[Neqn];
               double x[][]=new double[Nlimit][Neqn];
               double y[][]=new double[Nlimit][Neqn];
     for(int j=0;j<Nlimit;j++)
     {
               double yy=0;
               for(int i=0;i<Neqn;i++)
               {x1[i]=low[i]+Math.random()*(high[i]-low[i]);}
               x[j]=continuationRK6(f,x1);
               y[j]=f.func(x[j]);
               for(int i=0;i<Neqn;i++)
               {yy+=y[j][i];}
               System.out.println("j="+j+"yy="+yy);
               if(yy==0) {nbest=j;break;}
               if(Math.abs(yy)<ymin) {ymin=Math.abs(yy);nbest=j;}
   }
   double zz[][]={x[nbest],y[nbest]};
   return zz;
}
public double[][] findNewton(fi_xi f)
{    double ymin=3e40;
     int nbest=0;
     double x1[]=new double[Neqn];
               double x[][]=new double[Nlimit][Neqn];
               double y[][]=new double[Nlimit][Neqn];
     for(int j=0;j<Nlimit;j++)
     {
               double yy=0;
               for(int i=0;i<Neqn;i++)
               {x1[i]=low[i]+Math.random()*(high[i]-low[i]);}
               x[j]=newton(f,x1);
               y[j]=f.func(x[j]);
               for(int i=0;i<Neqn;i++)
               {yy+=y[j][i];}
               System.out.println("j="+j+"yy="+yy);
               if(yy==0) {nbest=j;break;}
     if(Math.abs(yy)<ymin) {ymin=Math.abs(yy);nbest=j;}
```

```
    }
    double zz[][]={x[nbest],y[nbest]};
    return zz;
}

public double[][] findNewtonContinuation(fi_xi f)
{    double ymin=3e40;
     int nbest=0;
     double x1[]=new double[Neqn];
                double x[][]=new double[Nlimit][Neqn];
                double y[][]=new double[Nlimit][Neqn];
     for(int j=0;j<Nlimit;j++)
     {
                double yy=0;
                for(int i=0;i<Neqn;i++)
                {x1[i]=low[i]+Math.random()*(high[i]-low[i]);}
                x[j]=newton_continuationRK6(f,x1);
                y[j]=f.func(x[j]);
                for(int i=0;i<Neqn;i++)
                {yy+=y[j][i];}
                System.out.println("j="+j+"yy="+yy);
                if(yy==0) {nbest=j;break;}
     if(Math.abs(yy)<ymin) {ymin=Math.abs(yy);nbest=j;}
    }
    double zz[][]={x[nbest],y[nbest]};
    return zz;
}
  public static void main(String arg[])
  {  fa fx=new fa();
     double low[]={0,0,0};
    double high[]={1,1,1};
    iterative_continuity itc=new iterative_continuity(30,low,high);
    double z[][]=itc.findContinuity(fx);
    String s[]={"x","y=f(x)"};
    Text.printT(z,s,"continuity");
    double z1[][]=itc.findNewton(fx);
    Text.printT(z1,s,"Newton");
    double z2[][]=itc.findNewtonContinuation(fx);
    Text.printT(z1,s,"Newton_continuation");
  }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" iterative_continuity
j=0yy=9.23383187689808E-8
j=1yy=0.0014655966934973286
j=2yy=2.7940106073209847E-7
j=3yy=8.762042850163354E-6
j=4yy=1.041124964018536E-10
j=5yy=2.4581519429967003E-5
j=6yy=4.661132704697479E-6
j=7yy=6.001758694473835E-7
j=8yy=2.3717842578818704E-8
j=9yy=2.4949303163435133E-5
j=10yy=8.468584794307787E-6
j=11yy=0.001434468568131675
j=12yy=1.720759090773072E-10
j=13yy=2.616555025624079E-4
j=14yy=1.0900877761266248E-4
j=15yy=1.993579393455569E-6
j=16yy=2.8659885866866297E-4
j=17yy=9.730782846162356E-9
j=18yy=5.629171173637815E-8
j=19yy=1.510731743028515E-4
j=20yy=0.0011197865506833438
j=21yy=1.289823708949811E-5
j=22yy=2.5530188223654715E-4
j=23yy=1.7026177970713707E-4
j=24yy=4.731952094161329E-5
j=25yy=9.743361971681352E-7
j=26yy=5.590070891955401E-4
j=27yy=5.725897835873184E-7
j=28yy=3.586605644041452E-4
j=29yy=9.07827877439038E-4
j=0yy=1.7763568394002505E-15
j=1yy=-1.7763568394002505E-15
```

```
j=2yy=-1.7763568394002505E-15
j=3yy=1.5543122344752192E-15
j=4yy=-1.7763568394002505E-15
j=5yy=-1.7763568394002505E-15
j=6yy=-1.7763568394002505E-15
j=7yy=1.7763568394002505E-15
j=8yy=1.5543122344752192E-15
j=9yy=1.7763568394002505E-15
j=10yy=1.7763568394002505E-15
j=11yy=1.7763568394002505E-15
j=12yy=1.7763568394002505E-15
j=13yy=1.7763568394002505E-15
j=14yy=1.7763568394002505E-15
j=15yy=-1.7763568394002505E-15
j=16yy=1.5543122344752192E-15
j=17yy=1.7763568394002505E-15
j=18yy=-1.7763568394002505E-15
j=19yy=1.5543122344752192E-15
j=20yy=-1.7763568394002505E-15
j=21yy=1.7763568394002505E-15
j=22yy=-1.7763568394002505E-15
j=23yy=1.7763568394002505E-15
j=24yy=1.7763568394002505E-15
j=25yy=-1.7763568394002505E-15
j=26yy=-1.7763568394002505E-15
j=27yy=1.5543122344752192E-15
j=28yy=1.7763568394002505E-15
j=29yy=1.7763568394002505E-15
j=0yy=1.5543122344752192E-15
j=1yy=-1.7763568394002505E-15
j=2yy=-1.7763568394002505E-15
j=3yy=1.7763568394002505E-15
j=4yy=1.7763568394002505E-15
j=5yy=-1.7763568394002505E-15
j=6yy=1.7763568394002505E-15
j=7yy=-1.7763568394002505E-15
j=8yy=-1.7763568394002505E-15
j=9yy=1.7763568394002505E-15
j=10yy=-1.7763568394002505E-15
j=11yy=1.7763568394002505E-15
j=12yy=-1.7763568394002505E-15
j=13yy=1.7763568394002505E-15
j=14yy=1.7763568394002505E-15
j=15yy=-1.7763568394002505E-15
j=16yy=-1.7763568394002505E-15
j=17yy=-1.7763568394002505E-15
j=18yy=1.998401444325281 8E-15
j=19yy=1.7763568394002505E-15
j=20yy=-1.7763568394002505E-15
j=21yy=-1.7763568394002505E-15
j=22yy=1.7763568394002505E-15
j=23yy=1.5543122344752192E-15
j=24yy=1.7763568394002505E-15
j=25yy=-1.7763568394002505E-15
j=26yy=1.7763568394002505E-15
j=27yy=1.7763568394002505E-15
j=28yy=1.7763568394002505E-15
j=29yy=1.5543122344752192E-15
> Terminated with exit code 0.
```

continuity

x	y=f(x)
0.4999991028979144	4.39648317751562E-14
-7.864957981283671E-5	3.765876499528531E-13
-0.5236008349935111	-2.9114488597770105E-12

Newton

x	y=f(x)
0.5	0.0
9.651666093172359E-18	-2.220446049250313E-16
-0.5235987755982988	1.7763568394002505E-15

Newton_continuation

x	y=f(x)
0.5	0.0
9.651666093172359E-18	-2.220446049250313E-16
-0.5235987755982988	1.7763568394002505E-15

JACOBİ, GAUSS-SEIDEL AND RELAXATION METHODS FOR NON-LINEAR SYSTEM OF EQUATIONS

Iterative solution methods of linear system of eqaution were investigated in the previous chapter. The same methods can also be applied to solve non-linear system of equations. There are no guarantee that these type of soutions will be converged. In this method the basic nonlinear set of eqaution first converged as

$f_j(x_i) = 0 \quad i = 0..n \quad j = 0..n$ (4.4.1) from the original form to

$x_j = g_j(x_i) \quad i = 0..n \quad j = 0..n$ or in an open form:

$x_0 = g_0(x_0, x_1, x_2, .., x_n)$

$x_1 = g_1(x_0, x_1, x_2, .., x_n)$

…..

$x_n = g_n(x_0, x_1, x_2, .., x_n)$

So out iterative set of eqaution will be

$x_0^{k+1} = (1-\alpha)x_0^k + \alpha g_0(x_0, x_1, x_2, .., x_n)$

$x_1^{k+1} = (1-\alpha)x_1^k + \alpha g_1(x_0, x_1, x_2, .., x_n)$

$x_2^{k+1} = (1-\alpha)x_2^k + \alpha g_2(x_0, x_1, x_2, .., x_n)$

…..

$x_j^{k+1} = (1-\alpha)x_j^k + \alpha g_j(x_0, x_1, x_2, .., x_n)$

.....

$x_n^{k+1} = (1-\alpha)x_n^k + \alpha g_n(x_0, x_1, x_2, .., x_n)$

The value could be between 0 and 1 or bigger than 1. Let us give an example problem as:

$3x_0 - \cos(x_1 x_2) - 0.5 = 0$

$x_0^2 - 81(x_1 + 0.01) + \sin(x_2) + 1.06 = 0$

$-\exp(-x_0 x_1) + 20x_2 + \frac{10\pi - 3}{3} = 0$

When we converged the system of equation, it will become:

$x_0 = [\cos(x_1 x_2) + 0.5]/3$

$x_1 = \frac{[x_0^2 + \sin(x_2) + 1.06]}{81} + 0.01$

$x_2 = (-\exp(-x_0 x_1) - \frac{10\pi - 3}{3})/20$

Program list is given as follows:

Program Jacobi-Gauss_seidel and relaxation method, Java version

```
 import java.io.*;
class fa extends fi_xi
{
```

```
public double[] func(double x[])
{
// çözümü istenen fonksiyon
double g[]=new double[3];
g[0]=(Math.cos(x[1]*x[2])+0.5)/3.0;
g[1]=Math.sqrt((x[0]*x[0]+Math.sin(x[2]+1.06))/81.0)-0.1;
g[2]=(-Math.exp(-x[0]*x[1])-(10.0*Math.PI-3)/3.0)/20.0;
return g;
} }
class SCO10X
{
  public static double [] gauss_seidel(fi_xi f,double x[],double alpha)
  {
  int n=x.length;
  double xx[]=new double[n];
  double xx1[]=new double[n];
  double delta=0;
  for(int i=0;i<n;i++) xx[i]=x[i];
  double g[]=new double[n];
  int iter=500;
  for(int k=0;k<iter;k++)
  {for(int i=0;i<n;i++) xx1[i]=xx[i];
   delta=0;
  for(int j=0;j<n;j++)
  {   g=f.func(xx);
                xx[j]=(1-alpha)*xx[j]+alpha*g[j];
                delta+=Math.abs(xx[j]-xx1[j]);
                System.out.print("xx["+j+"] = "+xx[j]);
  }
  System.out.println(" k = "+k+" delta = "+delta);
  if(delta<1e-15) break;
  }
  return xx;
  }
  public static void main(String args[]) throws IOException
  {
  double x[]={1.01,0.1,1.0121};
  fa ff=new fa();
  double alpha=0.1;
  Text.print(gauss_seidel(ff,x,alpha));
  }
}
```

Gauss_seidel interpolation soltion with java langage

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" SCO10X
xx[0] = 0.9588294212826883xx[1] = 0.09489183495398086xx[2] = 0.8589649657886227 k = 0 delta = 0.20941377797470812
xx[0] = 0.9128358124735219xx[1] = 0.09019866496669672xx[2] = 0.7211037819655458 k = 1 delta = 0.1885479626195275
xx[0] = 0.8714817470867342xx[1] = 0.085824605234896xx[2] = 0.5969938557799842 k = 2 delta = 0.16983805130414997
xx[0] = 0.8342898285193656xx[1] = 0.08169652937067469xx[2] = 0.4852640310069166 k = 3 delta = 0.1530498192046575
xx[0] = 0.8008346544741505xx[1] = 0.07776042319429945xx[2] = 0.3846796198127338 k = 4 delta = 0.1379756914157732
xx[0] = 0.7707362771620705xx[1] = 0.07397773129037596xx[2] = 0.2941288917316395 k = 5 delta = 0.12443179729709779
xx[0] = 0.743654758864458xx[1] = 0.07032203418833874xx[2] = 0.2126108822145333 k = 6 delta = 0.11225522491675591
xx[0] = 0.7192855573929079xx[1] = 0.06677618815104167xx[2] = 0.1392243959352476 k = 7 delta = 0.10130153378813286
xx[0] = 0.6973555611327479xx[1] = 0.06332995056219265xx[2] = 0.07315809120795819 k = 8 delta = 0.09144253857629844
xx[0] = 0.6776196472601156xx[1] = 0.05997805997870190 4xx[2] = 0.013681541973005897 k = 9 delta = 0.08256435369107533
xx[0] = 0.6598576713112414xx[1] = 0.05671871668388067xx[2] = -0.039862816917781055 k = 10 delta = 0.07456567813448237
…….
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921277xx[2] = -0.5236756641509748 k = 388 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921278xx[2] = -0.5236756641509748 k = 389 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921279xx[2] = -0.5236756641509748 k = 390 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921279 8xx[2] = -0.5236756641509748 k = 391 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921280 6xx[2] = -0.5236756641509748 k = 392 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921281 5xx[2] = -0.5236756641509748 k = 393 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921282 4xx[2] = -0.5236756641509748 k = 394 delta = 8.673617379884035E-19
xx[0] = 0.4999995683317302xx[1] = -0.003073182441921282 4xx[2] = -0.5236756641509748 k = 395 delta = 0.0
```

Program Jacobi-Gauss_seidel and relaxation method, Julia code

```
 #Non-linear gauss_seidel
#function set
function func(x)
# system of non-linear equation
g=zeros(Float64,3)
g[1]=(cos(x[2]*x[3])+0.5)/3.0;
g[2]=sqrt((x[1]*x[1]+sin(x[3]+1.06))/81.0)-0.1;
g[3]=(-exp(-x[1]*x[2])-(10.0*pi-3)/3.0)/20.0;
return g
end

function gauss_seidel(f,x,alpha)
global n=length(x)
global xx=zeros(Float64,n)
global xx1=zeros(Float64,n)
global g=zeros(Float64,n)
global delta=0.0
for i=1:n
  xx[i]=x[i]
end

println("Initial estimation xx= $xx alpha = $alpha")
g=zeros(Float64,n)
iter=500
for k=1:iter
  for i=1:n
    xx1[i]=xx[i]
  end #for i
  delta=0.0
  for j=1:n
      g=f(xx)
      zx1=alpha*g[j]
      zx2=(1.0-alpha)*xx[j]
      zx3=zx1.+zx2
                xx[j]=zx3
                xj1=       xx[j]-xx1[j]
                delta=delta+abs(xj1)
                z1=xx[j]
  end #for j
  if delta<1e-15
     println("iteration number = $k delta = $delta")
     break
  end # if
end # for k
return xx
end # function

x=[1.01 1.012 1.0121]
alpha=0.1
x=gauss_seidel(func,x,alpha)
print("iterative non-linear relaxation  method $x")
```

Gauss_seidel relaxation method

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_gauss_seidel.jl
Initial estimation xx= [1.01, 1.012, 1.0121] alpha = 0.1
iteration number = 331 delta = 9.32847549206528e-16
iterative non-linear relaxation  method [0.4999995683317303, -0.0030731824419162226, -0.523675664150972]
> Terminated with exit code 0.
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_gauss_seidel.jl
Initial estimation xx= [1.01, 1.012, 1.0121] alpha = 0.2
iteration number = 160 delta = 9.514958265732787e-16
iterative non-linear relaxation  method [0.49999956833173037, -0.0030731824419189787, -0.523675664150974]
> Terminated with exit code 0.
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_gauss_seidel.jl
Initial estimation xx= [1.01, 1.012, 1.0121] alpha = 0.5
iteration number = 54 delta = 6.951904329977054e-16
iterative non-linear relaxation  method [0.4999995683317301, -0.0030731824419208396, -0.523675664150975]
> Terminated with exit code 0.
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_gauss_seidel.jl
Initial estimation xx= [1.01, 1.012, 1.0121] alpha = 0.8
iteration number = 25 delta = 2.8796409701215e-16
iterative non-linear relaxation  method [0.49999956833173004, -0.0030731824419212824, -0.5236756641509752]
> Terminated with exit code 0.
```

```
---------- Capture Output ----------
> "C:\co\Julia\bin\julia.exe" NL_gauss_seidel.jl
Initial estimation xx= [1.01, 1.012, 1.0121] alpha = 1.0
iteration number = 7 delta = 4.163336342344337e-17
iterative non-linear relaxation  method [0.49999956833173004, -0.0030731824419213266, -0.5236756641509752]
> Terminated with exit code 0.
```

The value could be between 0 and 1 or bigger than 1. Let us give an example problem as:

$3x_0 - \cos(x_1 x_2) - 0.5 = 0$

$x_0^2 - 81(x_1 + 0.01) + \sin(x_2) + 1.06 = 0$

$-\exp(-x_0 x_1) + 20x_2 + \frac{10\pi - 3}{3} = 0$

When we converged the system of equation, it will become:

$x_0 = [\cos(x_1 x_2) + 0.5]/3$

$x_1 = \frac{[x_0^2 + \sin(x_2) + 1.06]}{81} + 0.01$

$x_2 = (-\exp(-x_0 x_1) - \frac{10\pi - 3}{3})/20$

Gauss-Seidel non-linear root finding, Octave-Matlab option

```
function retval = gauss_seidel (f,x,alpha)
  n=length(x);
  iter=500;
  xx=zeros(1,n);
  xx=x;
  for k=1:iter
    for i=1:n
      xx1(i)=xx(i);
    endfor
    delta=0.0;
    for j=1:n
      g=f(xx);
      xx(j)=(1.0-alpha)*xx(j)+alpha*g(j);
      delta=delta+abs(xx(j)-xx1(j));
    endfor
    if delta<1e-15
      break;
    end

  endfor
  retval=xx;
endfunction
```

```
##  M. Turhan Coban
function g = f4(x)
  g(1)=(cos(x(2)*x(3))+0.5)/3.0;
  g(2)=sqrt((x(1)*x(1)+sin(x(3))+1.06)/81.0)-0.1;
  g(3)=(-exp(-x(1)*x(2))-(10.0*pi-3.0)/3.0)/20.0;
endfunction
```

```
>> y=@f4;
>> x=[1 1 1];
>> x=gauss_seidel(y,x,0.5)
x =

   5.000000000000000e-01   2.486378468364878e-16  -5.235987755982988e-01
```

In matlab-octave built-in function fsolve is existed to solve non-linear system of equations

```
##  M. Turhan Coban
function y = f2(x)
  y(1)=3.0*x(1)-cos(x(2)*x(3))-0.5;
  y(2)=x(1)*x(1)-81.0*(x(2)+0.1)*(x(2)+0.1)+sin(x(3))+1.06;
  y(3)=exp(-x(1)*x(2))+20.0*x(3)+(10.0*pi-3.0)/3.0;
endfunction
```

```
>> y=@f2;
>> x=[1 1 1];
>> x=fsolve(y,x)
x =

   5.000000000001514e-01   1.684726857934463e-11  -5.235987755978659e-01

>>
```

SOLUTION OF THE NONLINEAR SYSTEM OF EQUATIONS BY USING OPTIMIZATION ALGORITHMS

Non-Linear system of equations can also be solved by using optimization methods through an adaptation function.

To solve function $f_i(x_j)=0$

$$g(x_j) = \sum_{i=0}^{n\_equation} f_i(x_j) * f_i(x_j)$$

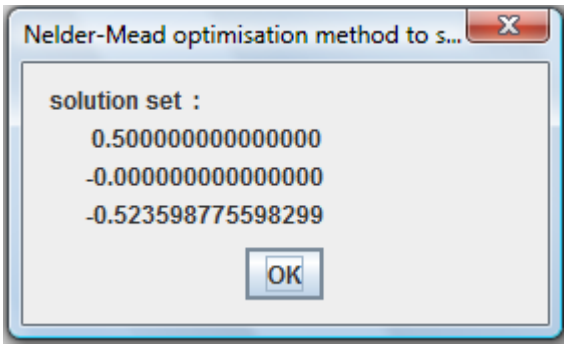
The derivative of this equation is the root of non-linear system of equation again became $f_i(x_j)=0$, therefore optimum of $g(x_j)$ is the same problem as solution of $f_i(x_j)=0$.

As the first adaptation algorithm Nelder-Mead Method will be given. Nelder_Mead method was defined at sub-chapter 5.10 as NA41:

Program Solution of Non-Linear system of equations by using Nelder-Mead optimization method, Java option

```
// Solution of system of equation by using Nelder-Mead Method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;


class f1 extends f_xj
{//adaptation function
fi_xi ff1;
public f1(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
public double[] func(double x[])
{
double ff[]=new double[3];
```

```
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
}
}

public class NA117
{
     public static void main (String args[])
    {
    String s="leave a space between initial root guess values: ";
    double [] x0=NA41.inputdata(s);
    double [] dx0=new double[x0.length];
    for(int i=0;i<x0.length;i++) dx0[i]=0.2*x0[i];
    fa ff=new fa();
    f1 f=new f1(ff);
    double [] r1= NA41.nelder(f,x0,dx0);
    s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Nelder-Mead optimization method to solve system of non-linear equation:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Nonlinear system of equation solution by using Nelder-Mead Algorithm Python code

```
from gauss import *;
from math import *;
from f_xj import *;
from fi_xi import *;
import numpy as np;
import matplotlib.pyplot as plt;
from mpl_toolkits.mplot3d import Axes3D
from random import *;

class f1(f_xj):
   def __init__(self,f):
      self.ff1=f
   def func(self,x):
      ffx=0.0
      fa=self.ff1.func(x)
      n=len(fa)
      for i in range(n):
         ffx=ffx+fa[i]*fa[i]
      return ffx

class fa(fi_xi):
   def __init__(self):
      self.name = 'fa'
   def func(self,x):
      ff=[0.0 for j in range(3)]
      ff[0]=3.0*x[0]-cos(x[1]*x[2])-0.5
      ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+sin(x[2])+1.06
```

```
        ff[2]=exp(-x[0]*x[1])+20.0*x[2]+(10.0*pi-3.0)/3.0
        return ff


def  nelder(fnelder,a,da,maxiteration,tolerance,printlist):
    #print("a",a)
    #print("da",da)
    n=len(a)
    m=n+1
    x=[[0.0 for j in range(n)] for i in range(n+1)]
    p=[[0.0 for j in range(m)] for i in range(m)]
    s=""
    NDIMS = len(x)-1
    NPTS = len(x)
    FUNC = NDIMS
    ncalls = 0;
    for i in range(m):
        for j in range(n):
            if i==j:
                x[i][j]=a[j]+da[j]
                p[i][j]=x[i][j]
            else:
                x[i][j]=a[j]
                p[i][j]=x[i][j]
        p[i][FUNC] = fnelder.func(p[i])
        #print("x[",i,"]=",x[i])
        #print("p[",i,"]=",p[i])
    #print("p",p)
    #Inlet variable definitions
    #fnelder : abstract multivariable function f(x)
    #x : independent variable set of n+1 simplex elements
    #maxiteration : maximum iteration number
    #tolerance :

    #///// construct the starting simplex //////////////////
    z=[0.0 for i in range(NDIMS)]
    best = 1.0E99;
    #/////////////// calculate the first function values for the simplex ////////////////
    iter=0;
    for iter in range(1,maxiteration):
        ########  define lo, nhi, hi (low high next_to_high //////////////
        ilo=0
        ihi=0
        inhi = -1
        # -1 means missing
        flo = p[0][FUNC];
        fhi = flo
        #print("flo=",flo,"fhi=",fhi)
        #double pavg,sterr;
        for i in range(1,NPTS):
            if p[i][FUNC] < flo:
                flo=p[i][FUNC]
                ilo=i
            if  p[i][FUNC] > fhi:
                fhi=p[i][FUNC]
                ihi=i
            #print("i=",i,"flo=",flo,"fhi=",fhi,"ilo=",ilo,"ihi=",ihi,"pifunc=",p[i][FUNC])
        fnhi = flo
        inhi = ilo
        #print("fnhi=",fnhi,"inhi=",inhi)
        for i in range(NPTS):
            if (i != ihi) and (p[i][FUNC] > fnhi):
                fnhi=p[i][FUNC]
                inhi=i
        ######### exit criteria //////////////
        if (iter % 4*NDIMS) == 0:
            #calculate the avarage (including maximum value)
            pavg=0
            for i in range(NPTS):
                pavg=pavg+p[i][FUNC]
            pavg=pavg/NPTS
            tot=0.0
            if printlist!=0:
                print(iter)
                for j in range(NDIMS):
                    print(p[ilo][j]," ")
```

```
            for i in range(NPTS):
                tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg)
            sterr=sqrt(tot/NPTS)
            for j in range(NDIMS):
                z[j]=p[ilo][j]
            best = p[ilo][FUNC];
        # end of if iter%4*NDMS)==0
        #//// calculate avarage without maximum value //////
        ave=[0.0 for i in range(NDIMS)]
        for j in range(NDIMS):
            ave[j] = 0
        for i in range(NPTS):
            if i != ihi:
                for j in range(NDIMS):
                    ave[j] = ave[j]+p[i][j];
        for j in range(NDIMS):
            ave[j]=ave[j]/(NPTS-1)
        #print("ave[",j,"]=",ave[j])
        #//////// reflect ////////////////
        #print("reflect")
        r=[0.0 for i in range(NDIMS)]
        for j in range(NDIMS):
            r[j] = 2.0*ave[j] - p[ihi][j]
        fr = fnelder.func(r)
        if (flo <= fr) and (fr < fnhi): #in zone: accept
            for j in range(NDIMS):
                p[ihi][j] = r[j]
            p[ihi][FUNC] = fr
            continue
        #///(////////////// expand
        if fr < flo:
            #print("expand")
            e=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                e[j] = 3.0*ave[j] - 2.0*p[ihi][j];
            fe = fnelder.func(e);
            if fe < fr:
                for j in range(NDIMS):
                    p[ihi][j] = e[j]
                p[ihi][FUNC] = fe
                continue
            else:
                for j in range(NDIMS):
                    p[ihi][j] = r[j]
                p[ihi][FUNC] = fr
                continue
        #////////////// shrink:
        if fr < fhi:
            #print("srink")
            c=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                c[j] = 1.5*ave[j] - 0.5*p[ihi][j]
            fc = fnelder.func(c);
            if fc <= fr:
                for j in range(NDIMS):
                    p[ihi][j] = c[j];
                p[ihi][FUNC] = fc
                continue
            else:   #////// shrink
                for i in range(NPTS):
                    if i != ilo:
                        for j in range(NDIMS):
                            p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j]
                        p[i][FUNC] = fnelder.func(p[i])
                continue
        #/////////////// contract
        if fr >= fhi:
            #print("contract")
            cc=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
            fcc = fnelder.func(cc)
            if fcc < fhi:
                for j in range(NDIMS):
                    p[ihi][j] = cc[j];
                p[ihi][FUNC] = fcc
```

```
            continue
        else:
            for i in range(NPTS):
                if i != ilo:
                    for j in range(NDIMS):
                        p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j]
                    p[i][FUNC] = fnelder.func(p[i]);
    return z

a=[0.5,0.5,0.5]
da=[0.1122,0.243,0.25]
ff=fa()
f=f1(ff)
r=nelder(f,a,da,100,1.0e-15,0)
print("r = ",r)
```

```
runfile('E:/okul/SCO1/nelder_NLE.py', wdir='E:/okul/SCO1')
Reloaded modules: gauss, f_xj, fi_xi
r =  [0.5000562603286207, 1.0137773003008005e-05, -0.5235979821174814]
```

Nonlinear system of equation solution by using Particle swarm Algorithm Python code

```
from gauss import *;
from math import *;
from f_xj import *;
from fi_xi import *;
import numpy as np;
import matplotlib.pyplot as plt;
from mpl_toolkits.mplot3d import Axes3D
from random import *;

class f1(f_xj):
    def __init__(self,f):
        self.ff1=f
    def func(self,x):
        ffx=0.0
        fa=self.ff1.func(x)
        n=len(fa)
        for i in range(n):
            ffx=ffx+fa[i]*fa[i]
        return ffx

class fa(fi_xi):
    def __init__(self):
        self.name = 'fa'
    def func(self,x):
        ff=[0.0 for j in range(3)]
        ff[0]=3.0*x[0]-cos(x[1]*x[2])-0.5
        ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+sin(x[2])+1.06
        ff[2]=exp(-x[0]*x[1])+20.0*x[2]+(10.0*pi-3.0)/3.0
        return ff

class particle_swarm1i():

#class variables
#int m,n
#double alpha,beta,beta0,teta,gamma
#m number of population
#n number of variables for each individual
#double x[][];  location for each population member
#double y[];    function value of each population member
#double yxmin;  current best minimum
#double ygmin;  global best minimum
#double xmin[]; current best x
#double gmin[]; global best x
#double v[][];  velocity vector
#int ngeneration;  number of generations in iteration


    def __init__(self,ff,mi, xxmin,xxmax,alphai,betai,ngenerationi):
        #initial population set
        self.teta=0.5
        self.gamma=0.5
        #initial population set
```

```
        self.n=len(xxmin);
        self.m=mi;
        self.alpha=alphai
        self.beta0=betai
        self.beta=self.beta0
        self.ngeneration=ngenerationi
        self.x= [[0 for j in range(self.n)] for i in range(self.m)]
        self.v= [[0 for j in range(self.n)] for i in range(self.m)]
        self.y=[0 for i in range(self.m)]
        self.xmin=[0 for i in range(self.n)]
        self.gmin=[0 for i in range(self.n)]
        self.ngeneration=ngenerationi
        yxmin=1.0e50
        for i in range(self.m):
            for j in range(self.n):
                self.x[i][j]=xxmin[j]+random()*(xxmax[j]-xxmin[j])
                self.v[i][j]=random()
            self.y[i]=ff.func(self.x[i]);
            if self.y[i]<yxmin:
                yxmin=self.y[i]
                ygmin=self.y[i]
                for k in range(self.n):
                    self.xmin[k]=self.x[i][k]
                    self.gmin[k]=self.x[i][k]
        t=0
        while t<self.ngeneration:
            yxmin=1.0e50
            for i in range(self.m):
                for j in range(self.n):
                    self.x[i][j]=self.x[i][j]+self.v[i][j]
                    if self.x[i][j]<xxmin[j]:
                        self.x[i][j]=xxmin[j]
                    if self.x[i][j]>xxmax[j]:
                        self.x[i][j]=xxmax[j]
                    self.y[i]=ff.func(self.x[i])
                    if self.y[i]<yxmin:
                        yxmin=self.y[i]
                        for k in range(self.n):
                            self.xmin[k]=self.x[i][k]
            for i in range(self.m):
                if self.y[i]<ygmin:
                    ygmin=self.y[i]
                    for k in range(self.n):
                        self.gmin[k]=self.x[i][k]
            for i in range(self.m):
                for j in range(self.n):
                    self.v[i][j]=self.teta*self.v[i][j]+self.alpha*random()*(self.gmin[j]-self.xmin[j])+self.beta*random()*(self.xmin[j]-
self.x[i][j])
            t=t+1
            self.beta=self.beta0*pow(self.gamma,0.01*t)

ff=fa()
f=f1(ff)
xmin=[-1.0,-1.0,-1.0];
xmax=[1.0,1.0,1.0];
m=500
alpha=2.0
beta=0.7
ngeneration=500
r1= particle_swarm1i(f,m,xmin,xmax,alpha,beta,ngeneration)
z=r1.xmin;
print("xmin = ",z,"fmin=",f.func(z));
```

```
runfile('E:/okul/SCO1/particle_swarm_NLE.py', wdir='E:/okul/SCO1')
Reloaded modules: gauss, f_xj, fi_xi
xmin =  [0.49814468458949124, -0.1996058955437799, -0.5288259775733875] fmin= 1.232595164407831e-32
```

The next example method to solve system of nonlinear equations will be Genetic algorithm method

Program 5.32-2 Solution of Non-Linear system of equations by using Genetic Algorithms optimization method, java option

```
//Genetic algorithms to solve system of equations
import java.io.*;

class f1 extends f_xj
{//adaptor class
fi_xi ff1;
public f1(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return 10-ff; //minimum değil maxsimum istendiğinden - ile çarptik
}
}

class fa extends fi_xi
{
public double[] func(double x[])
{
//çözümü istenen fonksiyon
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
}
}

class NA118
{
   public static void main(String args[]) throws IOException
   {
   int N=18;
            int POPSIZE=500;
   int MAXGENS=500;
   int NVARS=3;
   double PXOVER=0.2;
   double PMUTATION=0.05;
    fa ff=new fa();
    f1 f_x=new f1(ff);
   Genetic1 xx=new Genetic1(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   double low[]=new double[NVARS];
   double high[]=new double[NVARS];
   low[0]=-1.0;high[0]=1.0;
   low[1]=-1.0;high[1]=1.0;
   low[2]=-1.0;high[2]=1.0;
   xx.setPopulation(low,high,f_x);
   System.out.println("best=\n"+Matrix.toStringT(xx.calculate(f_x,false)));
   }
}
```

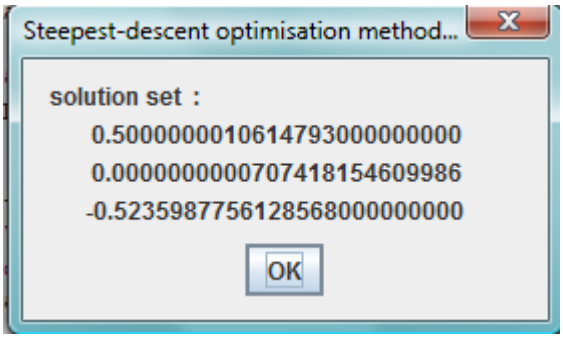
```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA118
best=
       0.499959945525915
      -0.000003814711818
      -0.523595899947739
       9.999999981442672
```

```
> Terminated with exit code 0.
```

As the next example steepest-descent method is used.

Program 5.32-3 Solution of Non-Linear system of equations by using Steepest descent optimization method, java option

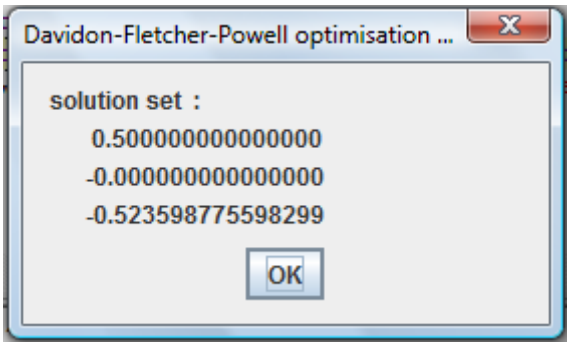
```
// Solution of system of equation by using Steepest descent Method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;


class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
public double[] func(double x[])
{
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
}
}

public class NA119
{
        public static void main (String args[])
    {
        String s="leave a space between initial root guess values: ";
    double [] x0=NA43.inputdata(s);
    fa ff=new fa();
    f2 f=new f2(ff);
    double [] r1= NA43.steepestdescent(f,x0);
    s=" solution set  : \n"+Matrix.toStringT(r1);
    String s2="Steepest-descent optimization method to solve system of non-linear equation:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```



Program 5.32-4 Solution of Non-Linear system of equations by using Davidon-Fletcher-Powell optimization method

```
// Solution of system of equation by using Davidon-Fletcher-Powell Method
//
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;


class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
public double[] func(double x[])
{
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
}
}

public class NA120
{
    public static void main (String args[])
    {  String s="leave a space between initial root guess values: ";
       double [] x0=NA43D.inputdata(s);
       fa ff=new fa();
       f2 f=new f2(ff);
       double [] r1= NA43D.davidon_fletcher_powell(f,x0);
       s=" solution set  : \n"+Matrix.toStringT(r1);
       String s2="Davidon-Fletcher-Powell optimization method to solve system of non-linear equation:";
       JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
       System.exit(0);
```

```
    }
}
```

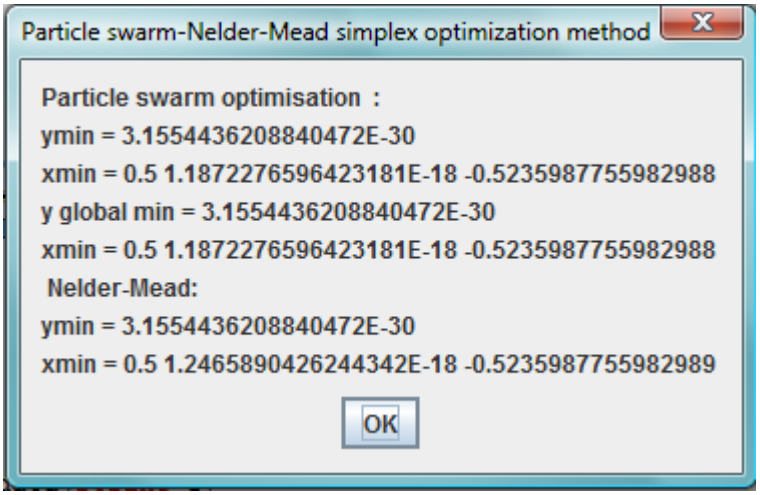


As a last example particle swarm optimisation is applied to system of non-linear equation solving. The same example is used as the perivious problems.

Program Solution of Non-Linear system of equations by using particle swarm optimization method

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
public double[] func(double x[])
{
double ff[]=new double[3];
ff[0]=3.0*x[0]-Math.cos(x[1]*x[2])-0.5;
ff[1]=x[0]*x[0]-81.0*(x[1]+0.1)*(x[1]+0.1)+Math.sin(x[2])+1.06;
ff[2]=Math.exp(-x[0]*x[1])+20.0*x[2]+(10.0*Math.PI-3.0)/3.0;
return ff;
}
}


public class particleswarmtest3
{
                public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
```

```
    return a;
    }

public static void main(String arg[])
{
String s="leave a space between initial root guess values: xmin0 xmin1.. ";
        double xmin[]=inputdata(s);
        s="leave a space between initial root guess values: xmax0 xmax1.. ";
        double xmax[]=inputdata(s);
        fa ff=new fa();
        f2 f=new f2(ff);

particle_swarm1 ps=new particle_swarm1(f,2000,xmin,xmax,2.0,0.7,2000);
double r1[]= particle_swarm1.nelder(f,ps.gmin());
s="Particle swarm optimisation  : \n"+ps.toString()+"\n Nelder-Mead: \n"+ps.toString(r1);
    String s2="Particle swarm-Nelder-Mead simplex optimization method ";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);

}
}
```



EXAMPLE APPLIED ENGINEERING PROBLEM : SOLUTION OF NON-LINEAR SYSTEM OF EQUATIONS :

All math algorithms heavily used in actual engineering and science problems. One of the purpose of this book is shown how it actually applied to actual science problems, following presentations by author is given as it is to reflect science application part of the nonlinear system of equations.

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08-10 Eylül 2021, GAZİANTEP

STOICHIOMETRIC CHEMİCAL EQUILIBRİUM ALGORİTHM BY USİNG SCHEREIBER AND PITZER REAL GAS EOS AND HOMOTOPHY(CONTINUATION) METHOD FOR SOLVING NON-LINEAR SYSTEM OF EQUATIONS

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Abstract In stoichiometric (based on K equilibrium constants) thermodynamic property algorithms, it is usually ideal gas EOS is used. The basic reason for this is simplicity of using ideal gas EOS. Real gas equation of states involves complicated mathematical procedures such as root finding processes, which is time consuming as well as requires long algorithms. Ideal gas based equilibrium calculations are usually satisfies requirements for chemical reactions at athmospheric pressure zone. An equation of state with better accuracy of thermodynamic properties will be required for extreme cases, such as gas turbine combustion chamber. In this study Scheireber-Pitzer real gas equation of state will be used to predict thermodynamic properties of gases. This is a generalised EoS for gases based on Pitzer's acentric factor. Stoichiometric formulations requires equlibrium formulations. Atomic balances gives equalibrium chemical moles as much as number of available atoms, for each additional chemical an additional equiibrium formulations is needed. When equations came together, they created quite a nonlinear system of equation. These set of equation can be solved by linearisation methods. In this study, it is prefer to solve directly in non-linear form by using homotophy(continuation) method. Continuation method is relatively less dependent to initial conditions. It is based on solution of a differential equation system by using a numerical soltion method. 6th degree Runge-Kutta method is used to solve differential equation to obtain non-linear equation solution by continuation method. All the codes are developed in java programming language. The program code are given to interested researchers as free access in www.turhancoban.com adress.
Keywords: Thermodynamic properties of gases, Schereiber-Pitzer EoS, chemical equilibrium, Stochiometric algorithm for chemical equilibrium

1. Formulation of Equation of State

In this paper, We will consider Scheiber-Pitzer equation of states for pure gases. Details of the Scheiber-PitzerEoS is given below.

$$Z=\frac{P}{\rho RT}=1+B(T_r,\rho_r)\rho_r+C(T_r,\rho_r)\rho_r^2+D(T_r,\rho_r)\rho_r^3+E(T_r,\rho_r)\rho_r^5+F(T_r,\rho_r)\rho_r^7+G(T_r,\rho_r)\rho_r^8+H(T_r,\rho_r)\rho_r^{10}+I(T_r,\rho_r)\rho_r^{12} \quad (1.1)$$

$$B(T_r,\rho_r)=c_1+\frac{c_2}{T_r}+\frac{c_3}{T_r^2}+\frac{c_4}{T_r^6}$$

$$C(T_r,\rho_r)=c_5+\frac{c_5}{T_r}+\frac{c_7}{T_r^3}+\frac{c_8}{T_r^4}\exp(-\rho_r^2)=C_1(T_r)+C_2(T_r)\exp(-\rho_r^2)$$

$$D(T_r,\rho_r)=c_9+\frac{c_{10}}{T_r}+\frac{c_{11}}{T_r^2}$$

$$E(T_r,\rho_r)=\frac{c_{12}}{T_r^2}+\frac{c_{13}}{T_r^3}$$

$$F(T_r,\rho_r)=\frac{c_{14}}{T_r^2}+\frac{c_{15}}{T_r^3}$$

$$G(T_r,\rho_r)=\frac{c_{16}}{T_r^3}+\left(\frac{c_{17}}{T_r^3}+\frac{c_{18}}{T_r^4}\right)\exp(-\rho_r^2)=G_1(T_r)+G_2(T_r)\exp(-\rho_r^2)$$

$$H(T_r,\rho_r)=\left(\frac{c_{19}}{T_r^3}+\frac{c_{20}}{T_r^4}\right)\exp(-\rho_r^2)=H_2(T_r)\exp(-\rho_r^2)$$

$$I(T_r,\rho_r)=\left(\frac{c_{21}}{T_r^3}+\frac{c_{22}}{T_r^4}\right)\exp(-\rho_r^2)=I_2(T_r)\exp(-\rho_r^2)$$

$$c_i=C_{i,0}+C_{i,1}\omega+C_{i,2}\omega^2$$

Where ωin Scheiber-Pitzer equation of states coefficient is called **Pitzer's accentric factor**. This factor is calculated as

$$\omega=-log_{10}P_{saturated\ vapor}(at\ T_r=0.7)-1 \quad (1.2)$$

Table 1.1 C[i][j] Coefficients of Scheiber-Pitzer EoS

	j=1	j=2	j=3
C1,j	0.4422590000	0.7256500000	0.0000000000
C2,j	-0.9809700000	0.2187140000	0.0000000000
C3,j	-0.6111420000	-1.2497600000	0.0000000000
C4,j	-0.0051562400	-0.1891870000	0.0000000000
C5,j	0.1513654000	2.3067060000	-10.4117400000
C6,j	-0.0438262500	4.6960680000	15.1414600000
C7,j	1.1026990000	3.1293840000	-9.5214090000
C8,j	-0.6361056000	0.3266766000	2.9046220000
C9,j	0.0087596260	-3.2040990000	8.0023380000
C10,j	0.3412103000	8.8721690000	-14.4038600000
C11,j	-0.8842722000	-6.6874710000	11.7685400000
C12,j	0.1375109000	0.2432806000	-0.5515101000
C13,j	-0.1443457000	1.2869320000	-2.1809880000
C14,j	-0.0059695540	0.0454196100	0.0000000000

C15,j	0.0245053700	-0.4158241000	0.7914067000
C16,j	-0.0041995900	0.0910596000	-0.1786378000
C17,j	0.0004665477	-1.2620280000	-2.8267720000
C18,j	-0.0194510100	0.7812220000	4.1900460000
C19,j	0.0408364300	1.3988440000	0.0000000000
C20,j	-0.0354691700	-1.4560410000	0.0000000000
C21,j	-0.0028779550	0.2104505000	0.0000000000
C22	0.0058962650	0.2191255000	0.0000000000

Derivatives of the equations:

$$\frac{\partial B(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[-\frac{c_2}{T_r^2}-2\frac{c_3}{T_r^3}-6\frac{c_4}{T_r^7}\right]$$
$$\frac{\partial C(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[-\frac{c_5}{T_r^2}-3\frac{c_7}{T_r^4}-4\frac{c_8}{T_r^5}\exp\left(-\rho_r^2\right)\right]$$
$$\frac{\partial D(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[-\frac{c_{10}}{T_r^2}-3\frac{c_{11}}{T_r^4}\right]$$
$$\frac{\partial E(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[-\frac{c_{12}}{T_r^2}-3\frac{c_{13}}{T_r^4}\right]$$
$$\frac{\partial F(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[-2\frac{c_{14}}{T_r^3}-3\frac{c_{13}}{T_r^4}\right]$$
$$\frac{\partial G(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[-3\frac{c_{16}}{T_r^4}+\left(-3\frac{c_{17}}{T_r^4}-4\frac{c_{18}}{T_r^5}\right)\exp\left(-\rho_r^2\right)\right]$$
$$\frac{\partial H(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[\left(-3\frac{c_{19}}{T_r^4}-4\frac{c_{18}}{T_r^5}\right)\exp\left(-\rho_r^2\right)\right]$$
$$\frac{\partial I(T_r,\rho_r)}{\partial T}=\frac{1}{T_c}\left[\left(-3\frac{c_{21}}{T_r^4}-4\frac{c_{22}}{T_r^5}\right)\exp\left(-\rho_r^2\right)\right] \quad (1.3)$$

Helmholtz energy equation will be used to predict other thermodynamic properties

$$dA=-SdT-Pdv \quad (1.4)$$
$$\left.\frac{\partial A}{\partial v}\right|_T=-P$$
$$dA=-Pdv=\frac{P}{\rho^2}d\rho$$
$$A-A_0=\int_{\rho_0}^{\rho}\frac{P}{\rho^2}d\rho=\int_{0}^{\rho}\frac{P}{\rho^2}d\rho+\int_{\rho_0}^{0}\frac{P}{\rho^2}d\rho=\int_{0}^{\rho}\frac{P}{\rho^2}d\rho+\int_{\rho_0}^{0}\frac{\rho RT}{\rho^2}d\rho$$

The second term (limits between low density ρ_0 and $density$ 0) can be defined as an ideal state case where $P=\rho RT$. Now we can add and substract ideal gas density term of the equation

$$A-A_0=\int_{\rho_0}^{\rho}\frac{P}{\rho^2}d\rho=\int_{0}^{\rho}\frac{P}{\rho^2}d\rho+\int_{\rho_0}^{0}\frac{\rho RT}{\rho^2}d\rho+\int_{0}^{\rho}\frac{\rho RT}{\rho^2}d\rho-\int_{0}^{\rho}\frac{\rho RT}{\rho^2}d\rho \quad (1.5)$$

Considering for real EoS $P=Z\rho RT$ equation becomes

$$A-A_0=\int_{\rho_0}^{\rho}\frac{P}{\rho^2}RT[B(T_r,\rho_r)\rho_r+C(T_r,\rho_r)\rho_r^2+D(T_r,\rho_r)\rho_r^3+E(T_r,\rho_r)\rho_r^5+F(T_r,\rho_r)\rho_r^7+G(T_r,\rho_r)\rho_r^8+H(T_r,\rho_r)\rho_r^{10}+I(T_r,\rho_r)\rho_r^{12}]d\rho$$
$$=\int_{0}^{\rho}\frac{P}{\rho^2}d\rho+\int_{\rho_0}^{0}\frac{\rho RT}{\rho^2}d\rho+\int_{0}^{\rho}\frac{\rho RT}{\rho^2}d\rho-\int_{0}^{\rho}\frac{\rho RT}{\rho^2}d\rho \quad (1.6)$$
$$A-A_0=\int_{0}^{\rho}\frac{Z\rho RT-\rho RT}{\rho^2}d\rho+\int_{\rho_0}^{\rho}\frac{\rho RT}{\rho^2}d\rho$$
$$A-A_0=\int_{0}^{\rho}\frac{RT(Z-1)}{\rho}d\rho+\int_{\rho_0}^{\rho}\frac{\rho RT}{\rho^2}d\rho$$
$$A-A_0=\int_{0}^{\rho_r}\frac{1}{\rho_r}d\rho_r+\int_{\rho_{r0}}^{\rho_r}\frac{RT}{\rho_r}d\rho_r$$
$$A-A_0=RT\int_{0}^{\rho_r}[B(T_r,\rho_r)+C(T_r,\rho_r)\rho_r+D(T_r,\rho_r)\rho_r^2+E(T_r,\rho_r)\rho_r^4+F(T_r,\rho_r)\rho_r^6+G(T_r,\rho_r)\rho_r^7+H(T_r,\rho_r)\rho_r^9+I(T_r,\rho_r)\rho_r^{11}]d\rho_r$$
$$+RTln\frac{\rho_r}{\rho_{r0}} \quad (1.7)$$

Some of the terms in this integration includes terms of exponential and power multiplications. This integrations are carried out as follows:

$$K(m,\rho_r)=\int_0^{\rho_r}\rho_r^m\exp(-\rho_r^2)d\rho_r=\int_0^{\rho_r}\sum_{n=0}^{\infty}(-1)^n\frac{\rho_r^{2n+m}}{n!}d\rho_r=\sum_{n=0}^{\infty}(-1)^n\frac{\rho_r^{2n+m+1}}{(2n+m+1)n!} \quad (1.8)$$

Handling these terms such as above Taylor series are much easier than taken numerical integration, then Helmholtz departure function becomes:

$$A-A_0=RT\left(B\rho_r+C_1\frac{\rho_r^2}{2}+C_2K(1,\rho_r)+D\frac{\rho_r^3}{3}+E\frac{\rho_r^5}{5}+F\frac{\rho_r^7}{7}+G_1\frac{\rho_r^8}{8}+G_2,K(8,\rho_r)+H_2K(10,\rho_r)+I_2K(12,\rho_r)\right)$$
$$+RTln\left(\frac{\rho_r}{\rho_{r0}}\right) \quad (1.9)$$

Entropy departure function:

$$S-S_0=-\left.\frac{\partial(A-A_0)}{\partial T}\right|_{\rho} \quad (1.10)$$

$$S-S_0=R\left(B\rho_r+C_1\frac{\rho_r^2}{2}+C_2K(1,\rho_r)+D\frac{\rho_r^3}{3}+E\frac{\rho_r^5}{5}+F\frac{\rho_r^8}{8}+G_1\frac{\rho_r^9}{9}+G_2,K(8,\rho_r)+H_2K(10,\rho_r)+I_2K(12,\rho_r)\right)$$
$$+RT\left(\frac{dB}{dT}\rho_r+\frac{dC_1}{dT}\frac{\rho_r^2}{2}+\frac{dC_2}{dT}K(1,\rho_r)+\frac{dD}{dT}\frac{\rho_r^3}{3}+\frac{dE}{dT}\frac{\rho_r^5}{5}+\frac{dF}{dT}\frac{\rho_r^8}{8}+\frac{dG_1}{dT}\frac{\rho_r^9}{9}+\frac{dG_2}{dT}K(8,\rho_r)+\frac{dH_2}{dT}K(10,\rho_r)\right.$$
$$\left.+\frac{dI_2}{dT}K(12,\rho_r)\right)+Rln\left(\frac{\rho_r}{\rho_{r0}}\right)$$

Enthalpy departure function:

$$H-H_0=(A-A_0)+T(S-S_0)+RT(Z-1) \quad (1.11)$$

Internal energy departure function:

$$U-U_0=(A-A_0)+T(S-S_0) \quad (1.12)$$

Gibbs energy departure function:

$$G-G_0=(A-A_0)+RT(Z-1) \quad (1.13)$$

The fugacity-pressure ratio:

$$ln\frac{f}{P}=\frac{(A-A_0)}{RT}+ln\frac{v}{v_0}+(Z-1)-\ln(Z) \quad (1.14)$$

Data is also needed to solve $C_p(T)$ value. In order to establish that, NIST tables given at the address https://janaf.nist.gov/ is used. The following partial continuous formulation is taken. the following partial difference curve fitting formula is used

$$C_{pi}(T)=A_i+B_i10^{-3}T+\frac{C_i10^5}{T^2}+D_i10^{-6}T^2T_{Li}\le T\le T_{Hi} \quad (1.15)$$

As an example case of Partial continuous curve fitting of Cp values Data for Nitrogen is given below.

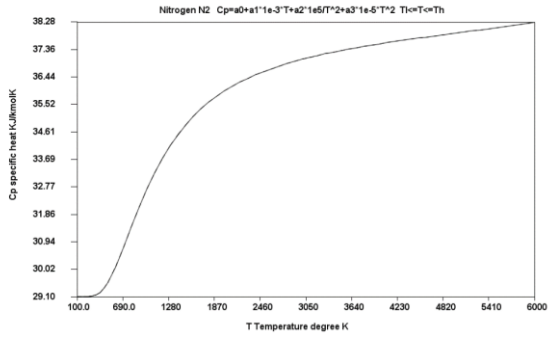


Figure 1.1 Cp of Nitrogen kJ/(kmolK)

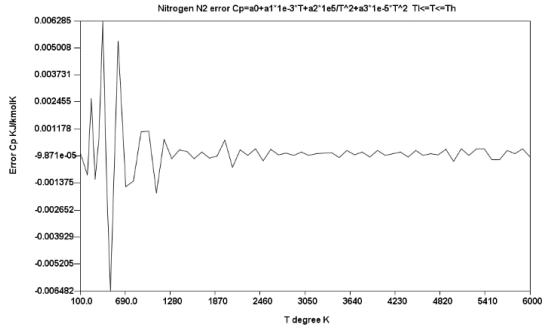


Figure 1.2 Error of Cp in Nitrogen kJ/(kmolK)

Table 1.2 Cp (kJ/kmol K) partial continious curve fitting equations for N_2 Nitrogen

i	A_i	B_i	C_i	D_i	T_{li} K	T_{hi} K
0	29.408631	-2.251447	-0.012473	4.520889	100	350
1	27.646169	0.882356	0.770074	4.764423	350	700
2	21.601706	14.878414	3.812808	-4.165467	700	1200
3	29.830766	5.421561	-15.043096	-1.089614	1200	1700
4	35.476742	0.973583	-42.547627	-0.097466	1700	2200
5	34.928203	1.319404	-38.184192	-0.159911	2200	2700
6	36.262526	0.581500	-50.898362	-0.045731	2700	3200
7	35.657341	0.766169	-34.665936	-0.059817	3200	3700
8	36.418045	0.432596	-44.184706	-0.020152	3700	4200
9	38.077688	-0.152960	-80.311808	0.036794	4200	4700
10	37.760284	-0.049949	-73.101156	0.027769	4700	5200
11	39.973855	-0.854555	-77.575938	0.101253	5200	6000

i	A_i	B_i	C_i	D_i	T_{li} K	T_{hi} K
0	29.4086307829	-2.2514470327	-0.0124732186	4.5208886188	100	350
1	27.6461690069	0.8823555268	0.7700742081	4.7644228675	350	700
2	21.6017064500	14.8784143146	3.8128084889	-4.1654669506	700	1200
3	29.8307659455	5.4215607907	-15.0430960215	-1.0896138268	1200	1700
4	35.4767415122	0.9735825946	-42.5476274875	-0.0974664401	1700	2200
5	34.9282028043	1.3194039653	-38.1841919451	-0.1599114820	2200	2700
6	36.2625256395	0.5815001033	-50.8983620805	-0.0457311313	2700	3200
7	35.6573409828	0.7661686027	-34.6659363416	-0.0598170521	3200	3700
8	36.4180454205	0.4325957723	-44.1847062013	-0.0201521727	3700	4200
9	38.0776880528	-0.1529603974	-80.3118075101	0.0367938172	4200	4700
10	37.7602843891	-0.0499492999	-73.1011559910	0.0277685442	4700	5200
11	39.9738552178	-0.8545553355	-77.5759376892	0.1012534665	5200	6000

For some gases only data is available is in polynomial curve fitting format, for this cases polynomial form is assumed.

2. Stoichiometric chemical equilibrium formulations

Actual products formed in a reaction is not a clear cut concept. The actual products formed in a chemical reactions can only be determined experimentally and as a function of the environmental values such as temperature, pressure and time. When these parameters changed actual products can change as well. One way of estimating what products will be formed is to use equilibrium concept. All physical systems are tend to minimize their energy levels when there are no any external force or energy to the system. For example a bubble takes spherical shape under the influence of surface tension which has the minimal area for the given volume. Chemical reactions when temperature and pressure is constant and long enough time is given, tends to minimize it energy level, specifically gibbs free energy level (or maximize its entropy level). This state is called chemical equilibrium. For example consider a closed system consisting initially of a gaseous mixture of carbondioxide , oxygen and carbonic acid. A reaction might take place is

$$CO_2 + H_2O \rightleftarrows H_2CO_3$$

At equilibrium the system will consist basically of three components, CO_2, H_2CO_3 and H_2O, for not all the components gases. This is what is called soda reaction. When soda bottle opened equailibrium condition (pressured in the system) is changed and rection slides to a new equilibrium states, so that some omount of CO_2 and H_2O is

formed and amount of carbonic acid is reduced . Changes in the amounts of these components during the opening of soda bottle:

$$dn_{CO_2} = -dn_{H_2CO_3}$$
$$dn_{H_2O} = -dn_{H_2CO_3} \quad (2.1)$$

where dn denotes the differential change in the representative component.

$$-\frac{dn_{CO_2}}{1} = -\frac{dn_{H_2O}}{1} = \frac{dn_{H_2CO_3}}{1} \quad (2.2)$$

Equilibrium is a condition of balance. When equilibrium is established amount of CO_2 and H_2O and H_2CO_3 tends to be balanced with each other. But when equilibrium is broken and a new conditions arise, like opening of a soda bottle, a new equilibrium will be established. And during these process total gibbs free energy of the system will be minimised.

$$dG(T,P) = 0 \quad (2.3)$$

This amount can be determined from chemical potential changes (see the first chapter for details)

$\mu_{CO_2} = \frac{\partial G(T,P,n)}{\partial n_{CO_2}} \quad \partial G(T,P,n) = \mu_{CO_2} \partial n_{CO_2}$

$\mu_{n_{H_2CO_3}} = \frac{\partial G(T,P,n)}{\partial n_{H_2CO_3}} \quad \partial G(T,P,n) = \mu_{H_2CO_3} \partial n_{H_2CO_3}$

$\mu_{H_2O} = \frac{\partial G(T,P,n)}{\partial n_{H_2O}} \quad \partial G(T,P,n) = \mu_{H_2O} \partial n_{H_2O}$

$$dG(T,P) = -1 * \mu_{CO_2} dn_{CO_2} - 1 * \mu_{H_2O} dn_{H_2O} + 1 * \mu_{H_2CO_3} dn_{H_2CO_3} = 0 \quad (2.4)$$
$$dG(T,P) = \left(-1 * \mu_{CO_2} - 1 * \mu_{H_2O} + 1 * \mu_{H_2CO_3}\right) dn_{H_2O} = 0 \quad (2.5)$$

This is called the equilibrium reaction.For a more general equilibrium equation such as:

$$\upsilon_A A + \upsilon_B B \rightleftarrows \upsilon_C C + \upsilon_D D \quad (2.6)$$

where υ 's are stoichiometric coefficients. The following equilibrium case is existed:

$$-\frac{dn_A}{\upsilon_A} = -\frac{dn_B}{\upsilon_B} = \frac{dn_C}{\upsilon_C} = \frac{dn_D}{\upsilon_D} = d\varepsilon \quad (2.7)$$

where $d\varepsilon$ is the proportionality factor. From which the following expressions are obtained.

$$dn_A = -\upsilon_A \, d\varepsilon$$
$$dn_B = -\upsilon_B \, d\varepsilon$$
$$dn_C = \upsilon_C \, d\varepsilon$$
$$dn_D = \upsilon_D \, d\varepsilon$$

(2.8)

For this case the Gibbs free energy equation takes form:

$$dG(T,P) = -\upsilon_A * \mu_A - \upsilon_B * \mu_B + \upsilon_C * \mu_C + \upsilon_D * \mu_D = 0 \quad (2.9)$$

For ideal gas mixtures chemical potential can be expressed as specific gibbs free energy as

$$h_{Ti}(T) = h_i(T) - h(298.15) + \Delta h_f(298.15)$$
$$g^0(T) = h_{Ti}(T) - Ts^0(T)$$
$$\mu_i = g_i^0(T) - RTln\left(\frac{P_i}{P_{ref}}\right) \quad (2.10)$$

where P_i is the partial pressure of the gas. For and ideal gas, partial pressure can be expressed as a function of components in the total components

$y_i = \frac{n_i}{n_{total}} \quad P_i = y_i P \quad P_i = \frac{n_i}{n_{total}} P \quad (2.11)$

Substituting these into the equation gives:

$$\mu_i = g_i^0(T) - RTln\left(\frac{n_i}{n_{total}} \quad \frac{P}{P_{ref}}\right) \quad (2.12)$$

For a real gas mixtures chemical potential can be expressed as specific gibbs free energy as

$$h_{Ti}(T,P) = h_i(T,P) - h(298.15,P) + \Delta h_f(298.15) \quad (2.13)$$
$$s(T,P) = s^0(T) - Rln\left(\frac{f_i}{P_{ref}}\right) \quad (2.14)$$
$$g(T,P) = g^0(T,P) - RTln\left(\frac{f_i}{P_{ref}}\right) \quad (2.15)$$
$$g^0(T,P) = h_{Ti}(T,P) - Ts^0(T) \quad (2.16)$$
$$\mu_i = g_i^0(T) - RTln\left(\frac{f_i}{P_{ref}}\right) \quad (2.17)$$

Note that $\lim_{P \to 0} \frac{f_i}{P_i} = 1 \quad (2.18)$ therefore $P_{ref} = f_{ref}$

where f_i is the partial fugacity of the gas. For a real gas, partial pressure can be expressed as a function of components in the total components

$$y_i = \frac{n_i}{n_{total}} \quad f_i = y_i f^* \quad f_i = \frac{n_i}{n_{total}} f^* \quad (2.19)$$

Where f^* fugacity that component i would have if the entire gas had that composition at the same temperature and pressure. $f^* = f(T,P)$ (2.20)

Substituting these into the equation gives:

$$\mu_i = g_i^0(T,P) - RTln\left(\frac{n_i}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) \quad (2.21)$$

Then the basic Gibbs free energy equation becomes

$$dG(T,P) = -\upsilon_A\left[g_A^0(T) - RTln\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] - \upsilon_B\left[g_B^0(T) - RTln\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] + \upsilon_C\left[g_C^0(T) - RTln\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right]$$
$$+ \upsilon_D\left[g_D^0(T) - RTln\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] = 0 \quad (2.22)$$

Arranging equation:

$$dG^0(T,P) = -\upsilon_A g_A^0(T,P) - \upsilon_B g_B^0(T,P) + \upsilon_C g_C^0(T,P) + \upsilon_D g_D^0(T,P)$$
$$dG(T,P) = dG^0(T,P) + RT\left[\upsilon_A ln\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) + \upsilon_B ln\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_C ln\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_D ln\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] = 0 \quad (2.23)$$

This equation can be written as:

$$-\frac{dG^0(T,P)}{RT} = \left[\upsilon_A ln\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) + \upsilon_B ln\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_C ln\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_D ln\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] \quad (2.24)$$

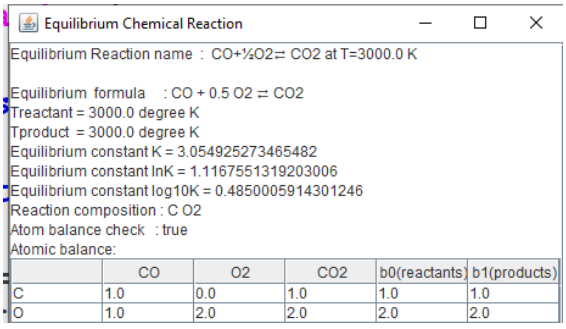
$$exp\left(-\frac{dG^0(T,P)}{RT}\right) = \left[\frac{\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)^{\upsilon_A}\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)^{\upsilon_B}}{\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)^{\upsilon_C}\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)^{\upsilon_D}}\right] = \left[\frac{(n_A)^{\upsilon_A}(n_B)^{\upsilon_B}}{(n_C)^{\upsilon_C}(n_D)^{\upsilon_D}}\right]\left(\frac{f^*}{n_{total}P_{ref}}\right)^{\upsilon_A+\upsilon_B-\upsilon_C-\upsilon_D} \quad (2.25)$$

The left hand side of the equation is called equilibrium constant, which is only function of temperature

$$K(T) = exp\left(-\frac{dG^0(T,P)}{RT}\right) \quad (2.26)$$

As an example let us evaluate equilibrium constant of the equilibrium reaction

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$



Equilibrium condition can be solved by solving chemical balance equations together with the equilibrium gibbs free energy minimisation equations. So mass balance is an important part of total set of equations to solve. Mass balance establishes as follows:

$$\sum_{j=1}^{NS} A_{ij} n_j - b_i^0 = 0 \quad (i = 1..na) \quad (2.27)$$

$$b_i = \sum_{j=1}^{NS} A_{ij} n_j \quad (2.28)$$

$$b_i - b_i^0 = 0 \quad (i = 1..na) \quad (2.29)$$

where na is the number of chemical elements. A_{ij} is number of kilogram atoms per kmole of species j. And b_i^0 is the assigned number of kilogram atoms element i per kmol of total reactants.

In order to explain this equation let us look at an example. If chemicals in the reaction and input moles re given as:

CH_4	1 kmol

H_2O	10 kmol
H_2	0 kmol
CO_2	0 kmol
CO	0 kmol
O_2	0 kmol

A_{ij} matrix will be

atom	CH_4	H_2O	H_2	CO_2	CO	O_2
H	4	2	2	0	0	0
C	1	0	0	1	1	0
O	0	1	0	2	1	2

b_i^0 vector is the multiplication of number of atoms with inlet mole numbers of the molecule

b_H^0=1*4+10*2+0*2+0*0+0*0+0*0=24

b_C^0=1*1+10*0+0*0+0*1+0*1+0*0=1

b_O^0=0*1+10*1+0*0+0*2+0*1+0*2=10

In this case initial matrix will be in the form of:

$$\begin{bmatrix} 4 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} n_{CH_4} \\ n_{H_2O} \\ n_{H_2} \\ n_{CO_2} \\ n_{CO} \\ n_{O_2} \end{Bmatrix} = \begin{Bmatrix} 24 \\ 1 \\ 10 \end{Bmatrix} \quad (2.30)$$

As it is seen from the example mass balance are given us 3 equations, but total number of equations(moles) are 6, remaining equations will be gibbs equations as described above. All togetger they construct a system of non-linear equations to solve. The results will be equilibrium balance.

In order to solve system of equations continuity method(it is also called homotophy method) is used.
This method is relatively less dependent to initial estimation of the system of equation solution, therefore a good selection for solving the system of non-linear equation. The method details is as follows:When a problem of system of nonlinear equations of the form F(x)=0 desired to be solved, assume that solution set to be found is x*.
Consider a parametric function G(λ,x) in the form of

G(λ,x) = λF(x) + (1-λ)[F(x) - F(x(0))] (2.31)

Where λ=0 coresponds to initial guess of the solution , x(0) ,and where λ=1 value corresponds the actual solution set x(1)= x*

It is desired to be found G(λ,x) = 0 therefore for λ=0 equation becomes

G(λ,x) =G(0,x) = F(x) - F(x(0)) and for λ=1 (2.32)

0=G(1,x) = F(x) (2.33)

Therefore at x(1)=x* solution set will be obtained. If a function G(λ,x) satisfies the above equation can be found, it will also find us the solution. Function G is called a homotopy between the function G(0,x) and G(1,x)=F(x). In order to find such a function, it is assumed to have a function

G(λ,x)=0 is existed and partial derivative of this function with respect to λ and x will also be zero

$$0 = \frac{\partial G(\lambda, x)}{\partial \lambda} + \frac{\partial G(\lambda, x)}{\partial x} x'(\lambda) \quad (2.34)$$

if x'(λ) is isolated form this equation, it becomes:

$$x'(\lambda) = -\left[\frac{\partial G(\lambda, x(\lambda))}{\partial x}\right]^{-1} \left[\frac{\partial G(\lambda, x(\lambda))}{\partial \lambda}\right] x'(\lambda) \quad (2.35)$$

If G(λ,x) = λF(x) + (1-λ)[F(x) - F(x(0))] equation is substituted into the differential equation

$$\left[\frac{\partial G(\lambda, x(\lambda))}{\partial x}\right] = \begin{bmatrix} \frac{\partial f_1(x(\lambda))}{\partial x_1} & \frac{\partial f_1(x(\lambda))}{\partial x_2} & \frac{\partial f_1(x(\lambda))}{\partial x_3} \\ \frac{\partial f_2(x(\lambda))}{\partial x_1} & \frac{\partial f_2(x(\lambda))}{\partial x_2} & \frac{\partial f_2(x(\lambda))}{\partial x_3} \\ \frac{\partial f_3(x(\lambda))}{\partial x_1} & \frac{\partial f_3(x(\lambda))}{\partial x_2} & \frac{\partial f_3(x(\lambda))}{\partial x_3} \end{bmatrix} = J(x(\lambda) \quad (2.36)$$

Forms a Jacobian matrix. and

$$\left[\frac{\partial G(\lambda, x(\lambda)}{\partial x}\right] = F(x(0)) \quad (2.37)$$

Differential equation becomes

$$x'(\lambda) = \frac{dx(\lambda)}{d\lambda} = -\left[J(x(\lambda))\right]^{-1} F(x(0)) \qquad 0 \le \lambda \le 1 \quad (2.38)$$

It is possible to solve such a differential equation by using initial value problem approaches, solution at x(1) will be given us the roots of the system of equation. Solutions of initial value problems will be given latter chapters in details, but A sixth order Runge-Kutta differential equation solution will be defined here to solve our homotopy problem. If equation

$$\frac{dx(\lambda)}{d\lambda} = f(\lambda, x(\lambda)) \quad (2.39)$$ is given the 6th order Runge-Kutta method to numerically solve this differential equation is defined as:

$y_{i+1} = y_i + (1/90)*(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)h$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + 0.25h , y_i + 0.25k_1h)$

$k_3 = f(x_i + 0.25h , y_i + 0.125k_1h + 0.125k_2h)$

$k_4 = f(x_i + 0.5h , y_i - 0.5k_2h + k_3h)$

$k_5 = f(x_i + 0.75h , y_i + (3/16)k_1h + (9/16)k_4h)$

$k_6 = f(x_i + h , y_i - (3/7)k_1h + (2/7)k_2h + (12/7)k_3h - (12/7)k_4h + (8/7)k_5h)$ (2.40)

This equation can be given as Buthcher tableu as:

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 0 & 0 & 0 \\ 2/4 & 1/4 & -1/4 & 1 & 0 & 0 \\ 3/4 & 3/16 & 0 & 0 & 9/16 & 0 \\ \hline 1 & -3/7 & 2/7 & 12/7 & -12/7 & 8/7 \\ & 1/90 & 7/90 & 32/90 & 12/90 & 7/90 \end{array} \quad (2.41)$$

In these equations h is finite difference step size. Solution starts by using the initial value $\lambda=0$, x0(0) and adds h into λ in each iteration step. The code given here uses 6th degree Runge-Kutta method to solve homotopy(Continuation problem). It should be note that Homotophy method is less dependent to initial value compare to methods such as Newton-Raphson therefore one possibility is to approach solution with a relatively rough estimate with homotophy following with a Newton-Raphson type of method, which is quite efficient when the estimation approaches the correct roots.

3. Computer code development

In order to calculate termodynamic properties of gases Schereiber and Pitzer real gas EoS is developed (Gas_SP.java). This equation of state has a subclass to calculate specific heat values as curvefitting values (Gas_Data.java). Equilibrium coefficients are calculated from ChemicalRaction_SP class. System of equations are set together in if_equilibrium_SP class. Continuatio method to solve non-linear system of equation is given in class iterative_continuity. And finally equilibrium codes are solve in equilibrium_SP class. In addition to this set an ideal gas equivalent is also prepared for comparison purposes. List of classes and their utilisation areas are given as a table below:

Program	Utilisation
Atom	Calculation atomic balances, atomic properties
Gas_Data	Gas data for approximately 600 gases
Gas_SP	Schereiber and Pitzer real gas EoS
Gas_PG	Perfect Gas EoS
ChemicalReaction_SP	Chemical reaction calculater for Gas_SP
If_equilibrium_SP	Defines non-linear system of equation for Gas_SP
Equilibrium_SP	Stoichiometric Equilibrium calculator for Gas_SP
ChemicalReaction_PG	Chemical reaction calculater for Gas_PG

If_equilibrium_PG	Defines non-linear system of equation for Gas_PG
Equilibrium_PG	Stoichiometric Equilibrium calculator for Gas_SP
Iterative_continuity	Continuation(homotophy) method for non-linear system of eqns.

Sample solutions:

One kmol of CO an d one kmol of O_2 established an equilibrium at 3000 K. The equilibrium reaction for this is as follows:

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$

The reaction will be

$$CO + O_2 \rightarrow n_0 CO + n_1 O_2 + n_2 CO_2$$

Find the equilibrium composition. System pressure is P=101.325 bar. (Pref=101.325 bar)

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
public class equilibrium_SP
{public double N[][][];
 public String s[];
 public double result[][];
 public double P;
 public chemicalReaction_SP r[];
 public if_equilibrium_SP fe;
 public equilibrium_SP(String si[],double Ni[][][],double Pi)
 {   N=Ni;
     s=si;
     P=Pi;
                int n=N.length;
     r=new chemicalReaction_SP[n];
     for(int i=0;i<n;i++)
     {r[i]=new chemicalReaction_SP("reaction:"+i,s,N[i]);}
 }
 public double[][] calculate(double Tproduct,double Treactant,double P,double n0[],double high[])
 {fe=new if_equilibrium_SP(n0,s,r,Treactant,Tproduct,P);
  int n_eqn=n0.length;
  double low[]=new double[n_eqn];
  int n_iteration=10;
  iterative_continuity itc=new iterative_continuity(n_iteration,low,high);
  result= itc.findContinuityRK4(fe);
  return result;
 }

public static void main(String arg[])
  {
     String s[]={"CO","O2","CO2"};
     double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0}};
     double N[][][]={N1};
     double P=101.325; //kPa
     equilibrium_SP eq=new equilibrium_SP(s,N,P);
     double Tproduct=3000;//degree K
     double Treactant=3000;//degree K
     //input moles
     //CO+H2O+N2-->
     double n0[]={1,1,0};
     //output mole first estimates
     double high[]={1.0,1.0,1.0};
     eq.calculate(Tproduct,Treactant,P,n0,high);
     String s1[]={"x initial guess","x","y=f(x)"};
     Text.printT(eq.result,s1,"Newton_continuation");
  }

 }
```

Newton_continuation

x initial guess	x	y=f(x)
0.3406696157029702	0.3406696157029702	4.222021930511133E-6
0.6703472942504498	0.6703472942504498	1.3546075638970478E-7
0.6593266528142379	0.6593266528142379	1.327369465564665E-5

To compare the results let us also run equlibrium_PG for the same conditions:

Newton_continuation

x initial guess	x	y=f(x)
0.049084885297797864	0.3421068814833881	0.0
0.4174016135657407	0.671053440741694	0.0
0.5199888966313939	0.657893118516612	0.0

For T=3000 K and P=1.01325 bar, if N2 is added to the reaction. Equilibrium reaction is still the same:

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$

The reaction will be

$CO+O_2+1.88N_2 \rightarrow n_0CO+n_1O_2+n_2CO_2+1.88N_2$

Find the equilibrium composition.

```
public static void main(String arg[])
 {
    String s[]={"CO","O2","CO2","N2"};
    double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{1.88,1.88}};
    double N[][][]={N1};
    double P=101.325; //kPa
    equilibrium_SP eq=new equilibrium_SP(s,N,P);
    double Tproduct=3000;//degree K
    double Treactant=3000;//degree K
    //input moles
    //CO+H2O+N2-->
    double n0[]={1,1,0,1.88};
    //output mole first estimates
    double high[]={1.0,1.0,1.0,2.0};
    eq.calculate(Tproduct,Treactant,P,n0,high);
    String s1[]={"x initial guess","x","y=f(x)"};
    Text.printT(eq.result,s1,"Newton_continuation");
 }
```

Newton_continuation

x initial guess	x	y=f(x)
0.4237119081525775	0.4237119081525775	3.5451650144402436E-6
0.7118707072048843	0.7118707072048843	2.302036426371501E-6
0.5762922534844812	0.5762922534844812	2.2552501910411138E-7
1.8799057131256034	1.8799057131256034	1.6311727337958715E-5

To compare the results let us also run equlibrium_PG for the same conditions:

Newton_continuation

x initial guess	x	y=f(x)
0.17901996274748555	0.42524996056811837	0.0
0.38504904730880185	0.7126249802840592	0.0
0.5567586223735005	0.5747500394318816	0.0
1.8905350157358185	1.88	0.0

For T=3000 K and P=1000 kPa, if N2 is added to the reaction. Equilibrium reaction is still the same:

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$ The reaction will be $CO + O_2 \rightarrow n_0CO + n_1O_2 + n_2CO_2$

Newton_continuation

x initial guess	x	y=f(x)
0.1471735996531205	0.1471735996531205	1.2957901418531037E-...
0.5735924677072117	0.5735924677072117	-6.902536369146617E-6
0.8528203359640902	0.8528203359640902	-2.799026342037436E-6

To compare the results let us also run equlibrium_PG for the same conditions:

Newton_continuation

x initial guess	x	y=f(x)
0.0	0.14799227192627473	0.0
0.0	0.5739961359631375	0.0
0.0	0.8520077280737253	0.0

Equilibrium Chemical Reaction

```
Treactant = 3000.0 degree K
Tproduct  = 3000.0 degree K
P         = 101.325 bar
========================================
Equilibrium Reaction  : CO + 0.5 O2 ⇄ CO2
Equilibrium constant K = 3.054925273465477
Equilibrium constant lnK = 1.1167551319202988
========================================
Equilibrium Reaction  : CO + H2O ⇄ CO2 + H2
Equilibrium constant K = 5.5591404532168915E-5
Equilibrium constant lnK = -9.7974819634039
========================================
```

name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.5	0.992611081...	0.496302593...
H2O	1.0	0.5	0.992587331...	0.496290718...
CO2	0.0	0.0	0.007388918...	0.003694437...
H2	0.0	0.0	0.007412668...	0.003706312...
O2	0.0	0.0	1.187503614...	5.937482819...
total	2.0	1	2.000011875...	1

Example case:

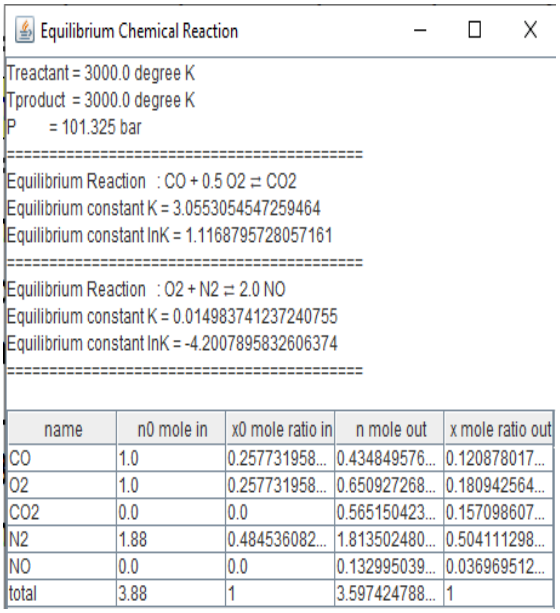
For T=3000 K and P=101.325 kPa, if N2 is added to the reaction. Equilibrium reaction is still the same:

$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2 \quad N_2 + 1/2O_2 \rightleftharpoons 2NO$

The reaction will be $CO + O_2 \rightarrow n_0 CO + n_1 O_2 + n_2 CO_2 + n_3 N_2 + n_4 NO$

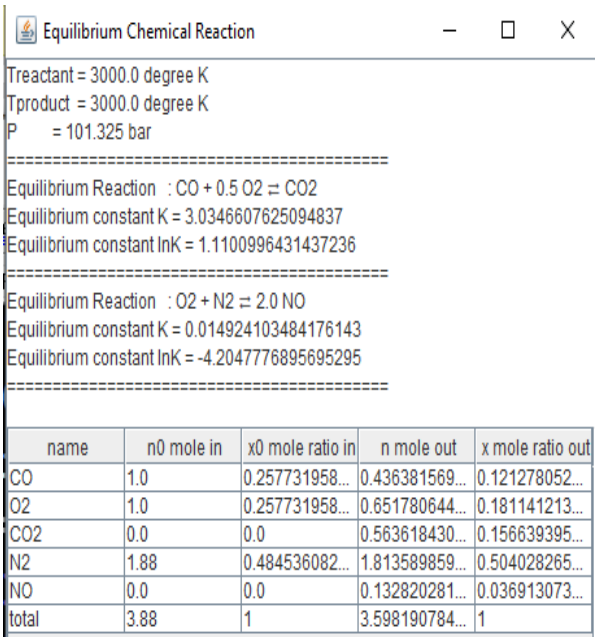
Sample program:

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
public class equilibrium1_SP
{  public static void main(String arg[])
  {    chemicalReaction_SP r[]=new  chemicalReaction_SP[2];
      String s[]={"CO","O2","CO2","N2","NO"};
      double N1[][]={{1.0,0.0},{0.5,0.0},{0,1},{0.0,0.0},{0.0,0.0}};
      r[0]=new  chemicalReaction_SP("r0",s,N1);
      double N2[][]={{0.0,0.0},{1.0,0.0},{0.0,0.0},{1.0,0.0},{0.0,2.0}};
      r[1]=new  chemicalReaction_SP("r1",s,N2);
      double Tproduct=3000.0;//degree K
      double Treactant=3000.0;//degree K
      double n0[]={1.0,1.0,0.0,1.88,0.0};
      double P=101.325;//kPa
                  if_equilibrium_SP fe=new if_equilibrium_SP(n0,s,r,Treactant,Tproduct,P);
                  double n[]={0.51,0.52,0.53,0.54,0.55};
                  //double r1[]=fe.func(n);
                  //double [] r1= continuity.continuationRK6(fe,n,4);
                  double [] r1= continuityi.newton_continuationRK6(fe,n);
                  String ss[]={"nCO","nH2O","nCO2","nH2","nO2"};
                  String ss1="Equilibrium reaction CO+"+'\u00BD'+"O2"+'\u21C4'+" CO2 \n";
                  ss1+="CO+"+"at T="+Tproduct+" K ";
                  fe.equilibrium_print(r1);
  }}
```



name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.257731958...	0.434849576...	0.120878017...
O2	1.0	0.257731958...	0.650927268...	0.180942564...
CO2	0.0	0.0	0.565150423...	0.157098607...
N2	1.88	0.484536082...	1.813502480...	0.504111298...
NO	0.0	0.0	0.132995039...	0.036969512...
total	3.88	1	3.597424788...	1

To compare the results let us also run equlibrium1_PG (Perfect gas) for the same conditions:



name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.257731958...	0.436381569...	0.121278052...
O2	1.0	0.257731958...	0.651780644...	0.181141213...
CO2	0.0	0.0	0.563618430...	0.156639395...
N2	1.88	0.484536082...	1.813589859...	0.504028265...
NO	0.0	0.0	0.132820281...	0.036913073...
total	3.88	1	3.598190784...	1

4. Results and conclusion

Stoichiometric chemical equilibrium algorithm is developed by using Schereiber and Pitzer real gas EoS. Schereiber and Pitzer EoS is relatively unknown equation of states that generalised gas relations by using Pitzer coefficients. Equilibrium calculations are based on atomic mass balances and minimisation of gibbs energy. Stoichimetric chemical equilibrium concepts is used in calculations, required equilibrum equations are defined as an input parameter, and chemical equilibrium coefficients are calculated by using only temperature dependent components of gibbs energy. In order to compare the results, perfect gas base calculations are also carried out. The most notable

difference of ideal gas based calculations and real gas based calulations are that ideal gas equations are based on pressure while real gas equations are based on fugacity. Results seems not deviates much, therefore one of the conclution is that for atmospheric reactions ideal gas approach will be sufficient for evaluation of chemical reactions. Several example cases runs with both EoS‘. The complete codes for this analysis is given in internet site www.turhancoban.com as zip fie SCO1.rar. Further analysis for direct gibbs free energy minimisation without Stoichiometric equations are also studied by our group. It will be presenrted as sperate papers.

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6. Nomenclature

Z	Compressibility factor
P	Pressure
ρ	density
R	Gas constant
T	Temperature
B,C,D,E,F,G	Schereiber-Pitzer EoS constants
T_r	Reduced pressure
T_c	Critical temperature
A	Helmholts energy
H, h	Enthalpy

G,g	Gibbs free energy
S,s	Entropy
f	Fugacity
v	Specific volume
C_p	Specific heat at constant pressure
μ	Chemical potential
A_{ij}	Atom matrix
λ	Homotophy(continuity) variable
n	Mole numbers
M	Molecular weight

EXAMPLE APPLIED ENGINEERING PROBLEM : SOLUTION OF SEMI-NON-LINEAR SYSTEM OF EQUATIONS :

All math algorithms heavily used in actual engineering and science problems. One of the purpose of this book is shown how it actually applied to actual science problems, following presentations by author is given as it is to reflect science application part of the nonlinear system of equations. An interesting point in the following paper is that system of eqaution is semi non linear, part of the equation is linear and this is utilised to simplify the solution

Stoichiometric chemical equilibrium algorithm by using Perfect gas EOS and selection acording to best zero approach of derivatives optimisation method for solving system of equations

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Keywords: Thermodynamic properties of gases, Perfect Gas EoS, chemical equilibrium, Stochiometric algorithm for chemical equilibrium, stochastic optimisation

1. Abstract In stoichiometric (based on K equilibrium constants) thermodynamic property algorithms, it is usually ideal gas EOS is used. In this study ideal gas equation of state will be used to predict thermodynamic properties of gases. Two version of ideal gas EoS is used with different set of specific heat equations. The first IGEoS is derived form partial continious Cp values. Data for this is taken from several sources and contains approximately 600 gases. The second IGEoS is utilises Cp equations defined by NASA. Stoichiometric formulations requires equlibrium formulations. Atomic balances gives equilibrium chemical moles as much as number of available atoms, for each additional chemical an additional equiibrium formulations is needed. When equations came together, they created quite a nonlinear system of equation. These set of equation can be solved by linearisation methods. In this study, it is prefer to solve directly in a semi-nonlinear form by using a selctive optimisation method. In optimisation methods acording to definitions derivatives should be equal to zero. In real numerical calculations they will approach to zero but will not be always exactly be zero due to numerical approximations. The optimisation will be carried out for several optimisation algorithms and the avarage retaing values compared and equation that is given minimum value will be selected as candidate to solve final values of system of semi-nonlinear equations. the codes are developed in java programming language. The program code are given to interested researchers as free access in www.turhancoban.com adress.

2. Formulation of Equations of State

NASA IDEAL GAS FORMULATION
Reference document: NASA/TP—2002-211556
https://shepherd.caltech.edu/EDL/PublicResources/sdt/refs/NASA-TP-2002-211556.pdf
NASA Glenn Coefficients for Calculating Thermodynamic Properties of Individual Species

A rich data collection is presented to calculate ideal gas thermodynamic properties.
Specific heat at constant temperature:

$$\frac{C_p^0(T)}{R} = a_1 T^{-2} + a_2 T^{-1} + a_3 + a_4 T + a_5 T^2 + a_6 T^3 + a_7 T^4$$

Enthalpy

$$\frac{H^0(T)}{RT} = -a_1 T^{-2} + a_2 \frac{\ln(T)}{T} + a_3 + a_4 \frac{T}{2} + a_5 \frac{T^2}{3} + a_6 \frac{T^3}{4} + a_7 \frac{T^4}{5} + b_1 T$$

Entropy

$$\frac{S^0(T)}{R} = -a_1 \frac{T^{-2}}{2} - a_2 T^{-1} + a_3 \ln(T) + a_4 T + a_5 \frac{T^2}{2} + a_6 \frac{T^3}{3} + a_7 \frac{T^4}{4} + b_2$$

b1 and b2 is integral constant for the equations. Coefficients are given as table format

$$S(T,P) = S^0(T) - R\ln\left(\frac{P}{P_0}\right)$$

Chemical enthalpy: $H_T(T) = H(T) - H(298.15) + \Delta h_f(298.15)$

Chemical entropy : $S_T(T,P) = S(T,P) - S(298.15, P0)$

P0=101.325 kPa

$$G^0(T) = H^0(T) - T * S^0(T)$$

Gibbs energy: $G(T,P) = h(T) - S(T,P) = G^0(T) - RT\ln\left(\frac{P}{P_0}\right)$

Chemical gibbs energy: $G_T(T,P) = H_T(T) - TS_T(T,P)$

As an alternative equation partial continious curve fitting base Cp equations are used. Partial continues data that is divided into someting like 10 subregion is usually considered more accurate compare to nasa formulation. We will use this set of program development for comparison purposes (Gas_Data and Gas_PG)

$C_{pi}(T) = A_i + B_i*10^{-3}*T + C_i*10^5/T^2 + D_i*10^{-6}*T^2$ $\quad T_{Li} >= T > T_{Hi}$ KJ/kmol K

$$h(T) = h_0 + \int_{T_0}^{T_{kN}>T_0} C_{pi}(T)dT + \sum_{k=kN}^{kM} \int_{T_{kN}}^{T_{kM}} C_{pi}(T)dT + \int_{T_{kN}<T}^{T} C_{pi}(T)dT \quad s(T)$$

$$= s_0 + h_0 + \int_{T_0}^{T_{kN}>T_0} \frac{C_{pi}(T)}{T}dT + \sum_{k=kN}^{kM} \int_{T_{kN}}^{T_{kM}} \frac{C_{pi}(T)dT}{T} + \int_{T_{kN}<T}^{T} \frac{C_{pi}(T)}{T}dT - R\ln\frac{P}{P_0}$$

Data is also needed to solve $C_p(T)$ value. In order to establish that, NIST tables given at the adress https://janaf.nist.gov/ is used. The following partial continious formulation is taken. the following partial difference curve fitting formula is used

$$C_{pi}(T) = A_i + B_i 10^{-3} T + \frac{C_i 10^5}{T^2} + D_i 10^{-6} T^2 \qquad T_{Li} \le T \le T_{Hi} \qquad (2.34)$$

3. Stoichiometric chemical equilibrium formulations

Actual products formed in a reaction is not a clear cut concept. The actual products formed in a chemical reactions can only be determined experimentally and as a function of the environmental values such as temperature, pressure and time. When these parameters changed actual products can change as well. One way of estimating what products will be formed is to use equilibrium concept. All physical systems are tend to minimize their energy levels when there are no external force or energy applied to the system. For example a bubble takes spherical shape under the influence of surface tension which has the minimal area for the given volume. Chemical reactions when temperature and pressure is constant and long enough time is given(for some reactions microseconds, for some reaction days) tends to minimize its energy level, specifically gibbs free energy level (or maximize its entropy level). This state is called chemical equilibrium. For example consider a closed system consisting initially of a gaseous mixture of carbondioxide , oxygen and carbonic acid. A reaction might take place is

$$CO_2 + H_2O \rightleftarrows H_2CO_3$$

At equilibrium the system will consist basically of three components, CO_2, H_2CO_3 and H_2O, for not all the components gases. This is what is called soda reaction. When soda bottle opened equailibrium condition (pressured in the system) is changed and rection slides to a new equilibrium states, so that some omount of CO_2 and H_2O is formed and amount of carbonic acid is reduced . Changes in the amounts of these components during the opening of soda bottle:

$$dn_{CO_2} = -dn_{H_2CO_3}$$
$$dn_{H_2O} = -dn_{H_2CO_3} \quad (3.1)$$

where dn denotes the differential change in the representative component.

$$-\frac{dn_{CO_2}}{1} = -\frac{dn_{H_2O}}{1} = \frac{dn_{H_2CO_3}}{1} \quad (3.2)$$

Equilibrium is a condition of balance. When equilibrium is established amount of CO_2 and H_2O and H_2CO_3 tends to be balanced with each other. But when equilibrium is broken and a new conditions arise, like opening of a soda bottle, a new equilibrium will be established. And during these process total gibbs free energy of the system will be minimised.

$$dG(T,P) = 0 \quad (3.3)$$

This amount can be determined from chemical potential changes (see the first chapter for details)

$$\mu_{CO_2} = \frac{\partial G(T,P,n)}{\partial n_{CO_2}} \qquad \partial G(T,P,n) = \mu_{CO_2}\partial n_{CO_2}$$
$$\mu_{n_{H_2CO_3}} = \frac{\partial G(T,P,n)}{\partial n_{H_2CO_3}} \qquad \partial G(T,P,n) = \mu_{H_2CO_3}\partial n_{H_2CO_3}$$
$$\mu_{H_2O} = \frac{\partial G(T,P,n)}{\partial n_{H_2O}} \qquad \partial G(T,P,n) = \mu_{H_2O}\partial n_{H_2O}$$

$$dG(T,P) = -1 * \mu_{CO_2} dn_{CO_2} - 1 * \mu_{H_2O} dn_{H_2O} + 1 * \mu_{H_2CO_3} dn_{H_2CO_3} = 0 \quad (3.4)$$

$$dG(T,P) = \left(-1 * \mu_{CO_2} - 1 * \mu_{H_2O} + 1 * \mu_{H_2CO_3}\right) dn_{H_2O} = 0 \quad (3.5)$$

This is called the equilibrium reaction.For a more general equilibrium equation such as:

$$\upsilon_A A + \upsilon_B B \rightleftarrows \upsilon_C C + \upsilon_D D \quad (3.6)$$

where υ 's are stoichiometric coefficients. The following equilibrium case is existed:

$$-\frac{dn_A}{\upsilon_A} = -\frac{dn_B}{\upsilon_B} = \frac{dn_C}{\upsilon_C} = \frac{dn_D}{\upsilon_D} = d\varepsilon \quad (3.7)$$

where $d\varepsilon$ is the proportionality factor. From which the following expressions are obtained.

$$dn_A = -\upsilon_A \, d\varepsilon$$
$$dn_B = -\upsilon_B \, d\varepsilon$$
$$dn_C = \upsilon_C \, d\varepsilon$$
$$dn_D = \upsilon_D \, d\varepsilon$$

(3.8)

For this case the Gibbs free energy equation takes form:

$$dG(T,P) = -\upsilon_A * \mu_A - \upsilon_B * \mu_B + \upsilon_C * \mu_C + \upsilon_D * \mu_D = 0 \quad (3.9)$$

For ideal gas mixtures chemical potential can be expressed as specific gibbs free energy as

$$h_{Ti}(T) = h_i(T) - h(298.15) + \Delta h_f(298.15)$$

$$g^0(T) = h_{Ti}(T) - Ts^0(T)$$

$$\mu_i = g_i^0(T) - RTln\left(\frac{P_i}{P_{ref}}\right) \quad (3.10)$$

where P_i is the partial pressure of the gas. For and ideal gas, partial pressure can be expressed as a function of components in the total components

$$y_i = \frac{n_i}{n_{total}} \quad P_i = y_i P \quad P_i = \frac{n_i}{n_{total}} P \quad (3.11)$$

Substituting these into the equation gives:

$$\mu_i = g_i^0(T) - RTln\left(\frac{n_i}{n_{total}} \quad \frac{P}{P_{ref}}\right) \quad (3.12)$$

For a real gas mixtures chemical potential can be expressed as specific gibbs free energy as

$$h_{Ti}(T,P) = h_i(T,P) - h(298.15,P) + \Delta h_f(298.15) \quad (3.13)$$

$$s(T,P) = s^0(T) - Rln\left(\frac{f_i}{P_{ref}}\right) \quad (3.14)$$

$$g(T,P) = g^0(T,P) - RTln\left(\frac{f_i}{P_{ref}}\right) \quad (3.15)$$

$$g^0(T,P) = h_{Ti}(T,P) - Ts^0(T) \quad (3.16)$$

$$\mu_i = g_i^0(T) - RTln\left(\frac{f_i}{P_{ref}}\right) \quad (3.17)$$

Note that $\lim_{P\to 0} \frac{f_i}{P_i} = 1 \quad (3.18)$ therefore $P_{ref} = f_{ref}$

where f_i is the partial fugacity of the gas. For a real gas, partial pressure can be expressed as a function of components in the total components

$$y_i = \frac{n_i}{n_{total}} \quad f_i = y_i f^* \quad f_i = \frac{n_i}{n_{total}} f^* \quad (3.19)$$

Where f^* fugacity that component i would have if the entire gas had that composition at the same temperature and pressure. $f^* = f(T,P) \quad (3.20)$

Substituting these into the equation gives:

$$\mu_i = g_i^0(T,P) - RTln\left(\frac{n_i}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) \quad (3.21)$$

Then the basic Gibbs free energy equation becomes

$$dG(T,P) = -\upsilon_A\left[g_A^0(T) - RTln\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] - \upsilon_B\left[g_B^0(T) - RTln\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right]$$
$$+ \upsilon_C\left[g_C^0(T) - RTln\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] + \upsilon_D\left[g_D^0(T) - RTln\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] = 0 \quad (3.22)$$

Arranging equation:

$$dG^0(T,P) = -\upsilon_A g_A^0(T,P) - \upsilon_B g_B^0(T,P) + \upsilon_C g_C^0(T,P) + \upsilon_D g_D^0(T,P) \quad (6.2.14)$$

$$dG(T,P) = dG^0(T,P)$$
$$+ RT\left[\upsilon_A ln\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) + \upsilon_B ln\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_C ln\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right.$$
$$\left. - \upsilon_D ln\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] = 0 \quad (3.23)$$

This equation can be written as:

$$-\frac{dG^0(T,P)}{RT} = \left[\upsilon_A ln\left(\frac{n_A}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) + \upsilon_B ln\left(\frac{n_B}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_C ln\left(\frac{n_C}{n_{total}} \quad \frac{f^*}{P_{ref}}\right) - \upsilon_D ln\left(\frac{n_D}{n_{total}} \quad \frac{f^*}{P_{ref}}\right)\right] \quad (3.24)$$

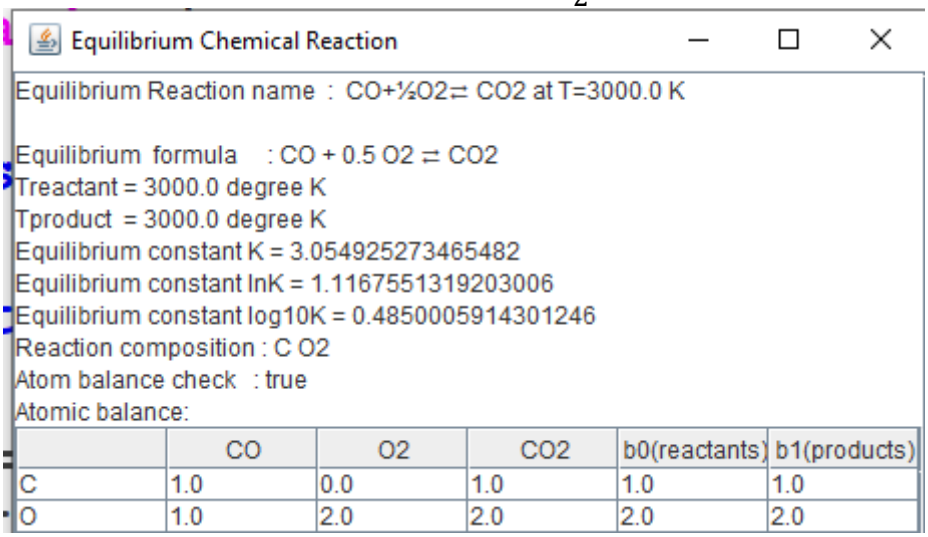
$$exp\left(-\frac{dG^0(T,P)}{RT}\right) = \frac{\left[\left(\frac{n_A}{n_{total}}\ \frac{f^*}{P_{ref}}\right)^{\upsilon_A}\left(\frac{n_B}{n_{total}}\ \frac{f^*}{P_{ref}}\right)^{\upsilon_B}\right]}{\left[\left(\frac{n_C}{n_{total}}\ \frac{f^*}{P_{ref}}\right)^{\upsilon_C}\left(\frac{n_D}{n_{total}}\ \frac{f^*}{P_{ref}}\right)^{\upsilon_D}\right]} = \left[\frac{(n_A)^{\upsilon_A}(n_B)^{\upsilon_B}}{(n_C)^{\upsilon_C}(n_D)^{\upsilon_D}}\right]\left(\frac{f^*}{n_{total}P_{ref}}\right)^{\upsilon_A+\upsilon_B-\upsilon_C-\upsilon_D} \quad (3.25)$$

The left hand side of the equation is called equilibrium constant, which is only function of temperature

$$K(T) = exp\left(-\frac{dG^0(T,P)}{RT}\right) \quad (3.26)$$

As an example let us evaluate equilibrium constant of the equilibrium reaction

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$



Equilibrium condition can be solved by solving chemical balance equations together with the equilibrium gibbs free energy minimisation equations. So mass balance is an important part of total set of equations to solve. Mass balance establishes as follows:

$$\sum_{j=1}^{NS} A_{ij}n_j - b_i^0 = 0 \quad (i = 1..na) \quad (3.27)$$

$$b_i = \sum_{j=1}^{NS} A_{ij}n_j \quad (3.28)$$

$$b_i - b_i^0 = 0 \quad (i = 1..na) \quad (3.29)$$

where na is the number of chemical elements. A_{ij} is number of kilogram atoms per kmole of species j. And b_i^0 is the assigned number of kilogram atoms element i per kmol of total reactants.

In order to explain this equation let us look at an example. If chemicals in the reaction and input moles re given as:

CH_4	1 kmol
H_2O	10 kmol
H_2	0 kmol
CO_2	0 kmol
CO	0 kmol
O_2	0 kmol

A_{ij} matrix will be

atom	CH_4	H_2O	H_2	CO_2	CO	O_2
H	4	2	2	0	0	0
C	1	0	0	1	1	0
O	0	1	0	2	1	2

b_i^0 vector is the multiplication of number of atoms with inlet mole numbers of the molecule
b_H^0=1*4+10*2+0*2+0*0+0*0+0*0=24
b_C^0=1*1+10*0+0*0+0*1+0*1+0*0=1
b_O^0=0*1+10*1+0*0+0*2+0*1+0*2=10
In this case initial matrix will be in the form of:

$$\begin{bmatrix} 4 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} n_{CH_4} \\ n_{H_2O} \\ n_{H_2} \\ n_{CO_2} \\ n_{CO} \\ n_{O_2} \end{Bmatrix} = \begin{Bmatrix} 24 \\ 1 \\ 10 \end{Bmatrix} \quad (3.30)$$

As it is seen from the example mass balance are given us 3 equations, but total number of equations(moles) are 6, remaining equations will be gibbs equations as described above. All togetger they construct a system of non-linear equations to solve. The results will be equilibrium balance.

4. Semi-nonlinear (semi-linear) system of equations

In a set of nonlinear system of equations, if some equations are linear, treditionally non-linear system of equation solutions are used. Solving linear system of equations are much simpler mathematical process compare to non-linear system of equations which involves iterative solution method. In semi-nonlinear system of equation, linearity of remaining linear equations can be used to simplify problem. We would like to explain the case with some equations

$$f_0(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$
$$f_1(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$
$$f_2(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$
$$f_3(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$
$$f_4(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$
$$f_4(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

Assume that first 3 equations of this set is in linear form, therefore, equations can be written as:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} \\ a_{10} & a_{11} & a_{12} & a_{12} & a_{14} & a_{15} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{23} & a_{25} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$f_3(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$f_4(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$f_5(x_0, x_0, x_0, x_0, x_0, x_0) = 0$$

Then equation can be converted into:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{03} & a_{04} & a_{05} \\ a_{12} & a_{14} & a_{15} \\ a_{23} & a_{23} & a_{25} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$f_3(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$f_4(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$f_5(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

Then:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_{03} & a_{04} & a_{05} \\ a_{12} & a_{14} & a_{15} \\ a_{23} & a_{23} & a_{25} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$f_3(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$f_4(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

$$f_5(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

In the first formulations variables x_0, $x_1, and\ x_2$ are linearly dependent to variables x_3, $x_4, and\ x_5$. Therefore nonlinear system reduced to a form with only 3 unknowns.

$$f(x_3, x_4, x_5)$$

A simple example case:

$$f_0(x_0, x_1, x_2) = x_0 + 0x_1 + x_2 - 1 = 0$$
$$f_1(x_0, x_1, x_2) = x_0 + 2x_1 + 2x_2 - 3 = 0$$
$$f_2(x_0, x_1, x_2) = \ln(3.1362952) - ln\left(\frac{x_2}{x_0 + x_1 + x_2}\right) + ln\left(\frac{x_0}{x_0 + x_1 + x_2}\right) + 0.5ln\left(\frac{x_1}{x_0 + x_1 + x_2}\right) = 0$$

Or

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [x_0] = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

f(x_0):

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [x_0]$$

$$f_2(x_0, x_1, x_2) = \ln(3.1362952) - ln\left(\frac{x_2}{x_0 + x_1 + x_2}\right) + ln\left(\frac{x_0}{x_0 + x_1 + x_2}\right) + 0.5ln\left(\frac{x_1}{x_0 + x_1 + x_2}\right) = 0$$

5. Stochastic optimisation methods

5a. Harris' hawks algorithm

Harris Hawks algorithm is a stachastic method based on hunting behavior of harris hawks[11]. Harris hawk, unlike other hawk species develop a group cooperative hunting behavior. Harris hawks shows evolved innovative team chasing capabilities in tracing, encircling, flushing out, and eventually attacking and catching their pray.

Their group behavior has Exploration and exploitation phases. In the exploration phase Haris' hawks track and detect their prey. But prey can not be seen directly all the time, in this case they fly randomly and perch to some position to detect the prays. Perching point will be selected on two different strategy, based on the position of the other members of the team and base on the possible position of the prey. As computer simulation of this can be given with the following formulations: Assume a random number q, and if q≥0.5 perch in a random position, if q<0.5 select a position based on position of the other members and possible prey location.

$$X(t+1) = \begin{cases} X_{rand}(t) - r_1[X_{rand}(t) - 2r_2X(t)] & if \quad \text{q} \geq 0.5 \\ \left(X_{prey}(t) - X_m(t)\right) - r_1[LB + r_4(UB - LB)] & if \quad \text{q} \geq 0.5 \end{cases}$$

Where $X(t+1)$ is the position vector of the hawks in the next iteration. $X_{prey}(t)$ is the position of the prey, $X(t)$ is the current position of hawks, r_1, r_2, r_3, r_4 and q are random numbers inside (0,1) range, which are updated in each iteration. LB and UB are search limits (upper and lower boundaries of variables). $X_{rand}(t)$ is a randomly selected hawk from the hawk group. $X_m(t)$ is the average position

of the hawks. The average position of the hawks group is calculated by using mathematical averahe formula $X_m(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$ where $X_i(t)$ is the location of the each hawk fort he iteration t, N is the total number of hawks.

In order to translate from Exploration to exploitation phase prey energy is observed. Energy of the prey is modelled as:

$E = 2E_0\left(1 - \frac{t}{T}\right)$ where E indicates escaping energy of the prey, T is yhe maximum number of iterations, and E_0 is the initial state of its energy. When E is positive, prey is stenthening, when E is negative the prey is weakening. Exploration happens when $|E| \geq 1$ and explotation happens when $|E| < 1$. In exploitation phase, attacks the intented prey. Howewer prey often attemps to escape from dangerous situation. Hence, different chasing styles occur in real situation. Suppose that r is the chance of a prey successfully escaping(r<0.5) or not successfully escaping(r≥0.5) before the surprise attack. The hawk behave two different mechanism to respond these, a hard or soft besiege.

When r≥ 0 and $|E| \geq 0.5$ soft besiege happens. In this case the prey has enough energy , and try to escape by some random movements. The Harris' hawks encircle it softly to make rabbit more exhausted the performs the surprise pounce. It is modelled as the following equations:

$$X(t+1) = \Delta X(t) - E\left[JX_{prey}(t) - X(t)\right]$$
$$\Delta X(t) = JX_{prey}(t) - X(t)$$
$$J = 2(1 - r_5)$$

Where $\Delta X(t)$ is the difference between the position vector of the rabbit and the current location in the iteration t, and r_5 is a random number. The hawks behave according to following pattern:

$$Y = X_{prey}(t) - E\left[JX_{prey}(t) - X(t)\right]$$
$$Z = Y + S * LF(D)$$

Where D is dimension of the problem aans S is a random vector of size 1xD and LF is the levy flight function.

$$LF(x) = 0.01\frac{u\sigma}{|v|^{\frac{1}{\beta}}}$$
$$\sigma = \left(\frac{\Gamma(1+\beta)sin\left(\frac{\beta\pi}{2}\right)}{\Gamma(\frac{1+\beta}{2})\beta\, 2^{\left(\frac{\beta-1}{2}\right)}}\right)^{\frac{1}{\beta}}$$

Where u,v are random numbers in (0,1) range, Γ(z) is gamma function which is defined as:

$\Gamma(z) = \int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)^{-1} e^{t/n}$

where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant

Hence the final strategy becomes:

$$X(t+1) = \begin{cases} Y & if\ F(Y) < F(X(t)) \\ Z & if\ F(Z) < F(X(t)) \end{cases}$$

When r< 0.5 and $|E| < 0.5$ the prey does not have enough energy so hard besiege happens. Hawks directly attempts the surprise pounce to kill the prey.

$$Y = X_{prey}(t) - E\left[JX_{prey}(t) - X_m(t)\right]$$
$$Z = Y + S * LF(D)$$
$$X(t+1) = \begin{cases} Y & if\ F(Y) < F(X(t)) \\ Z & if\ F(Z) < F(X(t)) \end{cases}$$

5b. Particle swarm algorithm

Particle swarm optimization (PSO) was developed by Kennedy and Eberhart in 1995[12], based on swarm behaviour such as fish and bird schooling in nature. PSO searches the space of an objective function by adjusting

the trajectories of individual agents, called particles. The movement of the particles has both stochastic and deterministic elements. Each particle is attracted toward the best location of the given time and the best location of the near past (a current global). It has also tendency to move randomly. When a particle finds a location that is better than any previously found locations, then it updates it as the new current best for particle *i*.It also remembers all the currents best values to update a current global value. The aim is to find the global best among all the current best values available until no more improvements are available.

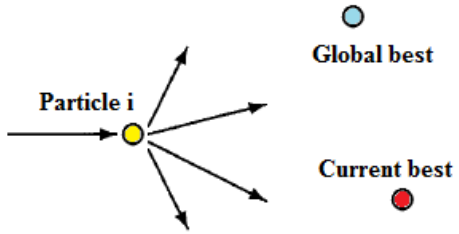


Figure 5.1 Particle will attract toward the global best, current best and also move randomly

The method is given in Table 5.23.1 as pseodocode. If x_i and v_i position vector and velocity for particle i, the new velocity vector is determined as:

$v_i^{t+1} = v_i^t + \alpha^* \varepsilon_1^* (g^* - x_i^t) + \beta^* \varepsilon_2^* (x_i^* - x_i^t)$

Where ε_1 and ε_1 are two random vectors between 0 and 1, g^* is the current global best x_i^* is local best. The parameters α and β are the learning parameters or acceleration constants. The initial velocity of the particles can be taken as zero, $v_i^{t=0} = 0$. The new position can then be updated by $x_i^{t+1} = x_i^t + v_i^{t+1}$. Although v_i can be any values, it is bounded in some range $[0, v_{max}]$.

There are many variants of the standart PSO algorithm. The most noticiable one is the version in which previous velocity function multiplied with an inertia function. This is called an accelerated PSO.

$v_i^{t+1} = \theta v_i^t + \alpha^* \varepsilon_1^* (g^* - x_i^t) + \beta^* \varepsilon_2^* (x_i^* - x_i^t)$

This is equivalent to introduce a virtual mass to stabilise the motion of the particles.

Table 5.1 Particle swarm algorithm

Objective function $f(P_i)$, $P_i = (x_0, ..., x_m)_i$ for particle i and m space dimension variable

Initialize locations P_i and velocity V_i of n particles by random selection

Use as velocity vector:

$V_i^{t+1} = \theta V_i^t + \alpha^* random()[GPmin - x_i^t] + \beta^* random()[Pmin - x_i^t]$

Where θ is inertia function(an iterative variable between 0 and 1)

α and β are learning parameters or acceleration constants

$\alpha = \alpha_0 \gamma^t$ where $0 \le \gamma \le 1$ $\alpha_0 \cong 0.5..1$ for t is the virtual time

Find Global best GPmin from min{ $f(P_0)$…$f(P_n)$} (at t = 0)

while (criterion)

t = t + 1 (pseudo time or iteration counter)

for loop over all n particles and all m dimensions

Generate new velocity V_i^{t+1}

Calculate new locations $P_i^{t+1} = P_i^t + V_i^{t+1}$

Evaluate objective functions at new locations

Find the current best for each particle Pmin

Find the global best for each particle GPmin

end for

end while

Output the final results Pmin and GPmin

5c. Firefly optimisation algorithm

You might notice amazing flashing lights of fireflys in the summer nights. The flashing light is produced by a

process of bioluminescence. The true main purpose of such flashes are still on debate but one of the fundamental function is attracting mate. The rithmic flash, the rate of flashing and the amount of time form part of the signal system that brings both sexes together. We know that the light intensity at a particular distance from the light source obeys the inverse square law. That is to say, the light intensity decreases as the distance increases. Furthermore, the air absorbs light which becomes weaker and weaker as the distance increases. This physical factors are utilised by fireflies to find each other. The same principle can be utilised to create an artificial firefly for optimisation .The Firefly Algorithm was developed by Xin-She Yang [13,14], based on the idealization of the flashing characteristics of fireflies. In creating artificial firefly it is assumed that they are unisex, i.e. any firefly in the population can be attracted to any other firefly, Attractiveness is function of brightness. Brightness will decrease as a function of distance between. The less brighter one will move through the brighter one. If there are no brighter fire flies around, it will fly randomly. The brightness of the artificial firefly will directly be determined by objective function value. By using these principles, a pseudocode of the firefly algorithm can be written as:

Objective function $f(X)$, $X = (x_0, ...,x_n)$
Generate initial population of fireflies X^i $(i = 1,2, ...,m)$ *randomly*
Light intensity Ii at x≫ is determined by $f(X^i)$
Define light absorption coefficient γ
while $(t <$MaxGeneration)
for $i = 1 : n$ all n fireflies
    for $j = 1 : n$ all n fireflies (inner loop)
        if $(Ii < Ij)$, Move firefly i towards j; end i f
        Vary attractiveness with distance r as

$$\beta = \beta_0 e^{-\gamma d^2} \text{ or } \beta = \frac{\beta_0}{1+\gamma d^2}$$

$$d_{ij} = \sqrt{\sum_{k=1}^{n} (x_{i,k} - x_{i,k})^2}$$

        Evaluate new solutions and update light intensity

$$X^i = (1-\beta) X^i + \beta X^j + \alpha\,(gaussrandom() - \frac{1}{2})$$

    end for j
end for i
Rank the fireflies and find the current global best *gt*
end while
Postprocess results and visualization

5d. Solution of the nonlinear system of equations by using optimization algorithms

Non-Linear system of equations can also be solved by using optimization methods through an adaptation function.
To solve function $f_i(x_j)=0$

$$g(x_j) = \sum_{i=0}^{n\_equation} f_i(x_j) * f_i(x_j)$$

The derivative of this equation is the root of non-linear system of equation again became $f_i(x_j)=0$, therefore optimum of $g(x_j)$ is the same problem as solution of $f_i(x_j)=0$.

5e. A heuristic algorithm with selection according to best zero approach of derivatives

In this section a variety of stochastic (heuristic) algorithms are investigated. All this algorithms are finding minimum with the limits of statistical approaches. If we can compare them with the same kind of referance point,we can decide which of the algorithm could give the best results for the specific optimisation problem in our hand. One of the possible method of comparision is to investigate derivatives of the optimisation function. Due to definition of optimisation derivatives with respect to all varşables should be equal to zero. For the optimisation function

$$f(x_0, x_1, x_2, x_3, .. x_n)$$

$$\frac{\partial f(x_0, x_1, x_2, x_3, .. x_n)}{\partial x_0} = 0$$
$$\frac{\partial f(x_0, x_1, x_2, x_3, .. x_n)}{\partial x_1} = 0$$
$$\ldots\ldots\ldots$$
$$\frac{\partial f(x_0, x_1, x_2, x_3, .. x_n)}{\partial x_n} = 0$$

In the real solution derivatives can be reached to zero or some small numbers (lets call them ε). So in our solution

$$\frac{\partial f(x_0, x_1, x_2, x_3, .. x_n)}{\partial x_0} = \varepsilon_0$$
$$\frac{\partial f(x_0, x_1, x_2, x_3, .. x_n)}{\partial x_1} = \varepsilon_1$$
$$\ldots\ldots\ldots$$
$$\frac{\partial f(x_0, x_1, x_2, x_3, .. x_n)}{\partial x_n} = \varepsilon_2$$

In the non-linear system of equation solutions, first derivatives will converge to system of non-linear equations

$$f_0(x_0, x_1, x_2, x_3, .. x_n) = \varepsilon_0$$
$$f_1(x_0, x_1, x_2, x_3, .. x_n) = \varepsilon_1$$
$$\ldots\ldots\ldots$$
$$f_n(x_0, x_1, x_2, x_3, .. x_n) = \varepsilon_n$$

Let us assume

$$\varepsilon_{average} = \frac{\sqrt{\sum_{i=0}^{n} \varepsilon_i^2}}{n+1}$$

When you compare $\varepsilon_{average}$ values of different stochastic equations the one that is closer to zero, ε_{min} will give us an equation to select for the given equation and given accuracy level (such as same population number and number of iterations)

5e. **Solution of the linear system of equations**

Singular value decomposition method is used to solve system of equations.

Let A be an m x n matrix such that the number of rows m is greater than or equal to the number of columns n. Then there exists:

1) an m x n column orthogonal matrix **U** (a matrix Q is orthogonal if its transpose is equal to its inverse)
2) an n x n diagonal matrix **S,** with positive or zero elements, and
3) an n x n orthogonal matrix V such that:

$$A = USV^T$$

This is the **Singular Value Decomposition (SVD) of matrix A.** If a matrix can be decomposed this way, its inverse value will be existed as:

$$A^{-1} = VS^{-1}U^T = V[diag(s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1}]U^T$$

Where

$$C = A^T A$$

If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ are eigemvalues of symmetric matrix C. Note that for symmetric matrices eigenvalues are real numbers.

$$S = I\sqrt{\lambda} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_2} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \sqrt{\lambda_n} \end{bmatrix}$$

So inverse of S will be:

$$S^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & & \\ & & \dots & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & \dots & 0 \\ & & \dots & 0 \\ 0 & 0 & & 1 \\ 0 & 0 & \dots & \frac{1}{\sqrt{\lambda_n}} \end{bmatrix}$$

Then a system of equation can be simply solve as

$$AX = B$$
$$X = A^{-1}B = (VS^{-1}U^T)B$$

6. Computer code development

In order to calculate termodynamic properties of gases Gas_PG and Gas_PG?nasa9 classes are developed. This ideal gas equation of states has a subclasses to calculate specific heat values as curvefitting values (Gas_Data.java for Gas_PG and nasa9_Data for Gas_PG_nasa9). Equilibrium coefficients are calculated from ChemicalReaction_PG and ChemicalReaction_PG_nasa9. System of equations are set together in if_equilibrium_PG and if_equilibrium_PG_nasa9 classes.
harris_hawks_optimizationi and particle_swarm1i methods to solve optimisation problem and class f_NL2 is used to adapt optimisation problem to system of on-linear equations problem. And finally equilibrium codes are solve in if_NL_equilibrium_PG and if_NL_equilibrium_PG_nasa9 classes.

Program	Utilisation
Atom	Calculation atomic balances, atomic properties
Gas_Data	Gas data for approximately 600 gases
Gas_PG	Perfect Gas EoS
ChemicalReaction_PG	Chemical reaction calculations for Gas_PG
Gas_PG_nasa9	Perfect Gas EoS based on nasa data
Nasa9_data	Perfect Gas data based on nasa data
if_NL_equilibrium_PG	Defines non-linear system of equation for Gas_PG and solve it
if_NL_equilibrium_PG_nasa9	Defines non-linear system of equation for Gas_PG_nasa9 and solve it
ChemicalReaction_PG_nasa9	Chemical reaction calculater for Gas_PG_nasa9
harris_hawks_optimizationi	Harris hawk optimisation
particle_swarm1i	Particle swarm optimisation
Thermal_Exchange_Optimizationi	Thermal exchange otimisation
fireflyi	Firefly optimisation
f_NL2	Adaptation function to calculate system of semi-nonlinear system of equations
if_xj	Interface function y=f(x[]) single function, multivariable
ifi_xj	Interface function y[]=f(x[]) multifunction, multivariable
SVD	Singular value decomposition method to solve linear system of equations. Eigenvalues are solved by using Jacobi symmetric method

Sample solutions:

One kmol of CO and one kmol of O_2 established an equilibrium at 3000 K. The equilibrium reaction for this is as follows:

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$

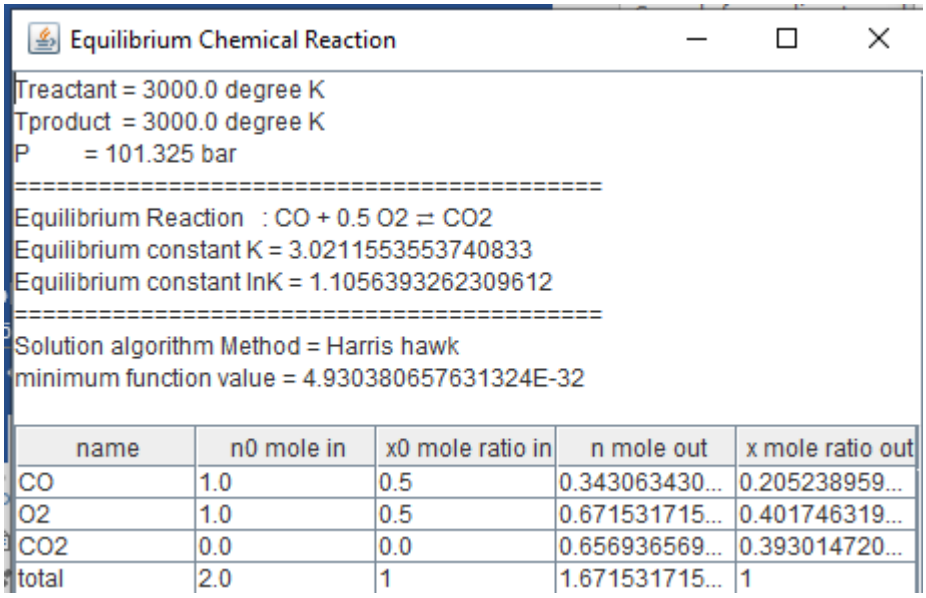
The reaction will be

$$CO + O_2 \rightarrow n_0 CO + n_1 O_2 + n_2 CO_2$$

Find the equilibrium composition. System pressure is P=101.325 bar. (Pref=101.325 bar)

```
            public static void main(String arg[])
            {String s[]={"CO","O2","CO2"};
    double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{0,0},{0,0}};
    chemicalReaction_PG_nasa9 r[]=new chemicalReaction_PG_nasa9[1];
    r[0]=new chemicalReaction_PG_nasa9("r1",s,N1);
    double Tproduct=3000.0;//degree K
    double Treactant=3000.0;//degree K
    double n0[]={1,1,0};
    double P=101.325;//kPa
            if_NL_equilibrium_PG_nasa9 fe=new if_NL_equilibrium_PG_nasa9(n0,s,r,Treactant,Tproduct,P);
            double n1[]={1.0};
            double n2[]=fe.solve(n1);
            fe.equilibrium_print(n2);
    }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" if_NL_equilibrium_PG_nasa9
Firefly =
      0.343063842226655          0.671531921113329          0.656936157773344
Particle swarm =
      0.343063430616319          0.671531715308160          0.656936569383680
Thermal Exchange optimisation =
      0.343063274982059          0.671531637491030          0.656936725017941
Harris hawk =
      0.343063430616319          0.671531715308160          0.656936569383680
Harris hawk
```



For T=3000 K and P=1.01325 bar, if N2 is added to the reaction. Equilibrium reaction is still the same:

$$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$$

The reaction will be

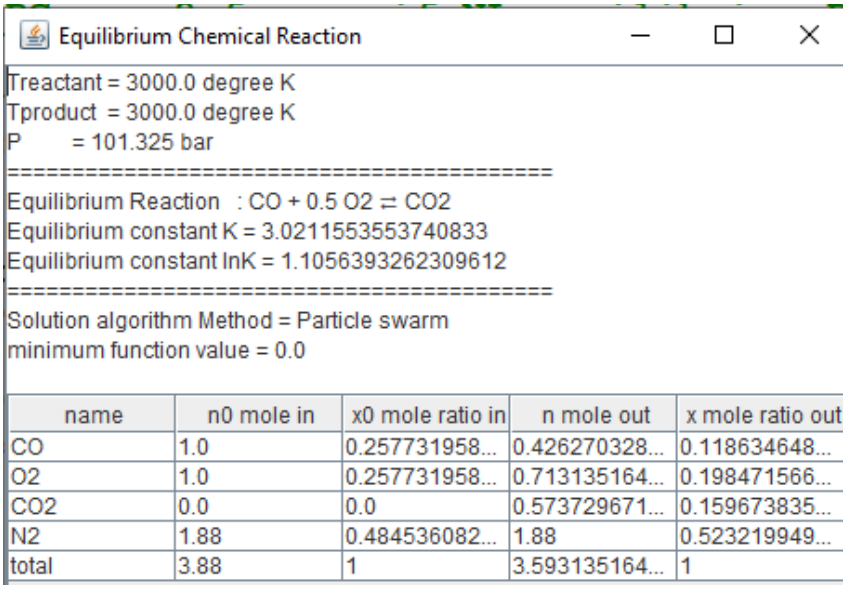
$CO+O_2+1.88N_2 \rightarrow n_0CO+n_1O_2+n_2CO_2+1.88N_2$

Find the equilibrium composition.

```
            public static void main(String arg[])
            {String s[]={"CO","O2","CO2","N2"};
    double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{0,0},{0,0},{0,0}};
    chemicalReaction_PG_nasa9 r[]=new chemicalReaction_PG_nasa9[1];
    r[0]=new chemicalReaction_PG_nasa9("r1",s,N1);
    double Tproduct=3000.0;//degree K
    double Treactant=3000.0;//degree K
    double n0[]={1,1,0,1.88};
    double P=101.325;//kPa
            if_NL_equilibrium_PG_nasa9 fe=new if_NL_equilibrium_PG_nasa9(n0,s,r,Treactant,Tproduct,P);
            double n1[]={1.0};
            double n2[]=fe.solve(n1);
```

```
            fe.equilibrium_print(n2);
    }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" if_NL_equilibrium_PG_nasa9
Firefly =
      0.426271106418998     0.713135553209500     0.573728893581002     1.880000000000000
Particle swarm =
      0.426270328591399     0.713135164295700     0.573729671408600     1.880000000000000
Thermal Exchange optimisation =
      0.426270328591399     0.713135164295700     0.573729671408600     1.880000000000000
Harris hawk =
      0.426270328591399     0.713135164295700     0.573729671408600     1.880000000000000
```



name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.257731958...	0.426270328...	0.118634648...
O2	1.0	0.257731958...	0.713135164...	0.198471566...
CO2	0.0	0.0	0.573729671...	0.159673835...
N2	1.88	0.484536082...	1.88	0.523219949...
total	3.88	1	3.593135164...	1

Example case:

For T=3000 K and P=101.325 kPa, if N2 is added to the reaction and NO apears in exhoust gases: .

Equilibrium reactions

$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$ $N_2 + 1/2O_2 \rightleftharpoons 2NO$

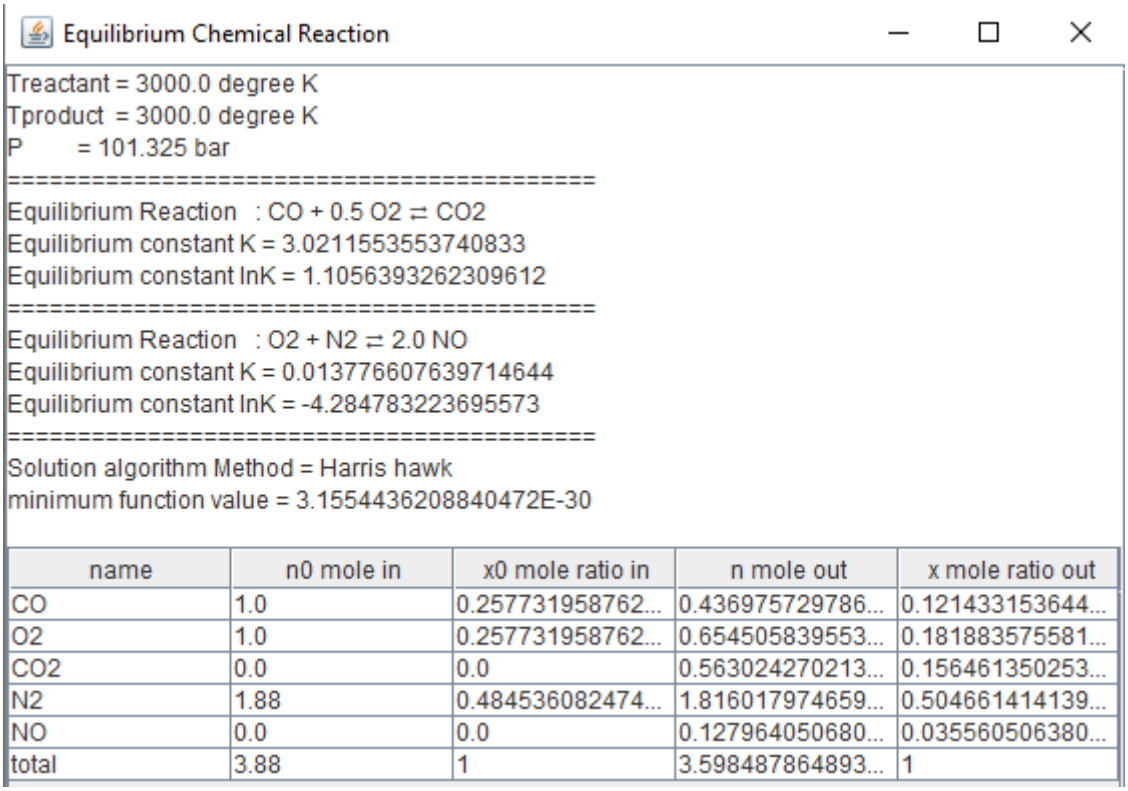
The reaction will be $CO + O_2 \rightarrow n_0 CO + n_1 O_2 + n_2 CO_2 + n_3 N_2 + n_4 NO$

Sample program: (**if_NL_equilibrium_PG_nasa9**)

```
            public static void main(String arg[])
            {String s[]={"CO","O2","CO2","N2","NO"};
    double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{0,0},{0,0}};
    chemicalReaction_PG_nasa9 r[]=new chemicalReaction_PG_nasa9[2];
    r[0]=new chemicalReaction_PG_nasa9("r1",s,N1);
    double N2[][]={{0.0,0.0},{1.0,0.0},{0.0,0.0},{1.0,0},{0,2}};
    r[1]=new chemicalReaction_PG_nasa9("r2",s,N2);
    double Tproduct=3000.0;//degree K
    double Treactant=3000.0;//degree K
    double n0[]={1,1,0,1.88,0};
    double P=101.325;//kPa
            if_NL_equilibrium_PG_nasa9 fe=new if_NL_equilibrium_PG_nasa9(n0,s,r,Treactant,Tproduct,P);
            double n1[]={1.0,1.0};
            double n2[]=fe.solve(n1);
            fe.equilibrium_print(n2);
    }
```

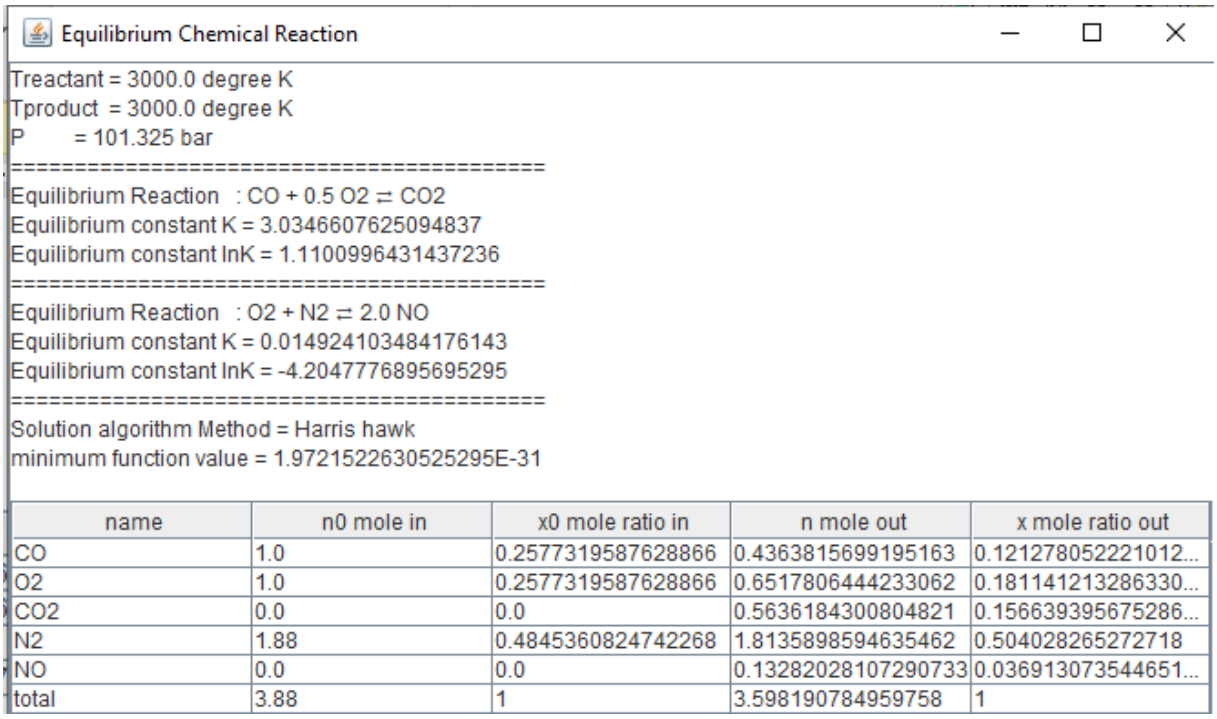
```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" if_NL_equilibrium_PG_nasa9
Firefly =
      0.436070713732631     0.654154577929363     0.563929286267367     1.816119221063045
0.127761557873909
Particle swarm =
      0.436975729786981     0.654505839553296     0.563024270213017     1.816017974659803
0.127964050680393
Thermal Exchange optimisation =
```

0.436927292469455	0.654480394744276	0.563072707530543	1.816016748509547
0.127966502980906			
Harris hawk =			
0.436975729786981	0.654505839553296	0.563024270213017	1.816017974659803
0.127964050680393			
Harris hawk			



name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.257731958762...	0.436975729786...	0.121433153644...
O2	1.0	0.257731958762...	0.654505839553...	0.181883575581...
CO2	0.0	0.0	0.563024270213...	0.156461350253...
N2	1.88	0.484536082474...	1.816017974659...	0.504661414139...
NO	0.0	0.0	0.127964050680...	0.035560506380...
total	3.88	1	3.598487864893...	1

To compare the results let us also run **if_NL_equlibrium_PG** for the same conditions:



name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.2577319587628866	0.4363815699195163	0.121278052221012...
O2	1.0	0.2577319587628866	0.6517806444233062	0.181141213286330...
CO2	0.0	0.0	0.5636184300804821	0.156639395675286...
N2	1.88	0.4845360824742268	1.8135898594635462	0.504028265272718
NO	0.0	0.0	0.13282028107290733	0.036913073544651...
total	3.88	1	3.598190784959758	1

Reaction Example case:
For T=1000 K and P=101.325 kPa, if N2 is added to the reaction and NO apears in exhoust gases: .
Equilibrium reactions

$CO+\frac{1}{2}O_2 \rightleftharpoons CO_2$ $N_2 + 1/2O_2 \rightleftharpoons 2NO$

The reaction will be $CO + O_2 \rightarrow n_0CO + n_1O_2 + n_2CO_2 + n_3N_2 + n_4NO$

Sample program: (**if_NL_equilibrium_PG_nasa9**)

```
            public static void main(String arg[])
            {String s[]={"CO","O2","CO2","N2","NO"};
    double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{0,0},{0,0}};
    chemicalReaction_PG_nasa9 r[]=new chemicalReaction_PG_nasa9[2];
    r[0]=new chemicalReaction_PG_nasa9("r1",s,N1);
    double N2[][]={{0.0,0.0},{1.0,0.0},{0.0,0.0},{1.0,0},{0,2}};
    r[1]=new chemicalReaction_PG_nasa9("r2",s,N2);
    double Tproduct=1000.0;//degree K
    double Treactant=1000.0;//degree K
    double n0[]={1,1,0,1.88,0};
    double P=101.325;//kPa
            if_NL_equilibrium_PG_nasa9 fe=new if_NL_equilibrium_PG_nasa9(n0,s,r,Treactant,Tproduct,P);
            double n1[]={1.0,1.0};
            double n2[]=fe.solve(n1);
            fe.equilibrium_print(n2);
  }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" if_NL_equilibrium_PG_nasa9
Firefly =
    0.000922518546788    0.500398251746144    0.999077481453210    1.879936992472748
0.000126015054504
Particle swarm =
    0.000000000158093    0.499962726652090    0.999999999841905    1.879962726573041
0.000074546853918
Thermal Exchange optimisation =
    0.000004196694065    0.499965899623453    0.999995803305933    1.879963801276418
0.000072397447164
Harris hawk =
    0.000000000158162    0.499527464793952    0.999999999841837    1.879527464714869
0.000945070570262
```

Equilibrium Chemical Reaction

```
Treactant = 1000.0 degree K
Tproduct  = 1000.0 degree K
P         = 101.325 bar
========================================
Equilibrium Reaction   : CO + 0.5 O2 ⇄ CO2
Equilibrium constant K = 1.6446644346304747E10
Equilibrium constant lnK = 23.523387302242682
========================================
Equilibrium Reaction   : O2 + N2 ⇄ 2.0 NO
Equilibrium constant K = 5.912508428790177E-9
Equilibrium constant lnK = -18.946195657541185
========================================
Solution algorithm Method = Particle swarm
minimum function value = 5.093400132963502
```

name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.2577319587628866	1.5816161079633095E-10	4.679337597415104E-11
O2	1.0	0.2577319587628866	0.4995274647939519	0.1477891907639836
CO2	0.0	0.0	0.9999999998418365	0.29585798811196445
N2	1.88	0.4845360824742268	1.8795274647148685	0.5560732143996727
NO	0.0	0.0	9.450705702623852E-4	2.7960667758587977E-4
total	3.88	1	3.3800000000790806	1

Reaction Example case:

For T=1000 K and P=101.325 kPa, if N2 is added to the reaction and NO apears in exhoust gases: .
Equilibrium reactions

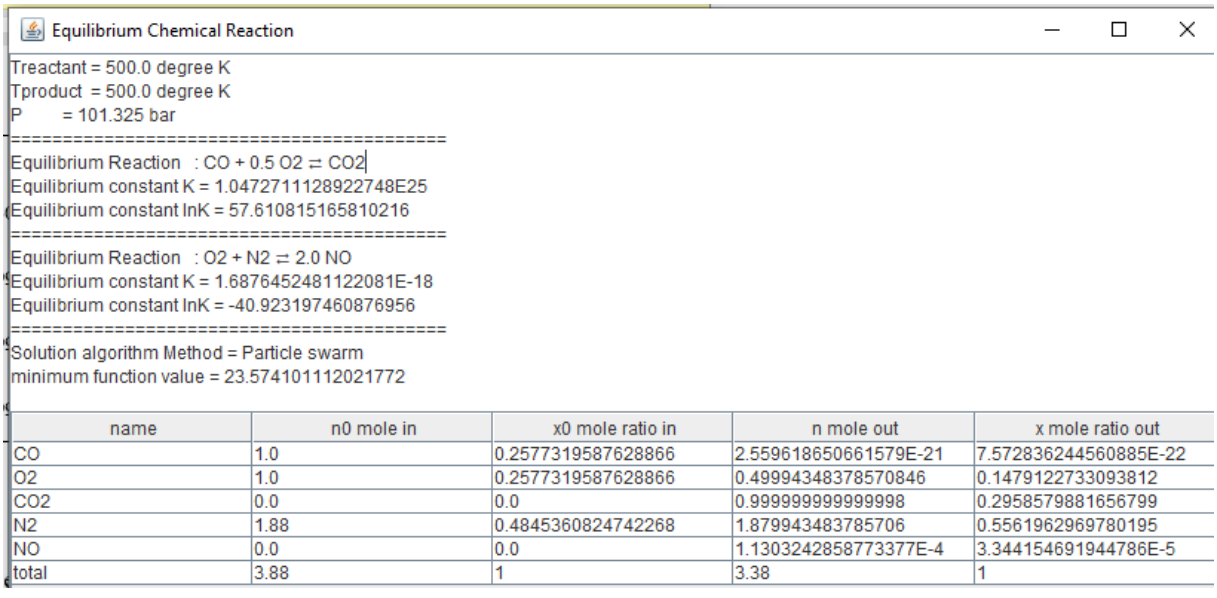
$CO + \frac{1}{2}O_2 \rightleftharpoons CO_2$ $N_2 + 1/2O_2 \rightleftharpoons 2NO$

The reaction will be $CO + O_2 \rightarrow n_0CO + n_1O_2 + n_2CO_2 + n_3N_2 + n_4NO$

Sample program: (**if_NL_equilibrium_PG_nasa9**)

```
            public static void main(String arg[])
            {String s[]={"CO","O2","CO2","N2","NO"};
    double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{0,0},{0,0}};
    chemicalReaction_PG_nasa9 r[]=new chemicalReaction_PG_nasa9[2];
    r[0]=new chemicalReaction_PG_nasa9("r1",s,N1);
    double N2[][]={{0.0,0.0},{1.0,0.0},{0.0,0.0},{1.0,0},{0,2}};
    r[1]=new chemicalReaction_PG_nasa9("r2",s,N2);
    double Tproduct=500.0;//degree K
    double Treactant=500.0;//degree K
    double n0[]={1,1,0,1.88,0};
    double P=101.325;//kPa
            if_NL_equilibrium_PG_nasa9 fe=new if_NL_equilibrium_PG_nasa9(n0,s,r,Treactant,Tproduct,P);
            double n1[]={1.0,1.0};
            double n2[]=fe.solve(n1);
            fe.equilibrium_print(n2);
    }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" if_NL_equilibrium_PG_nasa9
Firefly =
      0.016698120934418     0.927557777114353     0.983301879065580     2.299208716647143
-0.838417433294285
Particle swarm =
      0.000000000000000     0.499983997436283     0.999999999999998     1.879983997436280
0.000032005127439
Thermal Exchange optimisation =
      0.000000862921478     0.499986505386076     0.999999137078521     1.879986073925335
0.000027852149331
Harris hawk =
      0.000000000000000     0.499943483785708     0.999999999999998     1.879943483785706
0.000113032428588
```



name	n0 mole in	x0 mole ratio in	n mole out	x mole ratio out
CO	1.0	0.2577319587628866	2.559618650661579E-21	7.572836244560885E-22
O2	1.0	0.2577319587628866	0.49994348378570846	0.1479122733093812
CO2	0.0	0.0	0.999999999999998	0.2958579881656799
N2	1.88	0.4845360824742268	1.879943483785706	0.5561962969780195
NO	0.0	0.0	1.1303242858773377E-4	3.344154691944786E-5
total	3.88	1	3.38	1

```
import java.io.*;
import java.util.*;
import javax.swing.*;
import java.awt.*;
import java.awt.event.*;

//nn number of molecules
//n2 number of reactions
//n1 number of atoms nn-n2
```

```
class f_NL2 implements if_xj
{//adaptation function
ifi_xj ff1;
public f_NL2(ifi_xj ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}}

public class if_NL_equilibrium_PG_nasa9 implements ifi_xj
{ public double n0[];
  public double n[];
  int nn,n1,n2;
  String name[];
  double TR,TP;
  public chemicalReaction_PG_nasa9 cr;
  public double A[][];
  public double A1[][];
  public double A2[][];
  public double B[];
  public double B1[];
  public double B2[];
  public Gas_PG_nasa9 g[];
  public double Pref=101.325;//bar
  public double P;
  public chemicalReaction_PG_nasa9 chr[];
  public SVD q;
  public f_NL2 f2;
  public ql_new_heuristics qlnh;
  public fireflyi ffo;
  public particle_swarm1i pso;
  public harris_hawks_optimizationi hho;
  public Thermal_Exchange_Optimizationi teo;
  public double xmin[];
  public double xmax[];
  public int iN;
  public int iMaxiter;
  public int min_index=0;
  public double min=8.88e40;
  String method="";
  public if_NL_equilibrium_PG_nasa9(double n0i[],String  namei[], chemicalReaction_PG_nasa9 chri[],double T1i,double T2i,double Pi)
  { nn=n0i.length;
    int nn1=namei.length;
    if(nn!=nn1) System.out.println("array size error nn="+nn+"nn1="+nn1);
    P=Pi;
    n2=chri.length;
    n1=nn-n2;
   n0=new double[nn];
   n=new double[nn];
   g=new Gas_PG_nasa9[nn];
   name=new String[nn];
   TR=T1i;
   TP=T2i;
   chr=new chemicalReaction_PG_nasa9[n2];
   for(int i=0;i<n2;i++) {chr[i]=chri[i];}
   for(int i=0;i<nn;i++)
   {
                n0[i]=n0i[i];
    name[i]=namei[i];
    g[i]=new Gas_PG_nasa9(name[i]);
   }
   double N[][]=new double[nn][2];
   for(int i=0;i<nn;i++)
   {  N[i][0]=n0[i];
      N[i][1]=n[i];
   }
   cr=new chemicalReaction_PG_nasa9("base reaction",name,N);
   cr.conservation();
   A=cr.A;
   B=cr.b0;
   A1=new double[n1][n1];
```

```
 A2=new double[n1][n2];
 B1=new double[n1];
 B2=new double[nn];
 for(int i=0;i<n1;i++)
  {for(int j=0;j<n1;j++)
   {A1[i][j]=A[i][j+n2];}
  }
 for(int i=0;i<n1;i++)
  {for(int j=0;j<n2;j++)
   {A2[i][j]=A[i][j];}
  }
 q=new SVD(A1);
 f2=new f_NL2(this);
 xmin=new double[n2];
 xmax=new double[n2];
 iN=1000;iMaxiter=1000;
}

public double[] func(double x1[])
{
              double C[]=new double[n1];
  for(int i=0;i<n1;i++)
    {for(int j=0;j<n2;j++)
     {C[i]+=A2[i][j]*x1[j];
     }
    }
           for(int i=0;i<n1;i++)
             {B1[i]=B[i]-C[i];}
  double x2[]=q.solve(B1);
  for(int i=n2;i<nn;i++)
             {n[i]=x2[i-n2];}
           for(int i=0;i<n2;i++)
             {n[i]=x1[i];}
  double nt=0;
  for(int i=0;i<nn;i++)
  {nt+=n[i];}

  double lnK;
  double total;
  double mu[][]=new double[nn][2];
  double nx[];
  double y[]=new double[n2];
  for(int ii=n1;ii<nn;ii++)
           { int ij=ii-n1;
             int nn2=chr[ij].ngas;
             nx=new double[nn2];
             for(int i=0;i<nn;i++)
      {
                  for(int j=0;j<nn2;j++)
        {  if(name[i].equals(chr[ij].name[j]))
          {   mu[i][0]=chr[ij].N[j][0];mu[i][1]=chr[ij].N[j][1];
            nx[j]=n[i];

                     break;
                  }
        }
      }
             total=0;
             for(int i=0;i<chr[ij].ngas;i++)
   { if(nx[i]!=0)
     {total-=mu[i][0]*Math.log(nx[i]/nt*P/Pref);
      total+=mu[i][1]*Math.log(nx[i]/nt*P/Pref);
     }
             }
   lnK=chr[ij].lnK(TR,TP);
   y[ii-n1]=lnK-total;
           }
 return y;
}
  public  void equilibrium_print(double ni[])
  {
             double x[]=new double[nn];
             double x0[]=new double[nn];
             double nnt=0;
             double nn0t=0;
             for(int i=0;i<nn;i++){nnt+=ni[i];nn0t+=n0[i];}
```

```
            for(int i=0;i<nn;i++){x[i]=ni[i]/nnt;x0[i]=n0[i]/nn0t;}
 JTextArea jta;
 JPanel jpan=new JPanel();
 jpan.setLayout(new BorderLayout());
 String s="";
 s+="Treactant = "+TR+" degree K\n";
 s+="Tproduct  = "+TP+ " degree K\n";
 s+="P         = "+P+" bar\n";
 s+="=========================================\n";
 for(int i=0;i<n2;i++)
 {
 s+="Equilibrium Reaction   : "+chr[i].toStringE()+"\n";
 s+="Equilibrium constant K = "+chr[i].K(TR,TP)+"\n";
 s+="Equilibrium constant lnK = "+chr[i].lnK(TR,TP)+"\n";
 s+="=========================================\n";
 }
 s+="Solution algorithm Method = "+method+"\n";
 s+="minimum function value = "+min+"\n";
 //for(int j=0;j<nn;j++)
 //{s+="n_"+name[j]+" = ["+n0[j]+" ,  "+ni[j]+" ]\n";}
 jta=new JTextArea(s);

 String heading[]={"name","n0 mole in","x0 mole ratio in","n mole out","x mole ratio out"};
 String s1[][]=new String[nn+1][5];
 for(int i=0;i<nn;i++)
 {s1[i][0]=name[i];
  s1[i][1]=""+n0[i];
  s1[i][2]=""+x0[i];
  s1[i][3]=""+ni[i];
  s1[i][4]=""+x[i];
 }
 s1[nn][0]="total";
 s1[nn][1]=""+nn0t;
 s1[nn][2]=""+1;
 s1[nn][3]=""+nnt;
 s1[nn][4]=""+1;
            JTable jt;
            genelModel gm=new genelModel(s1,heading);
            jt=new JTable(gm);
             jpan.add(jta,BorderLayout.NORTH);
             jpan.add(new JScrollPane(jt),BorderLayout.SOUTH);
            String bb="Equilibrium Chemical Reaction";
            JFrame cerceve=new JFrame(bb);
 cerceve.addWindowListener(new BasicWindowMonitor());
 cerceve.getContentPane().add(new JScrollPane(jpan));
 cerceve.pack();
 cerceve.setVisible(true);
            }
            public double[] solve(double xx[])
 { xmax=xx;
   min_index=0;
   min=8.88e40;
 //Firefly
 // m number of fireflies
 // n number of variables for each fireflyi
 // I0 light intensity at source
 // gamma absorbtion coefficient
 // alpha size of the random step
 // beta attractiveness
 //QL new houristic algorith
// Firefly algorithm
double xfinal[][]=new double[4][nn];
double gamma=1.2143243;
double alpha0=0.5154656;
double alphan=0.0017857;
ffo=new fireflyi(f2,iN,n2,xmin,xmax,gamma,alpha0,alphan,iMaxiter);
double xx8[]=ffo.xmin();
double yy8[]=func(xx8);
double xaverage=0;
double xf[]=new double[nn];
xfinal[0]=this.n;
System.out.print("Firefly =\n"+Matrix.toString(xfinal[0]));
for(int i=0;i<xx8.length;i++)
{xaverage+=yy8[i]*yy8[i];}
if(xaverage<min) {min=xaverage;min_index=0;}
double x08[][]={xx8,yy8};
```

```
        pso=new particle_swarm1i(f2,iN,xmin,xmax,2.0,0.7,iMaxiter);
        double xx7[]=pso.xmin();
        double yy7[]=func(xx7);
        double x07[][]={xx7,yy7};
        xaverage=0;
        for(int i=0;i<xx7.length;i++)
        {xaverage+=yy7[i]*yy7[i];}
        if(xaverage<min) {min=xaverage;min_index=1;}
        xfinal[1]=this.n;
        System.out.print("Particle swarm =\n"+Matrix.toString(xfinal[1]));
        teo=new Thermal_Exchange_Optimizationi(f2,iN,xmin,xmax,iMaxiter);
        double[][] x01=teo.solution();
        x01[1]=func(x01[0]);
        xaverage=0;
        for(int i=0;i<x01[0].length;i++)
        {xaverage+=x01[1][i]*x01[1][i];}
        if(xaverage<min) {min=xaverage;min_index=2;}
        xfinal[2]=this.n;
        System.out.print("Thermal Exchange optimisation = \n"+Matrix.toString(xfinal[2]));
        min=Math.sqrt(xaverage)/xx8.length;
       hho=new harris_hawks_optimizationi(f2,iN,xmin,xmax,iMaxiter);
       double[][] x02=hho.solution();
       x02[1]=func(x02[0]);
       xaverage=0;
       xfinal[3]=this.n;
       System.out.print("Harris hawk =\n"+Matrix.toString(xfinal[3]));
       for(int i=0;i<x02[0].length;i++)
       {xaverage+=x02[1][i]*x02[1][i];}
       if(xaverage<min) {min=xaverage;min_index=3;}
       if(min_index==0)
       {method="firefly";xf= xfinal[0];}
       else if(min_index==1)
       {method="Particle swarm";; xf= xfinal[1];}
       else if(min_index==2)
       {method="thermal exchange";;xf= xfinal[2];}
       else if(min_index==3)
       {method="Harris hawk";System.out.println(method);xf= xfinal[3];}
       else xf= xfinal[1];
       return xf;
     }

                public static void main(String arg[])
                {String s[]={"CO","O2","CO2","N2","NO"};
     double N1[][]={{1.0,0.0},{0.5,0.0},{0.0,1.0},{0,0},{0,0}};
     chemicalReaction_PG_nasa9 r[]=new chemicalReaction_PG_nasa9[2];
     r[0]=new chemicalReaction_PG_nasa9("r1",s,N1);
     double N2[][]={{0.0,0.0},{1.0,0.0},{0.0,0.0},{1.0,0},{0,2}};
     r[1]=new chemicalReaction_PG_nasa9("r2",s,N2);
     double Tproduct=300.0;//degree K
     double Treactant=300.0;//degree K
     double n0[]={1,1,0,1.88,0};
     double P=101.325;//kPa
                 if_NL_equilibrium_PG_nasa9 fe=new if_NL_equilibrium_PG_nasa9(n0,s,r,Treactant,Tproduct,P);
                 double n1[]={1.0,1.0};
                 double n2[]=fe.solve(n1);
                 fe.equilibrium_print(n2);
     }
}
```

```
import java.io.*;
import java.util.*;
import javax.swing.*;
import java.awt.*;
import java.awt.event.*;

public class chemicalReaction_PG_nasa9
{String name[];
double N[][];
double n0[];
double n1[];
double T0,P0;
Gas_PG_nasa9 g[];
int natom,ngas;
int nrt,npt;
String reacName;
```

```
double A[][];
    public double[][] A1;
    public String[][] SA;
    public String[][] SA1;
double b0[];
double b1[];
Atom atomList[];
String atomnames[];
String atomsymbols[];
//combustion reaction

//Gas_PG_nasa9 names and input-output  moles
chemicalReaction_PG_nasa9(String crname,String si[],double Ni[][])
{ reacName=crname;
  ngas=si.length;
    P0=101.325;
      T0=273.15;
name=new String[ngas];
N=new double[ngas][2];
g=new Gas_PG_nasa9[ngas];
for(int i=0;i<ngas;i++)
{  name[i]=si[i];
   N[i][0]=Ni[i][0];
   N[i][1]=Ni[i][1];
   g[i]=new Gas_PG_nasa9(name[i]);
}
arrange_atoms();
conservation();
}
     public void arrange_atoms()
     {
     // atomic structure of the reactants
     int i,j;
        for(i=0;i<ngas;i++)
        {for(j=0;j<g[i].natom;j++)
                    {add_atom(i,j);}
        }
     }

     public int  add_atom(int i,int j)
     {
     //determine and order the atomic structure
     int k;
       for(k=0;k<natom;k++)
       {
       if(g[i].atomList[j].symbol.equals(atomList[k].symbol))
         {
         atomList[k]=new Atom(atomList[k].symbol,atomList[k].N+g[i].atomList[j].N*N[i][0]);
         return 1;
         }
       }
     Atom atomL[];
     atomL=new Atom[natom+1];
     for(k=0;k<natom;k++)
       atomL[k]=new Atom(atomList[k]);
     atomL[natom]=new Atom(g[i].atomList[j].symbol,g[i].atomList[j].N*N[i][0]);
     atomList=atomL;
     natom+=1;
     return 2;
     }
     public String toString()
     {
      // writes chemical reaction in chemistry norm
       String s="";
       for(int i=0;i<ngas;i++)
       {
       if(N[i][0]==1)
           { s+=g[i].toString(); }
       else if(N[i][0]!=0)
           { s+=N[i][0]+" "+g[i].toString(); }
       if(i!=(ngas-1) && N[i][0]!=0 ) s+=" + ";
       }
       if(s.endsWith(" + "))
         s=s.substring(0,s.length()-3);
       s+=" --> ";
       for(int i=0;i<ngas;i++)
```

```
  {
  if(N[i][1]==1)
     { s+=g[i].toString(); }
  else if(N[i][1]!=0)
     { s+=N[i][1]+" "+g[i].toString(); }
  if(i!=(ngas-1) && N[i][1]!=0 ) s+=" + ";
  }
  if(s.endsWith(" + "))
    s=s.substring(0,s.length()-3);
return s;
}
public String toStringE()
{
  // writes chemical reaction in chemistry equilibrium norm
  String s="";
  for(int i=0;i<ngas;i++)
  {
  if(N[i][0]==1)
     { s+=g[i].toString(); }
  else if(N[i][0]!=0)
     { s+=N[i][0]+" "+g[i]; }
  if(i!=(ngas-1) && N[i][0]!=0 ) s+=" + ";
  }
  if(s.endsWith(" + "))
    s=s.substring(0,s.length()-3);
  s+=" "+'\u21C4'+" ";
  for(int i=0;i<ngas;i++)
  {
  if(N[i][1]==1)
     { s+=g[i].toString(); }
  else if(N[i][1]!=0)
     { s+=N[i][1]+" "+g[i]; }
  if(i!=(ngas-1) && N[i][1]!=0 ) s+=" + ";
  }
  if(s.endsWith(" + "))
    s=s.substring(0,s.length()-3);
return s;
}
public String toString(String ch)
{
//return the c
String s="";
int i,j;
if(ch.equals("name"))
   s=s+reacName+"\n";
else if(ch.equals("formula"))
   {
   s=toString();
   }
else if(ch.equals("composition"))
   {
   for(i=0;i<natom;i++)
   s=s+atomList[i].toString()+" ";
   }
return s;
}
public double H(double TR,double TP)
{
// reaction enthalpy
 double HH=0;
 double h1,h2;
 for(int i=0;i<ngas;i++)
 {
  h1=g[i].ht(TP)*N[i][1];
  h2=g[i].ht(TR)*N[i][0];
              HH+=h1;
  HH-=h2;
 }
 return HH;
}
public double H(double TR[],double TP)
{
// reaction enthalpy
 double HH=0;
 double h1,h2;
 for(int i=0;i<ngas;i++)
```

```
 {
  h1=g[i].ht(TP)*N[i][1];
  h2=g[i].ht(TR[i])*N[i][0];
              HH+=h1;
  HH-=h2;
 }
 return HH;
}
public double S(double TR,double TP,double pt)
{
//reaction entropy
 double TP1,TR1,pt1;
 double s1,s2;
 TR1=TR;TP1=TP;pt1=pt;
 double SS=0;
 for(int i=0;i<ngas;i++)
 {
  if(N[i][1]!=0)
     SS+=g[i].s(TP1,N[i][1]/npt*pt1)*N[i][0];
  if(N[i][0]!=0)
     SS-=g[i].s(TR1,N[i][0]/nrt*pt1)*N[i][1];
 }
   return SS;
}
public double S(double TR[],double TP,double pt)
{
//reaction entropy
 double TP1,TR1,pt1;
 double s1,s2;
 pt1=pt;
 double SS=0;
 for(int i=0;i<ngas;i++)
 {
  if(N[i][1]!=0)

     SS+=g[i].s(TP,N[i][1]/npt*pt1)*N[i][0];
  if(N[i][0]!=0)
     SS-=g[i].s(TR[i],N[i][0]/nrt*pt1)*N[i][1];
 }
   return SS;
}
public double S(double TR,double TP)
{
//reaction entropy
 double TP1,TR1,pt1,pt;
 pt=1.0;TR1=TR;TP1=TP;pt1=pt;
 return S(TR,TP,pt);
}
public double G(double TR,double TP,double pt)
{
// reaction gibbs energy
 double TP1,TR1,pt1;
 TR1=TR;TP1=TP;pt1=pt;
 double GG=0;
 for(int i=0;i<ngas;i++)
 {
  if(N[i][1]!=0)
     GG+=g[i].gt(TP1,N[i][1]/npt*pt1)*N[i][1];
  if(N[i][0]!=0)
     GG-=g[i].gt(TR1,N[i][0]/nrt*pt1)*N[i][0];
 }
   return GG;
}
 public double G(double TR[],double TP,double pt)
{
// reaction gibbs energy
 double TP1,TR1,pt1;
 pt1=pt;
 double GG=0;
 for(int i=0;i<ngas;i++)
 {
  if(N[i][1]!=0)
     GG+=g[i].gt(TP,N[i][1]/npt*pt1)*N[i][1];
  if(N[i][0]!=0)
     GG-=g[i].gt(TR[i],N[i][0]/nrt*pt1)*N[i][0];
 }
```

```
   return GG;
}

public double G(double TR[],double TP)
{ // reaction gibbs energy
  double pt=P0;
  return G(TR,TP,pt);
}
public double G(double TR,double TP)
{ // reaction gibbs energy
 double pt=P0;
  return G(TR,TP,pt);
}


public double G0(double TR,double TP)
{
// reaction gibbs energy
 double GG=0;
 double h1=0;
 for(int i=0;i<ngas;i++)
 {
  if(N[i][1]!=0)
     { h1=g[i].gt(TP,P0)*N[i][1];GG+=h1;}
  if(N[i][0]!=0)
     { h1=g[i].gt(TR,P0)*N[i][0];GG-=h1;}
 }
  return GG;
}
public double G0(double TR[],double TP)
{
// reaction gibbs energy
 double GG=0;
 double h1=0;
 for(int i=0;i<ngas;i++)
 {
  if(N[i][1]!=0)
     { h1=g[i].gt0(TP)*N[i][1];GG+=h1;}
  if(N[i][0]!=0)
     { h1=g[i].gt0(TR[i])*N[i][0];GG-=h1;}
 }
  return GG;
}
public double K(double TR,double TP)
{
// Equilibrium costant. Usually TR and TP will be in
// the same temperature
return Math.exp(G0(TR,TP)/(-8.3145*TP));
}
public double lnK(double TR,double TP)
{
// Equilibrium costant. Usually TR and TP will be in
// the same temperature
return G0(TR,TP)/(-8.3145*TP);
}
public boolean conservation()
{                       //Gives atomic mass balance of the chemical reaction
        atomnames=new String[natom];
        atomsymbols=new String[natom];
                boolean bb=true;
        int nn=ngas;
                                int nat=g[0].atomList[0].atom_name.length;
                                double B[][]=new double[nat][nn];
                                int nk=0;
                                for(int i=0;i<nn;i++)
                                {
                                        for(nk=0;nk<g[i].atomList.length;nk++){
                                                B[g[i].atomList[nk].number-1][i]=g[i].atomList[nk].N;
                                                }
                                        }
                                int natcount=0;
                                boolean zeroflag[]=new boolean[nat];
                                for(int j=0;j<nat;j++)
                                {  zeroflag[j]=false;
                                   for(int i=0;i<nn;i++)
                                   {if(B[j][i]!=0){zeroflag[j]=true;}};
```

```
                                        if(zeroflag[j])
                                        {  String ss1=g[0].atomList[0].atom_name[j];
                                           String ss2=g[0].atomList[0].atom_symbol[j];
                                                   atomnames[natcount]=g[0].atomList[0].atom_name[j];
                                         atomsymbols[natcount]=g[0].atomList[0].atom_symbol[j];
                                         natcount++;}
                                      }
                                      A=new double[natcount][nn];
                                      int k=0;
                                      for(int j=0;j<nat;j++)
                                       {if(zeroflag[j]){A[k]=B[j];k++;}}
                                      natom=A.length;
                                      ngas=A[0].length;
                                      b0=new double[natom];
                                      for(int ii=0;ii<natom;ii++)
                                      {
                                                for(int ij=0;ij<ngas;ij++)
                                                {
                                                          double carpim=A[ii][ij]*N[ij][0];
                                                          b0[ii]+=carpim;
                                                }
                                      }
                                       b1=new double[natom];
                                      for(int ii=0;ii<natom;ii++)
                                      {
                                                for(int ij=0;ij<ngas;ij++)
                                                  {
                                                          double carpim=A[ii][ij]*N[ij][1];
                                                          b1[ii]+=carpim;
                                                  }
                                      }
                                      for(int ii=0;ii<natom;ii++)
                                                {
          if(b0[ii]==b1[ii]) bb=bb&&true;
          else                        bb=bb&&false;
                                                }
                                                return bb;
}



public  void equilibrium_print(double TR,double TP)
{
JTextArea jta;
JPanel jpan=new JPanel();
jpan.setLayout(new BorderLayout());
String s="Equilibrium Reaction name  : "+toString("name")+"\n";
s+="Equilibrium  formula     : "+toStringE()+"\n";
s+="Treactant = "+TR+" degree K\n";
s+="Tproduct  = "+TP+ " degree K\n";
s+="Equilibrium constant K = "+K(TR,TP)+"\n";
s+="Equilibrium constant lnK = "+lnK(TR,TP)+"\n";
 s+="Equilibrium constant log10K = "+Math.log10(K(TR,TP))+"\n";
s+="Reaction composition : "+toString("composition")+"\n";
s+="Atom balance check   : "+conservation()+"\n";
s+="Atomic balance:";
jta=new JTextArea(s);
natom=A.length;
ngas=A[0].length;
String bb="Equilibrium Chemical Reaction";
String baslik[]=new String[ngas+3];
String s1[][]=new String[natom][ngas+3];
int i,j;
for(i=0;i<ngas;i++)
{baslik[i+1]=g[i].Formula;}
baslik[i+1]="b0(reactants)";
baslik[i+2]="b1(products)";

for(j=0;j<natom;j++)
{s1[j][0]=atomsymbols[j];}
for(j=0;j<natom;j++)
 { for(i=0;i<ngas;i++)
    {s1[j][i+1]=""+A[j][i];}
 }
 for(j=0;j<natom;j++)
    {s1[j][i+1]=""+b0[j];
```

```
      s1[j][i+2]=""+b1[j];
     }
            JTable jt;
            int n=A.length;
            int m=A[0].length;
            genelModel gm=new genelModel(s1,baslik);
            jt=new JTable(gm);
             //JOptionPane.showMessageDialog(null,jt);
             jpan.add(jta,BorderLayout.NORTH);
             jpan.add(new JScrollPane(jt),BorderLayout.SOUTH);
JFrame cerceve=new JFrame(bb);
cerceve.addWindowListener(new BasicWindowMonitor());
cerceve.getContentPane().add(new JScrollPane(jpan));
cerceve.pack();
cerceve.setVisible(true);
            }
public  void print(double TR,double TP)
{
JTextArea jta;
JPanel jpan=new JPanel();
jpan.setLayout(new BorderLayout());
String s="Reaction name  : "+toString("name")+"\n";
s+="Reaction formula     : "+toString("formula")+"\n";
s+="Reaction composition : "+toString("composition")+"\n";
s+="Atom balance check   : "+conservation()+"\n";
s+="Treactant = "+TR+" degree K\n";
s+="Tproduct  = "+TP+ " degree K\n";
double Taf=Tadiabatic(TR);
s+="Adiabatic flame temperature = "+Taf+"\n";
double Q1=H(298.15,298.15);
s+="Heating value = "+Q1+" kJ/kmol\n";
double Q2=H(TR,TP);
s+="Q = "+Q2+" kJ/kmol \n";
s+="heating efficiency = "+Q2/Q1+"\n";
s+="\n";
s+="Atomic balance:";
jta=new JTextArea(s);
natom=A.length;
ngas=A[0].length;
String bb="Chemical Reaction";
String baslik[]=new String[ngas+3];
String s1[][]=new String[natom][ngas+3];
int i,j;
for(i=0;i<ngas;i++)
{baslik[i+1]=g[i].Formula;}
baslik[i+1]="b0(reactants)";
baslik[i+2]="b1(products)";

for(j=0;j<natom;j++)
{s1[j][0]=atomsymbols[j];}
for(j=0;j<natom;j++)
 { for(i=0;i<ngas;i++)
   {s1[j][i+1]=""+A[j][i];}
 }
 for(j=0;j<natom;j++)
    {s1[j][i+1]=""+b0[j];
     s1[j][i+2]=""+b1[j];
    }
            JTable jt;
            int n=A.length;
            int m=A[0].length;
            genelModel gm=new genelModel(s1,baslik);
            jt=new JTable(gm);
             //JOptionPane.showMessageDialog(null,jt);
             jpan.add(jta,BorderLayout.NORTH);
             jpan.add(new JScrollPane(jt),BorderLayout.SOUTH);
JFrame cerceve=new JFrame(bb);
cerceve.addWindowListener(new BasicWindowMonitor());
cerceve.getContentPane().add(new JScrollPane(jpan));
cerceve.pack();
cerceve.setVisible(true);
            }

public String atombalance()
{String s=" |";
 for(int i=0;i<name.length;i++)
```

```
      {  if(name[i].length()==2) s+=name[i]+"   ";
                 else if(name[i].length()==2) s+=name[i]+"  ";
        else if(name[i].length()==3) s+=name[i]+" ";
        else if(name[i].length()>=4) s+=name[i];
      }
     s+="|b0  b1 |\n";
     for(int j=0;j<atomList.length;j++)
     {s+=atomsymbols[j]+"|";
      for(int i=0;i<name.length;i++)
      {s+=A[j][i]+" ";}
      s+=" |"+b0[j]+" "+b1[j]+"|\n";
     }
    return s;
    }
    public double Tadiabatic(double TR)
    {     double Taf=Taf(TR,298.0,2000.0);
    if(Taf>1999.0&&Taf<=2000.0 ) Taf=Taf(TR,298.0,3000.0);
    if(Taf>2999.9&&Taf<=3000.0) Taf=Taf(TR,298.0,4000.0);
    if(Taf>3999.9&&Taf<=4000.0) Taf=Taf(TR,298.0,5000.0);
    return Taf;
    }
    public double Taf(double TR,double xl,double xu)
    {
     double TR1,xl1,xu1;
     TR1=TR;xl1=xl;xu1=xu;
    // Adyabatic flame temperature using Bisection method
    // bisection method to find roots of zero energy change
    // Q=0
    // defination of variables :
    // xl : lower guess
    // xu : upper guess
    // xr : root estimate
    // es : stopping criterion
    // ea :approximate error
    // maxit : maximum iterations
    // iter  : number of iteration
    double test;
    double xr=0;
    double es,ea;
    double fxl=0,fxr=0;
    int maxit=500,iter=0;
    es=0.000001;
    ea=1.1*es;
    while((ea>es)&&(iter<maxit))
    {
    xr=(xl1+xu1)/2.0;
    iter++;
    if((xl1+xu1)!=0)
       { ea=Math.abs((xu1-xl1)/(xl1+xu1))*100;}
    fxl= H(TR,xl1);
    fxr= H(TR,xr);
    test= fxl*fxr;
    if(test==0.0)       ea=0;
    else if(test<0.0)  xu1=xr;
    else
      {
      xl1=xr;
      }
    } //end of while
      return xr;
    }

    double Vi(double TR,double TP,int Ni)
    {return -G(TR,TP)/(96487*Ni);}

                double Vi0(double TR,double TP,int Ni)
    {return -G0(TR,TP)/(96487*Ni);}

}
```

7. Results and conclusion

Stoichiometric chemical equilibrium algorithm is developed by using ideal Gas gas EoS. Equilibrium calculations are based on atomic mass balances and minimisation of gibbs energy. Stoichimetric chemical equilibrium concepts is used in calculations, required equilibrum equations are defined as an input parameter, and chemical equilibrium coefficients are calculated by using only temperature dependent components of gibbs energy. Several example cases runs with both EoS‘. When result investigated, it is clear that when temperature getting down some species vanished, and mole numbers are coming close to zero. This caused a singularity problem with standart non-linear system of equation solution methods, but semi-nonlinear method applied in this solution seems to partially overcome this problem. The complete codes for this analysis is given in internet site www.turhancoban.com as zip fie SCO1.rar. Further analysis for direct gibbs free energy minimisation without Stoichiometric equations are also studied by our group. It will be presenrted as a seperate paper.

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PROBLEMS

PROBLEM 1

Solve the following system of non-linear equations

$4x_1^2-20x_1+0.25x_2^2+8=0$

$0.5x_1x_2^2+2x_1+8=0$

By using

a) Newton raphson method

b) Broyden sekant method

c) Continuity(Homotopy) method

PROBLEM 2

Solve the following system of non-linear equations

$3x_1^2-\cos(x_2x_2) - 0.5=0$

$4x_1^2-6.25x_2^2+2x_3-1=0$

$e^{x1\ x2} +20x_3+(10\pi-3)/3=0$

By using

a) Newton raphson method

b) Broyden sekant method

c) Continuity(Homotopy) method

PROBLEM 3

Solve te following system of non-linear equations

$x_1(1- x_1) + 4x_3 – 12 =0$

$(x_1-2)^2+(2x_2-3)^2-25 =0$

By using

a) Newton raphson method

b) Broyden sekant method

c) Continuity(Homotopy) method

PROBLEM 4

Solve the following system of non-linear equations

$\sin(4\pi x_1 x_2)-2x_2-x_1=0$

$\left(\frac{4\pi-1}{4\pi}\right)(e^{2x_1}-e)+4ex_2^2-2ex_1=0$

By using

a) Newton raphson method

b) Broyden sekant method

c) Continuity(Homotopy) method

PROBLEM 5

Solve the following system of non-linear equations

$10x_1-2x_2^2+x_2-2x_3-5=0$

$8x_2^2+4x_3^2-9=0$

$8x_2x_3+4=0$

By using

a) Newton raphson method

b) Broyden sekant method

c) Continuity(Homotopy) method

7 Curve fitting Methods

CURVE FITTING AND FUNCTION APPROXIMATIONS

Data is obtained by measurements in the scientific world. Scientits are generalized the measurement results by using various formulas and theories. Creating functions by using experimental numerical data is called curve fitting. Formulas are generally formed from the theories of physical laws or freely selected by the researchers. In the fitted equation usually a good fit to the experimental data is a primary requirement. Fitting means assigning values to unknown coefficients in the equations. The coefficients can be linearly or non-linearly dependent the the variables of the function(s). Interpolation is also similar process, but in the interpolation a function is not always required. Interpolation could be just a repeatetive calculation process by following a certain rule. Different curve fitting methods will be investigated in this chapter. At the end of the chapter Approximation of the functions will also be covered.

LINEAR CURVE FITTING : LEAST SQUARE METHODS

Least square methods are one of the most used curve fitting methods. Polinomial equation version is widely used. Assuming having a linear function f(a_j,x) where a_j are m linear coefficient and x are independent variable and $\phi_j(x)$ are m sub-funtions linearly multiplied with the coefficients

$$f(a_i, x) = \sum_{j=0}^{m} a_j^{(m)} \phi_j(x) \qquad (6.1\text{-}1)$$

It should be noted again that linearity is only for the coefficients, $\phi_j(x)$ functions does not have to be linear.

Sample functions: $f(a_i, x) = a_0 \sin(x) + a_1 \exp(x) + a_2 \ln(x) + a_3 x^2$

It is desired to fit data x_i, f_i, i=0..n into the equation so that the difference between data and fitted function dependent values for all points will be minimum. In order to establish this, the following function H will be minimized with respect to a_j

$$H(a_0^{(m)}, .., a_n^{(m)}) = \sum_{i=1}^{n} w(x_i) \left[f(x_i) - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x_i) \right]^2 \qquad (6.1-1)$$

Where $w(x_i)$ values in the equation are called weight function. In order to minimiz the function root of the derivative of the function can be calculated.

$$\frac{\partial H(a_0^{(m)}, .., a_n^{(m)})}{\partial a_k^{(m)}} = 2 \sum_{i=1}^{n} w(x_i) \left[f(x_i) - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x) \right] \phi_k(x) = 0 \quad k = 0, \dots, m \qquad (6.1-2)$$

equation in the open form can be written in the following form.

$$\begin{bmatrix} \sum_{i=1}^{n} w(x_i)\phi_0(x_i)\phi_0(x_i) & \sum_{i=1}^{n} w(x_i)\phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n} w(x_i)\phi_0(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)\phi_0(x_i)\phi_n(x_i) \\ \sum_{i=1}^{n} w(x_i)\phi_1(x_i)\phi_0(x_i) & \sum_{i=1}^{n} w(x_i)\phi_1(x_i)\phi_1(x_i) & \sum_{i=1}^{n} w(x_i)\phi_1(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)\phi_2(x_i)\phi_n(x_i) \\ \sum_{i=1}^{n} w(x_i)\phi_2(x_i)\phi_0(x_i) & \sum_{i=1}^{n} w(x_i)\phi_2(x_i)\phi_1(x_i) & \sum_{i=1}^{n} w(x_i)\phi_2(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)\phi_2(x_i)\phi_n(x_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} w(x_i)\phi_n(x_i)\phi_0(x_i) & \sum_{i=1}^{n} w(x_i)\phi_n(x_i)\phi_1(x_i) & \sum_{i=1}^{n} w(x_i)\phi_n(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)\phi_n(x_i)\phi_n(x_i) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{Bmatrix} =$$

$$\begin{Bmatrix} \sum_{i=1}^{n} w(x_i)\phi_0(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)\phi_1(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)\phi_2(x_i)f(x_i) \\ \cdots \\ \sum_{i=1}^{n} w(x_i)\phi_n(x_i)f(x_i) \end{Bmatrix} \quad (6.1\text{-}3)$$

For weight function to be taken equal to unity , $w(x_i)$, equation in the open form can be written in the following form.

$$\begin{bmatrix} \sum_{i=1}^{n} \phi_0(x_i)\phi_0(x_i) & \sum_{i=1}^{n} \phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n} \phi_0(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} \phi_0(x_i)\phi_n(x_i) \\ \sum_{i=1}^{n} \phi_1(x_i)\phi_0(x_i) & \sum_{i=1}^{n} \phi_1(x_i)\phi_1(x_i) & \sum_{i=1}^{n} \phi_1(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} \phi_2(x_i)\phi_n(x_i) \\ \sum_{i=1}^{n} \phi_2(x_i)\phi_0(x_i) & \sum_{i=1}^{n} \phi_2(x_i)\phi_1(x_i) & \sum_{i=1}^{n} \phi_2(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} \phi_2(x_i)\phi_n(x_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} \phi_n(x_i)\phi_0(x_i) & \sum_{i=1}^{n} \phi_n(x_i)\phi_1(x_i) & \sum_{i=1}^{n} \phi_n(x_i)\phi_2(x_i) & \cdots & \sum_{i=1}^{n} \phi_n(x_i)\phi_n(x_i) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} \phi_0(x_i)f(x_i) \\ \sum_{i=1}^{n} \phi_1(x_i)f(x_i) \\ \sum_{i=1}^{n} \phi_2(x_i)f(x_i) \\ \cdots \\ \sum_{i=1}^{n} \phi_n(x_i)f(x_i) \end{Bmatrix} \quad (6.1\text{-}4)$$

This equation is an m+1 linear system of equation. It can easily be solved by using a system of equation solving method. A special form of the above equation a form of a polinomial can be assumed. If $\phi_j(x) = x^j$, equation becomes

$$f(a_i, x) = \sum_{j=0}^{m} a_j^{(m)} x^j \qquad (6.1-5)$$

$f(a_i, x) = a_0 + a_1 x + a_2 x^2 + a_2 x^2 + \cdots + a_n x^n \quad (6.1.5a)$

In this case the minimisation equation becomes

$$\frac{\partial H(a_0^{(m)}, .., a_0^{(m)})}{\partial a_k^{(m)}} = 2\sum_{i=1}^{n} w(x_i)\left[f(x_i) - \sum_{j=0}^{m} a_j^{(m)} x^j\right] x^k = 0 \quad k = 0, \dots, m \qquad (6.1-6)$$

$$\begin{bmatrix} \sum_{i=1}^{n} w(x_i) & \sum_{i=1}^{n} w(x_i)x_i & \sum_{i=1}^{n} w(x_i)x_i^2 & \cdots & \sum_{i=1}^{n} w(x_i)x_i^n \\ \sum_{i=1}^{n} w(x_i)x_i & \sum_{i=1}^{n} w(x_i)x_i^2 & \sum_{i=1}^{n} w(x_i)x_i^3 & \cdots & \sum_{i=1}^{n} w(x_i)x_i^{n+1} \\ \sum_{i=1}^{n} w(x_i)x_i^2 & \sum_{i=1}^{n} w(x_i)x_i^3 & \sum_{i=1}^{n} w(x_i)x_i^4 & \cdots & \sum_{i=1}^{n} w(x_i)x_i^{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{i=1}^{n} w(x_i)x_i^n & \sum_{i=1}^{n} w(x_i)x_i^{n+1} & \sum_{i=1}^{n} w(x_i)x_i^{n+2} & \cdots & \sum_{i=1}^{n} w(x_i)x_i^{2n} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} w(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)x_i f(x_i) \\ \sum_{i=1}^{n} w(x_i)x_i^2 f(x_i) \\ \dots \\ \sum_{i=1}^{n} w(x_i)x_i^n f(x_i) \end{Bmatrix} \quad (6.1-7)$$

Or if $w(x_i) = 1$

$$\begin{bmatrix} n = \sum_{i=1}^{n} x_i^0 & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \cdots & \sum_{i=1}^{n} x_i^n \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \cdots & \sum_{i=1}^{n} x_i^{n+1} \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 & \cdots & \sum_{i=1}^{n} x_i^{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} x_i^n & \sum_{i=1}^{n} x_i^{n+1} & \sum_{i=1}^{n} x_i^{n+2} & \cdots & \sum_{i=1}^{n} x_i^{2n} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} f(x_i) \\ \sum_{i=1}^{n} x_i f(x_i) \\ \sum_{i=1}^{n} x_i^2 f(x_i) \\ \dots \\ \sum_{i=1}^{n} x_i^n f(x_i) \end{Bmatrix} \quad (6.1-8)$$

Linear polynomial regression

Consider the general polynomial curve fitting equation and take only first two column and row, equation will be for the linear polynomial $f(x)=a_0+a_1x$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} f(x_i) \\ \sum_{i=1}^{n} x_i f(x_i) \end{Bmatrix}$$

Python program for linear polynomial curve fitting

Sample data:

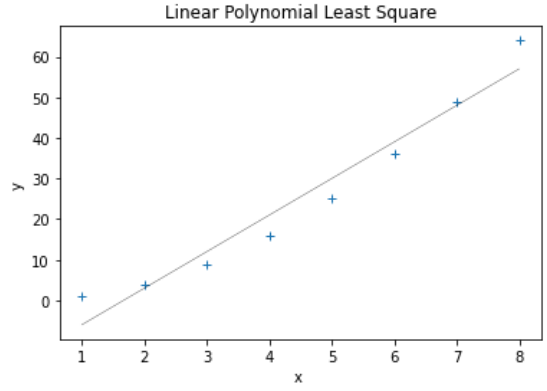
```
1 1
2 4
3 9
4 16
5 25
6 36
7 49
8 64
```

```
 # Linear least curve fitting (regression)  y=a0+a1x
from math import *
import numpy as np
import matplotlib.pyplot as plt
import gauss
def poly_1(x,y):
    n=len(x)
    x1=0
    x2=0
    y1=0
```

```
    y2=0
    for i in range(n):
        x1+=x[i]
        x2+=x[i]*x[i]
        y1+=y[i]
        y2+=x[i]*y[i]

    a= [[0 for i in range(2)] for j in range(2)];
    a[0][0]=n
    a[0][1]=x1
    a[1][0]=x1
    a[1][1]=x2;
    b=[0 for i in range(2)]
    b[0]=y1
    b[1]=y2
    x=gauss.gauss(a,b)
    return x

def func(e,x):
    # this function calculates the value of
    # least square curve fitting function
    ff=e[0]+e[1]*x
    return ff

def read_array2D(d):
    path = d
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    m=0
    while True:
        line=file1.readline()
        ee=line.split()
        nn+=1
        if  file1.tell()==endLocation:
            break
    m=len(ee)
    print("m=",m)
    a=[[0.0 for i in range(nn)] for j in range(m)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        for j in range(m):
            a[j][i]=float(ee[j])
    file1.close()
    return a

na=input("file name : ")
a=read_array2D(na)
n=len(a[0])
x =a[0]
print("x=",x)
y =a[1]
print("y=",y)
e=poly_1(x,y)
n1=100
x1=[0 for x in range(n1)]
y1=[0 for x in range(n1)]
xmax=max(x)
xmin=min(x)
dx=(xmax-xmin)/(n1-1)
for i in range(n1):
    x1[i]=xmin+i*dx
    y1[i]=func(e,x1[i])
print("x1=",x1)
print("y1=",y1)
plt.title('Linear Polynomial Least Square')
plt.xlabel('x ')
plt.ylabel('y ')
plt.plot(x,y,'+',x1,y1,'k',linewidth=0.3)
```

```
runfile('E:/okul/SCO1/d_regression1.py', wdir='E:/okul/SCO1')
```

```
Reloaded modules: gauss

file name : a.txt
m= 2
x= [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0]
y= [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0]
x1= [1.0, 1.0707070707070707, 1.1414141414141414, 1.2121212121212122, 1.2828282828282829, 1.3535353535353536,
1.4242424242424243, 1.4949494949494948, 1.5656565656565657, 1.6363636363636362, 1.7070707070707072, 1.7777777777777777,
1.8484848484848484, 1.9191919191919191, 1.9898989898989898, 2.0606060606060606, 2.1313131313131315, 2.202020202020202,
2.2727272727272725, 2.3434343434343434, 2.4141414141414144, 2.484848484848485, 2.5555555555555554, 2.6262626262626263,
2.696969696969697, 2.7676767676767673, 2.8383838383838382, 2.909090909090909, 2.9797979797979797, 3.0505050505050506,
3.121212121212121, 3.191919191919192, 3.2626262626262625, 3.333333333333333, 3.404040404040404, 3.4747474747474745,
3.5454545454545454, 3.616161616161616, 3.686868686868687, 3.7575757575757573, 3.8282828282828283, 3.898989898989899,
3.9696969696969697, 4.04040404040404, 4.111111111111111, 4.181818181818182, 4.252525252525253, 4.3232323232323235,
4.393939393939394, 4.4646464646464645, 4.535353535353535, 4.6060606060606055, 4.6767676767676765, 4.747474747474747,
4.818181818181818, 4.888888888888889, 4.959595959595959, 5.03030303030303, 5.101010101010101, 5.171717171717171,
5.242424242424242, 5.313131313131313, 5.383838383838384, 5.454545454545454, 5.525252525252525, 5.595959595959596,
5.666666666666666, 5.737373737373737, 5.808080808080808, 5.878787878787879, 5.949494949494949, 6.02020202020202,
6.090909090909091, 6.161616161616162, 6.232323232323232, 6.303030303030303, 6.373737373737374, 6.444444444444445,
6.515151515151515, 6.585858585858586, 6.656565656565657, 6.727272727272727, 6.797979797979798, 6.8686868686868685,
6.9393939393939394, 7.0101010101010095, 7.08080808080808, 7.151515151515151, 7.222222222222222, 7.292929292929292,
7.363636363636363, 7.434343434343434, 7.505050505050505, 7.575757575757575, 7.646464646464646, 7.717171717171717,
7.787878787878787, 7.858585858585858, 7.929292929292929, 8.0]
y1= [-6.000000000000027, -5.36363636363639, -4.727272727272753, -4.090909090909117, -3.45454545454548, -
2.818181818181843, -2.1818181818182047, -1.5454545454545716, -0.9090909090909314, -0.27272727272729647,
0.363636363636342, 0.9999999999999751, 1.6363636363636118, 2.272727272727252, 2.9090909090908887, 3.5454545454545254,
4.181818181818162, 4.818181818181799, 5.454545454545432, 6.090909090909072, 6.727272727272709, 7.3636363636363455,
7.999999999999979, 8.636363636363619, 9.272727272727252, 9.909090909090889, 10.545454545454529, 11.181818181818166,
11.818181818181802, 12.454545454545439, 13.090909090909076, 13.727272727272712, 14.363636363636349, 14.999999999999982,
15.636363636363622, 16.272727272727256, 16.909090909090896, 17.54545454545453, 18.18181818181817, 18.818181818181802,
19.454545454545443, 20.090909090909076, 20.727272727272716, 21.36363636363635, 21.99999999999999, 22.636363636363622,
23.272727272727263, 23.909090909090903, 24.545454545454536, 25.18181818181817, 25.818181818181802, 26.454545454545443,
27.090909090909083, 27.727272727272723, 28.363636363636356, 28.999999999999996, 29.63636363636363, 30.27272727272727,
30.909090909090903, 31.545454545454536, 32.18181818181817, 32.81818181818181, 33.45454545454545, 34.09090909090908,
34.72727272727272, 35.36363636363636, 35.999999999999986, 36.636363636363626, 37.272727272727266, 37.90909090909091,
38.54545454545455, 39.18181818181817, 39.81818181818181, 40.45454545454545, 41.09090909090909, 41.72727272727272,
42.36363636363636, 43.0, 43.63636363636364, 44.27272727272728, 44.90909090909091, 45.54545454545455, 46.18181818181819,
46.81818181818183, 47.45454545454545, 48.09090909090909, 48.727272727272734, 49.363636363636374, 50.000000000000014,
50.63636363636364, 51.27272727272728, 51.90909090909092, 52.54545454545456, 53.18181818181819, 53.81818181818183,
54.45454545454547, 55.09090909090909, 55.727272727272734, 56.363636363636374, 57.000000000000014]
```



Julia program for linear polynomial curve fitting

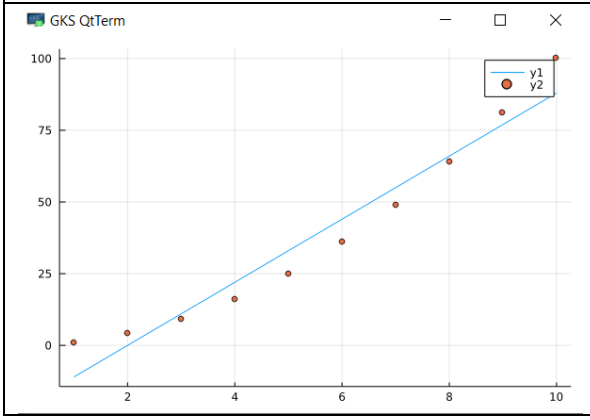
```
using SpecialFunctions
using Plots
 # Linear least curve fitting (regression)  y=a0+a1x
function gauss_pivot(A1,b)
            n=size(A1)[1]
            carpan=0.0
            toplam=0.0
            global x=Array{Float64}(undef,n)
            global a=Array{Float64}(undef,n,n)
            for i=1:n
              for j=1:n
                a[i,j]=A1[i,j]
              end
            end
            for k=1:n
            #pivoting
```

```
                p=k
                buyuk=abs(a[k,k])
                for ii=k+1:n
                  dummy=abs(a[ii,k])
                  if dummy > buyuk
                    buyuk=dummyp=ii
                  end #if dummy
                end #for ii
                if p!=k
                  for jj=k:n
                    dummy=a[p,jj]
                    a[p,jj]=a[k,jj]
                    a[k,jj]=dummy
                    end #for jj

                  dummy=b[p]
                  b[p]=b[k]
                  b[k]=dummy
                end #if p!
              # gauss elimination
   for i=k+1:n
                carpan=a[i,k]/a[k,k]
                a[i,k]=0.0
                for j=k+1:n
                  a[i,j]=a[i,j]-carpan*a[k,j]
                end #for j
                b[i]=b[i]-carpan*b[k]
              end #for i
              end #k
                          # back substituting
               x[n]=b[n]/a[n,n]
               n1=n-1
               for i=n1:-1:1
                 toplam=0.0
                 for j=i+1:n
                   toplam=toplam+a[i,j]*x[j]
                 end #for j
                 x[i]=(b[i]-toplam)/a[i,i]
               end #for i
              return x
end #function

function poly_1(x,y)
   n=length(x)
   x1=0.0
   x2=0.0
   y1=0.0
   y2=0.0
   for i=1:n
      x1+=x[i]
      x2+=x[i]*x[i]
      y1+=y[i]
      y2+=x[i]*y[i]
   end # for i=1:n

   a=zeros(Float64,2,2)
   a[1,1]=n
   a[1,2]=x1
   a[2,1]=x1
   a[2,2]=x2;
   b=zeros(Float64,2,2)
   b[1]=y1
   b[2]=y2
   x=gauss_pivot(a,b)
   return x
end

function func(e,x)
   # this function calculates the value of
   # least square curve fitting function
   ff=e[1]+e[2]*x
   return ff
end

function convert_to_float(column)
   new_column = map(x -> (
```

```
        x = ismissing(x) ? "" : x;  # no method matching tryparse(::Type{Float64}, ::Missing)
        x = tryparse(Float64, x);   # returns: Float64 or nothing
        isnothing(x) ? missing : x; # options: missing, or "", or 0.0, or nothing
        ), column)  # input
   # returns Array{Float64,1} OR Array{Union{Missing, Float64},1}
   return new_column
end

function readDouble(name)
  open(name) do f
  # read till end of file
  m=0
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    global m1=m
    global n1=n
  end  #while
  close(f)
  end  #open
  global x=Matrix{Float64}(undef,n1,m1)
  open(name) do f
  # read till end of file
  m=0
  global z=Array{Float64}(undef,n1,0)
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    x1=convert_to_float(s)
    z=hcat(z,x1)
    global x=z
  end  #while
  close(f)
  end  #open
  #transpose matrix
  global y=x'
  return y
  end

print("file name : ")
name=readline()
a=readDouble(name)
c=a'
n=size(a)[1]
m=size(a)[2]
x=c[1,:]
y=c[2,:]
x2=x;
y2=y;
println("x = $x2 \ny =  $y2")
e=poly_1(x,y)
print(" e = $e")
x1=zeros(Float64,n)
y1=zeros(Float64,n)
xmax=maximum(x2)
xmin=minimum(x2)

dx=(xmax-xmin)/(n-1)
print("xmin = $xmin xmax = $xmax n = $n dx = $dx")
for i=1:n
   x1[i]=xmin+(i-1.0)*dx
   xx=x1[i]
   y1[i]=func(e,xx)
end # for i
println("x1 0 $x1 \ny1 =  $y1")
plot(x1,y1)
plot!(x2,y2,seriestype=scatter)
```

```
 ---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" poly_least_linear.jl
file name : a.txt
```

```
x = [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
y =  [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0, 81.0, 100.0]
 e = [-22.0, 11.0]
xmin = 1.0 xmax = 10.0 n = 10 dx = 1.0x1 0 [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
y1 =  [-11.0, 0.0, 11.0, 22.0, 33.0, 44.0, 55.0, 66.0, 77.0, 88.0]
> Terminated with exit code 0.
```



Regression with quadratic polynomial

Consider the general polynomial curve fitting equation and take only first 3 column and row, equation will be for the quadratic polynomial $f(x)=a_0+a_1x+ a_2x^2$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} f(x_i) \\ \sum_{i=1}^{n} x_i f(x_i) \\ \sum_{i=1}^{n} x_i^2 f(x_i) \end{Bmatrix}$$

Python program for quadratic polynomial least square curve fitting

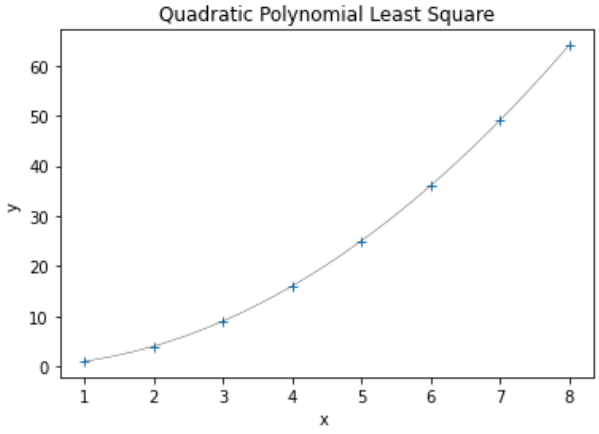
```
Sample data:
1 1
2 4
3 9
4 16
5 25
6 36
7 49
8 64
```

```
 # quadratic polynomial least square curve fitting
from math import *
import numpy as np
import matplotlib.pyplot as plt
import gauss

def poly_2(x,y):
    n=len(x)
    x1=0
    x2=0
    x3=0
    x4=0
    y1=0
    y2=0
    y3=0
    for i in range(n):
        x1+=x[i]
        x2+=x[i]*x[i]
        x3+=x[i]*x[i]*x[i]
        x4+=x[i]*x[i]*x[i]*x[i]
        y1+=y[i]
```

```
        y2+=x[i]*y[i]
        y3+=x[i]*x[i]*y[i]
    a= [[0 for i in range(3)] for j in range(3)];
    a[0][0]=n
    a[0][1]=x1
    a[0][2]=x2
    a[1][0]=x1
    a[1][1]=x2
    a[1][2]=x3
    a[2][0]=x2
    a[2][1]=x3
    a[2][2]=x4
    b=[0 for i in range(3)]
    b[0]=y1
    b[1]=y2
    b[2]=y3
    x=gauss.gauss(a,b)
    return x

def func(e,x):
    # this function calculates the value of
    # least square curve fitting function
    ff=e[0]+e[1]*x+e[2]*x*x
    return ff
def read_array2D(d):
    path = d
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    m=0
    while True:
        line=file1.readline()
        ee=line.split()
        nn+=1
        if  file1.tell()==endLocation:
            break
    m=len(ee)
    print("m=",m)
    a=[[0.0 for i in range(nn)] for j in range(m)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        for j in range(m):
            a[j][i]=float(ee[j])
    file1.close()
    return a
a=read_array2D('a.txt')
n=len(a[0])
x =a[0]
print("x=",x)
y =a[1]
print("y=",y)
e=poly_2(x,y)
n1=100
x1=[0 for x in range(n1)]
y1=[0 for x in range(n1)]
xmax=max(x)
xmin=min(x)
dx=(xmax-xmin)/(n1-1)
for i in range(n1):
    x1[i]=xmin+i*dx
    y1[i]=func(e,x1[i])
print("x1=",x1)
print("y1=",y1)
plt.title('Quadratic Polynomial Least Square')
plt.xlabel('x ')
plt.ylabel('y ')
plt.plot(x,y,'+',x1,y1,'k',linewidth=0.3)
```

```
runfile('E:/okul/SCO1/d_regression_2.py', wdir='E:/okul/SCO1')
Reloaded modules: gauss
m= 2
x= [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0]
```

```
y= [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0]
x1= [1.0, 1.0707070707070707, 1.1414141414141414, 1.2121212121212122, 1.2828282828282829, 1.3535353535353536,
1.4242424242424243, 1.4949494949494948, 1.5656565656565657, 1.6363636363636362, 1.7070707070707072, 1.7777777777777777,
1.8484848484848484, 1.9191919191919191, 1.9898989898989898, 2.0606060606060606, 2.1313131313131315, 2.202020202020202,
2.2727272727272725, 2.3434343434343434, 2.4141414141414144, 2.484848484848485, 2.5555555555555554, 2.6262626262626263,
2.696969696969697, 2.7676767676767673, 2.8383838383838382, 2.909090909090909, 2.9797979797979797, 3.0505050505050506,
3.121212121212121, 3.191919191919192, 3.2626262626262625, 3.333333333333333, 3.404040404040404, 3.4747474747474745,
3.5454545454545454, 3.616161616161616, 3.686868686868687, 3.7575757575757573, 3.8282828282828283, 3.898989898989899,
3.9696969696969697, 4.04040404040404, 4.111111111111111, 4.181818181818182, 4.252525252525253, 4.3232323232323235,
4.393939393939394, 4.4646464646464645, 4.535353535353535, 4.6060606060606055, 4.6767676767676765, 4.747474747474747,
4.818181818181818, 4.888888888888889, 4.959595959595959, 5.03030303030303, 5.101010101010101, 5.171717171717171,
5.242424242424242, 5.313131313131313, 5.383838383838384, 5.454545454545454, 5.525252525252525, 5.595959595959596,
5.666666666666666, 5.737373737373737, 5.808080808080808, 5.878787878787879, 5.949494949494949, 6.02020202020202,
6.090909090909091, 6.161616161616162, 6.232323232323232, 6.303030303030303, 6.373737373737374, 6.444444444444445,
6.515151515151515, 6.585858585858586, 6.656565656565657, 6.727272727272727, 6.797979797979798, 6.8686868686868685,
6.9393939393939394, 7.0101010101010095, 7.08080808080808, 7.151515151515151, 7.222222222222222, 7.292929292929292,
7.363636363636363, 7.434343434343434, 7.505050505050505, 7.575757575757575, 7.646464646464646, 7.717171717171717,
7.787878787878787, 7.858585858585858, 7.929292929292929, 8.0]
y1= [1.0, 1.146413631262116, 1.3028262422201817, 1.4692378328741966, 1.645648403224161, 1.8320579532700747,
2.028466483011938, 2.2348739924497494, 2.4512804815835123, 2.6776859504132227, 2.914090398938884, 3.1604938271604937,
3.416896235078053, 3.683297622691562, 3.95969799000102, 4.246097337006428, 4.542495663707785, 4.848892970105092,
5.165289256198346, 5.491684521987552, 5.828078767472708, 6.174471992653811, 6.530864197530863, 6.897255382103867,
7.273645546372818, 7.660034690337718, 8.056422813998571, 8.462809917355372, 8.87919600040812, 9.305581063156822,
9.74196510560147, 10.188348127742067, 10.644730129578614, 11.111111111111109, 11.587491072339557, 12.073870013263951,
12.570247933884298, 13.07662483420059, 13.593000714212835, 14.119375573921026, 14.655749413325172, 15.20212223242526,
15.758494031221304, 16.32486480971329, 16.90123456790123, 17.48760330578512, 18.083971023364963, 18.690337720640752,
19.306703397612484, 19.933068054280174, 20.569431690643803, 21.215794306703394, 21.87215590245893, 22.538516477910417,
23.214876033057852, 23.901234567901238, 24.597592082440563, 25.30394857667585, 26.020304050607084, 26.746658504234258,
27.48301193755739, 28.22936435057647, 28.985715743291504, 29.752066115702476, 30.528415467809406, 31.314763799612287,
32.11111111111111, 32.917457402305885, 33.73380267319661 4, 34.560146923783286, 35.39649015406591, 36.24283236404448,
37.099173553719005, 37.96551372308948, 38.8418528721559, 39.72819100091827, 40.62452810937659, 41.53086419753087,
42.44719926538108, 43.37353331292725, 44.30986634016937, 45.25619834710743, 46.21252933374145, 47.178859300071416,
48.15518824609734, 49.14151617181919, 50.13784307723701, 51.144168962350776, 52.160493827160494, 53.18681767166615,
54.22314049586777, 55.26946229976533, 56.32578308335884, 57.39210284664829, 58.46842158963371, 59.55473931231507,
60.651056014692365, 61.75737169676563, 62.87368635853484, 64.0]

In [192]:
```



Julia program for quadratic polynomial least square curve fitting

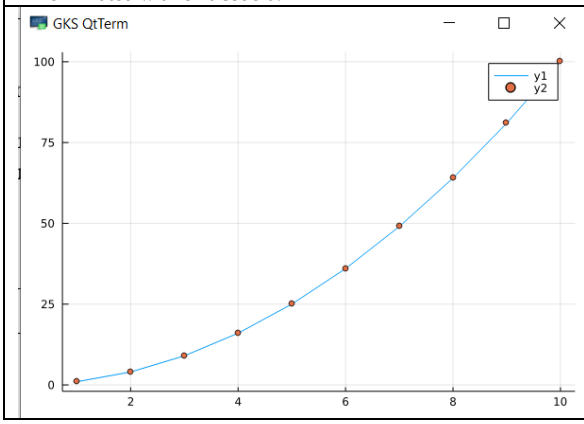
```
using SpecialFunctions
using Plots
 # quadratic least curve fitting (regression)  y=a0+a1*x+a2*x^2
function gauss_pivot(A1,b)
            n=size(A1)[1]
            carpan=0.0
            toplam=0.0
            global x=Array{Float64}(undef,n)
            global a=Array{Float64}(undef,n,n)
            for i=1:n
              for j=1:n
                a[i,j]=A1[i,j]
              end
            end
            for k=1:n
```

```
            #pivoting
               p=k
               buyuk=abs(a[k,k])
               for ii=k+1:n
                 dummy=abs(a[ii,k])
                 if dummy > buyuk
                    buyuk=dummyp=ii
                 end #if dummy
               end #for ii
               if p!=k
                 for jj=k:n
                   dummy=a[p,jj]
                   a[p,jj]=a[k,jj]
                   a[k,jj]=dummy
                   end #for jj

                 dummy=b[p]
                 b[p]=b[k]
                 b[k]=dummy
               end #if p!
            # gauss elimination
   for i=k+1:n
               carpan=a[i,k]/a[k,k]
               a[i,k]=0.0
               for j=k+1:n
                 a[i,j]=a[i,j]-carpan*a[k,j]
               end #for j
               b[i]=b[i]-carpan*b[k]
            end #for i
            end #k
                        # back substituting
              x[n]=b[n]/a[n,n]
              n1=n-1
              for i=n1:-1:1
                toplam=0.0
                for j=i+1:n
                  toplam=toplam+a[i,j]*x[j]
                end #for j
                x[i]=(b[i]-toplam)/a[i,i]
              end #for i
            return x
end #function

function poly_2(x,y)
   n=length(x)
   x1=0.0
   x2=0.0
   x3=0.0
   x4=0.0
   y3=0.0
   y1=0.0
   y2=0.0
   for i=1:n
      x1+=x[i]
      x2+=x[i]*x[i]
      x3+=x[i]*x[i]*x[i]
      x4+=x[i]*x[i]*x[i]*x[i]
      y1+=y[i]
      y2+=x[i]*y[i]
      y3+=x[i]*x[i]*y[i]
   end # for i=1:n
   a=zeros(Float64,3,3)
   a[1,1]=n
   a[1,2]=x1
   a[1,3]=x2
   a[2,1]=x1
   a[2,2]=x2
   a[2,3]=x3
   a[3,1]=x2
   a[3,2]=x3
   a[3,3]=x4
   b=zeros(Float64,3)
   b[1]=y1
   b[2]=y2
   b[3]=y3
   x=gauss_pivot(a,b)
```

```
    return x
end

function func(e,x)
    # this function calculates the value of
    # least square curve fitting function
    ff=e[1]+e[2]*x+e[3]*x*x
    return ff
end

function convert_to_float(column)
    new_column = map(x -> (
          x = ismissing(x) ? "" : x;  # no method matching tryparse(::Type{Float64}, ::Missing)
          x = tryparse(Float64, x);   # returns: Float64 or nothing
          isnothing(x) ? missing : x; # options: missing, or "", or 0.0, or nothing
          ), column)  # input
    # returns Array{Float64,1} OR Array{Union{Missing, Float64},1}
    return new_column
end

function readDouble(name)
  open(name) do f
  # read till end of file
  m=0
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    global m1=m
    global n1=n
  end  #while
  close(f)
  end  #open
  global x=Matrix{Float64}(undef,n1,m1)
  open(name) do f
  # read till end of file
  m=0
  global z=Array{Float64}(undef,n1,0)
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    x1=convert_to_float(s)
    z=hcat(z,x1)
    global x=z
  end  #while
  close(f)
  end  #open
  #transpose matrix
  global y=x'
  return y
  end

print("file name : ")
name=readline()
a=readDouble(name)
c=a'
n=size(a)[1]
m=size(a)[2]
x=c[1,:]
y=c[2,:]
x2=x;
y2=y;
println("x = $x2 \ny =  $y2")
e=poly_2(x,y)
print(" e = $e")
x1=zeros(Float64,n)
y1=zeros(Float64,n)
xmax=maximum(x2)
xmin=minimum(x2)

dx=(xmax-xmin)/(n-1)
print("xmin = $xmin xmax = $xmax n = $n dx = $dx")
for i=1:n
```

```
    x1[i]=xmin+(i-1.0)*dx
    xx=x1[i]
    y1[i]=func(e,xx)
end # for i
println("x1 0 $x1 \ny1 =  $y1")
plot(x1,y1)
plot!(x2,y2,seriestype=scatter)
```

```
 ---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" poly_least_quadratic.jl
file name : a.txt
x = [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
y =  [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0, 81.0, 100.0]
 e = [0.0, 0.0, 1.0]xmin = 1.0 xmax = 10.0 n = 10 dx = 1.0x1 0 [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
y1 =  [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0, 81.0, 100.0]
> Terminated with exit code 0.
```



Program Degree n polynomial least square curve fitting , java version

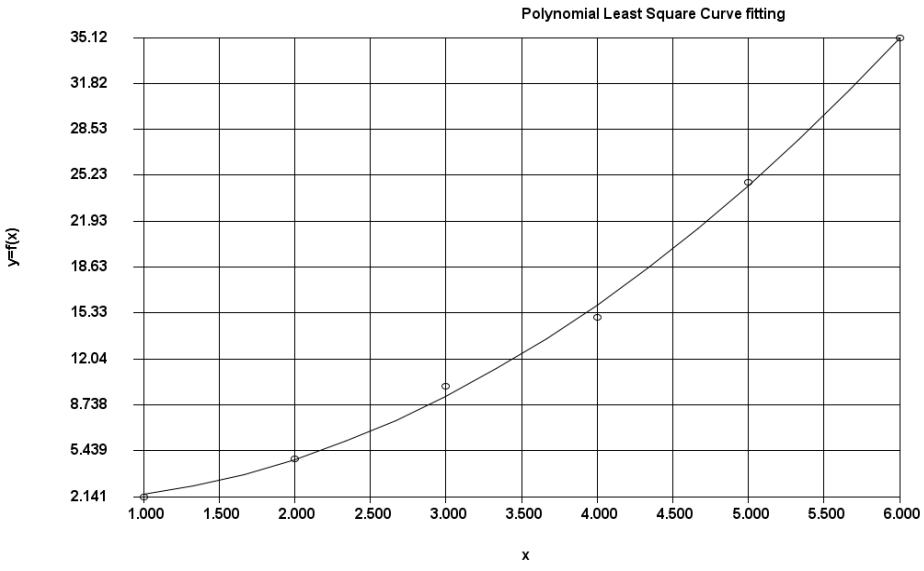
```
 import java.io.*;
import javax.swing.*;
import java.lang.*;
import java.util.Locale;

class NA44
{ // Polynomial least square
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
```

```
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] PolynomialLSQ(double xi[],double yi[],int n)
{ //Polynomial least square
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  for(j=0;j<n+1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= yi[k];
                  else      B[i] += Math.pow(xi[k],i)*yi[k];}
  }
 System.out.println("A="+Matrix.toString(A));
 System.out.println("B="+Matrix.toString(B));
X=gausswithpartialpivot(A,B);
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
return X;
}

public static double funcPolynomialLSQ(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
```

```
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcPolynomialLSQ(e,x[k]))*(y[k]-funcPolynomialLSQ(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcPolynomialLSQ(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
              //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
              // ara değer sayisi 2 dir
               int n=xi.length;
              int nn=(n-1)*(aradegersayisi+1)+1;
              double z[][]=new double[2][nn];
   double E[]=PolynomialLSQ(xi,yi,polinomkatsayisi);
   System.out.println("katsayilar :\n"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);
   return z;
 }
 public static String toString(double z[][])
 { String s="";
   for(int i=0;i<z.length;i++)
   {  for(int j=0;j<z[0].length;j++)
                {s+=String.format("%f ",z[i][j]);}
                s+="\n";
   }
   return s;
 }
 public static String toStringT(double z[][])
 { String s="";
   for(int i=0;i<z[0].length;i++)
   {  for(int j=0;j<z.length;j++)
                {s+=String.format(Locale.US,"%f ",z[j][i]);}
                s+="\n";
   }
   return s;
 }
 public static void main(String args[]) throws IOException
 {
 double x[];
 double y[];
 String s1=JOptionPane.showInputDialog("Data file name (input1.txt)");
 // JFileChooser fc=new JFileChooser();
 // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file = fc.getSelectedFile();s1=file.getName(); }
 double a[][]=Text.readDoubleT(s1);
 x=a[0];
 y=a[1];
 double z[][]=funcPolynomialLSQ(x,y,2,2);
 System.out.println("Polynomial Least Square Curve fitting\n"+toStringT(z));
 Plot pp=new Plot(a[0],a[1]);
 pp.setPlabel("Polynomial Least Square Curve fitting");
 pp.setXlabel("x");
 pp.setYlabel("y=f(x)");
 pp.setPlotType(0,22);
 pp.addData(z[0],z[1]);
 pp.setPlotType(1,0);
 pp.setGrid(1,1);
 pp.plot();
```

```
  }
}
```

```
 ---------- Capture Output ----------
> "E:\co\java\bin\java.exe" NA44
A=       6.000000000000000       21.000000000000000       91.000000000000000
      21.000000000000000       91.000000000000000      441.000000000000000
      91.000000000000000      441.000000000000000     2275.000000000000000

B=      91.921799999999990            436.484110000000000            2235.058210000000000

katsayilar :
           1.770298000000101
          -0.493580857142935
           1.007310000000011

Polynomial Least Square Curve fitting
1.000000 2.284027
1.333333 2.902964
1.666667 3.745747
2.000000 4.812376
2.333333 6.102853
2.666667 7.617176
3.000000 9.355345
3.333333 11.317362
3.666667 13.503225
4.000000 15.912935
4.333333 18.546491
4.666667 21.403894
5.000000 24.485144
5.333333 27.790240
5.666667 31.319183
6.000000 35.071973
```

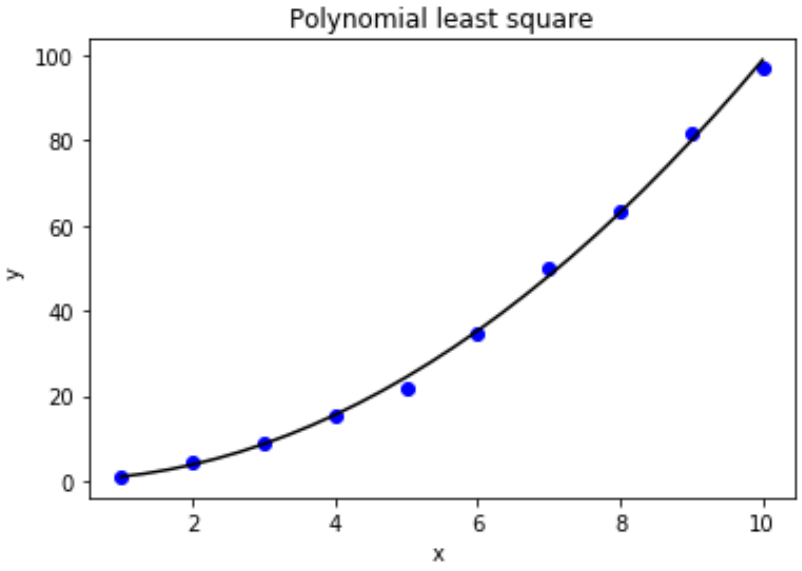


Program: degree n polynomial least square curve fitting (Python version)

```
 # -*- coding: utf-8 -*-
"""
Created on Sat Aug 25 09:31:06 2018

@author: Mustafa Turhan Çoban
"""
from math import *
from gauss import *;
import numpy as np;
import matplotlib.pyplot as plt;
```

```
def PolynomialLSQ(xi,yi,n):
    #Polynomial least square
    m=len(xi);
    print("l=",m);
    np1=n+1;
    print("np1=",np1);
    A = [[0 for x in range(np1)] for y in range(np1)];
    B =[0 for x in range(np1)];
    X =[0 for x in range(np1)];
    for i in range(np1):
        for j in range(np1):
            if i==0 and j==0:
                A[i][j]=m;
            else:
                for k in range(m):
                    pp=pow(xi[k],(i+j));
                    A[i][j]= A[i][j]+pow(xi[k],(i+j));
        for k in range(m):
            if i==0 and j==0:
                B[i]=  B[i]+yi[k];
            else:
                B[i]= B[i]+pow(xi[k],i)*yi[k];
    X=gauss(A,B);
    return X;

def funcPolynomialLSQ(e,x):
    # this function calculates the value of
    # least square curve fitting function
    n=len(e);
    if n!=0.0:
        ff=e[n-1];
        for i in range(n-2,-1,-1):
            ff=ff*x+e[i];
    else:
        ff=0;
    return ff;

def listPolynomialLSQ(E,xi,aradegersayisi):
     #aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                  # ara değer sayisi 2 dir
     n=len(xi);
     nn=(n-1)*(aradegersayisi+1)+1;
     z= [[0 for x in range(nn)] for y in range(2)];
     dx=0;
     k=0;
     for i in range(n-1):
         z[0][k]=xi[i];
         z[1][k]=funcPolynomialLSQ(E,z[0][k]);
         k=k+1;
         for j in range(aradegersayisi):
             dx=(xi[i+1]-xi[i])/(aradegersayisi+1.0);
             z[0][k]=z[0][k-1]+dx;
             z[1][k]=funcPolynomialLSQ(E,z[0][k]);
             k=k+1;
         z[0][k]=xi[i+1];z[1][k]=funcPolynomialLSQ(E,z[0][k]);
     return z;

a=np.loadtxt('a.txt');
print("a",a);
n=len(a);
x =[0 for x in range(n)];
y =[0 for x in range(n)];
for i in range(n):
    x[i]=a[i][0];
    y[i]=a[i][1];
print("x",x);
print("y",y);
E=PolynomialLSQ(x,y,2);
print("E=",E);
Z=listPolynomialLSQ(E,x,2);
print("Z=",Z[0]);
plt.title('Polynomial least square')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'bo',Z[0],Z[1],'k');
```

```
runfile('C:/Users/TURHAN/Desktop/turhan1/NA44.py', wdir='C:/Users/TURHAN/Desktop/turhan1')
Reloaded modules: gauss
a [[ 1.       1.2452 ]
 [ 2.       4.34534]
 [ 3.       8.9746 ]
 [ 4.      15.2    ]
 [ 5.      22.     ]
 [ 6.      34.8975 ]
 [ 7.      50.12378]
 [ 8.      63.61761]
 [ 9.      81.74675]
 [10.      97.     ]]
x [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
y [1.2452, 4.34534, 8.9746, 15.2, 22.0, 34.8975, 50.12378, 63.61761, 81.74675, 97.0]
l= 10
np1= 3
E= [0.2239984999998932, -0.1868977045453987, 1.005688749999995]
Z= [1.0, 1.3333333333333333, 1.6666666666666665, 2.0, 2.3333333333333335, 2.666666666666667, 3.0, 3.3333333333333335,
3.666666666666667, 4.0, 4.333333333333333, 4.666666666666666, 5.0, 5.333333333333333, 5.666666666666666, 6.0,
6.333333333333333, 6.666666666666666, 7.0, 7.333333333333333, 7.666666666666666, 8.0, 8.333333333333334,
8.666666666666668, 9.0, 9.333333333333334, 9.666666666666668, 10.0]
```



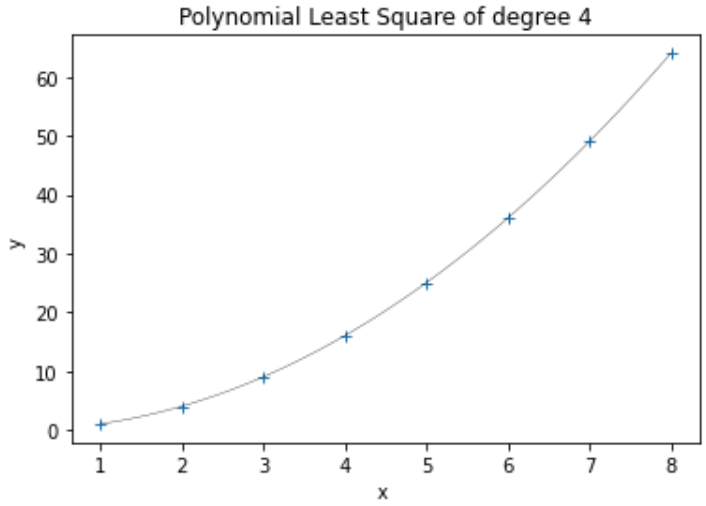
Program: degree n polynomial least square, Julia version

```
 # -*- coding: utf-8 -*-
"""
Created on Sat Aug 25 09:31:06 2018

@author: Mustafa Turhan Çoban
"""
from math import *
import numpy as np
import matplotlib.pyplot as plt
import gauss

def PolynomialLSQ(xi,yi,n):
    #Polynomial least square
    m=len(xi);
    print("l=",m);
    np1=n+1;
    print("np1=",np1);
    A = [[0 for x in range(np1)] for y in range(np1)];
    B =[0 for x in range(np1)];
    X =[0 for x in range(np1)];
    for i in range(np1):
        for j in range(np1):
            if i==0 and j==0:
                A[i][j]=m;
```

```
            else:
                for k in range(m):
                    pp=pow(xi[k],(i+j));
                    A[i][j]= A[i][j]+pow(xi[k],(i+j));
        for k in range(m):
            if i==0 and j==0:
                B[i]=  B[i]+yi[k];
            else:
                B[i]= B[i]+pow(xi[k],i)*yi[k];
    X=gauss.gauss(A,B);
    return X;

def func(e,x):
    # this function calculates the value of
    # least square curve fitting function
    n=len(e);
    if n!=0.0:
        ff=e[n-1];
        for i in range(n-2,-1,-1):
            ff=ff*x+e[i];
    else:
        ff=0;
    return ff;

def listPolynomialLSQ(E,xi,numberofmidpoints):
     #numberofmidpoints:  x--o--o--x--o--o--x
     n=len(xi)
     nn=(n-1)*(numberofmidpoints+1)+1;
     z= [[0 for x in range(nn)] for y in range(2)]
     dx=0.0
     k=0
     for i in range(n-1):
        z[0][k]=xi[i]
        z[1][k]=funcPolynomialLSQ(E,z[0][k])
        k=k+1
        for j in range(numberofmidpoints):
            dx=(xi[i+1]-xi[i])/(numberofmidpoints+1.0)
            z[0][k]=z[0][k-1]+dx
            z[1][k]=funcPolynomialLSQ(E,z[0][k])
            k=k+1
        z[0][k]=xi[i+1];z[1][k]=funcPolynomialLSQ(E,z[0][k])
     return z;

def read_array2D(d):
    path = d
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    m=0
    while True:
        line=file1.readline()
        ee=line.split()
        nn+=1
        if  file1.tell()==endLocation:
            break
    m=len(ee)
    print("m=",m)
    a=[[0.0 for i in range(nn)] for j in range(m)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        for j in range(m):
            a[j][i]=float(ee[j])
    file1.close()
    return a

a=read_array2D('a.txt')
n=len(a[0])
x =a[0]
print("x=",x)
y =a[1]
print("y=",y)
m=4
```

```
e=PolynomialLSQ(x,y,m)
n1=100
x1=[0 for x in range(n1)]
y1=[0 for x in range(n1)]
xmax=max(x)
xmin=min(x)
dx=(xmax-xmin)/(n1-1)
for i in range(n1):
   x1[i]=xmin+i*dx
   y1[i]=func(e,x1[i])
print("x1=",x1)
print("y1=",y1)
plt.title("Polynomial Least Square of degree "+str(m))
plt.xlabel('x ')
plt.ylabel('y ')
plt.plot(x,y,'+',x1,y1,'k',linewidth=0.3)
```

```
runfile('E:/okul/SCO1/d_regression_n.py', wdir='E:/okul/SCO1')
Reloaded modules: gauss
m= 2
x= [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0]
y= [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0]
l= 8
np1= 5
x1= [1.0, 1.0707070707070707, 1.1414141414141414, 1.2121212121212122, 1.2828282828282829, 1.3535353535353536,
1.4242424242424243, 1.4949494949494948, 1.5656565656565657, 1.6363636363636362, 1.7070707070707072, 1.7777777777777777,
1.8484848484848484, 1.9191919191919191, 1.9898989898989898, 2.0606060606060606, 2.1313131313131315, 2.202020202020202,
2.2727272727272725, 2.3434343434343434, 2.4141414141414144, 2.484848484848485, 2.5555555555555554, 2.6262626262626263,
2.696969696969697, 2.7676767676767673, 2.8383838383838382, 2.909090909090909, 2.9797979797979797, 3.0505050505050506,
3.121212121212121, 3.191919191919192, 3.2626262626262625, 3.333333333333333, 3.404040404040404, 3.4747474747474745,
3.5454545454545454, 3.616161616161616, 3.686868686868687, 3.7575757575757573, 3.8282828282828283, 3.898989898989899,
3.9696969696969697, 4.04040404040404, 4.111111111111111, 4.181818181818182, 4.252525252525253, 4.3232323232323235,
4.393939393939394, 4.4646464646464645, 4.535353535353535, 4.6060606060606055, 4.6767676767676765, 4.747474747474747,
4.818181818181818, 4.888888888888889, 4.959595959595959, 5.03030303030303, 5.101010101010101, 5.171717171717171,
5.242424242424242, 5.313131313131313, 5.383838383838384, 5.454545454545454, 5.525252525252525, 5.595959595959596,
5.666666666666666, 5.737373737373737, 5.808080808080808, 5.878787878787879, 5.949494949494949, 6.02020202020202,
6.090909090909091, 6.161616161616162, 6.232323232323232, 6.303030303030303, 6.373737373737374, 6.444444444444445,
6.515151515151515, 6.585858585858586, 6.656565656565657, 6.727272727272727, 6.797979797979798, 6.8686868686868685,
6.9393939393939394, 7.0101010101010095, 7.08080808080808, 7.151515151515151, 7.222222222222222, 7.292929292929292,
7.363636363636363, 7.434343434343434, 7.505050505050505, 7.575757575757575, 7.646464646464646, 7.717171717171717,
7.787878787878787, 7.858585858585858, 7.929292929292929, 8.0]
y1= [1.0, 1.146413631262116, 1.3028262422201817, 1.4692378328741966, 1.645648403224161, 1.8320579532700747,
2.028466483011938, 2.2348739924497494, 2.4512804815835123, 2.6776859504132227, 2.914090398938884, 3.1604938271604937,
3.416896235078053, 3.683297622691562, 3.959697990001102, 4.246097337006428, 4.542495663707785, 4.848892970105092,
5.165289256198346, 5.491684521987552, 5.828078767472708, 6.174471992653811, 6.530864197530863, 6.897255382103867,
7.273645546372818, 7.660034690337718, 8.056422813998571, 8.462809917355372, 8.87919600040812, 9.305581063156822,
9.74196510560147, 10.188348127742067, 10.644730129578614, 11.111111111111109, 11.587491072339557, 12.073870013263951,
12.570247933884298, 13.07662483420059, 13.593000714212835, 14.119375573921026, 14.655749413325172, 15.20212223242526,
15.758494031221304, 16.32486480971329, 16.90123456790123, 17.48760330578512, 18.083971023364963, 18.690337720640752,
19.306703397612484, 19.933068054280174, 20.569431690643803, 21.215794306703394, 21.87215590245893, 22.538516477910417,
23.214876033057852, 23.901234567901238, 24.597592082440563, 25.30394857667585, 26.020304050607084, 26.746658504234258,
27.48301193755739, 28.22936435057647, 28.985715743291504, 29.752066115702476, 30.528415467809406, 31.314763799612287,
32.11111111111111, 32.917457402305885, 33.733802673196614, 34.560146923783286, 35.39649015406591, 36.24283236404448,
37.099173553719005, 37.96551372308948, 38.8418528721559, 39.72819100091827, 40.62452810937659, 41.53086419753087,
42.44719926538108, 43.37353331292725, 44.30986634016937, 45.25619834710743, 46.21252933374145, 47.178859300071416,
48.15518824609734, 49.14151617181919, 50.13784307723701, 51.144168962350776, 52.160493827160494, 53.18681767166615,
54.22314049586777, 55.26946229976533, 56.32578308335884, 57.39210284664829, 58.46842158963371, 59.55473931231507,
60.651056014692365, 61.75737169676563, 62.87368635853484, 64.0]
```



Program: degree n polynomial least square curve fitting, Octave-matlab version

```
## Polynomial least square
## Author: M. Turhan COBAN
## Created: 2021-08-21

function retval = poly_least(x,y,n)
   #Polynomial least square
   ## this function calculates leat square polynomial coefficient a
   ## a=[a0 a1 a2 a3..]
   ## where f=a0+a1*x+a2*x^2+a3*x^3+.
   m=length(x);
   np1=n+1;
   A =zeros(np1,np1);
   B =zeros(np1);
   X =zeros(np1);
   x1=0.0;
   for i=1:np1
      for j=1:np1
         if i==1 && j==1
            A(i,j)=m;
         else
            for k=1:m
               x1=x(k);
               pp=x1^(i+j-2);
               A(i,j)= A(i,j)+pp;
            end # for k=1:m
         end # for if
       end # j=1:
      for k=1:m
         if i==1
            B(i)=  B(i)+y(k);
         else
            x1=x(k);
            B(i)= B(i)+x1^(i-1)*y(k);
         end # if i==0
      end # for k=1:m
   end # for i=1:np1
   retval=gauss1(A,B);
endfunction
```

```
## Polynomial evaluation function
## Author: M. Turhan COBAN
## Created: 2021-08-21
## this function calculates the value of polynomial
## a=[a0 a1 a2 a3..]
## f=a0+a1*x+a2*x^2+a3*x^3+.
function retval = poly_func(e,x)
## this function calculates the value of polynomial
## a=[a0 a1 a2 a3..]
## f=a0+a1*x+a2*x^2+a3*x^3+...
n=length(e);
if n!=0.0
```

```
ff=e(n);
for i=n-1:-1:1
ff=ff*x+e(i);
end # for i=n-1:-1:1
else
ff=0.0;
end # n!=0.0
retval= ff;
end
```

```
>> y=poly_func(a,4.0)
y = 16
>> x = [1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0];
>> y =  [1.0 4.0 9.0 16.0 25.0 36.0 49.0 64.0 81.0 100.0];
>> a=poly_least(x,y,4)
a =

  0   0   1   0   0

>> y1=poly_func(a,3.0)
y1 = 9
```

In matlab built in functions polyval and polyfit is existed. They are similar to above codes, but polynomial coefficients are given the inverse way

$$a=[a_m\ a_{m-1}\ a_{m-2}\ \ldots a_2\ a_1\ a_0]$$

$$f(a_i,x) = a_0 + a_1x + a_2x^2 + a_3x^3 + .. + a_mx^m$$

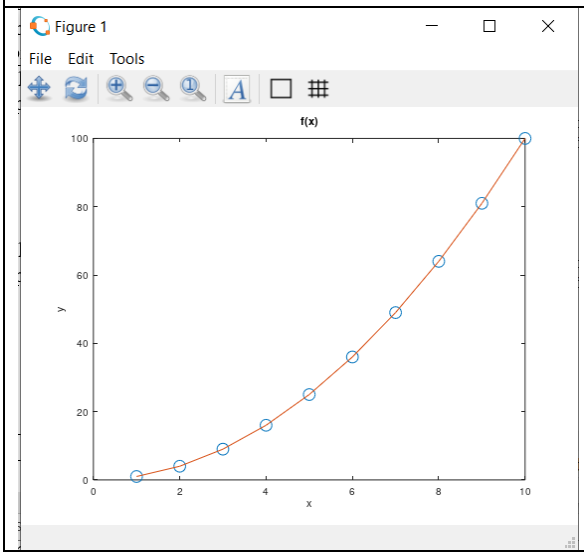
```
>> b=polyfit(x,y,6)
b =

 Columns 1 through 4:

  1.182289281649975e-17  -4.444153126043113e-16   6.568974224350917e-15  -4.816721246460687e-14

 Columns 5 through 7:

  1.000000000000181e+00  -3.194308323130954e-13   1.972028030392869e-13

>> y2=polyval(b,3.0)
y2 = 9.000000000000002
>>
```

A plotting program for polynomial curve fitting, Octave-matlab version

```
## Polynomial plotting
## Author: meral <meral@DESKTOP-7JLIIHR>
## Created: 2021-08-22
## e polynomial coefficients(poly_least)
## x,y original data
## x0,xn curve fitted data limits
## n number of urve fitted data

function retval = poly_plot (x,y,e,x0,xn,n,xlabel1,ylabel1,plabel1)
switch nargin
      case 6
        xlabel1='x';
        ylabel1='y';
        plabel1='f(x)';
      case 3
        xlabel1='x';
        ylabel1='y';
        plabel1='f(x)';
        n=length(x);
        x0=min(x);
        xn=max(x);
      case 2
        n1=input("degree of polynomial?")
        e=poly_least(x,y,n1)
        xlabel1='x';
        ylabel1='y';
```

```
        plabel1='f(x)';
        n=length(x);
        x0=min(x);
        xn=max(x);
end
x1=zeros(n);
y1=zeros(n);
xmax=max(x);
xmin=min(x);
dx=(xmax-xmin)/(n-1);
for i=1:n
   x1(i)=xmin+(i-1.0)*dx;
   xx=x1(i);
   y1(i)=poly_func(e,xx);
end # for i
plot(x,y,'o',x1,y1);
title(plabel1);
xlabel(xlabel1);
ylabel(ylabel1);
endfunction
```

```
>> poly_plot (x,y)
degree of polynomial?4
e =

  0   0   1   0   0

dx = 1
>> poly_plot (x,y)
degree of polynomial?4
n1 = 4
e =

  0   0   1   0   0

>>
```



A general function version of the least square program will look like :

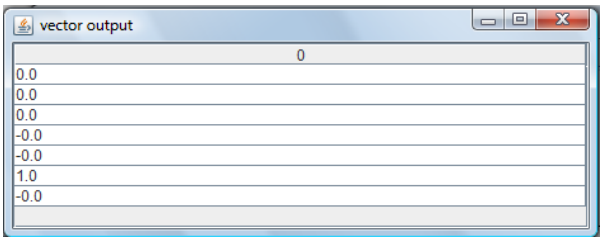
Program General least square curve fitting, Java version

```
import java.io.*;
import java.util.*;
import javax.swing.*;

//prototype function
abstract class f_xr
{
  abstract double func(double x,int equation_ref);
}

class fa extends f_xr
```

```
{
                double func(double T,int i)
                {
    double xx=1.0;
     if(i==0) xx=1.0;
     else if(i==1) xx=1.0/T;
     else if(i==2) xx=1.0/(T*T);
      else if(i==3) xx=1.0/(T*T*T);
     else if(i==4) xx=T;
     else if(i==5) xx=T*T;
     else if(i==6) xx=T*T*T;
  return xx;
                }
}

class NA45
{
//general least square curve fitting
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                  if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}


public static double[] GeneralLeastSquare(double xi[],double yi[],int n)
{
```

```
fa f=new fa();
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  B[i]=0;
                for(j=0;j<np1;j++)
    {
                A[i][j]=0.0;
    for(k=0;k<l;k++) A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);
    }
    for(k=0;k<l;k++) B[i]+= f.func(xi[k],i)*yi[k];
  }
//System.out.println("A = \n"+Matrix.toString(A));
//System.out.println("B = \n"+Matrix.toStringT(B));
X=gausswithpartialpivot(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
Text.printT(X);
return X;
}

public static double funcGeneralLeastSquare(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
fa f=new fa();
int n=e.length;
double ff=0;
if(n!=0.0)
  {
  for(int i=n-1;i>=0;i--)
   {ff+=e[i]*f.func(x,i);}
  }
return ff;
}

public static double hata(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcGeneralLeastSquare(e,x[k]))*(y[k]-funcGeneralLeastSquare(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

 public static double[][]  funcGeneralLeastSquare(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
                //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                 int n=xi.length;
                int nn=(n-1)*(aradegersayisi+1)+1;
                double z[][]=new double[2][nn];
   double E[]=GeneralLeastSquare(xi,yi,polinomkatsayisi);
   double dx=0;
```

```
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);
    return z;
  }

  public static void main(String args[]) throws IOException
  {
  String s1=JOptionPane.showInputDialog("file name : ");
  double c[][]=Text.readDoubleT(s1);
  double b[]=GeneralLeastSquare(c[0],c[1],6);
  System.out.println("General Least Square Coefficient : \n"+Matrix.toStringT(b));
  double z[][]=funcGeneralLeastSquare(c[0],c[1],6,2);
  System.out.println("General Least Square Solution \n"+Matrix.toStringT(z));
  Plot pp=new Plot(c[0],c[1]);
  pp.setPlabel("General Least Square");
  pp.setXlabel("x");
  pp.setYlabel("y=f(x)");
  pp.addData(z[0],z[1]);
  pp.plot();
  }

 }
```




```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA45
General Least Square Coefficient :
        0.000000000000000
        0.000000000000000
        0.000000000000000
       -0.000000000000000
       -0.000000000000000
        1.000000000000000
       -0.000000000000000

General Least Square Solution
   1.0000000000   1.0000000000
   1.3333333333   1.7777777778
   1.6666666667   2.7777777778
   2.0000000000   4.0000000000
   2.3333333333   5.4444444444
   2.6666666667   7.1111111111
   3.0000000000   9.0000000000
   3.3333333333  11.1111111111
   3.6666666667  13.4444444444
   4.0000000000  16.0000000000
   4.3333333333  18.7777777778
   4.6666666667  21.7777777778
   5.0000000000  25.0000000000
   5.3333333333  28.4444444444
```

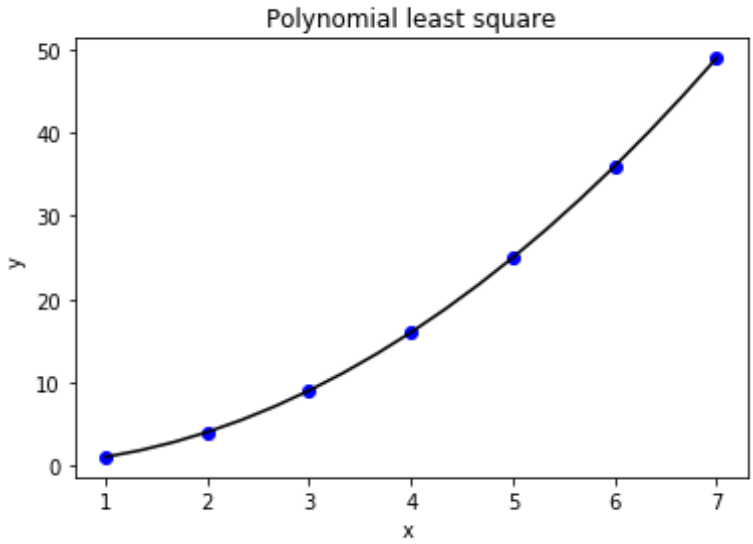
```
   5.6666666667  32.1111111111
   6.0000000000  36.0000000000
   6.3333333333  40.1111111111
   6.6666666667  44.4444444444
   7.0000000000  49.0000000000
   7.3333333333  53.7777777778
   7.6666666667  58.7777777778
   8.0000000000  64.0000000000
```

Program General least square curve fitting (Python version)

```
# -*- coding: utf-8 -*-
"""
Created on Sat Aug 25 09:31:06 2018

@author: Mustafa Turhan Çoban
"""
from math import *
from gauss import *;
import numpy as np;
import matplotlib.pyplot as plt;
from abc import ABC, abstractmethod
class f_xr(ABC):

    @abstractmethod
    def func(self,x,equation_ref):
        pass;

class f1(f_xr):
  def func(this,x,i):
      xx=0;
      if i==0: xx=1.0;
      elif i==1:xx=x;
      elif i==2:xx=x*x;
      elif i==3:xx=x*x*x;
      elif i==4:xx=x*x*x*x;
      elif i==5:xx=x*x*x*x*x;
      elif i==6:xx=x*x*x*x*x*x;
      return xx;

def PolynomialLSQ(f,xi,yi,n):
    #Polynomial least square
    m=len(xi);
    print("l=",m);
    np1=n+1;
    print("np1=",np1);
    A = [[0 for x in range(np1)] for y in range(np1)];
    B =[0 for x in range(np1)];
    X =[0 for x in range(np1)];
    for i in range(np1):
        for j in range(np1):
            for k in range(m):
                pp=pow(xi[k],(i+j));
                A[i][j]= A[i][j]+f.func(xi[k],i)*f.func(xi[k],j);
        for k in range(m):
            B[i]= B[i]+f.func(xi[k],i)*yi[k];
    X=gauss(A,B);
    return X;

def funcPolynomialLSQ(f,e,x):
    # this function calculates the value of
    # least square curve fitting function
    n=len(e);
    ff=0;
    if n!=0.0:
        for i in range(n-1,-1,-1):
            ff=ff+e[i]*f.func(x,i);
    return ff;

def listPolynomialLSQ(f,E,xi,aradegersayisi):
     #aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                 # ara değer sayisi 2 dir
     n=len(xi);
     nn=(n-1)*(aradegersayisi+1)+1;
```

```
    z= [[0 for x in range(nn)] for y in range(2)];
    dx=0;
    k=0;
    for i in range(n-1):
       z[0][k]=xi[i];
       z[1][k]=funcPolynomialLSQ(f,E,z[0][k]);
       k=k+1;
       for j in range(aradegersayisi):
          dx=(xi[i+1]-xi[i])/(aradegersayisi+1.0);
          z[0][k]=z[0][k-1]+dx;
          z[1][k]=funcPolynomialLSQ(f,E,z[0][k]);
          k=k+1;
       z[0][k]=xi[i+1];z[1][k]=funcPolynomialLSQ(f,E,z[0][k]);
    return z;

a=np.loadtxt('a.txt');
print("a",a);
n=len(a);
x =[0 for x in range(n)];
y =[0 for x in range(n)];
for i in range(n):
   x[i]=a[i][0];
   y[i]=a[i][1];
print("x",x);
print("y",y);
f=f1();
E=PolynomialLSQ(f,x,y,2);
print("E=",E);
Z=listPolynomialLSQ(f,E,x,2);
print("Z=",Z[0]);
plt.title('Polynomial least square')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'bo',Z[0],Z[1],'k');
```

```
runfile('D:/okul/SCO1/gen_poly_least_sqr.py', wdir='D:/okul/SCO1')
Reloaded modules: gauss
a [[ 1.  1.]
 [ 2.  4.]
 [ 3.  9.]
 [ 4. 16.]
 [ 5. 25.]
 [ 6. 36.]
 [ 7. 49.]]
x [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0]
y [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0]
l= 7
np1= 3
E= [0.0, -0.0, 1.0]
Z= [1.0, 1.3333333333333333, 1.6666666666666665, 2.0, 2.3333333333333335, 2.666666666666667, 3.0, 3.3333333333333335,
3.666666666666667, 4.0, 4.333333333333333, 4.666666666666666, 5.0, 5.333333333333333, 5.666666666666666, 6.0,
6.333333333333333, 6.666666666666666, 7.0]
```



Program General least square curve fitting , Julia version

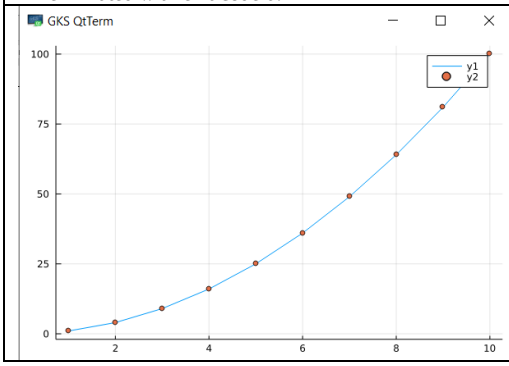
```
 using SpecialFunctions
using Plots
# Linear general least square curve fitting (regression)

function f1(x,i)
     xx=0
     if i==1
       xx=1.0
     elseif i==2
       xx=x
     elseif i==3
       xx=x*x
     elseif i==4
       xx=x*x*x
     elseif i==5
       xx=x*x*x*x
     elseif i==6
       xx=x*x*x*x*x
     elseif i==7
       xx=x*x*x*x*x*x
     end
     return xx
end

function func(f,e,x)
   # this function calculates the value of
   # least square curve fitting function
   n=length(e)
   ff=0.0
   if n!=0.0
      for i=n:-1:1
         ff=ff+e[i]*f(x,i)
      end
   end
   return ff
end

function PolynomialLSQ(f,xi,yi,n)
   #Polynomial least square
   m=length(xi)
   print("l= $m")
   np1=n+1
   print("np1= $np1")
   A =Array{Float64}(undef,np1,np1)
   B =Array{Float64}(undef,np1)
   X =Array{Float64}(undef,np1)
   for i=1:np1
      for j=1:np1
         for k=1:m
```

```
                pp=xi[k]^(i+j)
                A[i,j]= A[i,j]+f(xi[k],i)*f(xi[k],j)
            end # for k=1:m
        end # j=1:np1
        for k=1:m
            B[i]= B[i]+f(xi[k],i)*yi[k]
        end
    end
    print("\nA")
   show(IOContext(stdout, :limit=>false), MIME"text/plain"(), A)
   print("\nB")
   show(IOContext(stdout, :limit=>false), MIME"text/plain"(), B)
   print("\nX")
    X=gauss_pivot(A,B)
    show(IOContext(stdout, :limit=>false), MIME"text/plain"(), X)
    return X
end

function gauss_pivot(A1,b)
              n=size(A1)[1]
              carpan=0.0
              toplam=0.0
              global x=Array{Float64}(undef,n)
              global a=Array{Float64}(undef,n,n)
              for i=1:n
                for j=1:n
                  a[i,j]=A1[i,j]
                end
              end
              for k=1:n
              #pivoting
                 p=k
                 buyuk=abs(a[k,k])
                 for ii=k+1:n
                   dummy=abs(a[ii,k])
                   if dummy > buyuk
                     buyuk=dummyp=ii
                   end #if dummy
                 end #for ii
                 if p!=k
                   for jj=k:n
                     dummy=a[p,jj]
                     a[p,jj]=a[k,jj]
                     a[k,jj]=dummy
                     end #for jj

                   dummy=b[p]
                   b[p]=b[k]
                   b[k]=dummy
                 end #if p!
              # gauss elimination
    for i=k+1:n
                   carpan=a[i,k]/a[k,k]
                   a[i,k]=0.0
                   for j=k+1:n
                     a[i,j]=a[i,j]-carpan*a[k,j]
                   end #for j
                   b[i]=b[i]-carpan*b[k]
              end #for i
              end #k
                            # back substituting
                x[n]=b[n]/a[n,n]
                n1=n-1
                for i=n1:-1:1
                  toplam=0.0
                  for j=i+1:n
                    toplam=toplam+a[i,j]*x[j]
                  end #for j
                  x[i]=(b[i]-toplam)/a[i,i]
                end #for i
              return x
end #function

function convert_to_float(column)
    new_column = map(x -> (
          x = ismissing(x) ? "" : x;  # no method matching tryparse(::Type{Float64}, ::Missing)
```

```
        x = tryparse(Float64, x);   # returns: Float64 or nothing
        isnothing(x) ? missing : x; # options: missing, or "", or 0.0, or nothing
        ), column)  # input
   # returns Array{Float64,1} OR Array{Union{Missing, Float64},1}
   return new_column
end

function readDouble(name)
  open(name) do f
  # read till end of file
  m=0
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    global m1=m
    global n1=n
  end  #while
  close(f)
  end  #open
  global x=Matrix{Float64}(undef,n1,m1)
  open(name) do f
  # read till end of file
  m=0
  global z=Array{Float64}(undef,n1,0)
  while ! eof(f)
    m+=1
    line = readline(f)
    s=split(line," ")
    n=length(s)
    x1=convert_to_float(s)
    z=hcat(z,x1)
    global x=z
  end  #while
  close(f)
  end  #open
  #transpose matrix
  global y=x'
  return y
  end

print("file name : ")
name=readline()
a=readDouble(name)
c=a'
n=size(a)[1]
m=size(a)[2]
x=c[1,:]
y=c[2,:]
x2=x;
y2=y;
println("x = $x2 \ny =  $y2")
e=PolynomialLSQ(f1,x,y,2)
print(" e = $e")
x1=zeros(Float64,n)
y1=zeros(Float64,n)
xmax=maximum(x2)
xmin=minimum(x2)

dx=(xmax-xmin)/(n-1)
print("xmin = $xmin xmax = $xmax n = $n dx = $dx")
for i=1:n
   x1[i]=xmin+(i-1.0)*dx
   xx=x1[i]
   y1[i]=func(f1,e,xx)
end # for i
println("x1 0 $x1 \ny1 =  $y1")
plot(x1,y1)
plot!(x2,y2,seriestype=scatter)
```

```
 ---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" general_least_n.jl
file name : a.txt
x = [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
```

```
y =  [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0, 81.0, 100.0]
l= 10np1= 3
A
3Ã—3
Matrix{Float64}:
  10.0    55.0    385.0
  55.0   385.0   3025.0
 385.0  3025.0  25333.0
B3-element Vector{Float64}:
   385.0
  3025.0
 25333.0
X3-element Vector{Float64}:
 0.0
 0.0
 1.0 e = [0.0, 0.0, 1.0]
xmin = 1.0 xmax = 10.0 n = 10 dx = 1.0x1 0 [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]
y1 =  [1.0, 4.0, 9.0, 16.0, 25.0, 36.0, 49.0, 64.0, 81.0, 100.0]
> Terminated with exit code 0.
```



Program General least square curve fitting , Octave-matlab version

```
## a general function
## for general polynomial curve fitting
## Author: M. Turhan ÇOBAN
## Created: 2021-08-22

function retval = gen_f (x,i)
    xx=0;
    if i==1
      xx=1.0;
    elseif i==2
      xx=x;
    elseif i==3
      xx=x*x;
    elseif i==4
      xx=x*x*x;
    elseif i==5
      xx=x*x*x*x;
    elseif i==6
      xx=x*x*x*x*x;
    elseif i==7
      xx=x*x*x*x*x*x;
    end
    retval=xx;
endfunction
```

```
## General polynomial function
## Author: M. Turhan ÇOBAN
## Created: 2021-08-22

function retval = genpoly_func(f,e,x)
   # this function calculates the value of
   # least square curve fitting function
   n=length(e);
   ff=0.0;
   if n!=0.0
      for i=n:-1:1
         ff=ff+e(i)*f(x,i);
      end
```

```
    end
    retval =ff;
endfunction
```

```
## General polynomial least square curve fitting
## Author: M. Turhan ÇOBAN
## Created: 2021-08-22

function retval = genpoly_least(xi,yi,f,n)
    #Polynomial least square
    switch nargin
       case 3
         n=input("degree of polynomial ? ");
     case 2
         f=@gen_f;
         n=input("degree of polynomial ? ");
    end
    m=length(xi);
    np1=n+1;
    A =zeros(np1,np1);
    B =zeros(np1);
    X =zeros(np1);
    for i=1:np1
       for j=1:np1
          for k=1:m
                pp=xi(k)^(i+j);
                A(i,j)= A(i,j)+f(xi(k),i)*f(xi(k),j);
          end # for k=1:m
       end # j=1:np1
       for k=1:m
          B(i)= B(i)+f(xi(k),i)*yi(k);
       end
    end
    X=gauss1(A,B);
    retval=X;
endfunction
```

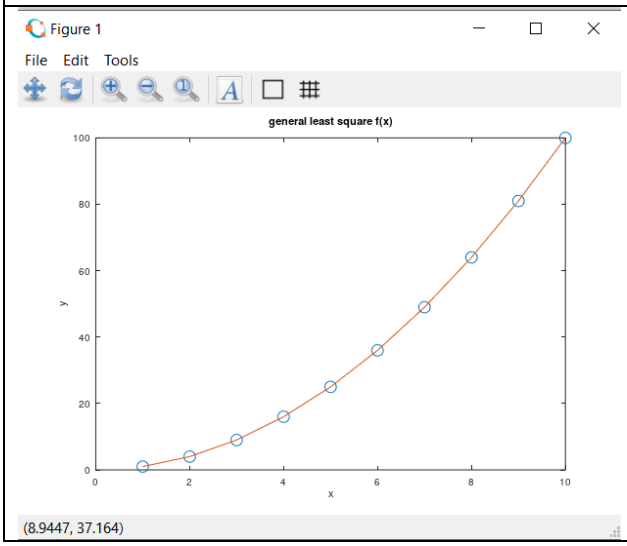
```
>> genpoly_least(x,y)
degree of polynomial ? 3
ans =

  0   0   1   0

>>
```

A plotting program for polynomial curve fitting, Octave-matlab

```
## General Polynomial least square plotting
## Author: M. Turhan ÇOBAN
## Created: 2021-08-22
## e polynomial coefficients(poly_least)
## x,y original data
## x0,xn curve fitted data limits
## n number of urve fitted data

function retval = genpoly_plot (x,y,f,e,x0,xn,n,xlabel1,ylabel1,plabel1)
switch nargin
       case 7
          xlabel1='x';
          ylabel1='y';
          plabel1='f(x)';
       case 4
          xlabel1='x';
          ylabel1='y';
          plabel1='f(x)';
          n=length(x);
          x0=min(x);
          xn=max(x);
       case 3
          n1=input("degree of polynomial?")
          e=genpoly_least(x,y,f,n1);
          xlabel1='x';
          ylabel1='y';
          plabel1='f(x)';
          n=length(x);
          x0=min(x);
```

```
        xn=max(x);
    case 2
        n1=input("degree of polynomial?");
        f=@gen_f;
        e=genpoly_least(x,y,f,n1);
        xlabel1='x';
        ylabel1='y';
        plabel1='general least square f(x)';
        n=length(x);
        x0=min(x);
        xn=max(x);
end
x1=zeros(n);
y1=zeros(n);
xmax=max(x);
xmin=min(x);
dx=(xmax-xmin)/(n-1);
for i=1:n
  x1(i)=xmin+(i-1.0)*dx;
  xx=x1(i);
  y1(i)=poly_func(e,xx);
end # for i
plot(x,y,'o',x1,y1);
title(plabel1);
xlabel(xlabel1);
ylabel(ylabel1);
endfunction
```

```
>> genpoly_plot (x,y)
degree of polynomial?3
>>
```



An excel version of polynomial least square curve fitting is given below. In order to carry out curve fitting, data (x,y) and x^2,x^3,x^4,.. should be defined and LINEST function (DOT in turkish version) should be used with ctrl-shift-enter

=DOT(G2:G9;B2:F9;DOĞRU;DOĞRU)

=LINEST(G2:G9;B2:F9;TRUE;TRUE)

x	x^2	x^3	x^4	x^5	y	f(x)=a0+a1*x+..
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1
2.0000	4.0000	4.0000	16.0000	32.0000	4.0000	4
3.0000	9.0000	9.0000	81.0000	243.0000	9.0000	9
4.0000	16.0000	16.0000	256.0000	1024.0000	16.0000	16
5.0000	25.0000	25.0000	625.0000	3125.0000	25.0000	25
6.0000	36.0000	36.0000	1296.0000	7776.0000	36.0000	36
7.0000	49.0000	49.0000	2401.0000	16807.0000	49.0000	49
8.0000	64.0000	64.0000	4096.0000	32768.0000	64.0000	64

a0	a1	a2	a3	a4	a5	turkish	english	
6.91E-19	-9.81031E-18	1	0	0	-	DOT	LINEST	ctrl-shift-enter

An excel version of polynomial least square curve fitting with our own gauss elimination linear system of equation solution is given below:

n	x	y	x^2	x^3	x^4	x*y	x^2*y
1	1	1.2452	1	1	1	1.2452	1.2452
1	2	4.34534	4	8	16	8.69068	17.38136
1	3	8.9746	9	27	81	26.9238	80.7714
1	4	15.2	16	64	256	60.8	243.2
1	5	22	25	125	625	110	550
1	6	34.8975	36	216	1296	209.385	1256.31
1	7	50.12378	49	343	2401	350.86646	2456.0652
1	8	63.61761	64	512	4096	508.94088	4071.527
1	9	81.74675	81	729	6561	735.72075	6621.4868
1	10	97	100	1000	10000	970	9700
10	55	379.15078	385	3025	25333	2982.5728	24997.987

GAUSS ELIMINATION 3 UNKNOWN

A B

	10	55	385	379.15078
	55	385	3025	2982.5728
	385	3025	25333	24997.987

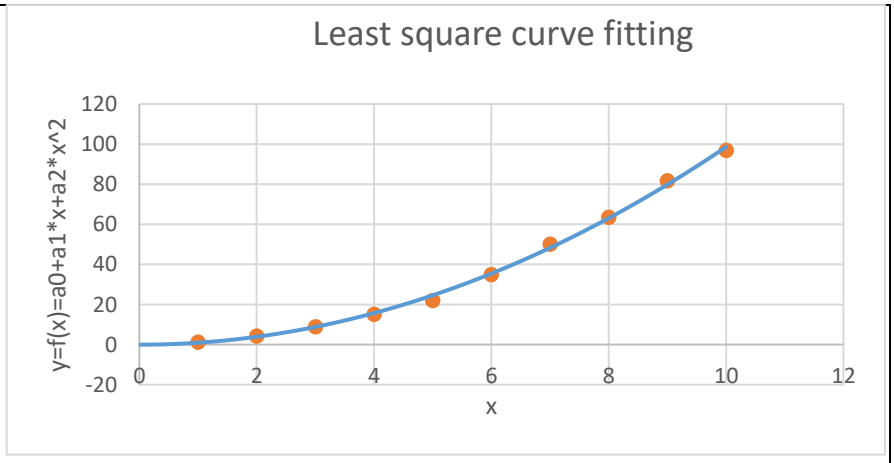
	10	55	385	379.15078
5.5	55	385	3025	2982.5728
38.5	385	3025	25333	24997.987

	10	55	385	379.15078
7.8571429	55	385	3025	2982.5728
	0	3025	25333	24997.987

x0	-0.0411622
x1	-0.0062948
x2	0.9867756

created data

x	y
0	-0.0411622
0.25	0.0189376
0.5	0.2023843
0.75	0.509178
1	0.9393186
1.25	1.4928062
1.5	2.1696408
1.75	2.9698223
2	3.8933507
2.25	4.9402261
2.5	6.1104485
2.75	7.4040178
3	8.820934
3.25	10.361197
3.5	12.024807
3.75	13.811765
4	15.722069



4.25	17.75572	

6.2 LINEAR CURVE FITTING : ORTOGONAL POLYNOMIAL LEAST SQUARE

In the previous subsection , polynomial least square method is investigated. In order to find coefficient of polynomial least square a matrix solution process should be carried out. Furthermore, the equation tends to create ill matrices, so that chance of having an error in the coefficient while solving the matrix is high. Orthogonal polynomials has the following usefull property

$$\begin{Bmatrix} p_i(x)p_j(x) = 0 & if\ \ i \neq j \\ p_i(x)p_j(x) = A & if\ \ i = j \end{Bmatrix} \quad (6.2-1)$$

Now Assume to have a jth order orthogonal polynomial$p_j(x)$. And assume that least square polynomial equation is

$$y_m(x) = \sum_{j=0}^{m} b_j^{(m)} p_j(x) \quad (6.2-2)$$

If x_i, f_i i=1,…,n is given as a data to be curve fit, The minimization function can be defined as

$$H\left(b_0^{(m)},..,b_0^{(m)}\right) = \sum_{i=1}^{n} w(x_i)\left[f(x_i) - y_m(x)\right]^2 = \sum_{i=1}^{n} w(x_i)\left[f(x_i) - \sum_{j=0}^{m} b_j^{(m)} p_j(x)\right]^2 \quad (6.2-3)$$

In order to minimize the function, derivative can be set as equal to zero

$$\frac{\partial H\left(b_0^{(m)},..,b_0^{(m)}\right)}{\partial b_k^{(m)}} = \sum_{i=1}^{n} w(x_i)[f(x_i) - y_m(x)]^2 = \sum_{i=1}^{n} w(x_i)\left[f(x_i) - \sum_{j=0}^{m} b_j^{(m)} p_j(x)\right] p_k(x) = 0 \ \ k = 0,..,m \quad (6.2-3)$$

for a shorter notation new variables can be defined as:

$$d_{jk} \sum_{i=1}^{n} w(x_i) p_j(x_i)\, p_k(x_i) \quad (6.2-4)$$

$$\omega_{jk} = \sum_{i=1}^{n} w(x_i) p_n(x_i) f(x_i) \quad (6.2-5)$$

$$\sum_{j=0}^{m} d_{jk}\left[b_j^{(m)}\right] = \omega_{jk} \quad k = 0,..,m \quad (6.2-6)$$

This equation came to the following matrix form

$$\begin{bmatrix} \sum_{i=1}^{n} w(x_i)p_0(x_i)\,p_0(x_i) & \sum_{i=1}^{n} w(x_i)p_0(x_i)\,p_1(x_i) & \sum_{i=1}^{n} w(x_i)p_0(x_i)\,p_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)p_0(x_i)\,p_n(x_i) \\ \sum_{i=1}^{n} w(x_i)p_1(x_i)\,p_0(x_i) & \sum_{i=1}^{n} w(x_i)p_1(x_i)\,p_1(x_i) & \sum_{i=1}^{n} w(x_i)p_1(x_i)\,p_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)p_1(x_i)\,p_n(x_i) \\ \sum_{i=1}^{n} w(x_i)p_2(x_i)\,p_0(x_i) & \sum_{i=1}^{n} w(x_i)p_2(x_i)\,p_1(x_i) & \sum_{i=1}^{n} w(x_i)p_2(x_i)\,p_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)p_2(x_i)\,p_n(x_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} w(x_i)p_n(x_i)\,p_0(x_i) & \sum_{i=1}^{n} w(x_i)p_n(x_i)\,p_1(x_i) & \sum_{i=1}^{n} w(x_i)p_n(x_i)\,p_2(x_i) & \cdots & \sum_{i=1}^{n} w(x_i)p_n(x_i)\,p_n(x_i) \end{bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdots \\ b_n \end{Bmatrix} =$$

$$\begin{Bmatrix} \sum_{i=1}^{n} w(x_i)p_0(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)p_1(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)p_2(x_i)f(x_i) \\ \cdots \\ \sum_{i=1}^{n} w(x_i)p_n(x_i)f(x_i) \end{Bmatrix} \quad (6.2\text{-}6a)$$

Writing the equation this way does not give any advantage, But considering that $p_k(x_i)$ is an orthogonal function, due to the propertiy of the orthogonal functions. In this case the equation that formed a matrix converted to a simple multiplication equation becomes

$$\begin{bmatrix} \sum_{i=1}^{n} w(x_i)p_0(x_i)\,p_0(x_i) & & & & \\ & \sum_{i=1}^{n} w(x_i)p_1(x_i)\,p_1(x_i) & & & \\ & & \sum_{i=1}^{n} w(x_i)p_2(x_i)\,p_2(x_i) & \cdots & \\ & & & & \sum_{i=1}^{n} w(x_i)p_n(x_i)\,p_n(x_i) \end{bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdots \\ b_n \end{Bmatrix} =$$

$$\begin{Bmatrix} \sum_{i=1}^{n} w(x_i)p_0(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)p_1(x_i)f(x_i) \\ \sum_{i=1}^{n} w(x_i)p_2(x_i)f(x_i) \\ \cdots \\ \sum_{i=1}^{n} w(x_i)p_n(x_i)f(x_i) \end{Bmatrix} \quad (6.2\text{-}7)$$

$$d_{kk}\left[b_j^{(m)}\right] = \omega_{kk} \quad k = 0, \ldots, m \quad (6.2-7a)$$

To solve $b_j^{(m)}$

$$\left[b_j^{(m)}\right] = \frac{\omega_{kk}}{d_{kk}} = \frac{\sum_{i=1}^{n} w(x_i)p_k(x_i)f(x_i)}{\sum_{i=1}^{n} w(x_i)p_k(x_i)\,p_k(x_i)} \quad k = 0, .., m \quad (6.2-8)$$

Several different orthogonal polynomials are available. The orthogonal polynomial that we will utilise in our first program has the following form. This equation is taken from "A First Course in Numerical Analysis, Anthony Ralston,Philip Rabinowitz, Mc Graw Hill ISBN 0-07-051158-6" [4]

$$p_{j+1}(x) = (x - \alpha_{j+1})p_j(x) - \beta_j p_{j-1}(x) \quad j = 0,1, \ldots \quad (6.2-9)$$

$p_0(x) = 1 \quad p_{-1}(x) = 0$

α_{j+1} and β_j are polynomial constants to be calculated.

$$\beta_k = \frac{\sum_{i=1}^{n} w(x_i)x_i p_{k-1}(x_i)\,p_k(x_i)}{\sum_{i=1}^{n} w(x_i)x_i p_{k-1}(x_i)\,p_{k-1}(x_i)} = \frac{\sum_{i=1}^{n} w(x_i)p_k(x_i)\,p_k(x_i)}{\sum_{i=1}^{n} w(x_i)p_{k-1}(x_i)\,p_{k-1}(x_i)} \quad (6.2-10)$$

$$\alpha_k = \frac{\sum_{i=1}^{n} w(x_i)x_i p_k(x_i)\,p_k(x_i)}{\sum_{i=1}^{n} w(x_i)p_{k-1}(x_i)\,p_{k-1}(x_i)} \quad (6.2-11)$$

Least square algorithm can be given as follows:

$$y_m(x) = \sum_{j=0}^{m} b_j^{(m)} p_j(x)$$

$$b_j = \frac{\omega_j}{\gamma_j}$$

$$\omega_j = \sum_{i=1}^{n} w(x_i)p_j(x_i)f(x_i)$$

$$\gamma_j = \sum_{i=1}^{n} w(x_i)p_j(x_i)p_j(x_i)$$

Because Orthogonal polynomial equations are not solving any system of equations, roundoff errors due to matrix solutions will not be existed. But the polynomial equation used has a more complex form compare to polynomial form.

Program Orthogonal polynomial least square, Ralston & Rabinowitz equation, Java version

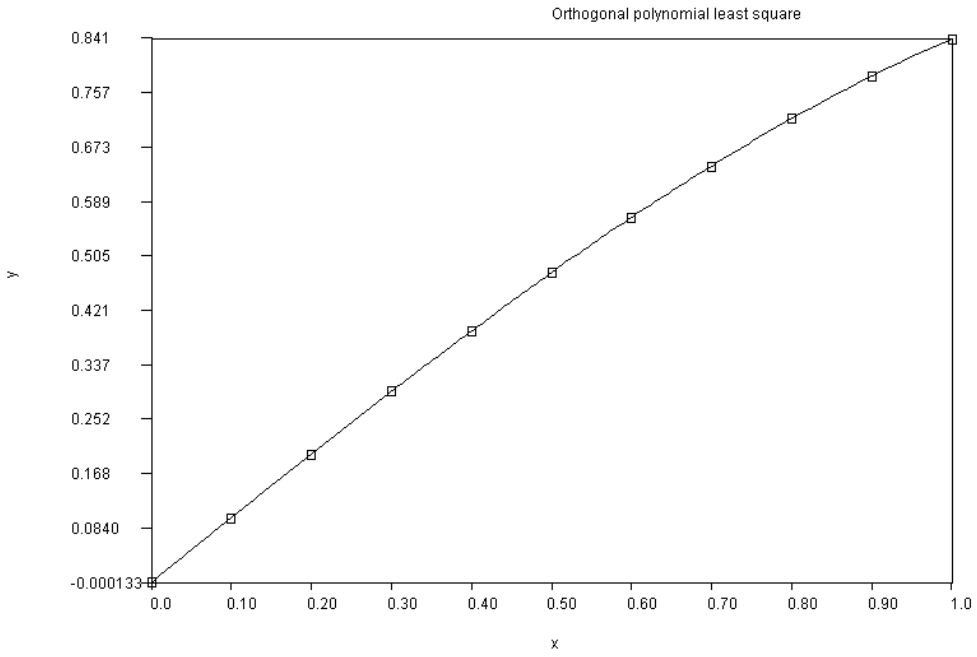
```
import java.io.*;

class NA121
{
public static double[][] OPEKK(double xi[],double fi[],int m)
//ortogonal polinomial least square method
//Referance : A First Course in Numerical Analysis
// Anthony Ralston,Philip Rabinowitz, Mc Graw Hill ISBN 0-07-051158-6
// m degree of polynomial xi fi input data
{
int i,j,k;
int n=xi.length;
int mp2=n+2;
int mp1=n+1;
double p[][]=new double[mp2][n];
```

```
double gamma[]=new double[mp1];
double beta[]=new double[mp1];
double omega[]=new double[mp1];
double alpha[]=new double[mp1];
double b[]=new double[mp1];
double wi[]=new double[n];
double a[][]=new double[3][mp1];
for(i=0;i<n;i++)
  {
   p[1][i]=1.0;
   p[0][i]=0.0;
   wi[i]=1.0;
  }
gamma[0]=0;
for(i=0;i<n;i++)
  {
  gamma[0]+=wi[i];
  }
beta[0]=0.0;
for(j=0;j<m+1;j++)
  {
  omega[j]=0;
  for(i=0;i<n;i++)
    {omega[j]+=wi[i]*fi[i]*p[j+1][i];}
  b[j]=omega[j]/gamma[j];
if( j != m)
    {
    alpha[j+1]=0;
    for(i=0;i<n;i++)
      { alpha[j+1]+=wi[i]*xi[i]*p[j+1][i]*p[j+1][i]/gamma[j];}
    for(i=0;i<n;i++)
      {p[j+2][i]=(xi[i]-alpha[j+1])*p[j+1][i]-beta[j]*p[j][i];}
      gamma[j+1]=0;
    for(i=0;i<n;i++)
      {gamma[j+1]+=wi[i]*p[j+2][i]*p[j+2][i];}
    beta[j+1]=gamma[j+1]/gamma[j];
    }
}//end of j
for(j=0;j<m+1;j++)
  {
  a[0][j]=b[j];
  a[1][j]=alpha[j];
  a[2][j]=beta[j];
  }
return a;
}

public static double funcOPEKK(double a[][],double x)
{
// polinom değerleri hesaplama fonksiyonu
double yy=0;
int k;
int m=a[0].length-1;
int mp2=m+2;
double q[];
q=new double[mp2];
//vector<double> q(m+2,0.0);
for(k=m-1;k>=0;k--)
  {
  q[k]=a[0][k]+(x-a[1][k+1])*q[k+1]-a[2][k+1]*q[k+2];
  yy=q[k];
  }
```

```
return yy;
}

public static double[][]  funcOPEKK(double xi[],double yi[],int polynomialcoefficient,int numberofmidpoints)
 {
            //number of midpoints:  x--o--o--x--o--o--x  x data points midpoints are 2
            int n=xi.length;
            int nn=(n-1)*(numberofmidpoints+1)+1;
            double z[][]=new double[2][nn];
   double E[][]=OPEKK(xi,yi,polynomialcoefficient);
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
     {z[0][k]=xi[i];z[1][k]=funcOPEKK(E,z[0][k]);k++;
     for(int j=0;j<numberofmidpoints;j++)
     {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
     z[0][k]=z[0][k-1]+dx;z[1][k]=funcOPEKK(E,z[0][k]);k++;}
     }
     z[0][k]=xi[i];z[1][k]=funcOPEKK(E,z[0][k]);
   return z;
 }

 public static void main(String args[]) throws IOException
 {
 double x[]={0.0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0};
 double y[]={0.0,0.099833,0.1986693,0.2955202,0.38941834,0.479425,0.5646424,0.6442176,0.717356,0.783326,0.84147};
 double z3[][]=funcOPEKK(x,y,3,8);
 Text.print(Text.T(z3),"Orthogonal polynomial least square ");
 Plot pp=new Plot(x,y);
 pp.setPlabel("Orthogonal polynomial least square ");
 pp.setPlotType(0,20);
 pp.addData(z3[0],z3[1]);
 pp.plot();
 }
}
```

x	y
0	-1.33E-04
0.011111111	0.011022274
0.022222222	0.022172031
0.033333333	0.033314774
0.044444444	0.044449318
0.055555556	0.055574478
0.066666667	0.066689069
0.077777778	0.077791906
0.088888889	0.088881804
0.1	0.099957578
0.111111111	0.111018044
0.122222222	0.122062016
0.133333333	0.13308831
0.144444444	0.14409574
0.155555556	0.155083123
0.166666667	0.166049271
0.177777778	0.176993002
0.188888889	0.18791313
0.2	0.19880847
0.211111111	0.209677837
0.222222222	0.220520046
0.233333333	0.231333912
0.244444444	0.242118251
0.255555556	0.252871877
0.266666667	0.263593606
0.277777778	0.274282253

0.288888889	0.284936632
0.3	0.295555559
0.311111111	0.306137849
0.322222222	0.316682317
0.333333333	0.327187778
0.344444444	0.337653048
0.355555556	0.34807694
0.366666667	0.358458271
0.377777778	0.368795855
0.388888889	0.379088508
0.4	0.389335044
0.411111111	0.399534279
0.422222222	0.409685028
0.433333333	0.419786106
0.444444444	0.429836327
0.455555556	0.439834508
0.466666667	0.449779463
0.477777778	0.459670007
0.488888889	0.469504955
0.5	0.479283123
0.511111111	0.489003325
0.522222222	0.498664376
0.533333333	0.508265092
0.544444444	0.517804288
0.555555556	0.527280779
0.566666667	0.53669338
0.577777778	0.546040905
0.588888889	0.555322171
0.6	0.564535991
0.611111111	0.573681182
0.622222222	0.582756559
0.633333333	0.591760935
0.644444444	0.600693128
0.655555556	0.60955195
0.666666667	0.618336219
0.677777778	0.627044748
0.688888889	0.635676353
0.7	0.644229849
0.711111111	0.652704051
0.722222222	0.661097774
0.733333333	0.669409833
0.744444444	0.677639043
0.755555556	0.68578422
0.766666667	0.693844178
0.777777778	0.701817733
0.788888889	0.709703699
0.8	0.717500892
0.811111111	0.725208127
0.822222222	0.732824219
0.833333333	0.740347982
0.844444444	0.747778233
0.855555556	0.755113785
0.866666667	0.762353455
0.877777778	0.769496058
0.888888889	0.776540407
0.9	0.783485319
0.911111111	0.790329609
0.922222222	0.797072091
0.933333333	0.803711581
0.944444444	0.810246894
0.955555556	0.816676844
0.966666667	0.823000248
0.977777778	0.82921592
0.988888889	0.835322674
1	0.841319327



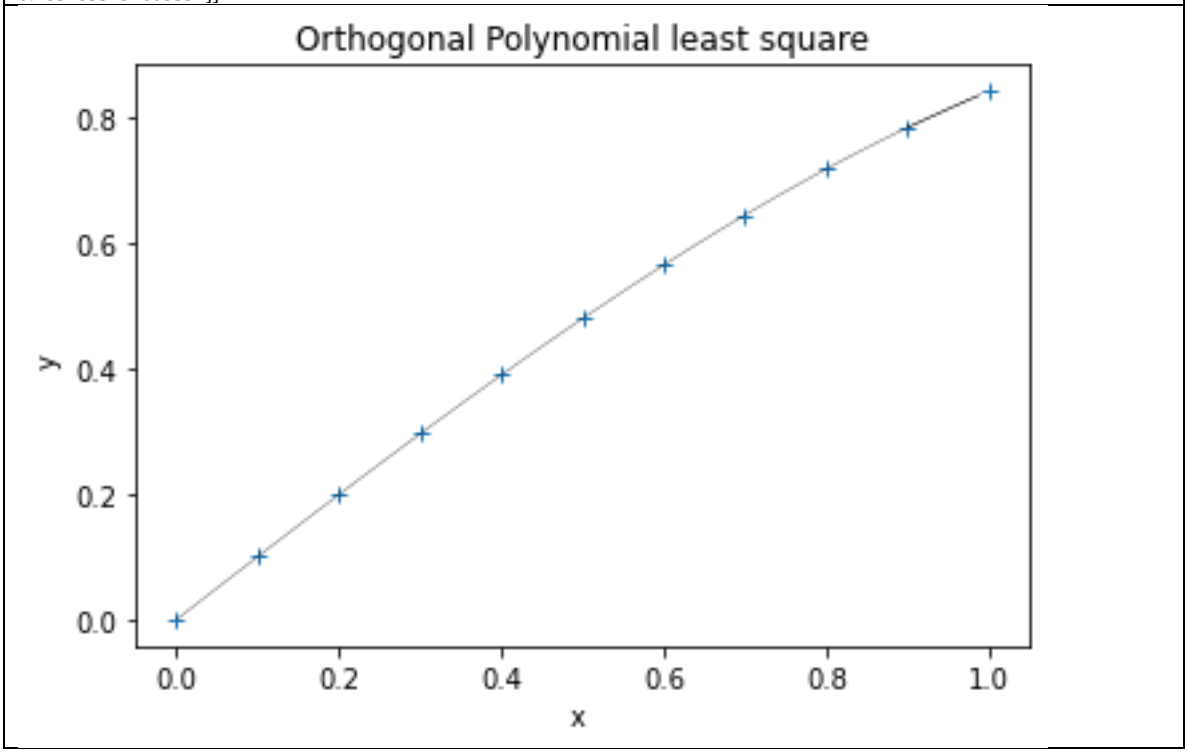
Orthogonal polynomial least square curve fitting Ralston & Rabinowitz formula, python version

```
from math import *
import numpy as np;
import matplotlib.pyplot as plt

def  OPEKK(xi,fi,m):
    #ortogonal polinomial least square method
    #Referance : A First Course in Numerical Analysis
    #Anthony Ralston,Philip Rabinowitz, Mc Graw Hill ISBN 0-07-051158-6
    #m degree of polynomial xi fi input data
    # int i,j,k;
    n=len(xi)
    mp2=n+2
    mp1=n+1
    p=[[0.0 for j in range(n)] for i in range(mp2)]
    gamma=[0.0 for j in range(mp1)]
    beta=[0.0 for j in range(mp1)]
    omega=[0.0 for j in range(mp1)]
    alpha=[0.0 for j in range(mp1)]
    b=[0.0 for j in range(mp1)]
    wi=[0.0 for j in range(n)]
    a=[[0.0 for j in range(mp1)] for i in range(3)]
    for i in range(n):
        p[1][i]=1.0
        p[0][i]=0.0
        wi[i]=1.0
    gamma[0]=0
    for i in range(n):
        gamma[0]=gamma[0]+wi[i]
    beta[0]=0.0
    for j in range(m+1):
        omega[j]=0;
        for i in range(n):
            omega[j]=omega[j]+wi[i]*fi[i]*p[j+1][i]
        b[j]=omega[j]/gamma[j]
        if j != m:
            alpha[j+1]=0
            for i in range(n):
                alpha[j+1]+=wi[i]*xi[i]*p[j+1][i]*p[j+1][i]/gamma[j]
            for i in range(n):
                p[j+2][i]=(xi[i]-alpha[j+1])*p[j+1][i]-beta[j]*p[j][i]
            gamma[j+1]=0
            for i in range(n):
                gamma[j+1]+=wi[i]*p[j+2][i]*p[j+2][i]
            beta[j+1]=gamma[j+1]/gamma[j]
```

```
    for j in range(m+1):
      a[0][j]=b[j]
      a[1][j]=alpha[j]
      a[2][j]=beta[j]
    return a


def func(a,x):
    #polinom değerleri hesaplama fonksiyonu
    yy=0
    k=0
    m=len(a[0])-1
    mp2=m+2;
    q=[0.0 for j in range(mp2)]
    for k in range(m-1,-1,-1):
       q[k]=a[0][k]+(x-a[1][k+1])*q[k+1]-a[2][k+1]*q[k+2]
       yy=q[k]
    return yy;

def funcOPEKK(xi,yi,polynomialcoefficient,numberofmidpoints):
    #number of midpoints:  x--o--o--x--o--o--x  x data points midpoints are 2
    n=len(xi)
    nn=(n-1)*(numberofmidpoints+1)+1
    z=[[0.0 for j in range(nn)] for i in range(2)]
    E=OPEKK(xi,yi,polynomialcoefficient);
    dx=0.0
    k=0
    for i in range(n-1):
       z[0][k]=xi[i]
       z[1][k]=func(E,z[0][k])
       k=k+1
       for j in range(numberofmidpoints):
          dx=(xi[i+1]-xi[i])/(float(numberofmidpoints)+1.0)
          z[0][k]=z[0][k-1]+dx
          z[1][k]=func(E,z[0][k])
          k=k+1
       z[0][k]=xi[i]
       z[1][k]=func(E,z[0][k])
    return z

x=[0.0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0]
y=[0.0,0.099833,0.1986693,0.2955202,0.38941834,0.479425,0.5646424,0.6442176,0.717356,0.783326,0.84147]
z3=funcOPEKK(x,y,3,8);
print("z3",z3)
plt.title('Orthogonal Polynomial least square')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'+',z3[0],z3[1],'k',linewidth=0.3,)
```

```
runfile('E:/okul/SCO1/OPEKK.py', wdir='E:/okul/SCO1')
z3 [[0.0, 0.011111111111111112, 0.022222222222222223, 0.03333333333333333, 0.044444444444444446, 0.05555555555555556,
0.06666666666666667, 0.07777777777777778, 0.08888888888888889, 0.1, 0.11111111111111112, 0.12222222222222223,
0.13333333333333333, 0.14444444444444443, 0.15555555555555553, 0.16666666666666663, 0.17777777777777773,
0.18888888888888883, 0.2, 0.2111111111111111, 0.2222222222222222, 0.2333333333333333, 0.2444444444444444,
0.25555555555555554, 0.26666666666666666, 0.2777777777777778, 0.2888888888888889, 0.3, 0.3111111111111111,
0.32222222222222224, 0.33333333333333337, 0.3444444444444445, 0.3555555555555556, 0.36666666666666675,
0.3777777777777779, 0.388888888888889, 0.4, 0.41111111111111115, 0.4222222222222223, 0.4333333333333334,
0.44444444444444453, 0.45555555555555566, 0.4666666666666668, 0.4777777777777779, 0.48888888888888904, 0.5,
0.5111111111111111, 0.5222222222222221, 0.5333333333333332, 0.5444444444444443, 0.5555555555555554,
0.5666666666666664, 0.5777777777777775, 0.5888888888888886, 0.6, 0.611111111111111, 0.6222222222222221,
0.6333333333333332, 0.6444444444444443, 0.6555555555555553, 0.6666666666666664, 0.6777777777777775,
0.6888888888888886, 0.7, 0.711111111111111, 0.7222222222222221, 0.7333333333333332, 0.7444444444444442,
0.7555555555555553, 0.7666666666666664, 0.7777777777777775, 0.7888888888888885, 0.8, 0.8111111111111111,
0.8222222222222222, 0.8333333333333333, 0.8444444444444443, 0.8555555555555554, 0.8666666666666665,
0.8777777777777775, 0.8888888888888886, 0.9, 0.9111111111111111, 0.9222222222222222, 0.9333333333333332,
0.9444444444444443, 0.9555555555555554, 0.9666666666666665, 0.9777777777777775, 0.9888888888888886, 0.9], [-
0.00013331314685310902, 0.011022273501971321, 0.02217203082505974, 0.03331477390831393, 0.04444931783763565,
0.05557447769892668, 0.06668906857808862, 0.0777919055610234, 0.0888818037336327, 0.09995757818181823,
0.1110180439914818, 0.1220620162485252, 0.13308831003885008, 0.14409574044835827, 0.15508312256295145,
0.16604927146853146, 0.17699300225099998, 0.1879131299962589, 0.19880846979020986, 0.20967783671875456,
0.22052004586779472, 0.23133391232323228, 0.24211825117096894, 0.25287187749690637, 0.2635936063869464,
0.2742822529269907, 0.2849366322029412, 0.2955555593006993, 0.30613784930616705, 0.31668231730524615,
0.32718777838383845, 0.3376530476278455, 0.3480769401231691, 0.35845827095571103, 0.3687958552113731,
0.379088507976057, 0.38933504433566435, 0.3995342793760972, 0.40968502818325714, 0.4197861058430459,
0.42983632744136524, 0.439834508064117, 0.4497794627972029, 0.4596700067265246, 0.46950495493798394,
0.4792831225174825, 0.48900332455092227, 0.49866437612420494, 0.5082650923232322, 0.5178042882339059,
```

```
0.5272807789421277, 0.5366933795337994, 0.5460409050948227, 0.5553221707110993, 0.5645359914685315,
0.5736811824530201, 0.5827565587504675, 0.5917609354467753, 0.6006931276278452, 0.6095519503795791,
0.6183362187878786, 0.6270447479386455, 0.6356763529177815, 0.6442298488111887, 0.6527040507047684,
0.6610977736844225, 0.6694098328360527, 0.6776390432455609, 0.6857842199988488, 0.693844178181818,
0.7018177328803704, 0.7097036991804079, 0.7175008921678322, 0.7252081269285447, 0.7328242185484475,
0.740347982113442, 0.7477782327094304, 0.7551137854223142, 0.7623534553379951, 0.7694960575423752,
0.7765404071213559, 0.7834853191608392, 0.7903296087467265, 0.7970720909649198, 0.8037115809013209,
0.8102468936418314, 0.8166768442723531, 0.8230002478787879, 0.8292159195470372, 0.8353226743630031,
0.7834853191608392]]
```



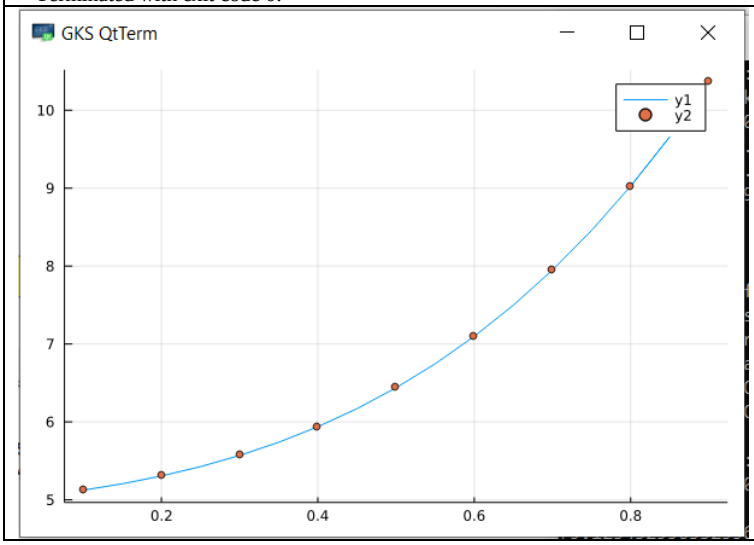
Orthogonal polynomial least square curve fitting Ralston & Rabinowitz formula, Julia version

```
 using SpecialFunctions
using Plots

function  OPEKK(xi,fi,m)
   #ortogonal polinomial least square method
   #Referance : A First Course in Numerical Analysis
   #Anthony Ralston,Philip Rabinowitz, Mc Graw Hill ISBN 0-07-051158-6
   #m degree of polynomial xi fi input data
   # int i,j,k;
   n=length(xi)
   mp2=m+2
   mp1=m+1
   p=zeros(Float64,mp2,n)
   gamma=zeros(Float64,mp1)
   beta=zeros(Float64,mp1)
   omega=zeros(Float64,mp1)
   alpha=zeros(Float64,mp1)
   ym=zeros(Float64,n)
   b=zeros(Float64,mp1)
   wi=ones(Float64,n)
   a=zeros(Float64,3,mp1)
   for i=1:n
      p[2,i]=1.0
      p[1,i]=0.0
      wi[i]=1.0
   end #for
   gamma[1]=0.0
   for i=1:n
      gamma[1]=gamma[1]+wi[i]
   end #for
   beta[1]=0.0
```

```
    for j=1:mp1
        omega[j]=0
        for i=1:n
            omega[j]=omega[j]+wi[i]*fi[i]*p[j+1,i]
        end # for i=1:n
        b[j]=omega[j]/gamma[j]
        if j!= (m+1)
            alpha[j+1]=0.0
            for i=1:n
                alpha[j+1]+=wi[i]*xi[i]*p[j+1,i]*p[j+1,i]/gamma[j]
            end # for i=1:n
            for i=1:n
                p[j+2,i]=(xi[i]-alpha[j+1])*p[j+1,i]-beta[j]*p[j,i]
            end # for i=1:n
            gamma[j+1]=0.0
            for i=1:n
                gamma[j+1]+=wi[i]*p[j+2,i]*p[j+2,i]
            end # for i=1:n
            beta[j+1]=gamma[j+1]/gamma[j]
        end #if j != m
     end #for j=1:(m+1)
    for j=1:(m+1)
       a[1,j]=b[j]
       a[2,j]=alpha[j]
       a[3,j]=beta[j]
    end #j=1:(m+1)
    return a
end # function

function func(a,x)
    #polynomial value calulation
    yy=0
    k=0
    a1=a[1,:]
    a2=a[2,:]
    a3=a[3,:]
    m=length(a1)
    mp2=m+1
    q=zeros(Float64,mp2)
    for k=m-1:-1:1
        q[k]=a1[k]+(x-a2[k+1])*q[k+1]-a3[k+1]*q[k+2]
        yy=q[k]
    end
    return yy
end # function
function funcOPEKK(xi,yi,polynomialcoefficient,numberofmidpoints)
    # number of midpoints:  x--o--o--x--o--o--x  x data points midpoints are 2
    n=length(xi)
    nn=(n-1)*(numberofmidpoints+1)+1
    z=zeros(Float64,2,nn)
    E=OPEKK(xi,yi,polynomialcoefficient)
    dx=0.0
    k=1
    for i=1:(n-1)
        z[1,k]=xi[i]
        z[2,k]=func(E,z[1,k])
        k=k+1
        for j=1:numberofmidpoints
            dx=(xi[i+1]-xi[i])/(float(numberofmidpoints)+1.0)
            z[1,k]=z[1,k-1]+dx
            z[2,k]=func(E,z[1,k])
            k=k+1
        end # for j=1:numberofmidpoints
        z[1,k]=xi[i]
        z[2,k]=func(E,z[1,k])
    end
    return z
end # function

using Plots
x=[0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9]
y=[5.1234,5.3057,5.5687,5.9378,6.4370,7.0978,7.9493,9.0253,10.3627]
z3=funcOPEKK(x,y,5,1);
x1=z3[1,:]
y1=z3[2,:]
```

```
println(x1)
println(y1)
plot(x1,y1)
plot!(x,y,seriestype=scatter)
```

```
 ---------- Capture Output ----------
> "D:\co\Julia-1.6.2\bin\julia.exe" OPEKK.jl
[0.1, 0.15000000000000002, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.6499999999999999, 0.7, 0.75, 0.8, 0.8500000000000001,
0.8]
[5.123432556332556, 5.205753532925408, 5.305574397824399, 5.42566194395882, 5.5689327894327905, 5.7384533775252535,
5.937439976689978, 6.169258680555558, 6.43742540792541, 6.745605902777779, 7.0976157342657356, 7.497420296717173,
7.949134809634811, 8.457024317696193, 9.025503690753693, 9.659137623834502, 9.025503690753693]
> Terminated with exit code 0.
```



A second example to orthogonal polynomial curve fitting will be **Chebyshev polynomial** curve fitting. Chebyshev polynomials defined at (-1, 1) region.General definition of Chebyshev polynomials

$T_n(x) = \cos(n.cos^{-1}(x)$ (6.2-12)

If this equation is written in open form

$T_0(x)=1$

$T_1(x)=x$

$T_2(x)=2x^2-x$ (6.2-12a)

…..

$T_{n+1}(x)=2x\ T_n(x)- T_{n-1}(x)$

It is clear that it is a serial equation. The weight function for the Chebchev polynomials

$w(x) = \frac{1}{\sqrt{1-x^2}}$ (6.2-13)

The root of Chebchev polynomials

$t_k = \cos\left(\pi \frac{2k+1}{2n+2}\right)$ k=0,…,n (6.2-13)

n in this equation indicates the degree of polynomial. In order to curve fit data y=f(x) using this set as an orthogonal polynomial data should be given at the root points of chebchev polynomials.

Furthermore due to regional limit of the data, to apply this equation to real data a conversion process is required. If the real data is in region (a,b) The convertion equations will be

$x_k = \left(\frac{b-a}{2}\right) t_k + \left(\frac{a+b}{2}\right)$ (6.2-14)

$t_k = \left(\frac{x_k-a}{b-1}\right) - 1$ (6.2-15)

If x_k and corresponding y_k ,is given

Coefficient of polynomial $P_n(x) = \sum_{k=0}^{n} C_k T_k(x)$ can be calculated by using least sqaure method. Due to orthogonality of the polynomials, matrix solution is not required, coefficients is found directly.

$C_0 = \frac{1}{n+1}\sum_{k=0}^{n} y_k T_0\,(t_k)$

$C_j = \frac{2}{n+1}\sum_{k=0}^{n} y_k T_j\,(t_k)\ \ j = 1,2,\ldots,n$ (6.2-16)

As an example problem f(x)=e^x function in the limit 0 to 2 will be investigated. If Chebchev polynomials are to be used in curve fitting, experiments should be design so that the data should be taken exactly in the roots of the Chebchev polynomials.

x_k	y_k
0.010178558	1.010231
0.090368005	1.094577
0.244250426	1.276664
0.459359183	1.583059
0.718267443	2.050877
1	2.718282
1.281732557	3.602877
1.540640817	4.66758
1.755749574	5.787784
1.909631995	6.750604
1.989821442	7.314228

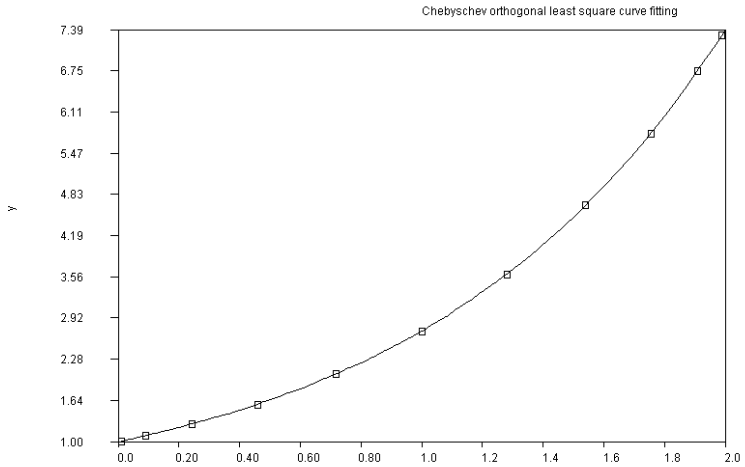
Program Orthogonal polynomial least square, Chebychev equation, Java version

```
import java.io.*;
//Chevychev orthogobnal polynomial least square

class NA122
{
        public static double Ti(double x,int equation_ref)
        {//Chebysev function
        double T[]=new double[equation_ref+1];
        T[0]=1.0;
        if(equation_ref>=1) T[1]=x;
        for(int i=2;i<=equation_ref;i++) {T[i]=2.0*x*T[i-1]-T[i-2];
   }
        return T[equation_ref];
        }
public static double[] xi(double a,double b,int N)
{
  int nn=N+1;
  double x[]=new double[nn];
  double t[]=new double[nn];
  for(int k=0;k<nn;k++)
    {t[k]=Math.cos((2.0*N+1-2.0*k)*Math.PI/(2.0*N+2.0));
     x[k]=(b-a)/2.0*t[k]+(a+b)/2.0;
    }
  return x;
}

public static double[] Chebyshev(double a,double b,double y[],int N)
{   //region a b
    int nn=N+1;
    System.out.println("N="+N);
    int k,j;
    double d=Math.PI/(2.0*N+2);
    double x[]=new double[nn];
    double t[]=new double[nn];
     // -1<=x<=1 region convertion
    for(k=0;k<nn;k++)
     {t[k]=Math.cos((2.0*N+1-2.0*k)*Math.PI/(2.0*N+2.0));
      x[k]=(b-a)/2.0*t[k]+(a+b)/2.0;
     }
     double C[]=new double[nn];
            double z=0;
```

```
    for(k=0;k<nn;k++)
       { z+=y[k];}
    C[0]=z/(N+1);
    for(j=1;j<=N;j++)
    {z=0;
     for(k=0;k<=N;k++)
     { z+=y[k]*Ti(t[k],j);}
    C[j]=2.0*z/(N+1);
    }
    return C;
}

public static double funcChebyshev(double C[],double a,double b,double x)
{
   double t=2.0*(x-a)/(b-a)-1;
   int n=C.length;
   double ff=0;
   if(n!=0.0)
   {  for(int i=0;i<n;i++)
      {ff+=C[i]*Ti(t,i);}
   }
   return ff;
}

public static double[][]  funcChebyshev(double a,double b,double y[],int polinomkatsayisi,int verisayisi)
 {
  int n=verisayisi+1;
  double z[][]=new double[2][n];
  double C[]=Chebyshev(a,b,y,polinomkatsayisi);
  double dx=(b-a)/(double)verisayisi;
  z[0][0]=a;
  for(int i=0;i<=verisayisi;i++)
     {if(i==0) z[0][i]=a;
      else z[0][i]=z[0][i-1]+dx;
                  z[1][i]=funcChebyshev(C,a,b,z[0][i]);         }
    return z;
 }
  public static void main(String args[]) throws IOException
  {
  double a=0;
  double b=2.0;
  double y[]={1.010230536,1.094577019,1.276664,1.583059208,2.05087687,2.718281828,
            3.60287651,4.66758038,5.787784491,6.750604088,7.314227631};
  double x[]= xi(a,b,10);
  double z[][]=funcChebyshev(a,b,y,10,50);
  Text.print(Text.T(z),"Chebyschev orthogonal least square curve fitting");
  Plot pp=new Plot(x,y);
  pp.setPlabel("Chebyschev orthogonal least square curve fitting");
  pp.setPlotType(0,20);
  pp.addData(z[0],z[1]);
  pp.plot();
   }
}
```

x	y
0	1
0.04	1.040810774
0.08	1.083287068
0.12	1.127496852
0.16	1.173510871
0.2	1.221402758
0.24	1.27124915

0.28	1.323129812
0.32	1.377127764
0.36	1.433329415
0.4	1.491824698
0.44	1.552707219
0.48	1.616074402
0.52	1.68202765
0.56	1.750672501
0.6	1.822118801
0.64	1.896480879
0.68	1.973877732
0.72	2.054433211
0.76	2.13827622
0.8	2.225540928
0.84	2.316366976
0.88	2.410899706
0.92	2.509290389
0.96	2.611696473
1	2.718281828
1.04	2.829217014
1.08	2.944679551
1.12	3.064854203
1.16	3.189933276
1.2	3.320116923
1.24	3.455613465
1.28	3.596639726
1.32	3.743421378
1.36	3.896193302
1.4	4.055199967
1.44	4.220695817
1.48	4.392945681
1.52	4.572225195
1.56	4.758821245
1.6	4.953032424
1.64	5.155169512
1.68	5.365555971
1.72	5.584528464
1.76	5.812437394
1.8	6.049647464
1.84	6.296538261
1.88	6.553504862
1.92	6.82095847
1.96	7.099327065
2	7.389056099



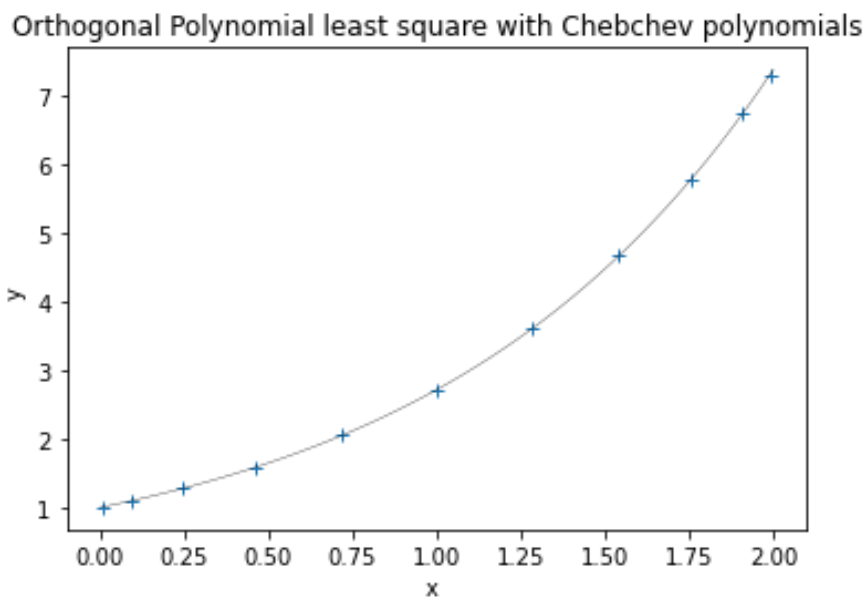
Orthogonal polynomial least square, Chebychev polynomials, Python version

```
#Checychev Orthogonal Polynomial Curve Fitting  chebchev_OPEKK.py
from math import *
```

```
import numpy as np;
import matplotlib.pyplot as plt

def Ti(x,equation_ref):
    #Chebysev function
    T=[0.0 for j in range(equation_ref+1)]
    T[0]=1.0
    if equation_ref>=1:
        T[1]=x
    for i in range(2,equation_ref+1):
        T[i]=2.0*x*T[i-1]-T[i-2]
    return T[equation_ref]

def xi(a,b,N):
    nn=N+1;
    x=[0.0 for j in range(nn)]
    t=[0.0 for j in range(nn)]
    for k in range(nn):
        t[k]=cos((2.0*N+1-2.0*k)*pi/(2.0*N+2.0))
        x[k]=(b-a)/2.0*t[k]+(a+b)/2.0
    return x

def Chebyshev(a,b,y,N):
    #region a b
    nn=N+1;
    #int k,j;
    d=pi/(2.0*N+2)
    x=[0.0 for j in range(nn)]
    t=[0.0 for j in range(nn)]
    # -1<=x<=1 region convertion
    for k in range(nn):
        t[k]=cos((2.0*N+1-2.0*k)*pi/(2.0*N+2.0))
        x[k]=(b-a)/2.0*t[k]+(a+b)/2.0
    C=[0.0 for j in range(nn)]
    z=0.0
    for k in range(nn):
        z+=y[k]
    C[0]=z/(N+1)
    for j in range(1,N+1):
        z=0
        for k in range(N+1):
            z=z+y[k]*Ti(t[k],j)
        C[j]=2.0*z/(N+1)
    return C

def func(C,a,b,x):
    t=2.0*(x-a)/(b-a)-1;
    n=len(C)
    ff=0
    if n!=0.0:
        for i in range(n):
            ff=ff+C[i]*Ti(t,i)
    return ff

def funcChebyshev(a,b,y,polinomkatsayisi,verisayisi):
    n=verisayisi+1
    z=[[0.0 for j in range(n)] for i in range(2)]
    C=Chebyshev(a,b,y,polinomkatsayisi)
    dx=(b-a)/float(verisayisi)
    z[0][0]=a;
    for i in range(verisayisi+1):
        if i==0:
            z[0][i]=a
        else:
            z[0][i]=z[0][i-1]+dx
        z[1][i]=func(C,a,b,z[0][i])
    return z

a=0
b=2.0
y=[1.010230536,1.094577019,1.276664,1.583059208,2.05087687,2.718281828,
        3.60287651,4.66758038,5.787784491,6.750604088,7.314227631]
#x values are roots of chebchev polynomials
x= xi(a,b,10)
z=funcChebyshev(a,b,y,10,50)
print("Chebyschev orthogonal least square curve fitting\n",z)
```

```
plt.title('Orthogonal Polynomial least square with Chebchev polynomials')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'+',z[0],z[1],'k',linewidth=0.3,)
```

```
runfile('E:/okul/SCO1/chebychev_OPEKK.py', wdir='E:/okul/SCO1')
Chebyschev orthogonal least square curve fitting
 [[0, 0.04, 0.08, 0.12, 0.16, 0.2, 0.24000000000000002, 0.28, 0.32, 0.36, 0.39999999999999997, 0.43999999999999995,
0.4799999999999999, 0.5199999999999999, 0.5599999999999999, 0.6, 0.64, 0.68, 0.7200000000000001, 0.7600000000000001,
0.8000000000000002, 0.8400000000000002, 0.8800000000000002, 0.9200000000000003, 0.9600000000000003,
1.0000000000000002, 1.0400000000000003, 1.0800000000000003, 1.1200000000000003, 1.1600000000000004,
1.2000000000000004, 1.2400000000000004, 1.2800000000000005, 1.3200000000000005, 1.3600000000000005,
1.4000000000000006, 1.4400000000000006, 1.4800000000000006, 1.5200000000000007, 1.5600000000000007,
1.6000000000000008, 1.6400000000000008, 1.6800000000000008, 1.7200000000000009, 1.760000000000001,
1.800000000000001, 1.840000000000001, 1.880000000000001, 1.920000000000001, 1.960000000000001, 2.000000000000001],
[1.0000000003363947, 1.0408107741052222, 1.0832870676174815, 1.1274968516551822, 1.1735108711582891,
1.2214027583411369, 1.2712491504601175, 1.323129812413029, 1.3771277643597053, 1.4333294145629911,
1.4918246976585994, 1.5527072185721034, 1.6160744023111264, 1.68202764987093, 1.7506725005018824, 1.822118800597927,
1.8964808794760928, 1.973877732328336, 2.054433210638661, 2.138276220370504, 2.2255409282418266, 2.316366976418341,
2.4108997059686983, 2.509290389439452, 2.6116964729221515, 2.718281828000004, 2.8292170139773067,
2.9446795508112373, 3.064854203182635, 3.1899332761602364, 3.3201169229313003, 3.455613465090903, 3.596639726002278,
3.7434213777615013, 3.8961933023216746, 4.055199967354418, 4.2206958174501725, 4.392945681283381,
4.572225195394 2035, 4.758821245265077, 4.9530324243980175, 5.155169512127402, 5.365555970932761, 5.584528464047143,
5.812437394188804, 6.049647464277367, 6.296538261030195, 6.553504862370688, 6.820958469617304, 7.099327065460759,
7.389056098776642]]
```



LINEAR CURVE FITTING : LEAST SQUARE METHOD WITH MULTI VARIABLES

Curve fitting of a multivariable function basically the same of general curve fitting of the single variable function. The only basic difference is definition of the function to be used. Assume that

$$f(x_0, x_1, .., x_n) = \sum_{j=0}^{m} a_j^{(m)}\, \phi_j(x_0, x_1, .., x_n) \qquad (6.3-1)$$

$$f(x_0, x_1, .., x_n) = a_0\phi_0(x_0, x_1, .., x_n) + a_0\phi_1(x_0, x_1, .., x_n) + ..$$

$$\phi_0(x_0, x_1) = \sin(x_0) * \cos(x_1)$$

$$\phi_1(x_0, x_1) = x_0 * x_0 + x_1 * x_1$$

j th degree function is given. It is desired to fit x_0, x_1, x_2,…, x_n ,f_i, i=0...n data set into this function. The best fitted $a_j^{(m)}$ values to be found. For this purpose Minimum value of

$$H(x_0, x_1, .., x_n) = \sum_{i=1}^{n} w(x_0, x_1, .., x_n)\left[f_i - \sum_{j=0}^{m} a_j^{(m)}\, \phi_j(x_0, x_1, .., x_n)\right]^2 \qquad (6.3-2)$$

function should be found. $w(x_0, x_{1,}.., x_n)$ is called weight function and it should satisfy the condition $w(x_0, x_{1,}.., x_n) \geq 0 \quad i = 1, \dots, n.$ The minimum of the function is the root of the derivative of the function.

$$\frac{\partial H(x_0, x_{1,}.., x_n)}{\partial a_k^{(m)}} = 2\sum_{i=1}^{n} w(x_0, x_{1,}.., x_n)\left[f_i - \sum_{j=0}^{m} a_j^{(m)}\,\phi_j(x_0, x_{1,}.., x_n)\right]\phi_k(x_0, x_{1,}.., x_n) = 0 \quad k = 0,..,m \quad (6.3-2)$$

If unit value is taken as weight function, equation becomes:

$$2\sum_{i=1}^{n}\left[f_i - \sum_{j=0}^{m} a_j^{(m)}\,\phi_j(x_0, x_{1,}.., x_n)\right]\phi_k(x_0, x_{1,}.., x_n) = 0 \qquad k = 0,..,m \quad (6.3-2a)$$

As an example a surface polinomial can be defined as
$f(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2 + a_{10}x^4 + a_{11}y^4 + a_{12}x^2y^2 + a_{13}x^3y + a_{14}xy^2$
This polynomial is defined in the example problem

Problem Multidimensional general least square curve fitting , java version

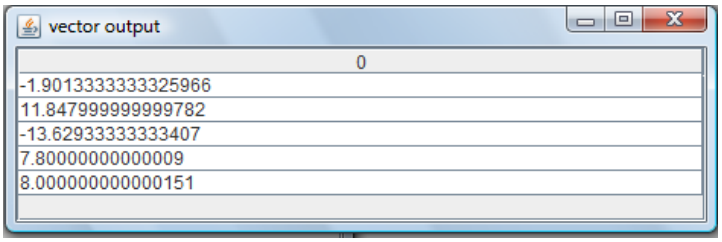
```
import java.io.*;
import java.util.*;
import javax.swing.*;
import java.awt.event.*;

// surface curve fitting
//f(x,y)=a0+a1*x+a2*y+a3*x^2+a4*y^2+a5*xy+a6*x^3+a7*y^3+a8*x^2y
//      +a9*xy^2+a10*x^4+a11*y^4+a12*x^2y^2+a13x^3y+a14*xy^3
abstract class f_xir
{
  abstract double func(double x[],int equation_ref);
}

class fa extends f_xir
{
            double func(double x[],int i)
            {//a general surface curve fitting model
   double xx=0.0;
   //density function
   if(i==0)      {xx=1.0;          }
   else if (i==1) {xx=x[0];         }
   else if (i==2) {xx=x[1];         }
   else if (i==3) {xx=x[0]*x[1];    }
   else if (i==4) {xx=x[1]*x[1];    }
   else if (i==5) {xx=x[0]*x[1];    }
   else if (i==6) {xx=x[0]*x[0]*x[0];}
   else if (i==7) {xx=x[1]*x[1]*x[1];}
   else if (i==8) {xx=x[0]*x[0]*x[1];}
   else if (i==9) {xx=x[0]*x[1]*x[1];}
   else if (i==10){xx=x[0]*x[0]*x[0]*x[0];}
   else if (i==11){xx=x[1]*x[1]*x[1]*x[1];}
   else if (i==12){xx=x[0]*x[0]*x[1]*x[1];}
   else if (i==13){xx=x[0]*x[0]*x[0]*x[1];}
   else if (i==14){xx=x[0]*x[1]*x[1]*x[1];}
   return xx;
            }
}
class NA46
{
//  General least square curve fitting with multivariable
            public static double[][] Transpose(double [][] left)
   { //transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
```

```
   int i,j;
   int n=left.length;
   int m=left[0].length;
   double b[][];
   b=new double[m][n];
   for(i=0;i<n;i++)
     {for(j=0;j<m;j++)  {b[j][i]=left[i][j];} }
   return b;
   }
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
              //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
              }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] GeneralLeastSquare(double c[][],int n)
{
int n1=c.length;
  int n2=c[0].length-1;
```

```
  System.out.println("n1="+n1+"n2="+n2);
  double xi[][]=new double[n1][n2];
  double yi[]=new double[n1];
  for(int i=0;i<n1;i++){for(int j=0;j<n2;j++){xi[i][j]=c[i][j];};yi[i]=c[i][n2];}
  return GeneralLeastSquare(xi,yi,n);
}

public static double[] GeneralLeastSquare(double xi[][],double yi[],int n)
{ // n dimensional surface general least square curve fitting
fa f=new fa();
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  B[i]=0;
              for(j=0;j<np1;j++)
    {
              A[i][j]=0.0;
    for(k=0;k<l;k++) A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);
    }
    for(k=0;k<l;k++) B[i]+= f.func(xi[k],i)*yi[k];
  }
X=gausswithpartialpivot(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
Text.printT(X);
return X;
}

public static double funcGeneralLeastSquare(double e[],double x[])
{
// multidimensional function calculation
fa f=new fa();
int n=e.length;
double ff=0;
if(n!=0.0)
  {
  for(int i=n-1;i>=0;i--)
   {ff+=e[i]*f.func(x,i);}
  }
return ff;
}

public static double[] funcGeneralLeastSquare(double e[],double xi[][])
{
// multidimensional function calculation
fa f=new fa();
int n=e.length;
double ff[]=new double[xi.length];
for(int k=0;k<xi.length;k++)
{
if(n!=0.0)
```

```
  {
  for(int i=n-1;i>=0;i--)
   {ff[k]+=e[i]*f.func(xi[k],i);}
  }
}
return ff;
}

public static double[][] cikti(double c[][],int polinomkatsayisi,int aradegersayisi)
{ int n1=c.length;
  int n2=c[0].length-1;
  System.out.println("n1="+n1+"n2="+n2);
  double xi[][]=new double[n1][n2];
  double yi[]=new double[n1];
  for(int i=0;i<n1;i++){for(int j=0;j<n2;j++){xi[i][j]=c[i][j];};yi[i]=c[i][n2];
  }
  return cikti(xi,yi,polinomkatsayisi,aradegersayisi);
}


public static double[][] cikti(double xi[][],double yi[],int polinomkatsayisi,int aradegersayisi)
{
              int n=xi.length;
              int nk=xi[0].length;
              int nn=(n-1)*(aradegersayisi+1)+1;
    double E[]=GeneralLeastSquare(xi,yi,polinomkatsayisi);
    double x[][]=new double[nn][nk];
    double c[][]=new double[nn][nk+1];
    double yy[]=new double[nn];
    double dx[]=new double[nk];
    int k=0;
    int i,j,w;
    for(i=0;i<(n-1);i++)
       {   for(w=0;w<nk;w++){x[k][w]=xi[i][w];dx[w]=(xi[i+1][w]-xi[i][w])/((double)aradegersayisi+1.0);};k++;
          for(j=0;j<aradegersayisi;j++)
          { for(w=0;w<nk;w++){x[k][w]=x[k-1][w]+dx[w];};k++;}
       }
    for(w=0;w<nk;w++){x[k][w]=xi[i][w];}
    yy=funcGeneralLeastSquare(E,x);
    for(i=0;i<x.length;i++)
      {for(w=0;w<nk;w++) c[i][w]=x[i][w];
       c[i][nk]=yy[i];
      }
    return c;
}

public static double funcGeneralLeastSquare(double e[],double x,double y)
{
// multidimensionaş surface fitting
double xx[]=new double[2];
xx[0]=x;
xx[1]=y;
return funcGeneralLeastSquare(e,xx);
}

public static double hata(double c[][],double e[])
{int n1=c.length;
  int n2=c[0].length-1;
  System.out.println("n1="+n1+"n2="+n2);
  double xi[][]=new double[n1][n2];
  double yi[]=new double[n1];
  for(int i=0;i<n1;i++){for(int j=0;j<n2;j++){xi[i][j]=c[i][j];};yi[i]=c[i][n2];
```

```
  }
return hata(xi,yi,e);
}

public static double hata(double x[][],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcGeneralLeastSquare(e,x[k]))*(y[k]-funcGeneralLeastSquare(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

  public static void main(String[] args)
  {
  String s1=JOptionPane.showInputDialog("file name : ");
  double c[][]=Text.readDouble(s1);
  double b[]=GeneralLeastSquare(c,4);
  double xi[][]=cikti(c,4,5);
  System.out.println("output \n"+Matrix.toString(xi));
  System.out.println("error \n"+Matrix.toString(hata(c,b)));

  }
}
```




```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA46
n1=90n2=2
n1=90n2=2
output
      0.000000000000000      0.000000000000000     -1.901333333332597
      0.000000000000000      0.066666666666667     -2.774399999999312
      0.000000000000000      0.133333333333333     -3.576355555554915
      0.000000000000000      0.200000000000000     -4.307199999999405
      0.000000000000000      0.266666666666667     -4.966933333332783
      0.000000000000000      0.333333333333333     -5.555555555555047
      0.000000000000000      0.400000000000000     -6.073066666666200
      0.000000000000000      0.466666666666667     -6.519466666666240
      0.000000000000000      0.533333333333333     -6.894755555555169
      0.000000000000000      0.600000000000000     -7.198933333332985
      0.000000000000000      0.666666666666667     -7.431999999999686
      0.000000000000000      0.733333333333333     -7.593955555555279
      0.000000000000000      0.800000000000000     -7.684799999999755
      0.000000000000000      0.866666666666667     -7.704533333333122
      0.000000000000000      0.933333333333333     -7.653155555555374
      0.000000000000000      1.000000000000000     -7.530666666666516
      0.000000000000000      1.066666666666667     -7.337066666666544
      0.000000000000000      1.133333333333333     -7.072355555555461
```

```
      0.000000000000000      1.200000000000000     -6.736533333333265
      0.000000000000000      1.266666666666667     -6.329599999999955
      0.000000000000000      1.333333333333333     -5.851555555555532
      0.000000000000000      1.400000000000000     -5.302400000000000
      0.000000000000000      1.466666666666667     -4.682133333333356
      0.000000000000000      1.533333333333333     -3.990755555555593
      0.000000000000000      1.600000000000000     -3.228266666666718
      3.200000000000000      3.133333333333334    150.057244444444650
      3.200000000000000      3.200000000000000    154.190400000000200
      3.200000000000000      3.266666666666667    158.394666666666840
      3.200000000000000      3.333333333333333    162.670044444444670
      3.200000000000000      3.400000000000000    167.016533333333570
      3.200000000000000      3.466666666666666    171.434133333333620
      3.200000000000000      3.533333333333332    175.922844444444730
      3.200000000000000      3.600000000000000    180.482666666667100

n1=90n2=2
error
     419.074496785476100
```

This polynomial is defined in the example problem

```
from abc import ABC, abstractmethod
from math import *

class f_xir(ABC):

    @abstractmethod
    def func(self,x,equation_ref):
        pass;
```

Data file a.txt

```
0       0       0
0       1       1
0       2       4
0       3       9
0       4       16
1       0       1
1       1       2
1       2       5
1       3       10
1       4       17
2       0       4
2       1       5
2       2       8
2       3       13
2       4       20
3       0       9
3       1       10
3       2       13
3       3       18
3       4       25
4       0       16
4       1       17
4       2       20
4       3       25
4       4       32
```

Mulitdimensional curve fitting, python version

```
from f_xir import *
from gauss import *
class f1(f_xir):
#f(x,y)=a0+a1*x+a2*y+a3*x^2+a4*y^2+a5*xy+a6*x^3+a7*y^3+a8*x^2y
#      +a9*xy^2+a10*x^4+a11*y^4+a12*x^2y^2+a13x^3y+a14*xy^3
```

```
    def func(self,x,i):
        xx=0.0
        if i==0: xx=1.0
        elif i==1:   xx=x[0]
        elif i==2:   xx=x[1]
        elif i==3:   xx=x[0]*x[0]
        elif i==4:   xx=x[1]*x[1]
        elif i==5:   xx=x[0]*x[1]
        elif i==6:   xx=x[0]*x[0]*x[0]
        elif i==6:   xx=x[0]*x[0]*x[0]
        elif i==7:   xx=x[1]*x[1]*x[1]
        elif i==8:   xx=x[0]*x[0]*x[1]
        elif i==9:   xx=x[0]*x[1]*x[1]
        elif i==10:   xx=x[0]*x[0]*x[0]*x[0]
        elif i==11:   xx=x[1]*x[1]*x[1]*x[1]
        elif i==12:   xx=x[0]*x[0]*x[1]*x[1]
        elif i==13:   xx=x[0]*x[0]*x[0]*x[1]
        elif i==14:   xx=x[0]*x[1]*x[1]*x[1]
        return xx

# General least square curve fitting with multivariable
def Transpose(left):
    # transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
    # int i,j;
    n=len(left)
    m=len(left[0])
    b=[[0 for j in range(n)] for i in range(m)]
    for i in range(n):
        for j in range(m):
            b[j][i]=left[i][j]
    return b

def GeneralLeastSquare(c,n):
    n1=len(c)
    n2=len(c[0])-1
    xi=[[0 for j in range(n2)] for i in range(n1)]
    yi=[0 for j in range(n1)]
    for i in range(n1):
        for j in range(n2):
            xi[i][j]=c[i][j]
        yi[i]=c[i][n2]
    return GLS(xi,yi,n)

def GLS(xi,yi,n):
# n dimensional surface general least square curve fitting
    f=f1()
    l=len(xi)
    np1=n+1
    A=[[0 for j in range(np1)] for i in range(np1)]
    B=[0 for j in range(np1)]
    X=[0 for j in range(np1)]
    for i in range(n+1):
        B[i]=0
        for j in range(np1):
            A[i][j]=0.0
            for k in range(l):
                A[i][j]=A[i][j]+f.func(xi[k],i)*f.func(xi[k],j)
        for k in range(l):
                B[i]=B[i]+f.func(xi[k],i)*yi[k]
    X=gauss(A,B)
    max=0
    for i in range(n+1):
        if abs(X[i]) > max: max=abs(X[i])
    for i in range(n+1):
        if abs(X[i])/max > 0 and abs(X[i]/max) < 1.0e-100: X[i]=0.0
    #print(X)
    return X

def func(e,x):
    #multidimensional function calculation
    f=f1()
    n=len(e)
    ff=0.0
    if n!=0:
        for i in range(n-1,-1,-1):
            ff=ff+e[i]*f.func(x,i)
```

```
    return ff

def funcGeneralLeastSquare(e,xi):
    # multidimensional function calcu0000000000 nnnmlation
    f=f1()
    n=len(e)
    m=len(xi)
    ff=[0 for j in range(m)]
    for k in range(m):
      if n!=0:
         for i in range(n-1,-1,-1):
          ff[k]=ff[k]+e[i]*f.func(xi[k],i)
    return ff

def read(name):
    path = name
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    while True:
       line=file1.readline()
       nn+=1
       if  file1.tell()==endLocation:
          break
    x=[0.0 for i in range(nn)]
    y=[0.0 for i in range(nn)]
    z=[0.0 for i in range(nn)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
       line=file1.readline()
       ee=line.split()
       x[i]=float(ee[0])
       y[i]=float(ee[1])
       z[i]=float(ee[2])
       #print(x[i]," ",y[i]," ",z[i])
    file1.close()
    a=[x,y,z]
    return Transpose(a)
def output1(c,polinomkatsayisi,aradegersayisi):
    n1=len(c)
    n2=len(c[0])-1
    xi=[[0 for j in range(n2)] for i in range(n1)]
    yi=[0 for j in range(n1)]
    for i in range(n1):
       for j in range(n2):
          xi[i][j]=c[i][j]
       yi[i]=c[i][n2]
    return output2(xi,yi,polinomkatsayisi,aradegersayisi);

def output2(xi,yi,polinomkatsayisi,aradegersayisi):
    n=len(xi)
    nk=len(xi[0])
    nn=(n-1)*(aradegersayisi+1)+1
    E=GLS(xi,yi,polinomkatsayisi)
    x=[[0 for j in range(nk)] for i in range(nn)]
    c=[[0 for j in range(nk+1)] for i in range(nn)]
    yy=[0.0 for i in range(nn)]
    dx=[0.0 for i in range(nk)]
    k=0
    # int i,j,w;
    for i in range(n-1):
       for w in range(nk):
          x[k][w]=xi[i][w]
          dx[w]=(xi[i+1][w]-xi[i][w])/(float(aradegersayisi)+1.0)
       k=k+1
       for j in range(aradegersayisi):
          for w in range(nk):
             x[k][w]=x[k-1][w]+dx[w]
          k=k+1
    for w in range(nk):
       x[k][w]=xi[i][w]
    yy=funcGeneralLeastSquare(E,x)
    for i in range(len(x)):
       for w in range(nk):
```

```
            c[i][w]=x[i][w]
        c[i][nk]=yy[i]
    return c

a=read('a.txt')
e=GeneralLeastSquare(a,9)
#print(e)
y=output1(a,9,1)
print(y)
```

```
runfile('E:/okul/SCO1/multidimensional_LSCF.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xir, gauss
[[0.0, 0.0, -2.1827872842550278e-14], [0.0, 0.5, 0.24999999999998757], [0.0, 1.0, 0.9999999999999939], [0.0, 1.5,
2.249999999999998], [0.0, 2.0, 4.0], [0.0, 2.5, 6.25], [0.0, 3.0, 9.0], [0.0, 3.5, 12.249999999999998], [0.0, 4.0, 15.999999999999996],
[0.5, 2.0, 4.249999999999998], [1.0, 0.0, 0.9999999999999859], [1.0, 0.5, 1.249999999999992], [1.0, 1.0, 1.999999999999996],
[1.0, 1.5, 3.249999999999998], [1.0, 2.0, 4.999999999999998], [1.0, 2.5, 7.249999999999997], [1.0, 3.0, 9.999999999999998], [1.0,
3.5, 13.249999999999996], [1.0, 4.0, 16.999999999999996], [1.5, 2.0, 6.249999999999999], [2.0, 0.0, 3.999999999999995], [2.0,
0.5, 4.249999999999998], [2.0, 1.0, 5.0], [2.0, 1.5, 6.25], [2.0, 2.0, 8.000000000000002], [2.0, 2.5, 10.25], [2.0, 3.0,
12.999999999999998], [2.0, 3.5, 16.25], [2.0, 4.0, 20.0], [2.5, 2.0, 10.250000000000002], [3.0, 0.0, 9.000000000000002], [3.0, 0.5,
9.250000000000004], [3.0, 1.0, 10.000000000000004], [3.0, 1.5, 11.250000000000004], [3.0, 2.0, 13.000000000000002], [3.0, 2.5,
15.25], [3.0, 3.0, 18.0], [3.0, 3.5, 21.25], [3.0, 4.0, 25.000000000000004], [3.5, 2.0, 16.249999999999996], [4.0, 0.0, 16.0], [4.0, 0.5,
16.25], [4.0, 1.0, 16.999999999999996], [4.0, 1.5, 18.249999999999996], [4.0, 2.0, 19.999999999999993], [4.0, 2.5,
22.249999999999993], [4.0, 3.0, 24.999999999999996], [4.0, 3.5, 28.25], [4.0, 3.0, 24.999999999999996]]
```

Multidimensional least square curve fitting, Octave-matlab version

```
## Multiple general polynomial function
## Author M. Turhan ÇOBAN
## Created 2021-08-22

function retval = mult_f(x,i,k)
#f(x,y)=a0+a1*x+a2*y+a3*x^2+a4*y^2+a5*xy+a6*x^3+a7*y^3+a8*x^2y
#      +a9*xy^2+a10*x^4+a11*y^4+a12*x^2y^2+a13x^3y+a14*xy^3
     xx=0.0;
     n=length(x);
     if i==1
       xx=1;
     elseif i==2
       xx=x(1,k);
     elseif i==3
       xx=x(2,k);
     elseif i==4
       xx=x(1,k)*x(1,k);
     elseif i==5
       xx=x(2,k)*x(2,k);
     elseif i==6
       xx=x(1,k)*x(2,k);
     elseif i==7
       xx=x(1,k)*x(1,k)*x(1,k);
     elseif i==8
       xx=x(2,k)*x(2,k)*x(2,k);
     elseif i==9
       xx=x(1,k)*x(1,k)*x(2,k);
     elseif i==10
       xx=x(2,k)*x(2,k)*x(1,k);
     elseif i==11
       xx=x(1,k)*x(1,k)*x(1,k)*x(1,k);
     elseif i==12
       xx=x(2,k)*x(2,k)*x(2,k)*x(2,k);
     elseif i==13
       xx=x(1,k)*x(1,k)*x(2,k)*x(2,k);
     elseif i==14
       xx=x(1,k)*x(1,k)*x(1,k)*x(2,k);
     elseif i==15
       xx=x(1,k)*x(2,k)*x(2,k)*x(2,k);
     endif
     retval= xx;
endfunction
```

```
## Multiple variable General polynomial function
## Author: M. Turhan ÇOBAN
## Created: 2021-08-22

function retval = multpoly_func(f,e,x,k)
   # this function calculates the value of
   # least square curve fitting function
```

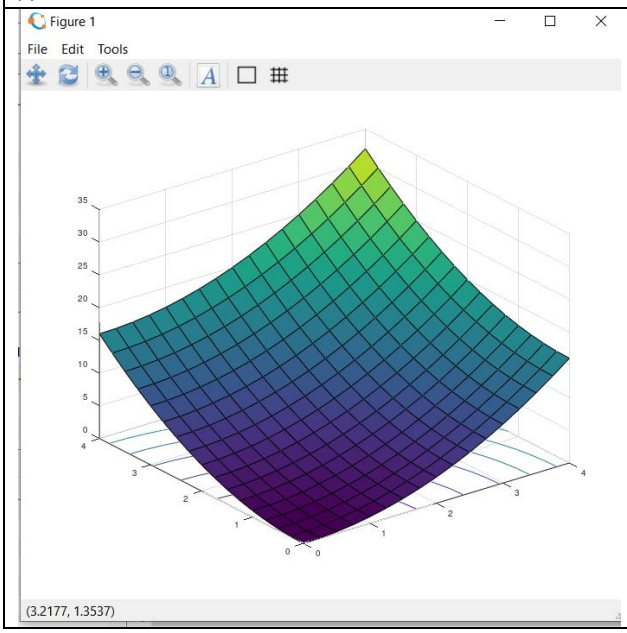
```
    n=length(e);
    ff=0.0;
    if n!=0.0
       for i=n:-1:1
          ff=ff+e(i)*f(x,i,k);
       end
    end
    retval =ff;
endfunction
```

```
# Multiple General polynomial least square curve fitting
# Author: M. Turhan ÇOBAN
# Created: 2021-08-22

function retval = multpoly_least(xi,yi,f,n)
    switch nargin
       case 3
         n=input("degree of polynomial ? ");
     case 2
         f=@mult_f;
         n=input("degree of polynomial ? ");
    end
    m=length(yi);
    np1=n+1;
    A =zeros(np1,np1);
    B =zeros(np1,1);
    X =zeros(np1);
    for i=1:np1
       for j=1:np1
          for k=1:m
                fi=f(xi,i,k);
                fj=f(xi,j,k);
                A(i,j)= A(i,j)+fi*fj;
          end # for k=1:m
       end # j=1:np1
       for k=1:m
          fi=f(xi,i,k);
          yyi=yi(k);
          B(i)= B(i)+fi*yyi;
       end
    end
    X=gauss1(A,B);
    retval=X;
end
```

```
## Multiple variable General polynomial least square plot
## Author: M. Turhan ÇOBAN
## Created: 2021-08-30

function retval = multpoly_plot(x,y,f,n)
    e=multpoly_least(x,y,f,n)
    n=length(y);
    xmin=0.0;
    xmax=4.0;
    ymin=0.0;
    ymax=4.0;
    m1=17;
    dx=(xmax-xmin)/(m1-1);
    x1=y1=linspace(xmin,xmax,m1);
    [xx,yy]=meshgrid (x1,y1);
    m2=m1*m1;
    for i=1:m1
      for j=1:m1
        xx1=xx(i,:);
        yy1=yy(j,:);
        xx2=[xx1;yy1];
        z(i,j)=multpoly_func(f,e,xx2,i);
      end
    end
   surfc(xx,yy,z);
   retval=z;
end
```

```
>> multpoly_plot(x,y,f,10)
e =

 Columns 1 through 9:
```

```
   8.2214e-14  -2.6024e-12  -9.0201e-14   1.0000e+00   1.0000e+00  -1.6825e-15  -1.4209e-12  -1.0826e-14   7.2006e-16

 Columns 10 and 11:

  -4.0602e-16   1.7621e-13

ans =

 Columns 1 through 9:

   8.2214e-14   6.2500e-02   2.5000e-01   5.6250e-01   1.0000e+00   1.5625e+00   2.2500e+00   3.0625e+00   4.0000e+00
   6.2500e-02   1.2500e-01   3.1250e-01   6.2500e-01   1.0625e+00   1.6250e+00   2.3125e+00   3.1250e+00   4.0625e+00
   2.5000e-01   3.1250e-01   5.0000e-01   8.1250e-01   1.2500e+00   1.8125e+00   2.5000e+00   3.3125e+00   4.2500e+00
   5.6250e-01   6.2500e-01   8.1250e-01   1.1250e+00   1.5625e+00   2.1250e+00   2.8125e+00   3.6250e+00   4.5625e+00
   1.0000e+00   1.0625e+00   1.2500e+00   1.5625e+00   2.0000e+00   2.5625e+00   3.2500e+00   4.0625e+00   5.0000e+00
   1.5625e+00   1.6250e+00   1.8125e+00   2.1250e+00   2.5625e+00   3.1250e+00   3.8125e+00   4.6250e+00   5.5625e+00
   2.2500e+00   2.3125e+00   2.5000e+00   2.8125e+00   3.2500e+00   3.8125e+00   4.5000e+00   5.3125e+00   6.2500e+00
   3.0625e+00   3.1250e+00   3.3125e+00   3.6250e+00   4.0625e+00   4.6250e+00   5.3125e+00   6.1250e+00   7.0625e+00
   4.0000e+00   4.0625e+00   4.2500e+00   4.5625e+00   5.0000e+00   5.5625e+00   6.2500e+00   7.0625e+00   8.0000e+00
   5.0625e+00   5.1250e+00   5.3125e+00   5.6250e+00   6.0625e+00   6.6250e+00   7.3125e+00   8.1250e+00   9.0625e+00
   6.2500e+00   6.3125e+00   6.5000e+00   6.8125e+00   7.2500e+00   7.8125e+00   8.5000e+00   9.3125e+00   1.0250e+01
   7.5625e+00   7.6250e+00   7.8125e+00   8.1250e+00   8.5625e+00   9.1250e+00   9.8125e+00   1.0625e+01   1.1563e+01
   9.0000e+00   9.0625e+00   9.2500e+00   9.5625e+00   1.0000e+01   1.0562e+01   1.1250e+01   1.2062e+01   1.3000e+01
   1.0562e+01   1.0625e+01   1.0812e+01   1.1125e+01   1.1562e+01   1.2125e+01   1.2812e+01   1.3625e+01   1.4562e+01
   1.2250e+01   1.2312e+01   1.2500e+01   1.2812e+01   1.3250e+01   1.3812e+01   1.4500e+01   1.5312e+01   1.6250e+01
   1.4062e+01   1.4125e+01   1.4312e+01   1.4625e+01   1.5062e+01   1.5625e+01   1.6312e+01   1.7125e+01   1.8062e+01
   1.6000e+01   1.6063e+01   1.6250e+01   1.6563e+01   1.7000e+01   1.7563e+01   1.8250e+01   1.9063e+01   2.0000e+01
Columns 10 through 17:

   5.0625e+00   6.2500e+00   7.5625e+00   9.0000e+00   1.0563e+01   1.2250e+01   1.4063e+01   1.6000e+01
   5.1250e+00   6.3125e+00   7.6250e+00   9.0625e+00   1.0625e+01   1.2312e+01   1.4125e+01   1.6062e+01
   5.3125e+00   6.5000e+00   7.8125e+00   9.2500e+00   1.0812e+01   1.2500e+01   1.4312e+01   1.6250e+01
   5.6250e+00   6.8125e+00   8.1250e+00   9.5625e+00   1.1125e+01   1.2812e+01   1.4625e+01   1.6562e+01
   6.0625e+00   7.2500e+00   8.5625e+00   1.0000e+01   1.1562e+01   1.3250e+01   1.5062e+01   1.7000e+01
   6.6250e+00   7.8125e+00   9.1250e+00   1.0562e+01   1.2125e+01   1.3812e+01   1.5625e+01   1.7562e+01
   7.3125e+00   8.5000e+00   9.8125e+00   1.1250e+01   1.2813e+01   1.4500e+01   1.6313e+01   1.8250e+01
   8.1250e+00   9.3125e+00   1.0625e+01   1.2063e+01   1.3625e+01   1.5313e+01   1.7125e+01   1.9063e+01
   9.0625e+00   1.0250e+01   1.1563e+01   1.3000e+01   1.4563e+01   1.6250e+01   1.8063e+01   2.0000e+01
   1.0125e+01   1.1313e+01   1.2625e+01   1.4063e+01   1.5625e+01   1.7313e+01   1.9125e+01   2.1063e+01
   1.1313e+01   1.2500e+01   1.3813e+01   1.5250e+01   1.6813e+01   1.8500e+01   2.0313e+01   2.2250e+01
   1.2625e+01   1.3813e+01   1.5125e+01   1.6563e+01   1.8125e+01   1.9813e+01   2.1625e+01   2.3562e+01
   1.4062e+01   1.5250e+01   1.6562e+01   1.8000e+01   1.9562e+01   2.1250e+01   2.3062e+01   2.5000e+01
   1.5625e+01   1.6812e+01   1.8125e+01   1.9562e+01   2.1125e+01   2.2812e+01   2.4625e+01   2.6562e+01
   1.7312e+01   1.8500e+01   1.9812e+01   2.1250e+01   2.2812e+01   2.4500e+01   2.6312e+01   2.8250e+01
   1.9125e+01   2.0312e+01   2.1625e+01   2.3062e+01   2.4625e+01   2.6312e+01   2.8125e+01   3.0062e+01
   2.1063e+01   2.2250e+01   2.3563e+01   2.5000e+01   2.6563e+01   2.8250e+01   3.0063e+01   3.2000e+01

>>
```



6.4 INTERPOLATION

We sometimes know the value of a function $f(x)$ at a set of points $x_1, x_2, \ldots, x_N$ (say, with $x_1 < \ldots. < x_N$), but we don't have an analytic expression for $f(x)$ that lets us calculate its value at an arbitrary point. For example, the $f(x_i)$'s might result from some physical measurement or from long numerical calculation that cannot be cast into a simple functional form. Often the x_i's are equally spaced, but not necessarily. The task now is to estimate $f(x)$ for arbitrary x by, in some sense, drawing a smooth curve through (and perhaps beyond) the x_i. If the desired x is in between the largest and smallest of the x_i's, the problem is called *interpolation*; if x is outside that range, it is called *extrapolation*,which is considerably more hazardous. Basic difference of interpolation compare to least square curve fitting is that interpolation is always passing through the data points. Some of the most common interpolation methods are given here:

Newton Interpolation Polynomial

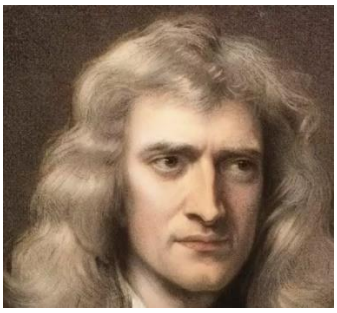


Figure Sir Isaac Newton

Newton interpolation Formula of degree n+1 is given as:

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)\big(x - x_1\big)..(x - x_n)\big)$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

…

$$b_n = f[x_n, x_{n-1}, .., x_2, x_1, x_0] \quad (6.4\text{-}1)$$

In this definition :

$$f\big[x_i, x_j\big] = \frac{f(x_i) - f(x_j)}{x_i - x_i} \quad (6.4.2a)$$

$$f\big[x_i, x_j, x_k\big] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k} \quad (6.4.2b)$$

Similarly nth degree divided difference Formula can be given as:

$$f[x_n, x_{n-1}, \ldots, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, .., x_1] - f[x_{n-1}, x_{n-2}, \ldots, \ldots, x_1, x_0]}{x_n - x_0} \quad (6.4.2c)$$

This equation called as Newton's interpolation polynomial.

An example problem :

Population of City A is given in the table below. Calculate population of the city for year 1945

year	1940	1950	1960	1970	1980	1990
Population	132165	151336	179323	203302	226542	249633

Newton's divided difference table :

Xi	yi	f[xi,xj]	f[xi,xj,xk]	f[xi,xj,xk,xl]	f[xi,xj,xk,xl,xn]	f[xi,xj,xk,xl,xn,xm]
1940	132165	1917.1	44.08	-2.137333333	0.067054167	-0.001564333
1950	151336	2798.7	-20.04	0.544833333	-0.0111625	

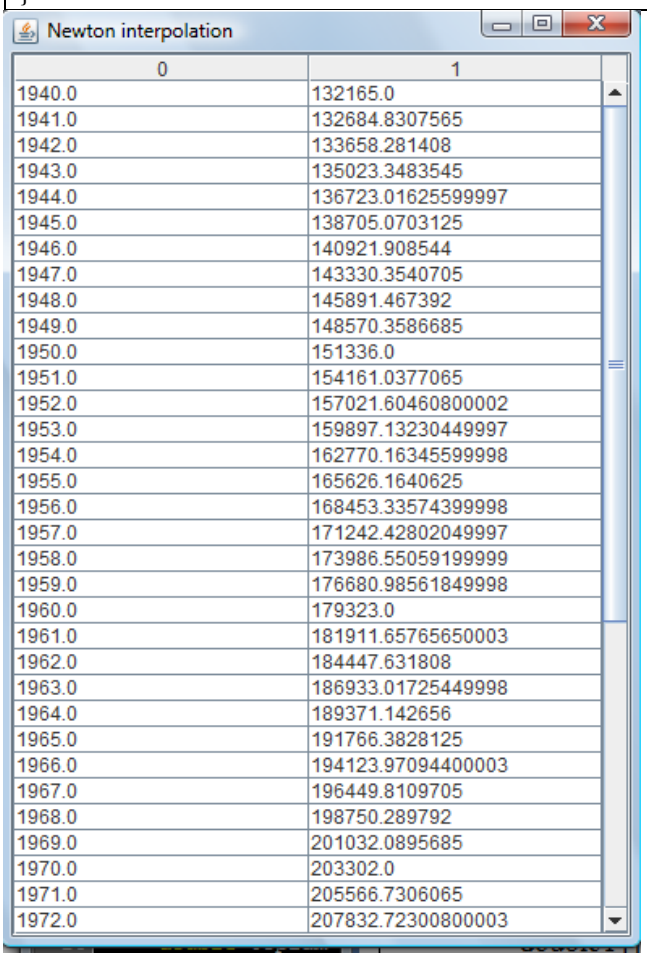
1960	179323	2397.9	-3.695	0.098333333		
1970	203302	2324	-0.745			
1980	226542	2309.1				
1990	249633					

Solution with increasing order can be given as:
132165+(1945-1940)*1917.1= 141750.5
141750.5+(1945-1940)* (1945-1950)* 44.08=140648.5
140648.5+(1945-1940)(1945-1950)*(1945-1960)* -2.137333333=139847
139847+(1945-1940)(1945-1950)*(1945-1960)*(1945-1970)* 0.067054167 =139218.4
139218.4+(1945-1940)(1945-1950)*(1945-1960)*(1945-1970)*(1945-1980)* -0.001564333=138705.1
The most important advantage of Newton intrepolation formula is the additive characters. In order to calculate (n+1) th degree of polynomial all you need is nth degree polynomial difference values. Example code is giving the result of the same problem

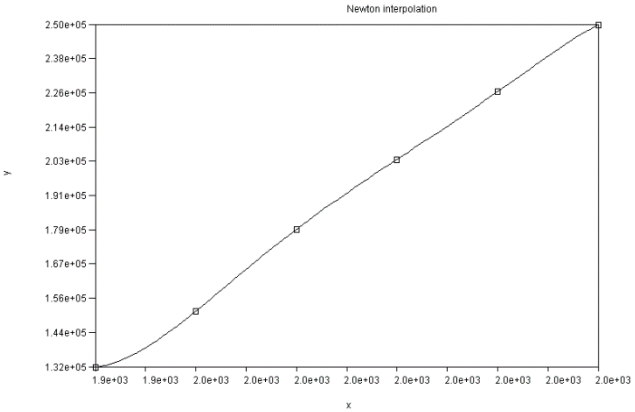
Problem Newton Interpolation formula, Java version

```
import java.io.*;
class NA47
{
  public static double[][]  Newton(double xi[],double yi[])
  {
              int n=xi.length;
              double F[][]=new double[n+1][n];
              for(int i=0;i<n;i++)
                 F[i][0]=yi[i];
    for(int i=1;i<n;i++)
    { for(int j=1;j<=i;j++)
       {F[i][j]=(F[i][j-1]-F[i-1][j-1])/(xi[i]-xi[i-j]);}
    }
    for(int i=0;i<n;i++)
    {F[n][i]=xi[i];}
    //System.out.println(Matrix.toString(F));
    return F;
  }

  public static double  funcNewton(double F[][],double x)
  {
              int n=F[0].length;
    double carpim=1.0;
    double toplam=0;
    for(int i=0;i<n;i++)
      {   carpim=F[i][i];
                  for(int j=0;j<=(i-1);j++)
        {carpim*=(x-F[n][j]);}
        toplam+=carpim;
      }
    return toplam;
  }

 public static double[][]  funcNewton(double xi[],double yi[],int numberofmidpoints)
 {
             //numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
             // numberofmidpoints is 2
             int n=xi.length;
             int nn=(n-1)*(numberofmidpoints+1)+1;
             double z[][]=new double[2][nn];
    double Q[][]=Newton(xi,yi);
    double dx=0;
    int k=0;
    int i;
```

```
    for(i=0;i<(n-1);i++)
      {  z[0][k]=xi[i];z[1][k]=funcNewton(Q,z[0][k]);k++;
        for(int j=0;j<numberofmidpoints;j++)
          {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
        z[0][k]=z[0][k-1]+dx;z[1][k]=funcNewton(Q,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcNewton(Q,z[0][k]);
    return z;
  }

  public static void main(String args[]) throws IOException
  {
  double x[]={1940,1950,1960,1970,1980,1990};
  double y[]={132165,151336,179323,203302,226542,249633};
  double z1[][]=funcNewton(x,y,9);
  System.out.println("Newton interpolation\n"+Matrix.toStringT(z1));
  Text.print(Text.T(z1),"Newton interpolation");
  Plot pp=new Plot(x,y);
  pp.setPlabel("Newton interpolation");
  pp.setPlotType(0,20);
  pp.addData(z1[0],z1[1]);
  pp.plot();
  }
}
```



0	1
1940.0	132165.0
1941.0	132684.8307565
1942.0	133658.281408
1943.0	135023.3483545
1944.0	136723.01625599997
1945.0	138705.0703125
1946.0	140921.908544
1947.0	143330.3540705
1948.0	145891.467392
1949.0	148570.3586685
1950.0	151336.0
1951.0	154161.0377065
1952.0	157021.60460800002
1953.0	159897.13230449997
1954.0	162770.16345599998
1955.0	165626.1640625
1956.0	168453.33574399998
1957.0	171242.42802049997
1958.0	173986.55059199999
1959.0	176680.98561849998
1960.0	179323.0
1961.0	181911.65765650003
1962.0	184447.631808
1963.0	186933.01725449998
1964.0	189371.142656
1965.0	191766.3828125
1966.0	194123.97094400003
1967.0	196449.8109705
1968.0	198750.289792
1969.0	201032.0895685
1970.0	203302.0
1971.0	205566.7306065
1972.0	207832.72300800003



---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA47
Newton interpolation
1940.0000000000132165.0000000000
1941.0000000000132684.8307565000
1942.0000000000133658.2814080000
1943.0000000000135023.3483545000
1944.0000000000136723.0162560000
1945.0000000000138705.0703125000
1946.0000000000140921.9085440000
1947.0000000000143330.3540705000
1948.0000000000145891.4673920000
1949.0000000000148570.3586685000
1950.0000000000151336.0000000000
1951.0000000000154161.0377065000
1952.0000000000157021.6046080000
1953.0000000000159897.1323045000
1954.0000000000162770.1634560000
1955.0000000000165626.1640625000
1956.0000000000168453.3357440000
1957.0000000000171242.4280205000
1958.0000000000173986.5505920000
1959.0000000000176680.9856185000
1960.0000000000179323.0000000000
1961.0000000000181911.6576565000
1962.0000000000184447.6318080000
1963.0000000000186933.0172545000
1964.0000000000189371.1426560000
1965.0000000000191766.3828125000
1966.0000000000194123.9709440000
1967.0000000000196449.8109705000
1968.0000000000198750.2897920000
1969.0000000000201032.0895685000
1970.0000000000203302.0000000000
1971.0000000000205566.7306065000
1972.0000000000207832.7230080000
1973.0000000000210105.9632045000
1974.0000000000212391.7938560000
1975.0000000000214694.7265625000
1976.0000000000217018.2541440000
1977.0000000000219364.6629205000
1978.0000000000221734.8449920000
1979.0000000000224128.1105185000
1980.0000000000226542.0000000000
1981.0000000000228972.0965565000
1982.0000000000231411.8382080001
1983.0000000000233852.3301545001

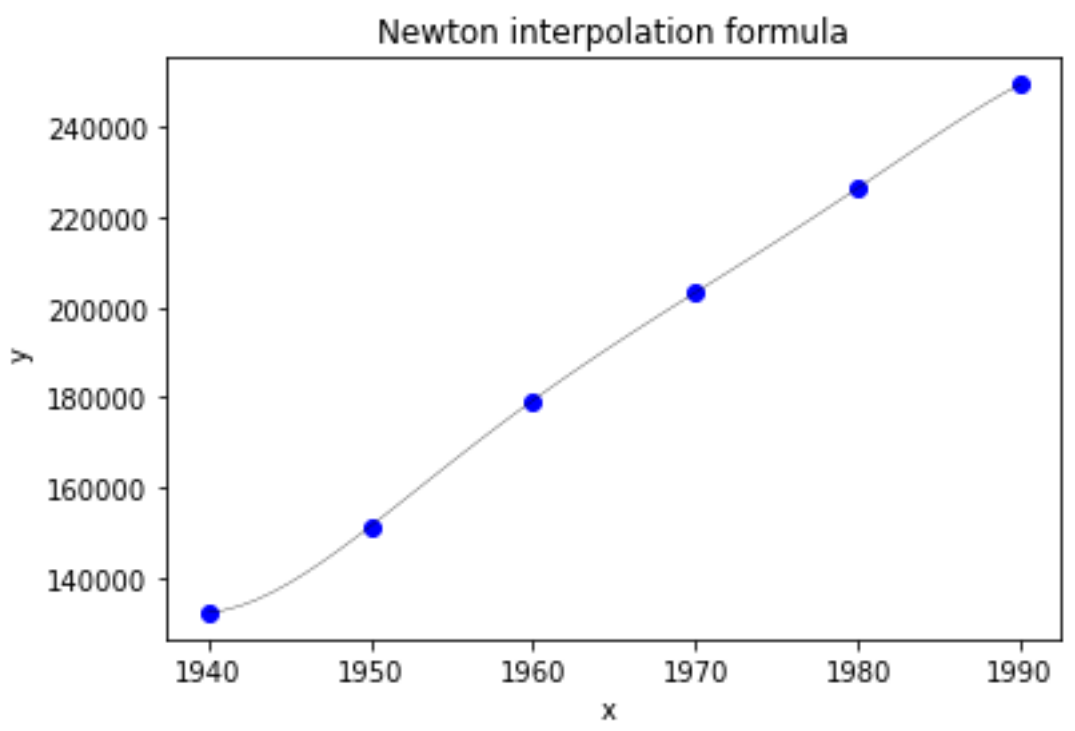
```
1984.0000000000236282.1570560000
1985.0000000000238687.1953125000
1986.0000000000241050.4253440000
1987.0000000000243351.7438705000
1988.0000000000245567.7761920000
1989.0000000000247671.6884685000
1990.0000000000249633.0000000000
> Terminated with exit code 1.
```

Newton interpolation formula, Python version

```
 from math import *
import numpy as np;
import matplotlib.pyplot as plt

def Newton(xi,yi):
    n=len(xi)
    F=[[0 for j in range(n)] for i in range(n+1)]
    for i in range(n):
        F[i][0]=yi[i]
    for i in range(n):
        for j in range(1,i+1):
            F[i][j]=(F[i][j-1]-F[i-1][j-1])/(xi[i]-xi[i-j])
    for i in range (n):
         F[n][i]=xi[i]
    return F

def func(F,x):
    n=len(F[0])
    carpim=1.0
    toplam=0
    for i in range(n):
        carpim=F[i][i]
        for j in range(i):
            carpim=carpim*(x-F[n][j])
        toplam=toplam+carpim;
    return toplam;

def funcNewton(xi,yi,numberofmidpoints):
    # numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
    # numberofmidpoints is 2
    n=len(xi)
    nn=(n-1)*(numberofmidpoints+1)+1
    z=[[0 for j in range(nn)] for i in range(2)]
    Q=Newton(xi,yi)
    dx=0.0
    k=0
    i=0
    for i in range(n-1):
        z[0][k]=xi[i];z[1][k]=func(Q,z[0][k])
        k=k+1
        for j in range(numberofmidpoints):
            dx=(xi[i+1]-xi[i])/(float(numberofmidpoints)+1.0)
            z[0][k]=z[0][k-1]+dx
            z[1][k]=func(Q,z[0][k])
            k=k+1
    z[0][k]=xi[n-1]
    z[1][k]=func(Q,z[0][k])
    return z
def Transpose(left):
    # transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
    # int i,j;
    n=len(left)
    m=len(left[0])
    b=[[0 for j in range(n)] for i in range(m)]
    for i in range(n):
        for j in range(m):
            b[j][i]=left[i][j]
    return b


x=[1940,1950,1960,1970,1980,1990]
y=[132165,151336,179323,203302,226542,249633]
```

```
print("x=",x,"\ny=",y)
z1=funcNewton(x,y,9)
print("Newton interpolation\n",Transpose(z1))
plt.title('Newton interpolation formula')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'bo',z1[0],z1[1],'k',linewidth=0.3,)
```

```
runfile('E:/okul/SCO1/newton_interpolation.py', wdir='E:/okul/SCO1')
x= [1940, 1950, 1960, 1970, 1980, 1990]
y= [132165, 151336, 179323, 203302, 226542, 249633]
Newton interpolation
 [[1940, 132165.0], [1941.0, 132684.8307565], [1942.0, 133658.281408], [1943.0, 135023.3483545], [1944.0, 136723.01625599997],
[1945.0, 138705.0703125], [1946.0, 140921.908544], [1947.0, 143330.3540705], [1948.0, 145891.467392], [1949.0, 148570.3586685],
[1950, 151336.0], [1951.0, 154161.0377065], [1952.0, 157021.60460800002], [1953.0, 159897.13230449997], [1954.0,
162770.16345599998], [1955.0, 165626.1640625], [1956.0, 168453.33574399998], [1957.0, 171242.42802049997], [1958.0,
173986.55059199999], [1959.0, 176680.98561849998], [1960, 179323.0], [1961.0, 181911.65765650003], [1962.0, 184447.631808],
[1963.0, 186933.01725449998], [1964.0, 189371.142656], [1965.0, 191766.3828125], [1966.0, 194123.97094400003], [1967.0,
196449.8109705], [1968.0, 198750.289792], [1969.0, 201032.0895685], [1970, 203302.0], [1971.0, 205566.7306065], [1972.0,
207832.72300800003], [1973.0, 210105.9632045], [1974.0, 212391.793856], [1975.0, 214694.7265625], [1976.0, 217018.254144],
[1977.0, 219364.6629205], [1978.0, 221734.844992], [1979.0, 224128.1105185], [1980, 226542.00000000003], [1981.0,
228972.09655649998], [1982.0, 231411.83820800006], [1983.0, 233852.33015450006], [1984.0, 236282.15705599997], [1985.0,
238687.1953125], [1986.0, 241050.42534399996], [1987.0, 243351.74387050004], [1988.0, 245567.776192], [1989.0,
247671.68846850004], [1990, 249633.0]]
```



Lagrange Interpolation Polynomial



Joseph-Loise Lagrange

Lagrange interpolation polynomial is the reformulated version of the Newton's interpolation polynomial. Lagrange Formula do not require calculation of divided differences.

$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$ (6.4-3) Where

$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_i}{x_i - x_j}$. (6.4-4) Where $\prod$ indicates multiplication process.

Open form of the equation;

for the first degree (linear) polinomial (n=1)

$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$ (6.4-5)

for the first degree (quadratic) polinomial (n=2)

$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_2)} f(x_2)$ (6.4-6)

Program Lagrange Interpolation formula

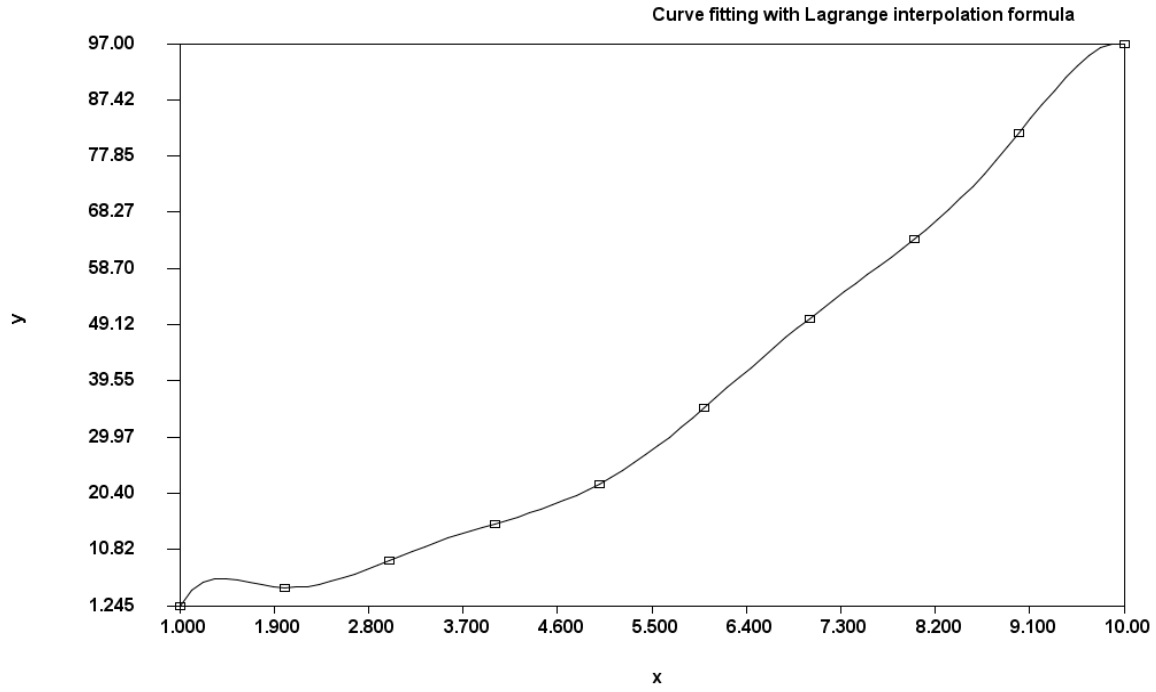
```
 //Lagrange interpolation polynomial

import java.io.*;
import javax.swing.*;
import java.lang.*;
import java.util.Locale;
class NA48
{
  public static double[][]  Lagrange(double xi[],double yi[])
  {
              int n=xi.length;
    double L[][]=new double[2][n];
    for(int i=0;i<n;i++)
    {L[0][i]=yi[i];
    for(int j=0;j<n;j++)
    {if(i!=j) L[0][i]/=(xi[i]-xi[j]);}
    }
    for(int i=0;i<n;i++)
    {L[1][i]=xi[i];}
    return L;
  }


  public static double  funcLagrange(double L[][],double x)
  {
              int n=L[0].length;
    double carpim=1.0;
    double toplam=0;
    for(int i=0;i<n;i++)
    {carpim=L[0][i];
    for(int j=0;j<n;j++)
    {if(i!=j) carpim*=(x-L[1][j]);}
    toplam+=carpim;
    }
    return toplam;
  }


 public static double[][]  funcLagrange(double xi[],double yi[],int numberofmidpoints)
 {
             //numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
             // numberofmidpoints is 2
             int n=xi.length;
             int nn=(n-1)*(numberofmidpoints+1)+1;
             double z[][]=new double[2][nn];
    double Q[][]=Lagrange(xi,yi);
    double dx=0;
    int k=0;
```

```
    int i;
    for(i=0;i<(n-1);i++)
      {  z[0][k]=xi[i];z[1][k]=funcLagrange(Q,z[0][k]);k++;
        for(int j=0;j<numberofmidpoints;j++)
          {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
        z[0][k]=z[0][k-1]+dx;z[1][k]=funcLagrange(Q,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcLagrange(Q,z[0][k]);
    return z;
  }
  public static String toString(double z[][])
  { String s="";
    for(int i=0;i<z.length;i++)
    {  for(int j=0;j<z[0].length;j++)
                {s+=String.format("%f ",z[i][j]);}
                s+="\n";
    }
    return s;
  }
  public static String toStringT(double z[][])
  { String s="";
    for(int i=0;i<z[0].length;i++)
    {  for(int j=0;j<z.length;j++)
                {s+=String.format(Locale.US,"%f ",z[j][i]);}
                s+="\n";
    }
    return s;
  }
  public static void main(String args[]) throws IOException
  {
double x[];
  double y[];
  String s1=JOptionPane.showInputDialog("Data file name (input1.txt)");
  //JFileChooser fc=new JFileChooser();
  // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  double z1[][]=funcLagrange(x,y,8);
  System.out.println("Lagrange interpolation\n"+toStringT(z1));
  Text.print(Text.T(z1),"Curve fitting with Lagrange interpolation formula");
  Plot pp=new Plot(x,y);
  pp.setPlabel("Curve fitting with Lagrange interpolation formula");
  pp.setPlotType(0,20);
  pp.addData(z1[0],z1[1]);
  pp.plot();
  }
}
```

```
 ---------- Capture Output ----------
> "E:\co\java\bin\java.exe" NA48
Lagrange interpolation
1.000000 1.245200
1.111111 3.844709
1.222222 5.220168
1.333333 5.766842
1.444444 5.792022
1.555556 5.529201
1.666667 5.150603
1.777778 4.778211
1.888889 4.493406
```

2.000000 4.345340
2.111111 4.358154
2.222222 4.537148
2.333333 4.873996
2.444444 5.351107
2.555556 5.945224
2.666667 6.630327
2.777778 7.379936
2.888889 8.168880
3.000000 8.974600
3.111111 9.778041
3.222222 10.564209
3.333333 11.322430
3.444444 12.046372
3.555556 12.733864
3.666667 13.386567
3.777778 14.009523
3.888889 14.610618
4.000000 15.200000
4.111111 15.789457
4.222222 16.391798
4.333333 17.020253
4.444444 17.687896
4.555556 18.407124
4.666667 19.189190
4.777778 20.043809
4.888889 20.978831
5.000000 22.000000
5.111111 23.110791
5.222222 24.312326
5.333333 25.603376
5.444444 26.980433
5.555556 28.437861
5.666667 29.968105
5.777778 31.561966
5.888889 33.208924
6.000000 34.897500
6.111111 36.615652
6.222222 38.351190
6.333333 40.092204
6.444444 41.827482
6.555556 43.546928
6.666667 45.241938
6.777778 46.905758
6.888889 48.533778
7.000000 50.123780
7.111111 51.676109
7.222222 53.193771
7.333333 54.682444
7.444444 56.150395
7.555556 57.608292
7.666667 59.068921
7.777778 60.546780
7.888889 62.057579
8.000000 63.617610
8.111111 65.243027
8.222222 66.949006
8.333333 68.748812
8.444444 70.652774
8.555556 72.667180
8.666667 74.793108
8.777778 77.025198

```
8.888889 79.350409
9.000000 81.746750
9.111111 84.182044
9.222222 86.612727
9.333333 88.982735
9.444444 91.222502
9.555556 93.248116
9.666667 94.960673
9.777778 96.245886
9.888889 96.973988
10.000000 97.000000

> Terminated with exit code 0.
```



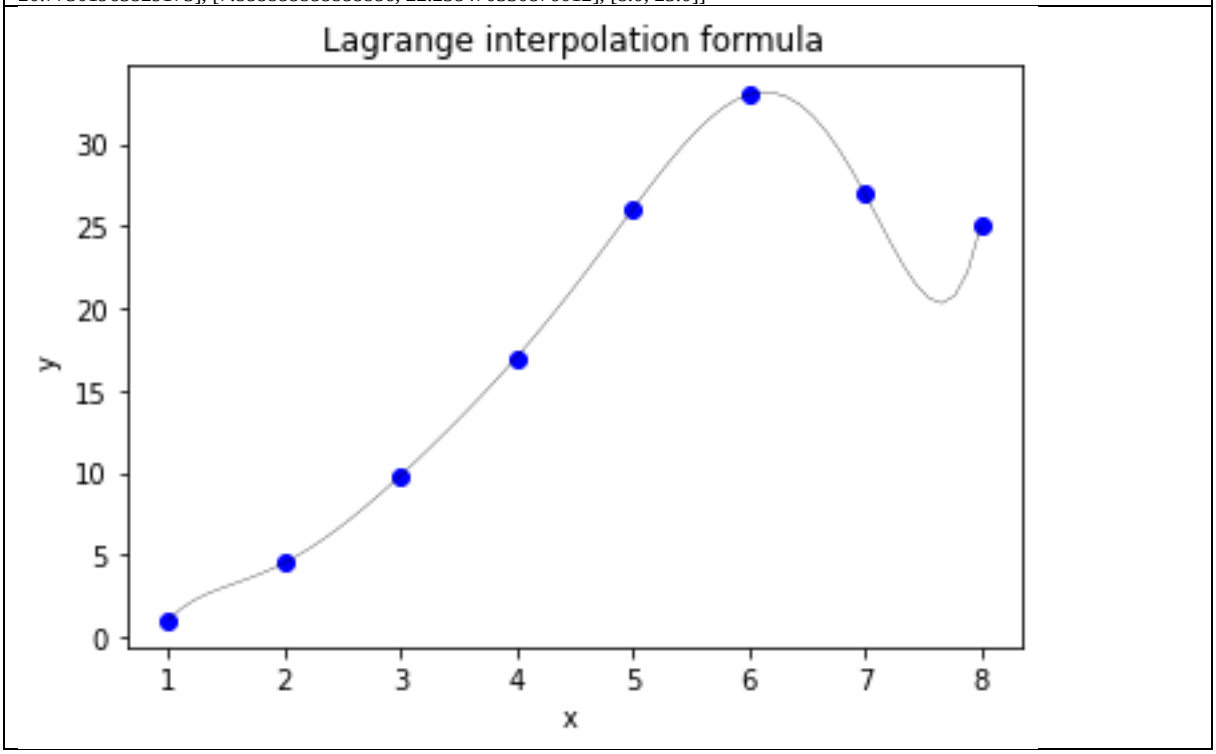
Lagrange interpolation formula, Python version

```
 from math import *
import numpy as np;
import matplotlib.pyplot as plt

def Lagrange(xi,yi):
    n=len(xi)
    L=[[0 for j in range(n)] for i in range(2)]
    for i in range(n):
        L[0][i]=yi[i]
        for j in range(n):
            if i!=j:
                L[0][i]=L[0][i]/(xi[i]-xi[j])
    for i in range (n):
         L[1][i]=xi[i]
    return L

def func(L,x):
    n=len(L[0])
    carpim=1.0
    toplam=0
    for i in range(n):
        carpim=L[0][i]
        for j in range(n):
            if i!=j:
                carpim=carpim*(x-L[1][j])
        toplam=toplam+carpim
    return toplam

def funcLagrange(xi,yi,numberofmidpoints):
```

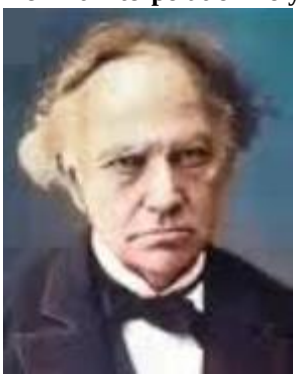
```
    # numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
    # numberofmidpoints is 2
    n=len(xi)
    nn=(n-1)*(numberofmidpoints+1)+1
    z=[[0 for j in range(nn)] for i in range(2)]
    Q=Lagrange(xi,yi)
    dx=0.0
    k=0
    i=0
    for i in range(n-1):
       z[0][k]=xi[i]
       z[1][k]=func(Q,z[0][k])
       k=k+1
       for j in range(numberofmidpoints):
          nx=float(numberofmidpoints+1.0)
          dx=(xi[i+1]-xi[i])/nx
          z[0][k]=z[0][k-1]+dx
          z[1][k]=func(Q,z[0][k])
          k=k+1
    z[0][k]=xi[n-1]
    z[1][k]=func(Q,z[0][k])
    return z
def read(name):
    path = name
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    while True:
       line=file1.readline()
       nn+=1
       if  file1.tell()==endLocation:
          break
    x=[0.0 for i in range(nn)]
    y=[0.0 for i in range(nn)]
    #z=[0.0 for i in range(nn)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
       line=file1.readline()
       ee=line.split()
       x[i]=float(ee[0])
       y[i]=float(ee[1])
       #z[i]=float(ee[2])
    file1.close()
    a=[x,y]
    return a
    #return Transpose(a)

def Transpose(left):
    # transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
    # int i,j;
    n=len(left)
    m=len(left[0])
    b=[[0 for j in range(n)] for i in range(m)]
    for i in range(n):
       for j in range(m):
          b[j][i]=left[i][j]
    return b

a=read('input1.txt')
x=a[0]
y=a[1]
print("x=",x,"'\ny=",y)
z=funcLagrange(x,y,8)
print("Lagrange interpolation\n",Transpose(z))
plt.title('Lagrange interpolation formula')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'bo',z[0],z[1],'k',linewidth=0.3,)
```

```
runfile('E:/okul/SCO1/Lagrange_interpolation.py', wdir='E:/okul/SCO1')
x= [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0]
y= [1.0, 4.5, 9.8, 17.0, 26.0, 33.0, 27.0, 25.0]
Lagrange interpolation
```

[[1.0, 1.0000000000000002], [1.1111111111111112, 1.70980181974836], [1.2222222222222223, 2.22388332435355], [1.3333333333333335, 2.6107910379515307], [1.4444444444444446, 2.924346363106264], [1.5555555555555558, 3.2057979050251073], [1.666666666666667, 3.4857796067672626], [1.7777777777777781, 3.786082786654062], [1.8888888888888893, 4.121250169089536], [2.0, 4.5], [2.111111111111111, 4.926488338101292], [2.2222222222222223, 5.4014176132021765], [2.3333333333333335, 5.922999542752632], [2.4444444444444446, 6.48778049784559], [2.555555555555556, 7.091337409880771], [2.666666666666667, 7.7288523090992305], [2.777777777777778, 8.395573586197193], [2.8888888888888893, 9.087172068227918], [3.0, 9.800000000000002], [3.111111111111111, 10.531261022180992], [3.2222222222222223, 11.279099237314735], [3.3333333333333335, 12.042615454961133], [3.4444444444444446, 12.821818707167038], [3.555555555555556, 13.617521125476673], [3.666666666666667, 14.431184270690446], [3.777777777777778, 15.264725006580646], [3.8888888888888893, 16.120289008772588], [4.0, 17.0], [4.111111111111111, 17.905692802943104], [4.222222222222221, 18.838638301858108], [4.333333333333332, 19.799268404206668], [4.444444444444443, 20.786909093494007], [4.555555555555554, 21.799529664524254], [4.666666666666664, 22.833516232281646], [4.777777777777775, 23.883477605646174], [4.888888888888886, 24.94209161715241], [5.0, 26.0], [5.111111111111111, 27.04575990352436], [5.222222222222221, 28.06586013833666], [5.333333333333332, 29.044810242341093], [5.444444444444443, 29.96531045883842], [5.555555555555554, 30.80851071792435], [5.666666666666664, 31.554366712391392], [5.777777777777775, 32.18210115934265], [5.888888888888886, 32.67077833872641], [6.0, 32.99999999999999], [6.111111111111111, 33.15073072813142], [6.222222222222221, 33.106260860147735], [6.333333333333332, 32.8533150434385], [6.444444444444443, 32.38331452702287], [6.555555555555554, 31.693801276989266], [6.666666666666664, 30.790032007315975], [6.777777777777775, 29.686750217281386], [6.888888888888886, 28.410144326672444], [7.0, 27.0], [7.111111111111111, 25.512054750929813], [7.222222222222221, 24.020562918137273], [7.333333333333332, 22.621079103795168], [7.444444444444443, 21.433468165902845], [7.555555555555554, 20.60514985566499], [7.666666666666664, 20.314586191129408], [7.777777777777775, 20.77501965829173], [7.888888888888886, 22.238470330876012], [8.0, 25.0]]



Hermit Interpolation Polynomial



Charles Hermite(1822-1901)

Hermit interpolation requires and uses derivative data in addition to the function data.

General formula

$H(x) = Q_{0,0}+ Q_{1,1}(x-x_0)+ Q_{2,2}(x-x_0)^2+ Q_{3,3}(x-x_0)^2(x-x_1)+ Q_{4,4}(x-x_0)^2(x-x_1)^2+\ldots+ Q_{2n+1,2n+1}(x-x_0)^2(x-x_1)^2\ldots(x-x_{n-1})^2(x-x_n)$

Calculation of the coefficients is explain at Algorithm 6.4-1 and Table 6.4-1.

Algorithm Hermit interpolation

INPUT $x_0,x_1,\ldots,x_n$ değerleri; $f(x_0), f(x_1), \ldots,f(x_n)$ değerleri; $f'(x_0), f'(x_1), \ldots,f'(x_n)$ değerleri

OUTPUT : $Q_{0,0},Q_{1,1},\ldots., Q_{2n+1,2n+1}$ değerleri

$H(x)= Q_{0,0}+ Q_{1,1}(x-x_0)+ Q_{2,2}(x-x_0)^2+ Q_{3,3}(x-x_0)^2(x-x_1)+ Q_{4,4}(x-x_0)^2(x-x_1)^2+\ldots+ Q_{2n+1,2n+1}(x-x_0)^2(x-x_1)^2\ldots(x-x_{n-1})^2(x-x_n)$

Step 1: for i=0,1,…,n

{

Step 2: $z_{2i}=x_i$;

$z_{2i+1}=x_i$;

$Q_{2i,0}=f(x_i)$;

$Q_{2i+1,0}=f(x_i)$;

$Q_{2i+1,0}=f'(x_i)$;

Step 3: if $i \neq 0$ then $Q_{2i,j} = \frac{Q_{2i,0}-Q_{2i-1,0}}{z_{2i}-z_{2i-1}}$

} End of Step 1 in sonu

Step 4: for i=2,3,…,2n+1

{ for j=2,3,…,i $Q_{i,j} = \frac{Q_{i,j-1}-Q_{i-1,j-1}}{z_i-z_{2i-j}}$ }

Table Hermit interpolation difference formulation

Z	f(z)	First difference equation	Second difference equation
$Z_0 = x_0$	$f[z_0]=f(x_0)$	$f[z_0, z_1] = f(x_0)$	$f[z_0, z_1, z_2] = \frac{f[z_1, z_2]- f[z_0, z_1]}{z_2 - z_0}$
$Z_1 = x_0$	$f[z_1]=f(x_0)$	$f[z_1, z_2] = \frac{f[z_2]- f[z_1]}{z_2 - z_1}$	$f[z_1, z_2, z_3] = \frac{f[z_2, z_3]- f[z_1, z_2]}{z_2 - z_1}$
$Z_2 = x_1$	$f[z_2]=f(x_1)$	$f[z2, z3] = f(x1)$	$f[z_2, z_3, z_4] = \frac{f[z_3, z_4]- f[z_2, z_3]}{z_4 - z_2}$
$Z_3 = x_1$	$f[z_3]=f(x_1)$	$f[z_3, z_4] = \frac{f[z_4]- f[z_3]}{z_4 - z_3}$	$f[z_3, z_4, z_5] = \frac{f[z_4, z_5]- f[z_3, z_4]}{z_5 - z_4}$
$Z_4 = x_2$	$f[z_4]=f(x_2)$	$f[z4, z5] = f(x2)$	
$Z_5 = x_2$	$f[z_5]=f(x_2)$		

An example problem :

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-05698959
2	1.9	0.2818186	-0.5811571

Difference table:

1.3	0.620086					
		-0.5220232				
1.3	0.620086		-0.08974267			
		-0.548946		0.06636556		
1.6	0.4554022		-0.069833		0.00266667	
		-0.5698959		0.06796556		-0.00277469
1.6	0.4554022		-0.02905367		0.00100185	
		-0.578612		0.06856667		
1.9	0.2818186		-0.00848367			
		-0.5811571				
1.9	0.2818186					

Interpolation value at x=1.5:

$H_5(1.5) = 0.620086+(1.5-1.3)*(-0.5220232)+ (1.5-1.3)^2*(1.5-1.6)*(0.06636556+ (1.5-1.3)^2* (1.5-1.6)^2*(0.00266667)+ 1.5-1.3)^2* (1.5-1.6)^2*(1.5-1.9) (-0.00277469)=0.5118277$

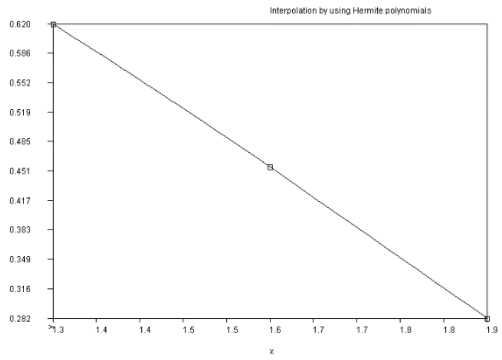
A program carrying out Hermit interpolation is given

Program Hermite polynomial interpolation formula, Java version

```
//Hermit Interpolation
import java.io.*;

class NA123
{
  public static double [] Hermite(double xi[],double fi[],double dfi[])
  {
  //m degree of polynomial
  int i,j;
  int m=xi.length;
  int n=m-1;
  double Q[][]=new double[2*n+2][2*n+2];
  double QQ[]=new double[2*n+2];
  double z[]=new double[2*n+2];
  for(i=0;i<=n;i++)
    {
            z[2*i]=xi[i];
            z[2*i+1]=xi[i];
            Q[2*i][0]=fi[i];
            Q[2*i+1][0]=fi[i];
            Q[2*i+1][1]=dfi[i];
            if(i!=0) { Q[2*i][1]=(Q[2*i][0]-Q[2*i-1][0])/(z[2*i]-z[2*i-1]);}
            }
            for(i=2;i<=(2*n+1);i++)
              {for(j=2;j<=i;j++){Q[i][j]=(Q[i][j-1]-Q[i-1][j-1])/(z[i]-z[i-j]);}
              }
            for(i=0;i<=(2*n+1);i++)
             {QQ[i]=Q[i][i];}
            //System.out.println(QQ.length+"\n"+Matrix.toStringT(QQ));
            return QQ;
  }

  public static double funcHermite(double QQ[],double xi[],double x)
  {
              //Hermit polynomial
  int i,j;
  int m=QQ.length-1;
  double xx=1;
  double toplam=QQ[0];
  for(i=1;i<m;i+=2)
    {   j=(i-1)/2;
              xx*=(x-xi[j]);
              toplam+=QQ[i]*xx;
              xx*=(x-xi[j]);
              toplam+=QQ[i+1]*xx;
    }
  return toplam;
  }

 public static double[][]  funcHermite(double xi[],double yi[],double dyi[],int aradegersayisi)
 {
            //number of midpoints:  x--o--o--x--o--o--x  x data points midpoints are 2
            int n=xi.length;
            int nn=(n-1)*(aradegersayisi+1)+1;
            double z[][]=new double[2][nn];
```

```
    double Q[]=Hermite(xi,yi,dyi);
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
      {  z[0][k]=xi[i];z[1][k]=funcHermite(Q,xi,z[0][k]);k++;
         for(int j=0;j<aradegersayisi;j++)
           {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
            z[0][k]=z[0][k-1]+dx;z[1][k]=funcHermite(Q,xi,z[0][k]);k++;
           }
      }
      z[0][k]=xi[i];z[1][k]=funcHermite(Q,xi,z[0][k]);
    return z;
  }

  public static void main(String args[]) throws IOException
  {
  double x[]={1.3,1.6,1.9};
  double y[]={0.620086,0.4554022,0.281818186};
  double dy[]={-0.5220232,-0.5698959,-0.5811571};
  double z1[][]=funcHermite(x,y,dy,8);
  Text.print(Text.T(z1),"Interpolation by using Hermite polynomials");
  System.out.println("Hermite polynomial interpolation\n"+Matrix.toStringT(z1));
  Plot pp=new Plot(x,y);
  pp.setPlabel("Interpolation by using Hermite polynomials");
  pp.setPlotType(0,20);
  pp.addData(z1[0],z1[1]);
  pp.plot();

  }
}
```

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" NA123
Hermite polynomial interpolation
   1.3000000000   0.6200860000
   1.3333333333   0.6025660584
   1.3666666667   0.5848174157
   1.4000000000   0.5668545838
   1.4333333333   0.5486921534
   1.4666666667   0.5303447937
   1.5000000000   0.5118272527
   1.5333333333   0.4931543568
   1.5666666667   0.4743410113
   1.6000000000   0.4554022000
   1.6333333333   0.4363529853
   1.6666666667   0.4172085082
   1.7000000000   0.3979839884
   1.7333333333   0.3786947244
   1.7666666667   0.3593560931
   1.8000000000   0.3399835500
   1.8333333333   0.3205926294
   1.8666666667   0.3011989443
   1.9000000000   0.2818181860
```



VANDERMONDE INTERPOLATION



Nicolaas van der Monde, Dutch

If the interpolating polynomial is written as

$$\boldsymbol{y = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N} \quad \textbf{(6.5-1)}$$

then the C_i's are required to satisfy the linear equation

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^2 \\ 1 & x_N & x_N^2 & \cdots & x_N^2 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{N-1} \\ c_N \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_1 \\ \cdots \\ y_{N-1} \\ y_N \end{Bmatrix} \quad (6.5\text{-}2)$$

This is a *Vandermonde matrix.* The matrix can be solved by using some of the Matrix solving method. In The next program, it is solved by using pivoted Gauss eliminaton method. Remember that Vandermonde systems can be quite ill-conditioned. In such a case, *no* numerical method is going to give a very accurate answer. So the method should be used for relatively small number of data points.

Program Vandermonde matrix interpolation with Gauss elimination solution , Java version

```
import java.io.*;
import java.util.*;
import javax.swing.*;

class Vandermonde
{
//Vandermonde interpolation
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
```

```
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
              //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
              }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}


public static double[] Vandermonde_coef(double xi[],double yi[])
{
int n=xi.length;
fa f=new fa();
int l=xi.length;
int i,j,k;
double A[][];
A=new double[n][n];
double B[];
B=new double[n];
double X[];
X=new double[n];
double cc=1.0;
for(i=0;i<n;i++)
  {  B[i]=yi[i];
     cc=1.0;
              for(j=0;j<n;j++)
     {
              A[i][j]=cc;
              cc*=xi[i];
     }
  }
X=gausswithpartialpivot(A,B);
return X;
}
```

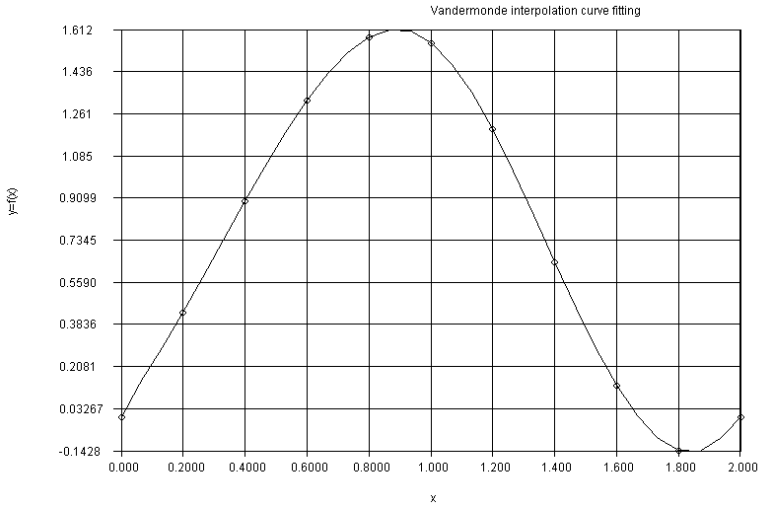
```
public static double func(double c[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=c.length;
double ff=0;
double xx=1.0;
if(n!=0.0)
  { xx=1.0;
  for(int i=0;i<n;i++)
   {ff+=c[i]*xx;
    xx*=x;
   }
  }
return ff;
}

public static double[][]  Vandermonde(double xi[],double yi[],int aradegersayisi)
 {
            //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
            // ara değer sayısı 2 dir
            int n=xi.length;
            int nn=(n-1)*(aradegersayisi+1)+1;
            double z[][]=new double[2][nn];
   double E[]=Vandermonde_coef(xi,yi);
   System.out.println("coefficients :\n"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=func(E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=func(E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=func(E,z[0][k]);
   return z;
 }

  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1=JOptionPane.showInputDialog("Data file name (input1.txt)");
  // JFileChooser fc=new JFileChooser();
  // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  double z[][]=Vandermonde(x,y,10);
  System.out.println("Vandermonde interpolation curve fitting\n"+Matrix.toStringT(z));
  Plot pp=new Plot(a[0],a[1]);
  pp.setPlabel("Vandermonde interpolation curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y=f(x)");
  pp.setPlotType(0,22);
  pp.addData(z[0],z[1]);
  pp.setPlotType(1,0);
  pp.setGrid(1,1);
  pp.plot();
  }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" Vandermonde
coefficients  :
```

0.000000000000000000000000
2.522238688245352000000000
-6.49968744938102400000000000
43.21953701155625500000000000
-139.17792182712440000000000000
260.28892984853250000000000000
-302.86673648874705000000000000
218.09728893170060000000000000
-94.06798950652913000000000000
22.26860946298418400000000000
-2.22686094623513360000000000

Vandermonde interpolation curve fitting

0.0000000000	0.0000000000
0.0666666667	0.1496357599
0.1333333333	0.2886288414
0.2000000000	0.4340028520
0.2666666667	0.5869147066
0.3333333333	0.7432097976
0.4000000000	0.8981438220
0.4666666667	1.0477593840
0.5333333333	1.1886496189
0.6000000000	1.3172323490
0.6666666667	1.4291826777
0.7333333333	1.5193154343
0.8000000000	1.5819574910
0.8666666667	1.6116896704
0.9333333333	1.6042547278
1.0000000000	1.5574077250
1.0666666667	1.4715139871
1.1333333333	1.3497637445
1.2000000000	1.1979574910
1.2666666667	1.0239080269
1.3333333333	0.8365900852
1.4000000000	0.6452323490
1.4666666667	0.4585755448
1.5333333333	0.2845001247
1.6000000000	0.1301438220
1.6666666667	0.0024690569
1.7333333333	-0.0910112194
1.8000000000	-0.1419971480
1.8666666667	-0.1427785660
1.9333333333	-0.0909568327
2.0000000000	-0.0000000000



The Vandermonde matrix can easily be inverted in terms of Lagrange basis polynomials: each *column* is the coefficients of the Lagrange basis polynomial, with terms in increasing order going down. The resulting solution to the interpolation problem is called the Lagrange polynomial. Remembering lagrange polynomial has the form

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_i}{x_i - x_j} = \sum_{k=0}^{N} A_{ik} x^k \quad (6.5-3)$$

Here the meaning of the last equality is to define the components of the matrix A_{ij} as the coefficients that arise when the product is multiplied out and like terms collected. The polynomial $P_j(x)$ is a function of x generally. But you will notice that it is specifically designed so that it takes on a value of zero at all x_i with $i \neq j$, and has a value of unity at $x = x_j$. In other words

$$L_i(x) = \delta_{ji} = \sum_{k=0}^{N} A_{ik} x^k$$

with

$$\delta_{ji} = \begin{cases} 0 \ if \ j \neq i \\ 1 \ if \ j = i \end{cases}$$

A_{ik} is exactly the inverse of the matrix of components x_i^k , which appears in Vandermonde Matrix, so

$$c_i = \sum_{k=0}^{N} A_{ki} x^k$$

A second version of the Vandermonde interpolation prosess is given as the next code, utilising Lagrange interpolation solution of the Matrix, instead of Gauss elimination

Program Vandermonde matrix interpolation with Lagrange interpolation formula solution

```
import java.io.*;
import java.util.*;
import javax.swing.*;

class Vandermonde1
{
//Vandermonde interpolation
public static double[] Vandermonde_coef(double x[], double y[])
{ // Based on the Lagrange interpolation method to find coefficient of
  // Vandermonde interpolation MAtrix
int n=x.length-1;
double cof[]=new double[n+1];
int k,j,i;
double phi,ff,b;
```

```
double s[]=new double[n+1];
s[n] = -x[0];
for (i=1;i<=n;i++) {
for (j=n-i;j<=(n-1);j++)
s[j] -= x[i]*s[j+1];
s[n] -= x[i];
}
for (j=0;j<=n;j++) {
phi=n+1;
for (k=n;k>=1;k--)
phi=k*s[k]+x[j]*phi;
ff=y[j]/phi;
b=1.0;
for (k=n;k>=0;k--)
{
cof[k] += b*ff;
b=s[k]+x[j]*b;
}
}
return cof;
}

public static double func(double c[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=c.length;
double ff=0;
double xx=1.0;
if(n!=0.0)
  { xx=1.0;
  for(int i=0;i<n;i++)
   {ff+=c[i]*xx;
    xx*=x;
   }
  }
return ff;
}

public static double[][]  Vandermonde(double xi[],double yi[],int aradegersayisi)
 {
              //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
              // ara değer sayısı 2 dir
              int n=xi.length;
              int nn=(n-1)*(aradegersayisi+1)+1;
              double z[][]=new double[2][nn];
    double E[]=Vandermonde_coef(xi,yi);
    System.out.println("katsayılar :\n"+Matrix.toStringT(E));
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
       {z[0][k]=xi[i];z[1][k]=func(E,z[0][k]);k++;
       for(int j=0;j<aradegersayisi;j++)
       {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=func(E,z[0][k]);k++;}
       }
       z[0][k]=xi[i];z[1][k]=func(E,z[0][k]);
    return z;
 }

  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1=JOptionPane.showInputDialog("Data file name (input1.txt)");
  // JFileChooser fc=new JFileChooser();
```

```
  // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  double z[][]=Vandermonde(x,y,2);
  System.out.println("Vandermonde interpolation curve fitting\n"+Matrix.toStringT(z));
  Plot pp=new Plot(a[0],a[1]);
  pp.setPlabel("Vandermonde interpolation curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y=f(x)");
  pp.setPlotType(0,22);
  pp.addData(z[0],z[1]);
  pp.setPlotType(1,0);
  pp.setGrid(1,1);
  pp.plot();
  }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" Vandermonde1
coefficients :
      -0.000000000087414483939 1412
       2.522239007660724400000 0000
      -6.499691833144510000000 0000
      43.219561040455530000000 0000
    -139.177992276292000000000 00000
     260.289053283083700000000 00000
    -302.866871907771950000000 00000
     218.097382711548050000000 00000
     -94.068029296310610000000 00000
      22.268618899567628000000 00000
      -2.226861903662786400000 0000

Vandermonde interpolation curve fitting
  0.0000000000  -0.0000000001
  0.0666666667   0.1496357675
  0.1333333333   0.2886288452
  0.2000000000   0.4340028519
  0.2666666667   0.5869147051
  0.3333333333   0.7432097966
  0.4000000000   0.8981438219
  0.4666666667   1.0477593843
  0.5333333333   1.1886496191
  0.6000000000   1.3172323488
  0.6666666667   1.4291826772
  0.7333333333   1.5193154337
  0.8000000000   1.5819574906
  0.8666666667   1.6116896704
  0.9333333333   1.6042547280
  1.0000000000   1.5574077250
  1.0666666667   1.4715139868
  1.1333333333   1.3497637436
  1.2000000000   1.1979574897
  1.2666666667   1.0239080255
  1.3333333333   0.8365900841
  1.4000000000   0.6452323486
  1.4666666667   0.4585755447
  1.5333333333   0.2845001244
  1.6000000000   0.1301438204
  1.6666666667   0.0024690536
  1.7333333333  -0.0910112234
  1.8000000000  -0.1419971501
  1.8666666667  -0.1427785627
  1.9333333333  -0.0909568244
  2.0000000000  -0.0000000025
```

The same data is entered as the previous version. Note the small differences in the coefficient vector and interpolation values.

CUBIC SPLINE INTERPOLATION POLYNOMIALS

Another way for the interpolation process is to fit a different polynomial in between each two point. If a third degree polinomial is considered:

$$r_k(x) = a_k(x - x_k)^3 + b_k(x - x_k)^2 + c_k(x - x_k)^3 + y_k \quad 1 \le k \le n \quad (6.6-1)$$

In the interpolation proses polinoms should be passing through all data points

$$r_k(x_{k+1}) = y_{k+1} \quad 1 \le k \le n \quad (6.6-2)$$

In the same time the first derivative of the polynomial should also be continious while passing from one polynomial to the next one at the data point

$$r'_{k-1}(x_k) = r'_k(x_k) \quad 1 \le k \le n \quad (6.6-3)$$

For the third degree polinomial second derivative of the polynomial should also be continious while passing from one polynomial to the next one at the data point

$$r''_{k-1}(x_k) = r''_k(x_k) \quad 1 \le k \le n \quad (6.6-4)$$

All these conditions are not enough to solve the coefficients of the polinomials. Two more conditions are required. This two additional conditions (A and B of the following equation) can be given by user

$r''_1(x_1) = A \qquad r''_{n-1}(x_n) = B$ (6.6-5)

They are the second derivatives at the both hand of the series of polinomials. If A and B values are taken equals to 0, it is called a **natural cubic spline**. Other end conditions such as the ones depends one the first derivatives can also be set to solve the system of equations.

Defining $h_k = x_{k+1} - x_k \quad 1 \le k \le n \quad (6.6-5)$

System of equations become:

$$a_k h_k^3 + b_k h_k^2 + b_k h_k = y_{k+1} - y_k \quad 1 \le k \le n \quad (6.6-6)$$
$$3a_{k-1}h_{k-1}^2 + 2b_{k-1}h_{k-1} + c_{k-1} - c_k = 0$$
$$6a_{k-1}h_{k-1} + 2b_{k-1} + 2b_k = 0$$
$$3b_0 = 0$$
$$6a_{n-1}h_{n-1} + 2b_{n-1} = 0$$

This set contains 3n-3 equations. This could a considerable load to the system of equation solving programs. To make calculation load simpler a special third degree polinomial can be considered. If our cubic polinomial is in the form of:

$$h_k = x_{k+1} - x_k \quad 1 \le k \le n$$

$$s_k(x) = a_k(x - x_k) + b_k(x_{k+1} - x) + [(x - x_k)^3 c_{k+1} + (x_{k+1} - x)^3 c_k]/(6h_k) \quad 1 \le k \le n \quad (6.6-7)$$

then derivative equations becomes

$$s'_k(x) = a_k - b_k + [(x - x_k)^2 c_{k+1} - (x_{k+1} - x)^2 c_k]/h_k \quad 1 \le k \le n \quad (6.6\text{-}8)$$
$$s''_k(x) = [(x - x_k)c_{k+1} - (x_{k+1} - x)c_k]/h_k \quad 1 \le k \le n$$

a_k ve b_k coefficients can be expressed as a function of c_k

$$b_k = \frac{[6y_k - h_k c_k]}{6h_k} \qquad 1 \le k \le n \qquad (6.6-9)$$

$$a_k = \frac{[6y_{k+1} - h_k^2 c_{k+1}]}{6h_k} \qquad 1 \le k \le n \qquad (6.6-10)$$

In this case only c_k terms left in the system of equations to be solved.

$$h_{k-1}c_{k-1} + 2(h_{k-1} - h_k)c_{k-1} + h_k c_{k+1} c_{k+1} = 6\left[\frac{y_{k+1} - y_k}{h_k} - \frac{y_k - y_{k-1}}{h_{k-1}}\right] \qquad 1 \le k \le n \qquad (6.6-10)$$

This system of equation has only n-2 terms to be solved. By making definition

$w_k = \frac{y_{k+1} - y_k}{h_k} = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$, $\quad 1 \le k \le n$ (6.6-11)

System of equation becomes

$$\begin{bmatrix} 1 & & & & & & \\ h_1 & 2(h_1 + h_2) & h_2 & & & & \\ & h_2 & 2(h_2 + h_3) & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & 2(h_{n-3} + h_{n-2}) & h_{n-2} & \\ & & & & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ & & & & & & 1 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdots \\ c_{n-2} \\ c_{n-1} \\ c_n \end{Bmatrix} = \begin{Bmatrix} A \\ 6(w_2 - w_1) \\ 6(w_3 - w_2) \\ \cdots \\ 6(w_{n-2} - w_{n-3}) \\ 6(w_{n-1} - w_{n-2}) \\ B \end{Bmatrix} \quad (6.6-12)$$

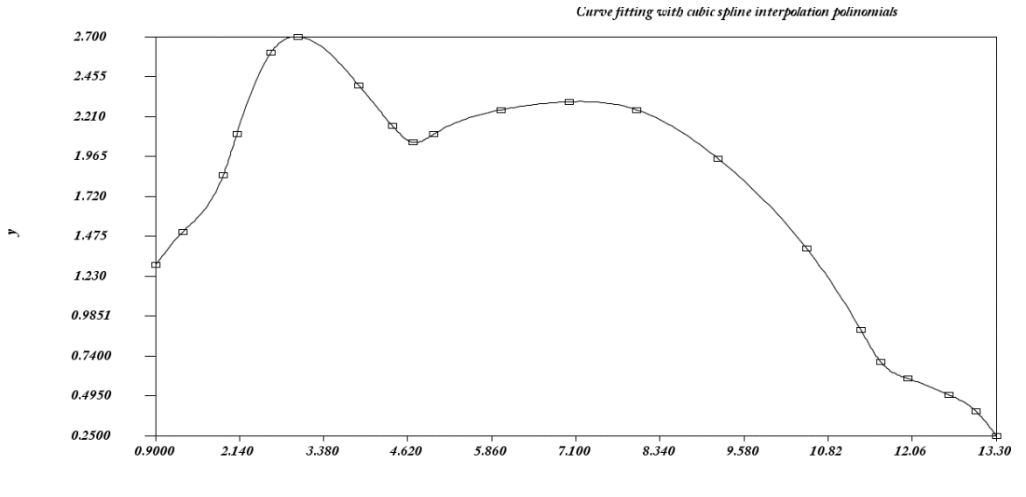
Where A and B are the second derivative end conditions. A and B should be defined by user. Another important property of the above matrix is that it is a band matrix, therefore less amount of calculation is required to solve it (by using band matrix algorithms such as Thomas algorithm) compare to gauss elimination type of solution methods.

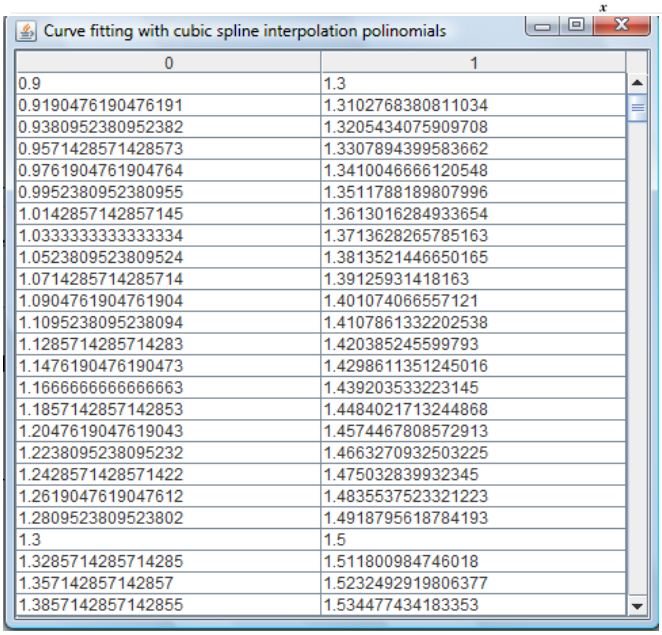
Program Cubic spline curve fitting

```
import java.io.*;

class NA49
{

 public static double [] thomas(double f[],double e[],double g[],double r[])
 {
 // 3 band matrix system of equation solving algorithm
 int n=f.length;
 double x[]=new double[n];
 for(int k=1;k<n;k++)
   {e[k]=e[k]/f[k-1];
    f[k]=f[k]-e[k]*g[k-1];
   }
 for(int k=1;k<n;k++)
   {r[k]=r[k]-e[k]*r[k-1];
   }
 x[n-1]=r[n-1]/f[n-1];
 for(int k=(n-2);k>=0;k--)
   {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
 return x;
 }


 public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
 {
 int n=xi.length;
 double h[]=new double[n];
 double w[]=new double[n];
 double f[]=new double[n];
 double e[]=new double[n];
 double g[]=new double[n];
 double d[]=new double[n];
 double x[]=new double[n];
 double S[][]=new double[4][n];
 int k;
 for(k=0;k<(n-1);k++)
   {h[k]=xi[k+1]-xi[k];
    w[k]=(yi[k+1]-yi[k])/h[k];
          }
 d[0]=c0;
 d[n-1]=cn;
 for(k=1;k<(n-1);k++)
   {d[k]=6.0*(w[k]-w[k-1]);}
 f[0]=1.0;
 f[n-1]=1.0;
 g[0]=0.0;
 g[n-1]=0.0;
 e[0]=0.0;
 e[n-1]=0.0;
 for(k=1;k<(n-1);k++)
 {f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
 S[2]=thomas(f,e,g,d);
 S[3]=xi;
 for(k=0;k<(n-1);k++)
  {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
```

```
    S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
   }
 return S;
 }

 public static double funcSpline(double S[][],double x)
 {
 int n=S[0].length;
 double xx1=0;
 double xx2=0;
 double y=0;
 double hk=0;
 for(int k=0;k<(n-1);k++)
    {if(S[3][k]<=x && x<=S[3][k+1])
     {hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      break;
      }
    }
      if(y==0 && S[3][n-2]<=x )
      {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      }
 return y;
 }

public static double[][]  funcSpline(double xi[],double yi[],int numberofmidpoints)
{
             //numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
             // numberofmidpoints is 2
             int n=xi.length;
             int nn=(n-1)*(numberofmidpoints+1)+1;
             double z[][]=new double[2][nn];
   double S[][]=cubic_spline(xi,yi,0,0);
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {  z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k++;
         for(int j=0;j<numberofmidpoints;j++)
            {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
         z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);
   return z;
}
 public static void main(String args[]) throws IOException
 {double x[]={0.9,1.3,1.9,2.1,2.6,3.0,3.9,4.4,4.7,5.0,6.0,7.0,8.0,9.2,10.5,11.3,11.6,12.0,12.6,13.0,13.3};
 double y[]={1.3,1.5,1.85,2.1,2.6,2.7,2.4,2.15,2.05,2.1,2.25,2.3,2.25,1.95,1.4,0.9,0.7,0.6,0.5,0.4,0.25};
 double z3[][]=funcSpline(x,y,20);
 System.out.println("Cubic spline \n"+Matrix.toStringT(z3));
 Text.print(Text.T(z3),"Curve fitting with cubic spline interpolation polinomials");
 Plot pp=new Plot(x,y);
 pp.setPlabel("Curve fitting with cubic spline interpolation polinomials");
 pp.setPlotType(0,20);
 pp.addData(z3[0],z3[1]);
```

```
pp.plot();
}}
```





0	1
0.9	1.3
0.9190476190476191	1.3102768380811034
0.9380952380952382	1.3205434075909708
0.9571428571428573	1.3307894399583662
0.9761904761904764	1.3410046666120548
0.9952380952380955	1.3511788189807996
1.0142857142857145	1.3613016284933654
1.0333333333333334	1.3713628265785163
1.0523809523809524	1.3813521446650165
1.0714285714285714	1.39125931418163
1.0904761904761904	1.401074066557121
1.1095238095238094	1.4107861332202538
1.1285714285714283	1.420385245599793
1.1476190476190473	1.4298611351245016
1.1666666666666663	1.439203533223145
1.1857142857142853	1.4484021713244868
1.2047619047619043	1.4574467808572913
1.2238095238095232	1.4663270932503225
1.2428571428571422	1.475032839932345
1.2619047619047612	1.4835537523321223
1.2809523809523802	1.4918795618784193
1.3	1.5
1.3285714285714285	1.511800984746018
1.357142857142857	1.5232492919806377
1.3857142857142855	1.534477434183353

Program Cubic spline curve fitting (python version)

```
# -*- coding: utf-8 -*-
"""
Created on Sun Jun  3 21:38:38 2018

@author: Mustafa Turhan Çoban
"""
from math import *
import numpy as np;
import matplotlib.pyplot as plt;

def thomas(f,e,g,r):
    n=len(f);
    x=[0.0 for i in range(n)];
    for k in range(0,n): e[k]=e[k]/f[k-1];f[k]=f[k]-e[k]*g[k-1];
    for k in range(1,n): r[k]=r[k]-e[k]*r[k-1];
    x[n-1]=r[n-1]/f[n-1];
    for k in range(n-2,-1,-1):  x[k]=(r[k]-g[k]*x[k+1])/f[k];
    return x;

def cubic_spline(xi,yi,c0,cn):
    n=len(xi);
    h=[0.0 for i in range(n)];
    w=[0.0 for i in range(n)];
    f=[0.0 for i in range(n)];
```

```
    e=[0.0 for i in range(n)];
    g=[0.0 for i in range(n)];
    d=[0.0 for i in range(n)];
    x=[0.0 for i in range(n)];
    S= [[0 for x in range(n)] for y in range(4)];
    for k in range(0,n-1):
        h[k]=xi[k+1]-xi[k];
        w[k]=(yi[k+1]-yi[k])/h[k];
    d[0]=c0;
    d[n-1]=cn;
    for k in range(0,n-1):
       d[k]=6.0*(w[k]-w[k-1]);
    f[0]=1.0;
    f[n-1]=1.0;
    g[0]=0.0;
    g[n-1]=0.0;
    e[0]=0.0;
    e[n-1]=0.0;
    for k in range(0,n-1):
        f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];
    S[2]=thomas(f,e,g,d);
    S[3]=xi;
    for k in range(0,n-1):
       S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
       S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
    return S;

def funcSpline(S,x):
    n=len(S[0]);
    xx1=0;
    xx2=0;
    y=0;
    hk=0;
    for k in range(0,n-1):
       if S[3][k]<=x and x<=S[3][k+1]:
            hk=(S[3][k+1]-S[3][k]);
            xx1=(x-S[3][k]);
            xx2=(S[3][k+1]-x);
            y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
            break;
    if y==0 and S[3][n-2]<=x:
         k=n-2;
         hk=(S[3][k+1]-S[3][k]);
         xx1=(x-S[3][k]);
         xx2=(S[3][k+1]-x);
         y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
    return y;

def listSpline(xi,yi,numberofmidpoints,c0,cn):
    #numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
                #numberofmidpoints is 2
    n=len(xi);
    nn=(n-1)*(numberofmidpoints+1)+1;
    z=[[0 for x in range(nn)] for y in range(2)];
    S= [[0 for x in range(nn)] for y in range(4)];
    S=cubic_spline(xi,yi,c0,cn);
    dx=0;
    k=0;
    for i in range(0,n-1):
       z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k=k+1;
       for j in range(numberofmidpoints):
          dx=(xi[i+1]-xi[i])/(numberofmidpoints+1.0);
          z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k=k+1;
    z[0][k]=xi[i+1];z[1][k]=funcSpline(S,z[0][k]);
    return z;

x=[0.9,1.3,1.9,2.1,2.6,3.0,3.9,4.4,4.7,5.0,6.0,7.0,8.0,9.2,10.5,11.3,11.6,12.0,12.6,13.0,13.3];
y=[1.3,1.5,1.85,2.1,2.6,2.7,2.4,2.15,2.05,2.1,2.25,2.3,2.25,1.95,1.4,0.9,0.7,0.6,0.5,0.4,0.25];
Z=listSpline(x,y,5,0,0);
print("Z=",Z);
plt.title('Cubic Spline')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'bo',Z[0],Z[1],'k');
```

```
runfile('D:/okul/SCO1/cubic_spline.py', wdir='D:/okul/SCO1')
Reloaded modules: gauss
Z= [[0.9, 0.9666666666666667, 1.0333333333333334, 1.1, 1.1666666666666667, 1.2333333333333334, 1.3, 1.4, 1.5, 1.6,
1.7000000000000002, 1.8000000000000003, 1.9, 1.9333333333333333, 1.9666666666666668, 2.0, 2.033333333333333,
2.0666666666666664, 2.1, 2.1333333333333336, 2.166666666666667, 2.3500000000000005, 2.433333333333334,
2.5166666666666675, 2.6, 2.666666666666667, 2.7333333333333334, 2.8, 2.8666666666666663, 2.9333333333333327, 3.0, 3.15, 3.3,
3.4499999999999997, 3.5999999999999996, 3.7499999999999996, 3.9, 3.9833333333333334, 4.066666666666666,
4.1499999999999995, 4.2333333333333325, 4.3166666666666655, 4.4, 4.45, 4.5, 4.55, 4.6, 4.6499999999999995, 4.7, 4.75, 4.8, 4.85,
4.8999999999999995, 4.949999999999999, 5.0, 5.166666666666667, 5.333333333333334, 5.500000000000001, 5.666666666666668,
5.833333333333335, 6.0, 6.166666666666667, 6.333333333333334, 6.500000000000001, 6.666666666666668, 6.833333333333335,
7.0, 7.166666666666667, 7.333333333333334, 7.500000000000001, 7.666666666666668, 7.833333333333335, 8.0, 8.2,
8.399999999999999, 8.599999999999998, 8.799999999999997, 8.999999999999996, 9.2, 9.416666666666666, 9.633333333333333,
9.85, 10.066666666666666, 10.283333333333333, 10.5, 10.633333333333333, 10.766666666666666, 10.899999999999999,
11.033333333333331, 11.166666666666664, 11.3, 11.350000000000001, 11.400000000000002, 11.450000000000003,
11.500000000000004, 11.550000000000004, 11.6, 11.666666666666666, 11.733333333333333, 11.799999999999999,
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2.0494053210004255, 2.0539631233699773, 2.062487449781682, 2.073792342908569, 2.086691845423666, 2.1,
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1.7939415032693384, 1.7068493242653617, 1.61267951764686, 1.510655328022263, 1.4, 1.3272597895357943,
1.2506677564782636, 1.169934830911663, 1.0847719429202491, 0.9948900225882772, 0.9000000000000001, 0.8632203068251879,
0.8266057525736611, 0.7910663910053992, 0.7575122758803815, 0.7268534609585878, 0.6999999999999998, 0.6710795558939062,
0.6490500148136773, 0.632419332721743, 0.6196954655805323, 0.6093863693524749, 0.6, 0.585262369387163,
0.5694620676711136, 0.5528173873631121, 0.5355466209744191, 0.517868061016295, 0.5, 0.48794696771124646,
0.47511033878706393, 0.46069372039065026, 0.4439007196852034, 0.42393494383392094, 0.4000000000000001,
0.3790954038292061, 0.3559569510242995, 0.33103141291210286, 0.30476556081943873, 0.27760616607312966, 0.25]]
```
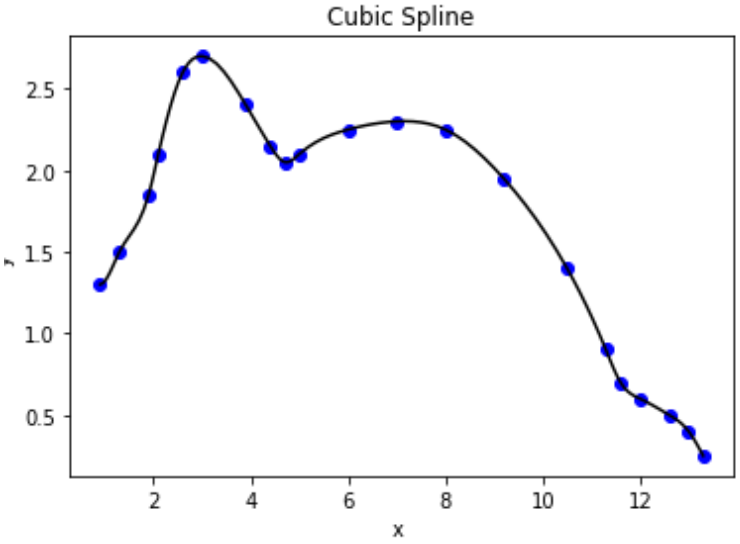



We can now investigate two dimensional cubic spline. Basically problem can be considered as spline of spline. If data is placed in a rectangular manner, data can be applied to create all the x axis data as one dimesional cubic spline curve fitting and then all created intermediate values can be used in a second cubic spline solution to find final interpolation value.
If x_i,y_i,z_i data set is given to find $z = f(x,y)$ first step is: to create $w_j = f(x)$ and then by using w_i,z_i find $z = f(x,y)$
Program for this is given below as two basic classes CS1 and CS2

Program Cubic spline surface fitting first step CS1.java

```
//Cubic spline curve fitting
import java.io.*;
import javax.swing.*;

class CS1
{  public double S[][]; // cubic spline coefficients
```

```
 public double c[][]; // data matrix
 public int nx,ny;    // x and y data column numbers
 public int kk;       // equation number that given x located
 public CS1(String name,int nxi,int nyi,double c0,double cn)
 {nx=nxi;ny=nyi;
  c=Text.readDoubleT(name);
  double x[]=c[nx];double y[]=c[ny];
  S=cubic_spline(x,y,c0,cn);
 }
 public CS1(double a[][],int nxi,int nyi,double c0,double cn)
 {c=a;nx=nxi;ny=nyi;
  double x[]=c[nx];double y[]=c[ny];
  S=cubic_spline(x,y,c0,cn);
 }
 public CS1(double xi[],double yi[],double c0,double cn)
 {double c1[][]={xi,yi};c=c1;S=cubic_spline(xi,yi,c0,cn);}
public static double [] thomas(double f[],double e[],double g[],double r[])
{
// Thomas algorithm
// 3 band matrix system of equation solving algorithm
int n=f.length;
double x[]=new double[n];
for(int k=1;k<n;k++)
   {e[k]=e[k]/f[k-1];
    f[k]=f[k]-e[k]*g[k-1];
   }
for(int k=1;k<n;k++)
   {r[k]=r[k]-e[k]*r[k-1];
   }
x[n-1]=r[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
   {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
return x;
}

public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
{
int n=xi.length;
double h[]=new double[n];
double w[]=new double[n];
double f[]=new double[n];
double e[]=new double[n];
double g[]=new double[n];
double d[]=new double[n];
double x[]=new double[n];
double S[][]=new double[4][n];
int k;
for(k=0;k<(n-1);k++)
   {h[k]=xi[k+1]-xi[k];
    w[k]=(yi[k+1]-yi[k])/h[k];
          }
d[0]=c0;
d[n-1]=cn;
for(k=1;k<(n-1);k++)
   {d[k]=6.0*(w[k]-w[k-1]);}
f[0]=1.0;
f[n-1]=1.0;
g[0]=0.0;
g[n-1]=0.0;
e[0]=0.0;
e[n-1]=0.0;
for(k=1;k<(n-1);k++)
{f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
S[2]=thomas(f,e,g,d);
S[3]=xi;
for(k=0;k<(n-1);k++)
  {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
   S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
  }
return S;
}

public double func(double x)
{
int n=S[0].length;
double xx1=0;
```

```
  double xx2=0;
  double y=0;
  double hk=0;
  int k=0;
  for(k=0;k<(n-1);k++)
     {if(S[3][k]<=x && x<=S[3][k+1])
      {hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       break;
       }
     }

       if(y==0 && S[3][n-2]<=x )
       {
                  k=n-2;
                  hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       }
     kk=k+1;
  return y;
  }

  // Türev formülleri
// =================


  public double dfunc(double x)
  {
  int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y=0;
  double hk=0;
  for(int k=0;k<(n-1);k++)
     {if(S[3][k]<=x && x<=S[3][k+1])
      {hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]-S[1][k]+(xx1*xx1*S[2][k+1]-xx2*xx2*S[2][k])/(2.0*hk);
       break;
       }
     }
       if(y==0 && S[3][n-2]<=x )
       {
                  int k=n-2;
                  hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       }
  return y;
  }

  public double integral(double a,double b)
  {
    //kübik şerit integral[S(x)]dx  formülü
  int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y1,y2;
  double hk=0;
  double toplam=0;
  y1=0;y2=0;
  for(int k=0;k<(n-1);k++)
     { hk=(S[3][k+1]-S[3][k]);
      if(a>S[3][k+1])
      {toplam=0;
      }
                 else if(S[3][k]<=a && a<=S[3][k+1] && b>S[3][k+1])//şart 2
       {
       xx1=(a-S[3][k]);
       xx2=(S[3][k+1]-a);
```

```
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=hk;
      xx2=0;
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      }
      else if(S[3][k]<=a && a<=S[3][k+1] && S[3][k]<=b && b<=S[3][k+1])//şart 3
      {
      xx1=(a-S[3][k]);
      xx2=(S[3][k+1]-a);
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=(b-S[3][k]);
      xx2=(S[3][k+1]-b);
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      }
      else if(a<S[3][k] && b>=S[3][k+1]) //şart 4
      {
      xx1=0;
      xx2=hk;
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=hk;
      xx2=0;
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      }
      else if(S[3][k]<=b && b<=S[3][k+1])//şart 5
      {xx1=0;
       xx2=hk;
       y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
       xx1=(b-S[3][k]);
       xx2=(S[3][k+1]-b);
       y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
       toplam+=(y2-y1);
       }
       else break;
    }
  return toplam;
  }


public double golden(double a,double b)
{
//  find the minimum of the function
//  note  maximum f(x) = minimum (-f(x))
    double epsilon=1.0e-10;
    double delta=1.0e-8;
    int print=0;
    double r1 = (Math.sqrt(5.0)-1.0)/2.0; // golden ratio
    double r2 = r1*r1;
    double h = b - a;
    double ya = -func(a);
    double yb = -func(b);
    double c = a + r2*h;
    double d = a + r1*h;
    double yc = -func(c);
    double yd = -func(d);
    int k = 1;
    double dp,dy,p,yp;
    while ((Math.abs(yb-ya)>epsilon) || (h>delta))
    {
      k++;
      if (yc<yd)
        {
        b = d;
        yb = yd;
        d = c;
        yd = yc;
        h = b - a;
        c = a + r2 * h;
        yc = -func(c);
        }
      else
       {
         a = c;
         ya = yc;
```

```
        c = d;
        yc = yd;
        h = b - a;
        d = a + r1 * h;
        yd = -func(d);
       }//end of if
    }//end of while
    dp = Math.abs(b-a);
    dy = Math.abs(yb-ya);
    p = a;
    yp = ya;
    if (yb<ya)
    {
      p = b;
      yp = yb;
    }
    if(print==1)
    {System.out.println("x min = "+p+"ymin = "+yp+"error of x ="+dp+"error of y"+dy); }
    return p;
}
public static void  printData(double c[][])
  {       String s3[]={"x[i]","y[i]"};
       Text.print(Text.T(c),s3,"Input data");
  }

  public static void main(String args[]) throws IOException
  { CS1 cs=new CS1("a100.txt",0,2,0.0,0.0);
    double T=300.0;
    double ro=cs.func(T);
    System.out.println("T="+T+"ro="+ro);
  }
}
```

Program Cubic spline surface fitting second step CS2.java

```
  //3D Cubic spline surface fitting
import java.io.*;
import javax.swing.*;

class CS2
{  public double S[][];
   String name[];
   public CS1 cs[];
   public int n;
   double xx[],yy[];
   public double c0,cn;
   public int nyy;
   public int nx,ny;
   public double d[][][];
   //use set of files each with a constant property

   // namei: name of datafile to read data
   // use one file with a repeated column nselect
   // c: data matrix with a repeated one column
   // nyi y selection column
   // nxi  x selection column
   // nzi  z selection column
   // c0x and cnx x axis cubic spline end conditions (second derivatives)
   // c0y and cny y axis cubic spline end conditions (second derivatives)
   public CS2(String namei,int nxi,int nyi,int nzi,double c0x,double cnx,double c0y,double cny)
   { double c[][]=Text.readDouble(namei);
     set(c,nxi,nyi,nzi,c0x,cnx,c0y,cny);
   }
   public CS2(double c[][],int nxi,int nyi,int nzi,double c0x,double cnx,double c0y,double cny)
   {  set(c,nxi,nyi,nzi,c0x,cnx,c0y,cny);}

   // use one file with a repeated column nselect
   // nyi y selection column
   // nxi  x selection column
   // nzi  z selection column
   // c0x and cnx x axis cubic spline end conditions (second derivatives)
   // c0y and cny y axis cubic spline end conditions (second derivatives)

   public void set(double c[][],int nxi,int nyi,int nzi,double c0x,double cnx,double c0y,double cny)
   { nyy=nyi;
     nx=nxi;
```

```
   ny=nzi;
   c0=c0y;
   cn=cny;
   d=readDouble(c,nyy);
   n=d.length;
   cs=new CS1[n];
   xx=new double[n];
   yy=new double[n];
   for(int i=0;i<n;i++)
   {   cs[i]=new CS1(Text.T(d[i]),nx,ny,c0x,cnx);}
 }

public static double[][][] readDouble(String name,int nc)
{double c[][]=Text.readDouble(name);
 int n=c.length;
 int m=c[0].length;
 double dd[][][]=new double[1000][n][m];
 int k=0;
 int i=0;
 int j=0;
 int nj[]=new int[100];
 for(i=0;i<n-1;i++)
 {if(c[i][nc]==c[i+1][nc]) {dd[k][j]=c[i];j++;}
  else {dd[k][j]=c[i];nj[k]=j;j=0;
        k++;
       }
 }
 double ee[][][]=new double[k][][];
 for(i=0;i<k;i++)
 {   ee[i]=new double[nj[i]+1][m];
     for(j=0;j<nj[i]+1;j++) ee[i][j]=dd[i][j];
 }
 return ee;
}
public static double[][][] readDouble(double c[][],int nc)
{int n=c.length;
 int m=c[0].length;
 double dd[][][]=new double[1000][n][m];
 int k=0;
 int i=0;
 int j=0;
 int nj[]=new int[1000];
 for(i=0;i<n-1;i++)
 {if(c[i][nc]==c[i+1][nc]) {dd[k][j]=c[i];j++;}
  else {dd[k][j]=c[i];nj[k]=j;j=0;
        k++;
       }
 }
 double ee[][][]=new double[k][][];
 for(i=0;i<k;i++)
 {
                ee[i]=new double[nj[i]+1][m];
     for(j=0;j<nj[i]+1;j++) ee[i][j]=dd[i][j];
 }
 return ee;
}
public static double [] thomas(double f[],double e[],double g[],double r[])
{
// 3 band matrix system of equation solving algorithm
int n=f.length;
double x[]=new double[n];
for(int k=1;k<n;k++)
   {e[k]=e[k]/f[k-1];
    f[k]=f[k]-e[k]*g[k-1];
   }
for(int k=1;k<n;k++)
   {r[k]=r[k]-e[k]*r[k-1];
   }
x[n-1]=r[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
   {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
return x;
}

public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
{
```

```
  int n=xi.length;
  double h[]=new double[n];
  double w[]=new double[n];
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double d[]=new double[n];
  double x[]=new double[n];
  double S[][]=new double[4][n];
  int k;
  for(k=0;k<(n-1);k++)
     {h[k]=xi[k+1]-xi[k];
      w[k]=(yi[k+1]-yi[k])/h[k];
                 }
  d[0]=c0;
  d[n-1]=cn;
  for(k=1;k<(n-1);k++)
     {d[k]=6.0*(w[k]-w[k-1]);}
  f[0]=1.0;
  f[n-1]=1.0;
  g[0]=0.0;
  g[n-1]=0.0;
  e[0]=0.0;
  e[n-1]=0.0;
  for(k=1;k<(n-1);k++)
  {f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
  S[2]=thomas(f,e,g,d);
  S[3]=xi;
  for(k=0;k<(n-1);k++)
    {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
     S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
    }
  return S;
  }
  public double func(double x1,double y1)
  { double xx1[]={x1,y1};
    return func(xx1);
  }
  public double func(double x[])
  { //create data set

                for(int i=0;i<n;i++)
                {  yy[i]=cs[i].func(x[0]);
                   xx[i]=cs[i].c[nyy][0];
                   //xx[i]=cs[i].c[nyy][cs[i].kk];
                }
                S=cubic_spline(xx,yy,c0,cn);
   int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y=0;
  double hk=0;
  int k=0;
  for(k=0;k<n-1;k++)
     {if(S[3][k]<=x[1] && x[1]<=S[3][k+1])
      {hk=(S[3][k+1]-S[3][k]);
       xx1=(x[1]-S[3][k]);
       xx2=(S[3][k+1]-x[1]);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       break;
       }
     }
       if(y==0 && S[3][n-2]<=x[1] )
       {
                   k=n-2;
                   hk=(S[3][k+1]-S[3][k]);
       xx1=(x[1]-S[3][k]);
       xx2=(S[3][k+1]-x[1]);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       }
  return y;
  }

   // Türev formülleri
// =================
```

```
  public double dfunc(double x1,double y1)
  {double xx1[]={x1,y1};
   return dfunc(xx1) ;
  }
  public double dfunc(double x[])
  {
  double xx1=0;
  double xx2=0;
  double y=0;
  double hk=0;
  double yy[]=new double[n];
              for(int i=0;i<=n;i++)
              {yy[i]=cs[i].func(x[0]);}
               S=cubic_spline(xx,yy,c0,cn);
  for(int k=0;k<(n-1);k++)
    {if(S[3][k]<=x[1] && x[1]<=S[3][k+1])
     {hk=(S[3][k+1]-S[3][k]);
      xx1=(x[1]-S[3][k]);
      xx2=(S[3][k+1]-x[1]);
      y=S[0][k]-S[1][k]+(xx1*xx1*S[2][k+1]-xx2*xx2*S[2][k])/(2.0*hk);
      break;
      }
    }
      if(y==0 && S[3][n-2]<=x[1] )
      {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
      xx1=(x[1]-S[3][k]);
      xx2=(S[3][k+1]-x[1]);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      }
  return y;
  }

  public static void main(String args[]) throws IOException
  { //String
name[]={"a10.txt","a50.txt","a90.txt","a95.txt","a100.txt","a101.txt","a110.txt","a150.txt","a200.txt","a500.txt","a1000.txt","a2000.txt","
a3000.txt"};
              //CS2 c2=new CS2(name,2,0,1,0.0,0.0,0.0,0.0);
              //column of P = 1
              //column of T 0 column of ro 2
              double c[][]=Text.readDouble("axy.txt");
              Text.print(c);
              CS2 c2=new CS2(c,1,0,2,0.0,0.0,0.0,0.0);
   double x=-1.0;
   double y=-1.0;
   double z=c2.func(x,y);
   System.out.println("x="+x+"y="+y+"z="+z);
  }
 }
```

Sample data:

x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z
-2	-2	8	-1.6	-2	6.5	-1.2	-2	5.4	-0.8	-2	4.6	-0.4	-2	4.16	0	-2	4.0
-2	-1.8	7.2	-1.6	-1.8	5.8	-1.2	-1.8	4.6	-0.8	-1.8	3.8	-0.4	-1.8	3.40	0	-1.8	3.2
-2	-1.6	6.5	-1.6	-1.6	5.1	-1.2	-1.6	4	-0.8	-1.6	3.2	-0.4	-1.6	2.72	0	-1.6	2.5
-2	-1.4	5.9	-1.6	-1.4	4.5	-1.2	-1.4	3.4	-0.8	-1.4	2.6	-0.4	-1.4	2.12	0	-1.4	1.9
-2	-1.2	5.4	-1.6	-1.2	4	-1.2	-1.2	2.8	-0.8	-1.2	2.0	-0.4	-1.2	1.60	0	-1.2	1.4
-2	-1	5	-1.6	-1	3.5	-1.2	-1	2.4	-0.8	-1	1.6	-0.4	-1	1.16	0	-1	1.0
-2	-0.8	4.6	-1.6	-0.8	3.2	-1.2	-0.8	2.0	-0.8	-0.8	1.2	-0.4	-0.8	0.80	0	-0.8	0.6
-2	-0.6	4.3	-1.6	-0.6	2.9	-1.2	-0.6	1.8	-0.8	-0.6	1	-0.4	-0.6	0.52	0	-0.6	0.3
-2	-0.4	4.1	-1.6	-0.4	2.7	-1.2	-0.4	1.6	-0.8	-0.4	0.8	-0.4	-0.4	0.32	0	-0.4	0.1
-2	-0.2	4.0	-1.6	-0.2	2.6	-1.2	-0.2	1.4	-0.8	-0.2	0.6	-0.4	-0.2	0.20	0	-0.2	0.0
-2	0	4	-1.6	0	2.5	-1.2	0	1.4	-0.8	0	0.6	-0.4	0	0.16	0	0	0.0
-2	0.2	4.0	-1.6	0.2	2.6	-1.2	0.2	1.4	-0.8	0.2	0.6	-0.4	0.2	0.20	0	0.2	0.0
-2	0.4	4.1	-1.6	0.4	2.7	-1.2	0.4	1.6	-0.8	0.4	0.8	-0.4	0.4	0.32	0	0.4	0.1
-2	0.6	4.3	-1.6	0.6	2.9	-1.2	0.6	1.8	-0.8	0.6	1	-0.4	0.6	0.52	0	0.6	0.3
-2	0.8	4.6	-1.6	0.8	3.2	-1.2	0.8	2.0	-0.8	0.8	1.2	-0.4	0.8	0.80	0	0.8	0.6
-2	1	5	-1.6	1	3.5	-1.2	1	2.4	-0.8	1	1.6	-0.4	1	1.16	0	1	1.0
-2	1.2	5.4	-1.6	1.2	4	-1.2	1.2	2.8	-0.8	1.2	2.0	-0.4	1.2	1.60	0	1.2	1.4
-2	1.4	5.9	-1.6	1.4	4.5	-1.2	1.4	3.4	-0.8	1.4	2.6	-0.4	1.4	2.12	0	1.4	1.9
-2	1.6	6.5	-1.6	1.6	5.1	-1.2	1.6	4	-0.8	1.6	3.2	-0.4	1.6	2.72	0	1.6	2.5
-2	1.8	7.2	-1.6	1.8	5.8	-1.2	1.8	4.6	-0.8	1.8	3.8	-0.4	1.8	3.40	0	1.8	3.2
-2	2	8	-1.6	2	6.5	-1.2	2	5.4	-0.8	2	4.6	-0.4	2	4.16	0	2	4.0
-1.8	-2	7.2	-1.4	-2	5.9	-1	-2	5	-0.6	-2	4.3	-0.2	-2	4.04	0.2	-2	4.0
-1.8	-1.8	6.4	-1.4	-1.8	5.2	-1	-1.8	4.2	-0.6	-1.8	3.6	-0.2	-1.8	3.28	0.2	-1.8	3.2
-1.8	-1.6	5.8	-1.4	-1.6	4.5	-1	-1.6	3.5	-0.6	-1.6	2.9	-0.2	-1.6	2.60	0.2	-1.6	2.6

-1.8	-1.4	5.2	-1.4	-1.4	3.9	-1	-1.4	2.9	-0.6	-1.4	2.3	-0.2	-1.4	2.00	0.2	-1.4	2.0
-1.8	-1.2	4.6	-1.4	-1.2	3.4	-1	-1.2	2.4	-0.6	-1.2	1.8	-0.2	-1.2	1.48	0.2	-1.2	1.4
-1.8	-1	4.2	-1.4	-1	2.9	-1	-1	2	-0.6	-1	1.3	-0.2	-1	1.04	0.2	-1	1.0
-1.8	-0.8	3.8	-1.4	-0.8	2.6	-1	-0.8	1.6	-0.6	-0.8	1	-0.2	-0.8	0.68	0.2	-0.8	0.6
-1.8	-0.6	3.6	-1.4	-0.6	2.3	-1	-0.6	1.3	-0.6	-0.6	0.7	-0.2	-0.6	0.40	0.2	-0.6	0.4
-1.8	-0.4	3.4	-1.4	-0.4	2.1	-1	-0.4	1.1	-0.6	-0.4	0.5	-0.2	-0.4	0.20	0.2	-0.4	0.2
-1.8	-0.2	3.2	-1.4	-0.2	2	-1	-0.2	1.0	-0.6	-0.2	0.4	-0.2	-0.2	0.08	0.2	-0.2	0.0
-1.8	0	3.2	-1.4	0	1.9	-1	0	1	-0.6	0	0.3	-0.2	0	0.04	0.2	0	0.0
-1.8	0.2	3.2	-1.4	0.2	2	-1	0.2	1.0	-0.6	0.2	0.4	-0.2	0.2	0.08	0.2	0.2	0.0
-1.8	0.4	3.4	-1.4	0.4	2.1	-1	0.4	1.1	-0.6	0.4	0.5	-0.2	0.4	0.20	0.2	0.4	0.2
-1.8	0.6	3.6	-1.4	0.6	2.3	-1	0.6	1.3	-0.6	0.6	0.7	-0.2	0.6	0.40	0.2	0.6	0.4
-1.8	0.8	3.8	-1.4	0.8	2.6	-1	0.8	1.6	-0.6	0.8	1	-0.2	0.8	0.68	0.2	0.8	0.6
-1.8	1	4.2	-1.4	1	2.9	-1	1	2	-0.6	1	1.3	-0.2	1	1.04	0.2	1	1.0
-1.8	1.2	4.6	-1.4	1.2	3.4	-1	1.2	2.4	-0.6	1.2	1.8	-0.2	1.2	1.48	0.2	1.2	1.4
-1.8	1.4	5.2	-1.4	1.4	3.9	-1	1.4	2.9	-0.6	1.4	2.3	-0.2	1.4	2.00	0.2	1.4	2.0
-1.8	1.6	5.8	-1.4	1.6	4.5	-1	1.6	3.5	-0.6	1.6	2.9	-0.2	1.6	2.60	0.2	1.6	2.6
-1.8	1.8	6.4	-1.4	1.8	5.2	-1	1.8	4.2	-0.6	1.8	3.6	-0.2	1.8	3.28	0.2	1.8	3.2
-1.8	2	7.2	-1.4	2	5.9	-1	2	5	-0.6	2	4.3	-0.2	2	4.04	0.2	2	4.0

Result:

```
---------- Capture Output ----------
> "D:\co\java\bin\java.exe" CS2
x=-1.0y=-1.0z=2.0
```

It should be note that repeated column of the data should be selected as y column so fort he given data column selection is done as 1,0,2, i.e first column is taken as y axis (axis calculated by second cubic spline in CS2). And data must be given this rectangualr form to utilise thissurface cubic spline interpolation process.

CUBIC B-SPLINE INTERPOLATION

B-spline interpolation will be investigated in this section. In general B-spline definition spline functions are defined in the region from minus infinity to plus infinity. We will specify cubic B-spline here for given $k_0,\ldots,k_N$ knots general curve fitting functions will be defined as

$$S(x) = \sum_{i=-3}^{N} a_i B_i(x) \quad (6.7-1)$$

$B_i(x)$ in the equation is called B-spline function. B-spline function for k_i, i=-3,-2,0,…,N,N+1,N+2 knots for the x value in the [a,b] interval , and with the limit values k_0=a, k_N=b can have the following property

$\sum_{i=-3}^{N} B_i(x) = 1 \quad (6.7-2)$

Interpolation equation S(x) has N+3 coefficient. There are N+1 data in the [a,b] data range so that additional equations will be needed to solve the data set. In order to define B-spline function B(x) a recursive calculation process will be defined. We first define B-splines that may be piecewise constant(n=1), linear(n=2), quadratic (n=3)and cubic (n=4). The condition that the sum of all B-spline function in all regions are still holds.

$\sum_{i=-3}^{N+3} B_i(x) = 1$ (n=1,2,3,4) (6.7-3)

The B-splines of order n are related to those of order n-1 by the recurrence relation

$$B_{i,n}(x) = \frac{(x-k_i)}{k_{i+n-1}-k_i} B_{i,n-1}(x) + \frac{(k_{i+n}-x)}{k_{i+n}-k_{i+1}} B_{i+1,n-1}(x) \quad (6.7\text{-}4)$$

Where i=-3,…N-1 range and n=(1,2,3,4) .In computing with this relation, B-spline of order one will be taken as a starting point. Therefore first order B spline should be independently defined.

$$B_{i,1}(x) = \begin{cases} 0 & x < k_i \\ 1 & k_i \le x < k_{i+1} \\ 0 & k_{i+1} \le x \end{cases} \quad \text{i=-3,-2,…,N+3} \quad (6.7\text{-}5)$$

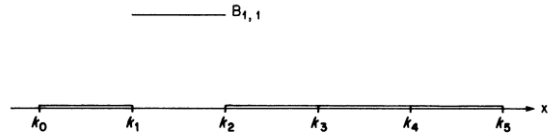


Figure 6.7-1 first order(constant) B-spline function

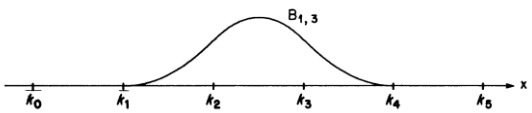


Figure 6.7-2 3th order(quadratic) B-spline function

By using first order equation and recurrence relation, linear, quadratic and cubic B spline functions can be calculated. First order(constant) and third order(quadratic) B-spline functions are shown in Figure 6.7-1 and 6.7-2 .

Now that B-spline functions are defined, we can return to solve coefficients of the iterations equation. By suppliying given k_j and f_j data points into the quation N+1 equations will be formed to solve

$\sum_{i=-3}^{N-1} a_i B_i(k_j) = f_j$ j=0,1,…,N (6.7-6)

In the jth equation only terms with B_{j-3}, B_{j-2}, B_{j-1} are different than zero the rest will be zero. In order to solve the data set two more equation will be needed. In order to established this, second derivative is at both end of the B-spline equation will be defined. If Natural cubic spline conditions are applied, this boundary values will be zero. Of course the problem can be solve by using any other type of boundary conditions.

$S"(k_0) = 0$

$S"(k_N) = 0$ (6.7-7)

When this condition applied to the curve fitting equations, they become

$a_{-3}B"_{-3}(k_0) + a_{-2}B"_{-2}(k_0) + a_{-1}B"_{-1}(k_0) = 0$

$a_{N-3}B"_{N-3}(k_N) + a_{N-2}B"_{N-2}(k_N) + a_{N-1}B"_{N-1}(k_N) = 0$ (6.7-8)

Second derivatives in the equation defined as:

$B"_{-3}(k_0) = \frac{6}{(k_1-k_{-1})(k_1-k_{-2})}$ (6.7-9)

$$B"_{-2}(k_0) = \frac{-6}{(k_1-k_{-1})}\left\{\frac{1}{(k_2-k_{-1})} + \frac{1}{(k_1-k_{-2})}\right\} \quad (6.7-10)$$

$$B"_{-1}(k_0) = \frac{6}{(k_1-k_{-1})(k_2-k_{-1})} \quad (6.7-11)$$

$$B"_{N-3}(k_0) = \frac{6}{(k_{N+1}-k_{N-1})(k_{N+1}-k_{N-2})} \quad (6.7-12)$$

$$B"_{N-2}(k_N) = \frac{-6}{(k_{N+1}-k_{N-1})}\left\{\frac{1}{(k_{N+2}-k_{N-1})} + \frac{1}{(k_{N+1}-k_{N-2})}\right\} \quad (6.7-13)$$

$$B"_{N-1}(k_0) = \frac{6}{(k_{N+1}-k_{N-1})(k_{N+2}-k_{N-1})} \quad (6.7-14)$$

It should be noted that k_{-3}, k_{-2}, k_{-1}, k_{N+1}, k_{N+2}, k_{N+3} values are not defined in the data set. This values can be set arbitrarily, but should satisfy $k_{-3}<k_{-2}<k_{-1}<k_0$ and $k_N<k_{N+1}<k_{N+2}<k_{N+3}$ conditions.

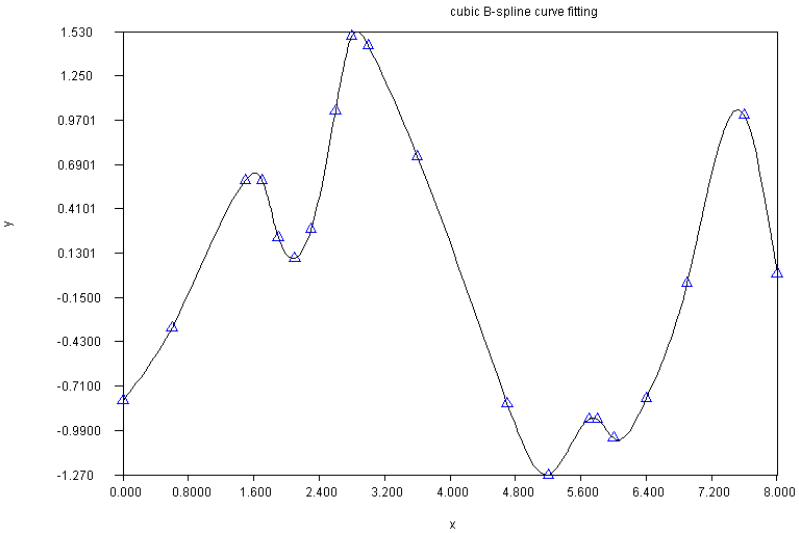
Program B-spline curve fitting

```
 import javax.swing.*;

public class B_spline
{ // cubic B-spline interpolation
  public double x[];
  public double f[];
  public double k[];
  int N;
  int n1;
  int m;
  double B[];
  double B2[];
  double a[];

  public B_spline(double xi[],double yi[])
  {   N=yi.length-1;
     n1=N+7;
            x=new double[N+1];
            f=new double[N+1];
            k=new double[n1];
     for(int i=0;i<=N;i++)
     {x[i]=xi[i];f[i]=yi[i];k[i+3]=x[i];
     }
```

```
      double dx=(x[N]-x[0]);
      k[0]=x[0]-6.0*dx;
      k[1]=x[0]-4.0*dx;
      k[2]=x[0]-2.0*dx;
      k[N+4]=x[N]+2.0*dx;
      k[N+5]=x[N]+4.0*dx;
      k[N+6]=x[N]+6.0*dx;
      m=5;
      B=new double[N+7];
      B2=new double[N+7];
      A();
  }

public  double[] B(double t)
{
double b[][]=new double[m][N+7];
for(int i=0;i<k.length-1;i++)
{if(t>=k[i] && t<k[i+1]) b[1][i]=1;
 else b[1][i]=0.0;
}
for(int n=2;n<m;n++)
{
for(int i=0;i<k.length-4;i++)
{
b[n][i]=(t-k[i])/(k[i+n-1]-k[i])*b[n-1][i]+(k[i+n]-t)/(k[i+n]-k[i+1])*b[n-1][i+1];
B[i]=b[n][i];
}
}
return B;
}

public void A()
{
B2[0]=6.0/(k[4]-k[2])/(k[4]-k[1]);
B2[1]=-6.0/(k[4]-k[2])*(1.0/((k[5]-k[2])+1.0/(k[4]-k[1]) ));
B2[2]=6.0/(k[4]-k[2])/(k[5]-k[2]);
B2[N]=6.0/(k[N+4]-k[N+2])/(k[N+4]-k[N+1]);
B2[N+1]=6.0/(k[N+4]-k[N+2])*(1.0/((k[N+5]-k[N+2])+1.0/(k[N+4]-k[N+1]) ));
B2[N+2]=6.0/(k[N+4]-k[N+2])/(k[N+5]-k[N+2]);
double AA[][]=new double[N+3][N+3];
double BB[]=new double[N+3];
AA[0][0]=B2[0];
AA[0][1]=B2[1];
AA[0][2]=B2[2];
BB[0]=0;
AA[N+2][N]=B2[N];
AA[N+2][N+1]=B2[N+1];
AA[N+2][N+2]=B2[N+2];
BB[N+2]=0;
for(int j=0;j<=N;j++)
{ B=B(k[j+3]);
  for(int i=0;i<=N+2;i++)
  { AA[j+1][i]=B[i];}
  BB[j+1]=f[j];
}
//Matrix is a 3-band matrix
a=thomas(AA,BB);
}
public static double [] thomas(double a[][],double r[])
  {
  int n=a.length;
  double f[]=new double[n];
```

```
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }
public double S(double x)
{ B=B(x);
  double ss=0;
  for(int i=0;i<=N+2;i++)
  {ss+=a[i]*B[i];}
  return ss;
}
public double func(double x)
{ return S(x);}

 public static  double[][]  funcB_Spline(double xi[],double yi[],int numberofmidpoints)
 {
    B_spline b1=new B_spline(xi,yi);
    double aa[][]=b1.funcB_Spline(numberofmidpoints);
    return aa;
 }

 public  double[][]  funcB_Spline(int numberofmidpoints)
 {
             // numberofmidpoints:  x--o--o--x--o--o--x in the curve fitting chain x
             // is experimental data and o is mid point (in this case equal to 2)
             // This function generates curve fitted data
             int n=x.length;
             int nn=(n-1)*(numberofmidpoints+1)+1;
             double z[][]=new double[2][nn];
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
     {  z[0][k]=x[i];z[1][k]=S(z[0][k]);k++;
        for(int j=0;j<numberofmidpoints;j++)
           {dx=(x[i+1]-x[i])/((double)numberofmidpoints+1.0);
        z[0][k]=z[0][k-1]+dx;z[1][k]=S(z[0][k]);k++;}
     }
     z[0][k]=x[i];z[1][k]=S(z[0][k]);
   return z;
 }
  public static void main(String[] arg)
  {
    double a[][]={{0.0,  0.6 ,1.5 ,1.7 ,1.9 ,2.1,2.3 ,2.6 ,2.8,3.0 ,3.6 ,4.7  ,5.2  ,5.7  ,5.8  , 6.0 , 6.4 ,6.9,7.6,8.0},
              {-0.8,-0.34,0.59,0.59,0.23,0.1,0.28,1.03,1.5,1.44,0.74,-0.82,-1.27,-0.92,-0.92,-1.04,-0.79,-0.06,1.0,0.0}};
    //String dosya=JOptionPane.showInputDialog("enter input file");
    //double a[][]=Matrix.T(Text.readDouble(dosya));
    int numberofmidpoints=10;
```

```
    double aa[][]=funcB_Spline(a[0],a[1],10);
    Plot pp=new Plot(a[0],a[1]);
    pp.setPlotType(0,24);
    pp.setColor(0,0,0,255);
    pp.addData(aa[0],aa[1]);
    pp.plot();
  }
}
```

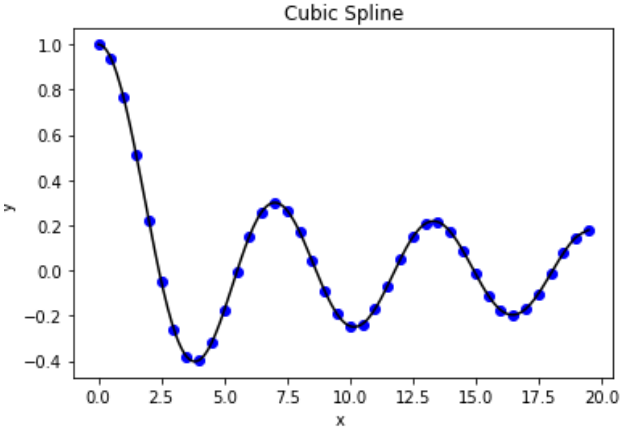


Program B-spline curve fitting python version

```
 # -*- coding: utf-8 -*-
"""
Created on Sun Jun  3 21:38:38 2018

@author: Mustafa Turhan Çoban
"""
from math import *
import matplotlib.pyplot as plt;

def thomas(f,e,g,r):
    n=len(f);
    x=[0.0 for i in range(n)];
    for k in range(0,n): e[k]=e[k]/f[k-1];f[k]=f[k]-e[k]*g[k-1];
    for k in range(1,n): r[k]=r[k]-e[k]*r[k-1];
    x[n-1]=r[n-1]/f[n-1];
    for k in range(n-2,-1,-1):  x[k]=(r[k]-g[k]*x[k+1])/f[k];
    return x;

def cubic_spline(xi,yi,c0,cn):
    n=len(xi);
    h=[0.0 for i in range(n)];
    w=[0.0 for i in range(n)];
    f=[0.0 for i in range(n)];
    e=[0.0 for i in range(n)];
    g=[0.0 for i in range(n)];
    d=[0.0 for i in range(n)];
    S= [[0 for x in range(n)] for y in range(4)];
    for k in range(0,n-1):
        h[k]=xi[k+1]-xi[k];
        w[k]=(yi[k+1]-yi[k])/h[k];
    d[0]=c0;
    d[n-1]=cn;
    for k in range(0,n-1):
       d[k]=6.0*(w[k]-w[k-1]);
    f[0]=1.0;
    f[n-1]=1.0;
    g[0]=0.0;
    g[n-1]=0.0;
    e[0]=0.0;
    e[n-1]=0.0;
    for k in range(0,n-1):
```

```
        f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];
    S[2]=thomas(f,e,g,d);
    S[3]=xi;
    for k in range(0,n-1):
       S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
       S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
    return S;


def funcSpline(S,x):
    n=len(S[0]);
    xx1=0;
    xx2=0;
    y=0;
    hk=0;
    for k in range(0,n-1):
       if S[3][k]<=x and x<=S[3][k+1]:
            hk=(S[3][k+1]-S[3][k]);
            xx1=(x-S[3][k]);
            xx2=(S[3][k+1]-x);
            y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
            break;
    if y==0 and S[3][n-2]<=x:
         k=n-2;
         hk=(S[3][k+1]-S[3][k]);
         xx1=(x-S[3][k]);
         xx2=(S[3][k+1]-x);
         y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
    return y;

def listSpline(xi,yi,numberofmidpoints,c0,cn):
    # numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
                # numberofmidpoints is 2
    n=len(xi);
    nn=(n-1)*(numberofmidpoints+1)+1;
    z=[[0 for x in range(nn)] for y in range(2)];
    S= [[0 for x in range(nn)] for y in range(4)];
    S=cubic_spline(xi,yi,c0,cn);
    dx=0;
    k=0;
    for i in range(0,n-1):
       z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k=k+1;
       for j in range(numberofmidpoints):
          dx=(xi[i+1]-xi[i])/(numberofmidpoints+1.0);
          z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k=k+1;
    z[0][k]=xi[i+1];z[1][k]=funcSpline(S,z[0][k]);
    return z;

def read_array2D(d):
    array2D = []
    f=open(d, 'r');
    lines=f.readlines();
    for line in lines:
       line.rstrip('\n');
       numbers_str=line.split();
       numbers_float=[float(x) for x in numbers_str]
       array2D.append(numbers_float)
    return T(array2D);

def T(a):
    n1=len(a);
    n2=len(a[0]);
    b = [[a[j][i] for j in range(n1)] for i in range(n2)];
    return b;

d=input("input file name (a.txt) : ");
a=read_array2D(d);
print(a);
x=a[0];
y=a[1];
Z=listSpline(x,y,5,0,0);
plt.title('Cubic Spline')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'bo',Z[0],Z[1],'k');
```

```
runfile('C:/Users/TURHAN/Desktop/NA_W8/cubic_spline.py', wdir='C:/Users/TURHAN/Desktop/NA_W8')

input file name (a.txt) : a.txt
[[0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0, 8.5, 9.0, 9.5, 10.0, 10.5, 11.0, 11.5, 12.0, 12.5, 13.0, 13.5,
14.0, 14.5, 15.0, 15.5, 16.0, 16.5, 17.0, 17.5, 18.0, 18.5, 19.0, 19.5], [1.0, 0.938469807240813, 0.7651976865579666,
0.5118276717359181, 0.22389077914123562, -0.048383776468198025, -0.2600519549019334, -0.38012773998726385, -
0.39714980986384796, -0.32054250898512326, -0.1775967713143408, -0.006843869417827212, 0.15064525725098935,
0.2600946055815953, 0.3000792705195197, 0.2663396578803943, 0.1716508071375183, 0.041939251842874115, -
0.09033361118311638, -0.19392874768828677, -0.24593576445249363, -0.23664819446450427, -0.17119030040887326, -
0.06765394812043596, 0.04768931078146386, 0.14688405468945884, 0.20692610234959996, 0.21498916577372518,
0.17107347599270106, 0.08754486788129101, -0.014224472951772184, -0.10923065060064199, -0.17489907408784924, -
0.19638069227528462, -0.16985425053436704, -0.10311039483282625, -0.013355803592061321, 0.07716482976695192,
0.14662945902756847, 0.17885385052619363]]
```



LEAST SQUARE CURVE FITTING OF NONLINEAR EQUATIONS

If curve fitting to a function $y = f(a; x)$ with non-linear coefficients is required. Process will be a little bit different compare to equations with linear coefficients.

$y = f(a; x)$ is given and data x_i, f_i, i=0...n will be fit into the equation. In this equation coefficients are placed in a non-linear form. For example : $y = a_0(1 - e^{-a_1 x})$ nonlinear

$y = a_0(1 - e^{-x}) + a_1(1 - e^{-2x}) = a_0\phi_0(x) + a_1\phi_1(x)$ linear

Least square error formula was defined as

$$H(a_j^{(m)}; x_0, x_1, .., x_n) = \sum_{i=1}^{n} w(x_0, x_1, .., x_n)\left[y_i - f(a_j^{(m)}; x_0, x_1, .., x_n)\right]^2 \quad (6.8-1)$$

This equation should be solved by using one of the multidimensional optimization methods such as Nelder-Mead, Newton-Raphson, Steepest descent, Generic algorithms

sample : (input3.txt)

```
5.0 16.0
10.0 25.0
15.0 32.0
20.0 33.0
25.0 38.0
30.0 36.0
```

A nonlinear function to fit the data:

double pi=ai[0]*(1.0-Math.exp(-ai[1]*x));

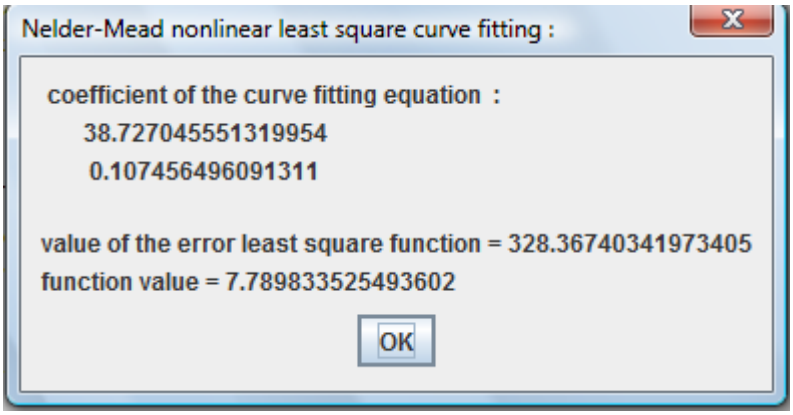
In this function ai are coefficient of the equations and x is independent variable.

Program 6.8-1 Nelder-Mead one independent variable least square curve fitting java version

```
import java.io.*;
import javax.swing.*;

abstract class f_xj
{
  // single function  multi independent variable
```

```
  // a single value is returned  (
  // example f=x[0]+sin(x[1])
  // func(x) returns the value of f
  abstract public double func(double x[]);
}
     class yy extends f_xj
     {
               double xi[]; // independent variable data set
     double yi[]; //dependent variable data set
     double a[];  //fit function coefficient set
     int nn;
               public yy(String filename,double ia[])
     {
       //read the data to curvefit
       //get the data file and initial fit coefficient when class is defined
                  xi=new double[500];
       yi=new double[500];
       int n=ia.length;
       a=new double[n];
       seta(ia);
       int i=-1;
       try{
                  BufferedReader fin=new BufferedReader(new FileReader(filename));
       try {
       while(fin != null)
       {
       i++;
       xi[i]=Text.readDouble(fin);
       yi[i]=Text.readDouble(fin);
       }
       } catch(EOFException e_eof) {System.out.println("end of file"); }
       }catch(IOException e_io) {System.out.println("dosya bulunamadi"); }
     nn=i;
     a=ia;
     }
     public void seta(double ia[])
     {
                  //assign new fit coefficient set
                  for(int ii=0;ii<nn;ii++)
            a[ii]=ia[ii];
               }
               public double[] geta()
     {
                  // return fit coefficient set
                  return a;
               }
               double Ps(double x,double ai[])
     {
     //sample specific function
               // saturation pressure as a function of saturation temperature
     a=ai;
     double pi=ai[0]*(1.0-Math.exp(-ai[1]*x));
     return pi; //kPa
     }

     public double func(double ai[])
     {
     double ff=0;
     double w;
     double yy;
      for(int i=0;i<nn;i++)
               {w=Ps(xi[i],ai);
```

```
                yy=yi[i];
       //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                w-=yy;
       ff+=w*w;
       //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                }
     return ff;
              }
}

class zz extends yy
{     public zz(String filename,double ia[])
      {super(filename,ia);}
      public double func(double ai[])
      {
      return -super.func(ai);
              }
}
public class NA50
{  public static void main(String args[]) throws IOException
   {
   //String in_name=JOptionPane.showInputDialog(" enter name of the input file : ");
   String in_name="input3.txt";
   double a[];
   a=new double[2];
   a[0] = 30.0;
   a[1] = 1.0;
   double da[];
   da=new double[2];
   da[0] =1.0;
   da[1] =0.2;

   yy f=new yy(in_name,a);
   int i=0;
   for(double t=-100;t<101;t+=1)
   {System.out.println("t="+t+"ti="+f.xi[i]+"Pi="+f.yi[i]+"P="+f.Ps(t,a));i++;};
   System.out.println("f input = "+f.func(a));
   double p[]=NA41.nelder(f,a,da,1000,1e-20,1);
   String s1=" coefficient of the curve fitting equation  : \n"+Matrix.toStringT(p)+"\n";
   s1+="value of the error least square function = "+f.func(a)+"\n";
   s1+="function value = "+f.func(p);
   String s2="Nelder-Mead nonlinear least square curve fitting : ";
   JOptionPane.showMessageDialog(null,s1,s2,JOptionPane.PLAIN_MESSAGE);
   }

}
```



The problem can be solved by using different optimization programs. For example id Davidon-Fletcher-Powell algorithms is used:

Program Nelder-Mead one independent variable least square curve fitting python version

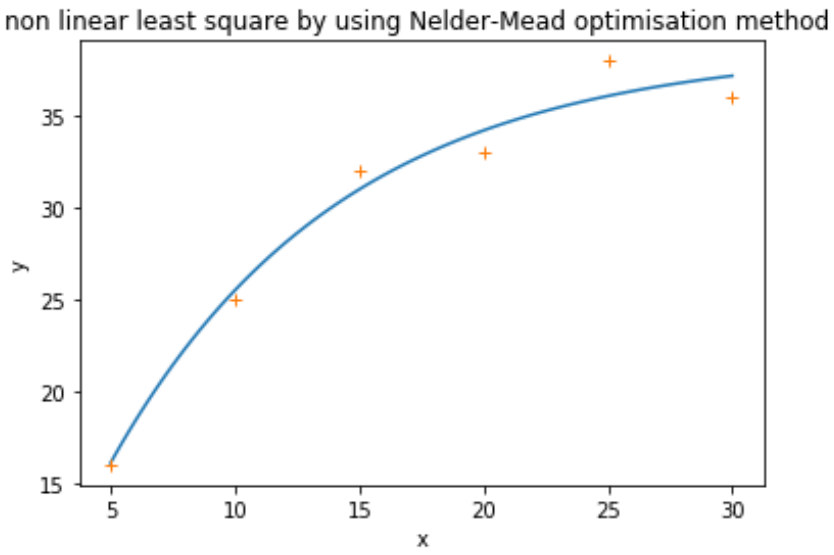
```
from math import *
from f_xj import *
from matplotlib import pyplot as plt
import numpy as np


class y(f_xj):
    def __init__(self,filename,ia):
        self.name = 'yy',
        self.filename=filename
        path = filename
        fin=open(path,'r')
        n=len(ia)
        self.a=ia
        path = self.filename
        file1 = open(path,'r')
        file1.seek(0,2)  #jumps to the end
        endLocation=file1.tell()
        file1.seek(0) #jumps to the start
        nn=0
        while True:
            line=file1.readline()
            nn+=1
            if  file1.tell()==endLocation:
                break
        xi=[0.0 for i in range(nn)]
        yi=[0.0 for i in range(nn)]
        file1.seek(0) #jumps to the start
        for i in range(nn):
            line=file1.readline()
            ee=line.split()
            xi[i]=float(ee[0])
            yi[i]=float(ee[1])
            #print(xi[i]," ",yi[i])
        file1.close()
        self.xi=xi
        self.yi=yi
        self.nn=nn

    def Ps(self,x,ai):
        pi=ai[0]*(1-exp(-ai[1]*x))
        return pi
    def func(self,ai):
        ff=0
        for i in range(self.nn):
            w=self.Ps(self.xi[i],ai)
            yy=self.yi[i]
            w=w-yy
            ff=ff+w*w
        return ff

def  nelder(fnelder,a,da,maxiteration,tolerance,printlist):
    #print("a",a)
    #print("da",da)
    n=len(a)
    m=n+1
    x=[[0.0 for j in range(n)] for i in range(n+1)]
    p=[[0.0 for j in range(m)] for i in range(m)]
    s=""
    NDIMS = len(x)-1
    NPTS = len(x)
    FUNC = NDIMS
    ncalls = 0;
    for i in range(m):
        for j in range(n):
            if i==j:
                x[i][j]=a[j]+da[j]
                p[i][j]=x[i][j]
            else:
                x[i][j]=a[j]
                p[i][j]=x[i][j]
```

```
        p[i][FUNC] = fnelder.func(p[i])
        #print("x[",i,"]=",x[i])
        #print("p[",i,"]=",p[i])
    #print("p",p)
    #Inlet variable definitions
    #fnelder : abstract multivariable function f(x)
    #x : independent variable set of n+1 simplex elements
    #maxiteration : maximum iteration number
    #tolerance :

    #///// construct the starting simplex //////////////////
    z=[0.0 for i in range(NDIMS)]
    best = 1.0E99;
    #/////////////// calculate the first function values for the simplex ////////////////
    iter=0;
    for iter in range(1,maxiteration):
        #////////  define lo, nhi, hi (low high next_to_high //////////////
        ilo=0
        ihi=0
        inhi = -1
        # -1 means missing
        flo = p[0][FUNC];
        fhi = flo
        #print("flo=",flo,"fhi=",fhi)
        #double pavg,sterr;
        for i in range(1,NPTS):
            if p[i][FUNC] < flo:
                flo=p[i][FUNC]
                ilo=i
            if  p[i][FUNC] > fhi:
                fhi=p[i][FUNC]
                ihi=i
            #print("i=",i,"flo=",flo,"fhi=",fhi,"ilo=",ilo,"ihi=",ihi,"pifunc=",p[i][FUNC])
        fnhi = flo
        inhi = ilo
        #print("fnhi=",fnhi,"inhi=",inhi)
        for i in range(NPTS):
            if (i != ihi) and (p[i][FUNC] > fnhi):
                fnhi=p[i][FUNC]
                inhi=i
        #////////// exit criteria //////////////
        if (iter % 4*NDIMS) == 0:
            #calculate the avarage (including maximum value)
            pavg=0
            for i in range(NPTS):
                 pavg=pavg+p[i][FUNC]
            pavg=pavg/NPTS
            tot=0.0
            if printlist!=0:
                print(iter)
                for j in range(NDIMS):
                    print(p[ilo][j]," ")
            for i in range(NPTS):
                tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg)
            sterr=sqrt(tot/NPTS)
            for j in range(NDIMS):
                z[j]=p[ilo][j]
            best = p[ilo][FUNC];
        # end of if iter%4*NDMS)==0
        #//// calculate avarage without maximum value //////
        ave=[0.0 for i in range(NDIMS)]
        for j in range(NDIMS):
            ave[j] = 0
        for i in range(NPTS):
            if i != ihi:
                for j in range(NDIMS):
                    ave[j] = ave[j]+p[i][j];
        for j in range(NDIMS):
            ave[j]=ave[j]/(NPTS-1)
        #print("ave[",j,"]=",ave[j])
        #//////// reflect ////////////////
        #print("reflect")
        r=[0.0 for i in range(NDIMS)]
        for j in range(NDIMS):
            r[j] = 2.0*ave[j] - p[ihi][j]
        fr = fnelder.func(r)
```

```
        if (flo <= fr) and (fr < fnhi): #in zone: accept
            for j in range(NDIMS):
                p[ihi][j] = r[j]
            p[ihi][FUNC] = fr
            continue
        #///(/////////// expand
        if fr < flo:
            #print("expand")
            e=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                e[j] = 3.0*ave[j] - 2.0*p[ihi][j];
            fe = fnelder.func(e);
            if fe < fr:
                for j in range(NDIMS):
                    p[ihi][j] = e[j]
                p[ihi][FUNC] = fe
                continue
            else:
                for j in range(NDIMS):
                    p[ihi][j] = r[j]
                p[ihi][FUNC] = fr
                continue
        #////////// shrink:
        if fr < fhi:
            #print("srink")
            c=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                c[j] = 1.5*ave[j] - 0.5*p[ihi][j]
            fc = fnelder.func(c);
            if fc <= fr:
                for j in range(NDIMS):
                    p[ihi][j] = c[j];
                p[ihi][FUNC] = fc
                continue
            else:   #////// shrink
                for i in range(NPTS):
                    if i != ilo:
                        for j in range(NDIMS):
                            p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j]
                        p[i][FUNC] = fnelder.func(p[i])
                continue
        #/////////////// contract
        if fr >= fhi:
            #print("contract")
            cc=[0.0 for i in range(NDIMS)]
            for j in range(NDIMS):
                cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
            fcc = fnelder.func(cc)
            if fcc < fhi:
                for j in range(NDIMS):
                    p[ihi][j] = cc[j];
                p[ihi][FUNC] = fcc
                continue
            else:
                for i in range(NPTS):
                    if i != ilo:
                        for j in range(NDIMS):
                            p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j]
                        p[i][FUNC] = fnelder.func(p[i]);
    return z

a=[30.0,1.0]
da=[1.0,0.2]
name="input3.txt"
f=y(name,a)
ai=nelder(f,a,da,100,1.0e-15,0)
print("ai = ",ai)
n1=100
xx1 =[0 for w in range(n1)]
yy1 =[0 for w in range(n1)]
dx=(30.0-5.0)/(n1-1)
for i in range(n1):
    xx1[i]=5.0+dx*i
    yy1[i]=f.Ps(xx1[i],ai)
plt.plot(xx1,yy1,f.xi,f.yi,"+")
plt.xlabel("x")
```

```
plt.ylabel("y")
plt.title("non linear least square by using Nelder-Mead optimisation method ")
plt.show()
```

```
runfile('E:/okul/SCO1/nelder_opt.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xj
ai =  [38.727045551319954, 0.1074564960913112]
```



Program Davison-Fletcher-Powell one independent variable nonlinear least square curve fitting

```
// Davidon-Fletcher-Powell optimization
// Method Referance : Numerical Analysis, 8th edition Richard L. Burden, J. Douglas Faires 3
// page 617
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;
     class fa extends f_xj
     { double xi[]; // independent variable data set
       double yi[]; //dependent variable data set
       double a[];  //fit function coefficient set
       int nn;
      public fa(String filename,double ia[])
     { //read the data to curvefit
       //get the data file and initial fit coefficient when class is defined
      xi=new double[500];
      yi=new double[500];
      int n=ia.length;
      a=new double[n];
      seta(ia);
      int i=-1;
      try { BufferedReader fin=new BufferedReader(new FileReader(filename));
      try {
      while(fin != null)
      { i++;
      xi[i]=Text.readDouble(fin);
      yi[i]=Text.readDouble(fin);
      }
      } catch(EOFException e_eof) {System.out.println("end of file"); }
      }catch(IOException e_io) {System.out.println("dosya bulunamadi"); }
```

```
      nn=i;
      a=ia;
      }

      public void seta(double ia[])
      {  //assign new fit coefficient set
         for(int ii=0;ii<nn;ii++)   a[ii]=ia[ii];}

      public double[] geta()
      {// return fit coefficient set
        return a;}
      double Ps(double x,double ai[])
      {
      //sample specific function
      // saturation pressure as a function of saturation temperature
      a=ai;
      double pi=ai[0]*(1.0-Math.exp(-ai[1]*x));
      //pi=Math.pow(10.0,pi);
      return pi; //kPa
      }

      public double func(double ai[])
      {
      double ff=0;
      double w;
      double yy;
       for(int i=0;i<nn;i++)
                  {w=Ps(xi[i],ai);
                  yy=yi[i];
        //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                  w-=yy;
        ff+=w*w;
        //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                  }
      return ff;
                }
}

public class NA125
{
public static double VT_V(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
              aa[i][j]=left[i]*left[j];
```

```
            }
  }
return aa;
}

public static double  VT_X(double [] left,double [] right)
{
//multiplies a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
return tot;
}

public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

public static double turev(f_xj f,double x[],int denklem_ref,int x_ref)
{
//
// verilen fonksiyonun ikinci türevlerinin matrisi
// fonksiyon f in denklem_ref sayili fonksiyonunun(sayilar 0 dan başlar)
// x[x_ref] değişkenine göre türevi (sayilar 0 dan başlar)
// df_denklem_ref(x)/d_x_ref
// bu metod newtond metodu içinde kullanilmak içindir.
double h0=0.256808;
int i,m;
int n=7;
double f1[];
f1=new double[x.length];
double f2[];
f2=new double[x.length];
double x1[];
x1=new double[x.length];
double x2[];
x2=new double[x.length];
for(i=0;i<x.length;i++)
{
x1[i]=x[i];
x2[i]=x[i];
}
//turev of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
//vector<double> h(n,0);
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
```

```
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
h[i]=h0*Math.pow(r,i);
}

for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i]*1.0094847;
f1=df(f,x1);
f2=df(f,x2);
T[i][0]=( f1[denklem_ref] - f2[denklem_ref])/(2.0*h[i]);
x1[x_ref]=x[x_ref];
x2[x_ref]=x[x_ref];
}
for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] - h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i]
  - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public static double[] df(f_xj f,double xxi[])
{
//Gredyen vektörü
// f(x0,x1,x2,...xn) founksiyonunun türev vektörü
// [df/dx0, df/dx1,df/dx2,....,df/dxn]
// df/dxj  j=0...x.length
//
// bu fonksiyon bilgisayar tarafindan hesaplanmaktadir
// kullanici tarafindan tanimlanan f(x) fonksiyonunun
// sayisal olarak alinan türevidir.

//
double a[]=new double[xxi.length];
for(int i=0;i<xxi.length;i++)
   {
   a[i]=turev(f,xxi,i);
   }
return a;
}
public static double turev(f_xj f,double xxi[],int x_ref)
{
// df/dxj
double h0=0.256808;
int i,m;
int n=7;
double x1[];
x1=new double[xxi.length];
double x2[];
x2=new double[xxi.length];
for(i=0;i<xxi.length;i++)
{
x1[i]=xxi[i];
x2[i]=xxi[i];
}
```

```
//turev of a simple function
double T[][];
T=new double[n][n];
double h[];
h=new double[n];
for(i=0;i<n;i++)
{
  h[i]=0;
  for(int j=0;j<n;j++)
    T[i][j]=0;
}
h[0]=h0;
double r=0.5;
for( i=1;i<n;i++)
{
//h[i]=h0*Math.pow(r,i);
h0/=2.0;
h[i]=h0;
}
//first turev (difference formula)
for(i=0;i<n;i++)
{
x1[x_ref]+=h[i];
x2[x_ref]-=h[i];
T[i][0]=(f.func(x1)-f.func(x2))/(2.0*h[i]);
x1[x_ref]=xxi[x_ref];
x2[x_ref]=xxi[x_ref];
}

for(m=1;m<n;m++)
{
  for(i=0;i<n-m;i++)
  {
  T[i][m]=(h[i]*h[i]*T[i+1][m-1] -
  h[i+m]*h[i+m]*T[i][m-1])/(h[i]*h[i] - h[i+m]*h[i+m]);
  }
}
double xx=T[0][n-1];
return xx;
}
public static double[] turev(f_xj f_deriv,double x[])
{
// df/dxj  j=0...x.length
// This method calculates turev of a function with more than one independent variable.
// Accuracy of method can be adjusted by changing variables h0 and n
// function input should be in the form given in abstract class
// f_xj,j=0...x.length = df/dx(x_ref)
double a[]=new double[x.length];
for(int x_ref=0;x_ref<x.length;x_ref++)
   {
   a[x_ref]=turev(f_deriv,x,x_ref);
   }
return a;
}

public static double SIGN(double a,double b) {return (b > 0.0 ? Math.abs(a) : -Math.abs(a));}
public static double MAX(double a,double b) {return (a > b ? a : b);}


public static double linmin(f_xj f,double p[],double xi[])
{
            double tol=2.0e-4;
```

```
   int n=p.length;
   double pcom[],xicom[];
            int j;
            double xx,xmin,bx,ax;
            f1dim f1=new f1dim(f,p,xi);
            ax=0.0;
            xx=1.0;
            bx=2.0;
            double aa[]=mnbrak(f1,ax,xx,bx);
            ax=aa[0];
            xx=aa[1];
            bx=aa[2];
            xmin=brent(f1,ax,xx,bx,tol);
            return xmin;
}

public static double[][] linminiter(f_xj f,double p[],double xi[])
{   // yeni iterasyon noktasini verir
   int n=p.length;
   double xmin=linmin(f,p,xi);
   double aa[][]=new double[2][n];
            for (int j=0;j<n;j++)
            {   xi[j] *= xmin;
                        p[j] += xi[j];
                        aa[0][j]=p[j];
                        aa[1][j]=xi[j];
            }
   return aa;
}
public  static double[] brentf(f_x f,double ax,double bx,double cx,double tol)
{
// ax,bx,cx üç adet tahmin noktasi (bxax ile cx arasinda yer almalidir)
// f : fonksiyon (f_x abstract sinifinda tanimlanmiş)
// tol tolerans
int ITMAX=100;
double CGOLD=(3.0-Math.sqrt(5))/2.0; //altin oran
double ZEPS=1.0e-10;
double xmin;
double aa[]=new double[2];
// SHFT(double a,double b,double c,d) (a)=(b);(b)=(c);(c)=(d);
            int iter;
            double a,b,d,etemp,fu,fv,fw,fx,p,q,r,tol1,tol2,u,v,w,x,xm;
            double e=0.0;
            d=0.0;
            a=((ax < cx) ? ax : cx);
            b=((ax > cx) ? ax : cx);
            x=w=v=bx;
            fw=fv=fx=f.func(x);
            for (iter=1;iter<=ITMAX;iter++) {
                        xm=0.5*(a+b);
                        tol2=2.0*(tol1=tol*Math.abs(x)+ZEPS);
                        if (Math.abs(x-xm) <= (tol2-0.5*(b-a))) {
                                    xmin=x;
                  aa[0]=xmin;
                  aa[1]=fx;
                  return aa;
                        }
                        if (Math.abs(e) > tol1) {
                                    r=(x-w)*(fx-fv);
                                    q=(x-v)*(fx-fw);
                                    p=(x-v)*q-(x-w)*r;
                                    q=2.0*(q-r);
```

```
                        if (q > 0.0) p = -p;
                        q=Math.abs(q);
                        etemp=e;
                        e=d;
                        if (Math.abs(p) >= Math.abs(0.5*q*etemp) || p <= q*(a-x) || p >= q*(b-x))
                                d=CGOLD*(e=(x >= xm ? a-x : b-x));
                        else {
                                d=p/q;
                                u=x+d;
                                if (u-a < tol2 || b-u < tol2)
                                        d=SIGN(tol1,xm-x);
                        }
                } else {
                        d=CGOLD*(e=(x >= xm ? a-x : b-x));
                }
                u=(Math.abs(d) >= tol1 ? x+d : x+SIGN(tol1,d));
                fu=f.func(u);
                if (fu <= fx) {
                        if (u >= x) a=x; else b=x;
                        {v=w;w=x;x=u;}
                        {fv=fw;fw=fx;fx=fu;}
                } else {
                        if (u < x) a=u; else b=u;
                        if (fu <= fw || w == x) {
                                v=w;
                                w=u;
                                fv=fw;
                                fw=fu;
                        } else if (fu <= fv || v == x || v == w) {
                                v=u;
                                fv=fu;
                        }
                }
        }
        System.out.println("BRENT metodunda maksimum iterasyon sayisi aşildi");
        xmin=x;
        //minumum değer a[0] ile fonksiyon değeri a[1] ile geridöndürülmektedir
        aa[0]=xmin;
        aa[1]=fx;
        return aa;
}
public  static double[] mnbrak(f_x f,double ax,double dx)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçilen uzaklik
return mnbrak(f,ax,(ax-dx),(ax+dx));
}

public  static double[] mnbrak(f_x f,double ax)
{
// dx ikinci derece polinom için gerekli iki ek noktayi oluşturmak için
// ana noktadan seçiln uzaklik
return mnbrak(f,ax,(ax-0.1),(ax+0.1));
}

public  static double[] mnbrak(f_x f,double ax,double bx,double cx)
{
// ax,bx,cx üç adet tahmin noktasi verildiğinde içinde minimum bulunan ax,bx,cx üç adetnoktayi bize arama
// sonucu bulur
// f : fonksiyon (f_x abstract sinifinda tanimlanmiş)
// tol tolerans
int ITMAX=100;
```

```
double xmin;
double GOLD=(Math.sqrt(5.0)+1.0)/2.0;;
double GLIMIT=100.0;
double TINY=1.0e-20;
double fa,fb,fc;
fa=0;
fb=0;
fc=0;
        double ulim,u,r,q,fu,dum;
        double aa[]=new double[3];
        fa=f.func(ax);
        fb=f.func(bx);
        if (fb > fa) {
                //SHFT(dum,*ax,*bx,dum)
                {dum=ax;ax=bx;bx=dum;}
                //SHFT(dum,*fb,*fa,dum)
                {dum=fb;fb=fa;fa=dum;}
        }
        cx=(bx)+GOLD*(bx-ax);
        fc=f.func(cx);
                aa[0]=ax;
                aa[1]=bx;
                aa[2]=cx;
        while (fb > fc) {
                r=(bx-ax)*(fb-fc);
                q=(bx-cx)*(fb-fa);
                u=(bx)-((bx-cx)*q-(bx-ax)*r)/
                        (2.0*SIGN(MAX(Math.abs(q-r),TINY),q-r));
                ulim=(bx)+GLIMIT*(cx-bx);
                if ((bx-u)*(u-cx) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                ax=bx;
                                bx=u;
                                fa=fb;
                                fb=fu;
                                return aa;
                        } else if (fu > fb) {
                                cx=u;
                                fc=fu;
                                return aa;
                        }
                        u=(cx)+GOLD*(cx-bx);
                        fu=f.func(u);
                } else if ((cx-u)*(u-ulim) > 0.0) {
                        fu=f.func(u);
                        if (fu < fc) {
                                //SHFT(bx,cx,u,cx+GOLD*(cx-bx))
                                {bx=cx;cx=u;u=cx+GOLD*(cx-bx);}
                                //SHFT(fb,fc,fu,f.func(u))
                                {fb=fc;fc=fu;fu=f.func(u);}
                        }
                } else if ((u-ulim)*(ulim-cx) >= 0.0) {
                        u=ulim;
                        fu=f.func(u);
                } else {
                        u=cx+GOLD*(cx-bx);
                        fu=f.func(u);
                }
                //SHFT(ax,bx,cx,u)
                {ax=bx;bx=cx;cx=u;}
                //SHFT(fa,fb,fc,fu)
```

```
                        {fa=fb;fb=fc;fc=fu;}
                        aa[0]=ax;
                        aa[1]=bx;
                        aa[2]=cx;
            }
                        return aa;
}
public  static double brent(f_x f,double ax,double bx,double cx,double tol)
{
 double aa[]=brentf(f,ax,bx,cx,tol);
 return aa[0];
}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=1.0;
return b;
}
// Davidon - Fletcher-Powell metodu
//_____________________________________________________________________
public static double[] davidon_fletcher_powell( f_xj f,double x[])
{
int i,j,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-12 ;
int n=x.length;
double alpha;
double dx[]=new double[n];
double g[] =new double[n];
double ge[]=new double[n];
double dg[]=new double[n];
double d[] =new double[n];
double r[] =new double[n];
double rdg,dxdg;
double beta;
double alpha_d;
double Q[][]=I(n);
double Q1[][]=new double[n][n];
double Q2[][]=new double[n][n];
int nn=15;
i=0;
j=0;k=0;
g=turev(f,x);
while( j++ < nmax && gamma > tolerance )
{
   for(int ii=0;ii<n;ii++)
   {
    d[ii]=0.0;
    for(int kk=0;kk<n;kk++)
                {d[ii]=d[ii]-Q[ii][kk]*g[kk];}
   }
   System.out.println("j="+j+"d="+Matrix.toString(d));
   alpha=linmin(f,x,d);
   System.out.println("alfa="+alpha);
   if(alpha==0) break;
   for(i=0;i<n;i++){dx[i]=alpha*d[i];x[i]+=dx[i];ge[i]=g[i];}
   System.out.println("j="+j+"dx=\n"+Matrix.toString(dx));
```

```
    System.out.println("j="+j+"x=\n"+Matrix.toString(x));
    g=turev(f,x);
    System.out.println("j="+j+"g=\n"+Matrix.toString(g));
    for(i=0;i<n;i++){dg[i]=g[i]-ge[i];}
    gamma=norm(g);
    for(int ii=0;ii<n;ii++)
    {r[ii]=0.0;
                for(int kk=0;kk<n;kk++)
                {r[ii]+=Q[ii][kk]*dg[kk];}
    }
    rdg=VT_X(r,dg);
    dxdg=VT_X(dx,dg);
    Q1=V_VT(r);
    Q2=V_VT(dx);
    for(int ii=0;ii<n;ii++)
      for(int jj=0;jj<n;jj++)
        Q[ii][jj]=Q[ii][jj]-Q1[ii][jj]/rdg+Q2[ii][jj]/dxdg;
    gamma=norm(dx);
    System.out.println("j="+j+"Q=\n"+Matrix.toString(Q));
j++;k++;
if(k>=nn) {k=0;Q=I(n);}
}
if(j >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
  " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}
public static double[] davidon_fletcher_powell_V1( f_xj f,double x[])
{
int i,j,k;
int nmax=500;
double gamma=1.0;
double tolerance=1.0e-12 ;
int n=x.length;
double alpha;
double dx[]=new double[n];
double g[] =new double[n];
double ge[]=new double[n];
double dg[]=new double[n];
double d[] =new double[n];
double r[] =new double[n];
double rdg,dxdg;
double beta;
double alpha_d;
double Q[][]=I(n);
double Q1[][]=new double[n][n];
double Q2[][]=new double[n][n];
int nn=15;
i=0;
j=0;k=0;
g=turev(f,x);
while( j++ < nmax && gamma > tolerance )
{

    for(int ii=0;ii<n;ii++)
    {
     d[ii]=0.0;
     for(int kk=0;kk<n;kk++)
                {d[ii]=d[ii]-Q[ii][kk]*g[kk];}
    }
    alpha=linmin(f,x,d);
    if(alpha==0) break;
    for(i=0;i<n;i++){dx[i]=alpha*d[i];x[i]+=dx[i];ge[i]=g[i];}
```

```
    g=turev(f,x);
    for(i=0;i<n;i++){dg[i]=g[i]-ge[i];}
    gamma=norm(g);
    for(int ii=0;ii<n;ii++)
    {  r[ii]=0.0;
                for(int kk=0;kk<n;kk++)
                {r[ii]+=Q[ii][kk]*dg[kk];}
    }
    rdg=VT_X(r,dg);
    dxdg=VT_X(dx,dg);
    beta=dxdg/rdg;
    Q1=V_VT(r);
    Q2=V_VT(dx);
    for(int ii=0;ii<n;ii++)
      for(int jj=0;jj<n;jj++)
        Q[ii][jj]=beta*(Q[ii][jj]-Q1[ii][jj]/rdg)+Q2[ii][jj]/dxdg;
    gamma=norm(dx);
j++;k++;
if(k>=nn) {k=0;Q=I(n);}
}
if(j >= nmax) JOptionPane.showMessageDialog(null,"Uyari maximum iterasyon sayisi aşildi \n"+
  " çözüm geçerli olmiyabilir","MAKSİMUM ITERASYON SAYISI UYARISI",JOptionPane.WARNING_MESSAGE);
return x;
}
                public static double[] verigir(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
                public static void main (String args[])
    {
                String s="write root estimates with one space in between: ";
    double [] x0=verigir(s);
    fa f_x=new fa("a.txt",x0);
    double [] r1= davidon_fletcher_powell(f_x,x0);
    double [] r2= davidon_fletcher_powell_V1(f_x,x0);
    s=" solution set DFP : \n"+Matrix.toStringT(r1);
    s+="solution set corrected DFP : \n"+Matrix.toStringT(r2);
    String s2="Davidon - Fletcher - Powell multivariable optimization:";
    JOptionPane.showMessageDialog(null,s,s2,JOptionPane.PLAIN_MESSAGE);
    System.exit(0);
    }
}
```

A real engineering problem will be defined here to show how actual need would arise and solution

As a research project, drying of sliced apple by a heat pump is investigated. The data includes t time(hour), slice thickness h(mm) and P is the retaining moisture in the apple slice(kg moisture/kg fresh apple moisture). When the previous researchs are investigated, it is found that the following equations are used.

$$\mathbf{a=a_0+a_1h}$$

$$\mathbf{k=a_2+a_3h}$$

$P=ae^{(-kt)}$

$$P = (a_0 + a_1 h)e^{(a_2 + a_3 h)}$$

In order to compare we would like to fit our data into the same equations.

The data (T,h,P)

0	6	1.000	3	8	0.332	4	10	0.328
0.5	6	0.736	3.5	8	0.276	4.5	10	0.287
1	6	0.603	4	8	0.229	5	10	0.251
1.5	6	0.482	4.5	8	0.192	5.5	10	0.218
2	6	0.390	5	8	0.160	6	10	0.185
2.5	6	0.307	5.5	8	0.128	6.5	10	0.161
3	6	0.245	6	8	0.102	7	10	0.138
3.5	6	0.192	6.5	8	0.083	7.5	10	0.117
4	6	0.143	7	8	0.067	8	10	0.101
4.5	6	0.108	7.5	8	0.055	8.5	10	0.086
5	6	0.081	8	8	0.046	9	10	0.071
5.5	6	0.059	0	10	1.000	9.5	10	0.063
6	6	0.044	0.5	10	0.824	10	10	0.056
0	8	1.000	1	10	0.718	10.5	10	0.051
0.5	8	0.765	1.5	10	0.618	11	10	0.045
1	8	0.649	2	10	0.542			
1.5	8	0.541	2.5	10	0.474			
2	8	0.456	3	10	0.424			
2.5	8	0.382	3.5	10	0.372			

Let us fit data into the above equation:

Program Nelder-Mead one independent variable nonlinear least square curve fitting, apple slice drying problem

```
import java.io.*;
import javax.swing.*;
    class fxa extends f_xj
    {
            double xi[][]; // m independent variable data set
    double yi[]; //dependent variable data set
    double a[];  //fit function coefficient set
    int nn;//number of data
    int m; //number of idependent variables xi
    double xmax;
            public fxa(String filename,double ia[])
    {
     //read the data to curvefit
     //last column is treated as independent data set
     //get the data file and initial fit coefficient when class is defined
     int n=ia.length;
     a=new double[n];
     seta(ia);
     xmax=-1.0e99;
     double b[][]=Text.readDoubleT(filename);
     m=b.length-1;
     yi=b[m];
     nn=yi.length;
     //System.out.println("m="+m);
     xi=new double[m][nn];
     for(int i=0;i<nn;i++) {
             xi[0][i]=b[0][i];if(xi[0][i]>xmax) {xmax=xi[0][i];}
     xi[1][i]=b[1][i];
             yi[i]=b[m][i];
```

```
        }
       //System.out.println("b=\n"+Matrix.toString(Matrix.T(b)));
       System.out.println("xi=\n"+Matrix.toString(xi));
       System.out.println("yi=\n"+Matrix.toString(yi));
     nn=yi.length;
     a=ia;
     }

     public void seta(double ia[])
     {
                 //assign new fit coefficient set
                 for(int ii=0;ii<nn;ii++)
           a[ii]=ia[ii];
              }

              public double[] geta()
     {
                // return fit coefficient set
                return a;
              }
      double Ps(double x[],double ai[])
     {
     // sample specific function m dimensional independent variable
     // independent variables : x[0],x[1]...x[m-1]
     // x[0] t
     // x[1] kalinlik
     // ai[0] a
     //
     a=ai;
     double t=x[0];
     double h=x[1];
     double a=ai[0]+ai[1]*h;
     double k=ai[2]+ai[3]*h;
     double pi=a*Math.exp(-k*t);
     return pi; //kPa
     }

     public double func(double ai[])
     {
     double ff=0;
     double w;
     double yy;
     double xx[]=new double[xi.length];
      for(int i=0;i<nn;i++)
                {   for(int j=0;j<m;j++) {xx[j]=xi[j][i];}
                                        w=Ps(xx,ai);
                yy=yi[i];
       //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                w-=yy;
       ff+=w*w;
       //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                }
     return ff;
                }
}

public class NA126
{//2. Nelder ve Mead method
//_____________________________________________________________________
public static double[] nelder(f_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
double x[][]=new double[a.length+1][a.length];
```

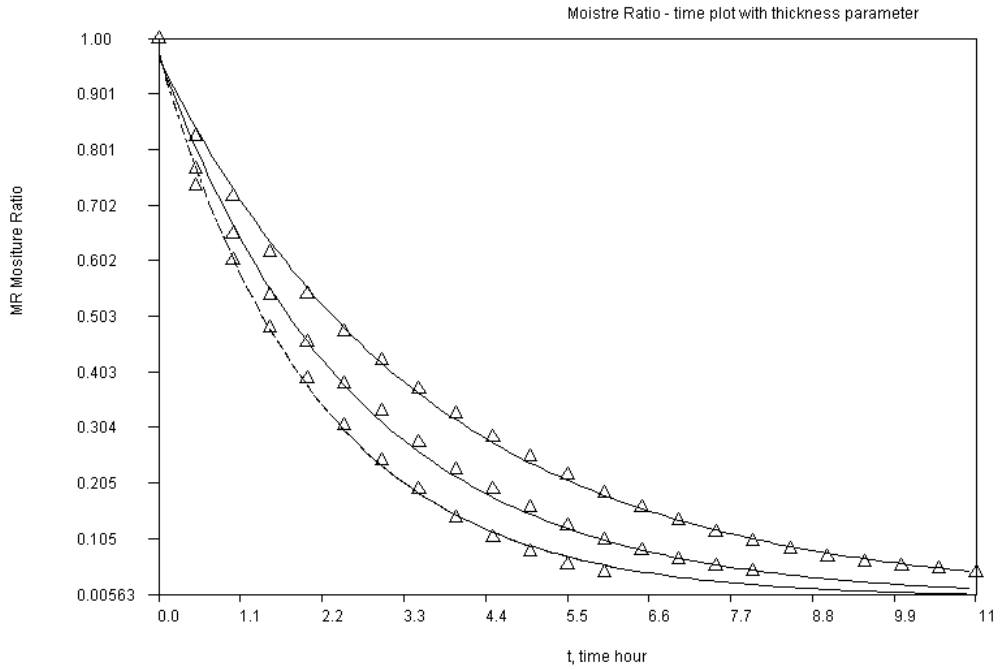
```
for(int i=0;i<x.length;i++)
  {for(int j=0;j<x[0].length;j++)
   {if(i==j){x[i][j]=a[j]+da[j];}
    else    {x[i][j]=a[j]; }
   }
  }
  //  Nelder mead multidimensional simplex minimization method
  //  Reference : Nelder & Mead 1965 Computer J, v.7, 308-313.
       int NDIMS = x.length-1;
       int NPTS = x.length;
       int FUNC = NDIMS;
       int ncalls = 0;
         ////// starting simplex ///////////////////
         double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
         double z[]=new double[NDIMS];
         double best = 1E99;
         ///////////////// calculate function values /////////////////
         for (int i=0; i<NPTS; i++)
         {
          for (int j=0; j<NDIMS; j++)
             {p[i][j] = x[i][j];}
          p[i][NDIMS] = fnelder.func(p[i]);
         }
         int iter=0;
         for (iter=1; iter<maxiteration; iter++)
         {
           ////////////  define lo, nhi, hi points //////////////
           int  ilo=0, ihi=0, inhi = -1; // -1 means missing
           double flo = p[0][FUNC];
           double fhi = flo;
           double pavg,sterr;
           for (int i=1; i<NPTS; i++)
           {
             if (p[i][FUNC] < flo)
               {flo=p[i][FUNC]; ilo=i;}
             if (p[i][FUNC] > fhi)
               {fhi=p[i][FUNC]; ihi=i;}
           }
           double fnhi = flo;
           inhi = ilo;
           for (int i=0; i<NPTS; i++)
            if ((i != ihi) && (p[i][FUNC] > fnhi))
              {fnhi=p[i][FUNC]; inhi=i;}
           //////////exit criterieas//////////////
           if ((iter % 4*NDIMS) == 0)
           {
              pavg=0;
              for(int i=0;i<NPTS;i++)
                pavg+=p[i][FUNC];
              pavg/=NPTS;
              double tot=0;
              if(printlist!=0)
              {  System.out.print(iter);
                 for (int j=0; j<=NDIMS; j++)
                    { System.out.print(p[ilo][j]+" ");}
                 System.out.println("");
              }
              for(int i=0;i<NPTS;i++)
              { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
              sterr=Math.sqrt(tot/NPTS);
              //if(sterr < tolerance)
                { for (int j=0; j<NDIMS; j++)
```

```
            { z[j]=p[ilo][j];}
          //break;
        }
      best = p[ilo][FUNC];
}

///// calculate ave[] vector except the largest value //////

double ave[] = new double[NDIMS];
for (int j=0; j<NDIMS; j++)
  ave[j] = 0;
for (int i=0; i<NPTS; i++)
  if (i != ihi)
    for (int j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (int j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);
//////// reflect ////////////////
double r[] = new double[NDIMS];
for (int j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
    for (int j=0; j<NDIMS; j++)
      p[ihi][j] = r[j];
    p[ihi][FUNC] = fr;
    continue;
}

if (fr < flo)  //// expand, else accept
{
    double e[] = new double[NDIMS];
    for (int j=0; j<NDIMS; j++)
      e[j] = 3*ave[j] - 2*p[ihi][j];
    double fe = fnelder.func(e);
    if (fe < fr)
    {
      for (int j=0; j<NDIMS; j++)
        p[ihi][j] = e[j];
      p[ihi][FUNC] = fe;
      continue;
    }
    else
    {
      for (int j=0; j<NDIMS; j++)
        p[ihi][j] = r[j];
      p[ihi][FUNC] = fr;
      continue;
    }
}
////////////// squeese:
if (fr < fhi)
{
    double c[] = new double[NDIMS];
    for (int j=0; j<NDIMS; j++)
      c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
    double fc = fnelder.func(c);
    if (fc <= fr)
    {
       for (int j=0; j<NDIMS; j++)
```

```
                    p[ihi][j] = c[j];
                  p[ihi][FUNC] = fc;
                  continue;
               }
               else   /////// squeese
               {
                  for (int i=0; i<NPTS; i++)
                   if (i != ilo)
                   {
                      for (int j=0; j<NDIMS; j++)
                        p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                      p[i][FUNC] = fnelder.func(p[i]);
                   }
                  continue;
               }
            }

            if (fr >= fhi)   ///
            {
               double cc[] = new double[NDIMS];
               for (int j=0; j<NDIMS; j++)
                 cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
               double fcc = fnelder.func(cc);
               if (fcc < fhi)
               {
                  for (int j=0; j<NDIMS; j++)
                    p[ihi][j] = cc[j];
                  p[ihi][FUNC] = fcc;
                  continue;
               }
               else    /////////
               {
                  for (int i=0; i<NPTS; i++)
                   if (i != ilo)
                   {
                      for (int j=0; j<NDIMS; j++)
                        p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                      p[i][FUNC] = fnelder.func(p[i]);
                   }
               }
            }
         }
     return z;
}
    public static double[] nelder(f_xj fnelder,double a[],double da[],double tolerance)
    {return nelder(fnelder,a,da,300,tolerance,0);}

    public static double[] nelder(f_xj fnelder,double a[],double da[])
    {return nelder(fnelder,a,da,500,1.0e-20,0);}

    public static double[] nelder(f_xj fnelder,double x0[])
    {  double [] dx0=new double[x0.length];
       for(int i=0;i<x0.length;i++) dx0[i]=0.2*x0[i];
       return nelder(fnelder,x0,dx0,500,1.0e-15,0);
    }

    public static double[] nelder(f_xj fnelder,double x0[],double tolerance)
    {
      double [] dx0=new double[x0.length];
      for(int i=0;i<x0.length;i++) dx0[i]=0.2*x0[i];
                return nelder(fnelder,x0,dx0,tolerance);
    }
```

```
 public static double[] nelder(f_xj fnelder,double x0[],int maxiteration,double tolerance)
 {
  double [] dx0=new double[x0.length];
  for(int i=0;i<x0.length;i++) dx0[i]=0.2*x0[i];
  return nelder(fnelder,x0,dx0,maxiteration,tolerance,0);
 }

public static void main(String args[]) throws IOException
{
//String in_name=JOptionPane.showInputDialog(" veri dosyasinin isminigiriniz : ");
String in_name="kalinlik1.txt";
double a[];
double x[][],y[]; //uydurulan veri
double xu1[][],yu1[]; //uydurulan veri
double xu2[][],yu2[];
double xu3[][],yu3[];
a=new double[4];
a[0] = 1.0;
a[1] = 1.0;
a[2] = 1.0;
a[3] = 1.0;
fxa f=new fxa(in_name,a);
int i=0,j=0;
double p[]=nelder(f,a);
f=new fxa(in_name,p);
double tmax=f.xmax;
double dt=0.1; //ZAMAN ARALIĞI
int n=(int)(tmax/dt);
System.out.println("length="+f.nn+"tmax="+tmax+"n="+n);
xu1=new double[f.m][n];
yu1=new double[n];
xu2=new double[f.m][n];
yu2=new double[n];
xu3=new double[f.m][n];
yu3=new double[n];
double z[]=new double[2];
for(i=0;i<n;i++)
{
xu1[0][i]=dt*i;
xu1[1][i]=6;
z[0]=xu1[0][i];
z[1]=xu1[1][i];
yu1[i]=f.Ps(z,p);
System.out.println("x0="+xu1[0][i]+"x1="+xu1[1][i]+"y="+yu1[i]);
}
for(i=0;i<n;i++)
{
xu2[0][i]=dt*i;
xu2[1][i]=8;
z[0]=xu2[0][i];
z[1]=xu2[1][i];
yu2[i]=f.Ps(z,p);
System.out.println("x0="+xu2[0][i]+"x1="+xu2[1][i]+"y="+yu2[i]);
}
for(i=0;i<n;i++)
{
xu3[0][i]=dt*i;
xu3[1][i]=10;
z[0]=xu3[0][i];
z[1]=xu3[1][i];
yu3[i]=f.Ps(z,p);
```

```
    System.out.println("x0="+xu3[0][i]+"x1="+xu3[1][i]+"y="+yu3[i]);
    }
    Plot pp=new Plot(f.xi[0],f.yi);
    pp.setPlabel("Moistre Ratio - time plot with thickness parameter");
    pp.setXlabel("t, time hour");
    pp.setYlabel("MR Mositure Ratio ");
    pp.setPlotType(0,24);
    pp.addData(xu1[0],yu1);
    pp.setPlotType(1,0);
    pp.addData(xu2[0],yu2);
    pp.setPlotType(1,1);
    pp.addData(xu3[0],yu3);
    pp.setPlotType(1,2);

    //pp.addData(x,y);
    //pp.setPlotType(1,24);// üçgen olarak çiz
    pp.plot();

    String s1=" Optimization value  : \n"+Matrix.toStringT(p)+"\n";
    s1+="Value of the original function = "+f.func(a)+"\n";
    s1+="Function vlue = "+f.func(p);
    s1+="standard deviation= "+Math.sqrt(f.func(p)/(f.nn-1));
    String s2="Nelder-Mead non linear least square curve fitting : ";
    System.out.println(s1);
    }
}
```



Resulting curve :

a_0= ***0.990155880325089***

a_1= ***-0.002517036051923***

a_2= ***0.765464843543500***

a_3= **-0.048774936274541**

In the next example harris' hawks optimisation is used to find non-linear curve fitting function.

Data:

```
5.0 16.0
10.0 25.0
15.0 32.0
20.0 33.0
```

```
25.0 38.0
30.0 36.0
```

Function to curvefit:

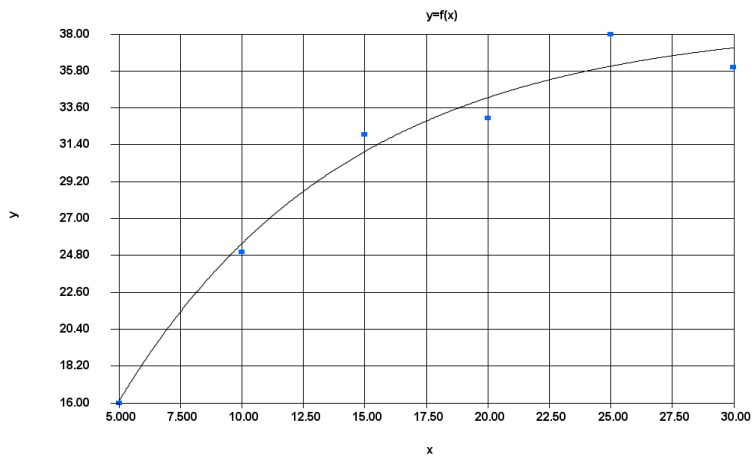
$f(x) = a_0[1 - \exp(-a_1 x)]$

Program Harris' hawks one independent variable nonlinear least square curve fitting

```
 import java.util.*;
import java.io.*;
    class yy implements if_xj
    {
            double xi[]; // independent variable data set
    double yi[]; //dependent variable data set
    double a[];  //fit function coefficient set
    int nn;
            public yy(String filename,int n)
    {
      //read the data to curvefit
      //get the data file and initial fit coefficient when class is defined
              xi=new double[5000];
      yi=new double[5000];
      a=new double[n];
      //seta(ia);
      int i=-1;
      try{
              BufferedReader fin=new BufferedReader(new FileReader(filename));
      try {
      while(fin != null)
      {
      i++;
      xi[i]=Text.readDouble(fin);
      yi[i]=Text.readDouble(fin);
      }
      } catch(EOFException e_eof) {System.out.println("end of file"); }
      }catch(IOException e_io) {System.out.println("dosya bulunamadi"); }
    nn=i;
    }
    public void seta(double ia[])
    {
              //assign new fit coefficient set
              for(int ii=0;ii<nn;ii++)
        a[ii]=ia[ii];
            }
            public double[] geta()
    {
              // return fit coefficient set
              return a;
            }
            double Ps(double x,double ai[])
    {
    //sample specific function
            // saturation pressure as a function of saturation temperature
    a=ai;
    double pi=ai[0]*(1.0-Math.exp(-ai[1]*x));
    return pi; //kPa
    }

    public double func(double ai[])
    {
    double ff=0;
    double w;
    double yy;
     for(int i=0;i<nn;i++)
              {w=Ps(xi[i],ai);
              yy=yi[i];
      //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
              w-=yy;
      ff+=w*w;
      //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
              }
    return ff;
            }
}

public class harris_hawks_optimization_test1
{
```

```
public static void main(String arg[])
   {   String in_name="input3.txt";
   int iN=100;
   yy f=new yy(in_name,2);
   double[] iLower={0.0,0.0};
   double[] iUpper={50,50};
   int iMaxiter=200;
   //harris_hawks_optimization(f_xj iff,int iN,double[] iLower,double[] iUpper, int iMaxiter)
            harris_hawks_optimizationi hho=new harris_hawks_optimizationi(f,iN,iLower,iUpper,iMaxiter);
   double[][] x0=hho.solution();
   String s="";
   s+="harris-hawks optimisation : \n"+Matrix.toStringT(x0[0]);
            //double [] r1= NA125.davidon_fletcher_powell_V1(f,x0[0]);
            //s+="solution set corrected DFP : \n"+Matrix.toStringT(r1);
            double [] r2=NA41.nelder(f,x0[0]);
            s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
            System.out.println(s);
            if_x ff=x->r2[0]*(1.0-Math.exp(-r2[1]*x));
            Plot pp=new Plot(ff,5.0,30.0);
            double a[][]=Text.readDoubleT("input3.txt");
            pp.addData(a[0],a[1]);
            pp.setPlotType(1,21);
            pp.plot();
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" harris_hawks_optimization_test1
end of file
harris-hawks optimisation :
        38.657896877219976
         0.108016570513192
Nelder-Mead solution :
        38.727045456466600
         0.107456496673013

> Terminated with exit code 0.
```



We will carry out the same optimisation by using Genetic algorithms and Thermal Exchange optimisation as well. It should be note that Genetic algorithms find maximum value, Programs integrates also with Nelder Mead optimisation methods which finds minimum value.

Program Genetic algorithm one independent variable nonlinear least square curve fitting

```
 import java.util.*;
import java.io.*;
    class yy1 implements if_xj
    { yy f;
            double xi[]; // independent variable data set
    double yi[]; //dependent variable data set
    double a[];  //fit function coefficient set
    int nn;
            public yy1(String filename,int n)
    {
```

```
      //read the data to curvefit
      //get the data file and initial fit coefficient when class is defined
                xi=new double[5000];
      yi=new double[5000];
      a=new double[n];
      //seta(ia);
      int i=-1;
      try{
                BufferedReader fin=new BufferedReader(new FileReader(filename));
      try {
      while(fin != null)
      {
      i++;
      xi[i]=Text.readDouble(fin);
      yi[i]=Text.readDouble(fin);
      }
      } catch(EOFException e_eof) { }
       }catch(IOException e_io) {System.out.println("dosya bulunamadi"); }
     nn=i;
     }
     public void seta(double ia[])
     {
                  //assign new fit coefficient set
                  for(int ii=0;ii<nn;ii++)
           a[ii]=ia[ii];
                }
                public double[] geta()
     {
                  // return fit coefficient set
                  return a;
                }
                double Ps(double x,double ai[])
     {
     //sample specific function
                // saturation pressure as a function of saturation temperature
     a=ai;
     double pi=ai[0]*(1.0-Math.exp(-ai[1]*x));
     return pi; //kPa
     }
     public double func(double ai[])
     {
     double ff=0;
     double w;
     double yy;
      for(int i=0;i<nn;i++)
                  {w=Ps(xi[i],ai);
                  yy=yi[i];
                  w-=yy;
       ff+=w*w;              }
     return -ff;
                }}

public class Genetic3_test1
{

public static void main(String arg[])
   {   String in_name="input3.txt";
   int N=63;
              int POPSIZE=500;
   int MAXGENS=500;
   int NVARS=2;
   double PXOVER=0.3;
   double PMUTATION=0.02;
   Genetic3i xx=new Genetic3i(POPSIZE,MAXGENS,NVARS,N,PXOVER,PMUTATION);
   yy1 fx=new yy1(in_name,2);
   yy fz=new yy(in_name,2);
   double[] iLower={0.0,0.0};
   double[] iUpper={50,50};
   int iMaxiter=200;
   xx.setPopulation(iLower,iUpper,fx);
   double[] x0=xx.solution(fx);
   String s="";
   s+="Genetic3 optimisation : \n"+Matrix.toStringT(x0);
              double [] r2=NA41_i.nelder(fz,x0);
              s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
              System.out.println(s);
```

```
            if_x ff=x->r2[0]*(1.0-Math.exp(-r2[1]*x));
            Plot pp=new Plot(ff,5.0,30.0);
            double a[][]=Text.readDoubleT("input3.txt");
            pp.addData(a[0],a[1]);
            pp.setPlotType(1,21);
            pp.plot();
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" Genetic3_test1
Genetic3 optimisation :
        38.800000088888890
         0.106943935273514
Nelder-Mead solution :
        38.727045433140990
         0.107456496707803

> Terminated with exit code 0.
```

Program Thermal exhange optimisation algorithm one independent variable nonlinear least square curve fitting

```
 import java.util.*;
import java.io.*;
public class Thermal_Exchange_Optimization_test3
{

public static void main(String arg[])
   {   String in_name="input3.txt";
   int iN=100;
   yy f=new yy(in_name,2);
   double[] iLower={0.0,0.0};
   double[] iUpper={50,50};
   int iMaxiter=200;
   Thermal_Exchange_Optimizationi wo=new   Thermal_Exchange_Optimizationi(f,iN,iLower,iUpper,iMaxiter);
   double[][] x0=wo.solution();
   String s="";
   s+="Thermal exchange optimisation  : \n"+Matrix.toStringT(x0[0]);
            double [] r2=NA41_i.nelder(f,x0[0]);
            s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
            System.out.println(s);
            if_x ff=x->r2[0]*(1.0-Math.exp(-r2[1]*x));
            Plot pp=new Plot(ff,5.0,30.0);
            double a[][]=Text.readDoubleT("input3.txt");
            pp.addData(a[0],a[1]);
            pp.setPlotType(1,21);
            pp.plot();
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" Thermal_Exchange_Optimization_test3
end of file
Thermal exchange optimisation  :
        38.558306736224620
         0.108852415410211
Nelder-Mead solution :
        38.727045546680750
         0.107456496004458
```

Now let us try drying of sliced apple problem we previously solved by using Hawkins' hawks algorith merged with Nelder Mead algorithm

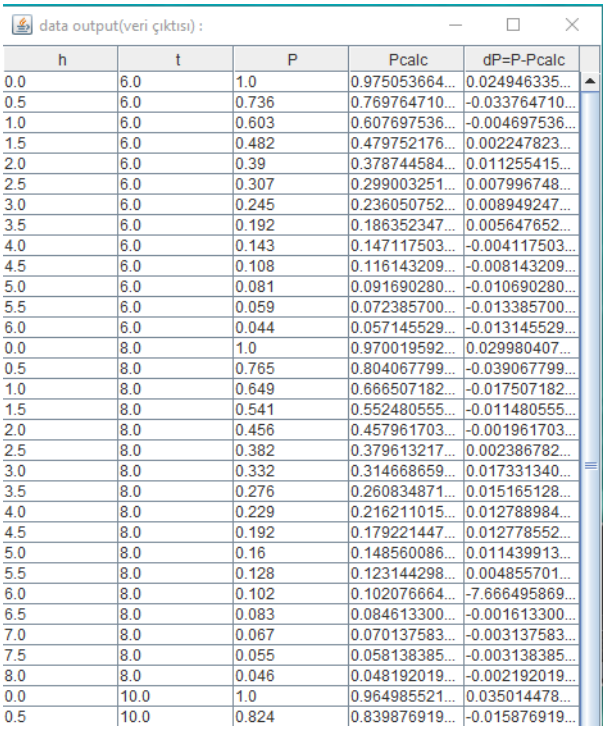
Program Harris' hawks algorithm with two independent variable nonlinear least square curve fitting

```
 import java.io.*;
import javax.swing.*;

    class ifxa implements if_xj
    {
            double xi[][]; // m independent variable data set
    double yi[]; //dependent variable data set
    double a[];  //fit function coefficient set
```

```
    int nn;//number of data
    int m; //number of idependent variables xi
    double xmax;
              public ifxa(String filename,double ia[])
    {
     //read the data to curvefit
     //last column is treated as independent data set
     //get the data file and initial fit coefficient when class is defined
     int n=ia.length;
     a=new double[n];
     seta(ia);
     xmax=-1.0e99;
     double b[][]=Text.readDoubleT(filename);
     m=b.length-1;
     yi=b[m];
     nn=yi.length;
     xi=new double[m][nn];
     for(int i=0;i<nn;i++) {
               xi[0][i]=b[0][i];if(xi[0][i]>xmax) {xmax=xi[0][i];}
     xi[1][i]=b[1][i];
               yi[i]=b[m][i];
     }
    nn=yi.length;
    a=ia;
    }
    public void seta(double ia[])
    {
                //assign new fit coefficient set
                for(int ii=0;ii<nn;ii++)
          a[ii]=ia[ii];
              }

              public double[] geta()
    {
                // return fit coefficient set
                return a;
              }
     double Ps(double x[],double ai[])
    {
    // sample specific function m dimensional independent variable
    // independent variables : x[0],x[1]...x[m-1]
    // x[0] t
    // x[1] kalınlık
    // ai[0] a
    //
    a=ai;
    double t=x[0];
    double h=x[1];
    double a=ai[0]+ai[1]*h;
    double k=ai[2]+ai[3]*h;
    double pi=a*Math.exp(-k*t);
    return pi; //kPa
    }

    public double func(double ai[])
    {
    double ff=0;
    double w;
    double yy;
    double xx[]=new double[xi.length];
     for(int i=0;i<nn;i++)
               {   for(int j=0;j<m;j++) {xx[j]=xi[j][i];}
                                        w=Ps(xx,ai);
                yy=yi[i];
                w-=yy;
      ff+=w*w;
               }
    return ff;
              }
}
class fx implements if_xj
{   double ai[];
             public fx(double aj[]) {ai=aj;}
   public double func(double x[])
   {// sample specific function m dimensional independent variable
     // independent variables : x[0],x[1]...x[m-1]
```

```
      // x[0] t
      // x[1] kalınlık
      // ai[0] a
      // ;
      double t=x[0];
      double h=x[1];
      double a=ai[0]+ai[1]*h;
      double k=ai[2]+ai[3]*h;
      double pi=a*Math.exp(-k*t);
      return pi; //kPa
    }
}

public class harris_hawks_optimization_test5
{

    public static void main(String args[]) throws IOException
    {
    //String in_name=JOptionPane.showInputDialog(" veri dosyasının isminigiriniz : ");
    String in_name="kalinlik1.txt";
    double a[];
    a=new double[4];
    a[0] = 1.0;
    a[1] = 1.0;
    a[2] = 1.0;
    a[3] = 1.0;
    double x[][],y[]; //curve fitting value
    double xu1[][],yu1[]; ////curve fitting value
    double xu2[][],yu2[];
    double xu3[][],yu3[];
    int iN=500;
    ifxa f=new ifxa(in_name,a);
    double[] iLower={-1.0,-1.0,-1.0,-1.0};
    double[] iUpper={2,2,2,2};
    int iMaxiter=500;
    harris_hawks_optimizationi hho=new harris_hawks_optimizationi(f,iN,iLower,iUpper,iMaxiter);
    double[][] x0=hho.solution();
    int i=0,j=0;
    double [] r2=NA41.nelder(f,x0[0]);
    String s="";
    s+="harris-hawks optimisation : \n"+Matrix.toStringT(x0[0]);
    s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
    System.out.println(s);
    double E[][]=Text.readDoubleT(in_name);
    fx f1=new fx(r2);
    double xx[]=E[0];
    double yy[]=E[1];
    double zz[]=E[2];
    double ww[]=new double[xx.length];
    double ee[]=new double[xx.length];
    for(int ii=0;ii<xx.length;ii++)
    {   double xx1[]={xx[ii],yy[ii]};
                ww[ii]=f1.func(xx1);
                ee[ii]=zz[ii]-ww[ii];
              }
              double qq[][]={xx,yy,zz,ww,ee};
              String s1[]={"h","t","P","Pcalc","dP=P-Pcalc"};
    Text.printT(qq,s1);
    }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" harris_hawks_optimization_test5
harris-hawks optimisation :
          0.963470015426018
          0.000756980450594
          0.758892779991807
         -0.047965637605179
Nelder-Mead solution :
          0.990155888634559
         -0.002517037118643
          0.765464844854249
         -0.048774936394621
```



data output(veri çıktısı) :

h	t	P	Pcalc	dP=P-Pcalc
0.0	6.0	1.0	0.975053664...	0.024946335...
0.5	6.0	0.736	0.769764710...	-0.033764710...
1.0	6.0	0.603	0.607697536...	-0.004697536...
1.5	6.0	0.482	0.479752176...	0.002247823...
2.0	6.0	0.39	0.378744584...	0.011255415...
2.5	6.0	0.307	0.299003251...	0.007996748...
3.0	6.0	0.245	0.236050752...	0.008949247...
3.5	6.0	0.192	0.186352347...	0.005647652...
4.0	6.0	0.143	0.147117503...	-0.004117503...
4.5	6.0	0.108	0.116143209...	-0.008143209...
5.0	6.0	0.081	0.091690280...	-0.010690280...
5.5	6.0	0.059	0.072385700...	-0.013385700...
6.0	6.0	0.044	0.057145529...	-0.013145529...
0.0	8.0	1.0	0.970019592...	0.029980407...
0.5	8.0	0.765	0.804067799...	-0.039067799...
1.0	8.0	0.649	0.666507182...	-0.017507182...
1.5	8.0	0.541	0.552480555...	-0.011480555...
2.0	8.0	0.456	0.457961703...	-0.001961703...
2.5	8.0	0.382	0.379613217...	0.002386782...
3.0	8.0	0.332	0.314668659...	0.017331340...
3.5	8.0	0.276	0.260834871...	0.015165128...
4.0	8.0	0.229	0.216211015...	0.012788984...
4.5	8.0	0.192	0.179221447...	0.012778552...
5.0	8.0	0.16	0.148560086...	0.011439913...
5.5	8.0	0.128	0.123144298...	0.004855701...
6.0	8.0	0.102	0.102076664...	-7.666495869...
6.5	8.0	0.083	0.084613300...	-0.001613300...
7.0	8.0	0.067	0.070137583...	-0.003137583...
7.5	8.0	0.055	0.058138385...	-0.003138385...
8.0	8.0	0.046	0.048192019...	-0.002192019...
0.0	10.0	1.0	0.964985521...	0.035014478...
0.5	10.0	0.824	0.839876919...	-0.015876919...

Program 9 Thermal Exchange algorithm with two independent variable nonlinear least square curve fitting

```
import java.io.*;
import javax.swing.*;

public class  Thermal_Exchange_Optimization_test5
{

   public static void main(String args[]) throws IOException
   {
   //String in_name=JOptionPane.showInputDialog(" veri dosyasının isminigiriniz : ");
   String in_name="kalinlik1.txt";
   double a[];
   a=new double[4];
   a[0] = 1.0;
   a[1] = 1.0;
   a[2] = 1.0;
   a[3] = 1.0;
   double x[][],y[]; //curve fitting value
   double xu1[][],yu1[]; ////curve fitting value
   double xu2[][],yu2[];
   double xu3[][],yu3[];
   int iN=500;
   ifxa f=new ifxa(in_name,a);
   double[] iLower={-1.0,-1.0,-1.0,-1.0};
   double[] iUpper={2,2,2,2};
   int iMaxiter=500;
    Thermal_Exchange_Optimizationi wo=new   Thermal_Exchange_Optimizationi(f,iN,iLower,iUpper,iMaxiter);
   double[][] x0=wo.solution();
   int i=0,j=0;
   double [] r2=NA41.nelder(f,x0[0]);
   String s="";
   s+="harris-hawks optimisation : \n"+Matrix.toStringT(x0[0]);
   s+="Nelder-Mead solution : \n"+Matrix.toStringT(r2);
   System.out.println(s);
   double E[][]=Text.readDoubleT(in_name);
   fx f1=new fx(r2);
   double xx[]=E[0];
   double yy[]=E[1];
   double zz[]=E[2];
   double ww[]=new double[xx.length];
   double ee[]=new double[xx.length];
   for(int ii=0;ii<xx.length;ii++)
   {   double xx1[]={xx[ii],yy[ii]};
                ww[ii]=f1.func(xx1);
```

```
                ee[ii]=zz[ii]-ww[ii];
              }
              double qq[][]={xx,yy,zz,ww,ee};
              String s1[]={"h","t","P","Pcalc","dP=P-Pcalc"};
    Text.printT(qq,s1);
    }
}
```

¶

```
 ---------- Capture Output ----------
> "E:\co\java\bin\javaw.exe" Thermal_Exchange_Optimization_test5
Thermal exchange optimisation :
          1.051951878301539
         -0.010213933271042
          0.764537767561333
         -0.048899008341797
Nelder-Mead solution :
          0.990155877760489
         -0.002517035624054
          0.765464840908594
         -0.048774935944010
```

USAGE OF PARAMETRIC VARIABLES IN CURVE FITTING

x and y values are given in Table 6.8-1. If it is tried to curve fit an equation to this data, it will fail no matter which method will be used. For the same x values two different y values are defined. In cases like this in order to curve fit the data an artificial parameter is added and x and y curvefit seperately with this artificial parametric variable.

Table 6.9-1 input data a.txt

i	0	1	2	3	4
T	0	1	2	3	4
X	-1	0	1	0	1
Y	0	1	0.5	0	-1

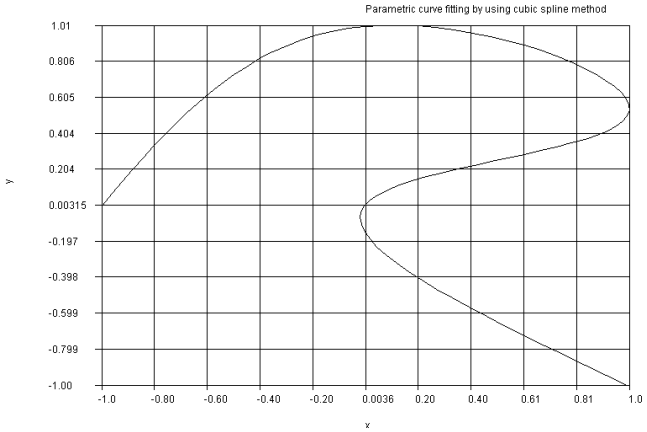
Program Cubic spline interpolation with indirect spline interpolation

```
import java.io.*;
import javax.swing.*;

class NA127
{
  public static double [] thomas(double a[][],double r[])
  {
  //
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }
```

```
public static double [] thomas(double f[],double e[],double g[],double r[])
{
int n=f.length;
double x[]=new double[n];
for(int k=1;k<n;k++)
   {e[k]=e[k]/f[k-1];
    f[k]=f[k]-e[k]*g[k-1];
   }
for(int k=1;k<n;k++)
   {r[k]=r[k]-e[k]*r[k-1];
   }
x[n-1]=r[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
   {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
return x;
}

public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
{
int n=xi.length;
double h[]=new double[n];
double w[]=new double[n];
double f[]=new double[n];
double e[]=new double[n];
double g[]=new double[n];
double d[]=new double[n];
double x[]=new double[n];
double S[][]=new double[4][n];
int k;
for(k=0;k<(n-1);k++)
   {h[k]=xi[k+1]-xi[k];
    w[k]=(yi[k+1]-yi[k])/h[k];
            }
d[0]=c0;
d[n-1]=cn;
for(k=1;k<(n-1);k++)
   {d[k]=6.0*(w[k]-w[k-1]);}
f[0]=1.0;
f[n-1]=1.0;
g[0]=0.0;
g[n-1]=0.0;
e[0]=0.0;
e[n-1]=0.0;
for(k=1;k<(n-1);k++)
{f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
S[2]=thomas(f,e,g,d);
S[3]=xi;
for(k=0;k<(n-1);k++)
  {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
   S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
  }
return S;
}

public static double funcSpline(double S[][],double x)
{
int n=S[0].length;
double xx1=0;
double xx2=0;
double y=0;
double hk=0;
```

```
 for(int k=0;k<(n-1);k++)
    {if(S[3][k]<=x && x<=S[3][k+1])
     {hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      break;
      }
    }
      if(y==0 && S[3][n-2]<=x )
      {
               int k=n-2;
               hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      }
 return y;
 }

 public static double[][]  funcSpline(double xi[],double yi[],int aradegersayisi)
 {
             //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
             // ara değer sayisi 2 dir
             int n=xi.length;
             int nn=(n-1)*(aradegersayisi+1)+1;
             double z[][]=new double[2][nn];
   double S[][]=cubic_spline(xi,yi,0,0);
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {  z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k++;
        for(int j=0;j<aradegersayisi;j++)
          {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
        z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k++;}
      }
     z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);
   return z;
 }

 public static void main(String args[]) throws IOException
 {
 double x[];
 double y[];
 String s1="a.txt";
 JFileChooser fc=new JFileChooser();
 if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file = fc.getSelectedFile();s1=file.getName();
}
 double a[][]=Text.readDoubleT(s1);
 x=a[0];
 y=a[1];
 int n=x.length;
 double t[]=new double [n];
 for(int i=0;i<n;i++) t[i]=i;
 double z1[][]=funcSpline(t,x,20);
 double z2[][]=funcSpline(t,y,20);
 System.out.println(" z1\n"+Matrix.toString(z1));
 System.out.println(" z2\n"+Matrix.toString(z1));
 Plot pp=new Plot(z1[1],z2[1]);
 pp.setPlabel("Parametric curve fitting by using cubic spline method");
 pp.setXlabel("x");
```

```
pp.setYlabel("y");
pp.setPlotType(0,0);
pp.addData(z1[1],z2[1]);
pp.plot();
}
}
```



MOVİNG LEAST SQUARES WITH SINGLE AND MULTI DIMENSIONAL VARIABLES (INTERPOLATION)

Polynomial least square equation was defined as

$$\frac{\partial H(a_0^{(m)},..,a_0^{(m)})}{\partial a_k^{(m)}} = 2\sum_{i=1}^{n} w(x_i)\left[f(x_i) - \sum_{j=0}^{m} a_j^{(m)} x^j\right] x^k = 0 \quad k = 0,\dots,m \qquad (6.1-6)$$

and this equation was solved for weight function $w(x_i)$=1. If it is desired that the accuracy of the equation to be improved, it will be seen that increasing coeefficients will not always give beter accuracy. To see this effect assume the following data and make 3rd 4th and 5th degree curve fitting into the data.

x	1.47	1.83	3.02	3.56	5.86	8.75	9.45
f	2.09	1.92	2.19	2.64	3.19	3.13	3.61

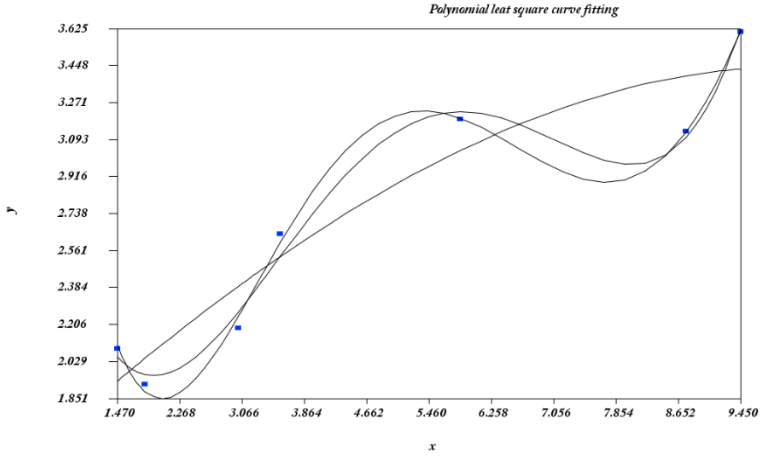
Program -1 3,4 ve 5th degree polynomial least square curve fitting

```
import java.io.*;
import javax.swing.*;

class NA128
{
//polynomial leat square
  public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivotlama
            p=k;
            buyuk=Math.abs(a[k][k]);
            for(ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[ii][k]);
              if(dummy > buyuk) {buyuk=dummy;p=ii;}
```

```
            }
            if(p!=k)
            { for(jj=k;jj<n;jj++)
              { dummy=a[p][jj];
                a[p][jj]=a[k][jj];
                a[k][jj]=dummy;
              }
              dummy=b[p];
              b[p]=b[k];
              b[k]=dummy;
            }
   for(i=k+1;i<n;i++)
   {  carpan=a[i][k]/a[k][k];
      a[i][k]=0;
      for(j=k+1;j<n;j++)
      {   a[i][j]-=carpan*a[k][j]; }
         b[i]   =b[i]   -carpan*b[k];
            }
  }
  //back substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] LSTSQR(double xi[],double yi[],int n)
{
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  for(j=0;j<n+1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= yi[k];
                  else      B[i] += Math.pow(xi[k],i)*yi[k];}
  }
System.out.println(Matrix.toString(A));
System.out.println(Matrix.toStringT(B));
X=pivotlugauss(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
return X;
}
```

```
public static double funcLSTSQR(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcLSTSQR(e,x[k]))*(y[k]-funcLSTSQR(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcLSTSQR(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
              //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
              // ara değer sayisi 2 dir
               int n=xi.length;
              int nn=(n-1)*(aradegersayisi+1)+1;
              double z[][]=new double[2][nn];
   double E[]=LSTSQR(xi,yi,polinomkatsayisi);
   System.out.println("coefficients :"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=funcLSTSQR(E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=funcLSTSQR(E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcLSTSQR(E,z[0][k]);
   return z;
 }


  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1="a.txt";
  //JFileChooser fc=new JFileChooser();
  //if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
```

```
  x=a[0];
  y=a[1];
  int n=x.length;
  double t[]=new double [n];
  for(int i=0;i<n;i++) t[i]=i;
  double z1[][]=funcLSTSQR(x,y,3,10);
  System.out.println("Least Square. z1\n"+Matrix.toString(z1));
   System.out.println("Least Square. z2\n"+Matrix.toString(z1));
  Plot pp=new Plot(z1[0],z1[1]);
  pp.addData(x,y);
  z1=funcLSTSQR(x,y,4,10);
  pp.addData(z1[0],z1[1]);
  z1=funcLSTSQR(x,y,5,10);
  pp.addData(z1[0],z1[1]);
  pp.setPlabel("Polynomial least square curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y");
  pp.setPlotType(0,0);
  pp.setPlotType(1,18);
  pp.plot();
  }
}
```



In here all the equations given different kind of results. Which one is correct or closer to the data then? It is not always easy to answer. As a different approach to this curve fitting, lets apply a continiously changing weight factor

$$w_i(x) = \frac{1}{|x - x_i|^k}$$

In this function when x value become equal to x_i the value of weight factor will go to infinity. $x \to x_i$ $w_i(x) \to \infty$. If it is assumed that for $x=x_i$, $f=f_i$ and solve the least square for the remaining points solution will be existed. But in this type of conditions due to the fact that w_i was changing in each region, least square curve fitting coefficients should be calculated for each point(region) separetely. This type of least square method is called **moving least squares.** An example program to moving least squares are presented below

Program Moving least square curve fitting , Java version

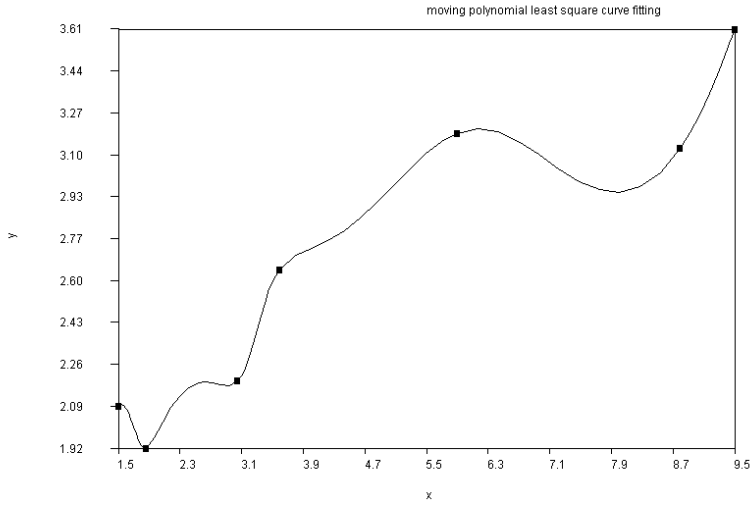
```
import java.io.*;
import javax.swing.*;

class NA129
{
//polynomial moving least square
  public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
```

```
  double dummy=0;
  //gauss eleme
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivotlama
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //gauss elemeyi çözme
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //geriye doğru yerine koyma
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double movingLSTSQR(double x,double xi[],double yi[],int n,int kk)
{
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
double w[]=new double[l];
for(i=0;i<l;i++) {if(x==xi[i]) return yi[i];}
for(i=0;i<l;i++) {w[i]=1.0/Math.pow(Math.abs(x-xi[i]),kk);}
for(i=0;i<np1;i++)
  {  for(j=0;j<np1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += w[k]*Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= w[k]*yi[k];
                        else      B[i] += w[k]*Math.pow(xi[k],i)*yi[k];}
  }
X=pivotlugauss(A,B);
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
```

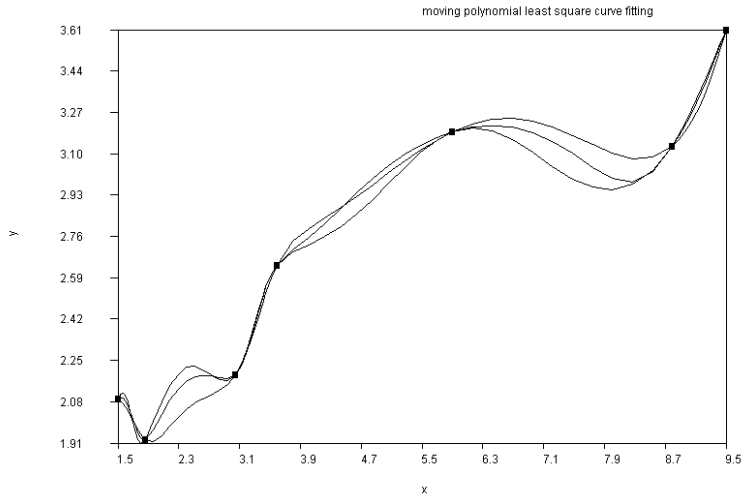
```
return funcEKK(X,x);
}

public static double funcEKK(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double[][]  funcEKK(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi,int kk)
 {
                int n=xi.length;
               int nn=(n-1)*(aradegersayisi+1)+1;
               double z[][]=new double[2][nn];
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
       {z[0][k]=xi[i];z[1][k]=movingLSTSQR(z[0][k],xi,yi,polinomkatsayisi,kk);k++;
       for(int j=0;j<aradegersayisi;j++)
       {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=movingLSTSQR(z[0][k],xi,yi,polinomkatsayisi,kk);k++;}
       }
       z[0][k]=xi[i];z[1][k]=movingLSTSQR(z[0][k],xi,yi,polinomkatsayisi,kk);
    return z;
 }
  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1="a.txt";
  //JFileChooser fc=new JFileChooser();
  //if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file = fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  int n=x.length;
  double t[]=new double [n];
  for(int i=0;i<n;i++) t[i]=i;
  double z1[][]=funcEKK(x,y,4,10,2);
  //System.out.println("E.K.K. z1\n"+Matrix.toString(z1));
  Plot pp=new Plot(z1[0],z1[1]);
  pp.setPlabel("moving polynomial least square curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y");
  pp.setPlotType(0,0);
  pp.addData(x,y);
   pp.setPlotType(1,21);
  pp.plot();
  }
}
```

The same data in the previous example will be now evaluated by moving least square of 4th order polynomial

With the weight factor $w_i(x) = \dfrac{1}{\left|x - x_i\right|^2}$



If this data is covered with different degree of polynomials(3rd, 4th and 5th degrees) :



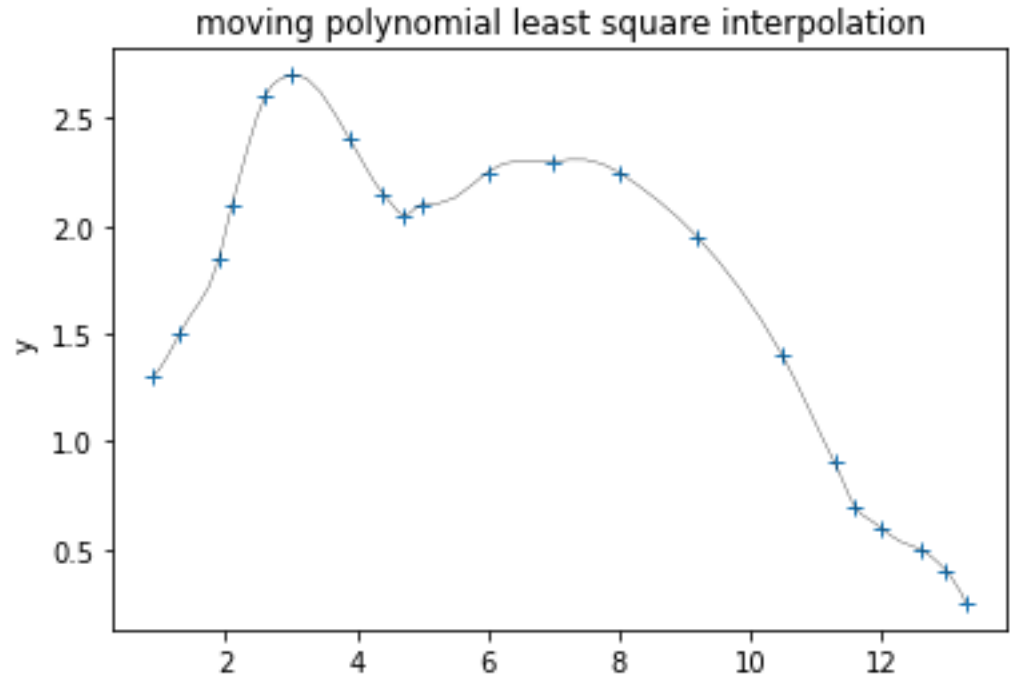
Program Moving least square curve fitting , Python version

```
# Moving least Square interpolation
from math import *
from gauss import *
import matplotlib.pyplot as plt

def movingLSTSQR(x,xi,yi,n,kk):
    # polynomial moving least square
    l=len(xi)
    np1=n+1
    A=[[0 for j in range(np1)] for i in range(np1)]
    B=[0 for j in range(np1)]
    X=[0 for j in range(np1)]
    w=[0 for j in range(l)]
    for i in range(l):
        if x==xi[i]:
            return yi[i]
    for i in range(l):
        w[i]=1.0/pow(abs(x-xi[i]),kk)
    for i in range(np1):
        for j in range(np1):
            if i==0 and j==0:
                A[i][j]=l;
            else:
                for k in range(l):
                    A[i][j] = A[i][j]+w[k]*pow(xi[k],(i+j));
        for k in range(l):
            if i==0:
                B[i]= B[i]+w[k]*yi[k];
            else:
```

```
            B[i]= B[i] +w[k]*pow(xi[k],i)*yi[k]
    X=gauss(A,B)
    #print("X=",X)
    max=0.0
    for i in range(n+1):
        if abs(X[i]) > max:
            max = abs(X[i])
    for i in range(n+1):
        if abs(X[i]/max) > 0 and abs(X[i]/max) < 1.0e-100:
            X[i]=0
    Y=funcEKK(X,x)
    #print("x=",x,"Y=",Y)
    return Y


def funcEKK(e,x):
    # this function calculates the value of
    # least square curve fitting function
    n=len(e)
    if n!=0:
        ff=e[n-1]
        for i in range(n-2,-1,-1):
            ff=ff*x+e[i]
    else:
        ff=0.0;
    return ff


def output(xi,yi,polinomkatsayisi,aradegersayisi,kk):
    n=len(xi)
    nn=(n-1)*(aradegersayisi+1)+1
    z=[[0 for j in range(nn)] for i in range(2)]
    dx=0.0
    k=0
    for i in range(n-1):
        z[0][k]=xi[i];z[1][k]=movingLSTSQR(z[0][k],xi,yi,polinomkatsayisi,kk)
        k=k+1
        for j in range(aradegersayisi):
            dx=(xi[i+1]-xi[i])/(float(aradegersayisi)+1.0)
            z[0][k]=z[0][k-1]+dx;z[1][k]=movingLSTSQR(z[0][k],xi,yi,polinomkatsayisi,kk)
            k=k+1
    z[0][k]=xi[n-1]
    z[1][k]=movingLSTSQR(z[0][k],xi,yi,polinomkatsayisi,kk)
    return z

def read(name):
    path = name
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    while True:
        line=file1.readline()
        nn+=1
        if  file1.tell()==endLocation:
            break
    x=[0.0 for i in range(nn)]
    y=[0.0 for i in range(nn)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        x[i]=float(ee[0])
        y[i]=float(ee[1])
        #print(x[i]," ",y[i]," ",z[i])
    file1.close()
    a=[x,y]
    #return Transpose(a)
    return a
def Transpose(left):
    # transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
    # int i,j;
    n=len(left)
    m=len(left[0])
    b=[[0 for j in range(n)] for i in range(m)]
```

```
    for i in range(n):
        for j in range(m):
            b[j][i]=left[i][j]
    return b

x=[0.9,1.3,1.9,2.1,2.6,3.0,3.9,4.4,4.7,5.0,6.0,7.0,8.0,9.2,10.5,11.3,11.6,12.0,12.6,13.0,13.3]
y=[1.3,1.5,1.85,2.1,2.6,2.7,2.4,2.15,2.05,2.1,2.25,2.3,2.25,1.95,1.4,0.9,0.7,0.6,0.5,0.4,0.25]
print("x=",x,"\ny=",y)
z1=output(x,y,10,3,2)
print("Moving least square interpolation \n",Transpose(z1))
plt.title('moving polynomial least square interpolation')
plt.ylabel('y ');
plt.plot(x,y,'+',z1[0],z1[1],'k',linewidth=0.3)
```

```
runfile('E:/okul/SCO1/movingLSTSQR.py', wdir='E:/okul/SCO1')
Reloaded modules: gauss
x= [0.9, 1.3, 1.9, 2.1, 2.6, 3.0, 3.9, 4.4, 4.7, 5.0, 6.0, 7.0, 8.0, 9.2, 10.5, 11.3, 11.6, 12.0, 12.6, 13.0, 13.3]
y= [1.3, 1.5, 1.85, 2.1, 2.6, 2.7, 2.4, 2.15, 2.05, 2.1, 2.25, 2.3, 2.25, 1.95, 1.4, 0.9, 0.7, 0.6, 0.5, 0.4, 0.25]
Moving least square interpolation
 [[0.9, 1.3], [1.0, 1.3399074194810645], [1.1, 1.3884269966852716], [1.2000000000000002, 1.4432593682824828], [1.3, 1.5], [1.45,
1.57801268377964], [1.5999999999999999, 1.643820985979865], [1.7499999999999998, 1.7204114787805167], [1.9, 1.85], [1.95,
1.9101641480772407], [2.0, 1.9757394093681733], [2.05, 2.040388880901058], [2.1, 2.1], [2.225, 2.2385686507389932], [2.35,
2.3760628693879484], [2.475, 2.5019715447208606], [2.6, 2.6], [2.7, 2.6493687856781385], [2.8000000000000003,
2.675628371156134], [2.9000000000000004, 2.690627045775627], [3.0, 2.7], [3.225, 2.687941544806875], [3.45,
2.621196612787153], [3.6750000000000003, 2.5172433271799317], [3.9, 2.4], [4.025, 2.333542777433391], [4.15,
2.266759243816306], [4.275, 2.2031290658102662], [4.4, 2.15], [4.4750000000000005, 2.123298654620633], [4.550000000000001,
2.0964777481116728], [4.625000000000001, 2.069080124159 49], [4.7, 2.05], [4.775, 2.0511411090741265], [4.8500000000000005,
2.0696495097577334], [4.925000000000001, 2.0891955432432976], [5.0, 2.1], [5.25, 2.1044708997799955], [5.5,
2.130582590521252], [5.75, 2.187517149684822], [6.0, 2.25], [6.25, 2.2905851628753586], [6.5, 2.3008898758714458], [6.75,
2.2977113105299214], [7.0, 2.3], [7.25, 2.3076216999052632], [7.5, 2.3072595475316318], [7.75, 2.288264584352389], [8.0, 2.25],
[8.3, 2.188014530103003], [8.600000000000001, 2.118324795204275], [8.900000000000002, 2.040457489270313], [9.2, 1.95],
[9.524999999999999, 1.836620236227467], [9.849999999999998, 1.7091493662389496], [10.174999999999997, 1.565955655881343],
[10.5, 1.4], [10.7, 1.282028995015975], [10.899999999999999, 1.153959493443806], [11.099999999999998, 1.0242851249841232],
[11.3, 0.9], [11.375, 0.8512700359358453], [11.45, 0.7980280248453694], [11.524999999999999, 0.7443948590388215], [11.6, 0.7],
[11.7, 0.6635742924114457], [11.799999999999999, 0.6430473518972486], [11.899999999999999, 0.6230474980782192], [12.0, 0.6],
[12.15, 0.5643837632309205], [12.3, 0.5356126866637697], [12.450000000000001, 0.5181854543312083], [12.6, 0.5], [12.7,
0.4820938600467023], [12.799999999999999, 0.4588386603926626], [12.899999999999999, 0.43150243883920625], [13.0, 0.4],
[13.075, 0.3725154872046411], [13.149999999999999, 0.33564367020276686], [13.224999999999998, 0.2996157341148143], [13.3,
0.25]]
```



The biggest disadvantage of this method is the requirement of matrix solution for each curve fitting point seperately. The equations are passing through each point. Therefore, this method is closer to interpolation methods compare to least square interpolations. Method can be applied also by using orthogonal polynomials,

instead of normal polynomials, and this way disadvantage of applying many matrix solutions will be removed from the equation set.

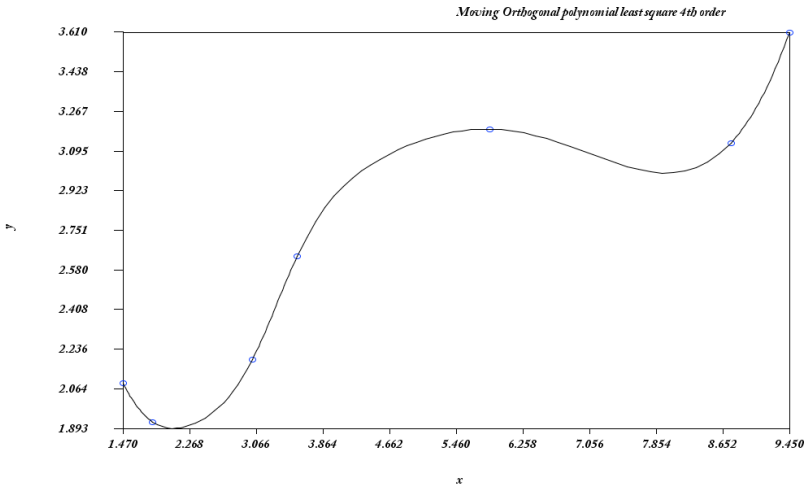
Program Moving least square curve fitting by using orthogonal polynomials, java version

```
import java.io.*;
import javax.swing.*;

class NA121A
{
public static double movingOPEKK(double x,double xi[],double fi[],int m,int kk)
//ortogonal polinomial moving least square method
//Referance : A First Course in Numerical Analysis
// Anthony Ralston,Philip Rabinowitz, Mc Graw Hill ISBN 0-07-051158-6
// m degree of polynomial xi fi input data
// kk weight factor power wi=1/|(x-xi)|^kk
{
int i,j,k;
int n=xi.length;
int mp2=n+2;
int mp1=n+1;
double p[][]=new double[mp2][n];
double gamma[]=new double[mp1];
double beta[]=new double[mp1];
double omega[]=new double[mp1];
double alpha[]=new double[mp1];
double b[]=new double[mp1];
double wi[]=new double[n];
double X[][]=new double[3][mp1];
for(i=0;i<n;i++) {if(x==xi[i]) return fi[i];}
for(i=0;i<n;i++)
  {
   p[1][i]=1.0;
   p[0][i]=0.0;
   wi[i]=1.0/Math.pow(Math.abs(x-xi[i]),kk);
  }
gamma[0]=0;
for(i=0;i<n;i++)
  {
  gamma[0]+=wi[i];
  }
beta[0]=0.0;
for(j=0;j<m+1;j++)
  {
  omega[j]=0;
  for(i=0;i<n;i++)
    {omega[j]+=wi[i]*fi[i]*p[j+1][i];}
  b[j]=omega[j]/gamma[j];
if( j != m)
    {
    alpha[j+1]=0;
    for(i=0;i<n;i++)
      { alpha[j+1]+=wi[i]*xi[i]*p[j+1][i]*p[j+1][i]/gamma[j];}
    for(i=0;i<n;i++)
      {p[j+2][i]=(xi[i]-alpha[j+1])*p[j+1][i]-beta[j]*p[j][i];}
      gamma[j+1]=0;
    for(i=0;i<n;i++)
      {gamma[j+1]+=wi[i]*p[j+2][i]*p[j+2][i];}
    beta[j+1]=gamma[j+1]/gamma[j];
    }
}//end of j
for(j=0;j<m+1;j++)
  {
  X[0][j]=b[j];
  X[1][j]=alpha[j];
  X[2][j]=beta[j];
  }
return funcOPEKK(X,x);
}

public static double funcOPEKK(double a[][],double x)
{
// polinom değerleri hesaplama fonksiyonu
double yy=0;
int k;
```

```
int m=a[0].length-1;
int mp2=m+2;
double q[];
q=new double[mp2];
//vector<double> q(m+2,0.0);
for(k=m-1;k>=0;k--)
  {
  q[k]=a[0][k]+(x-a[1][k+1])*q[k+1]-a[2][k+1]*q[k+2];
  yy=q[k];
  }
return yy;
}

public static double[][]  funcOPEKK(double xi[],double yi[],int polynomialcoefficient,int numberofmidpoints,int kk)
 {
              //number of midpoints:  x--o--o--x--o--o--x  x data points midpoints are 2
              int n=xi.length;
              int nn=(n-1)*(numberofmidpoints+1)+1;
              double z[][]=new double[2][nn];
   //double E[][]=OPEKK(xi,yi,polynomialcoefficient);
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=movingOPEKK(z[0][k],xi,yi,polynomialcoefficient,kk);
       k++;
      for(int j=0;j<numberofmidpoints;j++)
      {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=movingOPEKK(z[0][k],xi,yi,polynomialcoefficient,kk);k++;}
      }
      z[0][k]=xi[i];z[1][k]=movingOPEKK(z[0][k],xi,yi,polynomialcoefficient,kk);
   return z;
 }

 public static void main(String args[]) throws IOException
 {
 //double x[]={0.0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0};
 //double y[]={0.0,0.099833,0.1986693,0.2955202,0.38941834,0.479425,0.5646424,0.6442176,0.717356,0.783326,0.84147};
 String s1=JOptionPane.showInputDialog("file name : ");
 int n=Integer.parseInt(JOptionPane.showInputDialog("polinom derecesi (n)"));
 double c[][]=Text.readDoubleT(s1);
 double z3[][]=funcOPEKK(c[0],c[1],n,20,2);
 Text.print(Text.T(z3),"Moving Orthogonal polynomial least square ");
 Plot pp=new Plot(c[0],c[1]);
 pp.setPlabel("Moving Orthogonal polynomial least square ");
 pp.setPlotType(0,22);
 pp.addData(z3[0],z3[1]);
 pp.plot();
 }
}
```

Moving Orthogonal polynomial least square

0	1
1.47	2.09
1.487142857142857	2.07779704441685
1.5042857142857142	2.066029414208825
1.5214285714285714	2.054693812893271
1.5385714285714285	2.043786845502805
1.5557142857142856	2.0333050096474197
1.5728571428571427	2.023244686415547
1.5899999999999999	2.0136021311686294
1.607142857142857	2.004373464290527
1.624285714285714	1.9955546619634126
1.6414285714285712	1.987141547050553
1.6585714285714284	1.9791297801764047
1.6757142857142855	1.971514851104714
1.6928571428571426	1.9642920705252138
1.7099999999999997	1.9574565623699305
1.7271428571428569	1.9510032567899005
1.744285714285714	1.9449268839326925
1.7614285714285711	1.9392219686698753
1.7785714285714282	1.9338828264315624
1.7957142857142854	1.9289035603115945
1.8128571428571425	1.9242780596122262
1.83	1.92
1.8666666666666667	1.9082473193485172
1.9433333333333334	1.8999661657125702



Program Moving least square curve fitting by using orthogonal polynomials, Python version

```
 from f_xir import *
from gauss import *

class f1(f_xir):
#f(x,y)=a0+a1*x+a2*y+a3*x^2+a4*y^2+a5*xy+a6*x^3+a7*y^3+a8*x^2y
#      +a9*xy^2+a10*x^4+a11*y^4+a12*x^2y^2+a13x^3y+a14*xy^3
    def func(self,x,i):
        xx=0.0
        if i==0: xx=1.0
        elif i==1:   xx=x[0]
        elif i==2:   xx=x[1]
        elif i==3:   xx=x[0]*x[0]
        elif i==4:   xx=x[1]*x[1]
        elif i==5:   xx=x[0]*x[1]
        elif i==6:   xx=x[0]*x[0]*x[0]
        elif i==6:   xx=x[0]*x[0]*x[0]
        elif i==7:   xx=x[1]*x[1]*x[1]
        elif i==8:   xx=x[0]*x[0]*x[1]
        elif i==9:   xx=x[0]*x[1]*x[1]
        elif i==10:   xx=x[0]*x[0]*x[0]*x[0]
        elif i==11:   xx=x[1]*x[1]*x[1]*x[1]
        elif i==12:   xx=x[0]*x[0]*x[1]*x[1]
        elif i==13:   xx=x[0]*x[0]*x[0]*x[1]
```

```
        elif i==14:   xx=x[0]*x[1]*x[1]*x[1]
        return xx

# moving least square curve fitting with multivariable
#Euler distance
def d(x,xi):
    # Euler distance
    total=0
    n=len(x)
    for i in range(n):
        total=total+(x[i]-xi[i])*(x[i]-xi[i])
    return sqrt(total)
#weight function
def w(x,xi):
    d1=d(x,xi)
    eps=0.001
    alpha=1.0/16.0
    ww=exp(-alpha*d1*d1)/(d1*d1+eps)
    return ww

def Transpose(left):
    # transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
    # int i,j;
    n=len(left)
    m=len(left[0])
    b=[[0 for j in range(n)] for i in range(m)]
    for i in range(n):
        for j in range(m):
            b[j][i]=left[i][j]
    return b

def HEKK(x,xi,yi,n):
# n dimensional surface general least square curve fitting
# for the vector point x
    f=f1()
    l=len(xi)
    np1=n+1
    A=[[0 for j in range(np1)] for i in range(np1)]
    B=[0 for j in range(np1)]
    E=[0 for j in range(np1)]
    for i in range(n+1):
        B[i]=0
        for j in range(np1):
            A[i][j]=0.0
            for k in range(l):
                xx1=f.func(xi[k],i)
                xx2=f.func(xi[k],j)
                A[i][j]=A[i][j]+w(x,xi[k])*xx1*xx2
        for k in range(l):
                B[i]=B[i]+w(x,xi[k])*f.func(xi[k],i)*yi[k]
    E=gauss(A,B)
    max=0
    for i in range(n+1):
        if abs(E[i]) > max: max=abs(E[i])
    for i in range(n+1):
        if abs(E[i])/max > 0 and abs(E[i]/max) < 1.0e-100: E[i]=0.0
    X=func(E,x)
    return X

def func(e,x):
    #multidimensional function calculation
    f=f1()
    n=len(e)
    ff=0.0
    if n!=0:
        for i in range(n-1,-1,-1):
```

```
            ff=ff+e[i]*f.func(x,i)
    return ff

def HEKKn(x,xi,yi,n):
    # n dimension surface curve fitting for the array of vector points x
    m=len(x)
    z=[0 for j in range(m)]
    for i in range(m):
        z[i]= HEKK(x[i],xi,yi,n)
    return z;

def read(name):
    path = name
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    while True:
        line=file1.readline()
        nn+=1
        if  file1.tell()==endLocation:
            break
    x=[0.0 for i in range(nn)]
    y=[0.0 for i in range(nn)]
    z=[0.0 for i in range(nn)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        x[i]=float(ee[0])
        y[i]=float(ee[1])
        z[i]=float(ee[2])
        #print(x[i]," ",y[i]," ",z[i])
    file1.close()
    a=[x,y,z]
    return a
    #return Transpose(a)
def read1(name):
    path = name
    file1 = open(path,'r')
    file1.seek(0,2)  #jumps to the end
    endLocation=file1.tell()
    file1.seek(0) #jumps to the start
    nn=0
    while True:
        line=file1.readline()
        nn+=1
        if  file1.tell()==endLocation:
            break
    x=[0.0 for i in range(nn)]
    y=[0.0 for i in range(nn)]
    #z=[0.0 for i in range(nn)]
    file1.seek(0) #jumps to the start
    for i in range(nn):
        line=file1.readline()
        ee=line.split()
        x[i]=float(ee[0])
        y[i]=float(ee[1])
        #z[i]=float(ee[2])
        #print(x[i]," ",y[i]," ",z[i])
    file1.close()
    a=[x,y]
    #return a
    return Transpose(a)
```

```
def output(x,xi,yi,n):
    #gives curve fitting values together with the independent value set
    m=len(x)
    nn=len(x[0])
    c=[[0.0 for j in range(nn+1)] for i in range(m)]
    #z=[[0.0 for j in range(nn+1)] for i in range(m)]
    for i in range(m):
        for j in range(nn):
            c[i][j]=x[i][j]
    for i in range(m):
        c[i][nn]= HEKK(x[i],xi,yi,n)
    return c

def z(xmin,xmax,dx,ymin,ymax, dy):
    n=int((xmax-xmin)/dx+0.0001)
    m=int((ymax-ymin)/dy+0.0001)
    nn=(n+1)*(m+1)
    z=[[0 for j in range(2)] for i in range(nn)]
    x=0.0
    y=0.0
    for i in range(n+1):
        x=xmin+i*dx
        for j in range(m+1):
            y=ymin+j*dy
            z[i*(m+1)+j][0]=x
            z[i*(m+1)+j][1]=y
    return z
# Creating dataset


a=read('d.txt')
xy=z(0.0,4.0,0.5,0.0,4.0,0.5)
x=Transpose([a[0],a[1]])
y=a[2]
z=output(xy,x,y,12)
print(z)
```

```
 runfile('E:/okul/SCO1/moving_multidimensional_LSCF.py', wdir='E:/okul/SCO1')
Reloaded modules: f_xir, gauss
[[0.0, 0.0, -1.0620629998416146e-17], [0.0, 0.5, 0.25000000000000744], [0.0, 1.0, 0.9999999999999989], [0.0, 1.5,
2.2499999999999676], [0.0, 2.0, 3.999999999999995], [0.0, 2.5, 6.249999999999732], [0.0, 3.0, 9.000000000000027], [0.0, 3.5,
12.250000000000442], [0.0, 4.0, 15.999999999999977], [0.5, 0.0, 0.2499999999999984], [0.5, 0.5, 0.5000000000002016], [0.5, 1.0,
1.250000000000321], [0.5, 1.5, 2.4999999999998512], [0.5, 2.0, 4.249999999999982], [0.5, 2.5, 6.499999999999857], [0.5, 3.0,
9.250000000000014], [0.5, 3.5, 12.500000000010392], [0.5, 4.0, 16.250000000000547], [1.0, 0.0, 0.9999999999999979], [1.0, 0.5,
1.249999999999851], [1.0, 1.0, 2.0000000000000018], [1.0, 1.5, 3.2499999999999925], [1.0, 2.0, 5.00000000000002], [1.0, 2.5,
7.249999999999838], [1.0, 3.0, 9.999999999999963], [1.0, 3.5, 13.250000000011042], [1.0, 4.0, 17.00000000000005], [1.5, 0.0,
2.250000000000023], [1.5, 0.5, 2.499999999999966], [1.5, 1.0, 3.249999999999964], [1.5, 1.5, 4.500000000000141], [1.5, 2.0,
6.250000000000015], [1.5, 2.5, 8.500000000000169], [1.5, 3.0, 11.249999999999902], [1.5, 3.5, 14.499999999998959], [1.5, 4.0,
18.249999999999897], [2.0, 0.0, 4.000000000000005], [2.0, 0.5, 4.249999999999921], [2.0, 1.0, 5.000000000000009], [2.0, 1.5,
6.249999999999875], [2.0, 2.0, 8.000000000000053], [2.0, 2.5, 10.250000000000226], [2.0, 3.0, 13.00000000000017], [2.0, 3.5,
16.249999999996344], [2.0, 4.0, 19.999999999999957], [2.5, 0.0, 6.249999999999629], [2.5, 0.5, 6.499999999999341], [2.5, 1.0,
7.249999999999541], [2.5, 1.5, 8.50000000000014], [2.5, 2.0, 10.250000000000174], [2.5, 2.5, 12.500000000000274], [2.5, 3.0,
15.25000000000006], [2.5, 3.5, 18.5000000000023], [2.5, 4.0, 22.25000000000043], [3.0, 0.0, 9.000000000000007], [3.0, 0.5,
9.250000000000133], [3.0, 1.0, 10.000000000000012], [3.0, 1.5, 11.249999999999844], [3.0, 2.0, 13.000000000000135], [3.0, 2.5,
15.250000000000053], [3.0, 3.0, 18.00000000000015], [3.0, 3.5, 21.25000000000457], [3.0, 4.0, 25.000000000000234], [3.5, 0.0,
12.25000000000178], [3.5, 0.5, 12.500000000005063], [3.5, 1.0, 13.250000000004862], [3.5, 1.5, 14.499999999999472], [3.5, 2.0,
16.250000000001727], [3.5, 2.5, 18.49999999999745], [3.5, 3.0, 21.249999999990678], [3.5, 3.5, 24.499999999997236], [3.5, 4.0,
28.249999999999815], [4.0, 0.0, 16.000000000000004], [4.0, 0.5, 16.249999999999723], [4.0, 1.0, 17.000000000000043], [4.0, 1.5,
18.25000000000004], [4.0, 2.0, 19.999999999999986], [4.0, 2.5, 22.250000000000902], [4.0, 3.0, 25.00000000000011], [4.0, 3.5,
28.250000000000192], [4.0, 4.0, 32.00000000000008]]
```

Moving least square method can similarly be applied into multi-independent variable curve fitting problems. In principles number of independent variables in the equation does not effect the curve fitting process, except the definition of the functions. If a degree m polynomial

$$f(x_0, x_{1,}.., x_n) = \sum_{j=0}^{m} a_j^{(m)} \phi_j((x_0, x_{1,}.., x_n) \quad (6.3-1)$$

j th degree function is given. It is desired to fit x_0, $x_{1,}$ x_2,…, x_n ,$f_{i,}$ i=0...n data set into this function. The best fitted $a_j^{(m)}$ values to be found. For this purpose Minimum value of

$$H(x_0, x_{1,}.., x_n) = \sum_{i=1}^{n} w(x_0, x_{1,}.., x_n) \left[f_i - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x_0, x_{1,}.., x_n) \right]^2 \quad (6.3-2)$$

function should be found. $w(x_0, x_{1,}.., x_n)$ is called weight function and it should satisfy the condition $w(x_0, x_{1,}.., x_n) \geq 0 \quad i = 1, \dots, n.$ The minimum of the function is the root of the derivative of the function.

$$\frac{\partial H(x_0, x_{1,}.., x_n)}{\partial a_k^{(m)}} = 2 \sum_{i=1}^{n} w(x_0, x_{1,}.., x_n) \left[f_i - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x_0, x_{1,}.., x_n) \right] \phi_k(x_0, x_{1,}.., x_n) = 0 \quad k = 0,.., m \quad (6.3-2)$$

If unit value is taken as weight function, equation becomes:

$$2 \sum_{i=1}^{n} \left[f_i - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x_0, x_{1,}.., x_n) \right] \phi_k(x_0, x_{1,}.., x_n) = 0 \qquad k = 0,.., m \quad (6.3-2a)$$

This equation was solved in chapter 6.3 by using $w(x_0, x_{1,}.., x_n)$=1 as classical least square approach. In order to define weight factor function Euclidian distance between the points will be used. If two vectoral point is defined in Euclidian space as:

$x = (x_0, x_{1,}.., x_n)$ (6.10-1)

$x^i = (x_0^i, x_1^i, .., x_n^i)$ (6.10-2)

Euclidian distance between x and x^i will be given by

$$d(x, x^i) = \sqrt{\sum_{j=0}^{n} (x_j - x_j^i)^2} \quad (6.10-3)$$

$w_i(x) = w(d(x, x^i))$ (6.10-4)

Now the weight function can be defined as a function of Euclidian distance as:

.

As an example definition of an actual weight function the following equation can be given

$$w(d) = \frac{e^{-d^2/16}}{d^2 + 0.001}$$

In this equations x is the point that curve fitting value is desired and x^i is the neighbouring point. When this weight factor is applied, program in the chapter 6.3 will be converted to the moving least square form.

Program moving least square curve fitting of multidimensional independent variable function

```
 import java.io.*;
import java.util.*;
import javax.swing.*;
import java.awt.event.*;

//  Surface fitting by using moving least square model
//  f(x,y)=a0+a1*x+a2*y+a3*x^2+a4*y^2+a5*xy+a6*x^3+a7*y^3+a8*x^2y
//  +a9*xy^2+a10*x^4+a11*y^4+a12*x^2y^2+a13x^3y+a14*xy^3

abstract class f_xir
{ abstract double func(double x[],int equation_ref);}

//You can define your own fitting function
class fa extends f_xir
{
        double func(double x[],int i)
        {//A general two dimensional surface polynomial equation
         // of order (up to) 14
   double xx=0.0;
   //density function
   if(i==0)     {xx=1.0;        }
   else if (i==1) {xx=x[0];       }
   else if (i==2) {xx=x[1];       }
   else if (i==3) {xx=x[0]*x[0];    }
```

```
   else if (i==4) {xx=x[1]*x[1];     }
   else if (i==5) {xx=x[0]*x[1];     }
   else if (i==6) {xx=x[0]*x[0]*x[0];}
   else if (i==7) {xx=x[1]*x[1]*x[1];}
   else if (i==8) {xx=x[0]*x[0]*x[1];}
   else if (i==9) {xx=x[0]*x[1]*x[1];}
   else if (i==10){xx=x[0]*x[0]*x[0]*x[0];}
   else if (i==11){xx=x[1]*x[1]*x[1]*x[1];}
   else if (i==12){xx=x[0]*x[0]*x[1]*x[1];}
   else if (i==13){xx=x[0]*x[0]*x[0]*x[1];}
   else if (i==14){xx=x[0]*x[1]*x[1]*x[1];}
   return xx;
            }
}

class NA129H
{
            public static double d(double x[],double xi[])
            {// Euler distance
            double total=0;
            for(int i=0;i<x.length;i++)
            {total+=(x[i]-xi[i])*(x[i]-xi[i]);}
            return Math.sqrt(total);
   }

   public static double w(double x[],double xi[])
   {
            // Moving least square weight function
            double d=d(x,xi);
            double eps=0.001;
            double alpha=1.0/16.0;
            double ww=Math.exp(-alpha*d*d)/(d*d+eps);
            return ww;
            }

            public static double[][] Transpose(double [][] left)
   { //transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
   int i,j;
   int n=left.length;
   int m=left[0].length;
   double b[][];
   b=new double[m][n];
   for(i=0;i<n;i++)
     {for(j=0;j<m;j++)  {b[j][i]=left[i][j];} }
   return b;
   }

  public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
            p=k;
            buyuk=Math.abs(a[k][k]);
            for(ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[ii][k]);
              if(dummy > buyuk) {buyuk=dummy;p=ii;}
            }
            if(p!=k)
            { for(jj=k;jj<n;jj++)
              { dummy=a[p][jj];
                a[p][jj]=a[k][jj];
                a[k][jj]=dummy;
```

```
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
               }
               //gauss elimination process
   for(i=k+1;i<n;i++)
   {  carpan=a[i][k]/a[k][k];
      a[i][k]=0;
      for(j=k+1;j<n;j++)
      {   a[i][j]-=carpan*a[k][j]; }
         b[i]   =b[i]   -carpan*b[k];
               }
  }
  //back substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double HEKK(double x[],double xi[][],double yi[],int n)
{ // n dimension surface curve fitting for the vector point x[]
fa f=new fa();
int l=xi.length; //number of independent variables
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double E[];
E=new double[np1];
for(i=0;i<n+1;i++)
  {  B[i]=0;
                for(j=0;j<np1;j++)
    {
                A[i][j]=0.0;
    for(k=0;k<l;k++) A[i][j]+= w(x,xi[k])*f.func(xi[k],i)*f.func(xi[k],j);
    }
    for(k=0;k<l;k++) B[i]+= w(x,xi[k])*f.func(xi[k],i)*yi[k];
  }
E=pivotlugauss(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(E[i]) > max) max = Math.abs(E[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(E[i]/max) > 0) && (Math.abs(E[i]/max) < 1.0e-100)) E[i]=0;
double z=funcEKKgenel(E,x);
return z;
}


public static double funcEKKgenel(double e[],double x[])
{
// Moving least square function evaluation
fa f=new fa();
int n=e.length;
double ff=0;
if(n!=0.0)
  {
  for(int i=n-1;i>=0;i--)
   {ff+=e[i]*f.func(x,i);}
  }
```

```
return ff;
}

public static double[] HEKK(double x[][],double xi[][],double yi[],int n)
{
                // n dimension surface curve fitting for the array of vector points x[]
int m=x.length;
double z[]=new double[m];
for(int i=0;i<m;i++)
{
z[i]= HEKK(x[i],xi,yi,n);
}
return z;
}

public static double[][] output(double x[][],double xi[][],double yi[],int n)
{  //gives curve fitting values together with the independent value set
int m=x.length;
int nn=x[0].length;
double c[][]=new double[m][nn+1];
double z[]=new double[m];
for(int i=0;i<m;i++)
{for(int j=0;j<nn;j++)
                {c[i][j]=x[i][j];}
}
for(int i=0;i<m;i++)
{
c[i][nn]= HEKK(x[i],xi,yi,n);
}
return c;
}
  public static double[][] z(double xmin, double xmax, double dx,double ymin,double ymax, double dy)
  {int n=(int)((xmax-xmin)/dx+0.0001)+1;int m=(int)((ymax-ymin)/dy+0.0001)+1;
   double z[][]=new double[(n+1)*(m+1)][2];
   double x=0,y=0;
   for(int i=0;i<n;i++)
   {x=xmin+i*dx;
    for(int j=0;j<m;j++)
    {y=ymin+j*dy;
     z[i*m+j][0]=x;
     z[i*m+j][1]=y;
    }
  }
  return z;
  }
  public static void main(String[] args)
  {
  String s1=JOptionPane.showInputDialog("curve fitting input data : ");
  double c[][]=Text.readDouble(s1);
  int n=c[0].length;
  int m=n-1;
  int l=c.length;
  double x[][]=new double[l][m];
  double y[]=new double[l];
  for(int i=0;i<l;i++)
  {for(int j=0;j<m;j++)
   {x[i][j]=c[i][j];}
   y[i]=c[i][m];
  }
  //String s2=JOptionPane.showInputDialog("required curve fitting independent variable list data : ");
  //double xy[][]=Text.readDouble(s2);
  double xy[][]=z(0.0,4.0,1,0.0,4.0,1);
  //double xy[][]=z(0.0,3.0,1.0,0.0,15.0,1.0);
  Text.print(xy,"xy");
  double z[][]=output(xy,x,y,12);
  Text.print(z);
  //surfacePlot p=new surfacePlot(c);
  //p.plot();
  surfacePlot p1=new surfacePlot(z);
  p1.plot();
```

```
  }
}
```

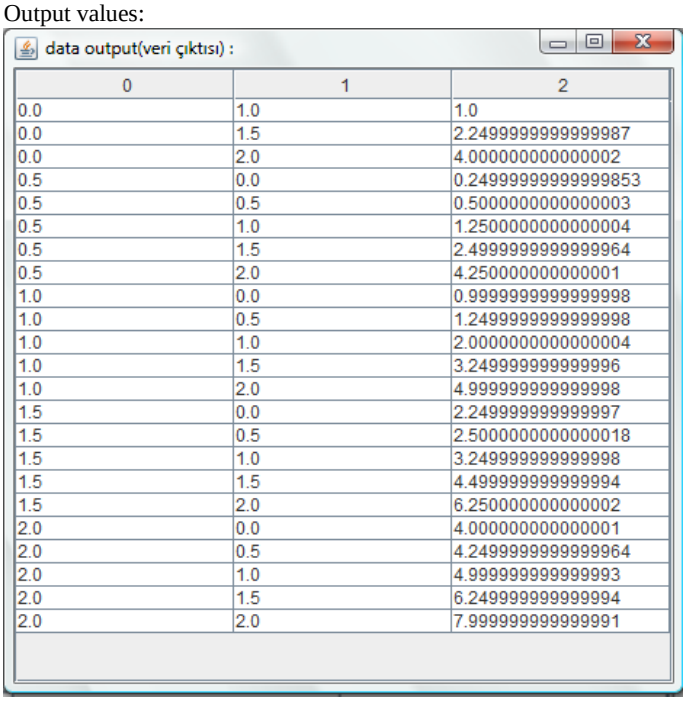
Example data for the program x_{ij},y_i sample data is generated by using the function $y=x_0^2+x_1^2$

0	0	0	3	0	9
0	1	1	3	1	10
0	2	4	3	2	13
0	3	9	3	3	18
0	4	16	3	4	25
1	0	1	4	0	16
1	1	2	4	1	17
1	2	5	4	2	20
1	3	10	4	3	25
1	4	17	4	4	32
2	0	4			
2	1	5			
2	2	8			
2	3	13			
2	4	20			

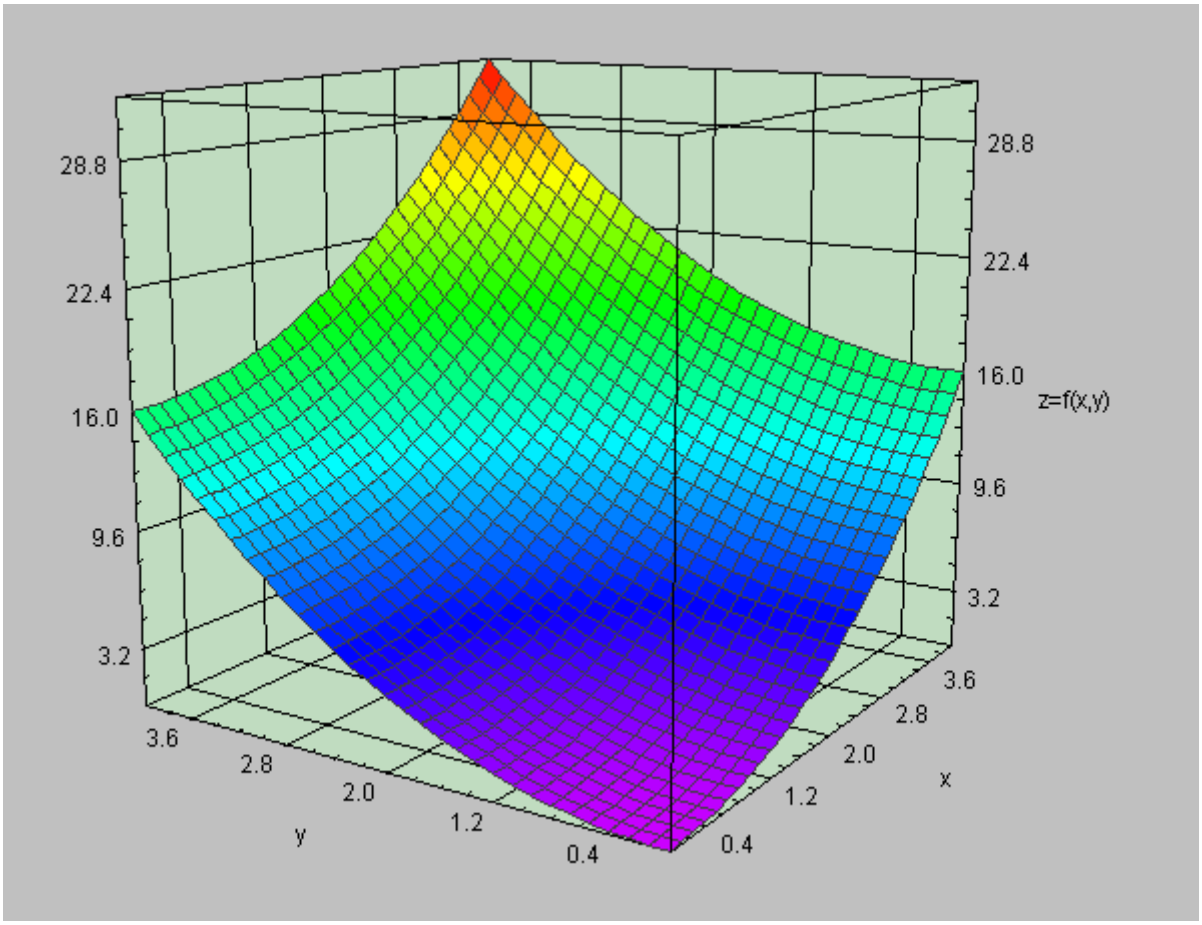
Independent variable data to curvefit

0	1	1.5	0
0	1.5	1.5	0.5
0	2	1.5	1
0.5	0	1.5	1.5
0.5	0.5	1.5	2
0.5	1	2	0
0.5	1.5	2	0.5
0.5	2	2	1
1	0	2	1.5
1	0.5	2	2
1	1		
1	1.5		
1	2		

Output values:



0	1	2
0.0	1.0	1.0
0.0	1.5	2.2499999999999987
0.0	2.0	4.000000000000002
0.5	0.0	0.24999999999999853
0.5	0.5	0.5000000000000003
0.5	1.0	1.2500000000000004
0.5	1.5	2.4999999999999964
0.5	2.0	4.250000000000001
1.0	0.0	0.9999999999999998
1.0	0.5	1.2499999999999998
1.0	1.0	2.0000000000000004
1.0	1.5	3.249999999999996
1.0	2.0	4.999999999999998
1.5	0.0	2.249999999999997
1.5	0.5	2.5000000000000018
1.5	1.0	3.249999999999998
1.5	1.5	4.499999999999994
1.5	2.0	6.250000000000002
2.0	0.0	4.000000000000001
2.0	0.5	4.2499999999999964
2.0	1.0	4.999999999999993
2.0	1.5	6.249999999999994
2.0	2.0	7.999999999999991



BERNSTEIN POLINOMIALS AND BEZIER CURVES



Figure Felix Bernstein, German Mathematician

Bernstein polynomials are a serial function approximation using the following form in a range of (a,b)

$$B_n(f;x) = \sum_{i=0}^{n} \binom{n}{i} (x-a)^i (b-x)^{n-i} f(x_i) \qquad (6.11-1)$$

x_i in the equation can be written for constant point differences $x_i=a+ih=a+(i/n)(b-a)$ i=0..n . If the same function is written in (0,1) range:

$$B_n(f;x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{n,i}(x) \quad (6.11-2)$$

$$B_{n,i}(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} \qquad (6.11-3)$$

When Bernstein polynomials are used for curve fitting approximations, it is seen that convertion rates are very low. Very high number of iterations are required for relatively good approximations. They will pass from starting and end points exactly but for the rest of the points it approximate slowly. But if mid-points are only taken as a reference. A practical method for drawing complicated shapes is obtained. This type of Bernstein polynomial applications have a special name: Bezier curves. Bezier curves uses parametric variables. For any (x_i , y_i) set with the parametric variable t

$$x(t) = \sum_{i=0}^{n} x_i \, B_{n,i}(t) \qquad (6.11-4)$$

$$y(t) = \sum_{i=0}^{n} y_i \, B_{n,i}(t) \qquad (6.11-5)$$

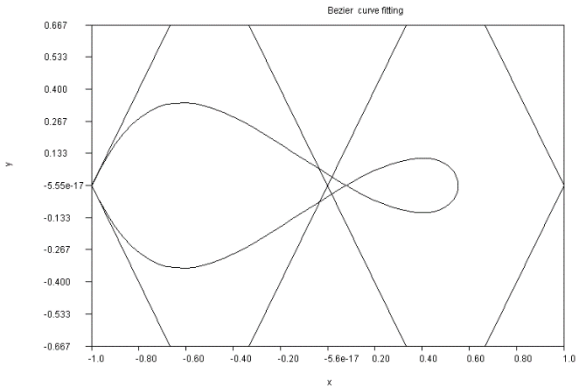
Let us use the following Example data to create th Bezier curve:

-1.0000000000 0.0000000000
-0.6666666667 0.6666666667
-0.3333333333 0.6666666667
0.3333333333 -0.6666666667
0.6666666667 -0.6666666667
1.0000000000 0.0000000000
0.6666666667 0.6666666667
0.3333333333 0.6666666667
-0.3333333333 -0.6666666667
-0.6666666667 -0.6666666667
-1.0000000000 0.0000000000

Program Bezier curve generator

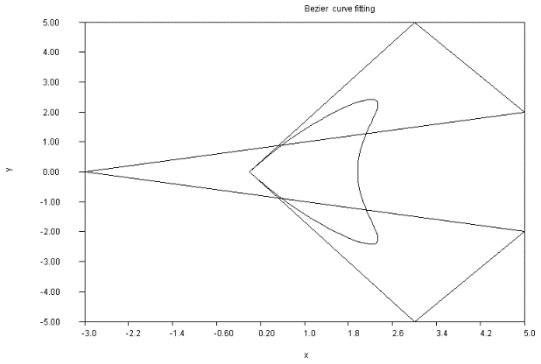
```
import java.io.*;
import javax.swing.*;

class NA130
{
//Bernstein polinomials and Bezier curves
  public static double[] B(int n,double t)
  { //  0<=t<=1
              double b[]=new double[n+1];
    b[0]=1.0;
    for(int i=1;i<=n;i++)
    {b[i]=t*b[i-1];
     for(int j=i-1;j>=1;j--)
     {b[j]=(1.0-t)*b[j]+t*b[j-1];}
     b[0]=(1.0-t)*b[0];
    }
  return b;
  }

  public static double funcB(double a[],double t)
  { double xx=0;
    int n=a.length-1;
    double b[]=B(n,t);
    for(int i=0;i<=n;i++)
```

```
    {xx+=a[i]*b[i];}
  return xx;
  }

  public static double[][] Bezier(double xi[],double yi[],int noktasayisi)
  {
            int n=xi.length;
            double z[][]=new double[2][noktasayisi];
   double dt=1.0/(double)noktasayisi;
   double t=0;
   int k=0;
   int i;
   for(i=0;i<noktasayisi;i++)
     {z[0][i]=funcB(xi,t);z[1][i]=funcB(yi,t);
      t=t+dt;
     }
   return z;
  }



  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1="a.txt";
  //JFileChooser fc=new JFileChooser();
  //if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  int n=x.length;
  //for(double t=0;t<=1.0;t+=0.1) {System.out.println(funcB(4,x,t)+" "+funcB(4,y,t));}
  double z1[][]=Bezier(x,y,100);
  //System.out.println("E.K.K. z1\n"+Matrix.toString(Matrix.T(z1)));
  Plot pp=new Plot(z1[0],z1[1]);
  pp.setPlabel("Bezier  curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y");
  pp.setPlotType(0,0);
  pp.addData(x,y);
  pp.addData(x,y);
  pp.setPlotType(2,18);
  pp.plot();
  }
}
```



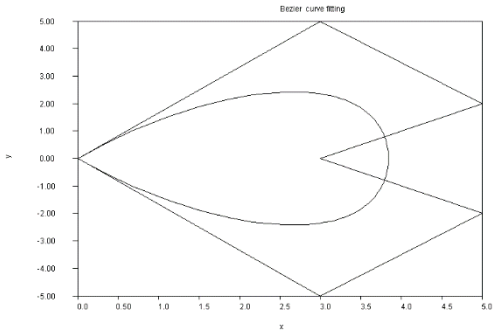
By changing reference points very complicated shapes can be drawn For example

0 0
3 5
5 2
-3 0
5 -2
3 -5
0 0



0 0
3 5
5 2
3 0
5 -2
3 -5

0 0



CURVE FITTING BY USING PARTIALLY CONTINOUS FUNCTIONS

One of the problems we faced in the real world least square curve fitting applications is that when the degree of equations are small, the curve fitting equation is unaccurate and when the degree of equation is increased round of error will considerably grow. If equation are defined for a smaller ranges, they tend to approach to the linear relations. So If the error level of a curve fitting equation can not be reduced to the desired level, one of the usefull way is defining partially continious functions for smaller regions of the curve. The method mathematically similar to the method of defining one curve, except that a family of the curves will be defined here instead of one curve. For each curve equation valid limits of the equations should also be given as additional coefficients. As an example, specific heat of perfect gases should be defined accurately for an accurate termodynamic model of the perfect gases. The specific heat of the gases will be defined with a partially continous function

$Cp=a_0 + a_1x10^{-3}T + a_2x10^5/T^2 + a_3x10^{-5}T^2$ where $T_L<T<T_H$. Nitrogen data will be taken as an example data. C_p values are given as (kJ/kg K) T unit is in degree Kelvin.

Table 6.12-1 Temperature – ideal gas Specific heat data for Nitrogen (N_2) gas

298.15 29.123	1900 35.794	3600 37.373
300 29.125	2000 35.969	3700 37.42
400 29.246	2100 36.126	3800 37.464

500 29.583	2200 36.267	3900 37.507
600 30.113	2300 36.395	4000 37.548
700 30.752	2400 36.511	4100 37.588
800 31.43	2500 36.616	4200 37.627
900 32.094	2600 36.713	4300 37.665
1000 32.696	2700 36.802	4400 37.701
1100 33.24	2800 36.884	4500 37.737
1200 33.723	2900 36.961	4600 37.773
1300 34.148	3000 37.031	4700 37.808
1400 34.519	3100 37.097	4800 37.842
1500 34.844	3200 37.159	4900 37.877
1600 35.128	3300 37.217	5000 37.911
1700 35.378	3400 37.272	
1800 35.598	3500 37.324	

Program Partially continious curve-fit example

```
import java.io.*;
import java.util.*;
import javax.swing.*;


abstract class f_xr
{ abstract double func(double x,int equation_ref);}

class fh extends f_xr
{
double func(double t,int i)
{
double T=t+273.15;
        double xx=0.0;
if(i==0) xx=T;
else if(i==1) xx=1.0e-3/2.0*T*T;
else if(i==2) xx=-1.0e5/T;
else if(i==3) xx=1.0e-6/3.0*T*T*T;
return xx;
}
}

class fcp extends f_xr
{
double func(double t,int i)
{ // Cp(T) = xa[i]+xb[i]*1e-3*T+xc[i]*1.0e5/T^2+xd[i]*1e-6*T^2
double T=t;
        double xx=0.0;
if(i==0) xx=1.0;
else if(i==1) xx=1.0e-3*T;
else if(i==2) xx=1.0e5/(T*T);
else if(i==3) xx=1.0e-6*T*T;
return xx;
}
}

class Gas_GEN_Data
{
//this class reads and store and proses ideal gas data
public double M;       // kg/kmol
public double h0;      // KJ/kmol at temperature a[0][0]=Tref
public double s0;      // KJ/kmolK at temperature a[0][0]=Tref
public double hf;      // KJ/kmol at 298 K
public double R=8.3145; // KJ/kgmolK
public double a[][];
public String GasName;
public String openName;
public String ekbilgi;
int fitoption;
f_xr f1;
f_xr f2;
```

```
public double cpl[][];  // ideal gas heat capacitiy
public double visl[][]; // saturated liquid viscosity
public double kl[][]; // saturated liquid thermal conductivity

//General partial continious least square method
public Gas_GEN_Data(int fitoptioni,String GasNamei,String openNamei,double ai[][],int ngroup,double Mi, double h0i,double
s0i,double hfi)
{
//fitoption : 0   Cp
//other         h
fitoption=fitoptioni;
GasName=GasNamei;
a=ai;
ekbilgi=" ";
openName=openNamei;
int n=a.length;
int i,j;
if(fitoption==0) {f1=new fcp(); }
else            {f1=new fh(); }
f2=new fcp();
cpl=EKKgeneral(f1,a[0],a[2],ngroup);
visl=EKKgeneral(f2,a[0],a[3],ngroup);
kl=EKKgeneral(f2,a[0],a[4],ngroup);
M=Mi;
h0=h0i;
s0=s0i;
hf=hfi;
}

public static double[] pivotlugauss(double a[][],double b[])
{ //gauss eliminiation with partial pivoting
int n=b.length;
double x[]=new double[n];
double carpan=0;
double toplam=0;
double buyuk;
double dummy=0;
//gausselimination
int i,j,k,p,ii,jj;
for(k=0;k<(n-1);k++)
{ //pivoting
p=k;
buyuk=Math.abs(a[k][k]);
for(ii=k+1;ii<n;ii++)
{  dummy=Math.abs(a[ii][k]);
            if(dummy > buyuk) {buyuk=dummy;p=ii;}
}
if(p!=k)
{ for(jj=k;jj<n;jj++)
            { dummy=a[p][jj];
            a[p][jj]=a[k][jj];
            a[k][jj]=dummy;
            }
            dummy=b[p];
            b[p]=b[k];
            b[k]=dummy;
}
//solving of gauss elimination
for(i=k+1;i<n;i++)
{  carpan=a[i][k]/a[k][k];
a[i][k]=0;
for(j=k+1;j<n;j++)
{   a[i][j]-=carpan*a[k][j]; }
b[i]   =b[i]   -carpan*b[k];
}
}
//back substitution
x[n-1]=b[n-1]/a[n-1][n-1];
for(i=n-2;i>=0;i--)
{
```

```
toplam=0;
for(j=i+1;j<n;j++)
{  toplam+=a[i][j]*x[j];}
x[i]=(b[i]-toplam)/a[i][i];
}
return x;
}

public static double[][] EKKgeneral(f_xr f, double xi[],double yi[],int ngroup)
{
//ngroup : number of data in each group
//negri  : degree of curve fitting
//groupnumber : number of group
int negri=3;
int n=negri;
double y[]=new double[xi.length];
for(int i=0;i<xi.length;i++)
        {y[i]=yi[i];}
if(ngroup<negri) ngroup=negri;
int Nxi=xi.length; //total number of data
int groupnumber=Nxi/ngroup;
int ngroup1=ngroup+Nxi%ngroup;
//
int i,j,k;
int np1=4;
int np5=np1+2;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[][];
X=new double[groupnumber][np5];
System.out.println("group number = "+ groupnumber);
double Y[]=new double[np1];
int l=0;
double max=0;
//
for(l=0;l<groupnumber-1;l++)
{
for(i=0;i<np1;i++)
{  B[i]=0;
        for(j=0;j<np1;j++)
{
        A[i][j]=0.0;
for(k=l*ngroup;k<=(l+1)*ngroup;k++) {A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);}
}
for(k=l*ngroup;k<=(l+1)*ngroup;k++) B[i]+= f.func(xi[k],i)*y[k];
}
Y=pivotlugauss(A,B);//pivotlugauss(A,B);

int ii;
for(ii=0;ii<Y.length;ii++) {X[l][ii]=Y[ii];}
X[l][ii]=xi[l*ngroup];
X[l][ii+1]=xi[(l+1)*ngroup];
for(i=0;i<n+1;i++)
if(Math.abs(X[l][i]) > max) max = Math.abs(X[l][i]);
for(i=0;i<n+1;i++)
if((Math.abs(X[l][i]/max) > 0) && (Math.abs(X[l][i]/max) < 1.0e-100))X[l][i]=0;
}
l=groupnumber-1;
int k1=l*ngroup;
int k2=(l+1)*ngroup+Nxi%ngroup;
for(i=0;i<np1;i++)
{  B[i]=0;
        for(j=0;j<np1;j++)
{
        A[i][j]=0.0;
for(k=k1;k<k2;k++) {A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);}
}
for(k=k1;k<k2;k++) {B[i]+= f.func(xi[k],i)*y[k];}
```

```
}
Y=pivotlugauss(A,B);
int ii;
for(ii=0;ii<Y.length;ii++) {X[l][ii]=Y[ii];}
X[l][ii]=xi[k1];
X[l][ii+1]=xi[k2-1];

return X;
}

public static double funcEKKgeneral(f_xr f,double e[][],double x)
{
// this function calculates the value of
// least square curve fitting function
int n1=e.length;
int n2=e[0].length;
int n3=n2-2;
System.out.println("n1="+n1+"n2="+n2+"n3="+n3);
double ff=0;
double xlow,xhigh;
for(int j=0;j<n1;j++)
{
xlow=e[j][n3];
xhigh=e[j][n3+1];
System.out.println("xlow="+xlow+"xhigh="+xhigh);
if((x>=xlow) && (x<=xhigh))
{
ff=0;
for(int i=0;i<n3;i++)
{ff+=e[j][i]*f.func(x,i);System.out.println("ff="+ff+"i="+i);}
break;
}
}
return ff;
}

public static double[][] xyfit(f_xr f,double x[],double yi[],double e[][])
{
//calculates absolute square root error of a least square approach
double n=x.length;
int k;
double total=0;
double yy=0;
double z[][]=new double[2][x.length];
for(k=0;k<n;k++)
{   z[0][k]=x[k];
            z[1][k]=funcEKKgeneral(f,e,x[k]);
}
return z;
}


public static double[][] dxyfit(f_xr f,double x[],double yi[],double e[][])
{
//calculates absolute square root error of a least square approach
double n=x.length;
int k;
double total=0;
double yy=0;
double z[][]=new double[2][x.length];
for(k=0;k<n;k++)
{   z[0][k]=x[k];
            z[1][k]=yi[k]-funcEKKgeneral(f,e,x[k]);
}
return z;
}

public static double hata(f_xr f,double x[],double yi[],double e[][])
{
//calculates absolute square root error of a least square approach
```

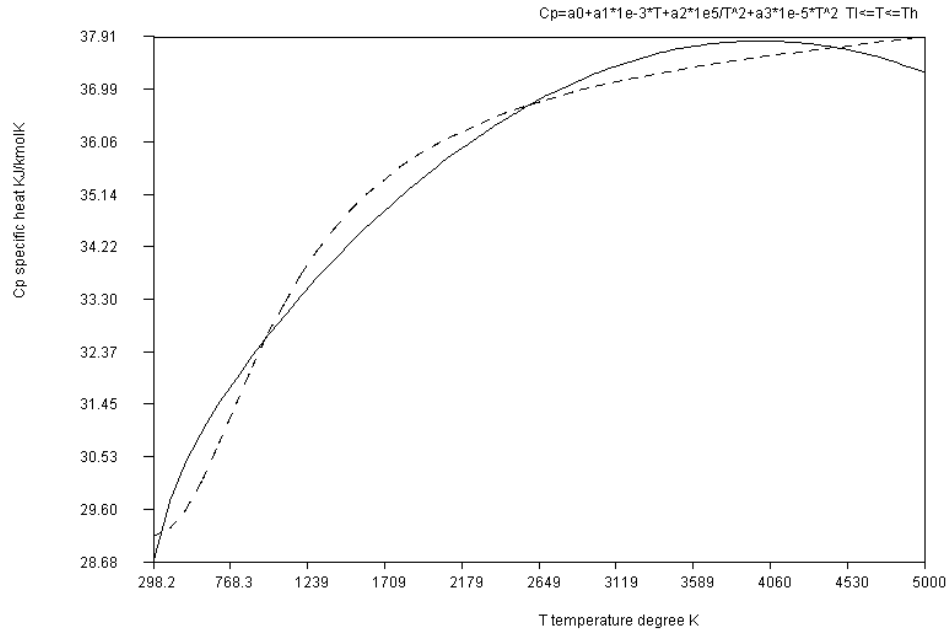
```
double n=x.length;
int k;
double total=0;
double yy=0;
for(k=0;k<n;k++)
{
        yy=yi[k]-funcEKKgeneral(f,e,x[k]);
total+=yy*yy;
}
total=Math.sqrt(total/(n-1));
return total;
}
public static double[][]  hataEKKPlot(f_xr f,double xi[],double yi[],int ngroup)
{
//ngroup : Number of data in each group
int n=xi.length;
int l;
double z[][];
double E[][];//=EKKgeneral(xi,yi,polinomkatnumber,ngroup);
double dx=0;
int k=0;
int i;
E=EKKgeneral(f,xi,yi,ngroup);
z=dxyfit(f,xi,yi,E);
Plot pp=new Plot(z[0],z[1]);
pp.setPlabel("Cp=a0+a1*1e-3*T+a2*1e5/T^2+a3*1e-5*T^2  Tl<=T<=Th");
pp.setXlabel("T degree K");
pp.setYlabel("Error Cp KJ/kmolK");
pp.plot();
return z;
}

public static double[][]  funcEKKPlot(f_xr f,double xi[],double yi[],int ngroup)
{
int n=xi.length;
int l;
double z[][];
double E[][];
double dx=0;
int k=0;
int i;
E=EKKgeneral(f,xi,yi,ngroup);
z=xyfit(f,xi,yi,E);
Plot pp=new Plot(z[0],z[1]);
pp.setPlabel("Cp=a0+a1*1e-3*T+a2*1e5/T^2+a3*1e-5*T^2  Tl<=T<=Th");
pp.setXlabel("T temperature degree K");
pp.setYlabel("Cp specific heat KJ/kmolK");
pp.addData(xi,yi,1);
pp.plot();
return z;
}

public static double[][]  funcEKKgeneral(f_xr f,double xi[],double yi[],int ngroup,int subintervalnumber)
{
int n=xi.length;
int nn=(n-1)*(subintervalnumber+1)+1;
double z[][]=new double[2][nn];
double E[][]=EKKgeneral(f,xi,yi,ngroup);
Text.print(E,"EfuncEKK");
double dx=0;
int k=0;
int i;
for(i=0;i<(n-1);i++)
{z[0][k]=xi[i];z[1][k]=funcEKKgeneral(f,E,z[0][k]);
for(int j=0;j<subintervalnumber;j++)
{dx=(xi[i+1]-xi[i])/((double)subintervalnumber+1.0);
k++;
z[0][k]=z[0][k-1]+dx;z[1][k]=funcEKKgeneral(f,E,z[0][k]);k++;}
}
z[0][k]=xi[i];z[1][k]=funcEKKgeneral(f,E,z[0][k]);
```

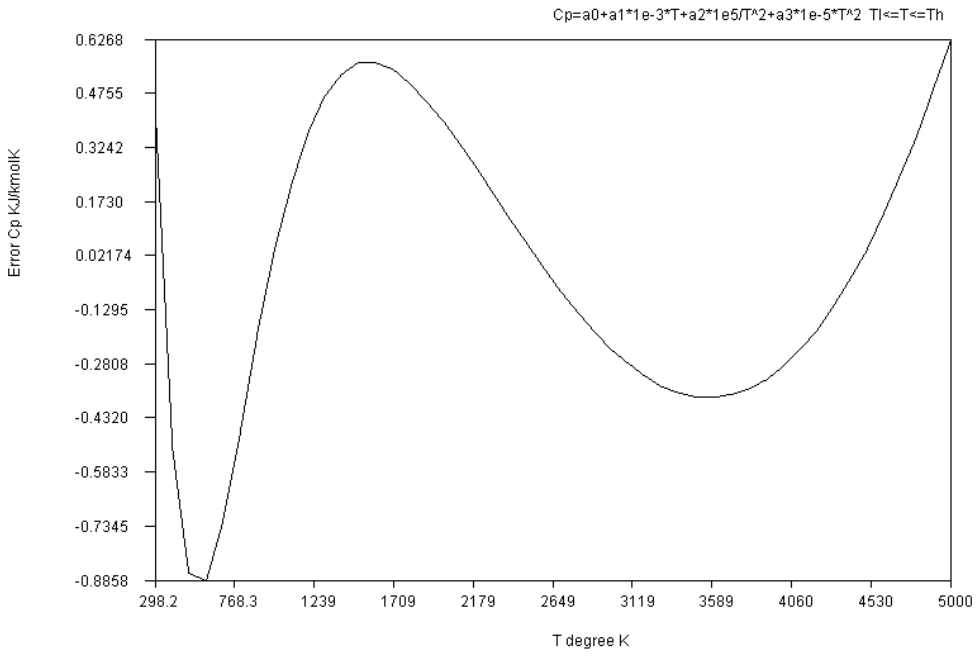
```
return z;
}

public double Cp_l(double t)
{ return funcEKKgeneral(f2,cpl,t);}

public double viscosity_l(double t)
{return funcEKKgeneral(f2,visl,t);}

public double k_l(double t)
{return funcEKKgeneral(f2,kl,t);}


public static void main(String args[]) throws IOException
{
String s1=JOptionPane.showInputDialog("name of the input file : ");
double c[][]=Text.readDoubleT(s1);
Text.print(Text.T(c));
double d[][]=EKKgeneral(new fcp(), c[0],c[1],10);
funcEKKPlot(new fcp(), c[0],c[1],10);
Text.print(d);
hataEKKPlot(new fcp(), c[0],c[1],4);
System.out.println(Matrix.toString(d));
}

}
```

Output 1 Least square curve fitting, one equation for the full range (n=1)

data output(veri çıktısı) :

0	1	2	3	4	5
28.875689238984958	4.485071684423939	-1.3183008985398887	-0.5604610967288349	298.15	5000.0





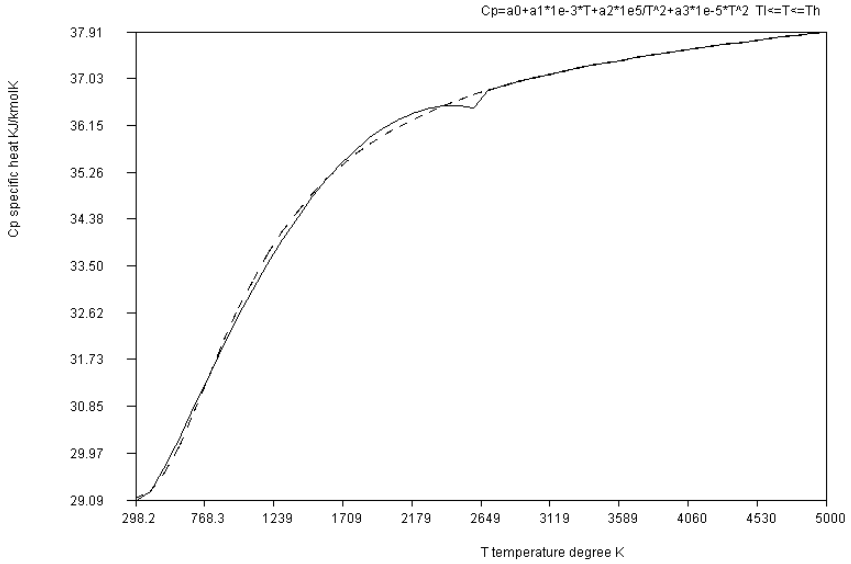
data output(veri çıktısı) :

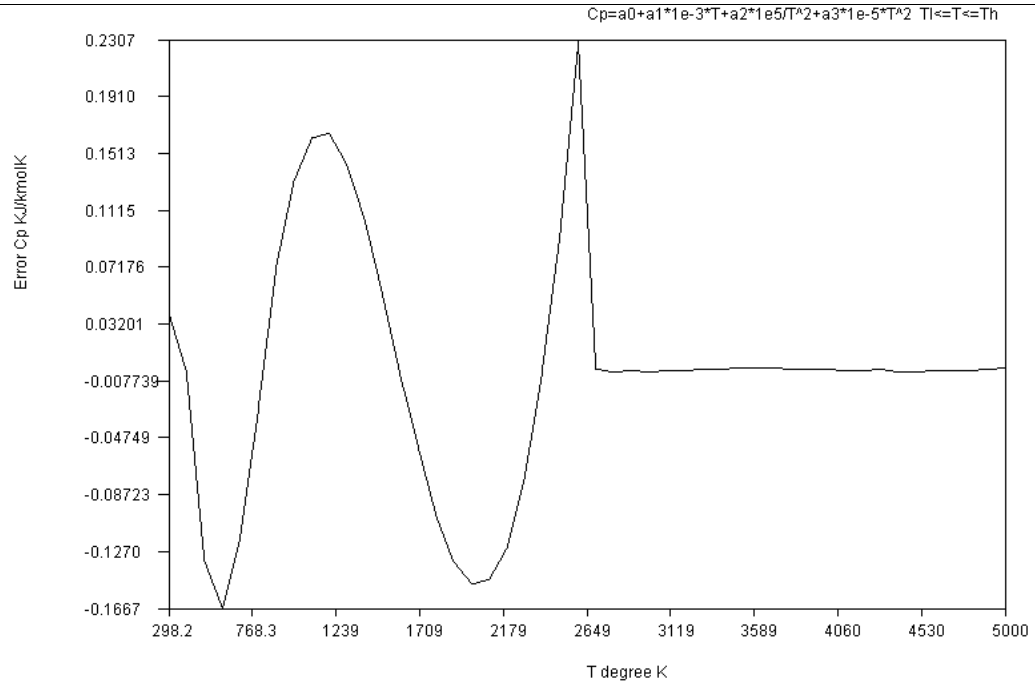
0	1
298.15	29.123
300.0	29.125
400.0	29.246
500.0	29.583
600.0	30.113
700.0	30.752
800.0	31.43
900.0	32.094
1000.0	32.696
1100.0	33.24
1200.0	33.723
1300.0	34.148
1400.0	34.519
1500.0	34.844
1600.0	35.128
1700.0	35.378
1800.0	35.598
1900.0	35.794
2000.0	35.969
2100.0	36.126
2200.0	36.267
2300.0	36.395
2400.0	36.511
2500.0	36.616
2600.0	36.713

Output Partially continious least square curve fitting, two equations for the full range (n=2)

data output(veri çıktısı) :

0	1	2	3	4	5
25.00311848...	9.321291108...	1.311097399...	-1.88988415...	298.15	2600.0
37.48845886...	0.064288598...	-70.8311804...	0.015320772...	2600.0	5000.0





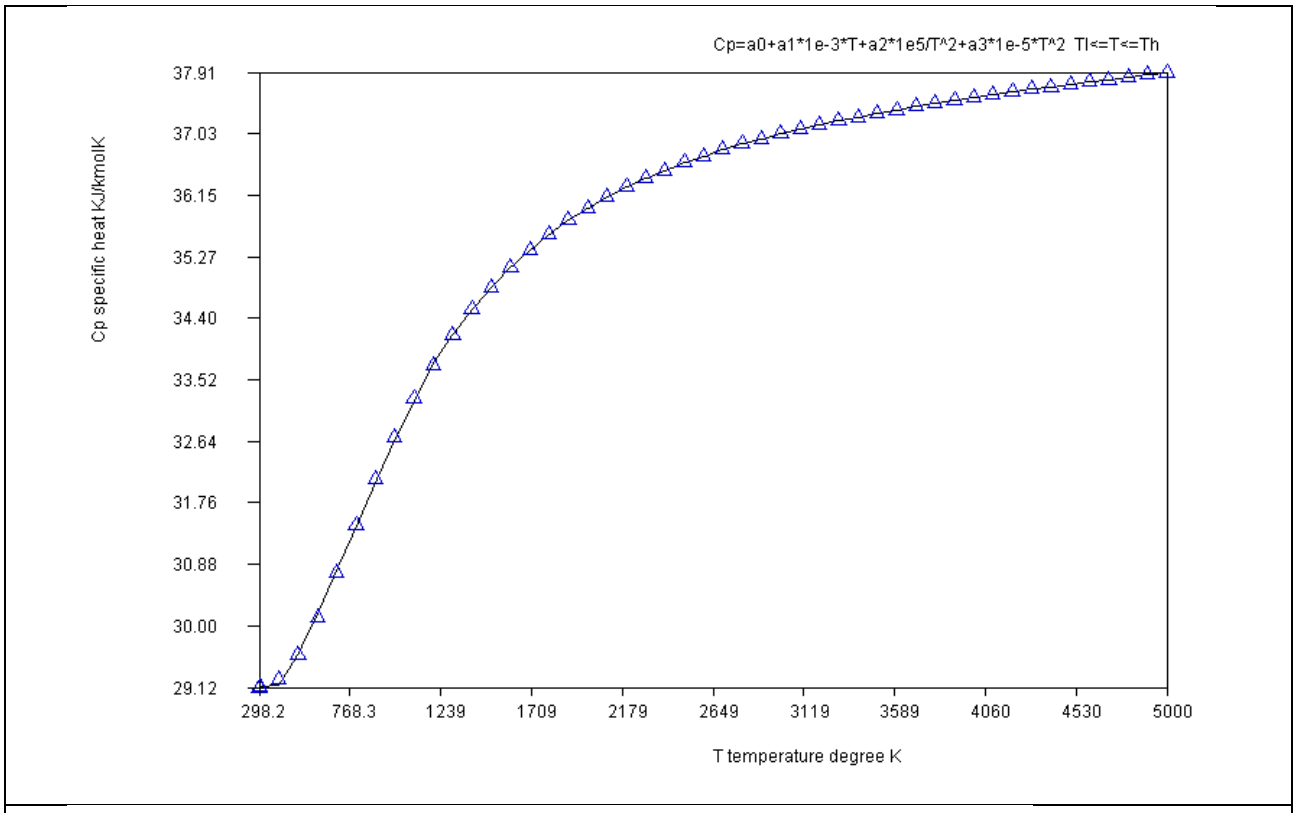
data output(veri çıktısı) :

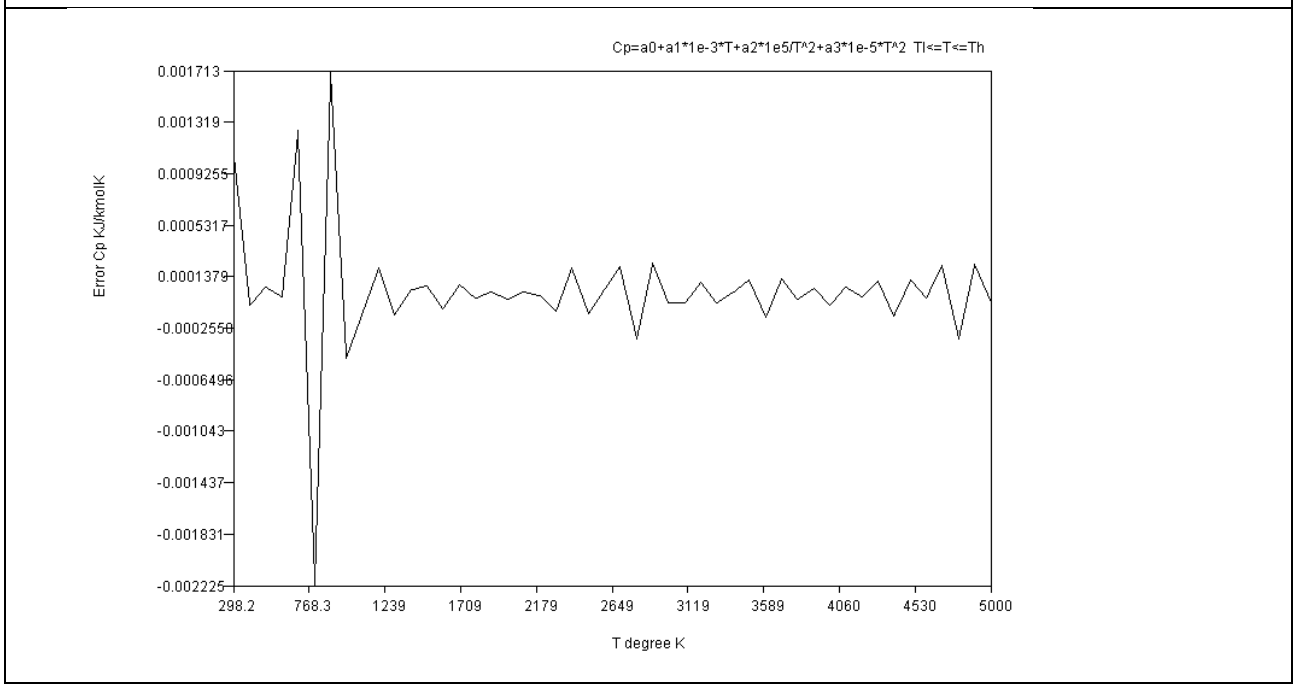
0	1
298.15	29.123
300.0	29.125
400.0	29.246
500.0	29.583
600.0	30.113
700.0	30.752
800.0	31.43
900.0	32.094
1000.0	32.696
1100.0	33.24
1200.0	33.723
1300.0	34.148
1400.0	34.519
1500.0	34.844
1600.0	35.128
1700.0	35.378
1800.0	35.598
1900.0	35.794
2000.0	35.969
2100.0	36.126
2200.0	36.267
2300.0	36.395
2400.0	36.511
2500.0	36.616
2600.0	36.713

Output Partially continious least square curve fitting, four equations for the full range (n=4)

data output(veri çıktısı) :

0	1	2	3	4	5
24.368546536...	10.007086647...	1.7415331317...	-1.8979238859...	298.15	1200.0
32.214041896...	3.2743928165...	-23.452661115...	-0.5513072397...	1200.0	2200.0
36.209932857...	0.6175206206...	-50.850743554...	-0.0518086414...	2200.0	3200.0
38.012874113...	-0.1102953868...	-84.323086856...	0.0314691118...	3200.0	5000.0


Cp=a0+a1*1e-3*T+a2*1e5/T^2+a3*1e-5*T^2 Tl<=T<=Th
Cp specific heat KJ/kmolK
37.91
37.03
36.15
35.27
34.40
33.52
32.64
31.76
30.88
30.00
29.12
298.2
768.3
1239
1709
2179
2649
3119
3589
4060
4530
5000
T temperature degree K


Cp=a0+a1*1e-3*T+a2*1e5/T^2+a3*1e-5*T^2 Tl<=T<=Th
Error Cp KJ/kmolK
0.001713
0.001319
0.0009255
0.0005317
0.0001379
-0.0002558
-0.0006496
-0.001043
-0.001437
-0.001831
-0.002225
298.2
768.3
1239
1709
2179
2649
3119
3589
4060
4530
5000
T degree K

data output(veri çıktısı) :

0	1
298.15	29.123
300.0	29.125
400.0	29.246
500.0	29.583
600.0	30.113
700.0	30.752
800.0	31.43
900.0	32.094
1000.0	32.696
1100.0	33.24
1200.0	33.723
1300.0	34.148
1400.0	34.519
1500.0	34.844
1600.0	35.128
1700.0	35.378
1800.0	35.598
1900.0	35.794
2000.0	35.969

RATIONAL FUNCTION APPROXIMATION : PADE APPROXIMATION METHOD



Figure 6.13.1 Henri Pade

The class of algebraic polynomials has some distinct advantages for use in approximations. The disadvantage of using polynomials for approximation is their tendency to oscilate. This often causes error bounds in polynomial approximation to significantly exceed the avarage approximation error. Another family of functions existed that spread the approximation error more evenly over the approximation interval. This family of functions are called **rational functions.** A rational function r of degree N has the form :

$r(x) = \frac{p(x)}{q(x)}$ (6.13-1)

Where p(x) and q(x) are polynomials whose degrees sum to N.

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n}{q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n} \qquad (6.13-2)$$

Pade′ rational approximation is based on Taylor expansion of the functions.

$$f(x) = f(x_0) + f'^{(x_0)}(x - x_0) + \frac{f"(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + .. + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad (6.13-3)$$

The special form of the Taylor expansion when x_0=0 was called MacLaurin expansion.

$$f(x) = f(0) + f'(0)x + \frac{f"(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + .. + \frac{f^{(n)}(0)}{n!}x^n \quad (6.13-4)$$

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \quad (6.13-5)$$

Now, if the function f(x) is opened to a MacLaurin series of degree N

$$r(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_N x^N$$

Consider the error term difference of the function f(x) and r(x) due to finite series approximation of the function.

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)} = \frac{\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^{\infty} q_i x^i - \sum_{i=0}^{\infty} p_i x^i}{q(x)} \quad (6.13-6)$$

We can assume that q_0=1. The object is to chose the constants $q_1, q_2, \ldots, q_m$ and $p_1, p_2, \ldots, p_n$ so that the right hand term of the above equation

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(1 + q_1 x + q_2 x^2 + \cdots + q_n x^m) - (p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n) \quad (6.13-7)$$

has no term of degree less than or equal to N. To simplify the notation, we define $p_{n+1}=p_{n+2}=\ldots p_N=0$ and define $q_{m+1}=q_{m+2}=\ldots q_m=0$. We can then express the coefficient of x^k in the expression as

$$\left(\sum_{i=0}^{k} a_i q_{k-i}\right) - p_k \quad (6.13-8)$$

So the rational function pade approximation results from the solution of n+1 linear equation

$$\left(\sum_{i=0}^{k} a_i q_{k-i}\right) = p_k \quad k = 0, \ldots, N$$

In the N+1 unknowns $q_1, q_2, \ldots, q_m, p_0, p_1, \ldots, p_m,$

As a first example take function $f(x) = e^x$

MacLaurin expansion of the function $f(x) = e^x$ for error $O(x^5)$,

$f(x) = e^x, \quad f(0) = 1$

$f'(x) = e^x, \quad f'(0) = 1$

$f"(x) = e^x, \quad f"(0) = 1$

$f'''(x) = e^x, \quad f'''(0) = 1$

$f""(x) = e^x, \quad f""(0) = 1$

Maclaurin equation then becomes;

$f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O(x^5)$

$p(x) = p_0 + xp_1 + x^2 p_2$

$q(x) = 1 + xq_1 + x^2 q_2$

$f(x)q(x) - p(x) = 0$

$$(1 - p_0) + (1 - p_1 + q_1)x + \left(\frac{1}{2} - p_2 + q_1 + q_2\right)x^2 + \left(\frac{1}{6} + \frac{q_1}{2} + q_2\right)x^3 + \left(\frac{1}{24} + \frac{q_1}{6} + \frac{q_2}{2}\right)x^4 + O(x^4) = 0$$

$(1 - p_0) = 0$

$(1 - p_1 + q_1) = 0$

$\left(\frac{1}{2} - p_2 + q_1 + q_2\right) = 0$

$\left(\frac{1}{6} + \frac{q_1}{2} + q_2\right) = 0$

$\left(\frac{1}{24} + \frac{q_1}{6} + \frac{q_2}{2}\right) = 0$

Solution of the equation set gives:

$p_0 = 1 \;\; p_1 = 1/2 \;\; p_1 = 1/12 \;\; q_1 = -1/2 \;\; q_2 = 1/12$

And $r(x) = \frac{1+\frac{1}{2x}+\frac{1}{12}x^2}{1-\frac{1}{2x}+\frac{1}{12}x^2}$

by simplifying

$r(x) = \frac{12+6x+x^2}{12-6x+x^2}$

Maximum error for x=1 is found as

$$f(x) - r(x) = e^x - \frac{12+6x+x^2}{12-6x+x^2} = 0.00399611$$

Let us solve the same equation, by developing the computer code. In the code MacLauring derivative terms are calculated up to term 8. But also another input possibility of directly inputting the polynomial coefficients are given. In case of higher degree of polynomials, coefficients can directly be inputted into the program.

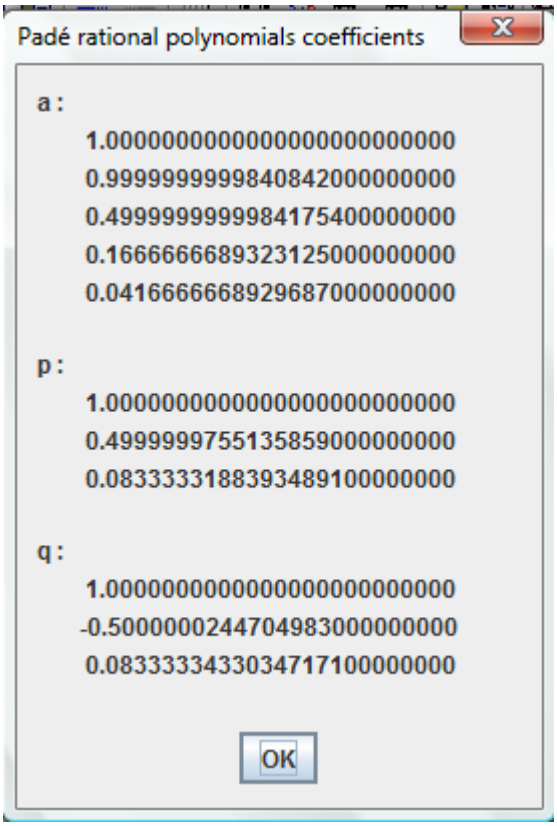
Program Pade rational function approximation program pade.java

```
import java.io.*;
import javax.swing.*;
class ff extends f_x
{
public double func(double x)
{return Math.exp(x);}
}

//Pade approximation
class pade
{
public double q[];
public double p[];
public double a[];

public static double[] Maclaurin(f_x f,double x,int n)
{
double a[]=new double[n+1];
double fact=1.0;
for(int i=0;i<=n;i++)
{a[i]=f.dfunc(x,i)/fact;fact*=(i+1);}
return a;
}

public pade(f_x f,int n,int m)
{
int N=m+n;
double a[]=Maclaurin(f,0.0,N);
initilize(a,n,m);
}

public pade(double a[],int n,int m)
{initilize(a,n,m);}

public void initilize(double ai[],int n,int m)
{
int N=m+n;
int NP1=N+1;
a=new double[NP1];
for(int i=0;i<NP1;i++)
{a[i]=ai[i];}
q=new double[m+1];
p=new double[n+1];
q[0]=1.0;
p[0]=a[0];
int i,j;
double b[][]=new double[N+1][N+1];
double bb[][]=new double[N][N];
double c[]=new double[N];
for(i=1;i<=N;i++)
{ for(j=1;j<(i-1);j++)
  {if(j<=n) b[i][j]=0;}
  if(i<=n) b[i][i]=1;
```

```
  for(j=i+1;j<=N;j++)
  {b[i][j]=0;}
  for(j=1;j<=i;j++)
  {if(j<=m)b[i][n+j]=-a[i-j];}
  for(j=n+i+1;j<=N;j++)
  {b[i][j]=0;}
  c[i-1]=a[i];
}
for(i=0;i<N;i++)
{for(j=0;j<N;j++)
 {bb[i][j]=b[i+1][j+1];}
}
double x[]=pivotlugauss(bb,c);
for(i=0;i<n;i++)
{p[i+1]=x[i];}
for(i=n;i<N;i++)
{q[i+1-n]=x[i];}
}

  public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
           b[i]   =b[i]   -carpan*b[k];
                }
  }
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }

public double func(double x)
{
return poly_func(p,x)/poly_func(q,x);
}
```

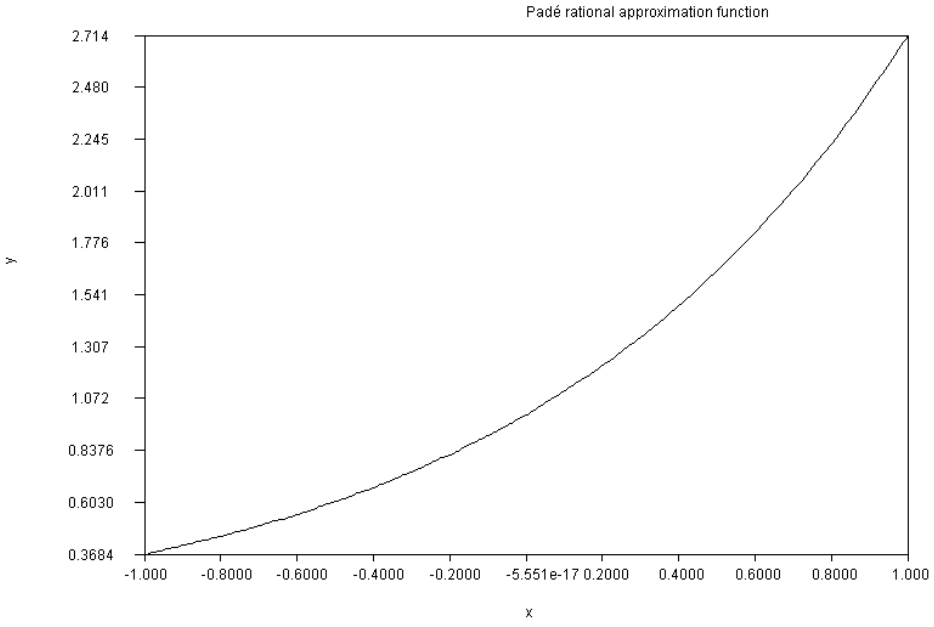
```
public static double poly_func(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public String toString()
{
                String s="a : \n"+Matrix.toStringT(a)+"\n";
                     s+="p : \n"+Matrix.toStringT(p)+"\n";
                     s+="q : \n"+Matrix.toStringT(q)+"\n";
                return s;
}

public String toString(f_x f,double a,double b,int n)
{

return Matrix.toStringT(result(f,a,b,n));
}

public double[][] result(f_x f,double a,double b,int n)
{   double xx[][]=new double[4][n+1];
   for(int i=0;i<=n;i++)
                { xx[0][i]=a+i*(b-a)/n;
                  xx[1][i]=func(xx[0][i]);
                  xx[2][i]=f.func(xx[0][i]);
                  xx[3][i]=xx[2][i]-xx[1][i];
                }
                return xx;
}
public void plot_func(f_x f,double a,double b)
{               double xx[]=new double[101];
   double yy[]=new double[101];
                for(int i=0;i<=100;i++)
                {xx[i]=a+i*(b-a)/100.0;yy[i]=func(xx[i]);}
                Plot pp=new Plot(xx,yy);
   pp.setPlabel("Pad"+'\u00E9'+" rational approximation function");
                pp.plot();}

                public void plot_error(f_x f,double a,double b)
   { double xx[]=new double[101];
     double yy[]=new double[101];
                for(int i=0;i<=100;i++)
                {xx[i]=a+i*(b-a)/100.0;yy[i]=f.func(xx[i])-func(xx[i]);}
                Plot pp=new Plot(xx,yy);
   pp.setPlabel("Pad"+'\u00E9'+" rational approximation error");
   pp.setYlabel("error");
                pp.plot();
}

public static void print(String s,String s1)
{JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);}

public static void main(String arg[])
{ ff f=new ff();
  //coefficients of function exp(x)
double a[]={1.000000000000000000000,
1.000000000000000000000,
0.500000000000000000000,
```

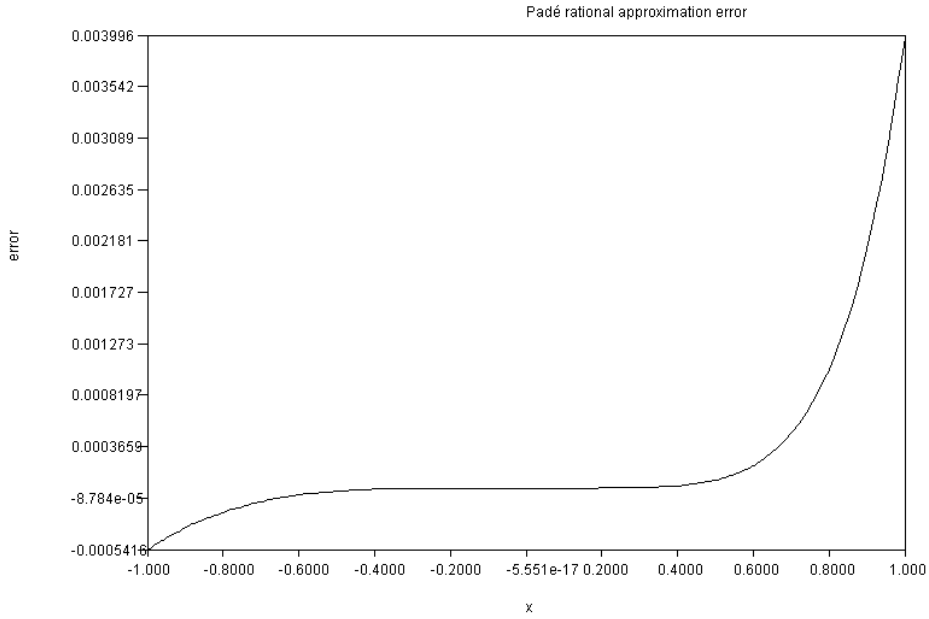
```
0.16666666666667000000,
0.04166666666666700000,
0.00833333333333330000,
0.00138888888888890000,
0.00019841269841269800,
0.00002480158730158730,
0.00000275573192239859,
0.00000027557319223985,
0.00000002505210838544,
0.00000000208767569878,
0.00000000016059043836,
0.00000000001147074559,
0.00000000000076471637,
0.00000000000004779477,
0.00000000000000281145,
0.00000000000000015619,
};
            int n=8;
            int m=8;
            pade Pa=new pade(f,n,m);
            //pade Pa1=new pade(a,n,m);
            print(Pa.toString(),"Pad"+'\u00E9'+" rational polynomials coefficients");
            String s1[]={"x","Pad"+'\u00E9'+" f(x)=p(x)/q(x)","f(x)","error"};
            Text.print(Text.T(Pa.result(f,-1.0,1.0,20)),s1,"Pad"+'\u00E9'+" rational polynomials n = "+n+" m = "+m);
            Pa.plot_func(f,-1.0,1.0);
            Pa.plot_error(f,-1.0,1.0);
}

}
```



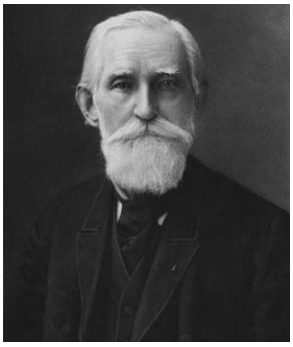
Padé rational polynomials n = 2 m = 2

x	Padé f(x)=p(x)/q(x)	f(x)	error
-1.0	0.3684210509286994	0.36787944117144233	-5.416097572570888E-4
-0.9	0.4069192738383368	0.4065696597405991	-3.496140977377049E-4
-0.8	0.4495412834707172	0.44932896411722156	-2.1231935349563802E-4
-0.7	0.4967046128965557	0.4965853037914095	-1.1930910514618143E-4
-0.6	0.5488721800340369	0.5488116360940265	-6.0543940010426844E-5
-0.5	0.6065573768024644	0.6065306597126334	-2.6717089830929908E-5
-0.4	0.6703296702018267	0.6703200460356393	-9.624166187327532E-6
-0.30000000000000004	0.7408207342878982	0.7408182206817178	-2.5136061804209575E-6
-0.19999999999999996	0.8187311178103238	0.8187307530779819	-3.647323418842774E-7
-0.09999999999999998	0.9048374306099658	0.9048374180359596	-1.2574006169607799E-8
0.0	1.0	1.0	0.0
0.10000000000000009	1.1051709027175882	1.1051709180756477	1.535805949082203E-8
0.19999999999999996	1.221402214037107	1.2214027581601699	5.441230628289162E-7
0.30000000000000004	1.3498542274610588	1.3498588075760032	4.5801149444191225E-6
0.3999999999999999	1.4918032788221807	1.4918246976412701	2.1418819089458196E-5
0.5	1.6486486489024128	1.6487212707001282	7.262179771538868E-5
0.6000000000000001	1.8219178086314263	1.822118800390509	2.0099175908283584E-4
0.7	2.013268999384152	2.0137527074704766	4.8370808632469675E-4
0.8	2.2244897966653	2.225540928492468	0.0010511318271677617
0.8999999999999999	2.457489879349433	2.4596031111569494	0.0021132318075163248
1.0	2.7142857149332364	2.7182818284590455	0.003996113525809086





RATIONAL FUNCTION APPROXIMATION : CHEBYSCHEV APPROXIMATION METHOD



Pafnuty Chebyshev, Russian Mathematician

Pade approximation based on Taylor expansion of the functions. Taylor expansion has a wide variation of accuracy. To obtain more uniformly accurate rational-function approximation Chebyschev polynomials can be used. In chapter 6.2 Chebyschev polynomials were defined and used as orthogonal least square curve fitting functions.

$T_n(x)=\cos(n \arccos(x))$ (6.14-1)

If this equation is written in open form

$T_0(x)=1$

$T_1(x)=x$

$T_2(x)=2x^2-x$

…..

$T_{n+1}(x)=2x\ T_n(x)- T_{n-1}(x)$ (6.14-2)

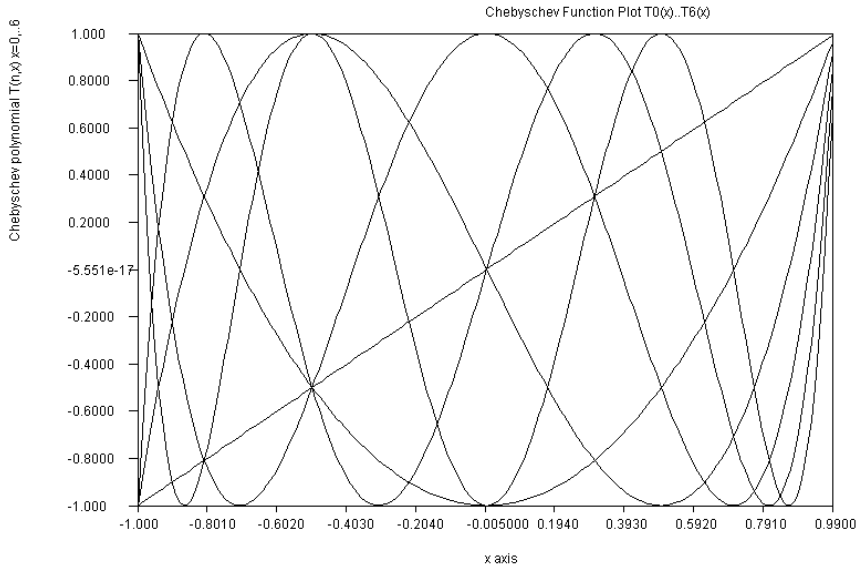


Figure Chebyschev polynomials $T_0(x)$ through $T_6(x)$

Now if x^k terms of the pade approximations is replaced by Chebyschev polynomial $T_k(x)$ Chebyschev approximation is constructed. Suppose we want to approximate the function f by an N-th degree rational function r written in the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{n} q_k T_k(x)} \quad (6.14\text{-}3)$$

where N=n+m and $q_0=1$

Writing f(x) in a series involving Chebyschev polynomials as:

$f(x) = \sum_{k=0}^{n} a_k T_k(x)$ gives

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{\sum_{k=0}^{n} a_k T_k(x) \sum_{k=0}^{n} q_k T_k(x) - \sum_{k=0}^{n} p_k T_k(x)}{\sum_{k=0}^{n} q_k T_k(x)} \quad (6.14-4)$$

The coefficients $q_1,q_2,\ldots,q_m$ and $p_1,p_2,\ldots,p_n$ are chosen so that the numerator on the right hand side of the equation has zero coefficients for $T_k(x)$ when k=0,1,..N. This implies that the series

$$(a_0T_0(x) + a_1T_1(x)+..)\big(T_0(x) + q_1T_1(x)+..+q_mT_m(x)\big) - \big(p_0T_0(x) + p_1T_1(x)+..+p_mT_m(x)\big) \quad (6.14-5)$$

Has no terms of degree less then or equal to N.

Solution of this equation involves few difficulties, the first one is the requirement of multiplication of Chebyschev terms. This problem is resolved by making use of the relationship

$$T_i(x)T_j(x) = 0.5\big[T_{i+1}(x) + T_{|i-j|}(x)\big] \quad (6.14-6)$$

Coefficients need to be calculated by using integration equations:

$$a_0 = \frac{1}{\pi}\int_{x=-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{1}{\pi}\int_{\theta=0}^{\pi} f(\cos(\theta))d\theta \quad (6.14-7)$$

$$a_k = \frac{1}{\pi}\int_{x=-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}dx = \frac{1}{\pi}\int_{\theta=0}^{\pi} f(\cos(\theta))\cos(k\theta)\,d\theta \quad (6.14-8)$$

These are difficult integrals to solve analytically. Usually Numerical methods are involved to solve the integrals. Please refer Chapter 7 for the details of the numerical integrals (everything is related with each other in mathematics and we should give topics in some order). In our example code, Gander and Gautschi adaptive Gauss-Lobatto integral formula defined in chapter 7.5-5 is used as an accurate integration formula to solve coefficients.

Program Chebyschev rational function approximation program Chebyshev.java

```
import java.io.*;
import javax.swing.*;
```

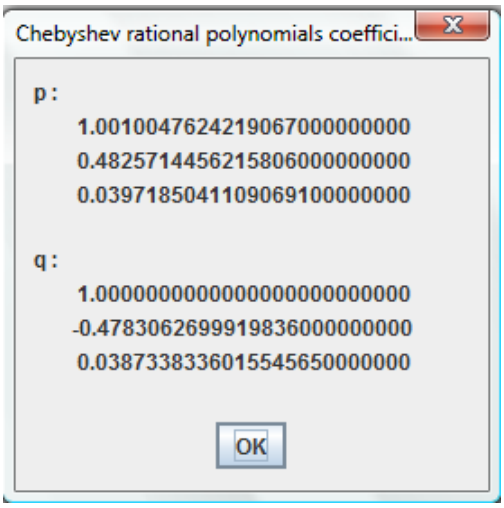
```
class ff extends f_x
{
public double func(double x)
{return Math.exp(x);}
}

//Chebyshev  rational approximation
class Chebyshev
{
public double q[];
public double p[];
public double a[];

            public static double Ti(double x,int k)
            {//Chebysev function
            double T[]=new double[k+1];
            T[0]=1.0;
            if(k>=1) T[1]=x;
            for(int i=2;i<=k;i++) {T[i]=2.0*x*T[i-1]-T[i-2];
   }
            return T[k];
            }

   public  double fk(f_x f,int k,double x)
   {   double a=0;
      double mult=2.0/Math.PI;
               a=mult*f.func(Math.cos(x))*Math.cos(k*x);
               return a;
            }

public double adaptive_Lobatto_integral(f_x f,double a,double b,int k)
{
            int i;
            double m=(a+b)/2.0; double h=(b-a)/2.0;
   double alpha=Math.sqrt(2.0/3.0);
   double beta=1.0/Math.sqrt(5.0);
   double x1,x2,x3;
   x1=0.942882415695479719056351758431857202 32;
   x2=0.641853342345781305781235541329031883 54;
   x3=0.236383199662149880282223773492052925 99;
   double A,B,C,D,E,F,G;
   A=0.015827191973480183087169986733305510591;
   B=0.094273840218850045531282505077108171960;
   C=0.155071987336585396253635979802102986 80;
   D=0.188821573960182454420005339372971671 25;
   E=0.199773405226858526792068022066488402 46;
   F=0.224926465333339527016017687996395080 76;
   G=0.242611071901407733799640957903256352 33;
            double x[]={a,(m-x1*h),(m-alpha*h),(m-x2*h),(m-beta*h),(m-
x3*h),m,(m+x3*h),(m+beta*h),(m+x2*h),(m+alpha*h),(m+x1*h),b};
            int n=x.length;
            double y[]=new double[n];
            for(i=0;i<n;i++) y[i]=fk(f,k,x[i]);
            double fa=y[0];
            double fb=y[12];
   double i2=(h/6.0)*(y[0]+y[12]+5.0*(y[4]+y[8]));
   double i1=(h/1470.0)*(77.0*(y[0]+y[12])+432.0*(y[2]+y[10])+
    625.0*(y[4]+y[8])+672.0*y[6]);
   double
is=h*(A*(y[0]+y[12])+B*(y[1]+y[11])+C*(y[2]+y[10])+D*(y[3]+y[9])+E*(y[4]+y[8])+F*(y[5]+y[7])+G*y[6]);
   int s=(int)(is/Math.abs(is));
   if(s==0) s=1;
```

```
    double erri1=Math.abs(i1-is);
    double erri2=Math.abs(i2-is);
    double R;
    R=erri1/erri2;
    //if(R>0 && R<1) tol=tol/R;
    //is=s*Math.abs(is)*tol/eps;
    if(is==0) is=b-a;
  double Q=0;
  Q=Lobatto(f,a,b,fa,fb,is,k);
  return Q;
}

public  double Lobatto(f_x f,double a,double b,double fa,double fb,double is,int k)
{
  double h=(b-a)/2;
  double m=(a+b)/2;
  double alpha=Math.sqrt(2.0/3.0);
  double beta=1.0/Math.sqrt(5.0);
  double mll=m-alpha*h;
  double ml=m-beta*h;
  double mr=m+beta*h;
  double mrr=m+alpha*h;
  double x[]={mll,ml,m,mr,mrr};
  int n=x.length;
  double y[]=new double[n];
  for(int i=0;i<n;i++) y[i]=fk(f,k,x[i]);
  double fmll=y[0];
  double fml=y[1];
  double fm=y[2];
  double fmr=y[3];
  double fmrr=y[4];
  double Q;
  double i2=(h/6)*(fa+fb+5*(fml+fmr));
  double i1=(h/1470)*(77*(fa+fb)+432*(fmll+fmrr)+625*(fml+fmr)+672*fm);
  boolean c1,c2,c3;
  double w1=i1-i2;;
  double w2=is;
  double w3=w2+w1;
  c1=(w3==w2);
  c2=(mll<=a);
  c3=(b<=mrr);
  if( c1|| c2 ||c3 )
  {
   if (((m <= a) || (b<=m)))
   {
      System.out.println("istenilen toleransa ulaşılamadı");
   }
   Q=i1;
  }
  else
  {
    Q=Lobatto(f,a,mll,fa,fmll,is,k) + Lobatto(f,mll,ml,fmll,fml,is,k)+
      Lobatto(f,ml,m,fml,fm,is,k)   + Lobatto(f,m,mr,fm,fmr,is,k)+
      Lobatto(f,mr,mrr,fmr,fmrr,is,k)+Lobatto(f,mrr,b,fmrr,fb,is,k);
  }
  return Q;
}
//=============
public Chebyshev(f_x f,int n,int m)
{
int N=m+n;
int Npm=N+m;
```

```
double ai[]=new double[Npm+1];
int k;
for(k=0;k<=(N+m);k++)
{ai[k]=adaptive_Lobatto_integral(f,0.0,Math.PI,k);
}
initilize(ai,n,m);
}

public Chebyshev(double ai[],int n,int m)
{initilize(ai,n,m);}

public void initilize(double ai[],int n,int m)
{
int N=m+n;
int NP1=N+1;
int NPm=N+m;
int mp1=m+1;
int np1=n+1;
a=new double[NPm+1];
for(int i=0;i<=NPm;i++)
{a[i]=ai[i];}
q=new double[mp1];
p=new double[np1];
q[0]=1.0;
int i,j;
double b[][]=new double[NP1][NP1];
double c[]=new double[NP1];
for(i=0;i<=N;i++)
{  for(j=0;j<=i;j++)
   {if(j<=n) b[i][j]=0;}
if(i<=n) b[i][i]=1;
   for(j=i+1;j<=n;j++)
   {b[i][j]=0;}
   int k1,k2;
   for(j=n+1;j<=N;j++)
   {k1=i+j-n;
    k2=Math.abs(i-j+n);
   if(i!=0) b[i][j]=-0.5*(a[k1]+a[k2]);
   else     b[i][j]=-0.5*a[j-n];
  }
  if(i!=0) c[i]=a[i];
  else     c[i]=0.5*a[i];
}
double x[]=pivotlugauss(b,c);
for(i=0;i<=n;i++)
{p[i]=x[i];}
for(i=n+1;i<=N;i++)
{q[i-n]=x[i];}
}

  public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
```

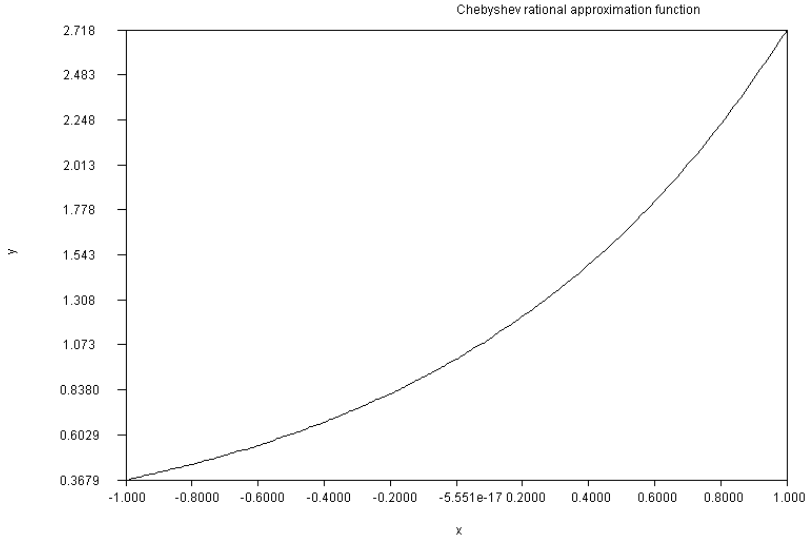
```
            p=k;
            buyuk=Math.abs(a[k][k]);
            for(ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[ii][k]);
              if(dummy > buyuk) {buyuk=dummy;p=ii;}
            }
            if(p!=k)
            { for(jj=k;jj<n;jj++)
              { dummy=a[p][jj];
                a[p][jj]=a[k][jj];
                a[k][jj]=dummy;
              }
              dummy=b[p];
              b[p]=b[k];
              b[k]=dummy;
            }
   for(i=k+1;i<n;i++)
   {  carpan=a[i][k]/a[k][k];
      a[i][k]=0;
      for(j=k+1;j<n;j++)
      {   a[i][j]-=carpan*a[k][j]; }
         b[i]   =b[i]   -carpan*b[k];
            }
  }
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }

public double func(double x)
{
return poly_func(p,x)/poly_func(q,x);
}

public static double poly_func(double a[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=a.length;
double b=0;
for(int k=0;k<n;k++)
{b+=a[k]*Ti(x,k);}
return b;
}

public void plot_func(f_x f,double a,double b)
{           double xx[]=new double[101];
   double yy[]=new double[101];
            for(int i=0;i<=100;i++)
            {xx[i]=a+i*(b-a)/100.0;yy[i]=func(xx[i]);}
            Plot pp=new Plot(xx,yy);
   pp.setPlabel("Chebyshev rational approximation function");
            pp.plot();}

            public void plot_error(f_x f,double a,double b)
   { double xx[]=new double[101];
```

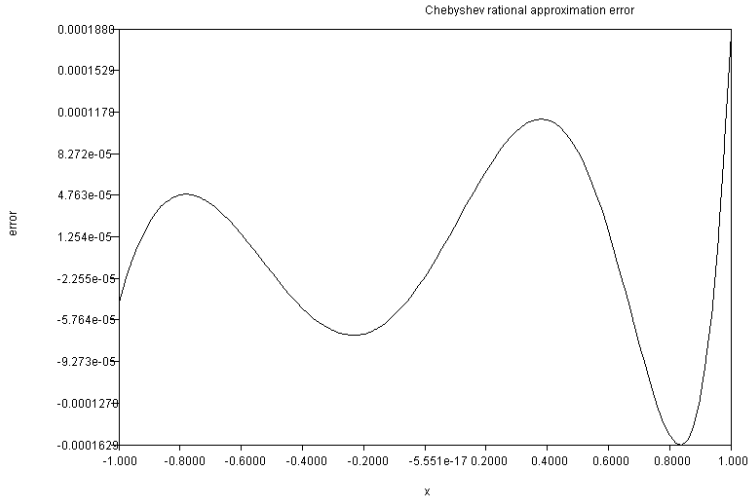
```
     double yy[]=new double[101];
            for(int i=0;i<=100;i++)
            {xx[i]=a+i*(b-a)/100.0;yy[i]=f.func(xx[i])-func(xx[i]);}
            Plot pp=new Plot(xx,yy);
   pp.setPlabel("Chebyshev rational approximation error");
   pp.setYlabel("error");
            pp.plot();
}

public String toString()
{
            String s="p : \n"+Matrix.toStringT(p)+"\n";
                s+="q : \n"+Matrix.toStringT(q)+"\n";
            return s;
}

public String toString(f_x f,double a,double b,int n)
{

return Matrix.toStringT(result(f,a,b,n));
}

public double[][] result(f_x f,double a,double b,int n)
{   double xx[][]=new double[4][n+1];
   for(int i=0;i<=n;i++)
            { xx[0][i]=a+i*(b-a)/n;
              xx[1][i]=func(xx[0][i]);
              xx[2][i]=f.func(xx[0][i]);
              xx[3][i]=xx[2][i]-xx[1][i];
            }
            return xx;
}
public static void print(String s,String s1)
{JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);}
public static void main(String arg[])
{
            ff f=new ff();
            int n=2;
            int m=2;
            Chebyshev Pa=new Chebyshev(f,n,m);
            print(Pa.toString(),"Chebyshev rational polynomials coefficients");
            String s1[]={"x","Chebyschev f(x)=p(x)/q(x)","f(x)","error"};
            Text.print(Text.T(Pa.result(f,-1.0,1.0,20)),s1,"Chebyshev rational polynomials n = "+n+" m = "+m);
            Pa.plot_func(f,-1.0,1.0);
            Pa.plot_error(f,-1.0,1.0);
}

}
```



Chebyshev rational polynomials n = 2 m = 2

x	Chebyschev f(x)=p(x)/q(x)	f(x)	error
-1.0	0.3679215991647767	0.36787944117144233	-4.215799333434633E-5
-0.9	0.4065450308521668	0.4065696597405991	2.4628888432298712E-5
-0.8	0.44928099031688123	0.44932896411722156	4.797380034032855E-5
-0.7	0.49654472373346964	0.4965853037914095	4.058005793988739E-5
-0.6	0.5487968820945643	0.5488116360940265	1.4753999462158163E-5
-0.5	0.6065487349499269	0.6065306597126334	-1.8075237293468938E-5
-0.4	0.670367949939593	0.6703200460356393	-4.790390395370547E-5
-0.30000000000000004	0.7408849752494372	0.7408182206817178	-6.675456771942923E-5
-0.19999999999999996	0.8188000534499897	0.8187307530779819	-6.930037200780781E-5
-0.09999999999999998	0.9048908805870234	0.9048374180359596	-5.346255106375430 4E-5
0.0	1.0000209015080908	1.0	-2.090150809075375E-5
0.10000000000000009	1.105148198077676	1.1051709180756477	2.2719997971654138E-5
0.19999999999999996	1.2213348769950267	1.2214027581601699	6.788116514311326E-5
0.30000000000000004	1.3497567929670753	1.3498588075760032	1.0201460892789171E-4
0.3999999999999999	1.4917133440587107	1.4918246976412701	1.1135358255942585E-4
0.5	1.6486369404047452	1.6487212707001282	8.433029538301717E-5
0.6000000000000001	1.822101564395447	1.822118800390509	1.7235995062092613E-5
0.7	2.0138295972158415	2.0137527074704766	-7.688974536490534E-5
0.8	2.225695768869679	2.225540928492468	-1.5484037721114063E-4
0.8999999999999999	2.4597266816072687	2.4596031111569494	-1.2357045031929914E-4
1.0	2.71809384667528	2.7182818284590455	1.8798178376533414E-4





TRIGONOMETRIC FUNCTION (FOURIER) APPROXIMATION



Figure Jean Baptiste Joseph Fourier 1768-1830, French Mathematician

Using of sine and cosine functions to represent data has a long history. For a function in the range of [-π,π] An approximation polynomial can be written as

$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1}[a_k \cos(kx) + b_k \sin(kx)]$ (6.15-1)

Since this set is orthogonal on [-π,π] range with respect to weight function w(x)=1, the selection of coefficients will be:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \quad k = 0,1,2,..,n \qquad (6.15-2)$$

$$b = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \quad k = 0,1,2,..,n \qquad (6.15-3)$$

The limit of $S_n(x)$ when $n \to \infty$ is called **Fourier series** of function f. A more general definition of the fourier series can be given in the range of [-l,l]

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\left(\frac{k\pi x}{l}\right)\right)\right] \qquad (6.15-4)$$

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi x}{l}\right)dx = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\left(\frac{k\pi x}{l}\right)dx \quad k = 0,1,2,..,n \quad (6.15-5)$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{k\pi x}{l}\right)dx = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\left(\frac{k\pi x}{l}\right)dx \quad k = 0,1,2,..,n \quad (6.15-6)$$

A Computer program developed to calculate Trigonometric polynomials. Integrations are again taken numerically in this program similar to Chebyschev polynomial rational approximations. The first sample function to be investigated will be f(x)=|x|

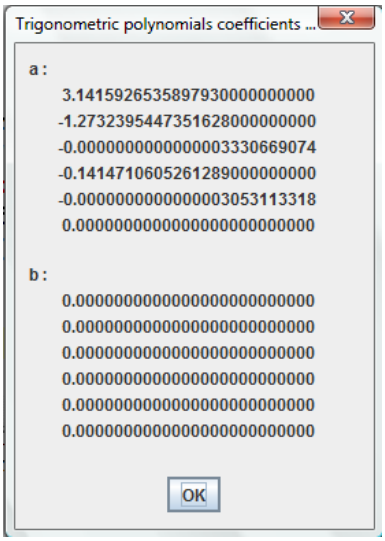
Program Trigonometric function approximation program Trigonometric.java

```
import java.io.*;
import javax.swing.*;
class ff extends f_x
{
public double func(double x)
{return Math.abs(x);}
}

// Trigonometric  rational approximation
class Trigonometric
{
public double a[];
public double b[];

public  double fak(f_x f,int k,double x)
   {   double a=0;
      double mult=1.0/Math.PI;
               a=mult*f.func(x)*Math.cos(k*x);
               return a;
            }

public  double fbk(f_x f,int k,double x)
   {   double a=0;
      double mult=1.0/Math.PI;
                a=mult*f.func(x)*Math.sin(k*x);
               return a;
            }

public double adaptive_Lobatto_integral(f_x f,double a,double b,int k,int eqn_type)
{
            int i;
            double m=(a+b)/2.0; double h=(b-a)/2.0;
   double alpha=Math.sqrt(2.0/3.0);
   double beta=1.0/Math.sqrt(5.0);
   double x1,x2,x3;
   x1=0.94288241569547971905635175843185720232;
   x2=0.64185334234578130578123554132903188354;
   x3=0.23638319966214988028222377349205292599;
   double A,B,C,D,E,F,G;
   A=0.015827191973480183087169986733305510591;
   B=0.094273840218850045531282505077108171960;
   C=0.155071987336585396253635979802102986 80;
   D=0.188821573960182454420005339372971671 25;
   E=0.199773405226858526792068022066488402 46;
   F=0.224926465333339527016017687996395080 76;
   G=0.242611071901407733799640957903256352 33;
            double x[]={a,(m-x1*h),(m-alpha*h),(m-x2*h),(m-beta*h),(m-
x3*h),m,(m+x3*h),(m+beta*h),(m+x2*h),(m+alpha*h),(m+x1*h),b};
            int n=x.length;
            double y[]=new double[n];
            for(i=0;i<n;i++)
            {
            if(eqn_type==0)
            {y[i]=fak(f,k,x[i]);}
            else
            {y[i]=fbk(f,k,x[i]);}
   }
            double fa=y[0];
            double fb=y[12];
   double i2=(h/6.0)*(y[0]+y[12]+5.0*(y[4]+y[8]));
   double i1=(h/1470.0)*(77.0*(y[0]+y[12])+432.0*(y[2]+y[10])+
```

```
     625.0*(y[4]+y[8])+672.0*y[6]);
    double
is=h*(A*(y[0]+y[12])+B*(y[1]+y[11])+C*(y[2]+y[10])+D*(y[3]+y[9])+E*(y[4]+y[8])+F*(y[5]+y[7])+G*y[6]);
    int s=(int)(is/Math.abs(is));
    if(s==0) s=1;
    double erri1=Math.abs(i1-is);
    double erri2=Math.abs(i2-is);
    double R;
    R=erri1/erri2;
    //if(R>0 && R<1) tol=tol/R;
    //is=s*Math.abs(is)*tol/eps;
    if(is==0) is=b-a;
  double Q=0;
  Q=Lobatto(f,a,b,fa,fb,is,k,eqn_type);
  return Q;
}

public  double Lobatto(f_x f,double a,double b,double fa,double fb,double is,int k,int eqn_type)
{
  double h=(b-a)/2;
  double m=(a+b)/2;
  double alpha=Math.sqrt(2.0/3.0);
  double beta=1.0/Math.sqrt(5.0);
  double mll=m-alpha*h;
  double ml=m-beta*h;
  double mr=m+beta*h;
  double mrr=m+alpha*h;
  double x[]={mll,ml,m,mr,mrr};
  int n=x.length;
  double y[]=new double[n];
  for(int i=0;i<n;i++)
              {
    if(eqn_type==0)
              {y[i]=fak(f,k,x[i]);}
              else
              {y[i]=fbk(f,k,x[i]);}
    }
  double fmll=y[0];
  double fml=y[1];
  double fm=y[2];
  double fmr=y[3];
  double fmrr=y[4];
  double Q;
  double i2=(h/6)*(fa+fb+5*(fml+fmr));
  double i1=(h/1470)*(77*(fa+fb)+432*(fmll+fmrr)+625*(fml+fmr)+672*fm);
  boolean c1,c2,c3;
  double w1=i1-i2;;
  double w2=is;
  double w3=w2+w1;
  c1=(w3==w2);
  c2=(mll<=a);
  c3=(b<=mrr);
  if( c1|| c2 ||c3 )
  {
   if (((m <= a) || (b<=m)))
   {
      System.out.println("istenilen toleransa ulaşılamadı");
   }
   Q=i1;
  }
  else
  {
    Q=Lobatto(f,a,mll,fa,fmll,is,k,eqn_type) + Lobatto(f,mll,ml,fmll,fml,is,k,eqn_type)+
      Lobatto(f,ml,m,fml,fm,is,k,eqn_type)   + Lobatto(f,m,mr,fm,fmr,is,k,eqn_type)+
      Lobatto(f,mr,mrr,fmr,fmrr,is,k,eqn_type)+Lobatto(f,mrr,b,fmrr,fb,is,k,eqn_type);
  }
  return Q;
```

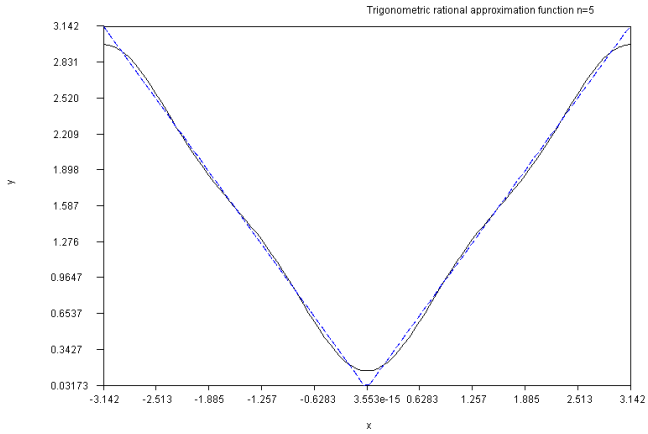
```
}

//=============


public Trigonometric(f_x f,int n)
{
double ai[]=new double[n+1];
double bi[]=new double[n+1];
for(int k=0;k<n;k++)
{ai[k]=adaptive_Lobatto_integral(f,-Math.PI,Math.PI,k,0);}
bi[0]=0.0;
for(int k=1;k<n;k++)
{bi[k]=adaptive_Lobatto_integral(f,-Math.PI,Math.PI,k,1);}
bi[n]=0;
initilize(ai,bi,n);
}

public Trigonometric(double ai[],double bi[],int n)
{initilize(ai,bi,n);}

public void initilize(double ai[],double bi[],int n)
{
a=new double[n+1];
b=new double[n+1];
for(int i=0;i<=n;i++)
{a[i]=ai[i];b[i]=bi[i];}
}

public  double func(double x)
{
int n=a.length-1;
double cc=a[0]/2.0+a[n]*Math.cos(n*x);
for(int k=1;k<n;k++)
{cc+=a[k]*Math.cos(k*x)+b[k]*Math.sin(k*x);}
return cc;
}

public void plot_func(f_x f,double x0,double xn)
{           double xx[]=new double[101];
   double yy[]=new double[101];
            for(int i=0;i<=100;i++)
            {xx[i]=x0+i*(xn-x0)/100.0;yy[i]=func(xx[i]);}
            Plot pp=new Plot(xx,yy);
            pp.addFunction(f,x0,xn,100);
   pp.setPlabel("Trigonometric rational approximation function n="+(a.length-1));
            pp.plot();
}

            public void plot_error(f_x f,double x0,double xn)
   { double xx[]=new double[101];
     double yy[]=new double[101];
            for(int i=0;i<=100;i++)
            {xx[i]=x0+i*(xn-x0)/100.0;yy[i]=f.func(xx[i])-func(xx[i]);}
            Plot pp=new Plot(xx,yy);
   pp.setPlabel("Trigonometric rational approximation error n = "+(a.length-1));
   pp.setYlabel("error");
            pp.plot();
}

public String toString()
{
            String s="a : \n"+Matrix.toStringT(a)+"\n";
                s+="b : \n"+Matrix.toStringT(b)+"\n";
            return s;
}
```

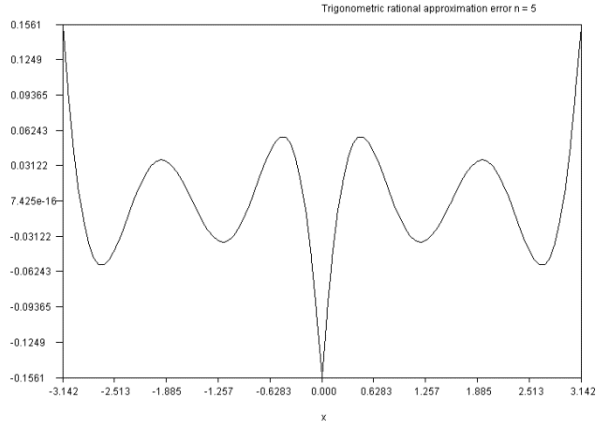
```
public String toString(f_x f,double a,double b,int n)
{

return Matrix.toStringT(result(f,a,b,n));
}

public double[][] result(f_x f,double a,double b,int n)
{   double xx[][]=new double[4][n+1];
   for(int i=0;i<=n;i++)
            { xx[0][i]=a+i*(b-a)/n;
              xx[1][i]=func(xx[0][i]);
              xx[2][i]=f.func(xx[0][i]);
              xx[3][i]=xx[2][i]-xx[1][i];
            }
            return xx;
}


public static void print(String s,String s1)
{JOptionPane.showMessageDialog(null,s,s1,JOptionPane.PLAIN_MESSAGE);}
public static void main(String arg[])
{
            ff f=new ff();
            int n=5;
            Trigonometric Pa=new Trigonometric(f,n);
            print(Pa.toString(),"Trigonometric polynomials coefficients n="+n);
            String s1[]={"x","Trigonometric f(x)","f(x)","error"};
            Text.print(Text.T(Pa.result(f,-Math.PI,Math.PI,20)),s1,"Trigonometric polynomials n = "+n);
            Pa.plot_func(f,-Math.PI,Math.PI);
            Pa.plot_error(f,-Math.PI,Math.PI);
}

}
```



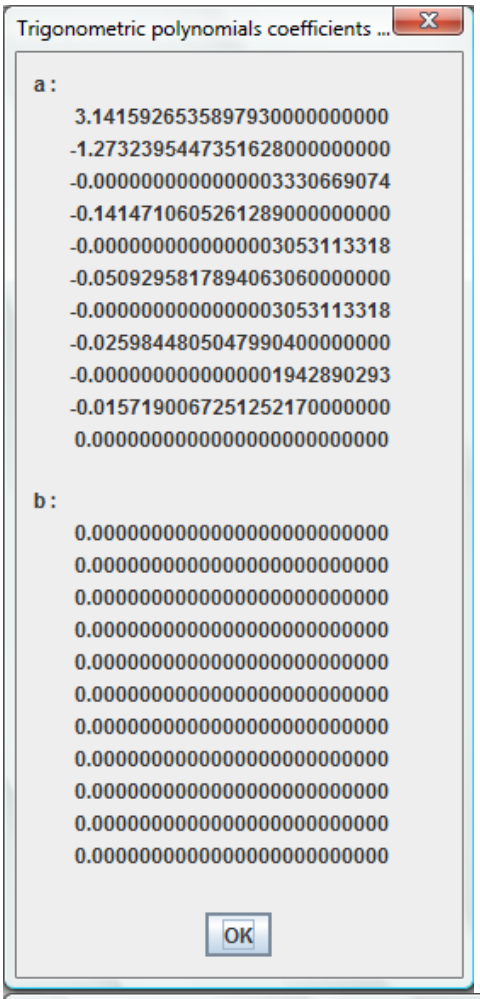
Trigonometric polynomials n = 5

x	Trigonometric f(x)	f(x)	error
-3.141592653589793	2.9855069320561873	3.141592653589793	0.15608572153360578
-2.827433388230814	2.864873695623382	2.827433388230814	-0.037440307392567984
-2.5132741228718345	2.5571517944810442	2.5132741228718345	-0.043877671609209745
-2.199114857512855	2.184640779845247	2.199114857512855	0.01447407766760822
-1.8849555921538759	1.849796491850398	1.8849555921538759	0.03515910030347791
-1.5707963267948966	1.5707963267948968	1.5707963267948966	-2.220446049250313E-16
-1.2566370614359172	1.2917961617393954	1.2566370614359172	-0.03515910030347813
-0.9424777960769379	0.9569518737445465	0.9424777960769379	-0.014474077667608554
-0.6283185307179586	0.5844408591087495	0.6283185307179586	0.04387767160920908
-0.3141592653589793	0.27671895796641055	0.3141592653589793	0.03744030739256876
0.0	0.15608572153360425	0.0	-0.15608572153360425
0.31415926535897887	0.27671895796641016	0.31415926535897887	0.037440307392568706
0.6283185307179586	0.5844408591087495	0.6283185307179586	0.04387767160920908
0.9424777960769379	0.9569518737445465	0.9424777960769379	-0.014474077667608554
1.2566370614359172	1.2917961617393954	1.2566370614359172	-0.03515910030347813
1.5707963267948966	1.5707963267948968	1.5707963267948966	-2.220446049250313E-16
1.8849555921538759	1.849796491850398	1.8849555921538759	0.03515910030347791
2.199114857512855	2.184640779845247	2.199114857512855	0.01447407766760822
2.5132741228718345	2.5571517944810442	2.5132741228718345	-0.043877671609209745
2.827433388230814	2.864873695623382	2.827433388230814	-0.037440307392567984
3.141592653589793	2.9855069320561873	3.141592653589793	0.15608572153360578



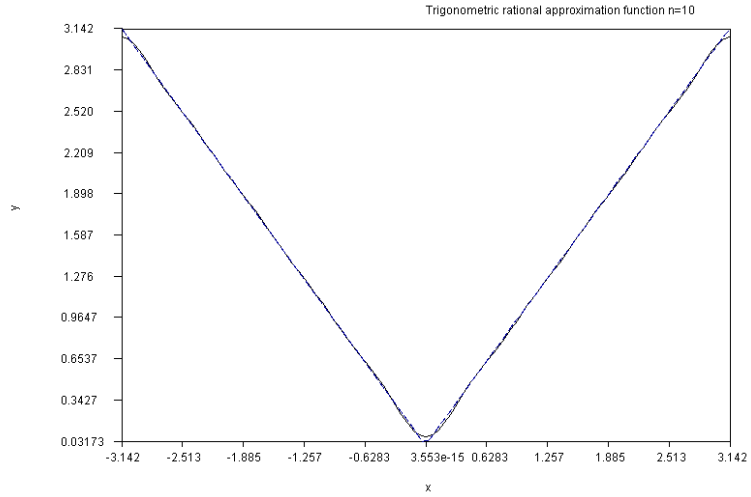


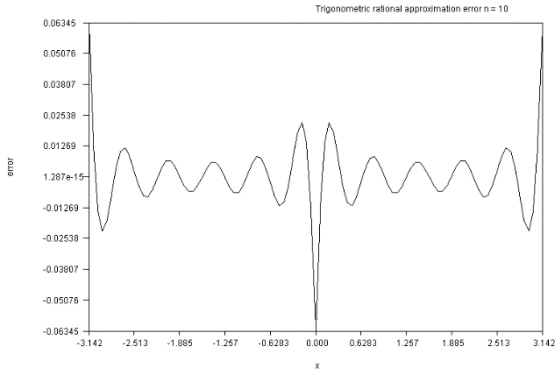
If number of the terms incresed, a better function fitting characteristics will be obtained.



Trigonometric polynomials n = 10

x	Trigonometric f(x)	f(x)	error
-3.141592653589793	3.0781400010755173	3.141592653589793	0.06345265251427579
-2.827433388230814	2.834650737418562	2.827433388230814	-0.007217349187748301
-2.5132741228718345	2.510909510200971	2.5132741228718345	0.0023646126708634085
-2.199114857512855	2.2001140890181654	2.199114857512855	-9.992315053102274E-4
-1.8849555921538759	1.8845616275341752	1.8849555921538759	3.9396461970064856E-4
-1.5707963267948966	1.5707963267948968	1.5707963267948966	-2.220446049250313E-16
-1.2566370614359172	1.2570310260556186	1.2566370614359172	-3.939646197013147E-4
-0.9424777960769379	0.9414785645716275	0.9424777960769379	9.992315053104495E-4
-0.6283185307179586	0.630683143388823	0.6283185307179586	-0.0023646126708644077
-0.3141592653589793	0.3069419161712307	0.3141592653589793	0.007217349187748634
0.0	0.06345265251427319	0.0	-0.06345265251427319
0.31415926535897887	0.3069419161712301	0.31415926535897887	0.007217349187748745
0.6283185307179586	0.630683143388823	0.6283185307179586	-0.0023646126708644077
0.9424777960769379	0.9414785645716275	0.9424777960769379	9.992315053104495E-4
1.2566370614359172	1.2570310260556186	1.2566370614359172	-3.939646197013147E-4
1.5707963267948966	1.5707963267948968	1.5707963267948966	-2.220446049250313E-16
1.8849555921538759	1.8845616275341752	1.8849555921538759	3.9396461970064856E-4
2.199114857512855	2.2001140890181654	2.199114857512855	-9.992315053102274E-4
2.5132741228718345	2.510909510200971	2.5132741228718345	0.0023646126708634085
2.827433388230814	2.834650737418562	2.827433388230814	-0.007217349187748301
3.141592653589793	3.0781400010755173	3.141592653589793	0.06345265251427579

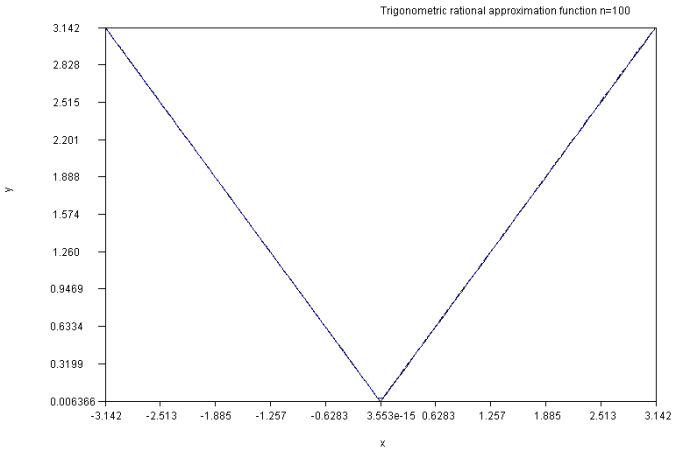


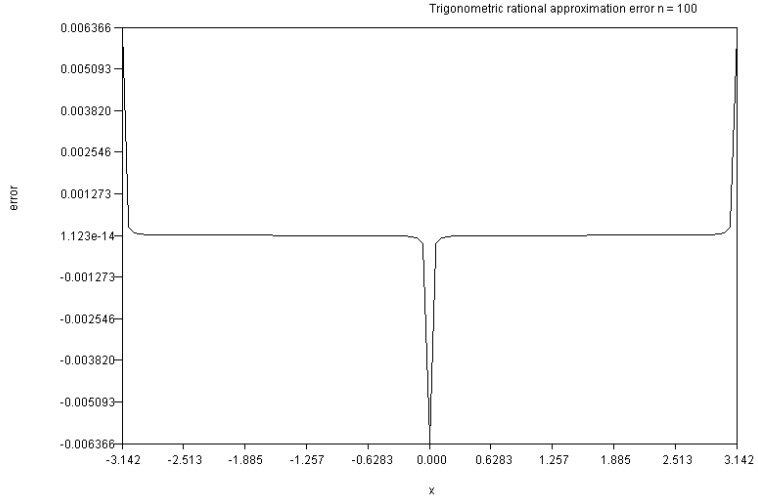


And finally error for 100 terms will be

Trigonometric polynomials n = 100

x	Trigonometric f(x)	f(x)	error
-3.141592653589793	3.135226668042987	3.141592653589793	0.006365985546806119
-2.827433388230814	2.8274208595836403	2.827433388230814	1.2528647173493113E-5
-2.5132741228718345	2.5132711510880275	2.5132741228718345	2.9717838070375535E-6
-2.199114857512855	2.1991137159337857	2.199114857512855	1.1415790694968564E-6
-1.8849555921538759	1.8849551576523567	1.8849555921538759	4.345015192086521E-7
-1.5707963267948966	1.5707963267948961	1.5707963267948966	4.440892098500626E-16
-1.2566370614359172	1.2566374959374371	1.2566370614359172	-4.345015198747859E-7
-0.9424777960769379	0.9424789376560113	0.9424777960769379	-1.141579073382637E-6
-0.6283185307179586	0.6283215025017691	0.6283185307179586	-2.971783810479245E-6
-0.3141592653589793	0.31417179400615136	0.3141592653589793	-1.2528647172049823E-5
0.0	0.006365985546783667	0.0	-0.006365985546783667
0.31415926535897887	0.3141717940061509	0.31415926535897887	-1.2528647172049823E-5
0.6283185307179586	0.6283215025017691	0.6283185307179586	-2.971783810479245E-6
0.9424777960769379	0.9424789376560113	0.9424777960769379	-1.141579073382637E-6
1.2566370614359172	1.2566374959374371	1.2566370614359172	-4.345015198747859E-7
1.5707963267948966	1.5707963267948961	1.5707963267948966	4.440892098500626E-16
1.8849555921538759	1.8849551576523567	1.8849555921538759	4.345015192086521E-7
2.199114857512855	2.1991137159337857	2.199114857512855	1.1415790694968564E-6
2.5132741228718345	2.5132711510880275	2.5132741228718345	2.9717838070375535E-6
2.827433388230814	2.8274208595836403	2.827433388230814	1.2528647173493113E-5
3.141592653589793	3.135226668042987	3.141592653589793	0.006365985546806119





There is a discreate analog that is useful for the discreate least square approximation and interpolation of the data. Assume that a collection of 2m paired data points (x_j,y_j) j=0..2m-1 is given. We assume that the interval is [-π,π]. If any other range is given, it should be converted by using linear conversion. Standart linear least square curve fitting method for w=1 can be applied here. For the trigonometric function

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} [a_k \cos(kx) + b_k \sin(kx)] \quad (6.15-7)$$

Error term will be:

$$E(S_n) = \sum_{j=0}^{2m-1} \left\{ y_j - \left[\frac{a_0}{2} + a_n \cos(nx_j) + \sum_{k=1}^{n-1} \left[a_k \cos(kx_j) + b_k \sin(kx_j) \right] \right] \right\} \quad (6.15-8)$$

We need to choose a_k and b_k coefficients so that error will be minimum. It should be note that cosine and sine functions are orthogonal and for orthogonal functions relation

$$\sum_{k=1}^{2m-1} \emptyset_k(x_j)\emptyset_l(x_j) = 0 \quad for\ k \neq l \quad (6.15-9)$$

Will held, So solution set will not require a matrix solution and the coeeficients will be:

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos(kx_j) \quad (6.15-10)$$

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin(kx_j) \quad (6.15-11)$$

If the function is defined in some other interval than [-π,π] then a range correction should apply
For example for the range $[x_0, x_n]$ ranges can be converted by using transfer functions

$z = \frac{2\pi(x-x_0)}{(x_n-x_0)} - \pi$ (6.15-12)

$x = \frac{(x_n-x_0)(x+\pi)}{2\pi} + x_0$ (6.15-13)

So basic equations becomes:

$S_n(z) = \frac{a_0}{2} + a_n \cos(nz) + \sum_{k=1}^{n-1}[a_k \cos(kz) + b_k \sin(kz)]$ (6.15-14)

$a_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \cos(kz_j)$ (6.15-16)

$b_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \sin(kz_j)$ (6.15-17)

Note that if a discreate set is given problem can be solved either as a function fitting or data fitting problem. As an example problem Let us find discreate least square approximation $S_3(x)$, for m=5 (2m-1=9). The function is $f(x) = x^4 - 3x^3 + 2x^2 \tan(x(x-1))$. The problem first solved as an excel datasheet problem

FOURIER DISCRETE LEAST SQUARE CURVE

	xj			*k*	*0*	*1*	*2*	*3*				
j	**j/5**	**z_i**		**f(x)**	**a_0**	**a_1**	**a_2**	**a_3**	**b_0**	**b_1**	**b_2**	**b_3**
0	0	-3.14159	-	0	0	0	0	0	0	0	0	0
1	0.2	-2.51327	-	0.43400285	0.0868006	-0.0702231	0.0268228	0.0268229	0	-0.0510201	0.0825523	-0.082552
2	0.4	-1.88496	-	0.89814382	0.1796288	-0.0555083	-0.1453227	0.1453227	0	-0.1708371	0.1055831	0.1055832
3	0.6	-1.25664	-	1.31723235	0.2634465	0.0814094	-0.2131327	-	0	-0.2505525	-0.15485	0.1548499
4	0.8	-0.62832	-	1.58195749	0.3163915	0.2559661	0.0977704	-	0	-0.1859703	-0.3009062	-0.300906
5	1	0	0	1.55740772	0.3114815	0.3114815	0.3114815	0.3114815	0	0	0	0
6	1.2	0.62832	0.62832	1.19795749	0.2395915	0.1938336	0.0740378	-	0	0.1408283	0.2278651	0.2278651
7	1.4	1.25664	1.25664	0.64523235	0.1290465	0.0398776	-0.1044008	-	0	0.1227305	0.0758516	-0.075852
8	1.6	1.88496	1.88496	0.13014382	0.0260288	-0.0080433	-0.0210577	0.0210577	0	0.0247548	-0.0152993	-0.015299
9	1.8	2.51327	2.51327	-0.1419971	-	0.0229756	-0.0087759	-	0	-0.0166928	0.0270095	-0.027009
10	2	3.14159	3.14159	0	0	0	0	0	0	0	0	0
				7.62008075	1.5524156	0.7717691	0.0174228	0.0065673	0	-0.386759	0.047806	-0.013321

x	**z**	**f(x)**	**$S_3(x)$**	**$\|f(x)-S_3(x)\|$**
0	-3.14159	0	0.01529	0.01529429
0.125	-2.74889	0.2644	0.2548	0.00959471
0.375	-1.9635	0.84081	0.86573	0.02492368
0.625	-1.1781	1.3615	1.37668	0.015174
0.875	-0.3927	1.61282	1.61826	0.00544731
1.125	0.392699	1.36672	1.38986	0.02313645
1.375	1.178097	0.71697	0.72965	0.01267567
1.625	1.963495	0.07909	0.08349	0.00439694

A java program is developed to solve discreate least square problem (**Fourier series problem**)

Program discreate trigonometric function approximation program Fourier.java

```
import java.io.*;
import javax.swing.*;
class ff extends f_x
{
public double func(double x)
{return x*x*x*x-3.0*x*x*x+2.0*x*x-Math.tan(x*(x-2.0));}
}

// Fourier approximation
class Fourier
{
public double a[];
public double b[];
public double xj[];
public double zj[];
public double yj[];
public double x0,xn;
public int m; //(2*m+1) data points
public int n; //degree n Fourier function

public double z_x(double x)
{double zz=2.0*Math.PI*(x-x0)/(xn-x0)-Math.PI;
```

```
 return zz;
}

public double x_z(double z)
{double xx=(z+Math.PI)/(2.0*Math.PI)*(xn-x0)+x0;
 return xx;
}

public Fourier(f_x f,double x0i,double xni,int mi,int ni)
{
m=mi;
n=ni;
a=new double[n+1];
b=new double[n+1];
xj=new double[2*m+1];
zj=new double[2*m+1];
yj=new double[2*m+1];
x0=x0i;
xn=xni;
for(int k=0;k<=2*m;k++)
{xj[k]=x0+(xn-x0)/(2*m)*k;
 yj[k]=f.func(xj[k]);
 zj[k]=z_x(xj[k]);
}
  for(int k=0;k<=n;k++)
  {
  a[k]=0;
  b[k]=0;
  for(int j=0;j<2*m;j++)
    { a[k]+=1.0/m*yj[j]*Math.cos(k*zj[j]);
      b[k]+=1.0/m*yj[j]*Math.sin(k*zj[j]);
    }
  }
}

public  double func(double x)
{
double cc=a[0]/2.0+a[n]*Math.cos(n*z_x(x));
for(int k=1;k<n;k++)
{cc+=a[k]*Math.cos(k*z_x(x))+b[k]*Math.sin(k*z_x(x));}
return cc;
}

public void plot_func(f_x f,double x0i,double xni,int ni)
{
            double xx[][]=new double[4][ni+1];
            double x1=0;
   for(int i=0;i<=ni;i++)
            { x1=x0i+(xni-x0i)*i/ni;
              xx[0][i]=x1;
              xx[1][i]=func(x1);
              xx[2][i]=f.func(x1);
              xx[3][i]=xx[2][i]-xx[1][i];
            }
            Plot pp=new Plot(xx[0],xx[1]);
            pp.addData(xx[0],xx[2]);
   pp.setPlabel("Discrete Trigonometric approximation function n="+(a.length-1));
            pp.plot();
}
            public void plot_error(f_x f,double x0i,double xni,int ni)
   { double xx[][]=new double[4][ni+1];
              double x1=0;
     for(int i=0;i<=ni;i++)
              { x1=x0i+(xni-x0i)*i/ni;
              xx[0][i]=x1;
              xx[1][i]=func(x1);
              xx[2][i]=f.func(x1);
```

```
                xx[3][i]=xx[2][i]-xx[1][i];
                }
            Plot pp=new Plot(xx[0],xx[3]);
    pp.setPlabel("Discrete Trigonometric approximation error n = "+(a.length-1));
    pp.setYlabel("error");
            pp.plot();
    }
public double[][] result(f_x f)
{   double xx[][]=new double[4][2*m];
    for(int i=0;i<2*m;i++)
            {
             xx[0][i]=xj[i];
             xx[1][i]=func(xj[i]);
             xx[2][i]=f.func(xj[i]);
             xx[3][i]=xx[2][i]-xx[1][i];
            }
            return xx;
}

public static void main(String arg[])
{
            ff f=new ff();
            int n=Integer.parseInt(JOptionPane.showInputDialog("number of terms in the Trigonometric function
approximation Sn(x): n="));
            int m=Integer.parseInt(JOptionPane.showInputDialog("number of data points (2m-1): m="));
            double x0=0;
            double xn=2.0;
            Fourier Pa=new Fourier(f,x0,xn,m,n);
    String s1[]={"x","Fourier f(x)","f(x)","error"};
            Text.print(Text.T(Pa.result(f)),s1,"Discrete Fourier curve fitting n = "+n+"m = "+m);
    Text.printT(Pa.a,"a");
    Text.printT(Pa.b,"b");
    Pa.plot_func(f,x0,xn,100);
    Pa.plot_error(f,x0,xn,100);
}

}
```

a

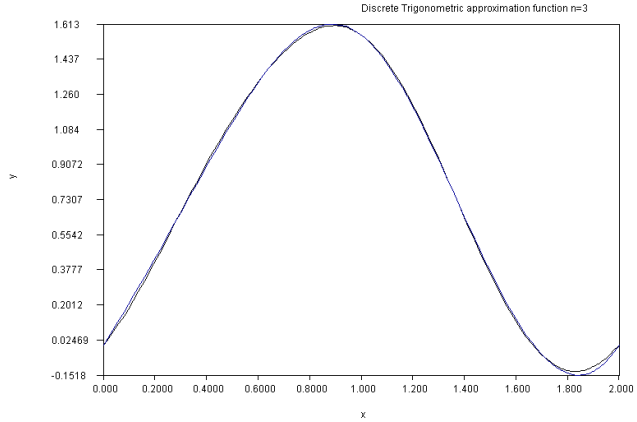
0
1.5240161502702365
0.771769039301839
0.017422790287928383
0.006567273148323184

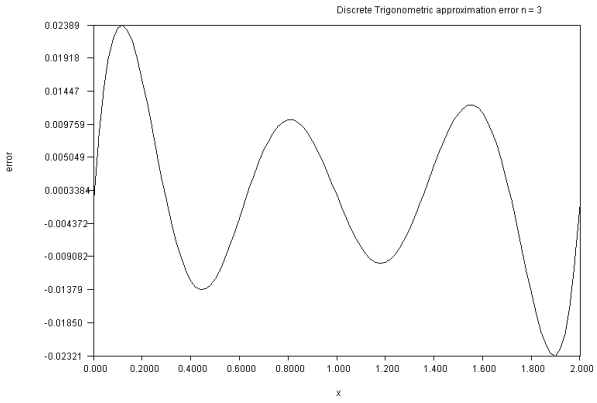
b

0
0.0
-0.3867590451331591
0.04780604706974956
-0.013320698468437413

Discrete Fourier curve fitting n = 3m = 5

x	Fourier f(x)	f(x)	error
0.0	0.0010945529728844348	0.0	-0.0010945529728844348
0.2	0.41784465940942667	0.434002851642027	0.016158192232600344
0.4	0.9106654279577352	0.8981438222209641	-0.01252160573677108
0.6000000000000001	1.3208194756677367	1.317232348561538	-0.0035871271061986842
0.8	1.571601893277591	1.5819574909236105	0.010355597646019454
1.0	1.557767177873209	1.5574077246549023	-3.594532183066601E-4
1.2000000000000002	1.2078718726056072	1.1979574909236106	-0.009914381681996653
1.4000000000000001	0.641359434323769	0.6452323485615388	0.003872914237769809
1.6	0.11880662886177322	0.130143822220963	0.01133719335918977
1.8	-0.12775037159854952	-0.1419971483579729	-0.014246776759423374

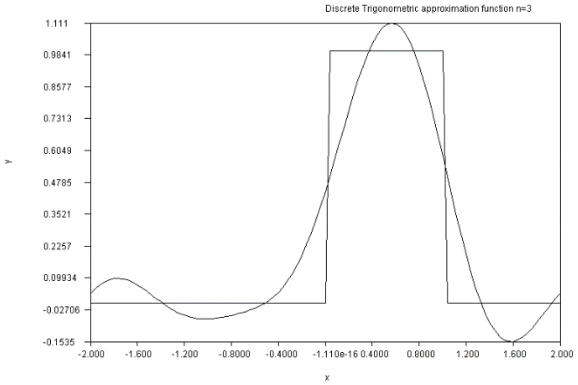


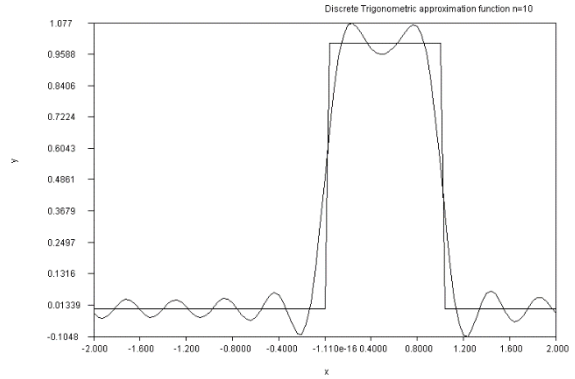


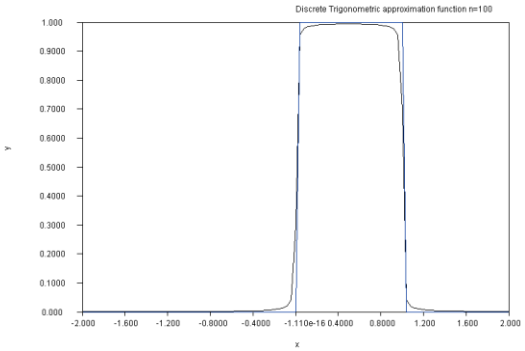
As another example, the following function is given

$$f(x) = \left\{ \begin{matrix} 0 \\ 1 \\ \end{matrix} \right\} \begin{matrix} for\ x < 0 \\ for\ \ 0 <= x < 1 \\ for\ x > 1 \end{matrix}$$

Fourier series of different order fits to this function: $S_3(x)$, $S_{10}(x)$ and $S_{100}(x)$ function plots are given below:







If data is given and a curve fitting is required, the same equations can applied.

Program discreate trigonometric data approximation(curve fitting) program Fourier1.java

```
import java.io.*;
import javax.swing.*;

// Fourier approximation
class Fourier1
{
public double a[];
public double b[];
public double xj[];
public double zj[];
public double yj[];
public double x0,xn;
public int m; //(2*m+1) data points
public int n; //degree n Fourier function

public void read_data(String s1)
{
double a[][]=Text.readDoubleT(s1);
int mm=a[0].length;
m=mm/2;
x0=a[0][0];
xn=a[0][mm-1];
xj=new double[mm];
zj=new double[mm];
yj=new double[mm];
for(int k=0;k<=2*m;k++)
{xj[k]=a[0][k];
 yj[k]=a[1][k];
 zj[k]=z_x(xj[k]);
}
}

public double z_x(double x)
{double zz=2.0*Math.PI*(x-x0)/(xn-x0)-Math.PI;
 return zz;
```

```
}

public double x_z(double z)
{double xx=(z+Math.PI)/(2.0*Math.PI)*(xn-x0)+x0;
 return xx;
}

public Fourier1(String name,int ni)
{
read_data(name);
n=ni;
a=new double[n+1];
b=new double[n+1];
  for(int k=0;k<=n;k++)
  {
  a[k]=0;
  b[k]=0;
  for(int j=0;j<2*m;j++)
    { a[k]+=1.0/m*yj[j]*Math.cos(k*zj[j]);
      b[k]+=1.0/m*yj[j]*Math.sin(k*zj[j]);
    }
  }
}

public  double func(double x)
{
double cc=a[0]/2.0+a[n]*Math.cos(n*z_x(x));
for(int k=1;k<n;k++)
{cc+=a[k]*Math.cos(k*z_x(x))+b[k]*Math.sin(k*z_x(x));}
return cc;
}

public void plot_func(double x0i,double xni,int ni)
{
            double xx[][]=new double[2][ni+1];
            double x1=0;
   for(int i=0;i<=ni;i++)
            { x1=x0i+(xni-x0i)*i/ni;
              xx[0][i]=x1;
              xx[1][i]=func(x1);
            }
            Plot pp=new Plot(xx[0],xx[1]);
            pp.addData(xj,yj);
            pp.setPlotType(1,32);
   pp.setPlabel("Discrete Trigonometric least square n="+(a.length-1));
            pp.plot();
}

public double[][] result()
{   double xx[][]=new double[4][2*m];
   for(int i=0;i<2*m;i++)
            {
              xx[0][i]=xj[i];
              xx[1][i]=yj[i];
              xx[2][i]=func(xj[i]);
              xx[3][i]=xx[2][i]-xx[1][i];
            }
            return xx;
}

public static void main(String arg[])
{
            int n=Integer.parseInt(JOptionPane.showInputDialog("number of terms in the Trigonometric function
approximation Sn(x): n="));
            String name=JOptionPane.showInputDialog("Data file name (a.txt)");
            Fourier1 Pa=new Fourier1(name,n);
   String s1[]={"x","Fourier f(x)","f(x)","error"};
```

```
            Text.print(Text.T(Pa.result()),s1,"Discrete Fourier curve fitting n = "+n+"m = "+Pa.m);
    Text.printT(Pa.a,"a");
    Text.printT(Pa.b,"b");
    Pa.plot_func(Pa.x0,Pa.xn,100);
}
}
```

Example data a.txt (generated from the function of the previous example)

```
0       0
0.2     0.434002852
0.4     0.898143822
0.6     1.317232349
0.8     1.581957491
1       1.557407725
1.2     1.197957491
1.4     0.645232349
1.6     0.130143822
1.8     -0.141997148
2       0
```

a

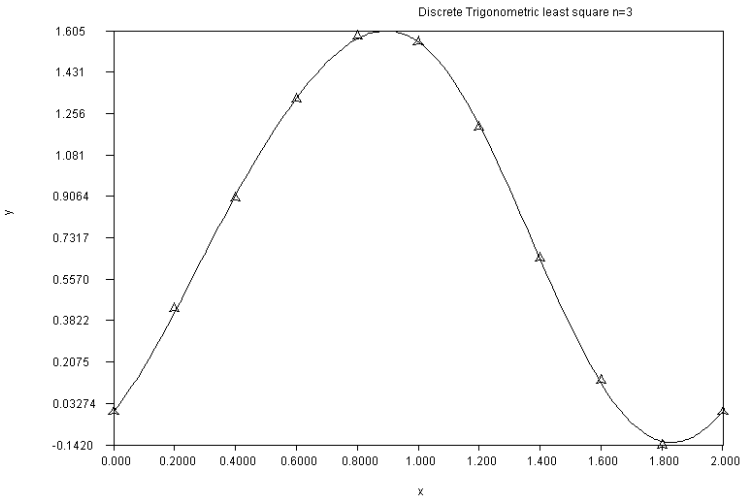
0
1.5240161506
0.7717690393612457
0.017422790340254395
0.006567273038754351

b

0
0.0
-0.38675904513315906
0.04780604706974935
-0.013320698468437184

Discrete Fourier curve fitting n = 3m = 5

x	Fourier f(x)	f(x)	error
0.0	0.0	0.001094553240254318	0.001094553240254318
0.2	0.434002852	0.4178446595085582	-0.016158192491441792
0.4	0.898143822	0.9106654279732834	0.012521605973283445
0.6	1.317232349	1.3208194758972864	0.0035871268972864634
0.8	1.581957491	1.5716018935405622	-0.010355597459437815
1.0	1.557407725	1.5577671780402547	3.5945304025464253E-4
1.2	1.197957491	1.2078718728685784	0.009914381868578515
1.4	0.645232349	0.6413594345533187	-0.0038729144466813636
1.6	0.130143822	0.11880662887732185	-0.01133719312267814
1.8	-0.141997148	-0.12775037149941762	0.01424677650058237

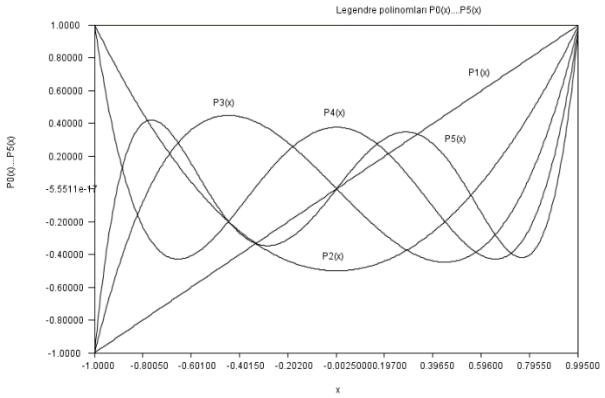


ORTHOGONAL FUNCTION APPROXIMATION: LEGENDRE POLYNOMIALS

Utilising sine and cosine functions as orthogonal function approximation is investigated in the previous sub-section. Similar tasks can be achieved by using other orthogonal polynomials. In this section we will be using Legendre polynomials as a means of function approximation. Definition of LEgendre polynomials are as follows:

$P_k(x) = \frac{(2k-1)}{k} x P_{k-1}(x) - \frac{(k-1)}{k} P_{k-2}(x)$ (6.16-1)

In this recursive equation first two terms are defined as: $P_0(x)=1$ and $P_1(x)=x$. The first 6 Legndre functions are shown in the plot below.



(bolum6_049.jpg,Legendre polinomları)

If the first 10 Legendre polynomials are listed in open form:

n	$P_n(x)$ -1<= x <=1
0	1
1	x
2	$(3x^2-1)/2$
3	$(5x^3-3x)/2$
4	$(35x^4-30x^2+3)/8$
5	$(63x^5-70x^3+15x)/8$
6	$(231x^6-315x^4+105x^2-5)/16$
7	$(429x^7-693x^5+315x^3-35x)/16$
8	$(6435x^8-12012x^6+6930x^4-1260x^2+35)/128$
9	$(12155x^9-25740x^7+18018x^5-4620x^3+315x)/128$
10	$(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)/256$

Equivalent form of a function f(x) in as Legendre function in the range [-1,1]

$f(x) = \sum_{k=0}^{m} b_k P_k(x) + error$

Basic definition of orthogonal form of Legendre function is based on [-1,1] region, if a function definition of a different range is concerned, coordinate system transfer is needed. For the region of [a,b] the following transfer function can be used.

$z = \left(\frac{b-a}{2}\right) x + \left(\frac{b+a}{2}\right) = \alpha x + \beta \qquad \alpha = \left(\frac{b-a}{2}\right) \quad \beta = \left(\frac{b+a}{2}\right)$ (6.16-2)

For the new coordinate system equation will be transferred as

$f(z) = \sum_{k=0}^{m} b_k q_k(z) + error$ (6.16-3) In this equation q_k converted Legendre function in the form of

$q_k(z) = q_k(\alpha x + \beta) = P_k(x)$ (6.16-4)

Due to orthogonality properties:

$\left[P_n(x)P_j(x)\right] = \int_{-1}^{1} P_n(x)P_j(x)dx$ where $\left[P_n(x)P_j(x)\right] = \begin{Bmatrix} 0 & if & n \neq j \\ \frac{2}{2n+1} & if & n = j \end{Bmatrix}$ (6.16-5)

so the coefficients of the function fitting will become $b_k = \frac{2k+1}{b-a} \int_a^b f(z) q_k(z) dz$ (6.16-6)

In order to achive numerical integration process, please referred to related section. As an example function let us assume **f(x)=|x|**

Program Legendre function approximation for a given function or data (converted to a function by cubic spline interpolation formula)

```
public class legendre
{
//cubic spline function or external input function
public if_x f;
//Legendre function
public if_x f_Legendre;
public double alfa,beta,xl,xh;
public double x[];
public double z[];
public double y[];
public double b[]; //katsayılar matrisi;
public int m;   //eğri uydurma derecesi

public static double P(int k, double x)
{ // Legendre polynomial  -1<=x<=1
if(k==0) return 1.0;
else if(k==1) return x;
else
{return ((2.0*k-1.0)*P((k-1),x)*x-P((k-2),x)*(double)(k-1))/(double)k;}
}

public  double q(int k, double z)
{  // converted Legendre polynomial xl<= z <=xh
  double x=(z-beta)/alfa;
  return P(k,x);
}
public  double fak(int k,double z)
   {   //integral fonksiyonu
                double a=(2.0*k+1)/(xh-xl)*f.func(z)*q(k,z);
                return a;
              }

public legendre(if_x f1,double xli,double xhi,int mi)
{   f=f1;
              setdata(xli,xhi,mi);
}
// Constructor from data using cubic spline interpolation
public legendre(double xy[][],double xli,double xhi,int mi)
{   double xi[]=xy[0];
   double yi[]=xy[1];
   double c0=0.0;
   double cn=0.0;
              double S[][]=cubic_spline(xi,yi,c0,cn);
              f=(x)->funcSpline(S,x);
              setdata(xli,xhi,mi);
}
public legendre(double xi[],double yi[],double xli,double xhi,int mi)
{             double c0=0.0;
   double cn=0.0;
              double S[][]=cubic_spline(xi,yi,c0,cn);
              f=(x)->funcSpline(S,x);
              setdata(xli,xhi,mi);
}

public void setdata(double xli,double xhi,int mi)
{  //z  xl, xh
     m=mi;
     xl=xli;
     xh=xhi;
     alfa=(xh-xl)/2.0;
     beta=(xh+xl)/2.0;
     b=new double[m+1];
     for(int j=0;j<=m;j++)
               {b[j]=integral(xl,xh,30,j);}
               f_Legendre=(x)->funcLegendre(x);
               System.out.println("b=\n"+Matrix.toStringT(b));
}
public  double funcLegendre(double z)
{
   double x=(z-beta)/alfa;
   double ff=0;
   double pk=0;
   for(int j=0;j<=m;j++)
```

```
      {   pk=P(j,x);
                  ff+=b[j]*pk;
               }
   return ff;
}

public static double[][] Gauss_Legendre_Coefficients(double x1,double x2,int n)
{
            // calculates Gauss-Legendre coefficients
            double EPS=3.0e-15;
            int m,j,i;
            double z1,z,xm,xl,pp,p3,p2,p1;
   double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
            m=(n+1)/2;
            xm=0.5*(x2+x1);
            xl=0.5*(x2-x1);
            for (i=1;i<=m;i++)  {
                        z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                        do {
                                    p1=1.0;
                                    p2=0.0;
                                    for (j=1;j<=n;j++) {
                                                p3=p2;
                                                p2=p1;
                                                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                                    }
                                    pp=n*(z*p1-p2)/(z*z-1.0);
                                    z1=z;
                                    z=z1-p1/pp;
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
                                    }
return a;
}

public  double integral(double xl,double xh,int n,int k)
{
// n : integral degree of polynomial
// This program calculates Gauss-Legendre coefficients in  [x1,x2] band
// and then calculates the integral in this band.
double a[][]=new double[2][n];
a=Gauss_Legendre_Coefficients(xl,xh,n);
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*fak(k,a[0][i]);}
return z;
}
// Cubic spline interpolation section
  public static double [] thomas(double a[][],double r[])
  {
  //
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

   public static double [] thomas(double f[],double e[],double g[],double r[])
```

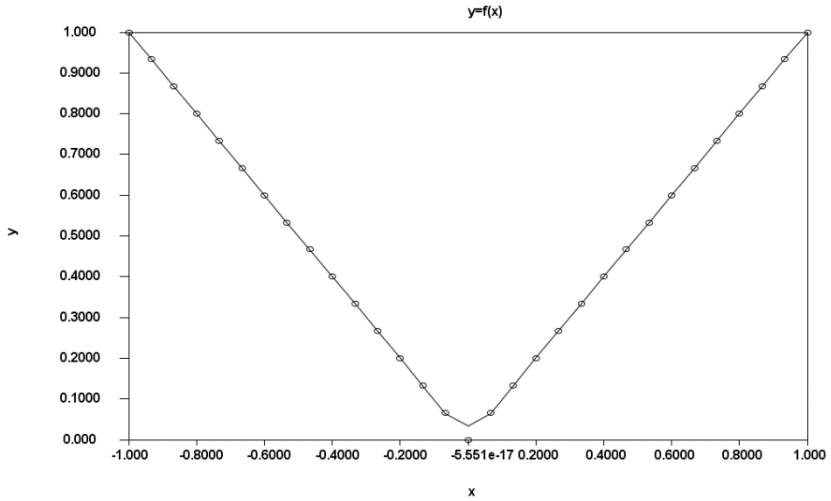
```
{
int n=f.length;
double x[]=new double[n];
for(int k=1;k<n;k++)
   {e[k]=e[k]/f[k-1];
    f[k]=f[k]-e[k]*g[k-1];
   }
for(int k=1;k<n;k++)
   {r[k]=r[k]-e[k]*r[k-1];
   }
x[n-1]=r[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
   {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
return x;
}


public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
{
// c0 second derivative of the first point in the first spline
// cn second derivative of the last point in last spline
int n=xi.length;
double h[]=new double[n];
double w[]=new double[n];
double f[]=new double[n];
double e[]=new double[n];
double g[]=new double[n];
double d[]=new double[n];
double x[]=new double[n];
double S[][]=new double[4][n];
int k;
for(k=0;k<(n-1);k++)
   {h[k]=xi[k+1]-xi[k];
    w[k]=(yi[k+1]-yi[k])/h[k];
              }
d[0]=c0;
d[n-1]=cn;
for(k=1;k<(n-1);k++)
   {d[k]=6.0*(w[k]-w[k-1]);}
f[0]=1.0;
f[n-1]=1.0;
g[0]=0.0;
g[n-1]=0.0;
e[0]=0.0;
e[n-1]=0.0;
for(k=1;k<(n-1);k++)
{f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
S[2]=thomas(f,e,g,d);
S[3]=xi;
for(k=0;k<(n-1);k++)
  {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
   S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
  }
return S;
}

public static double funcSpline(double S[][],double x)
{
int n=S[0].length;
double xx1=0;
double xx2=0;
double y=0;
double hk=0;
for(int k=0;k<(n-1);k++)
   {if(S[3][k]<=x && x<=S[3][k+1])
    {hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     break;
     }
   }
     if(y==0 && S[3][n-2]<=x )
     {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
```

```
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      }
  return y;
  }

  public static double dfSpline(double xi[],double yi[],double c0,double cn,double x)
  { //cubic spline formulations
              double S[][]=cubic_spline(xi,yi,c0,cn);
    return dfSpline(S,x);
  }
  public static double dfSpline(double S[][],double x)
  {
    //cubic spline derivative formulation
  int n=S[0].length;
  double xx1=0;
  double xx2=0;
  double y=0;
  double hk=0;
  for(int k=0;k<(n-1);k++)
     {if(S[3][k]<=x && x<=S[3][k+1])
      {hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]-S[1][k]+(xx1*xx1*S[2][k+1]-xx2*xx2*S[2][k])/(2.0*hk);
       break;
       }
     }
       if(y==0 && S[3][n-2]<=x )
       {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
       xx1=(x-S[3][k]);
       xx2=(S[3][k+1]-x);
       y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
       }
  return y;
  }
  public static double  dfSpline(double xi[],double yi[],double x)
  {
    //natural cubic spline derivative formula
              double S[][]=cubic_spline(xi,yi,0,0);
    return dfSpline(S,x);
  }
//end of cubic spline
}
```

```
public class legendre_test1A
{
 public static void main(String args[])
  { int n=30;//number of evaluation points
    if_x f=(x)->Math.abs(x);
    legendre l=new legendre(f,-1.0,1.0,n);
    Plot p1=new Plot(l.f_Legendre,l.xl,l.xh,n);
    p1.addFunction(f,l.xl,l.xh,n);
    p1.setPlotType(1,22);
    p1.plot();
  }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" legendre_test1
b=
           0.500442496686119
           0.000000000000000
           0.623887571165741
           0.000000000000000
          -0.185978548243092
           0.000000000000000
           0.099693276449802
           0.000000000000000
```

-0.064206907443573
-0.000000000000000
0.045318387689030
-0.000000000000000
-0.033735662852678
0.000000000000000
0.025936642703608
-0.000000000000000
-0.020306723986429
0.000000000000000
0.016004563758942
0.000000000000000
-0.012547099754516
-0.000000000000000
0.009633136817453
0.000000000000000
-0.007059268461882
-0.000000000000000
0.004675608843615
0.000000000000000
-0.002359881760763
0.000000000000000
0.000000000000002



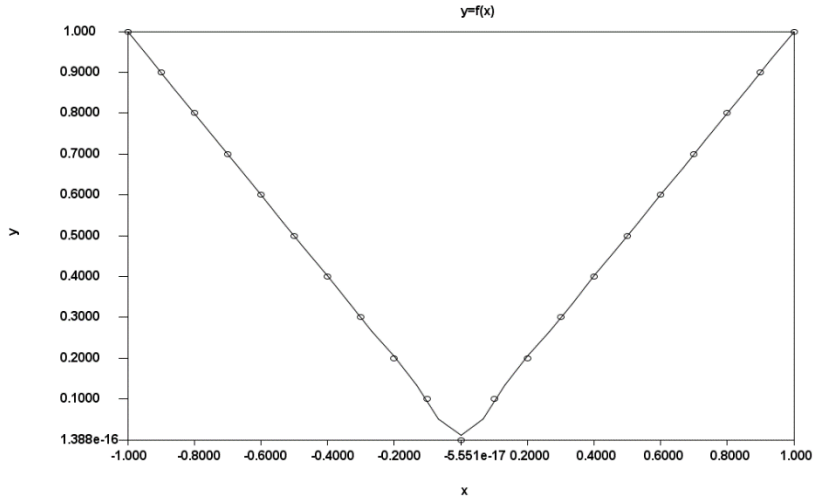
Now that we have a function fitting Legendre polynomial approach, we will apply this method to data fitting by creating a function from cubic spline interpolation and the apply Legendre function fitting method to find the curve fitting function

Data to curve-fit:

-1	1
-0.9	0.9
-0.8	0.8
-0.7	0.7
-0.6	0.6
-0.5	0.5
-0.4	0.4
-0.3	0.3
-0.2	0.2
-0.1	0.1
-1.38778E-16	1.38778E-16
0.1	0.1
0.2	0.2
0.3	0.3
0.4	0.4
0.5	0.5
0.6	0.6
0.7	0.7
0.8	0.8
0.9	0.9
1	1

```
public class legendre_test2A
```

```
{
 public static void main(String args[])
  { int n=30;//number of data points
    double a[][]=Text.readDoubleT("bb.txt");
    //x=a[0]  y=a[1]
    legendre l=new legendre(a,a[0][0],a[0][a[0].length-1],n);
    Plot p1=new Plot(l.f_Legendre,l.xl,l.xh,n);
    p1.addFunction(l.f,l.xl,l.xh,n);
    p1.setPlotType(1,22);
    p1.plot();
  }
}
```

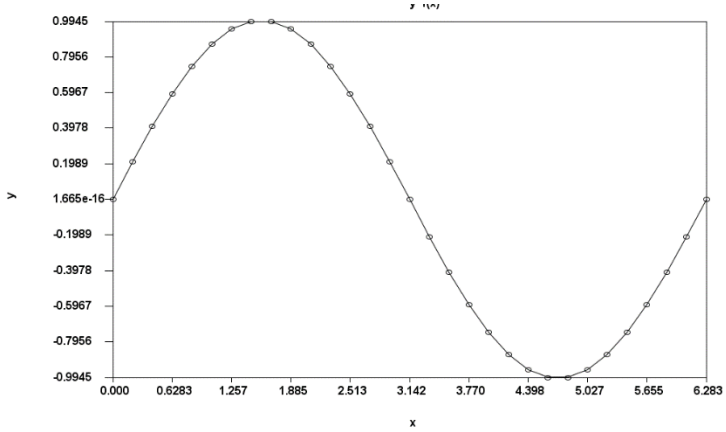


Now let us try a different function f=sin(x)

```
 public class legendre_test1
{public static void main(String args[])
  { int n=30;//number of evaluation points
    if_x f=(x)->Math.sin(x);
    legendre l=new legendre(f,0.0,2.0*Math.PI,n);
    Plot p1=new Plot(l.f_Legendre,l.xl,l.xh,n);
    p1.addFunction(f,l.xl,l.xh,n);
    p1.setPlotType(1,22);
    p1.plot();
  }}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" legendre_test1
b=       0.000000000000000
        -0.954929658551369
        -0.000000000000000
         1.158241912199413
         0.000000000000000
        -0.219289547216234
        -0.000000000000000
         0.016642266917960
        -0.000000000000000
        -0.000682206702639
         0.000000000000000
         0.000017538925948
        -0.000000000000000
        -0.000000309518242
        -0.000000000000000
         0.000000003983935
         0.000000000000000
        -0.000000000039074
         0.000000000000001
         0.000000000000307
         0.000000000000001
         0.000000000000004
        -0.000000000000001
         0.000000000000003
         0.000000000000001
        -0.000000000000002
        -0.000000000000001
```

```
-0.000000000000007
-0.000000000000002
-0.000000000000003
 0.000000000000003
```



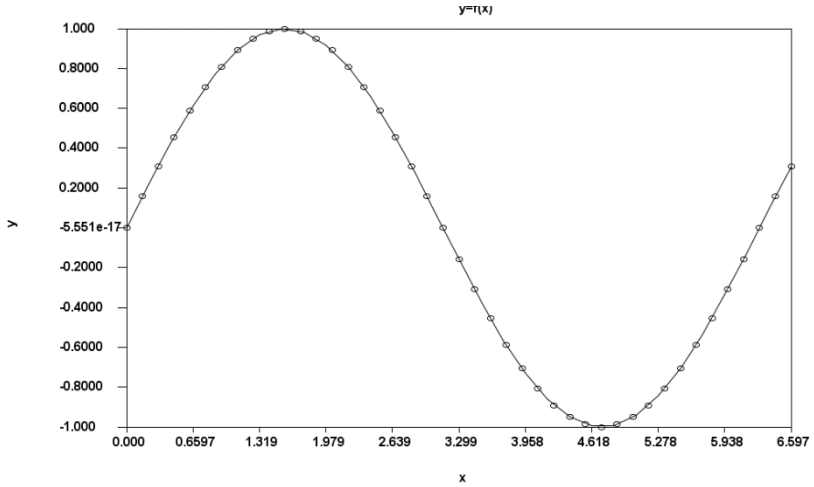
A data file of sinus function is created and Legendre polynomial function fitting is applied. Data file:

```
0.0000000000    0.000000000
0.1570796327    0.156434465
0.3141592654    0.309016994
0.4712388980    0.453990500
0.6283185307    0.587785252
0.7853981634    0.707106781
0.9424777961    0.809016994
1.0995574288    0.891006524
1.2566370614    0.951056516
1.4137166941    0.987688341
1.5707963268    1.000000000
1.7278759595    0.987688341
1.8849555922    0.951056516
2.0420352248    0.891006524
2.1991148575    0.809016994
2.3561944902    0.707106781
2.5132741229    0.587785252
2.6703537556    0.453990500
2.8274333882    0.309016994
2.9845130209    0.156434465
3.1415926536    0.000000000
3.2986722863    -0.156434465
3.4557519189    -0.309016994
3.6128315516    -0.453990500
3.7699111843    -0.587785252
3.9269908170    -0.707106781
4.0840704497    -0.809016994
4.2411500823    -0.891006524
4.3982297150    -0.951056516
4.5553093477    -0.987688341
4.7123889804    -1.000000000
4.8694686131    -0.987688341
5.0265482457    -0.951056516
5.1836278784    -0.891006524
5.3407075111    -0.809016994
5.4977871438    -0.707106781
5.6548667765    -0.587785252
5.8119464091    -0.453990500
5.9690260418    -0.309016994
6.1261056745    -0.156434465
6.2831853072    0.000000000
6.4402649399    0.156434465
6.5973445725    0.309016994
```

```
 public class legendre_test2
{
 public static void main(String args[])
  { int n=30;//number of data points
    double a[][]=Text.readDoubleT("aa.txt");
    //x=a[0]  y=a[1]
    legendre l=new legendre(a,a[0][0],a[0][a[0].length-1],n);
```

```
    Plot p1=new Plot(l.f_Legendre,l.xl,l.xh,n);
    p1.addFunction(l.f,l.xl,l.xh,n);
    p1.setPlotType(1,22);
    p1.plot();
  }}
```

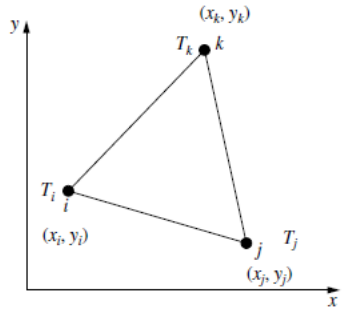
```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" legendre_test2
b=
          0.007414267330787
         -0.844614364918925
          0.239838145840113
          1.242916643035622
         -0.105404035077192
         -0.265311751791639
          0.013337733842588
          0.022383727555032
         -0.000851890424221
         -0.001058933715845
         -0.000003399532391
          0.000006612300055
         -0.000012688339823
          0.000000166584931
          0.000015221408603
          0.000029529923293
          0.000042576245059
          0.000054004185309
          0.000060956610402
          0.000063331749037
          0.000060975796011
          0.000052142373698
          0.000040400580168
          0.000025576784509
          0.000009029701868
         -0.000004223159282
         -0.000014649026394
         -0.000019613015178
         -0.000016868787182
         -0.000010864907456
         -0.000000000000003
```



FINITE ELEMENT (SIMPLEX) METHODS FOR SURFACE CURVE FITTING

The simplex elements were used in Nelder-Mead optimisation. By using a linear or non-linear simplex elements a surface curve fitting can be achieved. The simplest geometric shape that can be employed to approximate irregular surfaces is the triangle and it is one of the popular elements currently used in finite element calculations. The two-dimensional linear triangular element, also known as a *simplex element*, is represented by

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$



where the polynomial is linear in *x* and *y* and contains three coefficients. Since a linear triangle has three nodes, the values of α_1, α_2 and α_3 are determined from

$T_i=\alpha_1+\alpha_2 x_i+\alpha_3 y_i$

$T_j=\alpha_1+\alpha_2 x_j+\alpha_3 y_j$

$T_k=\alpha_1+\alpha_2 x_k+\alpha_3 y_k$

which results in the following:

$$\alpha_1 = \frac{1}{2A}\left[(x_j y_k - x_k y_j)T_i + (x_k y_i - x_i y_k)T_j + (x_i y_j - x_j y_i)T_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[(y_j - y_k)T_i + (y_k - y_i)T_j + (y_i - y_j)T_k\right]$$

$$\alpha_3 = \frac{1}{2A}\left[(x_k - x_j)T_i + (x_i - x_k)T_j + (x_j - x_i)T_k\right]$$

where '*A*' is the area of the triangle given by

$$2A = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = (x_i y_j - x_j y_i) + (x_k y_i - x_i y_k) + (x_j y_k - x_k y_j)$$

Substituting the values of α_1, α_2 and α_3 into the equation and collating the coefficients of T_i, T_j and T_k, we get

$$T = N_i T_i + N_j T_j + N_k T_k$$

$$N_i = \frac{1}{2A}\left[a_i + b_i x + c_i y\right]$$

$$N_j = \frac{1}{2A}\left[a_j + b_j x + c_j y\right]$$

$$N_k = \frac{1}{2A}\left[a_k + b_k x + c_k y\right]$$

$$a_i = (x_j y_k - x_k y_j) \qquad a_j = (x_k y_i - x_i y_k) \qquad a_k = (x_i y_j - x_j y_i)$$

$$b_i = (y_j - y_k) \qquad b_j = (y_k - y_i) \qquad b_k = (y_i - y_j)$$

$$c_i = (x_k - x_j) \qquad c_j = (x_i - x_k) \qquad c_k = (x_j - x_i)$$

If we evaluate N_i at node i, where the coordinates are (x_i,y_i), then we obtain $N_i(x_i,y_i)=1$ and similarly $N_i(x_j,y_j)=0$, $N_i(x_k,y_k)=0$

Thus, we see that the shape functions have a value of unity at the designated vertex and zero at all other vertices. It is possible to show that

$N_i+N_j+N_k=1$ everywhere in the element, including the boundaries. Linear simplex is given us a relatively simple way of interpolating any point in the surface (a linear interpolation). An interpolation program is given below

Program linear triangular finite simplex element

```
public class triangular_element1
{
public double a[],b[],c[];
public double dt;
public int P;
```

```
public double A;//Area
public double l[];
public double x[],y[];   //coordinates of data points m
public double xc,yc;     //center point m
public double N[],B[][]; // Shape factor and derivative
public double Ti[];      // Temperature nodes
public int n_point[];    // point numbers
public int n_element;    // element number

public triangular_element1(int n_elementi,int n_pointi[],double xi[],double yi[])
{dt=1.0;
 P=1;
 x=new double[3];
 y=new double[3];
 Ti=new double[3];
 l=new double[3];//0:lij,1:ljk,2:lik
 xc=0;yc=0;
 n_point=new int[3];
 n_element=n_elementi;
 for(int i=0;i<3;i++)
 { n_point[i]=n_pointi[i];
  x[i]=xi[i];y[i]=yi[i];
  xc+=x[i];yc+=y[i];
 }
 xc/=3.0;
 yc/=3.0;
 a=new double[3];
 b=new double[3];
 c=new double[3];
 N=new double[3];
 B=new double[2][3];
 a[0]=x[1]*y[2]-x[2]*y[1];
 a[1]=x[2]*y[0]-x[0]*y[2];
 a[2]=x[0]*y[1]-x[1]*y[0];
 b[0]=y[1]-y[2];
 b[1]=y[2]-y[0];
 b[2]=y[0]-y[1];
 c[0]=x[2]-x[1];
 c[1]=x[0]-x[2];
 c[2]=x[1]-x[0];
 A=0.5*((x[0]*y[1]-x[1]*y[0])+(x[2]*y[0]-x[0]*y[2])+(x[1]*y[2]-x[2]*y[1]));
 for(int i=0;i<3;i++)
 { B[0][i]=1.0/(2.0*A)*b[i];
   B[1][i]=1.0/(2.0*A)*c[i];
 }
 l[0]=Math.sqrt(b[2]*b[2]+c[2]*c[2]);
 l[1]=Math.sqrt(b[0]*b[0]+c[0]*c[0]);
 l[2]=Math.sqrt(b[1]*b[1]+c[1]*c[1]);
}
public triangular_element1(int n_elementi,int n_pointi[],double xi[],double yi[],double dti,int Pi)
{
 x=new double[3];
 y=new double[3];
 Ti=new double[3];
 l=new double[3];//0:lij,1:ljk,2:lik

 n_element=n_elementi;
 for(int i=0;i<3;i++)
 {n_point[i]=n_pointi[i];
  x[i]=xi[i];y[i]=yi[i];
  xc+=x[i];yc+=y[i];
 }
 xc/=3.0;
 yc/=3.0;
 a=new double[3];
 b=new double[3];
 c=new double[3];
 N=new double[3];
 B=new double[2][3];
 a[0]=x[1]*y[2]-x[2]*y[1];
 a[1]=x[2]*y[0]-x[0]*y[2];
 a[2]=x[0]*y[1]-x[1]*y[0];
 b[0]=y[1]-y[2];
 b[1]=y[2]-y[0];
 b[2]=y[0]-y[1];
 c[0]=x[2]-x[1];
```

```
 c[1]=x[0]-x[2];
 c[2]=x[1]-x[0];
 A=0.5*((x[0]*y[1]-x[1]*y[0])+(x[2]*y[0]-x[0]*y[2])+(x[1]*y[2]-x[2]*y[1]));
 for(int i=0;i<3;i++)
 { B[0][i]=1.0/(2.0*A)*b[i];
   B[1][i]=1.0/(2.0*A)*c[i];
 }
 l[0]=Math.sqrt(b[2]*b[2]+c[2]*c[2]);
 l[1]=Math.sqrt(b[0]*b[0]+c[0]*c[0]);
 l[2]=Math.sqrt(b[1]*b[1]+c[1]*c[1]);
}

public int setI(int cond,double L0,double L,double W0,double W)
{
   int i=100;
                if(cond==0) //x=0
                {if(x[0]==L0 && x[1]==L0) i=0;
                 else if(x[1]==L0 && x[2]==L0) i=1;
                 else if(x[0]==L0 && x[2]==L0) i=2;
                 else i=-1;
                }
                else if(cond==1) //x=L
                {if(x[0]==L && x[1]==L) i=0;
                 else if(x[1]==L && x[2]==L) i=1;
                 else if(x[0]==L && x[2]==L) i=2;
                 else i=-1;
                }
                else if(cond==2) //y=0
                {if(y[0]==W0 && y[1]==W0) i=0;
                 else if(y[1]==W0 && y[2]==W0) i=1;
                 else if(y[0]==W0 && y[2]==W0) i=2;
                 else i=-1;
                }
                else if(cond==3) //y=W
                {if(y[0]==W && y[1]==W) i=0;
                 else if(y[1]==W && y[2]==W) i=1;
                 else if(y[0]==W && y[2]==W) i=2;
                 else i=-1;
                }
                return i;
}


public void setN(double x,double y)
{for(int i=0;i<3;i++)
  {N[i]=1.0/(2.0*A)*(a[i]+b[i]*x+c[i]*y);}
}
public void setN(double z[])
{setN(z[0],z[1]);}

public void setT(double TT[])
{for(int i=0;i<3;i++)
  {Ti[i]=TT[i];}
}

public double[] getT()
{ return Ti;}

public double T(double x,double y)
{ //interpolation
                double TT=0;
 setN(x,y);
 for(int i=0;i<3;i++)
  {TT+=N[i]*Ti[i];}
   return TT;
}

public double T(double T[],double x,double y)
{ //set grid point temperatures and get interpolated value
                setT(T);
  setN(x,y);
  double TT=0;
 for(int i=0;i<3;i++)
  {TT+=N[i]*Ti[i];}
  return TT;
}
```

```
public double T(double T[],double z[])
{ return T(T,z[0],z[1]);}

}
```

Program linear triangular finite simplex grid (curve fitting)

```
 import javax.swing.*;
import org.math.plot.*;
import static java.lang.Math.*;
import static org.math.array.DoubleArray.*;
import visad.*;
import visad.java3d.DisplayImplJ3D;
import visad.java2d.DisplayImplJ2D;
import java.rmi.RemoteException;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;
public class triangular_grid1
{
public triangular_element1 te[];
public setElement se[];
public int N,M,K,K1,K2,Nm1,Mm1;
public double x[][];
public double y[][];
public double T[][];
public double Ti[];
public double xi[];
public double yi[];
public boolean bi[];
public double dx,dy;
public int Bx0[],BxM[],By0[],ByN[];
public double L,W;
public double min(double xi[][])
{ double min=1e200;
  for(int i=0;i<xi.length;i++)
   {for(int j=0;j<xi[0].length;j++)
     {if(xi[i][j]<min) min=xi[i][j];}
 }
 return min;
}
public double max(double xi[][])
{double max=-1e200;
 for(int i=0;i<xi.length;i++)
 {for(int j=0;j<xi[0].length;j++)
   {if(xi[i][j]<max) max=xi[i][j];}
 }
 return max;
}
public triangular_grid1(double xi[][],double yi[][],double zi[][])
{x=xi;
 y=yi;
 T=zi;
 N=x.length-1;
 M=x[0].length-1;
 L=max(xi);
 W=max(yi);
 setElements();
}

public void generateTriangularGrid(double Li, double Wi,int Ni, int Mi,double zi[][])
{int i,j;
 N=Ni;M=Mi;
 L=Li;
 W=Wi;
 x=new double[N+1][M+1];
 y=new double[N+1][M+1];
 T=zi;
 setElements();
}

public void setElements()
{ int i,j;
 dx=L/M;
 dy=W/N;
 K=M*N;
```

```
 K1=K*2;
 K2=K1+1;
 //grid points
 xi=new double[K2];
 yi=new double[K2];
 Ti=new double[K2];
 bi=new boolean[K2];
 se=new setElement[K2];
 for(i=0;i<K2;i++) {se[i]=new setElement();}
 te=new triangular_element1[K1];
 int k1=-1,k=0,n=0,m=0;
for(n=0;n<=N;n++)
{ for(m=0;m<=M;m++)
  {k=(N+1)*n+m;
   x[n][m]=m*dx;y[n][m]=n*dy;
   xi[k]=x[n][m];
   yi[k]=y[n][m];
   bi[k]=false;
   Ti[k]=T[n][m];
  }
}

for(n=0;n<N;n++)
{for(m=0;m<M;m++)
  { k=(N+1)*n+m;
              int kp1=(N+1)*(n+1)+m;
              int kp1p1= kp1+1;
              double xi1[]={x[n][m],x[n][m+1],x[n+1][m]};
              double yi1[]={y[n][m],y[n][m+1],y[n+1][m]};
    double zi1[]={T[n][m],T[n][m+1],T[n+1][m]};
              k1++;
              int n_point1[]={k,(k+1),kp1};
              se[k].add(k1,0);
              se[k+1].add(k1,1);
              se[kp1].add(k1,2);
              te[k1]=new triangular_element1(k1,n_point1,xi1,yi1);
              te[k1].setT(zi1);
              double xi2[]={x[n][m+1],x[n+1][m+1],x[n+1][m]};
              double yi2[]={y[n][m+1],y[n+1][m+1],y[n+1][m]};
              double zi2[]={T[n][m+1],T[n+1][m+1],T[n+1][m]};
              k1++;
              int n_point2[]={k+1,kp1p1,kp1};
    se[k+1].add(k1,0);
              se[kp1p1].add(k1,1);
              se[kp1].add(k1,2);
              te[k1]=new triangular_element1(k1,n_point2,xi2,yi2);
              te[k1].setT(zi2);
  }
}
}
public triangular_grid1(double Li, double Wi,int Ni, int Mi,double zz[][])
{generateTriangularGrid(Li,Wi,Ni,Mi,zz);}
public int find_element(double z[])
{ return find_element(z[0],z[1]);}
public int find_element(double x,double y)
{double R1=Math.sqrt(x*x+y*y);
 int kmin=0;
 double min=1.0e50;
 for(int k=0;k<K1;k++)
 {   te[k].setN(x,y);
     double r0=Math.sqrt((te[k].x[0]-x)*(te[k].x[0]-x)+(te[k].y[0]-y)*(te[k].y[0]-y));
     double r1=Math.sqrt((te[k].x[1]-x)*(te[k].x[1]-x)+(te[k].y[1]-y)*(te[k].y[1]-y));
     double r2=Math.sqrt((te[k].x[2]-x)*(te[k].x[2]-x)+(te[k].y[2]-y)*(te[k].y[2]-y));
     double dr=Math.sqrt(r0*r0+r1*r1+r2*r2);
               boolean b1=(x>=te[k].x[0]) && (x<=te[k].x[2]);
               boolean b2=(y>=te[k].y[0]) && (y<=te[k].y[1]);
               boolean b= b1 && b2;
     if(dr<min) {min=dr;kmin=k;}
 }
 return kmin;
}

public void ContourPlot()
{
ContourPlot pp=new ContourPlot( x,y,T,16,true,"Finite element","x","y");
pp.plot();
```

```
 }
public void Plot3D(double L0,double L,double W0,double W) throws RemoteException, VisADException
{
Plot3D.plot3D_CD1(T,L0,L,W0,W,200,"x","y","T degree C");
 }
 public void GridPlot(double L0,double L,double W0,double W)
 {Plot3DPanel plot = new Plot3DPanel("SOUTH");
  double dx=(L-L0)/M;
  double dy=(W-W0)/N;
  double[] xx = increment(L0, dx,L); // x = 0.0:0.1:1.0
  double[] yy = increment(W0,dy,W);
  plot.addGridPlot("Finite element Grid", xx, yy, T);
  JFrame frame = new JFrame("a plot panel");
  frame.setSize(600, 600);
  frame.setContentPane(plot);
  frame.setVisible(true);
 }
 public double getT(double x,double y)
 {double z[]={x,y};
  int n=find_element(z);
  double T1=te[n].T(x,y);
  return T1;
 }

 public static void main(String arg[]) throws RemoteException, VisADException
 {double L=1.0;
  double W=1.0;
  int N=2,M=2;
  double TT[][]={{80.0,90.0,100.0},{90.0,100.0,110.0},{100.0,110.0,120.0}};
  double hi=20.0;
  triangular_grid1 tg=new triangular_grid1(L,W,M,N,TT);
  double z[]={0.65,0.55};
  int n=tg.find_element(z);
  System.out.println("element = "+n);
  double T=tg.getT(z[0],z[1]);
  System.out.println("T = "+T);
 }
 }
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" triangular_grid1
element = 6
T = 104.00000000000001

> Terminated with exit code 0.
```

ARTIFICIAL NEURAL NETWORKS
(Note : This section and its programs is prepared by Dr. Mert Sinan Turgut)

Artificial Neural Networks (ANN) is a mathematical model inspired from animal central nervous systems, particularly brain.

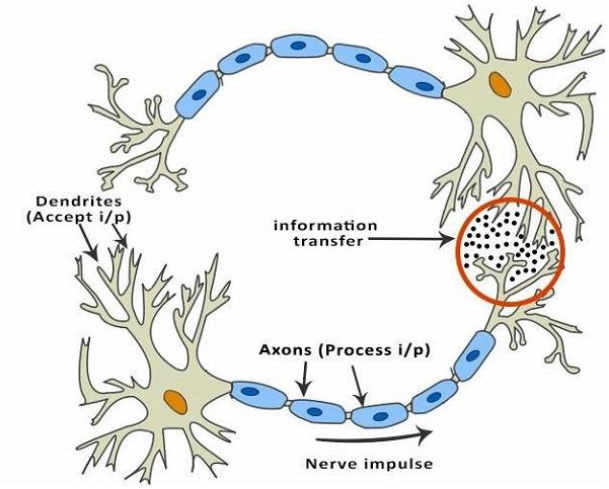


Figure Brain neuron structure

ANNs generally used to estimate or approximate functions for large number of inputs. Most ANNs consist of one input layer, one hidden layer and one output layer. An ANN with at least one hidden layer, in which each neuron is represented with a non-linear differentiable activation function is called “universal approximator” since it is shown that it can approximate any function. A one layer artificial neural network is only consisted input and outpu layers. The simlest one layer ANN is called ADALINE neoral Network. In figüre 6.7.2 such a network is shown.

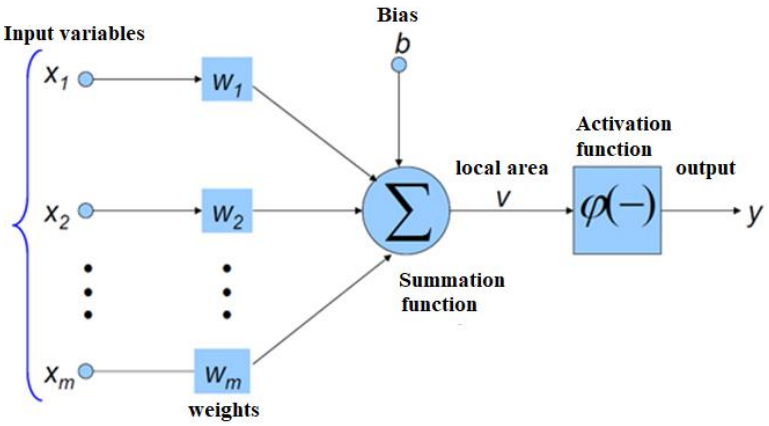
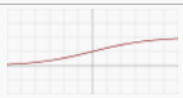
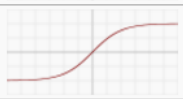
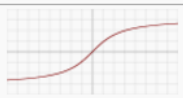
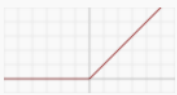


Figure An ADELINE network

The output y can be expressed as

$v = \sum_{j=1}^{n} x_j w_j + b$ $\qquad\qquad$ $y = \varphi(v) = \varphi\left(\sum_{j=1}^{n} x_j w_j + b \quad\right)$

Different activation functions can be defined in artificical neural networks. some of them are:

Name	Plot	Equation	Derivative
Identity		$f(x) = x$	$f'(x) = 1$
Binary step		$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$	$f'(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ ? & \text{for } x = 0 \end{cases}$
Logistic (a.k.a Soft step)		$f(x) = \frac{1}{1+e^{-x}}$	$f'(x) = f(x)(1-f(x))$
TanH		$f(x) = \tanh(x) = \frac{2}{1+e^{-2x}} - 1$	$f'(x) = 1 - f(x)^2$
ArcTan		$f(x) = \tan^{-1}(x)$	$f'(x) = \frac{1}{x^2+1}$
Rectified Linear Unit (ReLU)		$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$	$f'(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$
Parameteric Rectified Linear Unit (PReLU) [2]		$f(x) = \begin{cases} \alpha x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$	$f'(x) = \begin{cases} \alpha & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$
Exponential Linear Unit (ELU) [3]		$f(x) = \begin{cases} \alpha(e^x - 1) & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$	$f'(x) = \begin{cases} f(x) + \alpha & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$
SoftPlus		$f(x) = \log_e(1+e^x)$	$f'(x) = \frac{1}{1+e^{-x}}$

A graphical reprsentation of a Multi Layer Perceptron (MLP), a neural network with at least one hidden layer, consisting of one input, one hidden and one output layers is provided in Figure 6.17.1.

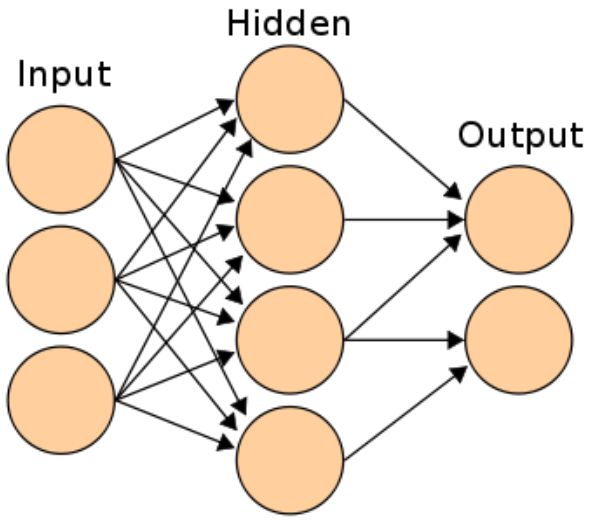


Figure A Multi Layer Perceptron [1]

To approximate a function, neural network modifies the biases, a real number without a variable parameter, of each and neuron weights of connections between each neuron in different layers. This

modifying process is called “training”. Various training procedures have been studied in the literature. One of the most common training procedure is called “backpropagation” [1].

Backpropagation means backpropagation of errors occuring in the output layer. Backpropagation is used with optimization methods such as gradient descent and the method employs the chain rule, which requires a non-linear differentiable function, which is called activation function, is assigned to the each neuron at least in the hidden layer. Hyperbolic tangent function and logistic sigmoid function are commonly used as activation functions in ANNs. Mathematical representations of logistic sigmoid, hyperbolic tangent functions and their derivatives, to be used later, are given below:

Logistic Sigmoid: $\varphi(x) = \frac{1}{1+e^{-x}}$

Derivative of logistic sigmoid: $\varphi(x)(1-\varphi(x))$

Hyperbolic Tangent: $\varphi(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

Derivative of hyperbolic tangent: $(1-\varphi(x))^2$

Backpropagation learning algorithm mainly consists of two phases: Propagation and weight update. In propagation phase, firstly, training input is forward propagated through the neurons in each layer to generate outputs. Mathematical representation of forward propagation is given below.

$net = x_1 w_1 + x_2 w_2$

y=f(net)

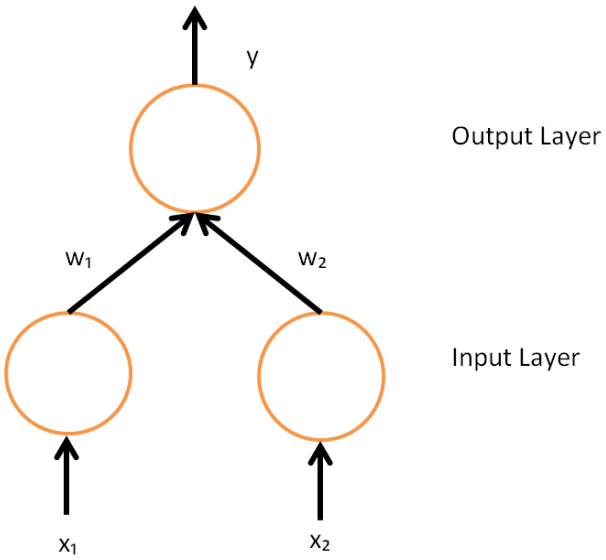


Figure 6.17.2. Forward propagation example [1]

where *x* represents in the inputs to the network, *y* shows the output, *w* represents the weights and *f()* represents the activation of the neuron. After the forward propagation finished and outputs of the network are generated, second step of the propagation phase begins, namely backpropagation. In the backpropagation step, difference between the desired output and calculated output, error, is minimized by a derivative-based optimization algorithm. One of the most commonly used derivative-based algorithm for **backpropagation** is gradient descent algorithm.

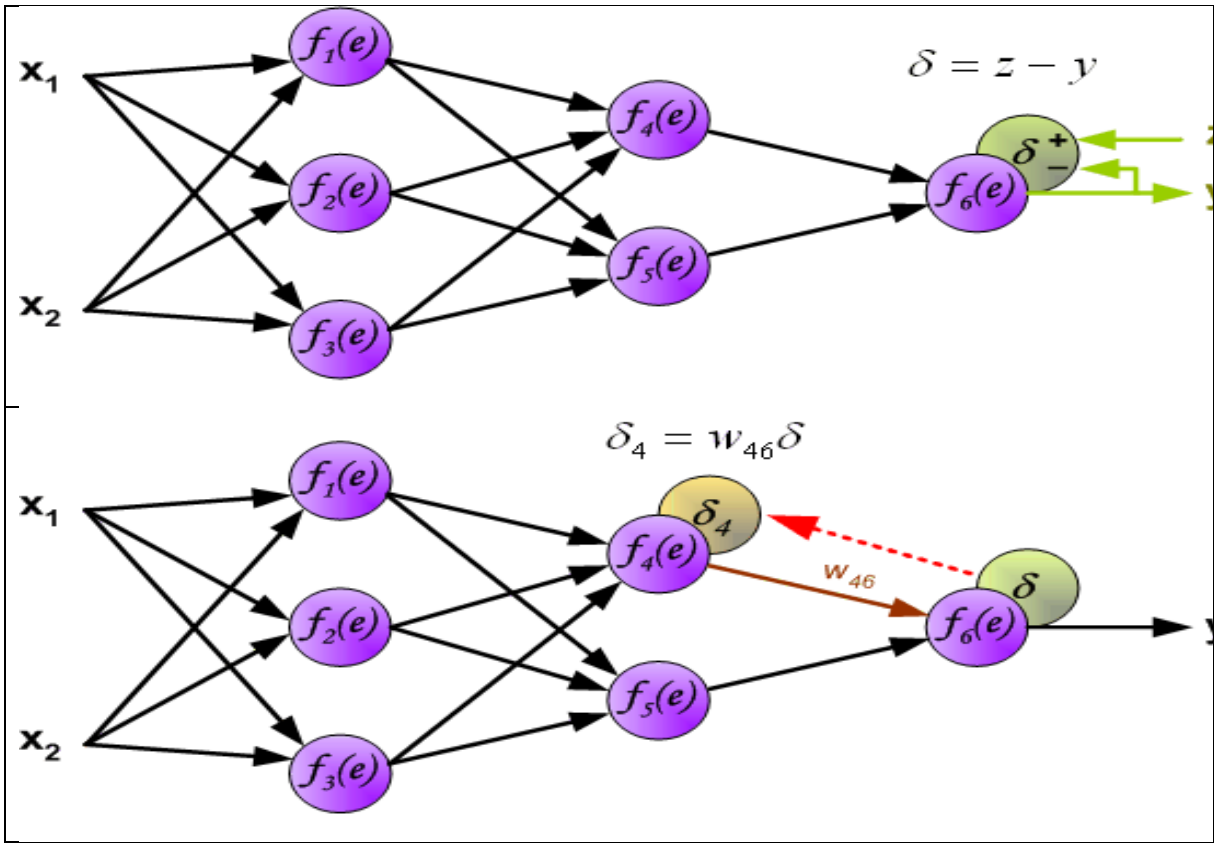


Figure 6.17.3 backpropogation

Generally, to measure the difference between the outputs, squared error function is used:

$$E = \frac{1}{2}(t - y)^2$$

where *t* is the target output, *y* is the actual output and *E* is the squared error. In backpropagation, partial derivative of the error with respect to a weight is calculated by using the chain rule twice.

$$\frac{\partial E}{\partial w_{ij}} = \frac{\partial E}{\partial o_j}\frac{\partial o_j}{\partial net_j}\frac{\partial net_j}{\partial w_{ij}}$$

$$\frac{\partial net_j}{\partial w_{ij}} = \frac{\partial}{\partial w_{ij}}\left(\sum_{k=1}^{n} w_{kj}\, o_k\right) = o_i$$

$$\frac{\partial o_j}{\partial net_j} = \frac{\partial}{\partial net_j}\varphi(net_j) = (1 - \varphi(net_j))^2 \quad \text{(assumed sigmoid function)}$$

$$\frac{\partial E}{\partial o_j} = \frac{\partial E}{\partial y} = \frac{\partial E}{\partial}\left[\frac{1}{2}(t - y)^2\right] = y$$

Since *E* is a function of all neurons:

$$\frac{\partial E}{\partial o_j} = \sum_{l \in L}\left(\frac{\partial E}{\partial net_l}\frac{\partial net_j}{\partial o_j}\right) = \sum_{l \in L}\left(\frac{\partial E}{\partial o_l}\frac{\partial o_l}{\partial net_j}w_{jl}\right)$$

After backpropagation of error finishes, the weight updating phase, the second phase, starts. For all iterations, each weight is updated by the following rule. α is the learning rate which determines the step size of each update.

$$\Delta w_{ij} = -\alpha \frac{\partial E}{\partial w_{ij}} = \begin{cases} -\alpha o_i (o_j - t_j) o_j (1 - o_j) \; if \; j \; is \; an \; output \; neuron \\ -\alpha o_i \left(\sum_{l \in L} (\delta_l w_{jl}) \right) o_j (1 - o_j) \; if \; j \; is \; an \; inner \; neuron \end{cases}$$

This procedure continues after a desired error value is reached or maximum number of iterations is accomplished. After the training finishes, test inputs can be forward propagated with the calculated weights and fitted, predicted, results can be found.

As a first programming example here an ADELINE network is given. Please note that result of one layer ADELINE is similar to linear interpolation process

Program ADELINE network of single layer

```
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;
class adalinenetwork{
int numout,sizeinput,numinput;
double error;
double weights[];
double bias[];
double x[][];
double trainedweights[];
double predicted;
double trainedbias[];
double d[];
double totalerror[];
double total[];
int imaxiteration;
double ilearningrate,output;

public adalinenetwork(double ix[][],double id[], int maxiteration, double learningrate)
{
x=ix;
d=id;
ilearningrate=learningrate;
imaxiteration=maxiteration;
numout=1;  //Number of output
sizeinput=x.length;
numinput=x[0].length;
weights=new double[numinput];
total=new double[numinput+1];
trainedweights=new double[numinput];
trainedbias=new double[numout];
bias=new double[numout];
totalerror=new double[imaxiteration];
initWeights();
}
public void initWeights()
{
for(int i=0;i<numinput;i++)
{
weights[i]=Math.random();
}
for(int i=0;i<numout;i++)
{
bias[i]=Math.random();
}
}
public double tanjanth(double x)
{
double p=(Math.exp(x)-Math.exp(-x))/(Math.exp(x)+Math.exp(-x));
return p;
}

public double sigmoid(double x)
{
double p=1.0/(1.0+Math.exp(-x));
return p;
```

```
}

public double[] train()
{
for(int i=0;i<imaxiteration;i++)
{
		error=0.0;
		for(int j=0;j<sizeinput;j++)
		{
			output=0.0;
			for(int k=0;k<numinput;k++)
			{
				output+=weights[k]*x[j][k];
			}
			output+=bias[0];
			error+=(d[j]-output);
		}
		totalerror[i]=error;
		for(int j=0;j<sizeinput;j++)
		{
			for(int k=0;k<numinput;k++)
			{
				weights[k]+=ilearningrate*error*x[j][k];
			}
		}
		bias[0]+=ilearningrate*error;
}
for(int k=0;k<numinput;k++)
{
		total[k]=weights[k];
}
total[numinput]=bias[0];
System.out.println("Final error is="+totalerror[imaxiteration-1]);
return total;
}


void writeMatrix(String filename) {
double[] total=train();
try {
BufferedWriter bw = new BufferedWriter(new FileWriter(filename));
for (int i = 0; i < total.length; i++) {
bw.write(total[i] + " ");
bw.newLine();
}
bw.flush();
} catch (IOException e) {}
}


public static double[][] readDouble(String filename)
{
//read data in matrix form
// data length is not limited
Vector<String> xi=new Vector<String>();
int nn;
int m=2;
int i=0;
BufferedReader fin;
try{
fin=new BufferedReader(new FileReader(filename));
try {
try{
try{
try{
String s1=Text.readStringLine(fin);
StringTokenizer token=new StringTokenizer(s1);
int n=token.countTokens()-1;
m=n+1;
while(token.hasMoreTokens())
{
String s2=token.nextToken();
xi.add(s2);
}
i=0;
while(fin != null)
```

```
{
i++;
String s11=Text.readStringLine(fin);
if(s11=="") break;
StringTokenizer token1=new StringTokenizer(s11);
while(token1.hasMoreTokens())
{
String s2=token1.nextToken();
xi.add(s2);
}
}
} catch(ArrayIndexOutOfBoundsException e_eof) { }
} catch(StringIndexOutOfBoundsException e_eof) { }
} catch(EOFException e_eof) {System.out.println("end of file"); }
} catch(NumberFormatException e_nfe) {System.out.println("non-number input"); }
} catch(IOException e_io) {System.out.println("file name is not existed"); }
nn=i;
double aa[][]=new double[nn][m];
Enumeration<String> nx=xi.elements();
for(i=0;i<nn;i++)
{
for(int j=0;j<m;j++)
{
String s1=nx.nextElement();
Double ax=new Double(s1);
aa[i][j]=ax.doubleValue();
}
}
return aa;
}


public double predict(double x[][])
{
double[][] totall=readDouble("weightsadaline.txt");
for(int k=0;k<numinput;k++)
{
        trainedweights[k]=totall[k][0];
}
trainedbias[0]=totall[numinput][0];
output=0.0;
for(int k=0;k<numinput;k++)
{
        output+=trainedweights[k]*x[0][k];
}
output+=trainedbias[0];
return output;
}
}
```

Program ADELINE network of single layer a test case

```
class adalinenetworktest
{
public static void main(String args[])
{
double[] d={2,5,7,8};
double[][] x={{1,1},{2,2},{3,3},{4,4}};
double[][] test={{2.5,2.5}};
int maxiteration=100;
double learningrate=0.001;
adalinenetwork nn=new adalinenetwork(x,d,maxiteration,learningrate);
nn.writeMatrix("weightsadaline.txt");
double predicts=nn.predict(test);
System.out.println("Predicted value is="+predicts);
}
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" adalinenetworktest
Final error is=2.2556898571224338E-9
Predicted value is=5.499999999551117

> Terminated with exit code 0.
```

In the second example we will have hidden layers in between given layers, therefore propagation and backpropogation will applied for all this layers.

Program neuralnetworkbackpropogation

```
import java.io.*;
import java.util.*;
class neuralnetworkbackprop{
int numout,sizeinput,numinput,numhid;
double error;
double weights[][];
double bias[];
double a[];
double totalerror[];
double fnet[];
double fnet1[];
double H1[];
double delta[];
double x[][];
double trainedweights[][];
double predicted[];
double trainedbias[];
double d[];
double H[];
double total[][];
int u,r,imaxiteration,inumhid,y,s;
double ilearningrate;
public neuralnetworkbackprop(double ix[][],double id[], int maxiteration, double learningrate, int numhid)
{
x=ix;
d=id;
ilearningrate=learningrate;
inumhid=numhid;
imaxiteration=maxiteration;
numout=1;
sizeinput=x.length;
numinput=x[0].length;
predicted=new double[numout];
fnet=new double[inumhid+numout];
fnet1=new double[inumhid+numout];
H1=new double[inumhid+numout];
delta=new double[inumhid+numout];
H=new double[inumhid+numout];
weights=new double[numout+numinput][inumhid];
trainedweights=new double[numout+numinput][inumhid];
trainedbias=new double[inumhid+numout];
a=new double[inumhid+numout];
bias=new double[inumhid+numout];
total=new double[inumhid+numout+numout+numinput][inumhid];
totalerror=new double[imaxiteration];
initWeights();
}
public void initWeights()
{
for(int i=0;i<numout+numinput;i++)
{
for(int j=0;j<inumhid;j++)
{
weights[i][j]=Math.random()-0.5;
}
}
for(int i=0;i<numout+inumhid;i++)
{
bias[i]=Math.random()-0.5;
}
}
public double tanjanth(double x)
{
double p=(Math.exp(x)-Math.exp(-x))/(Math.exp(x)+Math.exp(-x));
return p;
}
```

```
public double sigmoid(double x)
{
double p=1.0/(1.0+Math.exp(-x));
return p;
}
public double[][] train()
{
for(int i=0;i<imaxiteration;i++)
{
error=0.0;
for(int j=0;j<sizeinput;j++)
{
for(int k=0;k<inumhid+numout;k++)
{
delta[k]=0.0;
H[k]=0.0;
}
//hidden layer forward propagation
for(int k=0;k<inumhid;k++)
{
for(int l=0;l<numinput;l++)
{
H[k]+=x[j][l]*weights[l][k];
}
H[k]+=bias[k];
fnet[k]=tanjanth(H[k]);
}
//output layer forward propagation
u=0;
for(int k=inumhid;k<inumhid+numout;k++)
{
for(int l=0;l<inumhid;l++)
{
H[k]+=fnet[l]*weights[numinput+u][l];
}
H[k]+=bias[k];
fnet[k]=H[k];
u++;
}
//backpropagation starts here
//backpropagation in output layer
for(int k=inumhid;k<inumhid+numout;k++)
{
delta[k]=(d[j]-fnet[k]);
}
//backpropagation in hidden layer
for(int k=0;k<inumhid;k++)
{
for(int l=0;l<numout;l++)
{
delta[k]+=weights[numinput+l][k]*delta[inumhid+l];
}
delta[k]=(1.0-(fnet[k]*fnet[k]))*delta[k];
}
//bias update
for(int k=0;k<inumhid+numout;k++)
{
bias[k]+=ilearningrate*delta[k];
}
//hidden layer update
for(int k=0;k<numinput;k++)
{
for(int l=0;l<inumhid;l++)
{
weights[k][l]+=ilearningrate*x[j][k]*delta[l];
}
}
//output layer update
for(int k=0;k<numout;k++)
{
```

```
for(int l=0;l<inumhid;l++)
{
weights[numinput+k][l]+=ilearningrate*fnet[l]*delta[inumhid+k];
}
}
for(int k=0;k<numout;k++)
{
error+=(0.5*(d[j]-fnet[inumhid+k])*(d[j]-fnet[inumhid+k]));
}
}
totalerror[i]=error;
}
for(int i=0;i<numout+numinput;i++)
{
for(int j=0;j<inumhid;j++)
{
total[i][j]=weights[i][j];
}
}
y=0;
for(int i=numout+numinput;i<numout+numout+numinput+inumhid;i++)
{
total[i][0]=bias[y];
y++;
}
System.out.println("Final MSE error is="+totalerror[imaxiteration-1]);
return total;
}
void writeMatrix(String filename) {
double[][] total=train();
try {
BufferedWriter bw = new BufferedWriter(new FileWriter(filename));
for (int i = 0; i < total.length; i++) {
for (int j = 0; j < total[i].length; j++) {
bw.write(total[i][j] + " ");
}
bw.newLine();
}
bw.flush();
} catch (IOException e) {}
}
public static double[][] readDouble(String filename)
{
//read data in matrix form
// data length is not limited
Vector<String> xi=new Vector<String>();
int nn;
int m=2;
int i=0;
BufferedReader fin;
try{
fin=new BufferedReader(new FileReader(filename));
try {
try{
try{
try{
String s1=Text.readStringLine(fin);
StringTokenizer token=new StringTokenizer(s1);
int n=token.countTokens()-1;
m=n+1;
while(token.hasMoreTokens())
{
String s2=token.nextToken();
xi.add(s2);
}
i=0;
while(fin != null)
{
i++;
String s11=Text.readStringLine(fin);
```

```
if(s11=="") break;
StringTokenizer token1=new StringTokenizer(s11);
while(token1.hasMoreTokens())
{
String s2=token1.nextToken();
xi.add(s2);
}
}
} catch(ArrayIndexOutOfBoundsException e_eof) { }
} catch(StringIndexOutOfBoundsException e_eof) { }
} catch(EOFException e_eof) {System.out.println("end of file"); }
} catch(NumberFormatException e_nfe) {System.out.println("non-number input"); }
} catch(IOException e_io) {System.out.println("file name is not existed"); }
nn=i;
double aa[][]=new double[nn][m];
Enumeration<String> nx=xi.elements();
for(i=0;i<nn;i++)
{
for(int j=0;j<m;j++)
{
String s1=nx.nextElement();
Double ax=new Double(s1);
aa[i][j]=ax.doubleValue();
}
}
return aa;
}
public double[] predict(double x[][])
{
double[][] totall=readDouble("weights.txt");
for(int i=0;i<numout+numinput;i++)
{
for(int j=0;j<inumhid;j++)
{
trainedweights[i][j]=totall[i][j];
}
}
s=0;
for(int i=numout+numinput;i<numout+numout+numinput+inumhid;i++)
{
trainedbias[s]=totall[i][0];
s++;
}
for(int k=0;k<inumhid;k++)
{
for(int l=0;l<numinput;l++)
{
H1[k]+=x[0][l]*trainedweights[l][k];
}
H1[k]+=trainedbias[k];
fnet1[k]=tanjanth(H1[k]);
}
//output layer forward propagation
u=0;
for(int k=inumhid;k<inumhid+numout;k++)
{
for(int l=0;l<inumhid;l++)
{
H1[k]+=fnet1[l]*trainedweights[numinput+u][l];
}
H1[k]+=trainedbias[k];
fnet1[k]=H1[k];
u++;
}
u=0;
for(int k=inumhid;k<inumhid+numout;k++)
{
predicted[u]=fnet1[k];
u++;
}
```

```
return predicted;
}
}
```

Program neuralnetworkbackpropogationtest

```
class neuralnetworkbackproptest
{
public static void main(String args[])
{
double[] d={2,5,7,8};
double[][] x={{1,1},{2,2},{3,3},{4,4}};
double[][] test={{2,2}};
int maxiteration=10000;
int numhid=4;
double learningrate=0.1;
neuralnetworkbackprop nn=new neuralnetworkbackprop(x,d,maxiteration,learningrate,numhid);
nn.writeMatrix("weights.txt");
double[] predicts=nn.predict(test);
System.out.println("Predicted value is="+predicts[0]);
}
}
```

```
Result:

Final MSE error is=0.0
Predicted value is=5.0
```

Another way to find the desired weights and biases can be directly optimizing the error function with a metaheuristic algorithm after the forward propagation. Crow Search (CS) is a powerful metaheuristic algorithm good at global search, inspired by food searching mechanism of crow swarms [2]. In the following example crow search is used to train a neural network.

Program crow_search_algorithm

```
import java.io.*;
import java.util.*;
abstract class f_xj
{abstract double func(double x[]);}
public class crow_search_algorithm
{
double[] Lower;
double[] Upper;
double[] BEST;
double[][] X;
double[][] MEM;
double[][] XNEW;
double[] fitness;
double[] fitmem;
double[] VALS;
double AP;
double fl;
int D;
int N;
int Maxiter;
int[] num;
f_xj ff;
public crow_search_algorithm(f_xj iff,double[] iLower,double[] iUpper, int iMaxiter,int iN)
{
ff=iff;
Lower=iLower;
Upper=iUpper;
D=Lower.length;
Maxiter=iMaxiter;
N=iN;
BEST=new double[Maxiter];
```

```
X=new double[N][D];
MEM=new double[N][D];
XNEW=new double[N][D];
fitness=new double[N];
fitmem=new double[N];
num=new int[N];
VALS=new double[D];
AP=0.1;
fl=2.0;
}
void init()
{
for(int i=0;i<N;i++)
{for(int j=0;j<D;j++)
{X[i][j]=Lower[j]+((Upper[j]-Lower[j])*Math.random());}}
for(int i=0;i<N;i++)
{fitness[i]=ff.func(X[i]);}
}
double[] getminval_index(double[] a)
{
double m=0.0;
double b[]=new double[a.length];
for(int i=0;i<a.length;i++)
{b[i]=a[i];}
double minval=a[0];
for(int i=0;i<a.length;i++)
{if(a[i]<minval){minval=a[i];}}
for(int i=0;i<a.length;i++)
{if(b[i]==minval){m=i;break;}};
double[] dep=new double[2];
dep[0]=minval;
dep[1]=m;
return dep;
}
double[][] solution()
{
init();
for(int i=0;i<N;i++)
{for(int j=0;j<D;j++)
{MEM[i][j]=X[i][j];}
fitmem[i]=fitness[i];}
int iter=1;
while(iter<Maxiter)
{
for(int i=0;i<N;i++)
{num[i]=(int)Math.floor(Math.random()*(double)N);}
for(int i=0;i<N;i++)
{
if(Math.random()>AP)
{for(int j=0;j<D;j++)
{XNEW[i][j]=X[i][j]+fl*Math.random()*(MEM[num[i]][j] - X[i][j]);}
}
else
{
for(int j=0;j<D;j++)
{XNEW[i][j]=Lower[j]-((-Upper[j]+Lower[j])*Math.random());}
}
}
for(int i=0;i<N;i++)
{for(int j=0;j<D;j++)
{if((XNEW[i][j]>Upper[j])||(XNEW[i][j]<Lower[j]))
{XNEW[i][j]=Lower[j]+((Upper[j]-Lower[j])*Math.random());}}}
for(int i=0;i<N;i++)
{
fitness[i]=ff.func(XNEW[i]);
if( fitness[i]<fitmem[i] )
{
fitmem[i]=fitness[i];
for(int j=0;j<D;j++)
{MEM[i][j]=XNEW[i][j];X[i][j]=XNEW[i][j];}
```

```
}
}
double[] AA2=getminval_index(fitmem);
double fitval2=AA2[0];
BEST[iter]=fitval2;
iter++;
}
double[] plott=new double[Maxiter];
for(int i=0;i<Maxiter;i++)
{plott[i]=i;}
Plot pp=new Plot(plott,BEST);
// pp.plot();
double[] AA=getminval_index(fitmem);
double fitval=AA[0];
int mm=(int)AA[1];
for(int i=0;i<D;i++)
{VALS[i]=MEM[mm][i];}
double out[][]=new double[2][D];
out[0][0]=fitval;
for(int i=0;i<D;i++)
{out[1][i]=VALS[i];}
return out;
}
void trainandwrite()
{
double[][] out=solution();
double[] result=new double[D];
System.out.println("Optimized value = "+out[0][0]);
for(int i=0;i<D;i++)
{result[i]=out[1][i];
System.out.println("x["+i+"] = "+out[1][i]);}
writeMatrix("weightscrow.txt",result);
}
void writeMatrix(String filename,double[] total) {
try {
BufferedWriter bw = new BufferedWriter(new FileWriter(filename));
for (int i = 0; i < total.length; i++) {
bw.write(total[i] + " ");
}
bw.flush();
} catch (IOException e) {}
}
void toTxtfile()
{   String named="crow_f27_sche.txt";
String line="";
int lineNo;
try
{
FileWriter fw=new FileWriter(named,true);
BufferedWriter bw=new BufferedWriter(fw);
LineNumberReader lnr=new LineNumberReader(new FileReader(named));
lnr.setLineNumber(1);
for(int i=1;i<=lnr.getLineNumber();i++)
{bw.newLine();}
double[][] dd=solution();
Double d1d=new Double(dd[0][0]);
String s1=d1d.toString();
int nn=dd[1].length;
Double[] val=new Double[nn];
String[] str=new String[nn];
for(int i=0;i<nn;i++)
{val[i]=new Double(dd[1][i]);}
for(int i=0;i<nn;i++)
{str[i]=val[i].toString();}
bw.write(s1+" ");
for(int i=0;i<nn;i++)
{bw.write(str[i]+" ");}
bw.close();
lnr.close();
}catch(IOException e)
```

```
{e.printStackTrace();}
}
}
```

Program neuralnetwork

```
import java.io.*;
import java.util.*;
class neuralnetwork extends f_xj
{
int numout,sizeinput,numinput;
double error;
double weights[][];
double fnet[];
double s[][];
double d[];
double H[];
double predicted[];
double fnet1[];
double H1[];
int u,r,o,inumhid;
public neuralnetwork(double is[][],double id[],int numhid)
{
s=is;
d=id;
inumhid=numhid;
numout=1;
sizeinput=s.length;
numinput=s[0].length;
predicted=new double[numout];
fnet=new double[inumhid+numout];
fnet1=new double[inumhid+numout];
H1=new double[inumhid+numout];
H=new double[inumhid+numout];
}
public double tanjanth(double x)
{
double p=(Math.exp(x)-Math.exp(-x))/(Math.exp(x)+Math.exp(-x));
return p;
}
public double func(double x[])
{
error=0.0;
for(int j=0;j<sizeinput;j++)
{
for(int k=0;k<inumhid+numout;k++)
{
H[k]=0.0;
}
r=0;
//hidden layer forward propagation
for(int k=0;k<inumhid;k++)
{
for(int l=0;l<numinput;l++)
{
H[k]+=s[j][l]*x[r];
r++;
}
H[k]+=x[k+(numinput*inumhid)+(inumhid*numout)];
fnet[k]=tanjanth(H[k]);
}
//output layer forward propagation
u=0;
for(int k=inumhid;k<inumhid+numout;k++)
{
for(int l=0;l<inumhid;l++)
{
H[k]+=fnet[l]*x[l+(inumhid*numinput)];
}
```

```
H[k]+=x[(numinput*inumhid)+(inumhid*numout)+inumhid+u];
fnet[k]=H[k];
u++;
}
for(int k=0;k<numout;k++)
{
error+=(0.5*(d[j]-fnet[inumhid+k])*(d[j]-fnet[inumhid+k]));
}
}
return error;
}
public static double[][] readMatrix(String filename)
{
//read data in matrix form
// data length is not limited
Vector<String> xi=new Vector<String>();
int nn;
int m=2;
int i=0;
BufferedReader fin;
try{
fin=new BufferedReader(new FileReader(filename));
try {
try{
try{
try{
String s1=Text.readStringLine(fin);
StringTokenizer token=new StringTokenizer(s1);
int n=token.countTokens()-1;
m=n+1;
while(token.hasMoreTokens())
{
String s2=token.nextToken();
xi.add(s2);
}
i=0;
while(fin != null)
{
i++;
String s11=Text.readStringLine(fin);
if(s11=="") break;
StringTokenizer token1=new StringTokenizer(s11);
while(token1.hasMoreTokens())
{
String s2=token1.nextToken();
xi.add(s2);
}
}
} catch(ArrayIndexOutOfBoundsException e_eof) { }
} catch(StringIndexOutOfBoundsException e_eof) { }
} catch(EOFException e_eof) {System.out.println("end of file"); }
} catch(NumberFormatException e_nfe) {System.out.println("non-number input"); }
} catch(IOException e_io) {System.out.println("file name is not existed"); }

nn=i;
double aa[][]=new double[nn][m];
Enumeration<String> nx=xi.elements();
for(i=0;i<nn;i++)
{
for(int j=0;j<m;j++)
{
String s1=nx.nextElement();
Double ax=new Double(s1);
aa[i][j]=ax.doubleValue();
}
}
return aa;
}
public double[] predict(double s[][],double x[][])
{
```

```
r=0;
for(int k=0;k<inumhid;k++)
{
for(int l=0;l<numinput;l++)
{
H1[k]+=s[0][l]*x[0][r];
r++;
}
H1[k]+=x[0][k+(numinput*inumhid)+(inumhid*numout)];
fnet1[k]=tanjanth(H1[k]);
}
//output layer forward propagation
o=0;
for(int k=inumhid;k<inumhid+numout;k++)
{
for(int l=0;l<inumhid;l++)
{
H1[k]+=fnet1[l]*x[0][l+(inumhid*numinput)];
}
H1[k]+=x[0][(numinput*inumhid)+(inumhid*numout)+inumhid+o];
fnet1[k]=H1[k];
o++;
}
u=0;
for(int k=inumhid;k<inumhid+numout;k++)
{
predicted[u]=fnet1[k];
u++;
}
return predicted;
}
}
public class crow_search_algorithm_test
{
public static void main(String args[])
{
double[] d={2,5,7,8};
double[][] x={{1,1},{2,2},{3,3},{4,4}};
int numout=1;
int numinput=x[0].length;
double[][] test={{4,4}};
int numhid=4;
neuralnetwork nn=new neuralnetwork(x,d,numhid);
int num=((numout+numinput)*numhid)+(numhid+numout);
double[] Upper=new double[num];
double[] Lower=new double[num];
int Maxiter=100000; // Maximum iteration number
int N=50; // Number of individuals in the swarm
for(int i=0;i<num;i++)
{
Lower[i]=-20;
Upper[i]=20;
}
crow_search_algorithm mo=new crow_search_algorithm(nn,Lower, Upper,Maxiter,N);
mo.trainandwrite();
double[][] weights=nn.readMatrix("weightscrow.txt");
double predictedval[]=nn.predict(test,weights);
System.out.println("Predicted value is="+predictedval[0]);
}
}
```

```
Optimized value = 0.0
x[0] = -0.5364913717025673
x[1] = -5.599577914584505
x[2] = 0.2746411516819427
x[3] = -8.500643573331972
x[4] = 0.9461278732704546
x[5] = 5.2128555702178305
x[6] = 0.08398781880606487
```

```
x[7] = 2.008159771957661
x[8] = -9.347666061384377
x[9] = 2.1777320165608804
x[10] = 3.575094965521974
x[11] = 1.557195226098894
x[12] = -15.622863320944523
x[13] = 9.745906789502728
x[14] = -8.378299501090064
x[15] = -5.918923693975516
x[16] = -4.2791888524335775
Predicted value is=8.0
```

PROBLEMS

PROBLEM 1

Fit a linear line by using polynomial least square curve fitting

X	2	3	4	7	8	9	5	5
Y	9	6	5	10	9	11	2	3

PROBLEM 2

For R22 refrigerant liquid viscosity values are required. The values are defined as the table below. Fit a curve to input viscosity into computer as a function. Calculate the amount of error.

-101.1111111	8.6519780084E-04	-28.88888889	3.0093836551E-04	32.22222222	1.5253606713E-04
-95.55555556	7.7466826506E-04	-26.11111111	2.9143069737E-04	35	1.4798892150E-04
-90	6.9943367368E-04	-23.33333333	2.8233640610E-04	37.77777778	1.4302839899E-04
-84.44444444	6.3536025795E-04	-20.55555556	2.7365549171E-04	40.55555556	1.3848125336E-04
-78.88888889	5.8079451036E-04	-17.77777778	2.6538795420E-04	43.33333333	1.3393410773E-04
-73.33333333	5.3325616965E-04	-15	2.5712041668E-04	46.11111111	1.2938696209E-04
-70.55555556	5.1176057211E-04	-12.22222222	2.4926625605E-04	48.88888889	1.2483981646E-04
-67.77777778	4.9150510520E-04	-9.444444444	2.4182547228E-04	51.66666667	1.2070604770E-04
-65	4.7290314580E-04	-6.666666667	2.3479806540E-04	54.44444444	1.1615890207E-04
-62.22222222	4.5512794014E-04	-3.888888889	2.2777065851E-04	57.22222222	1.1202513331E-04
-59.44444444	4.3817948824E-04	-1.111111111	2.2074325162E-04	60	1.0747798768E-04
-56.66666667	4.2205779009E-04	1.666666667	2.1412922161E-04	62.77777778	1.0334421892E-04
-53.88888889	4.0717622256E-04	4.444444444	2.0792856848E-04	65.55555556	9.9210450167E-05
-51.11111111	3.9312140879E-04	7.222222222	2.0172791534E-04	68.33333333	9.5076681410E-05
-48.33333333	3.7947997189E-04	10	1.9552726220E-04	71.11111111	9.0529535778E-05
-45.55555556	3.6666528874E-04	12.77777778	1.8973998595E-04	73.88888889	8.6395767021E-05
-42.77777778	3.5426398247E-04	15.55555556	1.8395270969E-04	76.66666667	8.1848621388E-05
-40.81111111	3.4599644496E-04	18.33333333	1.7857881030E-04	79.44444444	7.7714852631E-05
-40	3.4268942995E-04	21.11111111	1.7320491092E-04	82.22222222	7.2754330123E-05
-37.22222222	3.3152825431E-04	23.88888889	1.6783101153E-04	85	6.8207184490E-05
-34.44444444	3.2078045554E-04	26.66666667	1.6287048902E-04	87.77777778	6.2833285106E-05
-31.66666667	3.1044603365E-04	29.44444444	1.5749658964E-04	90.55555556	5.7046008846E-05
				93.33333333	5.0018601959E-05

PROBLEM 3

Saturated vapor pressure for refrigerant R134a is required. These values are given below as table format. Fit a curve to this values that will minimize error.

-99	0.01	-49	2.06	0	32.73	50	212.61	100	786.93	150	2100.93
-98	0.01	-48	2.21	1	34.24	51	219.24	101	804.64	151	2137.75
-97	0.01	-47	2.37	2	35.81	52	226.02	102	822.65	152	2175.06
-96	0.01	-46	2.54	3	37.43	53	232.96	103	840.96	153	2212.87
-95	0.01	-45	2.72	4	39.12	54	240.07	104	859.56	154	2251.19
-94	0.02	-44	2.91	5	40.86	55	247.33	105	878.47	155	2290.01
-93	0.02	-43	3.11	6	42.67	56	254.77	106	897.69	156	2329.35
-92	0.02	-42	3.32	7	44.54	57	262.38	107	917.22	157	2369.21
-91	0.03	-41	3.54	8	46.48	58	270.15	108	937.07	158	2409.61
-90	0.03	-40	3.78	9	48.49	59	278.1	109	957.23	159	2450.54
-89	0.04	-39	4.03	10	50.57	60	286.23	110	977.71	160	2492.01
-88	0.04	-38	4.29	11	52.72	61	294.54	111	998.52	161	2534.04
-87	0.05	-37	4.57	12	54.94	62	303.03	112	1019.66	162	2576.63
-86	0.06	-36	4.86	13	57.23	63	311.7	113	1041.12	163	2619.8

-85	0.07	-35	5.17	14	59.61	64	320.57	114	1062.92	164	2663.54
-84	0.08	-34	5.5	15	62.06	65	329.62	115	1085.05	165	2707.88
-83	0.09	-33	5.84	16	64.59	66	338.86	116	1107.52	166	2752.81
-82	0.1	-32	6.2	17	67.2	67	348.3	117	1130.35	167	2798.36
-81	0.11	-31	6.58	18	69.9	68	357.93	118	1153.52	168	2844.54
-80	0.13	-30	6.98	19	72.68	69	367.77	119	1177.04	169	2891.36
-79	0.14	-29	7.4	20	75.55	70	377.81	120	1200.91	170	2938.83
-78	0.16	-28	7.84	21	78.52	71	388.05	121	1225.15	171	2986.97
-77	0.18	-27	8.3	22	81.57	72	398.5	122	1249.75	172	3035.81
-76	0.2	-26	8.78	23	84.71	73	409.17	123	1274.71	173	3085.29
-75	0.23	-25	9.29	24	87.95	74	420.04	124	1300.04	174	3135.59
-74	0.25	-24	9.82	25	91.29	75	431.13	125	1325.74	175	3186.64
-73	0.28	-23	10.38	26	94.73	76	442.45	126	1351.82	176	3238.47
-72	0.31	-22	10.96	27	98.27	77	453.98	127	1378.28	177	3291.12
-71	0.34	-21	11.58	28	101.92	78	465.74	128	1405.13	178	3344.62
-70	0.37	-20	12.22	29	105.66	79	477.72	129	1432.36	179	3399.02
-69	0.41	-19	12.88	30	109.52	80	489.93	130	1459.98	180	3454.38
-68	0.45	-18	13.58	31	113.49	81	502.38	131	1488	181	3510.78
-67	0.49	-17	14.32	32	117.57	82	515.06	132	1516.41	182	3568.29
-66	0.54	-16	15.08	33	121.76	83	527.98	133	1545.23	183	3627.05
-65	0.59	-15	15.88	34	126.07	84	541.15	134	1574.46		
-64	0.64	-14	16.71	35	130.49	85	554.55	135	1604.1		
-63	0.7	-13	17.58	36	135.04	86	568.21	136	1634.15		
-62	0.76	-12	18.48	37	139.71	87	582.11	137	1664.62		
-61	0.83	-11	19.43	38	144.51	88	596.27	138	1695.52		
-60	0.9	-10	20.41	39	149.43	89	610.68	139	1726.84		
-59	0.97	-9	21.44	40	154.48	90	625.35	140	1758.6		
-58	1.05	-8	22.5	41	159.66	91	640.28	141	1790.79		
-57	1.14	-7	23.61	42	164.98	92	655.47	142	1823.42		
-56	1.23	-6	24.77	43	170.43	93	670.94	143	1856.5		
-55	1.33	-5	25.97	44	176.03	94	686.67	144	1890.03		
-54	1.43	-4	27.22	45	181.76	95	702.68	145	1924.01		
-53	1.54	-3	28.52	46	187.63	96	718.96	146	1958.46		
-52	1.66	-2	29.87	47	193.65	97	735.53	147	1993.37		
-51	1.79	-1	31.27	48	199.82	98	752.38	148	2028.74		
-50	1.92			49	206.14	99	769.51	149	2064.6		

PROBLEM 4

Fit a parametric curve to the values given in the table P4

Table P.4 Girdi verisi

İ	0	1	2	3	4
T	0	1	2	3	4
X	-1	0	1	0	1
Y	0	1	0.5	0	-1

PROBLEM 5

X	y
0	-2.20
0.1	-1.90
0.2	0
0.3	-0.1
0.4	-0.01
0.5	-0.3
0.6	-0.2
0.7	-0.09
0.8	0.4

0.9	-0.1
1.0	-1.1
1.1	-1.3
1.2	-2
1.3	-1.8
1.4	-0.02
1.5	1

Data in the above table is given

a) Fit a polynomial least square curve
b) Use interpolation polynomials to fit the data

PROBLEM 6

X	y
0	0
1	1
2	4
3	8

Fit cubic spline to the data.

PROBLEM 7

X	y
0	0
1	1
2	0
1	-1

Fit cubic spline to the data.

PROBLEM 8

Ethylene gylicol-water mixture density values are given in the table below for different mixture percentages. By using formula

$\rho(x,t)= a_0+a_1*x+a_2*t+a_3*x^2+a_4*t^2+a_5*xt+a_6*x^3+a_7*t^3+a_8*x^2t$
$+a_9*xt^2+a_{10}*x^4+a_{11}*t^4+a_{12}*x^2t^2+a_{13}x^3t+a_{14}*xt^3$

	x %volume ratio ethylene gylicol concentration									
t	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
-35							1090	1105	1119	1132
-30							1089	1104	1117	1131
-25						1088	1102	1116	1129	1142
-20					1072	1087	1101	1115	1128	1140
-15					1071	1086	1100	1113	1126	1138
-10				1054	1070	1084	1098	1111	1124	1136
-5			1037	1053	1068	1083	1096	1109	1122	1134
0	999.8	1019	1036	1052	1067	1081	1095	1108	1120	1132
5	999.9	1018	1034	1050	1065	1079	1093	1105	1118	1129
10	999.7	1016	1033	1049	1063	1077	1091	1103	1115	1127
15	999.1	1015	1031	1047	1062	1075	1089	1101	1113	1124
20	998.2	1013	1030	1045	1060	1073	1086	1098	1110	1121
25	997	1012	1028	1043	1058	1071	1084	1096	1107	1118
30	995.6	1010	1026	1041	1055	1069	1081	1093	1105	1115
35	994	1008	1024	1039	1053	1066	1079	1090	1102	1112
40	992.2	1006	1022	1037	1051	1064	1076	1088	1098	1109
45	990.2	1004	1020	1034	1048	1061	1073	1084	1095	1106

50	988	1002	1017	1032	1045	1058	1070	1081	1092	1102
55	985.7	999.2	1015	1029	1043	1055	1067	1078	1088	1098
60	983.2	996.7	1012	1026	1040	1052	1064	1075	1085	1095
65	980.5	994.1	1009	1023	1037	1049	1060	1071	1081	1091
70	977.7	991.4	1006	1020	1033	1045	1057	1067	1077	1087
75	974.8	988.5	1003	1017	1030	1042	1053	1064	1073	1083
80	971.8	985.5	1000	1014	1027	1038	1049	1060	1069	1078
85	968.6	982.4	997	1011	1023	1035	1046	1056	1065	1074
90	965.3	979.2	993.6	1007	1019	1031	1042	1052	1061	1070
95	961.9	975.8	990.1	1003	1016	1027	1038	1047	1056	1065
100	958.3	972.3	986.5	999.7	1012	1023	1033	1043	1052	1060
105	954.7	968.7	982.8	995.8	1008	1019	1029	1038	1047	1055
110	950.9	965	978.9	991.8	1004	1014	1025	1034	1042	1050
115	947.1	961.2	974.9	987.7	999.3	1010	1020	1029	1038	1045
120	943.1	957.2	970.8	983.4	994.9	1005	1015	1024	1032	1040
125	939	953.1	966.6	979.1	990.4	1001	1010	1019	1027	1035

PROBLEM 9

Liquid flows through a sphere. Friction drag coefficient C_D is given as a function of Reynolds number. Fit a cubic spline interpolation polynomial to the data. Find C_D values for *Re* = 5, 50, 500 and 5000 . Note: Logarithmic scale can be used.

Re	0.2	2	20	200	2000	20 000
C_D	103	13.9	2.72	0.800	0.401	0.433

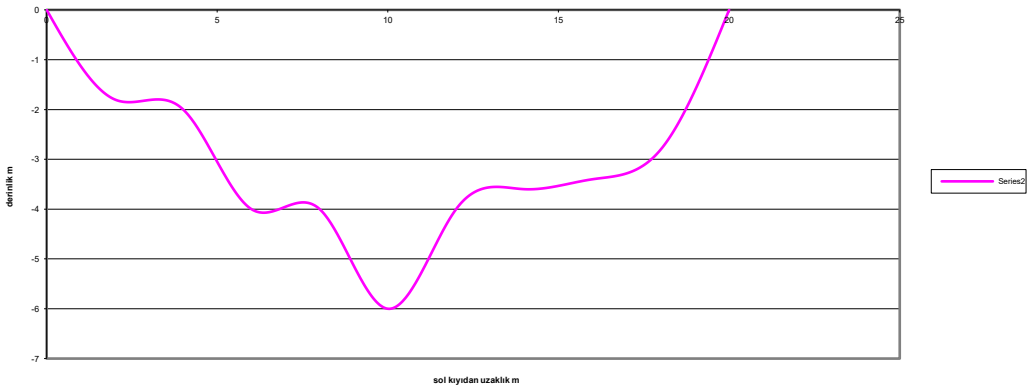
PROBLEM 10

Solve problem 9 by using 4 point Lagrange and Newton interpolation formula

PROBLEM 11

The depth and profile of a channel is given below

x, m	0	2	4	6	8	10	12	14	16	18	20	
H, m	0	1.8	2	4	4	6	4	3.6	3.4	2.8	0	
U m/s	0	0.03	0.045	0.055	0.065	0.12	0.07	0.06	0.05	0.04	0	



Find H(x) and U(x) curve fitting functions by using

a) Cubic spline interpolation formula
b) Polynomial least square
c) Lagrange interpolation formula

PROBLEM 12

Temperature distribution of a solid substance is given as

Time , t, s	**0**	**5**	**10**	**15**	**20**	**25**	**30**	**35**

Temperature T C	**90**	**55**	**40**	**28**	**24**	**22**	**21.5**	**20.6**

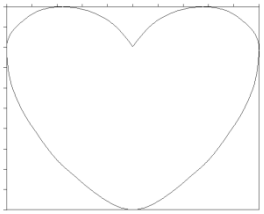
Find the temperature distribution at t=13 sn and t=23 sn t=33 sn, t=43sn ve t=53 sn deki sicaklik dağilimini bulunuz.

PROBLEM 13

A hard shape coordinates are given below, curvefit data to these coordinates

Xi	Yi	Xi	yi	Xi	yi	xi	Yi
241	127	331	157	215	246	149	120
258	107	323	178	199	232	160	106
283	96	312	197	184	217	175	97
307	97	298	217	170	197	199	96
322	106	283	232	159	178	224	107
333	120	267	246	151	157	241	127
334	138	241	258	148	138		

Note: the same values are repeated.



PROBLEM 14

Distance required to stop a car is function of velocity. This relation is given in the table below

Velocity (km/h)	**24**	**32**	**40**	**48**
Stopping distance (m)	**4.8**	**6.0**	**10**	**12**

If the vehicle travels with a velocity of 38 km/h, calculate stopping distance by using Newton interpolation formula.

PROBLEM 15

Thermal conductivity coefficient as a function of temperature is given in the table below.

Sicaklik ***T*** . [K]	275	290	305	320	340	360
Thermal conductivity coefficient ***k***. [W/(m·K)]	0.574	0.598	0.620	0.640	0.660	0.674

a-) Curve fit a quadratic polynomial to data

$$k = A \cdot T^2 + BT + C$$

b-) Estimate the thermal conductivity coefficient at 300 K

PROBLEM 16

For Etylenegylicol Prandtl (Pr) is given as a function of temperature in the below table. Derive a polynomial least square curve fitting and calculate Pr number for T= 42 °C

Temperature (K)	280	300	320	340
Pr number	400	151	73.5	42.8

PROBLEM 17

Saturated water temperature of water as a function of temperature t ^{o}C will be curvefit to the following equation

$$h_f = A + Bt + Ct^2 + Dt^3$$

Following table is taken from the thermodynamic property table

t, ^oC	10	30	50	70	90
h_f , kJ/kg	41.99	125.66	209.26	292.97	376.94

Calculate the coefficients by using the least square curve fitting

PROBLEM 18

According to Hooke's law Force applied to a linear spring is F=kx. In this equation k is spring constant, x is the displacement in the spring. Displacement versus force in a real spring is given by the below table find the k spring constant

X, m	0.02	0.04	0.06	0.08	0.1
F, kN	3.1	6.1	9.2	12	15.1

PROBLEM 19

Fit below values into function $f(x) = (Ax+B)^{-2}$

X	-1	0	1	2
Y	13.46	3.02	0.68	0.14

PROBLEM 20

2.7×10^7 W/m^3 heat is generated in 5 mm thick plate by a heat souce. Thermal conductivity of the plate is k=25 W/m K. Surfce temperatures of the plate (at x=0 mm) 161.5 °C, and (x=5 mm) 166.9 °C. Temperature distribution insde the plate is given in the table below. Give o polinomial function for the temperature distribution inside the plate. Use Newton interpolation Formula. Calculate temperature at x=4 mm.

x (mm)	0	0.5	2.5	4.5	5.0
T (°C)	161.5	166.9	177.7	171.22	166.9

PROBLEM 21

For very low pressure values (P<10 kPa) establish a Formula for the enthalpy for dry steam as h=A+BT

where A and B constants , h ,(kJ/kg) enthalpy and T , (^oC) is temperature.

h (kJ/kg)	2510.6	2519.8	2528.9	2538.1	2547.2	2556.3	2565.3
T (^oC)	5	10	15	20	25	30	35

PROBLEM 22

In a direct current electrical motor, Moment(F) and electrical current(I) is related with the relation

$I=A\ (F^B)$

Experimental measurements of F and I is given in the table below. Calculate A and B coefficients.

F (Nm)	100	200	300	400
I (Amper)	8	11.31	13.85	16

Bu ilişkiyi en küçük kareler yöntemi ile belirleyiniz.

PROBLEM 23

Derive and equation given the density of dry air ρ (kg/ m^3) at one athmosphere pressure by using Lagrange interpolation formula and then calculate the density at 22.5 °C.

t (°C)	0	20	40	60
ρ (kg/ m^3)	1,2930	1,2045	1,1267	1,0595

PROBLEM 24

For variable z the required function is given as

z=ax+by+cxy
By using least square method find a,b,c values and the function.

z	x	y
0.1	1	1
-0.9	1	2
2.0	2	2
-1.8	3	1

PROBLEM 25

Saturation pressure of water vapor(steam) changes as a function of temperature. One of the relation used for this rlation is known as "*Dupperet Formula"* is

$$P_{saturation} = A\left(\frac{t_{saturation}}{100}\right)^4$$

In this equation $t_{saturation}$:^oC is saturation temperature, $P_{saturation}$, kPa is saturation pressure. Find coefficient A.

t, ^oC	10	30	60	90	110	140	180	200	250	300	340	370
P kPa	1.2276	4.246	19.940	70.14	143.27	361.3	1002.1	1553.8	3973	8581	14586	21030

PROBLEM 26

The vibration(x) of a machine as a function of time(t) is measured and results are given as the table below.

i	**1**	**2**	**3**	**4**
t_i	0	2	4	6
x_i	5,0	3,7	2,7	2,0

For the vibration of this machine find an equation is in the form of

$$x(t) = ae^{-bt}$$

Where a and b are constants.

PROBLEM 27

In order to develop a new machine, shear stress (τ) is required to be measured. In the experiment the plate above is moved with a velocity of 2 m/s while the plate below remained constant. The distance between the plates are (h) 0.025 m . The shear stres is related to viscosity as follows:

$$\tau = \mu\frac{V}{h}$$

Viskosity (μ) N-s/m^2, is function of temperature and viscosity temperature relation is obtained in the laboratory as :

T (°C)	5	20	30	50	55
μ (N-s/m^2)	0,08	0,015	0,009	0,006	0,005

a) By using Lagrange interpolation find the shear stres at 38 °

b) By using a polynomial least square, obtain an eqaution for sheat stress

PROBLEM 28

A pump manufacturer is measured for the new pump the following data between volumetric flow rate(Q) and pump pressure (P)

Q (lt/s)	0,5	1,0	1,5	2,0	2,5	3,0	3,5

P (kPa)	275	290	280	255	229	205	178

d) Fit a second degree least square polynomial [$P=f(Q)=aQ^2+bQ+c$] b-) Pompa 2,2 lt/s debide su basarken oluşacak basinci tahmin ediniz.

b) Find the flow rate fort he maximum pressure?

PROBLEM 29

Yanma gazlari yoğunluğunun (ρ: kg/m^3), sicaklikla (T:°C) değişimi aşağida verilmiştir.

ρ: kg/m^3	0,950	0,748	0,617	0,525
T:°C	100	200	300	400

Newton enterpolasyon yöntemini kullanarak 1'den 4'ü dereceye kadar polinom Denklemlerini bulunuz. 285 °C'da yanma gazlarinin yoğunluğunu hesaplayiniz.

PROBLEM 30

An orifice is used to measure volumetric flow rate. Such a device is shown in the figure. Following data is used to design the orifice-meter.

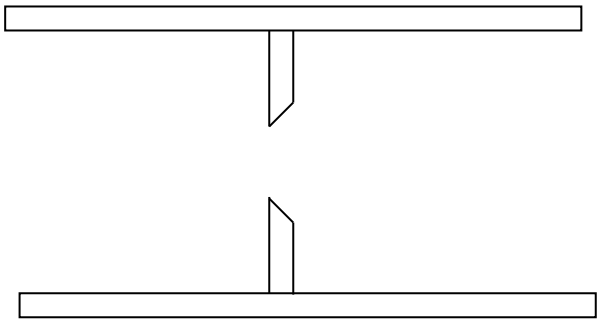
A : Orifice cross sectional area.

A_1 : pipe cross sectional area<.

A_2 : C_cA (shrinked area behind the orifice).

C_c is shrinkage coefficient is a function of (A/A_1) ratio. Experimental results fort his coefficient is given below: Find a least sqaure cirve fiitng equation for [$C_c=f(A/A_1)$]

A/A_1	**C_c**
0.1	0.62
0.2	0.63
0.3	0.64
0.4	0.66
0.5	0.68
0.6	0.71
0.7	0.76
0.8	0.81
0.9	0.89
1.00	1.00



PROBLEM 31

The life of the cutting tool for a lathe is T(minutes) and cutting speed V(m/minutes). The life test is carried out and the following data is obtained.

Cutting speed, V (m/minutes)	Life of the cutting tool,T (minutes)
140	80
160	70
180	55
200	50
220	35

If the general relation between cutting speed and life of the cutting tool is:

$$\mathbf{V=(b/T)^{(1/a)}}$$

Find the coefficients by using least square curve fitting

PROBLEM 32

Fit the data below the following equation by using least square curve fitting

$y=ax^b$

by using least square curve fitting

X	1	2	3	4	5
Y	1	1	2	4	4

PROBLEM 33

By using data in the following table answer the quaestions below.

X	1	2	3	5	6
f(X)	4,75	4	5,25	19,75	36

a) Obtain fourth order polynomial equation by using Newton interpolation formula
b) Obtain cubic spline for the second subregion
c) PROBLEM 34

Fit a line to following data by using least square method.

x	5	10	15	20	25	30	35	40	45	50
y	16	25	32	33	38	36	3	40	42	42

PROBLEM 35

Kinematic viscosities of oil SAE30 is given in the table below.

Sicaklik T(C)	5	10	15	20	25	30	35	40	45	50
Vizkozite μ (m^2/s)	16	25	32	33	38	36	3	40	42	42

a) Fit the data into $\mu=ae^{(b/T)}$ equation

b) Fit the data into $\mu = \frac{a\sqrt{T}}{1+\frac{b}{T}}$ equation

PROBLEM 36

For $f(x) = \frac{20}{1+2x^2}$ function in the range of -3<=x<=3 by using points x_i=-3,-2,-1,0,1,2,3

a) Find cubic spline interpolation formulas
e) Find quadratic B-spline formulas

PROBLEM 37

In a smooth sphere, the following relation is existed between Reynolds number(Re) and friction coefficient (C)

Re	0.1	1	10	100	1000	10000
C	210	30	4	1	0.5	0.4

Find an equation to represent the data properly

PROBLEM 38

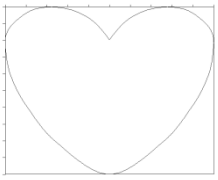
By using bezier curve construct the letter α.

PROBLEM 39

By using bezier curve construct the letter γ .

PROBLEM 40

By using two symmetric bezier curve construct the hard shape.



PROBLEM 41

The tuition for 12 units at *St. Anford* University has been increasing from 1998 to 2008 as shown in the table below:

Year Tuition per year

1998 $21,300
1999 $23,057
2000 $24,441
2001 $25,917
2002 $27,204
2003 $28,564
2004 $29,847
2005 $31,200
2006 $32,994
2007 $34,800
2008 $36,030

(a) Plot the given data points and intuitively interpolate (draw) a smooth curve through them.

(b) Interpolate the data with the Lagrange polynomial. Plot the polynomial and the data points. Use the polynomial to predict the tuition in 2010. This is an extrapolation problem; discuss the utility of Lagrange polynomials for extrapolation.

(c) Repeat with a cubic spline interpolation and compare your results.

PROBLEM 42

The concentration of a certain toxin in a system of lakes downwind of an industrial area has been monitored very accurately at intervals from 1993 to 2007 as shown in the table below. It is believed that the concentration has varied smoothly between these data points.

Year Toxin Concentration

1993 12.0
1995 12.7
1997 13.0
1999 15.2
2001 18.2
2003 19.8
2005 24.1
2007 28.1
2009 ???

(a) Interpolate the data with the Lagrange polynomial. Plot the polynomial and the data points. Use the polynomial to predict the condition of the lakes in 2009. Discuss this prediction.

(b) Interpolation may also be used to fill "holes" in the data. Say the data from 1997 and 1999 disappeared. Predict these values using the Lagrange polynomial fitted through the other known data points.

(c) Repeat (b) with a cubic spline interpolation. Compare and discuss your results.

PROBLEM 43

Fit Pade rational approximation to the function sin(x) between [-π,π] with polynomial

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{q_0 + q_1 x + q_2 x^2}$$

$p(x)$ is 3th degree polynomial and q(x) is 2nd degree polynomial.

PROBLEM 44

Fit Pade rational approximation to the function |x| between [-1,1] with polynomial

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{q_0 + q_1 x + q_2 x^2}$$

p(x) is 3th degree polynomial and q(x) is 3rd degree polynomial.

PROBLEM 45

Fit Chebyschev rational approximation to the function sin(x) between [-π,π] with polynomial

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 T_0(x) + p_1 T_1(x) + p_2 T_2(x) + p_3 T_3(x)}{q_0 T_0(x) + q_1 T_1(x) + q_2 T_2(x)}$$

p(x) is 3th degree polynomial and q(x) is 2nd degree polynomial.

PROBLEM 46

Fit Chebyschev rational approximation to the function sin(x) between [-1,1] with polynomial

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 T_0(x) + p_1 T_1(x) + p_2 T_2(x) + p_3 T_3(x)}{q_0 T_0(x) + q_1 T_1(x) + q_2 T_2(x) + q_3 T_3(x)}$$

p(x) is 3th degree polynomial and q(x) is 3th degree polynomial.

PROBLEM 47

Fit Trigonometric(Fourier) approximation to the function $f(x) = x^2 - 2x + 5$between [-π,π] with degree 5 function $S_5(x)$

PROBLEM 48

Fit Trigonometric(Fourier) approximation to the function $f(x) = |x|$ between [-1,1] with degree 10 function $S_{10}(x)$

PROBLEM 49

Average temperature in a city is measured as follows

Months x_i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Average Temperature T_i	**-6**	**-3**	**2**	**9**	**15**	**21**	**23**	**22**	**18**	**12**	**5**	**-2**

Fit a discrete Trigonometric (Fourier) least square curve to this data.

PROBLEM 50

For CO2 Specific heat , Cp (KJ/kmolK) data is given as a function of temperature (degree K) as

T (K)	**Cp(KJ/kmolK)**
298.15	37.132
300	37.217
400	41.326
500	44.625
600	47.323
700	49.563
800	51.434
900	52.999
1000	54.308
1100	55.412
1200	56.342
1300	57.13
1400	57.803
1500	58.381
1600	58.883
1700	59.321
1800	59.705
1900	60.046
2000	60.349
2100	60.62
2200	60.864
2300	61.086
2400	61.288

2500	61.473
2600	61.644
2700	61.804
2800	61.953
2900	62.094
3000	62.228

Find an equation for Cp. Investigate different kind of curve fitting equations for a best result.

PROBLEM 51

Saturation vapor pressure equation for HCFC-123 Refrigerant(2,2 dichloro-1,1,1-trifloroethane) is given as: $\log_{10} P_{sat} = A + B/T + C\log_{10} T + DT + E((F-T)/T)\log_{10}(F-T)$

Curve fit values are given in the book as:
A=1.656333e03 B=-2.480583e06,C=1.792522e1,D=-8.86838e-2,E=4.61781e2,F=1.666667e3
The data in the tables are given as T (degree K) and P_{sat}(kPa):

-80	0.13	18	69.95	50	212.46	84	540.19
-70	0.34	20	75.61	52	225.84	86	567.2
-60	0.81	22	81.63	54	239.85	88	595.2
-50	1.77	24	88.02	56	254.51	90	624.23
-40	3.58	26	94.8	58	269.85	92	654.3
-30	6.75	27.82	101.33	60	285.89	94	685.44
-20	12	28	101.98	62	302.64	96	717.68
-10	20.25	30	109.58	64	320.13	98	751.03
0	32.65	32	117.62	66	338.38	100	785.53
2	35.74	34	126.11	68	357.4	110	976.03
4	39.07	36	135.07	70	377.22	120	1199
6	42.64	38	144.52	72	397.87	130	1457.8
8	46.47	40	154.47	74	419.36	140	1756.3
10	50.57	42	164.95	76	441.71	150	2098.7
12	54.95	44	175.97	78	464.94	160	2490.1
14	59.63	46	187.55	80	489.09	170	2937.2
16	64.63	48	199.71	82	514.16	180	3450.6
						183.68	3661.8

Make a non-linear curve fit for a given data and check if the given coefficients are correct.

8 Ordinary differential equations & solution methods

Basic diffential equations are differential forms of function f(x). If only first derivative of the function existed, it is called first order, if a second derivative is existed it si called second order etc. There are several analytical solution methods for such equations let us start to look at some of these analytical solutions first.

8.1 First order homogeneous differential equations

If differential equation can be seperated into two parts with one variable in each part, then it can be solved by integration methods.

An example case for such an equation:

$$x\frac{dy}{dx} = \frac{y^2}{x} + y$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$$

If we apply variable change as

$$u = \frac{y}{x}$$

$$y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = u^2 + u$$

$$x\frac{du}{dx} = u^2$$

$$\frac{du}{u^2} = \frac{dx}{x}$$

$$\int \frac{du}{u^2} = \int \frac{dx}{x}$$

$$-\frac{1}{u} = ln\lfloor x \rfloor + C$$

$$y = -\frac{x}{ln\lfloor x \rfloor + C} =$$

Consider the first order dşfferential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

We can always write this equation in the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

Then in terms of differential, it can be written that

$M(x,y)dx + N(x,y)dy = 0$ Let us assign this differential equation eqaula to some dF

$$dF(x,y) = M(x,y)dx + N(x,y)dy = 0$$

So in solution of this F(x,y)=C

As an example case, consider:

$$(2xy^3 + 2)dx + (3x^2y^2 + e^y)dy = 0$$

$$\frac{dF(x,y)}{dx} = M(x,y) = (2xy^3 + 2)$$

$$\frac{dF(x,y)}{dy} = N(x,y) = (3x^2y^2 + e^y)$$

$$F(x,y) = \int (2xy^3 + 2)dx = x^2y^3 + 2x + k(y)$$

$$\frac{dF(x,y)}{dy} = 3x^2y^2 + k'(y) = (3x^2y^2 + e^y)$$

So

$$k'(y) = e^y$$

$$k(y) = e^y$$

$$F(x,y) = x^2y^3 + 2x + e^y$$

This type of equation will be given solution only if

$\frac{dM(x,y)}{dy} = \frac{dN(x,y)}{dy}$. This condition is called exactness condition.

If this exactness condition is not satisfied, one possibility is to find a multiplication factor $v(x,y)$ for the equation.

$$v(x,y)\, M(x,y)dx + v(x,y)\, N(x,y)dy = 0$$

General differential equation is still in the form of

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

Example case:

$$(y^2 - 6xy)dx + (3xy - 6x^2)dy = 0$$

$$M(x,y) = (y^2 - 6xy)$$

$$N(x,y) = (3xy - 6x^2)$$

$$\frac{dM(x,y)}{dy} = 2y - 6x$$

$$\frac{dN(x,y)}{dx} = 3y - 12x$$

So equation is not exact. Multiplying equation with $v(x,y) = y$

$$v(x,y)\, M(x,y) = (y^3 - 6xy^2)$$

$$v(x,y)\, N(x,y) = (3xy^2 - 6x^2y)$$

$$\frac{d[v(x,y)\, M(x,y)]}{dy} = 3y^2 - 12xy$$

$$\frac{d[v(x,y)N(x,y)]}{dx} = 3y^2 - 12xy$$

Assuming one existed, **Integrating factor** can be found from:

$$\frac{d[v(x,y)\, M(x,y)]}{dy} = \frac{d[v(x,y)N(x,y)]}{dx}$$

$$M(x,y)\frac{d[v(x,y)]}{dy} + v(x,y)\frac{d[\, M(x,y)]}{dy} = N(x,y)\frac{d[v(x,y)]}{dx} + v(x,y)\frac{d[N(x,y)]}{dx}$$

This is a partial differential equation for v(x,y). Not that solution of this equation could be more difficult than the original equation or sometimes no solution is existed.

The Bernoulli differential equation:

$$P(x)\frac{dy}{dx} + Q(x)y = R(x)y^{\alpha}$$

Is called **Bernaulli equation**. In this equation α is constant. In differential form eqaution becomes



Daniel Bernoulli, Swiss mathematician

$$[Q(x)y - R(x)y^{\alpha}]dx + P(x)dy = 0$$

If an integrating factor v(x,y) is found such that

$$\frac{d[v(x,y)\,(Q(x)y - R(x)y^{\alpha})]}{dy} = \frac{d[v(x,y)P(x)]}{dx}$$

Equation becomes

$$v(x,y)\frac{\partial}{\partial y}[Q(x)y - R(x)y^{\alpha}] + [Q(x)y - R(x)y^{\alpha}]\frac{\partial v(x,y)}{\partial y} = v(x,y)\frac{\partial [P(x)]}{\partial x} + P(x)\frac{\partial [v(x,y)]}{\partial x}$$

Assume that $v(x,y) = f(x)y^{b}$ In this case eqaution becomes

$$f(x)y^{b}\frac{\partial}{\partial y}[Q(x)y - R(x)y^{\alpha}] + [Q(x)y - R(x)y^{\alpha}]f(x)by^{b-1} = f(x)y^{b}\frac{dP(x)}{dx} + P(x)\frac{df(x)}{dx}y^{b}$$

Then

$$\frac{df(x)}{dx}y^{b}P(x) + f(x)\frac{dP(x)}{dx}y^{b} = f(x)Q(x)(b+1)y^{b} - R(x)f(x)(b+\alpha)y^{b+\alpha-1}$$

By choosing $b = -\alpha$ equation becomes:

$$\frac{df(x)}{dx}y^{-\alpha}\,P(x) + f(x)\frac{dP(x)}{dx}y^{-\alpha} = f(x)Q(x)(1-\alpha)y^{-\alpha}$$

Dividing by common factor $f(x)P(x)y^{-\alpha}$

$$\frac{1}{f(x)}\frac{df(x)}{dx} + \frac{1}{P(x)}\frac{\partial P(x)}{\partial x} = (1-\alpha)\frac{Q(x)}{P(x)}$$

$$\frac{f'(x)}{f(x)} = -\frac{P'(x)}{P(x)} + (1-\alpha)\frac{Q(x)}{P(x)}$$

Integrating of this equation yields:

$$\ln(|f(x)|) = -\ln(|P(x)|) + (1-\alpha)\int\frac{Q(x)}{P(x)}dx$$

Or

$$|f(x)| = \frac{1}{|P(x)|}\exp\left((1-\alpha)\int\frac{Q(x)}{P(x)}dx\right)$$

Let us try this in an actual differential equation

$$x^{2}\frac{dy}{dx} + xy = -y^{3/2}$$

Or

$$\left(xy + y^{\frac{3}{2}}\right)dx + x^{2}dy$$

$\alpha = -3/2 \quad P(x) = x^{2} \quad Q(x) = x \;\; R(x) = -1 \;\; b = -\alpha = 3/2$

$$|f(x)| = \frac{1}{|P(x)|}\exp\left((1-\alpha)\int\frac{Q(x)}{P(x)}dx\right)$$

$$f(x) = \frac{1}{x^{2}}\exp\left[\left(1+\frac{3}{2}\right)\int\frac{x}{x^{2}}dx\right] = \frac{1}{x^{2}}\exp\left[\frac{5}{2}\int\frac{dx}{x}\right] = \frac{1}{x^{2}}\exp\left[\frac{5}{2}lnx\right] = \frac{1}{x^{2}}\exp[lnx^{5/2}] = \frac{x^{5/2}}{x^{2}} = x^{1/2}$$

$$v(x,y) = f(x)y^b = x^{1/2}y^{3/2}$$

Multiplying equation with $v(x,y) = f(x)y^b = x^{1/2}y^{3/2}$

$$x^{1/2}y^{3/2}\left(xy + y^{\frac{3}{2}}\right)dx + x^{1/2}y^{3/2}x^2dy = 0$$

$$\left(x^{3/2}y^{5/2} + x^{1/2}\right)dx + x^{5/2}y^{3/2}dy = 0$$

This exact equation has potential function F(x,y)

$$\frac{\partial F(x,y)}{\partial x} = \left(x^{3/2}y^{5/2} + x^{1/2}\right)$$

$$F(x,y) = \int \left(x^{3/2}y^{5/2} + x^{1/2}\right)dx$$

$$F(x,y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2}$$

So the general solution

$$\frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2} = C$$

A first order differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called **linear differential equation**. Lineer differential equation is a Bernoulli equation with $\alpha = 0$ and P(x)=1. In order to sove equation multiplication factor used is:

$$v(x) = \exp\left(\int P(x)dx\right)$$

We obtain

$$\frac{dy}{dx}\exp\left(\int P(x)dx\right) + P(x)y\exp\left(\int P(x)dx\right) = Q(x)\exp\left(\int P(x)dx\right)$$

$$\frac{d[y\exp\left(\int P(x)dx\right)]}{dx} = Q(x)\exp\left(\int P(x)dx\right)$$

İntegrate we get

$$y\exp\left(\int P(x)dx\right) = \int Q(x)\exp\left(\int P(x)dx\right)dx + C$$

Solving equation for y:

$$y = \exp\left(-\int P(x)dx\right)\int Q(x)\exp\left(\int P(x)dx\right)dx + C\exp\left(-\int P(x)dx\right)$$

Numerical example:

$$y' + y = \sin(x)$$

P(x)=1 Q(x)=sin(x) $\int P(x)dx = \int dx = x$

$$\frac{dy}{dx}\exp(x) + y\exp(x) = \sin(x)\exp(x)$$

$$\frac{d[y\exp(x)]}{dx} = \sin(x)\exp(x)$$

To integrate:

$$y\exp(x) = \int \sin(x)\exp(x)dx + C = \frac{1}{2}[\sin(x) - \cos(x)]\exp(x) + C$$

$$y(x) = \frac{1}{2}[\sin(x) - \cos(x)] + C\exp(-x)$$

Numerical example:

$$\frac{dy}{dx} + \frac{y}{x} = 3\exp(-x^2)$$

$P(x) = \frac{1}{x}$ $Q(x) = 3\exp(-x^2)$ $\int P(x)dx = \int \frac{dx}{x} = \ln(x)$ $exp[\int P(x)dx] = \exp(\ln(x)) = x$

$$\frac{dy}{dx}x + \frac{xy}{x} = 3x\exp(-x^2)$$

$$\frac{d(xy)}{dx} = -\frac{d}{dx}\exp(-x^3) + C$$
$$xy = -\exp(-x^3) + C$$
$$y = -\frac{1}{x}\exp(-x^3) + \frac{C}{x}$$

The general solution of

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

This is a homogeneous second order differential equation.

If A and B is any real numbers, equation always has an exponential solution. Assume

$$y = \exp(rx)$$

$$\frac{dy}{dx} = r\exp(rx)$$

$$\frac{d^2y}{dx^2} = r^2\exp(rx)$$

Substituting into differential equation gives

$$r^2\exp(rx) + Ar\exp(rx) + B\exp(rx) = 0$$

$$[r^2 + Ar + B]\exp(rx) = 0$$

So

$$r = \frac{-A \mp \sqrt{A^2 - 4AB}}{2}$$

Note that roots of polynomial coul be real or complex, acording to these, solution can be written as:

$$y = C_1\exp(r_1x) + C_2\exp(r_2x)$$

Or

$$y = C_3\sin(r_1x) + C_2\cos(r_2x)$$

or

$$y = C_3\sinh(r_1x) + C_2\cosh(r_2x)$$

Now cnsider the general solution of

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = f(x)$$

This is a non-homogeneous second order differential equation. In order to look a solution for such an equation, we will consider superposition principles of solutions. Assume that this set of eqaution has two sun set one for the equation

$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$ which we will consider homegeneous part and has a soluton of y_h and due to non homegeneous part, we have also a solution y_p and total solution will be

$$y = y_h + y_p$$

Function f(x) will give us some hints what will be the partial solution of the eqaution. Let us look at this with some examples

$$\frac{d^2y}{dx^2} - 4y = 8x^2 - 2x$$

When we looked at the equation right hande side of the equation yilds to a polynomial. In the left hand side , a second derivative and y itself is existed, therefore, a second degree polynomial will give us a partial solution to the diferential equation as

$$y_p(x) = ax^2 + bx + c$$

Derivatives of the equation will give

$$\frac{dy_p}{dx} = 2ax + b$$

$$\frac{d^2y_p}{dx^2} = 2a$$

$$2a - 4(ax^2 + bx + c) = 8x^2 - 2x$$

$$2a - 4(ax^2 + bx + c) = 8x^2 - 2x$$

$$-4ax^2 = 8x^2$$

$$-8b = -2x$$

$$2a - 4c = 0$$

a=-2 b=1/2 c=-1

$$y_p(x) = -2x^2 + 0.5x - 1$$

$$y_h(x) = C_1 \exp(2x) + C_2 \exp(-2x)$$

Another example

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 8\exp(x)$$

$$y_p(x) = kxexp(x)$$

$$\frac{dy_p}{dx} = kexp(x) + kxexp(x)$$

$$\frac{d^2y_p}{dx^2} = 2kexp(x) + kxexp(x)$$

$$kexp(x)[2 + x + 2 + 2x - 3x] = 8exp(x)$$

$$4k = 8$$

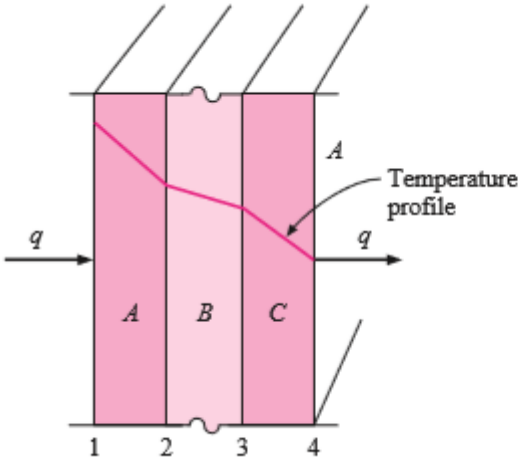
$$k = 2$$

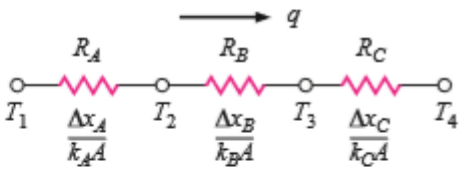
$$y_p(x) = 2xexp(x)$$

Some applied analytical solutions of one variable diferential equations from heat transfer area:

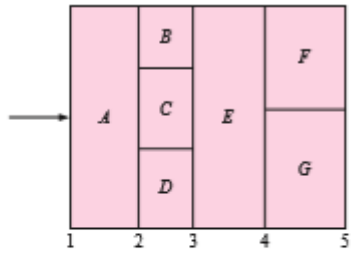
2.1 THE WALL

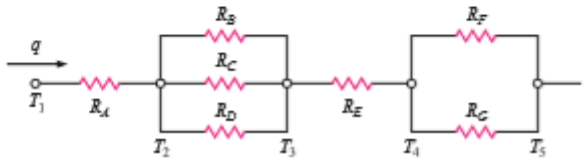
Let us consider a one dimensional steady state problem, general cartesian equation
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{\rho(T)C_p(T)}{k(T)}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ will reduce to
$\frac{\partial^2 T}{\partial x^2} = 0$ we will consider two boundary conditions. As the first condition let us assume that
x=0 T=T_1
$x = \Delta x_A$ T=T_2
Solving of the differential equation yields
$\frac{\partial T}{\partial x} = C_1$ (This equation indicates that heat transfer is constant $q_x = \frac{Q_x}{A} = -k_A\frac{dT}{dx}$)
$T = C_1x + C_2$ (This equation indicates that temperature profile is constant)
Applying boundary conditions:
$T_1 = C_2$
$T_2 = C_1\Delta x_A + T_1$
$\frac{T_2 - T_1}{\Delta x_A} = C_1$ Also note that $q_x = \frac{Q_x}{A} = -k_A\frac{dT}{dx} = -k_AC_1 = -k_A\frac{T_2 - T_1}{\Delta x_A} = \frac{T_1 - T_2}{\left(\frac{\Delta x_A}{k_A}\right)} = \frac{T_1 - T_2}{R_A} = constant$
Where R is called thermal resistance





If we have more than one layer of wall like 3 layers A,B and C shown in the figure and since $q_x = constant$
$q_x = \frac{Q_x}{A} = \frac{T_1 - T_2}{\left(\frac{\Delta x_A}{k_A}\right)} = \frac{T_2 - T_3}{\left(\frac{\Delta x_B}{k_B}\right)} = \frac{T_3 - T_4}{\left(\frac{\Delta x_C}{k_C}\right)}$
$q_x = \frac{Q_x}{A} = \frac{T_1 - T_2}{R_A} = \frac{T_2 - T_3}{R_B} = \frac{T_3 - T_4}{R_C} = \frac{T_1 - T_4}{R_A + R_B + R_C} = \frac{T_1 - T_4}{R}$
Thermal resistance concept works similar to electrical resistances, so when you have a wall in the figure below





R=$R_A + \frac{1}{\frac{1}{R_B}+\frac{1}{R_C}+\frac{1}{R_D}} + R_E + \frac{1}{\frac{1}{R_F}+\frac{1}{R_G}}$

$q_x = \frac{Q_x}{A} = \frac{T_1 - T_5}{R}$

Now consider convective heat transfer outside of the first layer.

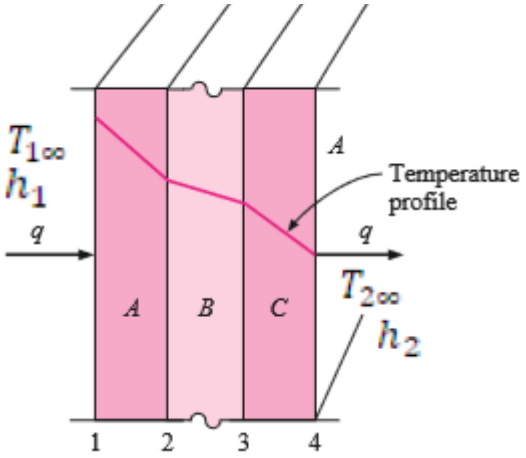
$\frac{\partial^2 T}{\partial x^2} = 0 \quad \frac{\partial T}{\partial x} = C_1 \quad T = C_1 x + C_2$

The boundary condition

x=0 $q_x = \frac{Q_x}{A} = -k_A \frac{dT}{dx} = h_1(T_1 - T_{1\infty}) = -k_A C_1 = \frac{(T_{1\infty} - T_1)}{\frac{1}{h_1}} = \frac{(T_{1\infty} - T_1)}{\frac{1}{h_1}} = \frac{(T_{1\infty} - T_1)}{R_1}$

$x = \Delta x_A$ T=T_2

Therefore for a wall with convective heat transfer in both sides



$q_x = \frac{Q_x}{A} = \frac{(T_{1\infty} - T_1)}{\left(\frac{1}{h_1}\right)} = \frac{T_1 - T_2}{\left(\frac{\Delta x_A}{k_A}\right)} = \frac{T_2 - T_3}{\left(\frac{\Delta x_B}{k_B}\right)} = \frac{T_3 - T_4}{\left(\frac{\Delta x_C}{k_C}\right)} = \frac{(T_4 - T_{2\infty})}{\left(\frac{1}{h_2}\right)}$

$q_x = \frac{Q_x}{A} = \frac{(T_{1\infty} - T_1)}{R_1} = \frac{T_1 - T_2}{R_A} = \frac{T_2 - T_3}{R_B} = \frac{T_3 - T_4}{R_C} = \frac{(T_4 - T_{2\infty})}{R_2} = \frac{T_{1\infty} - T_{2\infty}}{R_1 + R_A + R_B + R_C + R_2} = \frac{T_{1\infty} - T_{2\infty}}{R}$

or

$Q_x = \frac{T_{1\infty} - T_{2\infty}}{\left(\frac{1}{h_1 A}\right)+\left(\frac{\Delta x_A}{k_A A}\right)+\left(\frac{\Delta x_B}{k_B A}\right)+\left(\frac{\Delta x_C}{k_C A}\right)+\left(\frac{1}{h_2 A}\right)} = UA(T_{1\infty} - T_{2\infty})$

Where U is called overall heat transfer coefficient

Another case for one dimensional wall: Assume thermal conductivity is changing as a function of temperature

$k(T) = k_0(1 + \beta T)$

$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T(x)}{\partial x}\right) = 0$ B.C: x=0 T=T_1 $x = \Delta x$ T=T_2

$k(T)\frac{\partial T(x)}{\partial x} = C_1 \quad (1 + \beta T)\frac{\partial T(x)}{\partial x} = C_1 \quad \int_{T=T_1}^{T_2}(1 + \beta T)dT = C_1 \Delta x \quad (T_2 - T_1) + \frac{\beta}{2}(T_2^2 - T_1^2) = C_1 \Delta x$

$q_x = \frac{Q_x}{A} = -\frac{k_0}{2\Delta x}\left[(T_2 - T_1) + \frac{\beta}{2}(T_2^2 - T_1^2)\right]$

Another case for one dimensional wall: Assume a constant heat generation g_0 in the wall

$\frac{\partial^2 T}{\partial x^2} + \frac{g_0}{k} = 0$

$\frac{\partial T}{\partial x} = -\frac{g_0}{k}x + C_1$

$T(x) = -\frac{g_0}{k}\frac{x^2}{2} + C_1 x + C_2$

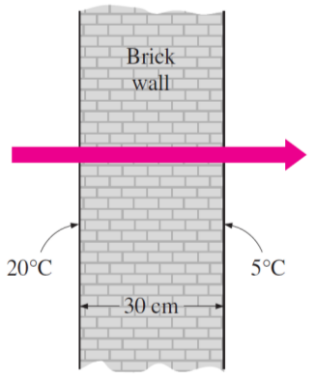
B.C. x=0 T=T_1

$x = \Delta x$ T=T_2

$T_1 = C_2 \quad T_2 = -\frac{g_0}{k}\frac{\Delta x^2}{2} + C_1\Delta x + T_1 \quad \frac{T_2 - T_1}{\Delta x} + \frac{g_0}{k}\frac{\Delta x}{2} = C_1 \quad R = \sum R_i$

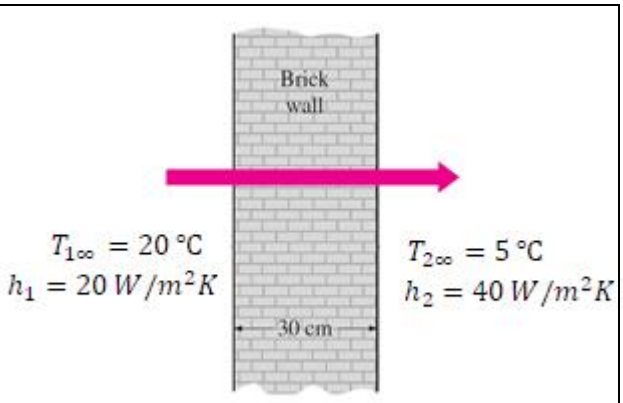
PROBLEM: The inner and outer surfaces of a 5 m x 5 m brick wall of thikness 0.3 m and thermal conductivity k=0.7 W/mK are maintained at temperatures of 20 C and 5 C, respectively. Determine the heat transfer through the wall.

A	25	m^2	
k	0.7	W/mK	
T1	20	C	
T2	5	C	
L	0.3	m	
Q	875	W	k*A*ΔT/L



PROBLEM: Surface of a brick wall of a 5 m x 5 m of thikness 0.3 m and thermal conductivity k=0.7 W/mK exposed of air $T_{1\infty}$ =20 C and h_1=20 W/m^{2}K on one side and $T_{2\infty}$ =5 C and h_1=40 W/m^{2}K on the other side. Determine the heat transfer through the wall.

A	25	m^2	
k	0.7	W/mK	
T1inf	20	C	
h1	20	W/m^2K	
T2inf	5	C	
h2	5	W/m^2K	
L	0.3	m	
R1	0.05	m^2K/W	1/h1
R2	0.4285714	m^2K/W	L/k
R3	0.2	m^2K/W	1/h2
R=R1+R2+R3	0.6785714	m^2K/W	
Q	552.63158	W	A/R*ΔT



2.2 CYLINDER – PIPE

Let us consider a one dimensional steady state problem, general cylindrical coordinates

$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r(T)r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k_\phi(T)\frac{\partial T}{\partial \phi}\right)++\frac{\partial}{\partial z}\left(k_z(T)\frac{\partial T}{\partial z}\right)+g=\rho C_p(T)\frac{\partial T}{\partial t}$

$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r(T)r\frac{\partial T}{\partial r}\right)=0$

$B.C.$

r=r$_1$ T=T_1

r=r$_2$ T=T_2

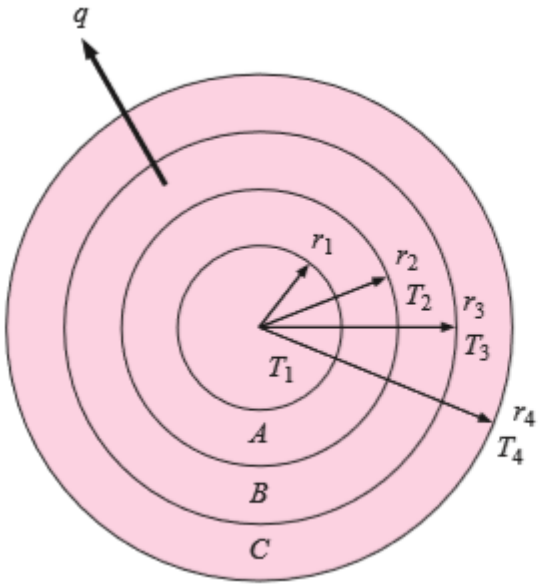
$k_r(T)r\frac{\partial T}{\partial r}=C_1 \quad \frac{\partial T}{\partial r}=\frac{C_1}{r} \quad T=C_1\ln(r)+C_2$

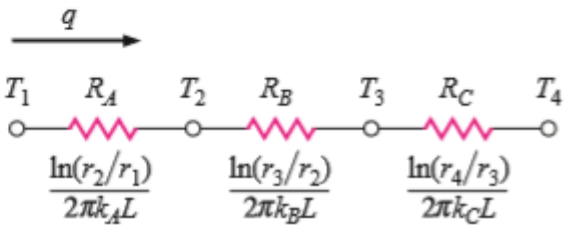
$T_1=C_1\ln(r_1)+C_2$

$T_2=C_1\ln(r_2)+C_2$

$\frac{(T_2-T_1)}{\ln\left(\frac{r_2}{r_1}\right)}=C_1$

$q=\frac{2\pi kL(T_2-T_1)}{\ln\left(\frac{r_2}{r_1}\right)}=\frac{(T_2-T_1)}{\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi kL}}=\frac{(T_2-T_1)}{R}$





For the serial case of resistances as shown in the figure

$q_r=\frac{(T_4-T_1)}{\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_A L}+\frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_B L}+\frac{\ln\left(\frac{r_4}{r_3}\right)}{2\pi k_C L}}=\frac{(T_4-T_1)}{R_A+R_B+R_C}$

Now consider convective heat transfer outside of the first layer. The boundary condition

$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r(T)r\frac{\partial T}{\partial r}\right)=0 \quad k_r(T)r\frac{\partial T}{\partial r}=C_1 \quad \frac{\partial T}{\partial r}=\frac{C_1}{r}$

$r=r_1 \quad Q_r=-k_A(2\pi rL)\frac{dT}{dr}=h_1(2\pi rL)(T_1-T_{1\infty})=\frac{(T_{1\infty}-T_1)}{\frac{1}{h_1(2\pi r_1 L)}}=\frac{(T_{1\infty}-T_1)}{R_1}$

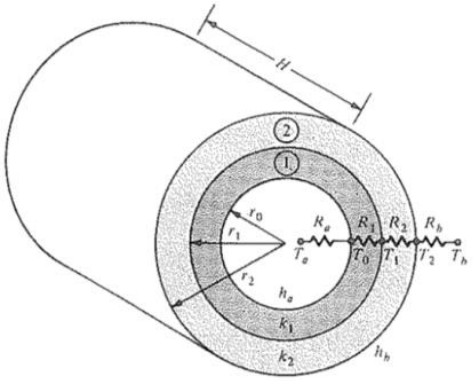
For the serial case of serial resistances

$q_r=\frac{(T_{1\infty}-T_{2\infty})}{\frac{1}{h_1(2\pi r_1 L)}+\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_A L}+\frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_B L}+\frac{\ln\left(\frac{r_4}{r_3}\right)}{2\pi k_C L}+\frac{1}{h_2(2\pi r_2 L)}}=\frac{(T_{1\infty}-T_{2\infty})}{R_1+R_A+R_B+R_C+R_2}$

It should be note that inverse of the overall thermal resistance is called **overall heat transfer coefficient**

$$UA = \frac{1}{\frac{1}{h_1(2\pi r_1 L)} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_A L} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_B L} + \frac{\ln\left(\frac{r_4}{r_3}\right)}{2\pi k_C L} + \frac{1}{h_2(2\pi r_2 L)}} = \frac{1}{R_1 + R_A + R_B + R_C + R_2}$$

PROBLEM : A steel tube with 5x10^{-2} m ID, 7.6 x10^{-2} m OD, and k=15 W/mK is covered with an insulative covering of thickness t=2 x10^{-2} m and k=0.2 W/mK. A hot gas T_a=330 C, h_a=400 W/m^{2}K flows inside the tube. The outer surface of the insulation is exposed to cooler air at T_b=30 C with h_b=60 W/m^{2}K. Calculate the heat loss from the tube to the air for H=10 m of the tube.



PROBLEM		
ID	5.0000000E-02	m
OD	7.6000000E-02	m
K1	1.5000000E+01	W/mK
t	2.0000000E-02	m
k2	2.0000000E-01	W/mK
Ta	3.3000000E+02	C
ha	4.0000000E+02	W/M^2K
Tb	3.0000000E+01	C
hb	6.0000000E+01	W/m^2K
H	1.0000000E+01	m
r0	2.5000000E-02	m
r1	3.8000000E-02	m
r2	5.8000000E-02	m
A0	1.5707963E+00	m^2
A2	3.6442475E+00	m^2
R0	1.5915494E-03	K/W
R1	4.4866505E-04	K/W
R2	3.3649879E-02	K/W
R3	4.5734179E-03	K/W
R	4.0263511E-02	K/W
Q	7.4509150E+03	W

A second example a solid cylinder with a constant heat generation where the surface temperature at r=R is T=T_w and thermal conductivity is constant.

$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r(T) r \frac{\partial T}{\partial r}\right) + g_0 = 0 \quad \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{g_0}{k} r$

B.C. T=T_w r=R $\quad \frac{\partial T}{\partial r} = 0$ r=0 (maximum temperature at the center due to heat generation)

$\left(r\frac{\partial T}{\partial r}\right) = -\frac{g_0}{k}\frac{r^2}{2} + C_1 \quad \frac{\partial T}{\partial r} = -\frac{g_0}{k}\frac{r}{2} + \frac{C_1}{r} \quad T = -\frac{g_0}{k}\frac{r^2}{4} + C_1 \ln(r) + C_2$

B.C $\quad \frac{\partial T}{\partial r} = 0$ r=0 $\xrightarrow{yields}$ $C_1 = 0$

B.C. T=T_w r=R $\quad \xrightarrow{yields} \quad T_w + \frac{g_0}{k}\frac{R^2}{4} = C_2$

$T = -\frac{g_0}{k}\frac{r^2}{4} + T_w + \frac{g_0}{k}\frac{R^2}{4} \quad T - T_w = \frac{g_0}{k}\frac{(R^2 - r^2)}{4}$

Another case: a hollow cylinder (pipe), $a \le r \le b$, has its boundary surfaces at r=a and r=b maintained at uniform temperatures T_1 and T_2, respectively Termal conductivity varies with temperature in the form: $k(T) = k_0(1 + \beta T)$ where k_0 and $\beta$are constants

$\frac{1}{r}\frac{\partial}{\partial r}\left(k(T)r\frac{\partial T}{\partial r}\right) = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}\left([k_0(1 + \beta T)]r\frac{\partial T}{\partial r}\right) = 0$

Boundary conditions:

$T = T_1$ at r=a

$T = T_2$ at r=b

$k(T)r\frac{\partial T}{\partial r} = C_1 \qquad [k_0(1 + \beta T)]r\frac{\partial T}{\partial r} = C_1$

$Q = (2\pi rL)\left(-k(T)\frac{\partial T}{\partial r}\right)$

$Q = -2\pi LC_1$

$[k_0(1 + \beta T)]r\frac{\partial T}{\partial r} = C_1$

$[k_0(1 + \beta T)]dT = \frac{C_1}{r}dr$

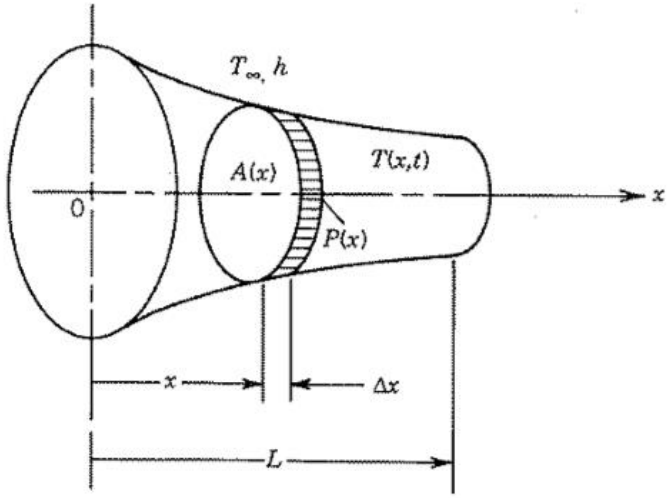
$\left[k_0(T + \beta\frac{T^2}{2})\right]_{T_1}^{T_2} = C_1 \ln(r)|_{r=a}^{b}$

$k_0(T_2 - T_1) + \frac{k_0\beta}{2}(T_2^2 - T_1^2)) = C_1 \ln(\frac{b}{a})$

$Q = \frac{2\pi Lk_0}{\ln(\frac{b}{a})}\left[1 + \frac{\beta}{2}(T_2 + T_1)\right](T_1 - T_2)$

2.3 LUMPED SOLUTION OF FINS

Instead of solving full differential equation by assuming some of the dimensions are not changing a simplified differential equations can be derived (again by assuming energy balance is hold for a given control volume). The following examples is assuming one dimensional steady state solution of a fin structure. Consider a heat transfer problem given as in the figure shown below. An extended surface has one dimensional conductive flow, and heat is removed by convection in each step.



A(x) = cross sectional area of the disk

P(x) = perimeter of the disk

L = Length of the fin

$\Delta S(x) = P(x)\Delta x$ =lateral surface area of the disk

h = convective heat transfer coefficient

k= thermal conductivity of solid

T_∞= ambient temperature

T_0 = base temperature (assumed to be constant)

Energy balance (First law of thermodynamics)

Net rate of heat gain by conduction +r ate of gain by convection from the lateral surfaces=rate of increase of internal energy

$$A(x)q_x - A(x+\Delta x)q_{x+\Delta x} + h(x)\Delta S(x)\big(T_\infty - T(x,t)\big) = \rho C_p \Delta x A(x)\frac{\partial T(x,t)}{\partial t}$$

$$A(x)q_x - A(x)q_x - \frac{\partial}{\partial x}(A(x)q_x)\Delta x + h(x)\Delta S(x)\big(T_\infty - T(x,t)\big) = \rho C_p \Delta x A(x)\frac{\partial T(x,t)}{\partial t}$$

$$-\frac{\partial}{\partial x}(A(x)q_x)\Delta x + h(x)\Delta S(x)\big(T_\infty - T(x,t)\big) = \rho C_p \Delta x A(x)\frac{\partial T(x,t)}{\partial t}$$

$$q_x = -k\frac{\partial T(x,t)}{\partial x}$$

$$\frac{\partial}{\partial x}(A(x)k\frac{\partial T(x,t)}{\partial x})\Delta x + h(x)\Delta S(x)\big(T_\infty - T(x,t)\big) = \rho C_p \Delta x A(x)\frac{\partial T(x,t)}{\partial t}$$

$$\frac{\partial}{\partial x}\Big(A(x)k\frac{\partial T(x,t)}{\partial x}\Big) - h(x)\frac{\Delta S(x)}{\Delta x}(T(x,t) - T_\infty) = \rho C_p A(x)\frac{\partial T(x,t)}{\partial t}$$

$$\theta(x,t) = \frac{T(x,t)-T_\infty}{T_0 - T_\infty}$$

$$\frac{\partial}{\partial x}\Big(A(x)k\frac{\partial \theta(x,t)}{\partial x}\Big) - h(x)\frac{\Delta S(x)}{\Delta x}\theta(x,t) = \rho C_p A(x)\frac{\partial \theta(x,t)}{\partial t}$$

k≠ $f(x)$

$$\frac{1}{A(x)}\frac{\partial}{\partial x}\Big(A(x)\frac{\partial \theta(x,t)}{\partial x}\Big) - \frac{h(x)}{A(x)k}\frac{\Delta S(x)}{\Delta x}\theta(x,t) = \frac{\rho C_p}{k}\frac{\partial \theta(x,t)}{\partial t} = \frac{1}{\alpha}\frac{\partial \theta(x,t)}{\partial t}$$

If steady state (independenty of time)

$$\frac{\partial}{\partial x}\Big(A(x)\frac{\partial \theta(x)}{\partial x}\Big) - \frac{h(x)}{k}\frac{\Delta S(x)}{\Delta x}\theta(x) = 0$$

Following normalisations and assumptions can be applied to this differential equation:

A_0= base cross-sectional area

P_0= base perimeter

L = Length of the fin

h_{av}=an average thermal convectivity

$X = \frac{x}{L}$ normalised length $0 \le X \le 1$

$K(X) = \frac{A(x)}{A_0}$ normalised cross-sectional area

$m^2 = \frac{h_{av}P_0}{kA_0}$

$W(X) = \frac{h(x)}{P_0 h_{av}}\frac{\Delta S(x)}{\Delta x}$ Note that $\frac{\Delta S(x)}{\Delta x} = P(x)$ also

$W(X) = \frac{h(x)}{h_{av}}\frac{P(x)}{P_0}$

$M = mL \qquad M^2 = m^2L^2 = \frac{h_{av}P_0}{kA_0}L^2$

Equation becomes

$$\frac{\partial}{\partial X}\Big(K(X)\frac{\partial \theta(X)}{\partial X}\Big) - M^2 W(X)\theta(X) = 0 \qquad 0 \le X \le 1$$

Boundary conditions:

$\theta(X) = 1$ X=0 $(T(x) = T_0$ x=0)

$\frac{d\theta(X)}{dX} = 0$ at X=0 $(Q(x) = -k * A(x)\frac{dT(x)}{dx} = 0$ x=L heat transfer at the edge is negligable)

Furthermore in order to simulate different geometries, some general polynomials are assigned with 3 different contants m,c and n

$K(X) = X^{1-2m}$ (change in the fin geometry)

$W(X) = c^2n^2X^{2c-2}K(X)$

Then the fin equation becomes:

$$\frac{\partial^2\theta(X)}{\partial X^2} + \frac{1-2m}{X}\frac{\partial \theta(X)}{\partial X} - M^2c^2n^2X^{2c-2}\theta(X) = 0$$

Solution of this differential equation for the boundary conditions given above:

$$\theta(X) = X^m \frac{I_{-\left(\frac{m}{c}\right)}(nMX^c)}{I_{-\left(\frac{m}{c}\right)}(nM)} \quad \frac{m}{c} < 1$$

$$\theta(X) = X^m \frac{I_{\left(\frac{m}{c}\right)}(nMX^c)}{I_{\left(\frac{m}{c}\right)}(nM)} \quad \frac{m}{c} > 1$$

I in the equation is called exponential Bessel function. It is given with the serial solution for $\nu=\frac{m}{c}$

$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}$ where Γ is gamma function

Where gamma function defined as $\Gamma(z)=\int_{z=0}^{\infty} t^{z-1}e^{-t}dt = \frac{e^{-\gamma t}}{t} \prod_{n=1}^{\infty}\left(1+\frac{t}{n}\right)^{-1} e^{t/n}$

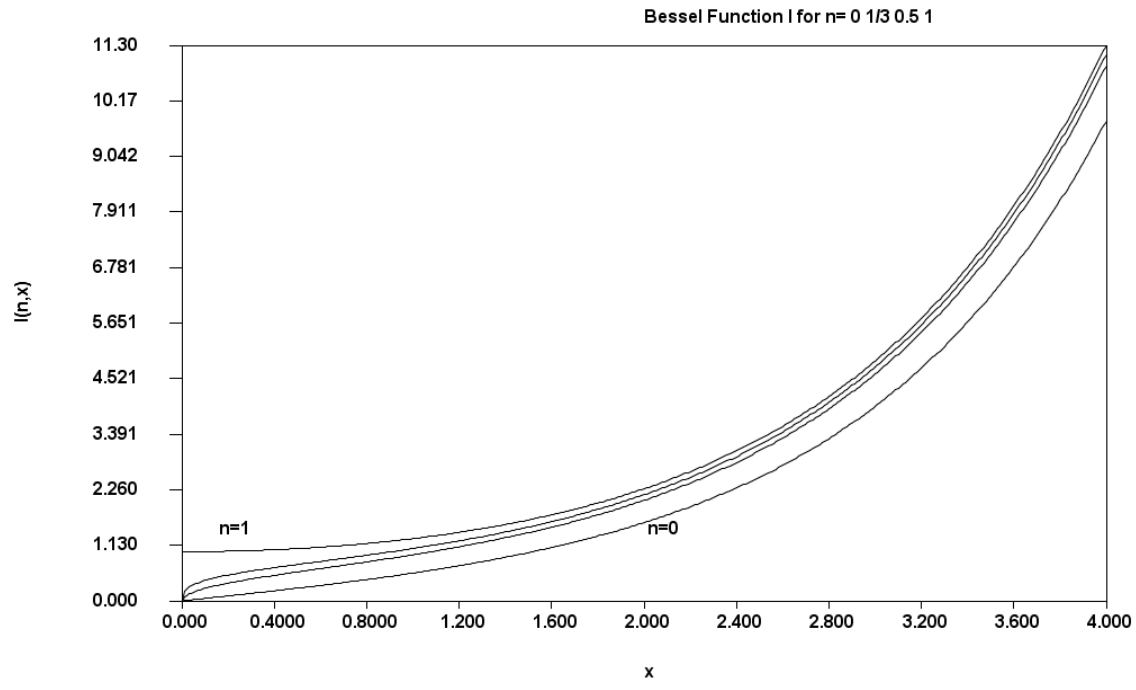
Where γ=0.57721566490153286... is the Euler-Mascheroni constant. A program is given to calculate I below:

```
public class bessel_I
{   //C Euler-Mascheroni constant
  public static double C=0.57721566490153286060651209008240243l;
  public static double I(double v,double x)
{ //I Bessel fonksiyonu
double JJ=0;double a=1.0;int fact=1;double b;
int vtest=(int)(v*10)/10;
if(v==0 && v==(double)vtest) JJ=bessi0(x);
else if(v==1 && v==(double)vtest) JJ=bessi1(x);
else if (v==(double)vtest) JJ=bessi((int)v,x);
else
{
double x1,x2,x3,x4;
long x5=1;
double lna=0.0;
x1=v*Math.log(0.5*x);
double lna1=Math.log(0.25*x*x);
for(int k=0;k<100;k++)
{
    x2=lna;//Math.log(a);
    x3=lngamma(k+1)[0];
    x4=lngamma(v+k+1)[0];
            b=x1+x2-x3-x4;
            b=Math.exp(b);
            lna+=lna1;
            JJ+=b;
}
if(v==0 && x==0) JJ=1;
}
return JJ;
}
//natural logarithm of gamma function
            public static double [] lngamma(double x)
   {//logaithm of gamma function
   //xx[0] value of logarithm of gamma function
   //xx[1] sign of gamma functon
            double xx[]=new double[2];
   double result;
   double a;
   double b;
   double c;
   double p;
   double q;
   double u;
   double w;
   double z;
   int i;
   double logpi;
   double ls2pi;
   double tmp;
   double sgngam = 1;
   logpi = 1.14472988584940017414;
   ls2pi = 0.91893853320467274178;
   if( x<-34.0 )
   {
      q = -x;
      double yy[]=lngamma(q);
      w = yy[0];
```

```
        tmp=yy[1];
        p = Math.floor(q);
        i = (int)Math.round(p);
        if( i%2==0 )
        {
           sgngam = -1;
        }
        else
        {
           sgngam = 1;
        }
        z = q-p;
        if( z>0.5 )
        {
           p = p+1;
           z = p-q;
        }
        z = q*Math.sin(Math.PI*z);
        result = logpi-Math.log(z)-w;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
     }
     if( x<13 )
     {
        z = 1;
        p = 0;
        u = x;
        while(u>=3)
        {
           p = p-1;
           u = x+p;
           z = z*u;
        }
        while(u<2)
        {
           z = z/u;
           p = p+1;
           u = x+p;
        }
        if( z<0 )
        {
           sgngam = -1;
           z = -z;
        }
        else
        {
           sgngam = 1;
        }
        if( u==2 )
        {
           result = Math.log(z);
           //return result;
           xx[0]=result;
           xx[1]=sgngam;
           return xx;
        }
        p = p-2;
        x = x+p;
        b = -1378.25152569120859100;
        b = -38801.6315134637840924+x*b;
        b = -331612.992738871184744+x*b;
        b = -1162370.97492762307383+x*b;
        b = -1721737.00820839662146+x*b;
        b = -853555.664245765465627+x*b;
        c = 1;
        c = -351.815701436523470549+x*c;
        c = -17064.2106651881159223+x*c;
        c = -220528.590553854454839+x*c;
        c = -1139334.44367982507207+x*c;
        c = -2532523.07177582951285+x*c;
        c = -2018891.41433532773231+x*c;
        p = x*b/c;
        result = Math.log(z)+p;
        //return result;
        xx[0]=result;
```

```
      xx[1]=sgngam;
      return xx;
   }
   q = (x-0.5)*Math.log(x)-x+ls2pi;
   if( x>100000000 )
   {
      result = q;
      //return result;
      xx[0]=result;
      xx[1]=sgngam;
      return xx;
   }
   p = 1/(x*x);
   if( x>=1000.0 )
   {
      q = q+((7.936507936507936507
```

```
        double BIGNI=1.0e-10;
        int j;
        double bi,bim,bip,tox,ans;

        if (n < 2) System.out.println("Index n less than 2 in bessi");
        if (x == 0.0)
                return 0.0;
        else {
                tox=2.0/Math.abs(x);
                bip=ans=0.0;
                bi=1.0;
                for (j=2*(n+(int) Math.sqrt(ACC*n));j>0;j--) {
                        bim=bip+j*tox*bi;
                        bip=bi;
                        bi=bim;
                        if (Math.abs(bi) > BIGNO) {
                                ans *= BIGNI;
                                bi *= BIGNI;
                                bip *= BIGNI;
                        }
                        if (j == n) ans=bip;
                }
                ans *= bessi0(x)/bi;
                if(x<0) return -ans;
                else return ans;
        }
}
public static void main(String arg[])
{   double x[]=new double[401];
   double y[][]=new double[4][401];
        for(int i=0;i<=400;i++)
        {x[i]=0.01*i;
         y[0][i]=I(0.0,x[i]);
         y[1][i]=I((1.0/3.0),x[i]);
         y[2][i]=I(0.5,x[i]);
         y[3][i]=I(1.0,x[i]);
        }
        Plot pp=new Plot(x,y[0]);
        pp.addData(x,y[1]);
        pp.addData(x,y[2]);
        pp.addData(x,y[3]);
        pp.plot();
}
}
```



DIFFERENTIAL EQUATIONS

EULER EQUATION



Leonhard Euler

It is desired the find solution of (y values) differential equation

$$\frac{dy}{dx} = f(x, y)$$

The first simplest method to solve differential equation numerically is the euler equation. Solution set in euler equation can be written as:

$$\frac{y_{i+1} - y_i}{h} \approx f(x, y)$$

$y_{i+1} = y_i + f(x_i,y_i)h$

In the equation h is step size. H can be calculated as $h = \Delta x = x_{i+1} - x_i$

Our example problem is solved for different step sizes as h=1, h=0.5 and h=0.1 The results clearly show that when h is getting smaller, euler solution and exact solution is getting closer to each other.

Program Euler equation

```
 //=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// and differentiation (derivative) functions
// Dr. Turhan Coban
// =====================================================
// Basit diferansiyel denklemler dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{ //diferansiyeli alinan fonksiyon
  double func(double x,double y)
  {return -2*x*x*x+12*x*x-20*x+8.5;}
}

class fxyKC extends f_x
{ //Gerçek çözüm fonksiyonu
  public double func(double x)
  {return -0.5*x*x*x*x+4.0*x*x*x-10.0*x*x+8.5*x+1.0;}
}

class diferansiyel
{
  public static double[][] euler(f_xy f,double x0,double y0,double xn,double h)
```

```
  {
  int n=(int)((xn-x0)/h);
  System.out.println("n="+n);
  double a[][]=new double[2][n+1];
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
  for(double xx=x0;xx<xn;xx+=h)
  for(i=0;i<n;i++)
   { xx=x0+i*h;
   a[0][i+1]=xx+h;
   a[1][i+1]=a[1][i]+f.func(a[0][i],a[1][i])*h;
   }
  return a;
  }

  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   fxy f1=new fxy();
   fxyKC fKC=new fxyKC();
   double a[][]=euler(f1,x0,y0,4.0,0.01);
   Text.print(Text.T(a));
   Plot p1=new Plot(a);
   //p1.setPlotType(0,24);
   //p1.addSpline(a,5);
   p1.addFunction(fKC,0.0,4.0,200);
   p1.plot();
  }
 }
```

Output Euler equation h=1

0.0 1.0

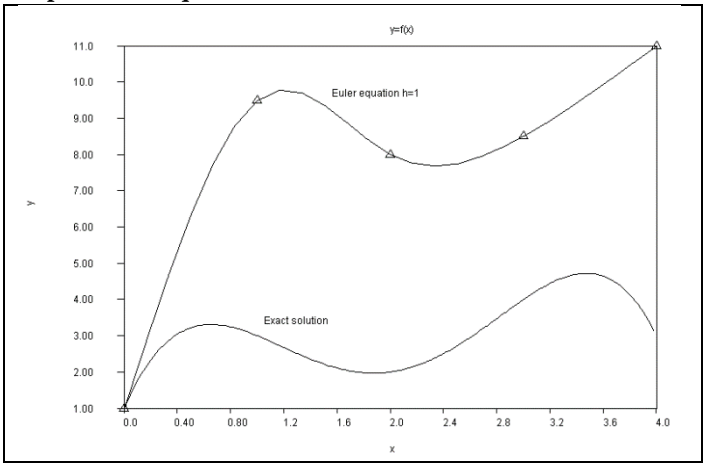
1.0 9.5

2.0 8.0

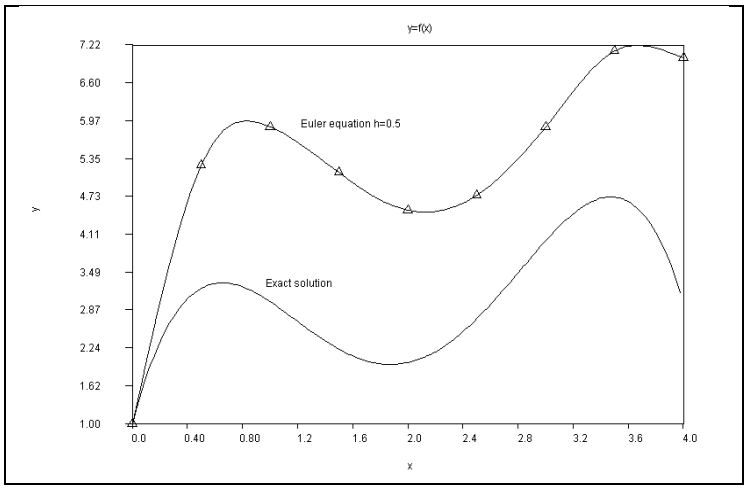
3.0 8.5

4.0 11.0

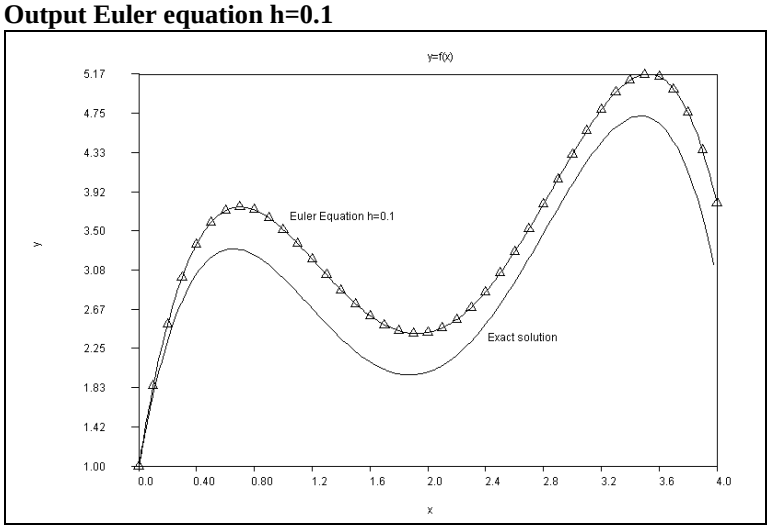
Output Euler equation h=1



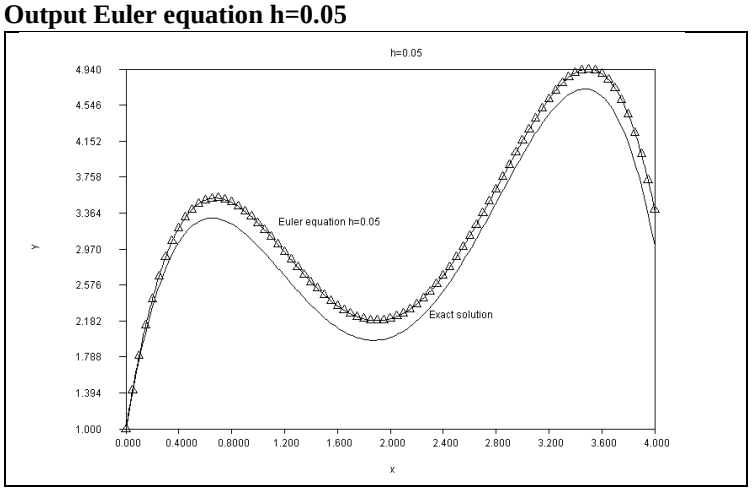
Output Euler equation h=0.5



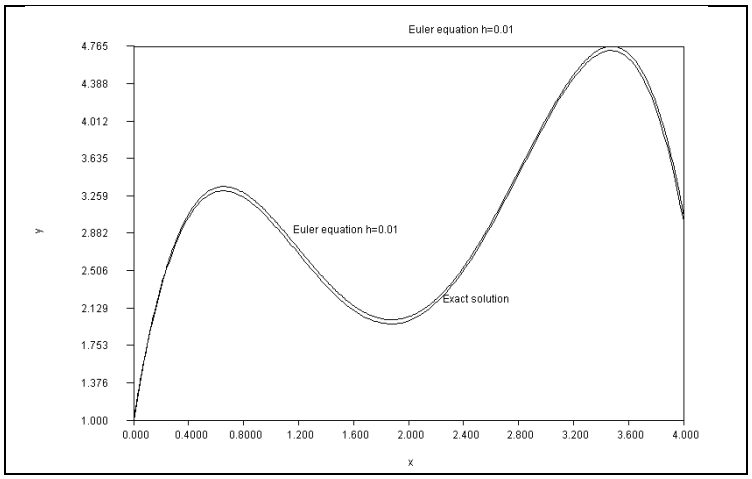
Output Euler equation h=0.1



Output Euler equation h=0.05



Output Euler equation h=0.01



HEUN EQUATION

Heun equation is similar to euler equation. It is added a correction term into the euler equation

$y^0_{i+1}=y_i+f(x_i,y_i)h$

Corection term

$$y_{i+1}^{k+1} = y_i + \frac{f(x_i, y_i^k) + f(x_{i+1}, y_{i+1}^k)}{2} h \quad , k = 0...n$$

The term y^k_{i+1} can be calculated iteratively. Number of iteration can be changed.

Program Heun equation

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// and differentiation (derivative) functions
// Dr. Turhan Coban
// =====================================================
// Basit diferansiyel denklemler dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}


class fxy extends f_xy
{
  double func(double x,double y)
  {return -2*x*x*x+12*x*x-20*x+8.5;}
}

class fxyKC extends f_x
{
  double func(double x)
  {return -0.5*x*x*x*x+4.0*x*x*x-10.0*x*x+8.5*x+1.0;}
}

class fxy1 extends f_xy
{
  double func(double x,double y)
  {return 4.0*Math.exp(0.8*x)-0.5*y;}
}
```

```
class diferansiyelA
{

  public static double[][] heun(f_xy f,double x0,double y0,double xn,double h)
  {
  int n=(int)((xn-x0)/h);
  System.out.println("n="+n);
  double a[][]=new double[2][n+1];
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
  double yi0=0;
  for(double xx=x0;xx<xn;xx+=h)
   {a[0][i+1]=xx+h;
    yi0=a[1][i]+f.func(a[0][i],a[1][i])*h;
    for(int j=0;j<5;j++)
    {yi0=a[1][i]+(f.func(a[0][i],a[1][i])+f.func(a[0][i+1],yi0) )/2.0; }
     a[1][i+1]=a[1][i]+(f.func(a[0][i],a[1][i])+f.func(a[0][i+1],yi0) )/2.0;
   i++;
   }
  return a;
  }

  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   fxy f1=new fxy();
   fxyKC fKC=new fxyKC();
   double a[][]=heun(f1,x0,y0,4.0,0.1);
   Text.print(Text.T(a));
   Plot p1=new Plot(a);
   p1.setPlotType(0,24);
   p1.addSpline(a,8);
   p1.addFunction(fKC,0.0,4.0,200);
   p1.plot();
  }
 }
```

Output Heun equation, step h=1 created data

0.0 1.0

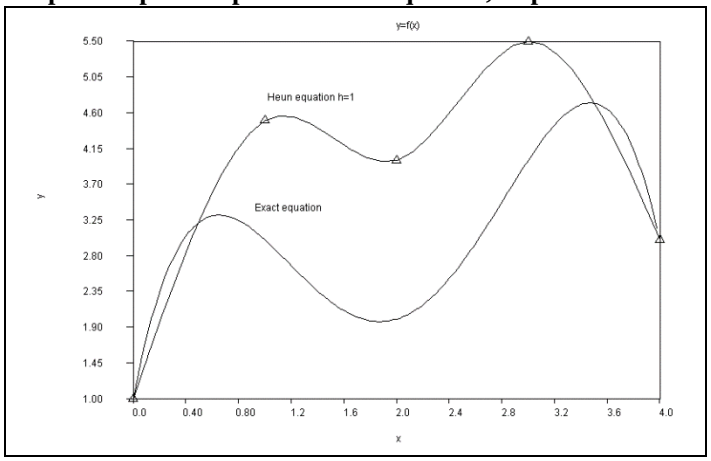
1.0 4.5

2.0 4.0

3.0 5.5

4.0 3.0

Output Graphic output for Heun equation, step h=1



MIDPOINT METHOD

In this method a midpoint is calculated first, then by using derivative at the mid point the real point is calculated.

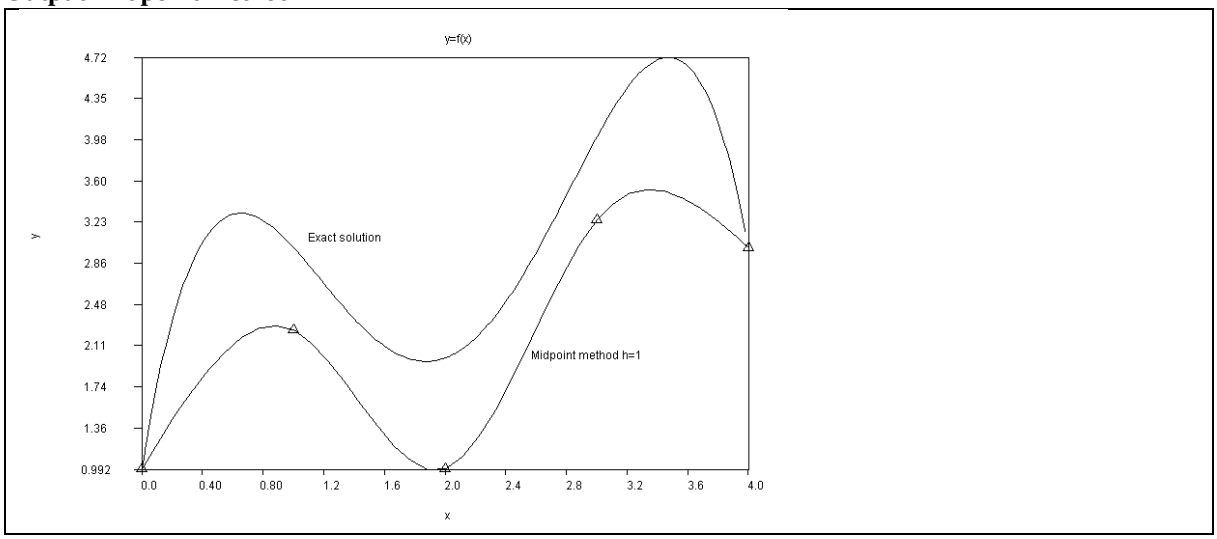
$y_{i+1/2} = y_i + f(x_i , y_i)h/2$

$y_{i+1} = y_i + f(x_{i+1/2} , y_{i+1/2})h$

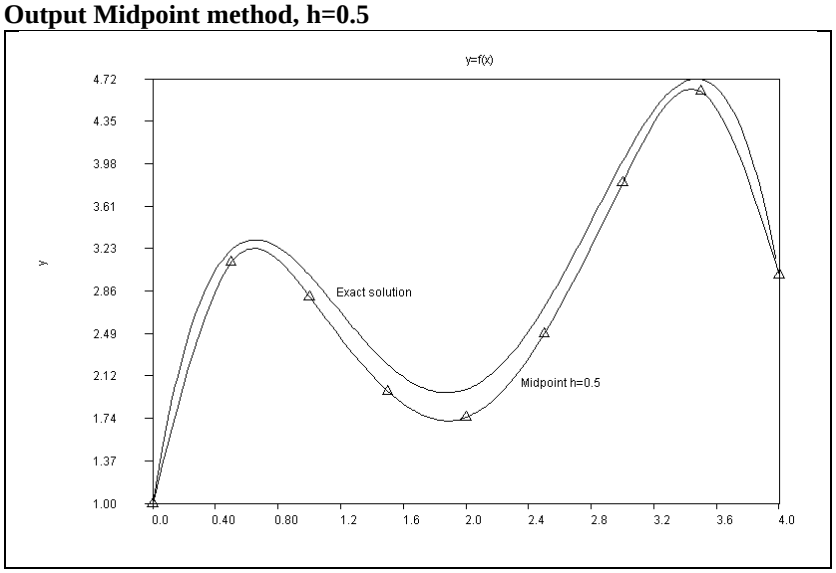
Program 8.3-1 Mid Point equation

```
/=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// and differentiation (derivative) functions
// Dr. Turhan Coban
// =====================================================
// Basit diferansiyel denklemler dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{
  double func(double x,double y)
  {return -2*x*x*x+12*x*x-20*x+8.5;}
}


class NA84
{
  public static double[][] polygon(f_xy f,double x0,double y0,double xn,double h)
  {
  int n=(int)((xn-x0)/h);
  System.out.println("n="+n);
  double a[][]=new double[2][n+1];
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
  double yi0=0;
  double xi0;
  for(double xx=x0;xx<xn;xx+=h)
   {a[0][i+1]=xx+h;
    xi0=xx+h/2.0;
   yi0=a[1][i]+f.func(a[0][i],a[1][i])*h/2.0;
   a[1][i+1]=a[1][i]+f.func(xi0,yi0)*h;
   i++;
   }
  return a;
  }

  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   fxy f1=new fxy();
   fxyKC fKC=new fxyKC();
   double a[][]=polygon(f1,x0,y0,4.0,1.0);
   Text.print(Text.T(a));
   Plot p1=new Plot(a);
   p1.setPlotType(0,24);
   p1.addSpline(a,8);
   p1.addFunction(fKC,0.0,4.0,200);
   p1.plot();
  }
}
```

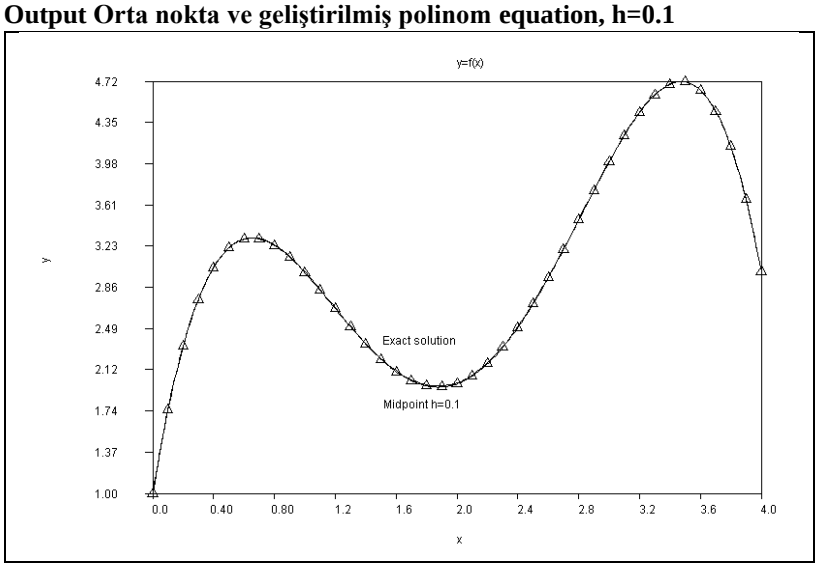
Output Midpoint method h=1



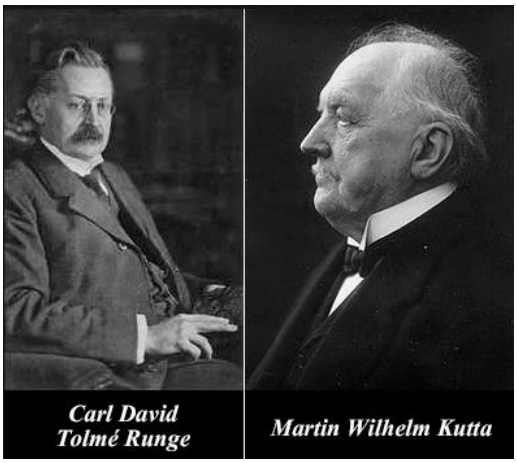
Output Midpoint method, h=0.5



Output Orta nokta ve geliştirilmiş polinom equation, h=0.1



RUNGE KUTTA METHODS



Carl Davis Tolme Runge and Martin Wilhelm Kutta

Runge-Kutta method as in most general form can be given as:

For differential form $\frac{dy}{dx} = f(x, y)$

$y_{n+1} = y_n + h*\phi(x_i,y_i,h)$

$$\phi(x_1, y_1, h) = \sum_{i=1}^{n} b_i k_i$$

In this equation a is constant values depends on polynamial degree of Runge-Kutta equation and coefficients k, can be defined as:

$k_1=f(x_i,y_i)$

$k_2=f(x_i+c_2h,y_i+a_{21}k_1h)$

$k_3=f(x_i+c_3h,y_i+a_{31}k_1h+ q_{32}k_2h)$

….

$k_n=f(x_i+c_nh,y_i+a_{n,1}k_1h+ \ldots+ a_{n,n-1}k_{n-1}h)$

In more generalized form, these equations can be written as:

$x_i = x_n + c_i h$ i=1…s

$$y_i = y_n + h\sum_{j=1}^{i-1} a_{ij} k_j$$

$$k_i = f(x_i, y_i)$$

$y_{n+1} = y_n + h\sum_{j=1}^{s} b_i k_i$ where s is the degree of coefficients of equations. In numerical analysis books, coefficients are usually shown as Buthcher tableu which puts the coefficients of the method in a table as follows:

$$\begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & a_{13} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \ldots & a_{2s} \\ c_3 & a_{31} & a_{32} & a_{33} & \ldots & a_{3s} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ c_s & a_{s1} & a_{s2} & a_{s3} & \ldots & a_{s4} \\ \hline & b_1 & b_2 & b_3 & \ldots & b_s \end{array}$$

As it is seen from the equation the definition is very general. Depends on polynamial degree of Runge-Kutta equation accuracy will improve. For example Runge-Kutta equation derived by using **a second degree polynomial (RK2)** has the following form

$y_{i+1} = y_i + (0.5k_1 + 0.5k_2)h$

$k_1=f(x_i,y_i)$
$k_2=f(x_i+h,y_i+k_1h)$
This equation can be given as Buthcher tableu as:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array}$$

Second degree Ralston method:
$y_{i+1} = y_i + (1.0/3.0k_1 + 2.0/3.0k_2)h$
$k_1=f(x_i,y_i)$
$k_2=f(x_i+(3.0/4.0)h,y_i+(3.0/4.0)k_1h)$
This equation can be given as Buthcher tableu as:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 3/4 & 3/4 & 0 \\ \hline & 1/3 & 2/3 \end{array}$$

Runge-Kutta with third degree polynomial solution RK3:
$y_{i+1} = y_i + ((1/6)\ k_1 + (4/6)k_2 + (1/6)k_3)h$
$k_1=f(x_i,y_i)$
$k_2=f(x_i+0.5h,y_i+0.5k_1h)$
$k_3=f(x_i+h,y_i - k_1h + 2k_2h)$
This equation can be given as Buthcher tableu as:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline & 1/6 & 4/6 & 1/6 \end{array}$$

Runge-Kutta with fourth degree polynomial solution RK4:
$y_{i+1} = y_i + (1/6)*(\ k_1 + 2k_2 + 2k_3 + k_4)h$
$k_1=f(x_i,y_i)$
$k_2=f(x_i+0.5h,y_i+0.5k_1h)$
$k_3=f(x_i+0.5h,y_i+0.5k_2h)$
$k_4=f(x_i+h,y_i + k_3h)$
This equation can be given as Buthcher tableu as:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

Runge-Kutta with sixth degree polynomial solution RK6:
$y_{i+1} = y_i + (1/90)*(\ 7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)h$
$k_1=f(x_i,y_i)$
$k_2=f(x_i+0.25h\ ,\ y_i+0.25k_1h)$
$k_3=f(x_i+0.25h\ ,\ y_i+0.125k_1h+0.125k_2h)$
$k_4=f(x_i+0.5h\ ,\ y_i - 0.5k_2h+k_3h)$
$k_5=f(x_i+0.75h\ ,\ y_i + (3/16)k_1h+(9/16)k_4h)$
$k_6=f(x_i+h\ ,\ y_i - (3/7)k_1h+(2/7)k_2h+(12/7)k_3h - (12/7)k_4h+(8/7)k_5h)$

This equation can be given as Buthcher tableu as:

0	0	0	0	0	0
1/4	1/4	0	0	0	0
1/4	1/8	1/8	0	0	0
2/4	1/4	−1/4	1	0	0
3/4	3/16	0	0	9/16	0
1	−3/7	2/7	12/7	−12/7	8/7
	1/90	7/90	32/90	12/90	7/90

Runge-Kutta-Verner methods with different degree of accuricies

For the degrees a general formula of Runge_Kutta equations can be written as:

$$x_i = x_n + c_i h \quad \text{i=1…s}$$

$$y_i = y_n + h\sum_{j=1}^{i-1} a_{ij} k_j$$

$$k_i = f(x_i, y_i)$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s} b_i k_j \text{ or}$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s} bh_i k_j$$

Verner[42-46] given coefficient of a pair of solutions (b and bh). For the constant step size any of these equations can be used, in the coming sections utilisation of these type of equations as a base of variable step size calculations will also be introduced. Due to growing sizes of equations Buthcher tableus will be given directly as table values for the remaining equations.

Table 9th degree Runge-Kutta-Verner coefficients

c[1] = 0
c[2] = 3/50
c[3] = 1439/15000
c[4] = 1439/10000
c[5] = 4973/10000
c[6] = 389/400
c[7] = 1999/2000
c[8] = 1
c[9] = 1

a[2,1] = 3/50

a[3,1] = 519479/27000000
a[3,2] = 2070721/27000000

a[4,1] = 1439/40000
a[4,2] = 0
a[4,3] = 4317/40000

a[5,1] = 109225017611/82828840000
a[5,2] = 0
a[5,3] = -417627820623/82828840000
a[5,4] = 43699198143/10353605000

a[6,1] = -80368152926439073494525521723 69/
191934985946683241245914401600
a[6,2] = 0
a[6,3] = 246134619571490020064824665/154381649665540511760236 8
a[6,4] = -13880495956885686234074067279/113663489566254201783474344
a[6,5] = 7550050577777889947341 29/136485922925633667082436

a[7,1] = -16632998415661020971805066664988809342302 61/30558424506156170307020957791311384232000

a[7,2] = 0
a[7,3] = 13083812419528549179904362881109303/631862949514135618861563657970240
a[7,4] = -3287100453856023634160618787153901962873/
207243149153767556291357110268514092 00
a[7,5] = 27718267901403321408652425203692 41/396438716042723436917079980147600
a[7,6] = -1799166916139193/96743806114007800

a[8,1] = -832144750039369683895428386437986853923637763/1522297455006960074876365184466761994520 4887
a[8,2] = 0
a[8,3] = 818622075710363565982285196611368750/393657623790372815185607239534 3129
a[8,4] = -9818985165491658464841194581385463434793741875/616425979626589940698693709231964635818660 11
a[8,5] = 317966921418485587204257110425481347693 75/45302540335000459755578580160063086280 92
a[8,6] = -1406454211884383007 5/76692874826430685364 4
a[8,7] = -1424670304836288125/2782839104764768088217

a[9,1] = 382735282417/11129397249634
a[9,2] = 0
a[9,3] = 0
a[9,4] = 5535620703125000/21434089949505429
a[9,5] = 13867056347656250/32943296570459319
a[9,6] = 626271188750/142160006043
a[9,7] = -51160788125000/289890548217
a[9,8] = 163193540017/946795234

b[1] = 382735282417/11129397249634
b[2] = 0
b[3] = 0
b[4] = 5535620703125000/21434089949505429
b[5] = 13867056347656250/32943296570459319
b[6] = 626271188750/142160006043
b[7] = -51160788125000/289890548217
b[8] = 163193540017/946795234
b[9] = 0

bh[1] = 124310637869885675646798613/2890072468789466426596827670
bh[2] = 0
bh[3] = 0
bh[4] = 265863151737164990361330921875/11131972714633723039403193695 79
bh[5] = 307549355717403080653630295312 5/68437499220423238765469496998 76
bh[6] = 67798000008733879813263055/29532792147666737550036372
bh[7] = -10994365851553908462383263 75/150557064964464088591961 67
bh[8] = 26171252653086373181571802/36879447889073234603350 5
bh[9] = 1/30

Table 10th degree Runge-Kutta-Verner coefficients

c[1] = 0
c[2] = 1/200
c[3] = 49/450
c[4] = 49/300
c[5] = 91/200
c[6] = 34704460/57271701
c[7] = 167/200
c[8] = 183/200
c[9] = 1
c[10] = 1

a[2,1] = 1/200

a[3,1] = -4361/4050
a[3,2] = 2401/2025

a[4,1] = 49/1200
a[4,2] = 0
a[4,3] = 49/400

a[5,1] = 1781/2800
a[5,2] = 0
a[5,3] = -13689/5600
a[5,4] = 507/224

a[6,1] = -2757769235683215548894853133264 60/10878254403907579741558876498209 9
a[6,2] = 0
a[6,3] = 9576001512158705648097438730000/92976533366731450782554499984 7
a[6,4] = -2217853846564295490229045868960 0/27892960010019435234766349995 41
a[6,5] = 1232368675010928345385491377920 0/15540363434153685345084109283157

a[7,1] = 8221508014471276464993/8206108584945090400000
a[7,2] = 0
a[7,3] = -9852144759099/2364568872400
a[7,4] = 153225509327783989072 99/3996134347464283007200
a[7,5] = -1833846312189852000 4/36506562121215938675
a[7,6] = 233404755446021251193075113735193834 99/34957329425893779598660175543672800000

a[8,1] = 810886430227405450337307801 69/29751821102319371401528000 00
a[8,2] = 0
a[8,3] = -2837794586103/67559110640
a[8,4] = -1416757560688131609503834 1141/13447191885934685533378360 00
a[8,5] = 2395552232834276839307772/ 29760062864388068268875
a[8,6] = -4076715891031001341580357765362043260356514682697/60535801523558513633981092635987721507186400000
a[8,7] = 3655152735545995780 8/2801171464968864455

a[9,1] = -3347747115771808477876752833599/11013275913079015494642110732 80
a[9,2] = 0
a[9,3] = 121470487847762512 5/119815105452943264
a[9,4] = -6558111861786403812445645 2415/10200342297342072539428709672
a[9,5] = -1333730829115754792732984060 95/84070826916189821955373830704
a[9,6] = 6225156836540393863837014637589522674477368410502819501376 93/
3289942188601405845401861424555681176690770946533324320857 60
a[9,7] = 4616918867155144139901376312 5/23436927041568412756289303582 08
a[9,8] = 1888086786587759749309187 5/34696641481969116434755335 04

a[10,1] = -7430981552872219612752603 7/51427190037752065777334640
a[10,2] = 0
a[10,3] = 913722369253875/113761793498234
a[10,4] = -4406582271592926020603968 90/58109996881684093545238767
a[10,5] = 3729088829393576334181438 0/10411746696360295961914239
a[10,6] = -6451528881135810657803603921051793104524963268 47/26473542512180481489813104213136732045148791096 0
a[10,7] = 4737574861900868386818925 00/55632110080294263936045988 7
a[10,8] = 0
a[10,9] = 0

b[1] = 9420080774669597/198627609019792680
b[2] = 0
b[3] = 0
b[4] = 18658605936510000/72821569629535727
b[5] = 296950875175030000/1101802245630054969
b[6] =
188752769802742126868248355661431511898505538963300 09/14878094713960970610439415759664835799457557703622444 0
b[7] = 18663850606812500/74993696164706319
b[8] = 179884749312500/58508928482581269
b[9] = 349315176247648/7273791403140339
b[10] = 0

bh[1] = 7362904929137/155056681514280
bh[2] = 0
bh[3] = 0
bh[4] = 7505178129270000/29317774785916981
bh[5] = 1851744839320000/6843492208882329
bh[6] = 75088277818981843751281040783964505112808 9/60043632957155362707357899923191763222092 40
bh[7] = 27902602073000000/110704980052661709
bh[8] = 0
bh[9] = 0
bh[10] = 331667036438/6791588611709

Table 13th degree Runge-Kutta-Verner coefficients

c[1] = 0
c[2] = 1/20
c[3] = 341/3200
c[4] = 1023/6400
c[5] = 39/100
c[6] = 93/200
c[7] = 31/200
c[8] = 943/1000
c[9] = 7067558016280/7837150160667
c[10] = 909/1000
c[11] = 47/50
c[12] = 1
c[13] = 1

a[2,1] = 1/20

a[3,1] = -7161/1024000
a[3,2] = 116281/1024000

a[4,1] = 1023/25600
a[4,2] = 0
a[4,3] = 3069/25600

a[5,1] = 4202367/11628100
a[5,2] = 0
a[5,3] = -3899844/2907025
a[5,4] = 3982992/2907025

a[6,1] = 5611/114400
a[6,2] = 0
a[6,3] = 0
a[6,4] = 31744/135025
a[6,5] = 923521/5106400

a[7,1] = 21173/343200
a[7,2] = 0
a[7,3] = 0
a[7,4] = 8602624/76559175
a[7,5] = -26782109/689364000
a[7,6] = 5611/283500

a[8,1] = -1221101821869329/690812928000000
a[8,2] = 0
a[8,3] = 0
a[8,4] = -125/2
a[8,5] = -1024030607959889/168929280000000
a[8,6] = 1501408353528689/265697280000000
a[8,7] = 6070139212132283/92502016000000

a[9,1] = -14725142644862158038813847088772642463460444333307094\
20782905197804453180113305715512468948016200320011570596216439860248033015583 9\
34879004404536361680460696864366 08
a[9,2] = 0
a[9,3] = 0
a[9,4] = -5172294311085668458375175655246981230039025336933699\
11413831527077231937246928000 0/12461938100480914589727863057121529836525707941 0\
23625292185093674907648713299519 1
a[9,5] = -1207067925846925480797893644173318794948457151612046\
99665345142964068916526149703 75/27220311547616572217104781845311006994972840850\
48389015085076961673446140398628096
a[9,6] = 78012515584389364132309055253043103656779559256849718\
27014606748031267701114816 25/18311042541273197219788987450715878685922610298086\
18595052414430736291431008053 76
a[9,7] = 66411312295991164213478213583910646992814032816057703\
53571553403929500094925118 75/15178465598586248136333023107295349175279765150089\
07830113994325301687782317 0816
a[9,8] =
10332848184452015604056836767286656859124007796970668046446015775000000/
13127035500360336480738342487407279145379720286389501652495827336793937 83

a[10,1] = -2905557336033741508853861844223103644131406051 1/
22674759891089577691327962602370597632000000000
a[10,2] = 0
a[10,3] = 0
a[10,4] = -2046274952459104910540336523906 9/
45425191349989346959623126875 0
a[10,5] = -1802692598031722811637246632249810 97/
38100922558256871086579832832000000
a[10,6] = 2112767021417280287012828699200394081065522148 9/
46794738779978929061458226979767086336737280 00
a[10,7] = 3186072351736493124051512658496608699276534144254 13/
67147167155589653031329380729354654239109120000 00
a[10,8] = 2120832024345190822818422455358 94/
20022426044775672563822865371173879
a[10,9] = -\
26984049294008425187211664850871297985622698482295177937034139512267145 83/
46954567491393431507700044208087114188467603590271755032561672817587500000 0

a[11,1] = -\
23426598458140868369512071400656091790738384762429439 17/

135848096135105677702223140013915876085753216279552000 0
a[11,2] = 0
a[11,3] = 0
a[11,4] = -9962860301325381596139308896 52/
163530688859961649054643256 75
a[11,5] = -2605308595925653415258808936384 1/
437755280456568306101129994240 0
a[11,6] =
209808223450967602922240867949781053126445339 25634933539/
377588999200755080387872783911549464197221296 21741568 00
a[11,7] =
890722993756379186418929622095833835264322635782294899/
139212420013951126575019419555940138228301198037647 36
a[11,8] = 161021426143124178389075121929246710833125/
109972077221310346506670413643464226943714 43
a[11,9] = 3007606697681025178342324975654524349466722661958764\
963718742623926848522439253598648849625 13/465544333750134645558506533660450560 3\
760824779615521285751892810315680492364106674524398280000
a[11,10] = -3115523743711173066592320687 5/
392862141594230515010338956291

a[12,1] = -\
286655699182566397177829532910103388753491278772403436 3/
868226711619262703011213925016143612030669233795338240
a[12,2] = 0
a[12,3] = 0
a[12,4] = -1695708871417146867638705435895475400 0/
143690415119654683326368228101570221
a[12,5] = -4583493974484572912949314673356033540575/
451957703655250747157313034270335135744
a[12,6] =
234630538855340425865625847344618441915474017251994957 5/
256726716407895402892744978301151486254183185289662464
a[12,7] =
165712155931984680217128369091361069858625657348480866 2625/
134314804112551464772591551049560935053616444320881090 56
a[12,8] = 345685379554677052215495825476969226377187500/
747711674369300772216672031795513475463620 89
a[12,9] = -320589096271707254279143431215272753400810277402321\
024057136157075724905616701523016035208704867454219601 1/947569549683965814783 01\
512445127360498465774712725761537244920597319265730601723910349107473832403325 9\
120
a[12,10] = 402795458327062334331004385884589332109375 00/
889646084279948284691697212637733894721510 1
a[12,11] = -612293360107076959161309399399335887725 0/
105051700151023551319824672130202767595 3

a[13,1] = -6186759055354825006728008593445384103586601538996 37
/203544282118214047100119475340667684874292102389760
a[13,2] = 0
a[13,3] = 0
a[13,4] = -4411194916804718600478400319122931000/
403730539024699674507614912696330 19
a[13,5] = -1673471140944929253453942253172852022 5/
180124371529008866930720392721023795 2
a[13,6] =
135137519757054679098042184152749677761254751865630525/
160295877944862895977713263619118951127037165939834 88
a[13,7] =
389375683674098760125485519034921961379297104315848 75/
340956454090191606099548798001469306974758443147264
a[13,8] = -6748865855011993037732355335815350667265625/
700288039571742462121356540671508776477035 7
a[13,9] = -1756005520307450928195422767042525091954178296002 78\
830892656319352366240473977978973268567 1/348767814578469983605688098046186480 90\
460727802103054073533386208706157493415494283006232 0
a[13,10] = 5338102458923561108401389767418162929687 5/
895935758479569452487496959850859294414 1
a[13,11] = 0
a[13,12] = 0

b[1] = 4490186773775461685197 3/10140464099802310133806 80
b[2] = 0
b[3] = 0
b[4] = 0
b[5] = 0

```
b[6] =  791638675191615279648100000/22356047250899731264115123 19
b[7] =  384774949086898034811950000/155170450621382716181412375 17
b[8] = -13734512432397741476562500000/875132892924995907746928783
b[9] =  122747654703131968784288120377406350503192342760069863982 9444\
3554969616342274215316330684448207141/489345147493715517650385834143510934888 82\
928068660965448289652679652335305216675729945285216604 0
b[10] = -979836368457773944531250000 0/308722986341456031822630699
b[11] =  282035543183190840068750/122954076298730404259 91
b[12] = -306814272936976936753/1299331183183744997286
b[13] =  0

bh[1] =  108354017394070194065 77/244521829356935137978320
bh[2] =  0
bh[3] =  0
bh[4] =  0
bh[5] =  0
bh[6] =  139081897783218954913750 00/392211355278942653756405 67
bh[7] =  734879475270272434876250 00/296504045773342769773399443
bh[8] =  682931406412576496093750 00/153532086478069457499461 19
bh[9] =  220606479489966786110177113799745788605220182089497215594485\
602033384376260221427763 81/111154200926232587451295918579572721575901057756573 6\
079641376621381577236680929558640
bh[10] = -54797122949564245820312 5000/232372140257009916425636 01
bh[11] =  0
bh[12] =  0
bh[13] = -28735456870978964189/79783493704265043693
```

Table 16th degree Runge-Kutta-Verner coefficients

```
c[1] = 0
c[2] = 1731/50000
c[3] = 7630049/53810000-983539/53810000*6^(1/2)
c[4] = 22890147/107620000-2950617/107620000*6^(1/2)
c[5] = 561/1000
c[6] = 387/1000-129/2000*6^(1/2)
c[7] = 387/1000+129/2000*6^(1/2)
c[8] = 129/200
c[9] = 387/800
c[10] = 6757/100000
c[11] = 1/4
c[12] = 142797165095125837 2/2166662646162554701
c[13] = 4103/5000
c[14] = 2253/2500
c[15] = 1
c[16] = 1

a[2,1] =  1731/50000

a[3,1] =  -177968356965557/1002427673820000+14180534491313/
250606918455000*6^(1/2)
a[3,2] =  64021741529527/200485534764000-7504450763411/
100242767382000*6^(1/2)

a[4,1] =  22890147/430480000-2950617/430480000*6^(1/2)
a[4,2] =  0
a[4,3] =  68670441/430480000-8851851/430480000*6^(1/2)

a[5,1] =  592203994261020339/513126355505556250+
730386990293623641/205250542202222 5000*6^(1/2)
a[5,2] =  0
a[5,3] =  -871215388418279490 3/2052505422022225000-\
284342135919585153 3/2052505422022225000*6^(1/2)
a[5,4] =  187369836222329544 3/513126355505556250+
528258592225556973/513126355505556250*6^(1/2)

a[6,1] =  11380823631/157617812000-339148869/39404453000*6^(1/2
)
a[6,2] =  0
a[6,3] =  0
a[6,4] =  -2355345717024309/58864341808507450*6^(1/2)+
161932328870918 31/58864341808507450
a[6,5] =  165912282616977/4179075230308000-33181894472511/
2089537615154000*6^(1/2)

a[7,1] =  26523528363/231790900000+863255358/123138915625*6^(1/
```

```
2)
a[7,2] =  0
a[7,3] =  0
a[7,4] =  -382087481780164848177 87/842517966262441068418750-\
861187885562823698228 07/842517966262441068418750*6^(1/2)
a[7,5] =  92362336407446913/290322814529044000-\
232039320950012997/2467743923496874000*6^(1/2)
a[7,6] =  -362925891/1690350537500+857800423623/3380701075000*6
^(1/2)

a[8,1] =  43/600
a[8,2] =  0
a[8,3] =  0
a[8,4] =  0
a[8,5] =  0
a[8,6] =  43/150+43/2400*6^(1/2)
a[8,7] =  -43/2400*6^(1/2)+43/150

a[9,1] =  7353/102400
a[9,2] =  0
a[9,3] =  0
a[9,4] =  0
a[9,5] =  0
a[9,6] =  22833/102400+8901/204800*6^(1/2)
a[9,7] =  -8901/204800*6^(1/2)+22833/102400
a[9,8] =  -3483/102400

a[10,1] =  3767087424722149887008 53/778845602812500000000000000,
`:`
a[10,2] =  0
a[10,3] =  0
a[10,4] =  0
a[10,5] =  0
a[10,6] =  1879146667539568401952 79/259615200937500000000000000-\
210440846556290693268911/155769120562500000000000000*6^(1/2)
a[10,7] =  210440846556290693268911/155769120562500000000000000*
6^(1/2)+1879146667539568401952 79/259615200937500000000000000
a[10,8] =  -1855266722189674422664 7/865384003125000000000000000,
`:`
a[10,9] =  -3167799860072183913409/30423656359863281250000

a[11,1] =  -426968570497/54394415898750-92754382349/
12087647977500*6^(1/2)
a[11,2] =  0
a[11,3] =  0
a[11,4] =  0
a[11,5] =  0
a[11,6] =  1/30
a[11,7] =  -2865012129681958/114898584332330625-\
12962517687655099/229797168664661250*6^(1/2)
a[11,8] =  4389715333607/309890657317500+92754382349/
11477431752500*6^(1/2)
a[11,9] =  371017529396/9306344041875*6^(1/2)+4990058173976/
83757096376875
a[11,10] =  1099523524595993125000/6257667909869756018891+
1009573480379896875 00/6257667909869756018891*6^(1/2)

a[12,1] =  1838203110479840386993853900915465658752149857359559\
5063164077882800315372787284683238439478955141517997198007108623761931447163756
/139742569444997243449189609938909336141610253229704500479326889980950085286208\
2123960473460811129176944470618749780786917955084132937 5+4078857781851586092107\
9389251758259530589647075646761263679625961149140826089641344688345089135162291\
4818800693274034252252905536/2808492638860122607362409616917500295697019157645 5\
1106332267651411613722940986932751171812393853121981375088465359331278371679268\
75*6^(1/2)
a[12,2] =  0
a[12,3] =  0
a[12,4] =  0
a[12,5] =  0
a[12,6] =  -3338813117898494119715734728681282814382022107217 23\
1232517421453677345828875773955477782287601740687580861343899520155634039 04/227\
0872004608103037127689848604039623086639035441372934050180593816493796129405349\
9141489814607142022329887277387784945577276 35+4819272892477768171373308666720 68\
9121421091953625792970278044071549950640195056472955523769829034800621890424847\
009130000000/2316289444700265097870243645576120415548371816150200392731184205 69\
2823672051993456912431961089928486277648502293554064448882187 7*6^(1/2)
```

a[12,7] = 1698450855653613368055560092963943745276369523793889\
6102606662872515552183276208687563236699647756792865753591219139615556676545782\
6139904/159397991371817326062367942596607749779797818314659613579779527210763 76\
200986140874348737329339373150944891873144740452931192009996095868 75*6^(1/2)-13\
666660749646362227013560886377207644362546879813948039042674099302480394698176 3\
209348364716108721312822619845726151693667598437699964416/371928646534240427478\
858532725418082819528242734205765019485563491782111356343287068137204351252040 1\
887141437067106105683944802332422369375
a[12,8] = 561098789927327852541196052808144290219856759480976 4\
379756195673673265700551076812883925583370253765702553235594764427173637673766 2\
08/928815981981440330182788047406263341354233567916395981093588677703616092328 4\
601262673233245084426429384045657495603634963319733636187 5-558747641349532341 38\
464916783230492507657050788557207210520035563218001131629645677655267245390633 2\
760025754374347992126373843 2/365303089362201664516413596925286161494473575337 11\
529625051175285972810886869692961402480325512278540323235981796528873956555062 5
*6^(1/2)
a[12,9] = 545985398180836152335661486022032448966969589107343 3\
975406527098543350794516270773775946921467448080727221064814847749923878327625 9\
328/301247919092298852634886875129959310794662932014184499827145075851637298698\
312074030567479239502011693447423026416040794479934024058125-652617245096253774\
737270228028132152489434353210348180218874015378386253217434261515013521426162 5\
966637100811092384548036046488576/864909328430372818360283879213205026685796531\
766248922845664874681703412857628693742657132470577122289541840443342063722308 1\
6544375*6^(1/2)
a[12,10] = -810882514508508810434472104816632522517372949568 93\
646964267201611120124142277523289697206589873156541798737603577252357340003994 4\
0000000/265558296661064021061274102756797754810572081529546692522040189640618 20\
669308150613320650066298300104729878761982741137567571658328380175 7*6^(1/2)+939\
166734840458401095542221032870712500612066161106190888975080561941878582094800 2\
455890360939221912190524731087070645107486913457760000000/581572669687730206124\
190285037387083035152858549707256623268015312953872657848498431722236451932772 2\
935843448874220309127298193173915258478 3
a[12,11] = 123461712659887915177271339396606860810479028777869\
348014870450606260914019560285661288212498128400476015695960341952/281629106670\
320674754245209358840703704235147307838896741075511220826056829047205614324978 2\
532261762750789227161324 61

a[13,1] = -\
560427726753220421392276299780425863306336227060533639467661444169336 31/
588085407723231905255901226132234305073521185345573426660156250000000 00+
281404579734699232141455524604487724159024972527/
147800994483274318045231620407718841552734375000 0*6^(1/2)
a[13,2] = 0
a[13,3] = 0
a[13,4] = 0
a[13,5] = 0
a[13,6] = -102716390022975035656123823794722533267562151 7/
179261894431132664078747698292867431640625000-\
274529239164120252537310397933681351337232 1/
117022164684643403110606497445583859375000 00*6^(1/2)
a[13,7] = -\
157229999853748227305165773364426925282378072238332930121/
366999073679854585732732040943307160339634132385253906 25+
575760644280279509531898606731783790418427865066459025210 1/
352319110732660402303422759305574873926048767089843750000 0*6^(1/2)
a[13,8] = -\
931144816859393414601596501990401360213380294332581834662278128590705 7/
425597084901012421719313544966873998540131336300557615936279296875000 0-\
844213739204097696424366573813463172477074917581/
421018835994657833697686816496616302490234375000 0*6^(1/2)
a[13,9] =
885774233856672590222951867695327816457340130391639153070521335485617578/
301098541380295011015469248465465290112505656143757799934635162353515625-\
281404579734699232141455524604487724159024972527/
284481916364737983221402322504830303192138671875*6^(1/2)
a[13,10] =
315479116729780153956412124052199685097744239386639023787359107959254802182/
134481850506505848012587842215515574380212543200894932329128471154748828125-\
294039645364787227664606877659229222973765193793462 3/
734546505878198371079583742953078477724528652070312 5*6^(1/2)
a[13,11] =
225099616340654537861653203901884658621763159945382254 1/
382491303797095993563304148204275636433504028320312500
a[13,12] = 268934095730769185329490238833445400395937814695752\
986623352925198635939233604415170894972095880974797051436629345842427217402449 3
/959516386019578808500569114780871708466894752280482835105408027815194895319055\

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4438427822271021204939608056495755617968750000000000

a[14,1] =  4734200384802439149870797684768889301308307444115977\
946571986362505166893988770263031 9/448025468739260507304012226366568557608024 19\
9938520602646153208014853924560546875000 00-\
866369530987077991125562402829092187100493209601/
33255223758736721560177114591736739349365234375 00*6^(1/2)
a[14,2] =  0
a[14,3] =  0
a[14,4] =  0
a[14,5] =  0
a[14,6] =  871779321807802447463310035318238762878527157/
13444642082334949805906077371965057373046875 0+
10764126848099939608184897527184985799481 8/
10970827939185319041619359135523486816406 25*6^(1/2)
a[14,7] =
49610378635186229280003480511419070548480074351335411 7014/
11009972210395637571981961228299214810189023971557617 1875-\
13299384126061974857693125993903073511915408915993748 31099/
66059833262373825431891767369795288861134143829345703 1250*6^(1/2)
a[14,8] =  40774077277747636354598451708891165494123131383777 23\
52295386119893921751932859942664 71/15264290546248162101058985941588079518256741\
25537703173635794612571352470397949218750 0+
12376707585529685587508034326129888387149902994 3/
45109160999427625039037873196066032409667968750 0*6^(1/2)
a[14,9] =  -10522038608500556459828649038302068473735749030796 3\
72764961618751973793724796364606986664/389941742500542225403457400039738286223 5\
89282965337583519734091827155605550765991210937 5+
34654781239483119645022496113163687484019728384 04/
25603372472826418489926209025434727287292480468 75*6^(1/2)
a[14,10] =  -27843764471262693189365201135620670490328475323282\
82021947485162169389576952709433468710898 4/12257041066285164222002594300605593 9\
29434139193022166317802121412999357024704596261133984375+
57477430027199859868387311410547201669924149505529 2/
10493521512545691015422624899329692538921837886718 75*6^(1/2)
a[14,11] =  -\
34241134351848245624232809437676889009431930503529853032576417589898516/
56133478243586519811009850090242810076032300624399426827131652832031 25
a[14,12] =  -34320443758939323781023685680522865010338509105169\
99202088532705211633432793920547702800961532438008401883737341854688972639605 33\
46001639386102688557057427640726 09/11431741063416822609716476904105672921439261\
98650927778920823267461111371275907599801714870165813394147519068210931766844 49\
49946165802584355181814345751955 12500000
a[14,13] =  47469308760239193350794516127267176492182641999 84/
18592065538407049755200144388134089346432755594877

a[15,1] =  -25188329249258825443748527038142409879923012133738 9\
85313265430932280250855708601/11370641325574469312056961874077298550827642308 77\
46473169957170363475580642862 50+
12342730589818601701795925985355086313430825355498819 56/
21056337714696287445183906429685521440698988458958081 25*6^(1/2)
a[15,2] =  0
a[15,3] =  0
a[15,4] =  0
a[15,5] =  0
a[15,6] =  -54821142119685055562477216205428613949905430396088/
39594398370094612890855877467480979473931012780 95-\
15112767538259828560728914695044712566649759250 00/
40386286337496505148672995016830599063409633036569*6^(1/2)
a[15,7] =  -\
60922424274061599918603524049390657305431262635197540405697952/
64848617474890321697745846247599531485315640324174619095168 75+
84558575751635978733109961893984238786929550462615375699341616/
19454585242467096509323753874279859445594692097252385728550625*6^(1/2)
a[15,8] =  -11611814757504516973322287583571995533433479819145 9\
87978212353488939046793510977 2/88106269019548352456722751312958708925037139575 1\
21706814533008149884176424931 25-\
17632472271169431002565608550507266162044036222141170 8/
28561940671982910748577120704204013346542014996455562 5*6^(1/2)
a[15,9] =  17769448722513898342276837490665097286927607247073 33\
56185669871434672949001830332 16/25512170081378896150563421460845618671224851635\
96619283719957742418751029506356875-\
19748368943709762722873481576568138101489320568798111296/
64845542623222590712865459359971291351118136871756506 25*6^(1/2)
a[15,10] =  97659266139124074818193264801929547781659926543786 3\
81510190954184218570746215033823993530000000/18560076654469706205963482908787 05\
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685081230820560312732685536096172760824279655110118208003359 9-\
852970846117821224749111313630789000588880252246079137450000 00/
692106594502013938431667467229540363263383556499153838517339 11*6^(1/2)
a[15,11] =
473389749049752963256114649231353822492912259509649519870869 750525/
354124408823603417997988424283654229412165081213226224792608 46291
a[15,12] =  333514392451584382480734940567841440978729127734 159\
045364007283876903345639683941147024141088075051581063851164687328534582028999 6\
674848871853154570655914289590314484876463 7/2316611025327287427714802011322252 8\
860907939049899006215923656276490975781021635721905022324254906067733123106655 9\
342498274574429937128559858829860608854337674205464481896 6
a[15,13] =  -\
387149926569584133897432527260168975992839116829452556366435546875000 00/
485404949269715874992945893825722120361691354298779017023475213004217 67
a[15,14] =
148002502009403237171246161756412612351192957957688147178039550781 25/
335655771251418777602873805886324212234331940781569482984884711604 89

a[16,1] =
230578569608639756108085818693989717364564133108504131394438984998658410128 7/
617508244345282265819087370078275122671246164669900462139876057008239440000 -\
854046233055897126321659052339741831376078991407 19/
124822287169084833758410283469525117460541643292500*6^(1/2)
a[16,2] =  0
a[16,3] =  0
a[16,4] =  0
a[16,5] =  0
a[16,6] =  102903996961580448264190625267026062654799259083/
504639808489000485748162999967332043881948473 0+
413209254873042193133002720521283745670811281 25/
514732604658780495463126259966678684759587442 46*6^(1/2)
a[16,7] =
627984433498764575067189208439756613999495645980184881444 66/
413255349878257332405826358255371522077705135978014138062 5-\
723088070819329615544257110897167710135714199506573007291 03/
123976604963477199721747907476611456623311540793404241418 75*6^(1/2)
a[16,8] =
179490914212648256439084852292422555322146901975147054495929761465466129337 7/
525964811939942644356016261097529886746796916442754567166339757859786725 00+
122006604722271018045951293191391690196582713058 17/
169315614565599591152077093440565782633977606025 00*6^(1/2)
a[16,9] =  -27752447327801096673428456123947393191156626363714 7\
730045574702242327047590725 6/
228417153675584029725018045422706955827996328208181619436454383447149337555625 +
341618493222358850528663620935896732550431596562876/
961013383787733574692452119549115054475510972056 25*6^(1/2)
a[16,10] =  -27680554659769016623530979176727448251292244310769 \
996015342190819068970556083063125000/329955777742964896057656138225660684467725\
843879707295534158135405137503652223147143 7+
442655212757989537347967035610017975994476655814173031250 0/
307711373866732070774887719980463674649497700065896798767 7*6^(1/2)
a[16,11] =  -\
292603171929706291053929402159930330736639136252680853622275/
154736228262791611502270768872902624435105509642758581439 64
a[16,12] =  -98157171295691069885693021932209993438249320845820\
936475960869317546660986625941530952589885163051657947397448735398290696172035 2\
350913668221693302043 1/2864769911709341530761466410944021718019372500685965429 3\
102867866950176225328769329439768932779738811385458811343006393940507197909254 7\
998950955940992
a[16,13] =
272949114470983790579914876665078253290605029897140651852416992187 5/
215811588862213947314277581210944780292065614924312730925368695146 9
a[16,14] =  0
a[16,15] =  0

b[1] =  819816036620317341111994371150033 1/
561057579384085860167277847128765528
b[2] =  0
b[3] =  0
b[4] =  0
b[5] =  0
b[6] =  0
b[7] =  0
b[8] =  -455655493073428838813281446213740000000/
116380801115091056124046422583731249786 9
b[9] =  199651636487060080811350757469156147200000 00/
```

```
863944041905370868683946862057824325165445 99
b[10] =  892311079199814187055669708043437500000000000000000000000/
699979870988335674445594679856445060562597693583175985391
b[11] =  47104273954945906713184913871143492/
209684639122339601934631113492763467
b[12] =  208450044214045004640105847407967506508321767983703830842263\
51294730731196673647311062330972740734737279503119387627146381678677156136042 52\
41393119074828028440 83/366708498911363730202382253282651002506051447185019263 05\
14096658675805484760468146633610316928475598775354232120246237155412059385814 97\
55539878561976786592389608
b[13] =  605303728214230650979591128690917968750000000 0/
10389925735051806345529007757377516273972512698 9
b[14] =  917401104920993498360358406096725463867187500/
67242498159113466533157907374536073829895514 63
b[15] =  258544955766526895137169959649395 7/
84574345160764140163208606048427531
b[16] =  0

bh[1] =  552562031208180939317806684253/
27669654257734667858523344041464
bh[2] =  0
bh[3] =  0
bh[4] =  0
bh[5] =  0
bh[6] =  0
bh[7] =  0
bh[8] =  221223388631423597589898601690000000/
10094613679858709005468507466712 7461
bh[9] =  101835408791305297984657812561920000000/
11497638332007437599765066502413121001 39
bh[10] =  131372030907763001445323984375000000000000000000/
11518201923215510989126466531107437037395719117133
bh[11] =  48336112327014405045080861517 28/
19081321241454145230196661524503
bh[12] =  -212966237458232464810691979570337364535311827306674223072 4\
17273102581396471247364714401059920666982538271935911319623885770902551234058 99\
57/10355437392723670808851905462010972188912687281182073325925959875548518829 72\
29267088179417838009771658312306348528743579365742588923308056 8
bh[13] =  10847615917536408558443580639648437500000 00/
31828954860312490719385496913205024887334 23
bh[14] =  0
bh[15] =  0
bh[16] =  18391900710606498871278951007 84/
38045139523510634351420875415397
```

Runge-Kutta-Fehlberg methods

Similar to Verner methods, Fehlberg method is also given a pair of solutions. Later, it will be used in variable step calculations, but it can also utilised as single step solution of a good accuracy. Runge-Kutta-Fehlberg Buthcher tableu is given as:

0	0	0	0	0	0	0
1/4	1/4	0	0	0	0	0
3/8	3/32	9/32	0	0	0	0
12/13	1932/2197	−7200/2197	7296/2197	0	0	0
1	439/2197	−8	3680/513	−845/4104	0	0
1/2	−8/27	2	−3544/2565	1859/4104	−11/40	0
	25/216	0	1408/2565	2197/4104	−1/5	0
	16/135	0	6656/12825	28561/56430	−9/50	2/55

Table The same coefficients in table format :

```
c[1] =  0
c[2] =  1/4
c[3] =  3/8
c[4] =  12/13
c[5] =  1
c[6] = 1/2


a[2,1] =  1/4
```

a[3,1] = 3/32
a[3,2] = 9/32

a[4,1] = 1932/2197
a[4,2] = -7200/2197
a[4,3] = 7296/2197

a[5,1] = 439/216
a[5,2] = -8
a[5,3] = 3680/513
a[5,4] = -845/4104

a[6,1] = -8/27
a[6,2] = 2
a[6,3] = -3544/2565
a[6,4] = 1859/4104
a[6,5] = -11/40

b[1] = 25/216
b[2] = 0
b[3] = 1408/2565
b[4] = 2197/4104
b[5] = -1/5
b[5] = 0

bh[1] = 16/135
bh[2] = 0
bh[3] = 6656/12825
bh[4] = 28561/56430
bh[5] = -9/50
bh[5] = 2/55

6th degree Runge-Kutta- Cash-Karp coefficients are as below:

Runge-Kutta-Cash-Karp method(RKCK)

Runge-Kutta-Cash-Karp is a modified form of the Runge-Kutta_fehlberg method (different coefficients). Similar to RKF, this method also have a pair of solutions

0	0	0	0	0	0	0
1/5	1/5	0	0	0	0	0
3/10	3/40	9/40	0	0	0	0
3/5	3/10	−9/10	6/5	0	0	0
1	−11/54	5/2	−70/27	35/27	0	0
7/8	1631/55296	175/512	575/13824	44275/110592	253/4096	0
	37/378	0	250/621	125/594	0	512/1771
	2825/27648	0	18575/48384	13525/55296	277/14336	1/4

The same coefficients is also given in table form below

Table 6th degree Runge-Cash-Karp coefficients

c[1] = 0
c[2] = 0.2
c[3] = 0.3
c[4] = 0.6
c[5] = 1
c[6] = 7/8

a[2,1] = 0.2

a[3,1] = 3/40
a[3,2] = 9/40

a[4,1] = 0.3
a[4,2] = -9/10
a[4,3] = 6/5

a[5,1] = -11/54
a[5,2] = 5/2
a[5,3] = -70/27
a[5,4] = 35/27

```
a[6,1] =  1631/55296
a[6,2] =  175/512
a[6,3] =  575/13824
a[6,4] =  44275/110592
a[6,5] =  253/4096


b[1] =  37/378
b[2] =  250/621
b[3] =  125/591
b[4] =  512/1771
b[5] =  0

bh[1] =  2825/27648
bh[2] =  18575/48384
bh[3] =  13525/55296
bh[4] =  277/14336
bh[5] =  1/4
```

Table 7th degree Runge-Kutta-Dormant-Prince coefficients

```
    c[1]=0.0e0;
    c[2]=0.2e0;
    c[3]=0.3e0;
    c[4]=0.8e0;
    c[5]=8.0e0/9.0e0;
    c[6]=1;
    c[7]=1;
    a[2][1]=0.2e0;
    a[3][1]=3.0e0/40.0e0;
    a[3][2]=9.0e0/40.0e0;
    a[4][1]=44.0e0/45.0e0;
    a[4][2]=-56.0e0/15.0e0;
    a[4][3]=32.0e0/9.0e0;
    a[5][1]=19372.0e0/6561.0e0;
    a[5][2]=-25360.0e0/2187.0e0;
    a[5][3]=64448.0e0/6561.0e0;
    a[5][4]=-212.0e0/729.0e0;
    a[6][1]=9017.0e0/3168.0e0;
    a[6][2]=-355.0e0/33.0e0;
    a[6][3]=46732.0e0/5247.0e0;
    a[6][4]=49.0e0/176.0e0;
    a[6][5]=-5103.0e0/18656.0e0;
    a[7][1]=35.0e0/384.0e0;
    a[7][3]=500.0e0/1113.0e0;
    a[7][4]=125.0e0/192.0e0;
    a[7][5]=-2187.0e0/6784.0e0;
    a[7][6]=11.0e0/84.0e0;

    b[1]=71.0e0/57600.0e0;
    b[3]=-71.0e0/16695.0e0;
    b[4]=71.0e0/1920.0e0;
    b[5]=-17253.0e0/339200.0e0;
    b[6]=22.0e0/525.0e0;
    b[7]=0.0e0;
    bh[1]=-12715105075.0e0/11282082432.0e0;
    bh[3]=87487479700.0e0/32700410799.0e0;
    bh[4]=-10690763975.0e0/1880347072.0e0;
    bh[5]=701980252875.0e0/199316789632.0e0;
    bh[6]=-1453857185.0e0/822651844.0e0;
    bh[7]=69997945.0e0/29380423.0e0;
```

Program of Runge-Kutta is given in general form. For coefficients of different formulations a Butcher class is created and then a general Runge_Kutta method reads coefficients from this class.

Program Butcher coefficients for Runge_Kutta class

```
 public class Buthcher
{public double a[][];
 public double b[][];
 public double c[];
```

```
 public Buthcher(double ai[][],double bi[][],double ci[])
 {set(ai,bi,ci); }

 public void set(double ai[][],double bi[][],double ci[])
 {a=ai;b=bi;c=ci;}

 public Buthcher(String isim)
 {if(isim.equals("RK2"))
   {double a2[][]={{0,0},{1,0}};
   double b2[][]={{0.5,0.5}};
   double c2[]={0,1};
   set(a2,b2,c2);
   }
  else if(isim.equals("Raltston"))
  {double a2[][]={{0,0},{3.0/4.0,0}};
   double b2[][]={{1.0/3.0,2.0/3.0}};
   double c2[]={1.0/6.0,4.0/6.0,1.0/6.0};
   set(a2,b2,c2);
  }
  else if(isim.equals("RK3"))
  {double a2[][]={{0,0,0},{0.5,0,0},{-1,2,0}};
   double b2[][]={{0,0.5,1}};
   double c2[]={1.0/6.0,4.0/6.0,1.0/6.0};
   set(a2,b2,c2);
  }
  else if(isim.equals("RK4"))
  {double a2[][]={{0,0,0,0},{0.5,0,0,0},{0,0.5,0,0},{0,0,1,0}};
   double b2[][]={{1.0/6.0,2.0/6.0,2.0/6.0,1.0/6.0}};
   double c2[]={0,0.5,0.5,1};
   set(a2,b2,c2);
  }
  else if(isim.equals("RK6"))
  {double a2[][]={{0,0,0,0,0,0},{0.25,0,0,0,0,0},{0.125,0.125,0,0,0,0},{0,-0.5,1,0,0,0},{3.0/16.0,0,0,9.0/16.0,0,0},{-
3.0/7.0,2.0/7.0,12.0/7.0,-12.0/7.0,8.0/7.0,0}};
   double b2[][]={{7.0/90.0,0,32.0/90.0,12.0/90.0,32.0/90.0,7.0/90.0}};
   double c2[]={0,0.25,0.25,0.5,0.75,1.0};
   set(a2,b2,c2);
  }
  else if(isim.equals("RKV9"))
  { //Runge-Kutta-Verner 9uncu derece sabit adımlı metodu
    int s=10;
    double c[]=new double[s];
    double a[][]=new double[s][s];
    double bh[]=new double[s];
    double bh1[]=new double[s];
c[0]=0;
c[1] =  0.;
c[2] =  .6e-1;
c[3] =  .95933333333333333333333333333333333333333333e-1;
c[4] =  .1439;
c[5] =  .4973;
c[6] =  .9725;
c[7] =  .9995;
c[8] =  1.;
c[9] =  1.;
a[2][1] =  .6e-1;
a[3][1] =  .19239962962962
```

```
a[8][1] = -54.663741787281976802412156480503869593 51;
a[8][2] =  0.;
a[8][3] =  207.952806255389373451582481669983424423 8;
a[8][4] = -159.288957474499507150895980587142665421 6;
a[8][5] =  7.018743740796944434698170760964252490817;
a[8][6] = -.183387859050457230647278200514173826836 1e-1;
a[8][7] = -.511948499788209907787543249724516839584 0e-3;

a[9][1] =  .343895786835703600927882012472832238652 0e-1;
a[9][2] =  0.;
a[9][3] =  0.;
a[9][4] =  .258262455563350340465955809858612085876 7;
a[9][5] =  .420937118967353715064255151406980196703 2;
a[9][6] =  4.405396469669310170148836816197095664891;
a[9][7] = -176.483119024298657615174094249900212502 9;
a[9][8] =  172.364133401415073029402258271190241331 5;

bh1[1] =  .343895786835703600927882012472832238652 0e-1;
bh1[2] =  0.;
bh1[3] =  0.;
bh1[4] =  .258262455563350340465955809858612085876 7;
bh1[5] =  .420937118967353715064255151406980196703 2;
bh1[6] =  4.405396469669310170148836816197095664891;
bh1[7] = -176.483119024298657615174094249900212502 9;
bh1[8] =  172.364133401415073029402258271190241331 5;
bh1[9] =  0.;

bh[ 1] =  .490996764838248973090685492797122583647 9e-1;
bh[ 2] =  0.;
bh[ 3] =  0.;
bh[ 4] =  .225111222951652415340139532053987532948 5;
bh[ 5] =  .469468225302956203943194852504738741255 3;
bh[ 6] =  .806579224998886770763416180899521798144 3;
bh[ 7] =  0.;
bh[ 8] = -.607119489177795979767295146525621712248 8;
bh[ 9] =  .568611394404756924114760317876613815359 4e-1;
double b[][]={bh1,bh};
set(a,b,c);
  }
  else if(isim.equals("RKCK45"))
  { // Runge-Kutta-Fehlberg, Cash-Karp, Dormant-Price, Verner
  int s=7;
double dydx=1.0;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh1[]=new double[s];
double bh2[]=new double[s];
c[2] =  0.2;
c[3] =  0.3;
c[4] =  0.6;
c[5] =  1;
c[6] = 7.0/8.0;

a[1][1] = 0;

a[2][1] =  0.2;

a[3][1] =  3.0/40.0;
a[3][2] =  9.0/40.0;

a[4][1] =  0.3;
a[4][2] =  -9.0/10.0;
a[4][3] =  6.0/5.0;

a[5][1] =  -11.0/54.0;
a[5][2] =  0.25;
a[5][3] =  -70.0/27.0;
a[5][4] =  35.0/27.0;

bh1[1] =  37.0/378.0;
bh1[2] =  250.0/621.0;
bh1[3] =  125.0/591.0;
bh1[4] =  512.0/1771.0;
bh1[5] =  0.0;
```

```
bh2[1] =  2825.0/27648.0;
bh2[2] =  18575.0/48384.0;
bh2[3] =  13525.0/55296.0;
bh2[4] =  277.0/14336.0;
bh2[5] =  1.0/4.0;
double b[][]={bh1,bh2};
set(a,b,c);
  }
  else if(isim.equals("RKDP45"))
  {// Runge-Kutta Dormant-Prince order 4-5
  int s=8;
double dydx=1.0;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh1[]=new double[s];
double bh2[]=new double[s];
c[1]=0.0e0;
      c[2]=0.2;
      c[3]=0.3;
      c[4]=0.8;
      c[5]=8.0/9.0;
      c[6]=1;
      c[7]=1;
      a[2][1]=0.2;
      a[3][1]=3.0/40.0;
      a[3][2]=9.0/40.0;
      a[4][1]=44.0/45.0;
      a[4][2]=-56.0/15.0;
      a[4][3]=32.0/9.0;
      a[5][1]=19372.0/6561.0;
      a[5][2]=-25360.0/2187.0;
      a[5][3]=64448.0/6561.0;
      a[5][4]=-212.0/729.0;
      a[6][1]=9017.0/3168.0;
      a[6][2]=-355.0/33.0;
      a[6][3]=46732.0/5247.0;
      a[6][4]=49.0/176.0;
      a[6][5]=-5103.0/18656.0;
      a[7][1]=35.0/384.0;
      a[7][3]=500.0/1113.0;
      a[7][4]=125.0/192.0;
      a[7][5]=-2187.0/6784.0;
      a[7][6]=11.0/84.0;
      bh1[1]=5179.0/57600.0;
      bh1[2]=0;
      bh1[3]=7571.0/16695.0;
      bh1[4]=393.0/640.0;
      bh1[5]=-92097.0/339200.0;
      bh1[6]=187.0/2100.0;
      bh1[7]=1.0/40.0;
      bh2[1]=35.0/384.0;
      bh2[2]=0;
      bh2[3]=500.0/1113.0;
      bh2[4]=125.0/192.0;
      bh2[5]=-2187.0/6784.0;
      bh2[6]=11.0/84.0;
      bh2[7]=0.0;
      double b[][]={bh1,bh2};
      set(a,b,c);
  }
  else if(isim.equals("RKV13"))
  { //Runge-Kutta-Verner 13üncü derece sabit adımlı metodu
int s=14;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
c[0]=0;
c[1] =  0.;
c[2] =  .5e-1;
a[2][1] =  .5e-1;
c[3] =  .1065625;
a[3][1] = -.69931640625e-2;
a[3][2] =  .1135556640625;
c[4] =  .15984375;
a[4][1] =  .399609375e-1;
a[4][2] =  0.;
```

```
a[4][3] =  .1198828125;
c[5] =  .39;
a[5][1] =  .36139756280045751240529407211840283451 29;
a[5][2] =  0.;
a[5][3] = -1.34152406670049277181998778820271583491 7;
a[5][4] =  1.37012650390003525941469371608431300040 4;
c[6] =  .465;
a[6][1] =  .49047202797202797202797202797202797202 80e-1;
a[6][2] =  0.;
a[6][3] =  0.;
a[6][4] =  .23509720422144047398629883354934271431 22;
a[6][5] =  .18085559298135672881090396365345448848 50;
//
c[7] =  .155;
a[7][1] =  .61692890442890442890442890442890442890 44e-1;
a[7][2] =  0.;
a[7][3] =  0.;
a[7][4] =  .11236568314640276622625570351300154423 03;
a[7][5] = -.38850460714513667670490481081112445674 56e-1;
a[7][6] =  .19791887125220458553791887125220458553 79e-1;
c[8] =  .943;
a[8][1] = -1.76763024022232687573559711957214558671 4;
a[8][2] =  0.;
a[8][3] =  0.;
a[8][4] = -62.5;
a[8][5] = -6.06188937737666910082136145965933199975 8;
a[8][6] =  5.65082319822276313856129803060084017420 1;
a[8][7] =  65.62169641937623283799566054863063741227;
c[9] =  .90180204173585695825970790406783721499560;
a[9][1] = -1.18094506655497079982511628262828297957882;
a[9][2] =  0.;
a[9][3] =  0.;
a[9][4] = -41.50473441114320841606641502701994225874;
a[9][5] = -4.43443831910372501122516922984610021177 6;
a[9][6] =  4.26040818858613302481219371074469324076 1;
a[9][7] =  43.75364022446171584987676829438379303004;
a[9][8] =  .78714254899123106874464750442263075508 60e-2;
c[10] =  .909;
a[10][1] = -1.28140599944148840545951029118205424626 6;
a[10][2] =  0.;
a[10][3] =  0.;
a[10][4] = -45.04713996013986630220754257136007322267;
a[10][5] = -4.73136206944957647731146426549128281094 3;
a[10][6] =  4.51496701659380784118585158459724099621 4;
a[10][7] =  47.44909557172985134869022392235929015114;
a[10][8] =  .10592282971116611356873939555165428752 28e-1;
a[10][9] = -.57468422638446162544323184782862962320 21e-2;
c[11] =  .94;
a[11][1] = -1.72447013426248519175670981748448186173 1;
a[11][2] =  0.;
a[11][3] =  0.;
a[11][4] = -60.92349008483054016518434619253765246063;
a[11][5] = -5.95151837622239245520283276706185486829 0;
a[11][6] =  5.55652373069845623597979165084359249683 9;
a[11][7] =  63.98301198033305336837536378635995939281;
a[11][8] =  .14642028250414961592759213917594526760 03e-1;
a[11][9] =  .64604087723582036036218651449776507148 92e-1;
a[11][10] = -.79303231690088789840244525486933732914 47e-1;
c[12] =  1.;
a[12][1] = -3.30162266774707901635399478979098362556 9;
a[12][2] =  0.;
a[12][3] =  0.;
a[12][4] = -118.01127235975250856669233039578988685 10;
a[12][5] = -10.14142238845611248642783916034510897595;
a[12][6] =  9.13931133223205792354401227355682700061 9;
a[12][7] =  123.37594282840426836848471809865018943 64;
a[12][8] =  4.62324437887458047483980762506763092479 2;
a[12][9] = -3.38327773806820192365255097153681124081 4;
a[12][10] =  4.52759210032461818945126533935112903532 5;
a[12][11] = -5.82849548581162296319308801916298570375 5;
c[13] =  1.;
a[13][1] = -3.03951503376630903004010285182120025105 6;
a[13][2] =  0.;
a[13][3] =  0.;
a[13][4] = -109.26086808941762546864441923221646233 52;
a[13][5] = -9.29064249740029344971766554265689754915 8;
```

```
a[13][6] =  8.43050498176491114213429925383616780345 4;
a[13][7] =  114.20100103783313135574240410955234274 76;
a[13][8] = -.96372713421454793581623756589879016527 62;
a[13][9] = -5.03488408880218979119868033618333232311 8;
a[13][10] =  5.95813082400292317754040216538817207279 4;
a[13][11] =  0.;
a[13][12] =  0.;
bh[1] =  .44312615229089792124864365102090297648 93e-1;
bh[2] =  0.;
bh[3] =  0.;
bh[4] =  0.;
bh[5] =  0.;
bh[6] =  .35460956423432264478631793508950550388 55;
bh[7] =  .24784804313666530696199867215044586600 16;
bh[8] =  4.44813473247578449272512831715964887131 2;
bh[9] =  19.84688636611873369930932399297687935291;
bh[10] = -23.58162337746561841969517960870394965085;
bh[11] =  0.;
bh[12] =  0.;
bh[13] = -.36016794372897751621245367377462024091 10;
double b[][]={bh};
set(a,b,c);
  }
else if(isim.equals("RKF6"))
  { //Runge-Kutta-Fehlberg 6ıncı derece sabit adımlı metodu
int s=7;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
c[1] =  1;
c[2] =  0.2;
c[3] =  0.3;
c[4] =  0.6;
c[5] =  1;
c[5] = 7.0/8.0;

a[1][1] =  1;

a[2][1] =  0.2;

a[3][1] =  3.0/40.0;
a[3][2] =  9.0/40.0;

a[4][1] =  0.3;
a[4][2] =  -9.0/10.0;
a[4][3] =  6.0/5.0;

a[5][1] =  -11.0/54.0;
a[5][2] =  0.25;
a[5][3] =  -70.0/27.0;
a[5][4] =  35.0/27.0;

bh[1] =  2825.0/27648.0;
bh[2] =  18575.0/48384.0;
bh[3] =  13525.0/55296.0;
bh[4] =  277.0/14336.0;
bh[5] =  1.0/4.0;
double b[][]={bh};
set(a,b,c);
  }

}
}
```

Program Runge_Kutta class

```
import java.io.*;
import java.text.*;
import java.util.Locale;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}
```

```
abstract class f_xi
{
  // çok fonksiyonlu ve çok denklemli fonksiyon
  // denklemlerin sadece bir tanesi denklem_ref
  // tarafından seçilir.
  // örnek   f[0]=x[0]+sin(x[1])
  //         f[1]=x[0]*x[0]-x[1]
  // func(x,1)  f[1] değerini verir
  // func(x,0)  f[0] değerini verir

  abstract double func(double x[],int denklem_ref);
}
public class Runge_Kutta
{
double a[][];
double b[][];
double c[];

public Runge_Kutta(double ai[][],double bi[][],double ci[])
{a=ai;b=bi;c=ci;}

public Runge_Kutta(Buthcher bu)
{a=bu.a;b=bu.b;c=bu.c;}
public Runge_Kutta(String isim)
{Buthcher bu=new Buthcher(isim);
 a=bu.a;b=bu.b;c=bu.c;
}
public double[][] hesapla(f_xy f,double x0,double xn,double y0,double h)
{return hesapla(f,x0,xn,y0,h,0);}

public double[][] hesapla(f_xy f,double x0,double xn,double y0,double h,int bk)
{
  int n=(int)((xn-x0)/h);
  double x[]=new double[n+1];
  double y[]=new double[n+1];
  int s=c.length;
  double xi;
  double yi;
  double ki[]=new double[s];
  x[0]=x0;y[0]=y0;
  for(int ni=0;ni<n;ni++)
  {for(int i=0;i<s;i++)
   { xi=x[ni]+c[i]*h;
     yi=y[ni];
    for(int j=0;j<i;j++)
    {yi+=h*a[i][j]*ki[j];//System.out.println("xi(0)="+xi+"xi(j)="+yi+"i="+i+"j="+j+"ki="+ki[j]);
    }
    ki[i]=f.func(xi,yi);
    //System.out.println("ki="+ki[i]);
   }
   y[ni+1]=y[ni];
   for(int i=0;i<s;i++)
    {y[ni+1]+=h*b[bk][i]*ki[i];}
   x[ni+1]=x[ni]+h;
  }
  double XY[][]=new double[2][n+1];
  XY[0]=x;
  XY[1]=y;
  return XY;
}
public static double[] tasi(double x1,double y1[])
{
int n=y1.length;
double z[]=new double[n+1];
z[0]=x1;
for(int i=1;i<=n;i++)
{z[i]=y1[i-1];}
return z;
}
//çok boyutlu RK girişler N(toplam veri sayısı) veya h(step boyu) üzeriinden verilebilmektedir.
public double[][] hesapla(f_xi f,double x0,double xn,double y0[],double h)
{return hesapla(f,x0,xn,y0,h,0);}
public double[][] hesapla(f_xi f,double x0,double xn,double y0[],int ni)
{double h=(xn-x0)/ni;
return hesapla(f,x0,xn,y0,h,0);
}
```

```
public double[][] hesapla(f_xi f,double x0,double xn,double y0[],int ni,int bk)
{double h=(xn-x0)/ni;
return hesapla(f,x0,xn,y0,h,bk);
}
public double[][] hesapla(f_xi f,double x0,double xn,double y0[],double h,int bk)
{
  int n=(int)((xn-x0)/h);
  int m=y0.length;
  double x[]=new double[n+1];
  double y[][]=new double[n+1][m];
  int s=c.length;
  double xi;
  double yi;
  double z[]=new double[m+1];
  double ki[]=new double[s];

  x[0]=x0;
  for(int ni=0;ni<=n;ni++)
  {x[ni]=x0+ni*h;}
  y[0]=y0;
  for(int ni=0;ni<n;ni++)
  { for(int k=0;k<m;k++)
                {
     for(int i=0;i<s;i++)
     { xi=x[ni]+c[i]*h;
       z=tasi(xi,y[ni]);
       for(int j=0;j<i;j++)
          {z[k+1]+=h*a[i][j]*ki[j];//System.out.println("xi(0)="+z[0]+"xi(j)="+z[k+1]+"i="+i+"j="+j+"ki="+ki[j]);
          }
       ki[i]=f.func(z,k);
       //System.out.println("ki="+ki[i]);
     }
   y[ni+1][k]=y[ni][k];
   for(int i=0;i<s;i++)
    {y[ni+1][k]+=h*b[bk][i]*ki[i];}
  }
  }
  double XY[][]=new double[m+1][n+1];
  XY[0]=x;
  for(int k=0;k<m;k++)
   { for(int ni=0;ni<=n;ni++)
     {XY[k+1][ni]=y[ni][k];}
   }
  return XY;
}

public static String toString(double left, int w, int d)
// converts a double to a string with given width and decimals.
{
                NumberFormat df=NumberFormat.getInstance(Locale.US);
                df.setMaximumFractionDigits(d);
    df.setMinimumFractionDigits(d);
    df.setGroupingUsed(false);
    String s = df.format(left);
    while (s.length() < w)
      s = " " + s;
    if (s.length() > w)
    {
       s = "";
       for (int i=0; i<w; i++)
         s = s + "-";
    }
    return s;
}
public String toString()
 {int m=b[0].length;
  int n=c.length;
  int k=b.length;
  String s="Buthcher tablosu n = "+n+"\n";
  for(int i=0;i<n;i++)
  {s+=toString(c[i],15,8)+" | ";
  for(int j=0;j<i;j++)
  {s+=toString(a[i][j],15,8)+" ";}
  s+="\n";
  }
  for(int j=0;j<m;j++)
```

```
  {s+="------------------";}
  s+="\n                 ";
  for(int i=0;i<k;i++)
  {
  for(int j=0;j<m;j++)
  {s+=toString(b[i][j],15,8)+" ";}
   s+="\n                 ";
  }
   return s;
 }

}
```

Now, we can call desired Runge-Kutta formula as a small program

Program second degree Runge-Kutta (Ralston) equation by using class Runge_Kutta

```
 import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{ //diferansiyeli alınan fonksiyon
  public double func(double x,double y)
  {return x*y;}
}

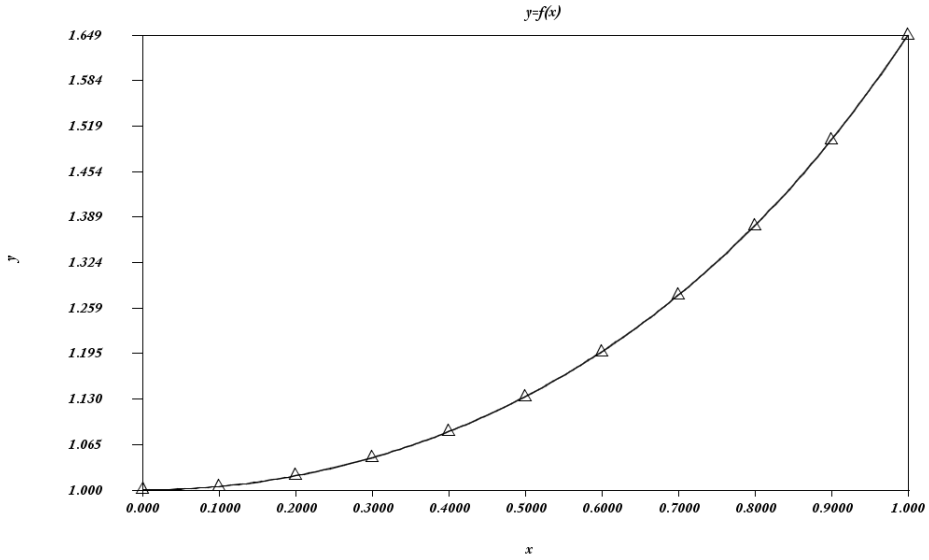
class fxyKC extends f_x
{ //Gerçek çözüm fonksiyonu
  public double func(double x)
  {return Math.exp(x*x/2.0);}
}

class dif1
{
  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   double xn=1.0;
   fxy f1=new fxy();
   fxyKC f2=new fxyKC();
   Runge_Kutta rk=new Runge_Kutta("RK2");
   System.out.println(rk.toString());
   double XY[][]=rk.hesapla(f1,x0,xn,y0,0.1);
   System.out.println(Matrix.toString(Matrix.Transpose(XY)));
   Plot p1=new Plot(XY);
   p1.addFunction(f2,x0,xn,200);
   p1.setPlotType(0,24);
   p1.plot();
  }
 }
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" dif1
Buthcher table n = 2
    0.00000000 |
    1.00000000 |     1.00000000
---------------------------------------
                   0.50000000     0.50000000

       0.000000000000000       1.000000000000000
       0.100000000000000       1.005000000000000
       0.200000000000000       1.020175500000000
       0.300000000000000       1.045985940150000
       0.400000000000000       1.083223039619340
       0.500000000000000       1.133051299441830
       0.600000000000000       1.197068697860293
       0.700000000000000       1.277392007486719
       0.800000000000000       1.376773105669185
```

0.900000000000000	1.498755202831475
1.000000000000000	1.647881345513207



The same programming can be done by defining the coefficients inside the specific Runge-Kutta program

Program second degree Runge-Kutta (Ralston) equation

```
 //=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// A second degree Runge-Kutta RK2 (Ralston equation)
// Dr. Turhan Coban
// =====================================================
//  dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{
  double func(double x,double y)
  {return x*y;}
}

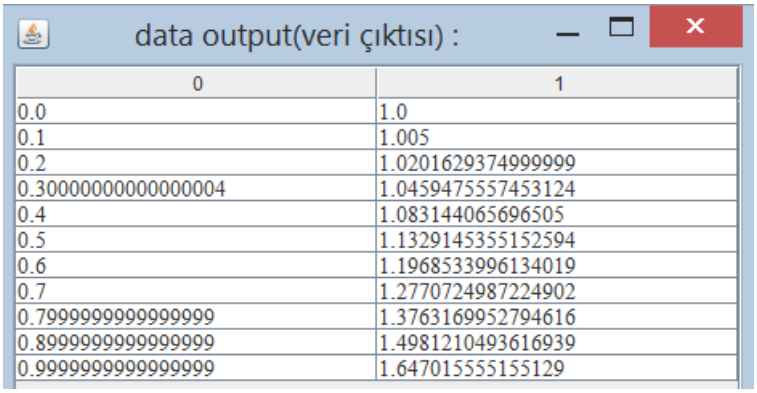
class fxyKC extends f_x
{
  public double func(double x)
  {return Math.exp(x*x/2.0);}
}

class NA85
{


  public static double[][] RK2(f_xy f,double x0,double y0,double xn,double h)
  {
                //Ralston 2/3
  int n=(int)((xn-x0)/h);
  double a[][]=new double[2][n+1];
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
```

```
  double k1,k2;
  double xx=x0;
  for(i=0;i<n;i++)
   {a[0][i+1]=xx+h;
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i]+3.0/4.0*h,a[1][i]+3.0/4.0*k1*h);
   a[1][i+1]=a[1][i]+(1.0/3.0*k1+2.0/3.0*k2)*h;
   xx+=h;
   }
  return a;
  }


  public static void main(String args[]) throws IOException
  {double x0=0;
   double y0=1.0;
   fxy f1=new fxy();
   fxyKC fKC=new fxyKC();
   double a[][]=RK2(f1,x0,y0,1.0,0.1);
   Text.print(Text.T(a));
  }
}
```



0	1
0.0	1.0
0.1	1.005
0.2	1.0201629374999999
0.30000000000000004	1.0459475557453124
0.4	1.083144065696505
0.5	1.1329145355152594
0.6	1.1968533996134019
0.7	1.2770724987224902
0.7999999999999999	1.3763169952794616
0.8999999999999999	1.4981210493616939
0.9999999999999999	1.647015555155129

Program Fourth degree Runge-Kutta Equation

```
 import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{ //diferansiyeli alınan fonksiyon
  public double func(double x,double y)
  {return x*y;}
}

class fxyKC extends f_x
{ //Gerçek çözüm fonksiyonu
  public double func(double x)
  {return Math.exp(x*x/2.0);}
}

class dif1
{
  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   double xn=1.0;
   fxy f1=new fxy();
```

```
   fxyKC f2=new fxyKC();
   Runge_Kutta rk=new Runge_Kutta("RK4");
   System.out.println(rk.toString());
   double XY[][]=rk.hesapla(f1,x0,xn,y0,0.1);
   System.out.println(Matrix.toString(Matrix.Transpose(XY)));
   Plot p1=new Plot(XY);
   p1.addFunction(f2,x0,xn,200);
   p1.setPlotType(0,24);
   p1.plot();
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" dif1
Buthcher table n = 4
    0.00000000 |
    0.50000000 |     0.50000000
    0.50000000 |     0.00000000     0.50000000
    1.00000000 |     0.00000000     0.00000000     1.00000000
-------------------------------------------------------------------------------
                  0.16666667     0.33333333     0.33333333     0.16666667

       0.000000000000000       1.000000000000000
       0.100000000000000       1.005012520833333
       0.200000000000000       1.020201339758369
       0.300000000000000       1.046027858885970
       0.400000000000000       1.083287064820495
       0.500000000000000       1.133148446117537
       0.600000000000000       1.197217347412631
       0.700000000000000       1.277621279469110
       0.800000000000000       1.377127694901942
       0.900000000000000       1.499302362448324
       1.000000000000000       1.648721007053397

> Terminated with exit code 0.
```

Program Sixth order Runge-Kutta (RK6) equation

```
 import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{ //diferansiyeli alınan fonksiyon
  public double func(double x,double y)
  {return x*y;}
}

class fxyKC extends f_x
{ //Gerçek çözüm fonksiyonu
  public double func(double x)
  {return Math.exp(x*x/2.0);}
}

class dif1
{
  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   double xn=1.0;
   fxy f1=new fxy();
   fxyKC f2=new fxyKC();
   Runge_Kutta rk=new Runge_Kutta("RK6");
   System.out.println(rk.toString());
   double XY[][]=rk.hesapla(f1,x0,xn,y0,0.1);
   System.out.println(Matrix.toString(Matrix.Transpose(XY)));
```

```
   Plot p1=new Plot(XY);
   p1.addFunction(f2,x0,xn,200);
   p1.setPlotType(0,24);
   p1.plot();
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" dif1
Buthcher table n = 6
    0.00000000 |
    0.25000000 |     0.25000000
    0.25000000 |     0.12500000     0.12500000
    0.50000000 |     0.00000000    -0.50000000     1.00000000
    0.75000000 |     0.18750000     0.00000000     0.00000000     0.56250000
    1.00000000 |    -0.42857143     0.28571429     1.71428571    -1.71428571     1.14285714
-------------------------------------------------------------------------------------------------------------
                  0.07777778     0.00000000     0.35555556     0.13333333     0.35555556     0.07777778

      0.000000000000000      1.000000000000000
      0.100000000000000      1.005012521378270
      0.200000000000000      1.020201341079585
      0.300000000000000      1.046027861534459
      0.400000000000000      1.083287069933695
      0.500000000000000      1.133148456035439
      0.600000000000000      1.197217366887088
      0.700000000000000      1.277621317852590
      0.800000000000000      1.377127769933579
      0.900000000000000      1.499302506626659
      1.000000000000000      1.648721278178056
```

Butcher table and general Runge_Kutta method based on Butcher table and example case is is given below

Program Butcher table class (Python code)

```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 10:22:54 2018

@author: Mustafa Turhan Çoban
"""
from math import *

class  Buthcher():
   def set(self,ai,bi,ci):
       self.a=ai;
       self.b=bi;
       self.c=ci;
   def __init__(self,namei):
      self.name = namei;
      if str(self.name) ==("RK2"):
          a=[[0,0],[1,0]];
          b=[[0.5,0.5]];
          c=[0,1];
          self.set(a,b,c);
      elif str(self.name) == ("Raltston"):
          a=[[0,0],[3.0/4.0,0]];
          b=[[1.0/3.0,2.0/3.0]];
          c=[1.0/6.0,4.0/6.0,1.0/6.0];
          self.set(a,b,c);
      elif str(self.name) == ("RK3"):
          a=[[0,0,0],[0.5,0,0],[-1,2,0]];
          b=[[0,0.5,1]];
          c=[1.0/6.0,4.0/6.0,1.0/6.0];
          self.set(a,b,c);
      elif str(self.name) == ("RK4"):
          a=[[0,0,0,0],[0.5,0,0,0],[0,0.5,0,0],[0,0,1,0]];
          b=[[1.0/6.0,2.0/6.0,2.0/6.0,1.0/6.0]];
          c=[0,0.5,0.5,1];
          self.set(a,b,c);
      elif str(self.name) == ("RK6"):
          a=[[0,0,0,0,0,0],[0.25,0,0,0,0,0],[0.125,0.125,0,0,0,0],[0,-0.5,1,0,0,0],[3.0/16.0,0,0,9.0/16.0,0,0],[-3.0/7.0,2.0/7.0,12.0/7.0,-
12.0/7.0,8.0/7.0,0]];
          b=[[7.0/90.0,0,32.0/90.0,12.0/90.0,32.0/90.0,7.0/90.0]];
```

```
        c=[0,0.25,0.25,0.5,0.75,1.0];
        self.set(a,b,c);
    elif str(self.name) == ("RKV9"):
        #Runge-Kutta-Verner 9th degree formula
        n=10;
        c=[0.0 for i in range(n)];
        a = [[0 for x in range(n)] for y in range(n)];
        bh=[0.0 for i in range(n)];
        bh1=[0.0 for i in range(n)];
        c[0]=0;
        c[1] =  0.;
        c[2] =  .6e-1;
        c[3] =  .95933333333333333333333333333333333333333333e-1;
        c[4] =  .1439;
        c[5] =  .4973;
        c[6] =  .9725;
        c[7] =  .9995;
        c[8] =  1.;
        c[9] =  1.;
        a[2][1] =  .6e-1;
        a[3][1] =  .19239962962962962962962962962962962962962296e-1;
        a[3][2] =  .76693370370370370370370370370370370370370037e-1;
        a[4][1] =  .35975e-1;
        a[4][2] =  0.;
        a[4][3] =  .107925;
        a[5][1] =  1.3186834152331482609197472764317356128 61;
        a[5][2] =  0.;
        a[5][3] = -5.042058063628562225427761634715637693344;
        a[5][4] =  4.220674648395413964508014358283902080483;
        a[6][1] = -41.872591664327514618037577806443468129 05;
        a[6][2] =  0.;
        a[6][3] =  159.43256216313749177003656690703468304 53;
        a[6][4] = -122.11921356501003092025162033892421406 63;
        a[6][5] =  5.53174306620005376825263123833299915007 6;
        a[7][1] = -54.43015693531650433250642051294142461271;
        a[7][2] =  0.;
        a[7][3] =  207.06725136501846442736571738665098359 87;
        a[7][4] = -158.61081378458999918287424243650585994 69;
        a[7][5] =  6.99181658595024232199259728079179390709 6;
        a[7][6] = -.18597231062203233977651717995492946236 92e-1;
        a[8][1] = -54.66374178728197680241215648050386959351;
        a[8][2] =  0.;
        a[8][3] =  207.95280625538937345158248166998342442 38;
        a[8][4] = -159.28895747449950715089598058714266542 16;
        a[8][5] =  7.01874374079694443469817076096425249081 7;
        a[8][6] = -.18338785905045723064727820051417382683 61e-1;
        a[8][7] = -.51194849978820990778754324972451683958 40e-3;
        a[9][1] =  .34389578683570360092788201247283223865 20e-1;
        a[9][2] =  0.;
        a[9][3] =  0.;
        a[9][4] =  .25826245556335034046595580985861208587 67;
        a[9][5] =  .42093711896735715064255151406980196703 2;
        a[9][6] =  4.40539646966931017014883681619709566489 1;
        a[9][7] = -176.48311902429865761517409424990021250 29;
        a[9][8] =  172.36413340141507302940225827119024133 15;
        bh1[1] =  .34389578683570360092788201247283223865 20e-1;
        bh1[2] =  0.;
        bh1[3] =  0.;
        bh1[4] =  .25826245556335034046595580985861208587 67;
        bh1[5] =  .42093711896735715064255151406980196703 2;
        bh1[6] =  4.40539646966931017014883681619709566489 1;
        bh1[7] = -176.48311902429865761517409424990021250 29;
        bh1[8] =  172.36413340141507302940225827119024133 15;
        bh1[9] =  0.;
        bh[ 1] =  .49099676483824897309068549279712258364 79e-1;
        bh[ 2] =  0.;
        bh[ 3] =  0.;
        bh[ 4] =  .22511122295165241534013953205398753294 85;
        bh[ 5] =  .46946822530295620394319485250473874125 53;
        bh[ 6] =  .80657922499888677076341618089952179814 43;
        bh[ 7] =  0.;
        bh[ 8] = -.60711948917779597976729514652562171224 88;
        bh[ 9] =  .56861139440475692411476031787661381535 94e-1;
        b=[bh1,bh];
        self.set(a,b,c);
    elif str(self.name) == ("RKCK45"):
```

```
        #Runge-Kutta-Fehlberg, Cash-Karp, Dormant-Price, Verner
        n=7;
        dydx=1.0;
        c=[0.0 for i in range(n)];
        a = [[0 for x in range(n)] for y in range(n)];
        bh1=[0.0 for i in range(n)];
        bh2=[0.0 for i in range(n)];
        c[2] =  0.2;
        c[3] =  0.3;
        c[4] =  0.6;
        c[5] =  1;
        c[6] = 7.0/8.0;
        a[1][1] = 0;
        a[2][1] =  0.2;
        a[3][1] =  3.0/40.0;
        a[3][2] =  9.0/40.0;
        a[4][1] =  0.3;
        a[4][2] =  -9.0/10.0;
        a[4][3] =  6.0/5.0;
        a[5][1] =  -11.0/54.0;
        a[5][2] =  0.25;
        a[5][3] =  -70.0/27.0;
        a[5][4] =  35.0/27.0;
        bh1[1] =  37.0/378.0;
        bh1[2] =  250.0/621.0;
        bh1[3] =  125.0/591.0;
        bh1[4] =  512.0/1771.0;
        bh1[5] =  0.0;
        bh2[1] =  2825.0/27648.0;
        bh2[2] =  18575.0/48384.0;
        bh2[3] =  13525.0/55296.0;
        bh2[4] =  277.0/14336.0;
        bh2[5] =  1.0/4.0;
        b=[bh1,bh2];
        self.set(a,b,c);
    elif str(self.name) == ("RKDP45"):
        #Runge-Kutta Dormant-Prince order 4-5
        n=8;
        dydx=1.0;
        c=[0.0 for i in range(n)];
        a = [[0 for x in range(n)] for y in range(n)];
        bh1=[0.0 for i in range(n)];
        bh2=[0.0 for i in range(n)];
        c[1]=0.0e0;
        c[2]=0.2;
        c[3]=0.3;
        c[4]=0.8;
        c[5]=8.0/9.0;
        c[6]=1;
        c[7]=1;
        a[2][1]=0.2;
        a[3][1]=3.0/40.0;
        a[3][2]=9.0/40.0;
        a[4][1]=44.0/45.0;
        a[4][2]=-56.0/15.0;
        a[4][3]=32.0/9.0;
        a[5][1]=19372.0/6561.0;
        a[5][2]=-25360.0/2187.0;
        a[5][3]=64448.0/6561.0;
        a[5][4]=-212.0/729.0;
        a[6][1]=9017.0/3168.0;
        a[6][2]=-355.0/33.0;
        a[6][3]=46732.0/5247.0;
        a[6][4]=49.0/176.0;
        a[6][5]=-5103.0/18656.0;
        a[7][1]=35.0/384.0;
        a[7][3]=500.0/1113.0;
        a[7][4]=125.0/192.0;
        a[7][5]=-2187.0/6784.0;
        a[7][6]=11.0/84.0;
        bh1[1]=5179.0/57600.0;
        bh1[2]=0;
        bh1[3]=7571.0/16695.0;
        bh1[4]=393.0/640.0;
        bh1[5]=-92097.0/339200.0;
        bh1[6]=187.0/2100.0;
```

```
        bh1[7]=1.0/40.0;
        bh2[1]=35.0/384.0;
        bh2[2]=0;
        bh2[3]=500.0/1113.0;
        bh2[4]=125.0/192.0;
        bh2[5]=-2187.0/6784.0;
        bh2[6]=11.0/84.0;
        bh2[7]=0.0;
        b=[bh1,bh2];
        self.set(a,b,c);
    elif str(self.name) == ("RKV13"):
        #Runge-Kutta-Verner 13th degree
        n=14;
        c=[0.0 for i in range(n)];
        a = [[0 for x in range(n)] for y in range(n)];
        bh=[0.0 for i in range(n)];
        c[0]=0;
        c[1] =  0.;
        c[2] =  .5e-1;
        a[2][1] =  .5e-1;
        c[3] =  .1065625;
        a[3][1] = -.69931640625e-2;
        a[3][2] =  .1135556640625;
        c[4] =  .15984375;
        a[4][1] =  .399609375e-1;
        a[4][2] =  0.;
        a[4][3] =  .1198828125;
        c[5] =  .39;
        a[5][1] =  .36139756280045751240529407211840283 45129;
        a[5][2] =  0.;
        a[5][3] = -1.34152406670049277181998778820271583 4917;
        a[5][4] =  1.37012650390003525941469371608431300 0404;
        c[6] =  .465;
        a[6][1] =  .49047202797202797202797202797202797 20280e-1;
        a[6][2] =  0.;
        a[6][3] =  0.;
        a[6][4] =  .23509720422144047398629883354934271 43122;
        a[6][5] =  .18085559298135672881090396365345448 84850;
        c[7] =  .155;
        a[7][1] =  .61692890442890442890442890442890442 89044e-1;
        a[7][2] =  0.;
        a[7][3] =  0.;
        a[7][4] =  .11236568314640276622625570351301500 15442303;
        a[7][5] = -.38850460714513667670490481081112445 67456e-1;
        a[7][6] =  .19791887125220458553791887125220458 55379e-1;
        c[8] =  .943;
        a[8][1] = -1.76763024022232687573559711957214558 6714;
        a[8][2] =  0.;
        a[8][3] =  0.;
        a[8][4] = -62.5;
        a[8][5] = -6.06188937737666910082136145965933199 9758;
        a[8][6] =  5.65082319822276313856129803060084017 4201;
        a[8][7] =  65.62169641937623283799566054863063741227;
        c[9] =  .90180204173585695825970794067837214995 60;
        a[9][1] = -1.18094506655497079982511628262829795 7882;
        a[9][2] =  0.;
        a[9][3] =  0.;
        a[9][4] = -41.50473441114320841606641502701994225874;
        a[9][5] = -4.43443831910372501122516922984610021 1776;
        a[9][6] =  4.26040818858613302481219371074469324 0761;
        a[9][7] =  43.75364022446171584987676829438379303004;
        a[9][8] =  .78714254899123106874464750442263075508 60e-2;
        c[10] =  .909;
        a[10][1] = -1.28140599944148840545951029118205424 6266;
        a[10][2] =  0.;
        a[10][3] =  0.;
        a[10][4] = -45.04713996013986630220754257136007322267;
        a[10][5] = -4.73136206944957647731146426549128281 0943;
        a[10][6] =  5.51496701659380784118585158459724099 6214;
        a[10][7] =  47.44909557172985134869022392235929015114;
        a[10][8] =  .10592282971116611356873939555165428752 28e-1;
        a[10][9] = -.57468422638446162544323184782862962320 21e-2;
        c[11] =  .94;
        a[11][1] = -1.72447013426248519175670981748448186 1731;
        a[11][2] =  0.;
        a[11][3] =  0.;
```

```
        a[11][4] = -60.923490084830540165184346192537652460 63;
        a[11][5] = -5.951518376222392455202832767061854868290;
        a[11][6] =  5.556523730698456235979791650843592496839;
        a[11][7] =  63.983011980333053368375363786359959392 81;
        a[11][8] =  .146420282504149615927592139175945267600 3e-1;
        a[11][9] =  .646040877235820360362186514497765071489 2e-1;
        a[11][10] = -.793032316900887898402445254869337329144 7e-1;
        c[12] =  1.;
        a[12][1] = -3.301622667747079016353994789790983625569;
        a[12][2] =  0.;
        a[12][3] =  0.;
        a[12][4] = -118.011272359752508566692330395789886851 0;
        a[12][5] = -10.141422388456112486427839160345108975 95;
        a[12][6] =  9.139311332232057923544012273556827000619;
        a[12][7] =  123.375942828404268368484718098650189436 4;
        a[12][8] =  4.623244378874580474839807625067630924792;
        a[12][9] = -3.383277738068201923652550971536811240814;
        a[12][10] =  4.527592100324618189451265339351129035325;
        a[12][11] = -5.828495485811622963193088019162985703755;
        c[13] =  1.;
        a[13][1] = -3.039515033766309030040102851821200251056;
        a[13][2] =  0.;
        a[13][3] =  0.;
        a[13][4] = -109.260868089417625468644419232216462335 2;
        a[13][5] = -9.290642497400293449717665542656897549158;
        a[13][6] =  8.430504981764911142134299253836167803454;
        a[13][7] =  114.201001037833131355742404109552342747 6;
        a[13][8] = -.963727134214547935816237565898790165276 2;
        a[13][9] = -5.034884088802189791198680336183332323118;
        a[13][10] =  5.958130824002923177540402165388172072794;
        a[13][11] =  0.;
        a[13][12] =  0.;
        bh[1] =  .443126152290897921248643651020902976489 3e-1;
        bh[2] =  0.;
        bh[3] =  0.;
        bh[4] =  0.;
        bh[5] =  0.;
        bh[6] =  .354609564234322644786317935089505503885 5;
        bh[7] =  .247848043136665306961998672150445866001 6;
        bh[8] =  4.448134732475784492725128317159648871312;
        bh[9] =  19.846886366118733699309323992976879359 1;
        bh[10] = -23.581623377465618419695179608703949650 85;
        bh[11] =  0.;
        bh[12] =  0.;
        bh[13] = -.360167943728977516212453673774620240911 0;
        b=[bh];
        self.set(a,b,c);
    elif str(self.name) == ("RKF6"):
        #Runge-Kutta-Fehlberg 6th degree
        n=7;
        c=[0.0 for i in range(n)];
        a = [[0 for x in range(n)] for y in range(n)];
        bh=[0.0 for i in range(n)];
        c[1] =  1;
        c[2] =  0.2;
        c[3] =  0.3;
        c[4] =  0.6;
        c[5] =  1;
        c[5] = 7.0/8.0;
        a[1][1] =  1;
        a[2][1] =  0.2;
        a[3][1] =  3.0/40.0;
        a[3][2] =  9.0/40.0;
        a[4][1] =  0.3;
        a[4][2] =  -9.0/10.0;
        a[4][3] =  6.0/5.0;
        a[5][1] =  -11.0/54.0;
        a[5][2] =  0.25;
        a[5][3] =  -70.0/27.0;
        a[5][4] =  35.0/27.0;
        bh[1] =  2825.0/27648.0;
        bh[2] =  18575.0/48384.0;
        bh[3] =  13525.0/55296.0;
        bh[4] =  277.0/14336.0;
        bh[5] =  1.0/4.0;
        b=[bh];
```

```
        self.set(a,b,c);
```

Program Runge_Kutta class in python to calculate single or set differential equations

```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 13:10:28 2018

@author: Mustafa Turhan Çoban
"""
from  Butcher import *;
from f_xi import *;

class  Runge_Kutta():
  def __init__(self,namei):
     self.name=namei;
     self.bu=Buthcher(namei);
     self.a=self.bu.a;
     self.b=self.bu.b;
     self.c=self.bu.c;
  def calc_1(self,f,x0,xn,y0,hi,bk):
     ff=float((xn-x0)/hi);
     n=int(ff);
     self.h=hi;
     xx=[0 for i in range(2)];
     x=[0 for i in range(n+1)];
     y=[0 for i in range(n+1)];
     s=len(self.c);
     ki=[0 for i in range(n+1)];
     x[0]=x0;y[0]=y0;
     for ni in range(n):
        for i in range(s):
           xi=x[ni]+self.c[i]*self.h;
           yi=y[ni];
           for j in range(i):
              yi+=self.h*self.a[i][j]*ki[j];
           xx[0]=xi;
           xx[1]=yi;
           ki[i]=f.func(xx);
        y[ni+1]=y[ni];
        for i in range(s):
           y[ni+1]+=self.h*self.b[bk][i]*ki[i];
        x[ni+1]=x[ni]+self.h;
     XY=[[0 for i2 in range(n+1)] for j2 in range(2)];
     XY[0]=x;
     XY[1]=y;
     return XY;

  def  tasi(self,x1,y1):
     n1=len(y1);
     z=[0 for i in range(n1+1)];
     z[0]=x1;
     for i in range(1,n1+1):
        z[i]=y1[i-1];
     print("z=",z);
     return z;

  def calc_n(self,f,x0,xn,y0,hi,bk):
     #bk index of k if more than one equation is existed
     self.h=hi;
     ff=float((xn-x0)/hi+1e-30);
     n=int(ff);
     m=len(y0);
     x=[0 for i in range(n+1)];
     y=[[0 for i2 in range(m)] for j2 in range(n+1)];
     print(y[0]);
     s=len(self.c);
     z=[0 for i in range(m+1)];
     ki=[0 for i in range(s)];
     x[0]=x0;
     for ni in range(n+1):
        x[ni]=x0+ni*self.h;
     y[0]=y0;
     for ni in range(n):
        for k in range(m):
           for i in range(s):
```

```
                xi=x[ni]+self.c[i]*self.h;
                z[0]=xi;
                for ix in range(m):
                    z[ix+1]=y[ni][ix];
                for j in range(i):
                    z[k+1]+=self.h*self.a[i][j]*ki[j];
                ki[i]=f.func(z,k);
            y[ni+1][k]=y[ni][k];
            for i in range(s):
                y[ni+1][k]+=self.h*self.b[bk][i]*ki[i];
    XY=[[0 for i2 in range(n+1)] for j2 in range(m+1)];
    XY[0]=x;
    for k in range(m):
        for ni in range(n+1):
            XY[k+1][ni]=y[ni][k];
    return XY;
```

Program Sixth order Runge-Kutta (RK6) equation using Buthher class (Python code)

```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 13:10:28 2018

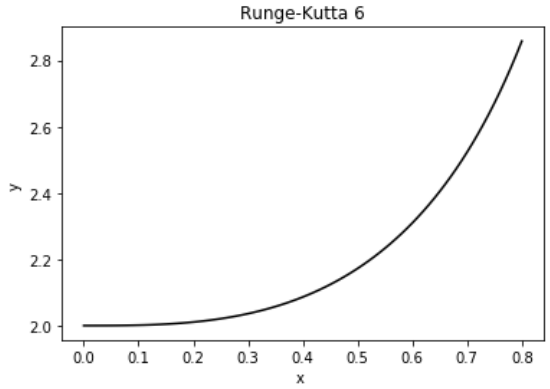
@author: Mustafa Turhan Çoban
"""
from Runge_Kutta import *;
from f_x import *;
import numpy as np;
import matplotlib.pyplot as plt;

class f1(f_x):
    def func(self,X):
        x=X[0];
        y=X[1];
        F=y*y*sin(x)*sin(x);
        return F;


rk=Runge_Kutta("RK6");
f=f1();
XY=rk.calc_1(f,0.0,0.8,2.0,0.01,0);
print(XY);
x=XY[0];
y=XY[1];
plt.title('Runge-Kutta 6')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'k');
```

```
 runfile('D:/okul/SCO1/Runge_Kutta_test_RK6_1.py', wdir='D:/okul/SCO1')
Reloaded modules: Runge_Kutta, Butcher, f_xi, f_x
[[0.0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.060000000000000005, 0.07, 0.08, 0.09, 0.09999999999999999, 0.10999999999999999,
0.11999999999999998, 0.12999999999999998, 0.13999999999999999, 0.15, 0.16, 0.17, 0.18000000000000002, 0.19000000000000003,
0.20000000000000004, 0.21000000000000005, 0.22000000000000006, 0.23000000000000007, 0.24000000000000007,
0.25000000000000006, 0.26000000000000006, 0.2700000000000001, 0.2800000000000001, 0.2900000000000001,
0.3000000000000001, 0.3100000000000001, 0.3200000000000001, 0.3300000000000001, 0.34000000000000014,
0.35000000000000014, 0.36000000000000015, 0.37000000000000016, 0.38000000000000017, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002, 0.4300000000000002, 0.4400000000000002,
0.45000000000000023, 0.46000000000000024, 0.47000000000000025, 0.48000000000000026, 0.49000000000000027,
0.5000000000000002, 0.5100000000000002, 0.5200000000000002, 0.5300000000000002, 0.5400000000000003, 0.5500000000000003,
0.5600000000000003, 0.5700000000000003, 0.5800000000000003, 0.5900000000000003, 0.6000000000000003, 0.6100000000000003,
0.6200000000000003, 0.6300000000000003, 0.6400000000000003, 0.6500000000000004, 0.6600000000000004, 0.6700000000000004,
0.6800000000000004, 0.6900000000000004, 0.7000000000000004, 0.7100000000000004, 0.7200000000000004, 0.7300000000000004,
0.7400000000000004, 0.7500000000000004, 0.7600000000000005, 0.7700000000000005, 0.7800000000000005, 0.7900000000000005,
0.8000000000000005], [2.0, 2.0000013333075586, 2.0000106658702514, 2.0000359941683423, 2.0000853096695534,
2.0001665972293488, 2.00028783412938, 2.0004569897517466, 2.0006820258861464, 2.00097089766665, 2.001331555134674,
2.001771945424804, 2.002300155703943, 2.0029237159263467, 2.0036510042071387, 2.004489850139076, 2.005448240726798,
2.0065341861353527, 2.007755726190629, 2.009120937502593, 2.0106379412176634, 2.0123149114086223, 2.0141600841127394,
2.0161817670313047, 2.0183883499064637, 2.020788315594225, 2.0233902518557123, 2.0262028638922014, 2.029234987653242,
2.032495603951213, 2.035993853420053, 2.0397390523606527, 2.043740709520522, 2.048008543860918, 2.0525525033706464,
2.057382784992311, 2.062509855733902, 2.0679444750463984, 2.07369771855652, 2.0797810032530317, 2.0862061142351456,
2.092985233142692, 2.1001309683999394, 2.1076563874184013, 2.1155750509187783, 2.1239010495485644, 2.1326490429899403,
2.1418343017726333, 2.151472752028696, 2.161581023450873, 2.172176500743812, 2.183277378888052, 2.194902722571101,
2.2070725301782668, 2.2198078027789916, 2.2331306185927375, 2.2470642134728376, 2.2616330680079257, 2.2768630019096614,
```

2.292781276433536, 2.3094167056680117, 2.3267997776275444, 2.34496278619912, 2.3639399751217804, 2.383767695326836, 2.4044845771358694, 2.4261317190077967, 2.4487528947490995, 2.4723947813578184, 2.4971072099676364, 2.5229434427002473, 2.549960478630286, 2.5782193925271737, 2.60778571057397, 2.638729827888825, 2.6711274734069064, 2.7050602285404257, 2.7406161070467014, 2.77789020472977, 2.8169854290175724, 2.858013320140403]]



EXAMPLE :

Solve differential equation

$\frac{dy}{dx} = yx^2 - y$ at the range x=0 ve x=2 , with initial condition y(0)=1.

a) solve analytically from x=0 to x=2.

b) solve numerically by hand with h=0.5 step size with 4th order Runge_Kutte and calculate y(2)

c) solve numerically by computer with h=0.1 step size with 4th order Runge-Kutta

d) Plot to results and compare with analytical solution

Analytical solution :

$$\frac{dy}{dx} = yx^2 - y = y(x^2 - 1)$$

$$\frac{dy}{y} = (x^2 - 1)dx$$

$$\int \frac{dy}{y} = \int (x^2 - 1)dx$$

$$\ln(y) = (\frac{x^3}{3} - x) + C$$

Boundary condition x=0 y=1➔ C=0

$$y = \exp(\frac{x^3}{3} - x)$$

Solution with h=0.5 and h=0.1 by using spreadsheet:

h	0.5														
	x1	y1	k1	x2	y2	k2	x3	y3	k3	x4	y4	k4	y	y exact	error
	0	1	-1	0.25	0.75	-0.703	0.25	0.8242	-0.773	0.5	0.6136	-0.46	1	1	0
	0.5	0.6323	-0.474	0.75	0.5138	-0.225	0.75	0.5761	-0.252	1	0.5063	0	0.6323	0.6323	-5.4E-06
	1	0.5133	0	1.25	0.5133	0.2888	1.25	0.5855	0.3294	1.5	0.678	0.8475	0.5133	0.5134	7.02E-05
	1.5	0.687	0.8587	1.75	0.9017	1.8597	1.75	1.1519	2.3758	2	1.8749	5.6248	0.687	0.6873	0.000294
	2	1.9332	5.7996	2.25	3.3831	13.744	2.25	5.3692	21.812	2.5	12.839	67.407	1.9332	1.9477	0.01452

h	0.1														
	x1	y1	k1	x2	y2	k2	x3	y3	k3	x4	y4	k4	y	y exact	error
	0	1	-1	0.05	0.95	-0.948	0.05	0.9526	-0.95	0.1	0.905	-0.896	1	1	0

	0.1	0.9051	-0.896	0.15	0.8603	-0.841	0.15	0.8631	-0.844	0.2	0.8208	-0.788	0.9051	0.9051	-7E-08
	0.2	0.8209	-0.788	0.25	0.7815	-0.733	0.25	0.7843	-0.735	0.3	0.7474	-0.68	0.8209	0.8209	-1.2E-07
	0.3	0.7475	-0.68	0.35	0.7135	-0.626	0.35	0.7162	-0.628	0.4	0.6847	-0.575	0.7475	0.7475	-1.5E-07
	0.4	0.6848	-0.575	0.45	0.656	-0.523	0.45	0.6586	-0.525	0.5	0.6322	-0.474	0.6848	0.6848	-1.6E-07
	0.5	0.6323	-0.474	0.55	0.6086	-0.425	0.55	0.6111	-0.426	0.6	0.5897	-0.377	0.6323	0.6323	-1.6E-07
	0.6	0.5898	-0.377	0.65	0.5709	-0.33	0.65	0.5733	-0.331	0.7	0.5567	-0.284	0.5898	0.5898	-1.5E-07
	0.7	0.5567	-0.284	0.75	0.5425	-0.237	0.75	0.5449	-0.238	0.8	0.5329	-0.192	0.5567	0.5567	-1.4E-07
	0.8	0.5329	-0.192	0.85	0.5234	-0.145	0.85	0.5257	-0.146	0.9	0.5184	-0.098	0.5329	0.5329	-1.3E-07
	0.9	0.5184	-0.098	0.95	0.5135	-0.05	0.95	0.5159	-0.05	1	0.5134	0	0.5184	0.5184	-1.2E-07
	1	0.5134	0	1.05	0.5134	0.0526	1.05	0.516	0.0529	1.1	0.5187	0.1089	0.5134	0.5134	-1.2E-07
	1.1	0.5188	0.1089	1.15	0.5242	0.1691	1.15	0.5272	0.17	1.2	0.5358	0.2357	0.5188	0.5187	-1.2E-07
	1.2	0.5358	0.2358	1.25	0.5476	0.308	1.25	0.5512	0.31	1.3	0.5668	0.3911	0.5358	0.5358	-1.2E-07
	1.3	0.5668	0.3911	1.35	0.5864	0.4823	1.35	0.591	0.4861	1.4	0.6155	0.5908	0.5668	0.5668	-1.1E-07
	1.4	0.6155	0.5909	1.45	0.645	0.7112	1.45	0.651	0.7178	1.5	0.6873	0.8591	0.6155	0.6155	-6.9E-08
	1.5	0.6873	0.8591	1.55	0.7302	1.0242	1.55	0.7385	1.0357	1.6	0.7909	1.2337	0.6873	0.6873	7.08E-08
	1.6	0.7908	1.2337	1.65	0.8525	1.4685	1.65	0.8643	1.4887	1.7	0.9397	1.776	0.7908	0.7908	5.25E-07
	1.7	0.9396	1.7758	1.75	1.0284	2.121	1.75	1.0456	2.1566	1.8	1.1552	2.5877	0.9396	0.9396	1.9E-06
	1.8	1.1549	2.5869	1.85	1.2842	3.111	1.85	1.3104	3.1745	1.9	1.4723	3.8428	1.1549	1.1549	5.82E-06
	1.9	1.4716	3.8408	1.95	1.6636	4.6622	1.95	1.7047	4.7773	2	1.9493	5.8479	1.4716	1.4716	1.67E-05
	2	1.9477	5.8431	2.05	2.2398	7.1731	2.05	2.3063	7.3861	2.1	2.6863	9.1603	1.9477	1.9477	4.59E-05

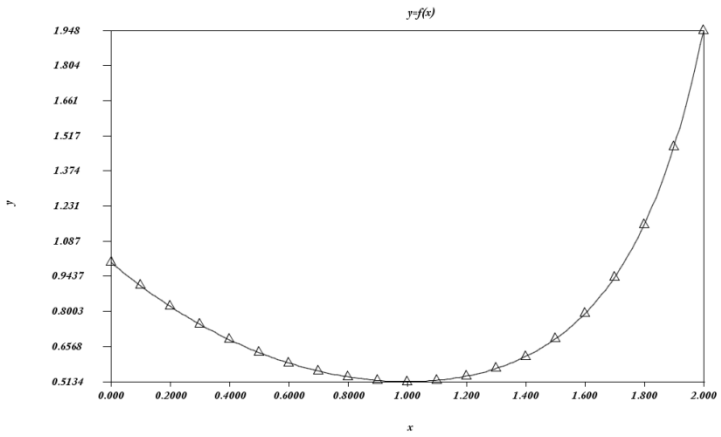
Solution by java:

Program solution of example problem with fourth order Runge-Kutta (RK4) equation

```
  import java.io.*;
abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}
class fxy extends f_xy
{
  public double func(double x,double y)
  {return y*x*x-y;}
}


class fxyKC extends f_x
{
  public double func(double x)
  {return Math.exp(x*x*x/3.0-x);}
}

class dif1A
{
  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=1.0;
   double xn=2.0;
   fxy f2=new fxy();
   fxyKC f3=new fxyKC();
   Runge_Kutta rk=new Runge_Kutta("RK4");
    System.out.println(rk.toString());
   double XY[][]=rk.hesapla(f2,x0,xn,y0,0.1);
   Plot p1=new Plot(XY);
   p1.addFunction(f3,x0,xn,200);
   p1.setPlotType(0,24);
   p1.plot();
  }
}
```




```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" dif1A
Buthcher table n = 4
    0.00000000 |
    0.50000000 |     0.50000000
    0.50000000 |     0.00000000     0.50000000
    1.00000000 |     0.00000000     0.00000000     1.00000000
--------------------------------------------------------------------------
                  0.16666667     0.33333333     0.33333333     0.16666667

       0.000000000000000       1.000000000000000
       0.100000000000000       0.905139151280990
       0.200000000000000       0.820917069099996
       0.300000000000000       0.747515827045378
       0.400000000000000       0.684773992072904
       0.500000000000000       0.632336819274610
       0.600000000000000       0.589783505270557
       0.700000000000000       0.556734717986668
       0.800000000000000       0.532947107134396
       0.900000000000000       0.518404336705643
       1.000000000000000       0.513417235958808
       1.100000000000000       0.518750050786687
       1.200000000000000       0.535797072713138
       1.300000000000000       0.566846645557574
       1.400000000000000       0.615492067374456
       1.500000000000000       0.687289208016086
       1.600000000000000       0.790833891701643
       1.700000000000000       0.939567747881861
       1.800000000000000       1.154878288457116
       1.900000000000001       1.471558461178071
       2.000000000000000       1.947688122952552
```

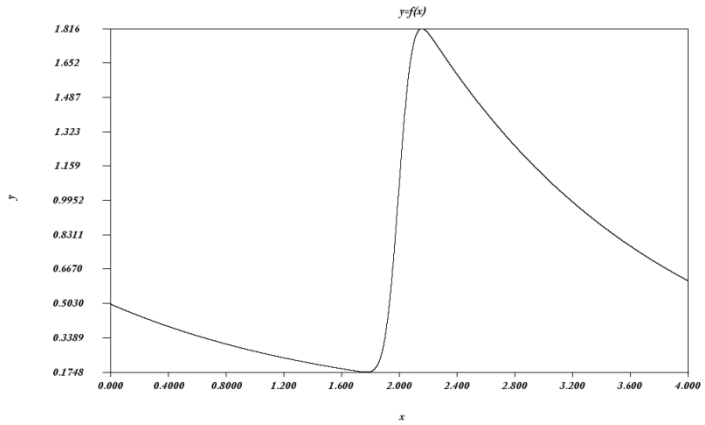
Program Runge-Kutta Fehlberg and Runge-Kutta-Verner constant step size codes

```
 import java.io.*;
class fxy2 extends f_xy
{
  double func(double x,double y)
  {return -0.6*y+10.0*Math.exp(-(x-2)*(x-2)/(2*0.075*0.075));}
}

class fxyKC extends f_x
{
  public double func(double x)
  {return Math.exp(x*x*x/3.0-x);}
}

class dif1B
{
  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=0.5;
   double xn=4.0;
```

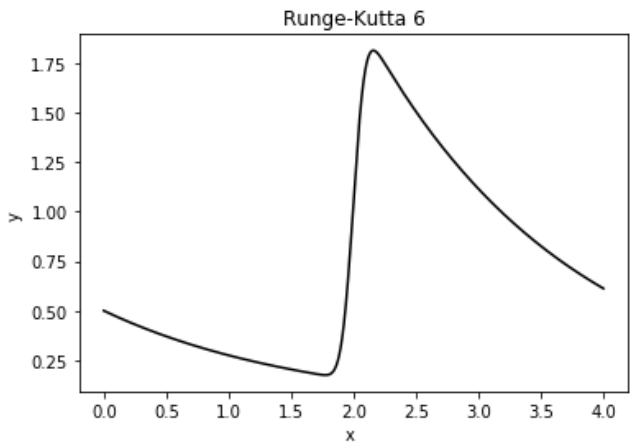
```
   fxy2 f2=new fxy2();
   fxyKC f3=new fxyKC();
   Runge_Kutta rk=new Runge_Kutta("RKF6");
   System.out.println(rk.toString());
   double XY[][]=rk.hesapla(f2,x0,xn,y0,0.001);
   //System.out.println(Matrix.toString(Matrix.Transpose(XY)));
   Plot p1=new Plot(XY);
   p1.plot();
  }
}
```




```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" dif1B
Buthcher table n = 7
    0.00000000 |
    1.00000000 |    0.00000000
    0.20000000 |    0.00000000    0.20000000
    0.30000000 |    0.00000000    0.07500000    0.22500000
    0.60000000 |    0.00000000    0.30000000   -0.90000000    1.20000000
    0.87500000 |    0.00000000   -0.20370370    0.25000000   -2.59259259    1.29629630
    0.00000000 |    0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    0.00000000
-------------------------------------------------------------------------------------------------------------
                0.00000000    0.10217737    0.38390790    0.24459274    0.01932199    0.25000000    0.00000000
```

Program Runge-Kutta Fehlberg constant step size Python codes

```
class f1(f_xj):
    def func(self,X):
        x=X[0];
        y=X[1];
        F=-0.6*y+10.0*exp(-(x-2)*(x-2)/(2*0.075*0.075));
        return F;
rk=Runge_Kutta("RKF6");
f=f1();
XY=rk.calc_1(f,0.0,4.0,0.5,0.001,0);
print(XY);
x=XY[0];
y=XY[1];
plt.title('Runge-Kutta 6')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(x,y,'k');
```



MULTISTEP (OPEN) METHODS (PREDICTOR-CORRECTOR METHODS)

If more than one step of equations are used for calculation of one step solution, it is called multistep method. Some people claims that this type of equations gives more stable solutions. **Milne** Formula is the first multistep method to be investigated.

Milne 1. Step formula (first prediction)

$$y_{i+1}^{p} = y_{i-3} + \frac{4h}{3}\left[2f(x_i, y_i) - f(x_{i-1}, y_{i-1}) + 2f(x_{i-2}, y_{i-2})\right]$$

Milne 2. Step Formula (first correction)

$$y_{i+1}^{(0)} = y_{j+1}^{p} + \frac{28}{29}\left[y_i - y_i^{p}\right]$$

Milne 3. Step formula (second correction)

$$y_{i+1}^{(k+1)} = y_{j-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(k)}) + 4f(x_i, y_i) + f(x_{i-1}, y_{i-1})\right] \quad k=0,1,\ldots.$$

Due to using a first step to predict a guess and then corrected it with one or multistep correctors, Multistep methods are also called **Predictor-Corrector** methods. In order to apply this method, previously calculated points are needed. Single step methods like Runge-Kutta methods can serve us for calculating this first points. As an example problem

$y' = \frac{1}{1+x^2} - 2y^2$ diferential equation with the initial conditions y(0)=0 will be solved. Exact analytical solution of this diferential equation

$$y = \frac{x}{1+x^2}$$

Program Milne Predictor-corrector formula

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// and differentiation (derivative) functions
// Dr. Turhan Coban
// =====================================================
// Basit diferansiyel denklemler dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
```

```
}

class fxy extends f_xy
{
  double func(double x,double y)
  {return 1.0/(1.0+x*x)-2.0*y*y;}
}

class fxyKC extends f_x
{
  double func(double x)
  {return x/(1+x*x);}
}

class diferansiyelN
{
  public static double[][] RK6M(f_xy f,double x0,double y0,double xn,double h)
  {
  // Milne method
  int n=(int)((xn-x0)/h);
  double a[][]=new double[2][n+1];
  double yp[]=new double[n+1];
  double y[][]=new double[6][n+1];
  // Runge-Kutta 6
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
  double x2=x0+3.0*h;
  double k0,k1,k2,k3,k4,k5,k6;
  for(double xx=x0;xx<x2;xx+=h)
   {a[0][i+1]=xx+h;
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i]+0.25*h,a[1][i]+0.25*k1*h);
   k3=f.func(a[0][i]+0.25*h,a[1][i]+1.0/8.0*k1*h+1.0/8.0*k2*h);
   k4=f.func(a[0][i]+0.5*h,a[1][i]-0.5*k2*h+k3*h);
   k5=f.func(a[0][i]+3.0/4.0*h,a[1][i]+3.0/16.0*k1*h+9.0/16.0*k4*h);
   k6=f.func(a[0][i]+h,a[1][i]-3.0/7.0*k1*h+2.0/7.0*k2*h+12.0/7.0*k3*h-12.0/7.0*k4*h+8.0/7.0*k5*h);
   a[1][i+1]=a[1][i]+(7.0*k1+32.0*k3+12.0*k4+32.0*k5+7*k6)/90.0*h;
   i++;
   }
   for(double xx=x2;xx<xn;xx+=h)
   { a[0][i+1]=xx+h;
   //Milne formülü
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i-1],a[1][i-1]);
   k3=f.func(a[0][i-2],a[1][i-2]);
   yp[i+1]=a[1][i-3]+4.0*h/3.0*(2.0*k1-k2+2.0*k3);
   y[0][i+1]=yp[i+1]+28.0/29.0*(a[1][i]- yp[i]);
   int k;
   for(k=0;k<5;k++)
   {
                k0=f.func(a[0][i+1],y[k][i+1]);
      y[k+1][i+1]=a[1][i-1]+h/3.0*(k0+4.0*k1+k2);
   }
   a[1][i+1]=y[k][i+1];
   i++;
   }
  return a;
  }

  public static void main(String args[]) throws IOException
  {double x0=0;
   double y0=0.0;
   double yn=10.0;
   double h=0.0625;
   fxy f2=new fxy();
   fxyKC fKC=new fxyKC();
   double a[][]=RK6M(f2,x0,y0,yn,h);
   Text.print(Text.T(a));
   Plot p1=new Plot(a);
```

```
    p1.setPlotType(0,24);
    p1.addFunction(fKC,0.0,10.0,50);
    p1.plot();
   }
}
```

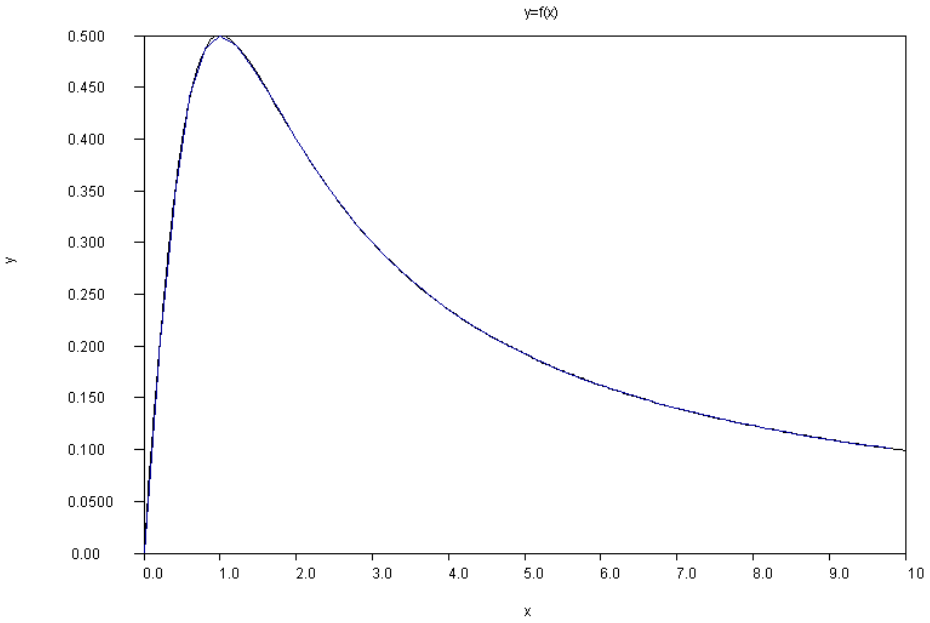
Output 8.5-1 Milne Predictor-Corrector method table output with error term

	Milne solution	Exact solution	Error
0	**0**	**0**	**0**
0.0625	**0.062256809**	**0.062256809**	**1.29392E-10**
0.125	**0.123076922**	**0.123076923**	**6.07481E-10**
0.1875	**0.181132074**	**0.181132075**	**1.36894E-09**
0.25	0.235294614	0.235294118	-4.9588E-07
0.3125	0.284697577	0.284697509	-6.78819E-08
0.375	0.328767367	0.328767123	-2.43534E-07
0.4375	0.367212702	0.367213115	4.1304E-07
0.5	0.399999711	0.4	2.8879E-07
0.5625	0.427298769	0.427299703	9.34563E-07
0.625	0.449437527	0.449438202	6.75003E-07
0.6875	0.466842287	0.466843501	1.2147E-06
0.75	0.479999216	0.48	7.8413E-07
0.8125	0.489410522	0.489411765	1.24273E-06
0.875	0.495574537	0.495575221	6.83803E-07
0.9375	0.498959376	0.498960499	1.1231E-06
1	0.499999514	0.5	4.86142E-07
1.0625	0.499081613	0.499082569	9.56231E-07
1.125	0.496551452	0.496551724	2.72191E-07
1.1875	0.492705844	0.492706645	8.01101E-07
1.25	0.487804796	0.487804878	8.16887E-08
1.3125	0.482065316	0.482065997	6.81141E-07
1.375	0.475675749	0.475675676	-7.3537E-08
1.4375	0.468789209	0.468789809	5.99558E-07
1.5	0.461538657	0.461538462	-1.95371E-07
1.5625	0.454028961	0.454029512	5.51033E-07
1.625	0.446352222	0.446351931	-2.9049E-07
1.6875	0.438578152	0.43857868	5.28107E-07
1.75	0.430769597	0.430769231	-3.6604E-07
1.8125	0.422971217	0.422971741	5.23921E-07
1.875	0.415225342	0.415224913	-4.28026E-07
1.9375	0.40755904	0.407559573	5.33046E-07
2	0.400000481	0.4	-4.8098E-07
2.0625	0.392564504	0.392565056	5.51496E-07
2.125	0.38526965	0.385269122	-5.28128E-07
2.1875	0.378122313	0.37812289	5.76463E-07
2.25	0.371134592	0.371134021	-5.7169E-07
2.3125	0.364307086	0.364307692	6.06014E-07
2.375	0.357647672	0.357647059	-6.13158E-07
2.4375	0.351152991	0.35115363	6.38839E-07
2.5	0.34482824	0.344827586	-6.53519E-07
2.5625	0.338667369	0.338668043	6.74055E-07
2.625	0.332673961	0.332673267	-6.93416E-07
2.6875	0.326840144	0.326840855	7.11066E-07
2.75	0.321168616	0.321167883	-7.33265E-07
2.8125	0.315650281	0.31565103	7.49474E-07
2.875	0.310287451	0.310286678	-7.73331E-07
2.9375	0.305070205	0.305070994	7.89007E-07
3	0.300000814	0.3	-8.13778E-07
3.0625	0.295068798	0.295069627	8.29482E-07
3.125	0.290276617	0.290275762	-8.54704E-07
3.1875	0.28561341	0.285614281	8.70771E-07
3.25	0.281081977	0.281081081	-8.96169E-07

3.3125	0.276671192	0.276672104	9.12786E-07
3.375	0.272384293	0.272383354	-9.38201E-07
3.4375	0.268209956	0.268210911	9.55464E-07
3.5	0.264151924	0.264150943	-9.80814E-07
3.5625	0.260198716	0.260199715	9.98759E-07
3.625	0.256354615	0.256353591	-1.02401E-06
3.6875	0.252608002	0.252609045	1.04264E-06
3.75	0.248963723	0.248962656	-1.06778E-06
3.8125	0.245410027	0.245411114	1.08707E-06
3.875	0.241952332	0.24195122	-1.11212E-06
3.9375	0.23857875	0.238579882	1.13204E-06
4	0.235295275	0.235294118	-1.15702E-06
4.0625	0.232089874	0.232091051	1.17754E-06
4.125	0.228969112	0.22896791	-1.20245E-06
4.1875	0.2259208	0.225922023	1.22353E-06
4.25	0.222952068	0.22295082	-1.24842E-06
4.3125	0.220050554	0.220051824	1.27002E-06
4.375	0.217223948	0.217222653	-1.29491E-06
4.4375	0.214459699	0.214461016	1.31699E-06
4.5	0.211766048	0.211764706	-1.3419E-06
4.5625	0.209130238	0.209131603	1.36443E-06
4.625	0.206561054	0.206559665	-1.38939E-06
4.6875	0.204045518	0.204046931	1.41233E-06
4.75	0.201592949	0.201591512	-1.43735E-06
4.8125	0.199190132	0.199191593	1.46069E-06
4.875	0.196846912	0.196845426	-1.48579E-06
4.9375	0.194549822	0.194551331	1.50949E-06
5	0.192309227	0.192307692	-1.53468E-06
5.0625	0.190111394	0.190112953	1.55873E-06
5.125	0.1879672	0.187965616	-1.58403E-06
5.1875	0.185862632	0.185864241	1.6084E-06
5.25	0.183809074	0.18380744	-1.63382E-06
5.3125	0.181792219	0.181793878	1.65849E-06
5.375	0.179823953	0.179822269	-1.68403E-06
5.4375	0.177889665	0.177891374	1.70899E-06
5.5	0.176001735	0.176	-1.73467E-06
5.5625	0.174145238	0.174146998	1.7599E-06
5.625	0.172333045	0.172331259	-1.78573E-06
5.6875	0.170549905	0.170551716	1.81122E-06
5.75	0.168809177	0.168807339	-1.83719E-06
5.8125	0.167095274	0.167097136	1.86293E-06
5.875	0.165422039	0.16542015	-1.88905E-06
5.9375	0.16377354	0.163775455	1.91503E-06
6	0.162164103	0.162162162	-1.94131E-06
6.0625	0.160577443	0.16057941	1.96752E-06
6.125	0.159028363	0.159026369	-1.99395E-06
6.1875	0.157500217	0.157502237	2.02038E-06
6.25	0.156008287	0.15600624	-2.04697E-06
6.3125	0.154535557	0.15453763	2.07362E-06
6.375	0.153097785	0.153095685	-2.10037E-06
6.4375	0.151677578	0.151679705	2.12722E-06
6.5	0.150291171	0.150289017	-2.15413E-06
6.5625	0.148920787	0.148922968	2.18119E-06
6.625	0.147583134	0.147580926	-2.20826E-06
6.6875	0.146260046	0.146262281	2.23552E-06
6.75	0.144968706	0.144966443	-2.26275E-06
6.8125	0.14369055	0.14369284	2.29019E-06
6.875	0.142443237	0.142440919	-2.31759E-06
6.9375	0.1412078	0.141210146	2.34522E-06
7	0.140002373	0.14	-2.37278E-06
7.0625	0.13880758	0.138809981	2.40059E-06

7.125	0.13764203	0.137639602	-2.42831E-06
7.1875	0.136485935	0.136488391	2.4563E-06
7.25	0.135358377	0.135355893	-2.48418E-06
7.3125	0.134239151	0.134241664	2.51235E-06
7.375	0.133147815	0.133145275	-2.54039E-06
7.4375	0.132063742	0.132066311	2.56873E-06
7.5	0.131006964	0.131004367	-2.59693E-06
7.5625	0.129956427	0.129959052	2.62544E-06
7.625	0.128932641	0.128929987	-2.65379E-06
7.6875	0.12791412	0.127916802	2.68247E-06
7.75	0.126921851	0.12691914	-2.71098E-06
7.8125	0.125933914	0.125936654	2.73981E-06
7.875	0.124971774	0.124969006	-2.76848E-06
7.9375	0.124013071	0.124015868	2.79748E-06
8	0.123079749	0.123076923	-2.8263E-06
8.0625	0.122149006	0.122151861	2.85546E-06
8.125	0.121243267	0.121240382	-2.88443E-06
8.1875	0.120339281	0.120342194	2.91375E-06
8.25	0.119459956	0.119457014	-2.94287E-06
8.3125	0.118581592	0.118584564	2.97234E-06
8.375	0.117727579	0.117724577	-3.00161E-06
8.4375	0.116873761	0.116876792	3.03124E-06
8.5	0.116044016	0.116040956	-3.06066E-06
8.5625	0.11521373	0.11521682	3.09043E-06
8.625	0.114407265	0.114404145	-3.12E-06
8.6875	0.113599547	0.113602697	3.14993E-06
8.75	0.112815428	0.112812248	-3.17964E-06
8.8125	0.112029367	0.112032577	3.20971E-06
8.875	0.111266707	0.111263467	-3.23957E-06
8.9375	0.110501439	0.110504709	3.26979E-06
9	0.109759397	0.109756098	-3.29979E-06
9.0625	0.109014103	0.109017433	3.33015E-06
9.125	0.108291882	0.108288522	-3.3603E-06
9.1875	0.107565784	0.107569174	3.3908E-06
9.25	0.106862627	0.106859206	-3.42109E-06
9.3125	0.106154984	0.106158436	3.45173E-06
9.375	0.105470172	0.10546669	-3.48216E-06
9.4375	0.104780284	0.104783797	3.51294E-06
9.5	0.104113133	0.104109589	-3.5435E-06
9.5625	0.10344033	0.103443905	3.57443E-06
9.625	0.102790189	0.102786584	-3.60513E-06
9.6875	0.102133838	0.102137474	3.63619E-06
9.75	0.101500089	0.101496422	-3.66703E-06
9.8125	0.100859582	0.10086328	3.69823E-06
9.875	0.100241636	0.100237906	-3.72919E-06
9.9375	0.099616398	0.099620159	3.76053E-06
10	0.099013693	0.099009901	-3.79163E-06

Output 8.5-2 Milne Predictor-Corrector method graphic output



As a second method, Hamming Formula will be investigated. This method is very close to Milne method. As amound of error it is a little worst than Milne method, but its stability is much better.

Hamming 1. step formula (first prediction)

$$y_{i+1}^{p} = y_{i-3} + \frac{4h}{3}\left[2f(x_i, y_i) - f(x_{i-1}, y_{i-1}) + 2f(x_{i-2}, y_{i-2})\right]$$

Hamming 2. Step formula (first correction)

$$y_{i+1}^{(0)} = y_{j+1}^{p} + \frac{112}{121}\left[y_i - y_i^{p}\right]$$

Hamming 3. Step formula (second correction)

$$y_{i+1}^{(k+1)} = \frac{1}{8}(9y_i - y_{i-2}) + \frac{3h}{8}\left[f(x_{i+1}, y_{i+1}^{(k)}) + 2f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right] \quad \text{k=0,1,....}$$

Program Hamming Predictor-corrector Formula

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Hamming Predictor-Corrector method
// Dr. Turhan Coban
// =====================================================
// dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{
  double func(double x,double y)
  {return 1.0/(1.0+x*x)-2.0*y*y;}
}

class fxyKC extends f_x
```

```
{
  double func(double x)
  {return x/(1+x*x);}
}

class NA90
{
  public static double[][] RK6M(f_xy f,double x0,double y0,double xn,double h)
  {
  // Hamming formülü
  int n=(int)((xn-x0)/h);
  double a[][]=new double[2][n+1];
  double yp[]=new double[n+1];
  double y[][]=new double[6][n+1];
  // Runge-Kutta 6
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
  double x2=x0+3.0*h;
  double k0,k1,k2,k3,k4,k5,k6;
  for(double xx=x0;xx<x2;xx+=h)
   {a[0][i+1]=xx+h;
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i]+0.25*h,a[1][i]+0.25*k1*h);
   k3=f.func(a[0][i]+0.25*h,a[1][i]+1.0/8.0*k1*h+1.0/8.0*k2*h);
   k4=f.func(a[0][i]+0.5*h,a[1][i]-0.5*k2*h+k3*h);
   k5=f.func(a[0][i]+3.0/4.0*h,a[1][i]+3.0/16.0*k1*h+9.0/16.0*k4*h);
   k6=f.func(a[0][i]+h,a[1][i]-3.0/7.0*k1*h+2.0/7.0*k2*h+12.0/7.0*k3*h-12.0/7.0*k4*h+8.0/7.0*k5*h);
   a[1][i+1]=a[1][i]+(7.0*k1+32.0*k3+12.0*k4+32.0*k5+7*k6)/90.0*h;
   i++;
   }
   for(double xx=x2;xx<xn;xx+=h)
   { a[0][i+1]=xx+h;
   //Milne formülü
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i-1],a[1][i-1]);
   k3=f.func(a[0][i-2],a[1][i-2]);
   yp[i+1]=a[1][i-3]+4.0*h/3.0*(2.0*k1-k2+2.0*k3);
   y[0][i+1]=yp[i+1]+112.0/121.0*(a[1][i]- yp[i]);
   int k;
   for(k=0;k<5;k++)
   {
                k0=f.func(a[0][i+1],y[k][i+1]);
      y[k+1][i+1]=1.0/8.0*(9.0*a[1][i]-a[1][i-2])+3.0*h/8.0*(k0+2.0*k1-k2);
   }
   a[1][i+1]=y[k][i+1];
   i++;
   }
  return a;
  }

  public static void main(String args[]) throws IOException
  {double x0=0;
   double y0=0.0;
   double yn=10.0;
   double h=0.0625;
   fxy f2=new fxy();
   fxyKC fKC=new fxyKC();
   double a[][]=RK6M(f2,x0,y0,yn,h);
   Text.print(Text.T(a));
   Plot p1=new Plot(a);
   p1.setPlotType(0,24);
   p1.addFunction(fKC,0.0,10.0,50);
   p1.plot();
  }
}
```

Output Hamming Predictor-corrector Formula calculated values

	Hamming solution	Exact solution	Error

0	**0**	**0**	**0**
0.0625	**0.062256809**	**0.062256809**	**1.29392E-10**
0.125	**0.123076922**	**0.123076923**	**6.07481E-10**
0.1875	**0.181132074**	**0.181132075**	**1.36894E-09**
0.25	0.235295447	0.235294118	-1.32917E-06
0.3125	0.284699363	0.284697509	-1.85455E-06
0.375	0.328768754	0.328767123	-1.63052E-06
0.4375	0.367213781	0.367213115	-6.6612E-07
0.5	0.399999285	0.4	7.14995E-07
0.5625	0.427297497	0.427299703	2.20588E-06
0.625	0.449434654	0.449438202	3.5483E-06
0.6875	0.466838916	0.466843501	4.58534E-06
0.75	0.479994747	0.48	5.25346E-06
0.8125	0.489406203	0.489411765	5.56192E-06
0.875	0.495569657	0.495575221	5.56428E-06
0.9375	0.498955165	0.498960499	5.33375E-06
1	0.499995055	0.5	4.94536E-06
1.0625	0.499078103	0.499082569	4.46539E-06
1.125	0.496547778	0.496551724	3.94663E-06
1.1875	0.492703217	0.492706645	3.42766E-06
1.25	0.487801944	0.487804878	2.93423E-06
1.3125	0.482063515	0.482065997	2.48172E-06
1.375	0.475673598	0.475675676	2.07773E-06
1.4375	0.468788085	0.468789809	1.7244E-06
1.5	0.461537041	0.461538462	1.42031E-06
1.5625	0.45402835	0.454029512	1.16193E-06
1.625	0.446350987	0.446351931	9.44654E-07
1.6875	0.438577917	0.43857868	7.63473E-07
1.75	0.430768617	0.430769231	6.13446E-07
1.8125	0.422971251	0.422971741	4.89946E-07
1.875	0.415224525	0.415224913	3.88796E-07
1.9375	0.407559266	0.407559573	3.06316E-07
2	0.399999761	0.4	2.3933E-07
2.0625	0.392564871	0.392565056	1.85132E-07
2.125	0.38526898	0.385269122	1.41441E-07
2.1875	0.378122784	0.37812289	1.06351E-07
2.25	0.371133942	0.371134021	7.82796E-08
2.3125	0.364307636	0.364307692	5.59194E-08
2.375	0.357647021	0.357647059	3.81956E-08
2.4375	0.351153605	0.35115363	2.42269E-08
2.5	0.344827573	0.344827586	1.3293E-08
2.5625	0.338668039	0.338668043	4.80608E-09
2.625	0.332673269	0.332673267	-1.71236E-09
2.6875	0.326840862	0.326840855	-6.65133E-09
2.75	0.321167894	0.321167883	-1.03267E-08
2.8125	0.315651043	0.31565103	-1.29944E-08
2.875	0.310286693	0.310286678	-1.4862E-08
2.9375	0.30507101	0.305070994	-1.6097E-08
3	0.300000017	0.3	-1.68349E-08
3.0625	0.295069645	0.295069627	-1.7185E-08
3.125	0.290275779	0.290275762	-1.72351E-08
3.1875	0.285614298	0.285614281	-1.70561E-08
3.25	0.281081098	0.281081081	-1.67044E-08
3.3125	0.276672121	0.276672104	-1.62255E-08
3.375	0.27238337	0.272383354	-1.56554E-08
3.4375	0.268210926	0.268210911	-1.50227E-08
3.5	0.264150958	0.264150943	-1.435E-08
3.5625	0.260199728	0.260199715	-1.36552E-08
3.625	0.256353604	0.256353591	-1.29519E-08
3.6875	0.252609057	0.252609045	-1.22508E-08
3.75	0.248962667	0.248962656	-1.15601E-08
3.8125	0.245411125	0.245411114	-1.08858E-08
3.875	0.24195123	0.24195122	-1.02323E-08
3.9375	0.238579891	0.238579882	-9.60278E-09
4	0.235294127	0.235294118	-8.99929E-09
4.0625	0.23209106	0.232091051	-8.42314E-09
4.125	0.228967918	0.22896791	-7.87496E-09
4.1875	0.225922031	0.225922023	-7.35492E-09
4.25	0.222950827	0.22295082	-6.86278E-09
4.3125	0.22005183	0.220051824	-6.39803E-09

4.375	0.217222659	0.217222653	-5.95992E-09
4.4375	0.214461021	0.214461016	-5.54758E-09
4.5	0.211764711	0.211764706	-5.16002E-09
4.5625	0.209131607	0.209131603	-4.79616E-09
4.625	0.206559669	0.206559665	-4.4549E-09
4.6875	0.204046935	0.204046931	-4.13513E-09
4.75	0.201591516	0.201591512	-3.83572E-09
4.8125	0.199191596	0.199191593	-3.55556E-09
4.875	0.196845429	0.196845426	-3.29356E-09
4.9375	0.194551334	0.194551331	-3.04869E-09
5	0.192307695	0.192307692	-2.81991E-09
5.0625	0.190112956	0.190112953	-2.60627E-09
5.125	0.187965618	0.187965616	-2.40683E-09
5.1875	0.185864243	0.185864241	-2.22071E-09
5.25	0.183807442	0.18380744	-2.04707E-09
5.3125	0.18179388	0.181793878	-1.88511E-09
5.375	0.17982227	0.179822269	-1.73409E-09
5.4375	0.177891375	0.177891374	-1.59329E-09
5.5	0.176000001	0.176	-1.46205E-09
5.5625	0.174146999	0.174146998	-1.33975E-09
5.625	0.17233126	0.172331259	-1.22579E-09
5.6875	0.170551717	0.170551716	-1.11963E-09
5.75	0.16880734	0.168807339	-1.02075E-09
5.8125	0.167097137	0.167097136	-9.28673E-10
5.875	0.16542015	0.16542015	-8.42937E-10
5.9375	0.163775456	0.163775455	-7.63125E-10
6	0.162162163	0.162162162	-6.88839E-10
6.0625	0.160579411	0.16057941	-6.19713E-10
6.125	0.15902637	0.159026369	-5.55399E-10
6.1875	0.157502238	0.157502237	-4.95579E-10
6.25	0.156006241	0.15600624	-4.39949E-10
6.3125	0.154537631	0.15453763	-3.88231E-10
6.375	0.153095685	0.153095685	-3.40162E-10
6.4375	0.151679706	0.151679705	-2.95498E-10
6.5	0.150289018	0.150289017	-2.54013E-10
6.5625	0.148922968	0.148922968	-2.15491E-10
6.625	0.147580926	0.147580926	-1.79737E-10
6.6875	0.146262281	0.146262281	-1.46565E-10
6.75	0.144966443	0.144966443	-1.15802E-10
6.8125	0.14369284	0.14369284	-8.72877E-11
6.875	0.142440919	0.142440919	-6.08722E-11
6.9375	0.141210146	0.141210146	-3.64138E-11
7	0.14	0.14	-1.3784E-11
7.0625	0.138809981	0.138809981	7.14004E-12
7.125	0.137639602	0.137639602	2.64724E-11
7.1875	0.136488391	0.136488391	4.43198E-11
7.25	0.135355893	0.135355893	6.07798E-11
7.3125	0.134241664	0.134241664	7.59459E-11
7.375	0.133145275	0.133145275	8.99049E-11
7.4375	0.132066311	0.132066311	1.02736E-10
7.5	0.131004367	0.131004367	1.14516E-10
7.5625	0.129959052	0.129959052	1.25316E-10
7.625	0.128929987	0.128929987	1.35198E-10
7.6875	0.127916802	0.127916802	1.44227E-10
7.75	0.12691914	0.12691914	1.52459E-10
7.8125	0.125936654	0.125936654	1.59948E-10
7.875	0.124969006	0.124969006	1.66743E-10
7.9375	0.124015868	0.124015868	1.72892E-10
8	0.123076923	0.123076923	1.78437E-10
8.0625	0.122151861	0.122151861	1.83421E-10
8.125	0.121240382	0.121240382	1.8788E-10
8.1875	0.120342194	0.120342194	1.91852E-10
8.25	0.119457013	0.119457014	1.95369E-10
8.3125	0.118584564	0.118584564	1.98462E-10
8.375	0.117724577	0.117724577	2.0116E-10
8.4375	0.116876792	0.116876792	2.03492E-10
8.5	0.116040955	0.116040956	2.05481E-10
8.5625	0.11521682	0.11521682	2.07154E-10
8.625	0.114404145	0.114404145	2.0853E-10
8.6875	0.113602697	0.113602697	2.09633E-10

8.75	0.112812248	0.112812248	2.10481E-10
8.8125	0.112032577	0.112032577	2.11091E-10
8.875	0.111263467	0.111263467	2.11482E-10
8.9375	0.110504709	0.110504709	2.1167E-10
9	0.109756097	0.109756098	2.1167E-10
9.0625	0.109017433	0.109017433	2.11494E-10
9.125	0.108288522	0.108288522	2.11156E-10
9.1875	0.107569174	0.107569174	2.1067E-10
9.25	0.106859206	0.106859206	2.10047E-10
9.3125	0.106158436	0.106158436	2.09296E-10
9.375	0.10546669	0.10546669	2.08428E-10
9.4375	0.104783796	0.104783797	2.07452E-10
9.5	0.104109589	0.104109589	2.06378E-10
9.5625	0.103443904	0.103443905	2.05214E-10
9.625	0.102786584	0.102786584	2.03965E-10
9.6875	0.102137474	0.102137474	2.02642E-10
9.75	0.101496421	0.101496422	2.01251E-10
9.8125	0.10086328	0.10086328	1.99795E-10
9.875	0.100237906	0.100237906	1.98283E-10
9.9375	0.099620159	0.099620159	1.96719E-10
10	0.099009901	0.099009901	1.9511E-10

First points calculated by Runge Kutta -6 method is given with bold font. As the third method **Adams-Bashford-Moulton formulas.** This formulas like Runge-Kutta sets can be given in different orders. They are given as two sets. The first set (Adams-Bashford) is used as a predictor, and the second set is used as a corrector step.

Predictor Formula :

Adams-Bashford 1 step formula

$$y_{i+1} = y_i + hf(x_i, y_i)$$

Adams-Bashford 2 step formula

$$y_{i+1} = y_i + \frac{h}{2}\left(3f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right)$$

Adams-Bashford 3 step formula

$$y_{i+1} = y_i + \frac{h}{12}\left(23f(x_i, y_i) - 16f(x_{i-1}, y_{i-1}) + 5f(x_{i-2}, y_{i-2})\right)$$

Adams-Bashford 4 step formula

$$y_{i+1} = y_i + \frac{h}{24}\left(55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) - 9f(x_{i-3}, y_{i-3})\right)$$

Adams-Bashford 5 step formula

$$y_{i+1} = y_i + \frac{h}{720}\left(1901f(x_i, y_i) - 2774f(x_{i-1}, y_{i-1}) + 2616f(x_{i-2}, y_{i-2}) - 1274f(x_{i-3}, y_{i-3}) + 251f(x_{i-4}, y_{i-4})\right)$$

Corrector Formula :

Adams-Moulton 0 step formula

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

Adams-Moulton 1 step formula

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_{i+1}, y_{i+1}) + f(x_i, y_i)\right)$$

Adams-Moulton 2 step formula

$$y_{i+1} = y_i + \frac{h}{12}\left(5f(x_{i+1}, y_{i+!}) + 8f(x_i, y_i) - f(x_{i-1}, y_{i-!})\right)$$

Adams-Moulton 3 step formula

$$y_{i+1} = y_i + \frac{h}{24}\left(9f(x_{i+1}, y_{i+1}) + 19f(x_i, y_i) - 5f(x_{i-1}, y_{i-1}) + f(x_{i-2}, y_{i-2})\right)$$

Adams-Moulton 4 step formula

$$y_{i+1} = y_i + \frac{h}{720}\left(251 f(x_{i+1}, y_{i+1}) + 646 f(x_i, y_i) - 264 f(x_{i-1}, y_{i-1}) + 106 f(x_{i-2}, y_{i-2}) - 19 f(x_{i-3}, y_{i-3})\right)$$

Program list is given as the following program. In this program two method is defined. In the first method Adams-Bashford 4 step Formula is used together with Adams-Milton 3 step Formula. In the second method Adams-Bashford 5 step Formula is used together with Adams-Milton 4 step Formula.

Program Adams-Bashford-Milton Predictor-corrector method.

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Adams-BAshford-Milton Predictor Corrector formula
// with Adams-Bashford 4 step predictor and Adams-Milford 3 step corrector
// also with Adams-Bashford 5 step predictor and Adams-Milford 4 step corrector
// Dr. Turhan Coban
// =====================================================
// dy/dx=f(x,y)

import java.io.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy extends f_xy
{
  double func(double x,double y)
  {return 1.0/(1.0+x*x)-2.0*y*y;}
}

class fxyKC extends f_x
{
  double func(double x)
  {return x/(1+x*x);}
}

class NA91
{
  public static double[][] RK6AB4AM3(f_xy f,double x0,double y0,double xn,double h)
  {
  // Adams-Bashford (4 adim formülü)  Adams-Moulton (3 adim formülü) predictor corrector method
  //with RK6 initial step
  int n=(int)((xn-x0)/h);
  double a[][]=new double[2][n+1];
  // Runge-Kutta 6
  int i=0;
  a[0][0]=x0;
  a[1][0]=y0;
  double x3=x0+3.0*h;
  double k0,k1,k2,k3,k4,k5,k6;
  for(double xx=x0;xx<x3;xx+=h)
   {a[0][i+1]=xx+h;
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i]+0.25*h,a[1][i]+0.25*k1*h);
   k3=f.func(a[0][i]+0.25*h,a[1][i]+1.0/8.0*k1*h+1.0/8.0*k2*h);
   k4=f.func(a[0][i]+0.5*h,a[1][i]-0.5*k2*h+k3*h);
   k5=f.func(a[0][i]+3.0/4.0*h,a[1][i]+3.0/16.0*k1*h+9.0/16.0*k4*h);
   k6=f.func(a[0][i]+h,a[1][i]-3.0/7.0*k1*h+2.0/7.0*k2*h+12.0/7.0*k3*h-12.0/7.0*k4*h+8.0/7.0*k5*h);
   a[1][i+1]=a[1][i]+(7.0*k1+32.0*k3+12.0*k4+32.0*k5+7*k6)/90.0*h;
   i++;
   }
   for(double xx=x3;xx<xn;xx+=h)
   {a[0][i+1]=xx+h;
   //Adams-Bashford
   k1=f.func(a[0][i],a[1][i]);
   k2=f.func(a[0][i-1],a[1][i-1]);
```

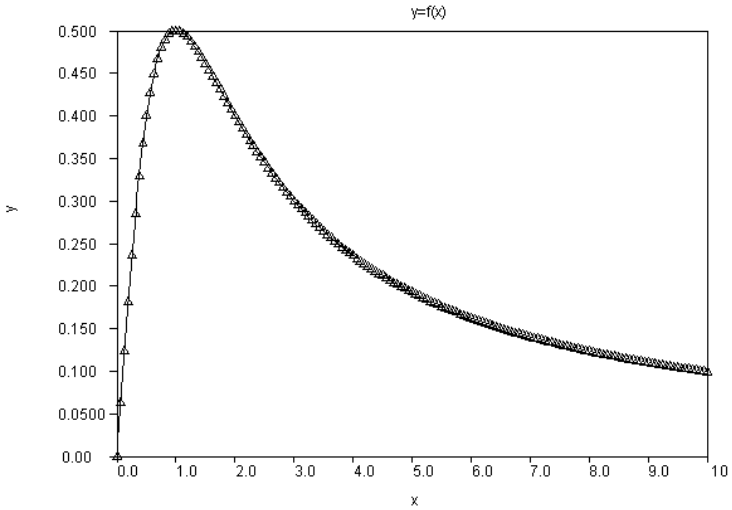
```
 k3=f.func(a[0][i-2],a[1][i-2]);
 k4=f.func(a[0][i-3],a[1][i-3]);
 a[1][i+1]=a[1][i]+h/24.0*(55.0*k1-59.0*k2+37.0*k3-9.0*k4);
 //Adams-Moulton j 1 den büyük olabilir
 for(int j=0;j<5;j++)
 {k0=f.func(a[0][i+1],a[1][i+1]);
 a[1][i+1]=a[1][i]+h/24.0*(9.0*k0+19.0*k1-5.0*k2+k3);
 }
 i++;
 }
return a;
}

public static double[][] RK6AB5AM4(f_xy f,double x0,double y0,double xn,double h)
{
// Adams-Bashford (5 adim formülü)  Adams-Moulton (4 adim formülü) predictor corrector method
//with RK6 initial step
int n=(int)((xn-x0)/h);
double a[][]=new double[2][n+1];
// Runge-Kutta 6
int i=0;
a[0][0]=x0;
a[1][0]=y0;
double x4=x0+4.0*h;
double k0,k1,k2,k3,k4,k5,k6;
for(double xx=x0;xx<x4;xx+=h)
 {a[0][i+1]=xx+h;
 k1=f.func(a[0][i],a[1][i]);
 k2=f.func(a[0][i]+0.25*h,a[1][i]+0.25*k1*h);
 k3=f.func(a[0][i]+0.25*h,a[1][i]+1.0/8.0*k1*h+1.0/8.0*k2*h);
 k4=f.func(a[0][i]+0.5*h,a[1][i]-0.5*k2*h+k3*h);
 k5=f.func(a[0][i]+3.0/4.0*h,a[1][i]+3.0/16.0*k1*h+9.0/16.0*k4*h);
 k6=f.func(a[0][i]+h,a[1][i]-3.0/7.0*k1*h+2.0/7.0*k2*h+12.0/7.0*k3*h-12.0/7.0*k4*h+8.0/7.0*k5*h);
 a[1][i+1]=a[1][i]+(7.0*k1+32.0*k3+12.0*k4+32.0*k5+7*k6)/90.0*h;
 i++;
 }
 for(double xx=x4;xx<xn;xx+=h)
 {a[0][i+1]=xx+h;
 //Adams-Bashford
 k1=f.func(a[0][i],a[1][i]);
 k2=f.func(a[0][i-1],a[1][i-1]);
 k3=f.func(a[0][i-2],a[1][i-2]);
 k4=f.func(a[0][i-3],a[1][i-3]);
 k5=f.func(a[0][i-4],a[1][i-4]);
 a[1][i+1]=a[1][i]+h/720.0*(1901.0*k1-2774.0*k2+2616.0*k3-1274.0*k4+251*k5);
 //Adams-Moulton j 1 den büyük olabilir
 for(int j=0;j<5;j++)
 {k0=f.func(a[0][i+1],a[1][i+1]);
 a[1][i+1]=a[1][i]+h/720.0*(251.0*k0+646.0*k1-264.0*k2+106.0*k3-19*k4);
 }
 i++;
 }
return a;
}

public static void main(String args[]) throws IOException
{double x0=0;
 double y0=0.0;
 double yn=10.0;
 double h=0.0625;
 fxy f2=new fxy();
 fxyKC fKC=new fxyKC();
 double a[][]=RK6AB4AM3(f2,x0,y0,yn,h);
 Text.print(Text.T(a));
 Plot p1=new Plot(a);
 p1.setPlotType(0,24);
 //p1.addSpline(a,10);
 p1.addFunction(fKC,0.0,4.0,200);
 p1.plot();
}
```

```
}
```

Output 8.5-4 Adams-Bashford 4 step Adams-Milton 3 step Predictor-corrector formüla graphic output



In order to check the error created solution and exact solution is given as a table. The first steps calculated with Runge –Kutta 6 method is given as bold font.

Output 8.5-5 Adams-Bashford 4 step Adams-Milton 3 step Predictor-corrector formüla calculated output with error

x	y_calculated	y_exact	Error
0	**0**	**0**	**0**
0.0625	**0.062256809**	**0.062256809**	**1.29392E-10**
0.125	**0.123076922**	**0.123076923**	**6.07481E-10**
0.1875	**0.181132074**	**0.181132075**	**1.36894E-09**
0.25	0.235295557	0.235294118	-1.43921E-06
0.3125	0.284699363	0.284697509	-1.8537E-06
0.375	0.328768525	0.328767123	-1.40145E-06
0.4375	0.367213479	0.367213115	-3.64545E-07
0.5	0.399999069	0.4	9.31283E-07
0.5625	0.427297499	0.427299703	2.20423E-06
0.625	0.449434941	0.449438202	3.26136E-06
0.6875	0.466839495	0.466843501	4.0062E-06
0.75	0.479995578	0.48	4.42175E-06
0.8125	0.489407221	0.489411765	4.54356E-06
0.875	0.495570788	0.495575221	4.43364E-06
0.9375	0.498956338	0.498960499	4.16066E-06
1	0.499996212	0.5	3.78762E-06
1.0625	0.499079203	0.499082569	3.36586E-06
1.125	0.496548791	0.496551724	2.93349E-06
1.1875	0.492704129	0.492706645	2.51644E-06
1.25	0.487802747	0.487804878	2.13058E-06
1.3125	0.482064213	0.482065997	1.78422E-06
1.375	0.475674195	0.475675676	1.48028E-06
1.4375	0.468788591	0.468789809	1.21822E-06
1.5	0.461537466	0.461538462	9.95355E-07
1.5625	0.454028704	0.454029512	8.0791E-07
1.625	0.44635128	0.446351931	6.51661E-07
1.6875	0.438578158	0.43857868	5.22374E-07
1.75	0.430768815	0.430769231	4.16056E-07
1.8125	0.422971412	0.422971741	3.29088E-07
1.875	0.415224655	0.415224913	2.58278E-07
1.9375	0.407559372	0.407559573	2.00868E-07
2	0.399999845	0.4	1.54505E-07

In order to see the error comparisons, the same problem is also solved with Adam Bashford 5 step predictor and Adams-Milton 4 step corrector

Output Adams-Bashford 5 step predictor and Adams-Milton 4 step corrector and the error created compare to exact solution.

x	y_calculated	y_exact solution	Error
0.0000000000000000000	**0.0000000000000000000**	**0.0000000000000000000**	**0.0000000000000000000E+00**
0.0625000000000000000	**0.0622568092091299000**	**0.0622568093385214000**	**1.29391504455345000000E-10**
0.1250000000000000000	**0.1230769224694420000**	**0.1230769230769230000**	**6.07481079328842000000E-10**
0.1875000000000000000	**0.1811320741027570000**	**0.1811320754716980000**	**1.36894109936669000000E-09**

0.25000000000000000000	**0.23529411533906500000**	**0.23529411764705900000**	**2.30799382383573000000E-09**
0.31250000000000000000	0.28469685475736000000	0.28469750889679700000	6.54139437172852000000E-07
0.37500000000000000000	0.32876594499773400000	0.32876712328767100000	1.17828993723146000000E-06
0.43750000000000000000	0.36721161302808700000	0.36721311475409800000	1.50172601132148000000E-06
0.50000000000000000000	0.39999839935601100000	0.40000000000000000000	1.60064398901927000000E-06
0.56250000000000000000	0.42729819196822300000	0.42729970326409500000	1.51129587194765000000E-06
0.62500000000000000000	0.44943690583998300000	0.44943820224719100000	1.29640720797353000000E-06
0.68750000000000000000	0.46684247928276000000	0.46684350132626000000	1.02204349994306000000E-06
0.75000000000000000000	0.47999925820049200000	0.48000000000000000000	7.41799507986141000000E-07
0.81250000000000000000	0.48941127391795700000	0.48941176470588200000	4.90787925355729000000E-07
0.87500000000000000000	0.49557493472326900000	0.49557522123893800000	2.86515669023490000000E-07
0.93750000000000000000	0.49896036576082800000	0.49896049896049900000	1.33199671015216000000E-07
1.00000000000000000000	0.49999997320325300000	0.50000000000000000000	2.67967469880226000000E-08
1.06250000000000000000	0.49908260956179300000	0.49908256880733900000	-4.07544535341309000000E-08
1.12500000000000000000	0.49655180272928400000	0.49655172413793100000	-7.85913529810500000000E-08
1.18750000000000000000	0.49270674027182300000	0.49270664505672600000	-9.52150969180110000000E-08
1.25000000000000000000	0.48780497575024000000	0.48780487804878000000	-9.77014595338055000000E-08
1.31250000000000000000	0.48206608863152900000	0.48206599713056000000	-9.15009694679725000000E-08
1.37500000000000000000	0.47567575623482700000	0.47567567567567600000	-8.05591512942350000000E-08
1.43750000000000000000	0.46878987649136200000	0.46878980891719700000	-6.75741645417283000000E-08
1.50000000000000000000	0.46153851582225100000	0.46153846153846200000	-5.42837894634829000000E-08
1.56250000000000000000	0.45402955364521000000	0.45402951191827500000	-4.17269353514271000000E-08
1.62500000000000000000	0.44635196178861500000	0.44635193133047200000	-3.04581428967055000000E-08
1.68750000000000000000	0.43857870091491100000	0.43857868020304600000	-2.07118653339222000000E-08
1.75000000000000000000	0.43076924329127700000	0.43076923076923100000	-1.25220462332543000000E-08
1.81250000000000000000	0.42297174691765600000	0.42297174111212400000	-5.80553199958089000000E-09
1.87500000000000000000	0.41522491391281700000	0.41522491349481000000	-4.18007350955207000000E-10
1.93750000000000000000	0.40755956890978500000	0.40755957271980300000	3.81001780125345000000E-09
2.00000000000000000000	0.39999999294883700000	0.40000000000000000000	7.05116304233755000000E-09

METHODS WITH VARIABLE STEP SIZES

One way of reducing the error, is changing step sizes by using error estimation methods. In order to achive this, two equations with different error level is required. Runge Kutta equations were defined in general form as

$$x_i = x_n + c_i h \quad \text{i=1…s}$$

$$y_i = y_n + h\sum_{j=1}^{i-1} a_{ij} k_j$$

$$k_i = f(x_i, y_i)$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s} b_i k_j \text{ or}$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s} bh_i k_j$$

Runge Kutta-Fehlberg is one of the best candidate for variable step size due to fact that two different degree equations can be given with the same coefficients. Runge Kutta Fehlberg coefficients were given in Table 8.4-6. If equations are written with coefficients it becomes :

$$y_{i+1} = y_i + (\frac{37}{378}k_1 + \frac{250}{621}k_3 + \frac{125}{591}k_4 + \frac{512}{1771}k_6)h \qquad \text{(4. order Runge-Kutta- Fehlberg)}$$

$$y_{i+1} = y_i + (\frac{2825}{27648}k_1 + \frac{18575}{48384}k_3 + \frac{13525}{55296}k_4 + \frac{277}{14336}k_5 + \frac{1}{4}k_6)h \qquad \text{(5. order Runge-Kutta- Fehlberg)}$$

$k_1=f(x_i,y_i)$
$k_2=f(x_i+0.2h\ ,\ y_i+0.2k_1h)$
$k_3=f(x_i+0.3h\ ,\ y_i+(3/40)k_1h+(9/40)k_2h)$
$k_4=f(x_i+(3/5)h\ ,\ y_i + 0.3k_1h - (9/10)k_2h +(6/5)k_3h)$
$k_5=f(x_i+h\ ,\ y_i - (11/54)k_1h+0.25k_2h-(70/27)k_3h+(35/27)k_4h)$
$k_6=f(x_i+(7/8)h\ ,\ y_i + (1631/55296)k_1h+(175/512)k_2h+(575/13824)k_3h + (44275/110592)k_4h+(253/4096)k_5h)$

The error term

$$h_{new} = h_{previous}\left|\frac{\Delta_{new}}{\Delta_{previous}}\right|^{\alpha}$$

$$\Delta_{new} = \varepsilon\, y_{measure}$$

ε is a general tolerance level. $y_{measure}$ indicates how error will be measured. For example $y_{measure}$ = y can be choosen. In our program, $y_{measure}$ will be taken as

$y_{measure} = |y| + \left|y\frac{dy}{dx}\right|$. The value of α can be taken as 0.2 if $(\Delta_{previous} \le \Delta_{new})$; as 0.25 if $(\Delta_{previous} \ge \Delta_{new})$.

As an example differential equation the following equation will be taken

$$\frac{dy}{dx} = -0.6y + 10e^{-(x-2)^2/[2(0.075)^2]}$$

Solution of this differential equation has a jump point around x=2 . Because of this, solving it with constant step size is relatively difficult.

Program Runge-Kutta-Fehlberg variable step method with variable step size

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta-Fehlberg, Cash-Karp, Dormant-Price, Verner
//  variable step methods
// Dr. Turhan Coban
// =====================================================
// dy/dx=f(x,y)

import java.io.*;
import java.util.*;

abstract class f_xy
{
  //single function single independent variable
  // example f=x*x
  abstract double func(double x,double y);
}

class fxy2 extends f_xy
{
  double func(double x,double y)
  {return -0.6*y+10.0*Math.exp(-(x-2)*(x-2)/(2*0.075*0.075));}
}

class NA92H1
{

public static double[][] RKCK45(f_xy f,double x0,double y0,double xn,double h,double eps)
{
   double hmax=(xn-x0)/40;
   double hmin=hmax/100;
// Runge-Kutta Cash-Karp order 4-5
//eps bağıl hata
int s=7;
double dydx=1.0;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
double b[]=new double[s];
c[2] =  0.2;
c[3] =  0.3;
```

```
c[4] =  0.6;
c[5] =  1;
c[6] = 7.0/8.0;

a[1][1] = 0;

a[2][1] =  0.2;

a[3][1] =  3.0/40.0;
a[3][2] =  9.0/40.0;

a[4][1] =  0.3;
a[4][2] =  -9.0/10.0;
a[4][3] =  6.0/5.0;

a[5][1] =  -11.0/54.0;
a[5][2] =  0.25;
a[5][3] =  -70.0/27.0;
a[5][4] =  35.0/27.0;

b[1] =  37.0/378.0;
b[2] =  250.0/621.0;
b[3] =  125.0/591.0;
b[4] =  512.0/1771.0;
b[5] =  0.0;

bh[1] =  2825.0/27648.0;
bh[2] =  18575.0/48384.0;
bh[3] =  13525.0/55296.0;
bh[4] =  277.0/14336.0;
bh[5] =  1.0/4.0;

  double alfa;
  //double a[][]=new double[2][n+1];
  int i=0;
  double delta=h;
  double deltayeni=eps*y0;
  double k1,k2,k3,k4,k5,k6;
  Vector<Double> xi=new Vector<Double>();
  Vector<Double> yi=new Vector<Double>();
  double x,y,yp1_5,yp1_6,xeski,yeski;
  x=x0;
  y=y0;
  xeski=x;
  yeski=y;
  Double Xi=new Double(x);
  Double Yi=new Double(y);
  xi.add(Xi);
  yi.add(Yi);
  while((x+h)<=xn)
   {
   xeski=x;
   yeski=y;
   k1=f.func(x,y);
   k2=f.func(x+c[2]*h,y+a[2][1]*k1*h);
   k3=f.func(x+c[3]*h,y+a[3][1]*k1*h+a[3][2]*k2*h);
   k4=f.func(x+c[4]*h,y+a[4][1]*k1*h+a[4][2]*k2*h+a[4][3]*k3*h);
   k5=f.func(x+c[5]*h,y+a[5][1]*k1*h+a[5][2]*k2*h+a[5][3]*k3*h+a[5][4]*k4*h);
   k6=f.func(x+c[6]*h,y+a[6][1]*k1*h+a[6][2]*k2*h+a[6][3]*k3*h+a[6][4]*k4*h+a[6][5]*k5*h);
   yp1_5=y+(b[1]*k1+b[2]*k2+b[3]*k3+b[4]*k4+b[5]*k5+b[6]*k6)*h;
   yp1_6=y+(bh[1]*k1+bh[2]*k2+bh[3]*k3+bh[4]*k4+bh[5]*k5+bh[6]*k6)*h;
   y=yp1_6;
   x=x+h;
   Xi=new Double(x);
   Yi=new Double(y);
   xi.add(Xi);
   yi.add(Yi);
   delta=Math.abs(yp1_6-yp1_5);
   dydx=f.func(x,y);
   deltayeni=Math.abs(eps*h*dydx);
```

```
  if(delta<=deltayeni) alfa=0.2;
  else                 alfa=0.25;
  h=h*0.9437436*Math.pow((deltayeni/delta),alfa);
  if(h>hmax) h=hmax;
  if(h<hmin) h=hmin;
  i++;
  }
  if(x<xn)//son bir step daha
  {
  h=xn-x;
  k1=f.func(x,y);
  k2=f.func(x+c[2]*h,y+a[2][1]*k1*h);
  k3=f.func(x+c[3]*h,y+a[3][1]*k1*h+a[3][2]*k2*h);
  k4=f.func(x+c[4]*h,y+a[4][1]*k1*h+a[4][2]*k2*h+a[4][3]*k3*h);
  k5=f.func(x+c[5]*h,y+a[5][1]*k1*h+a[5][2]*k2*h+a[5][3]*k3*h+a[5][4]*k4*h);
  k6=f.func(x+c[6]*h,y+a[6][1]*k1*h+a[6][2]*k2*h+a[6][3]*k3*h+a[6][4]*k4*h+a[6][5]*k5*h);
  yp1_5=y+(b[1]*k1+b[2]*k2+b[3]*k3+b[4]*k4+b[5]*k5+b[6]*k6)*h;
  yp1_6=y+(bh[1]*k1+bh[2]*k2+bh[3]*k3+bh[4]*k4+bh[5]*k5+bh[6]*k6)*h;
  y=yp1_6;
  //System.out.println("x="+x+"h="+h+"y="+yp1_6);
  x=xn;
  Xi=new Double(x);
  Yi=new Double(y);
  xi.add(Xi);
  yi.add(Yi);
  i++;
  }
  //vektöre aktarılan değerleri double matrise aktar
  int n=i;
      double aa[][]=new double[2][n+1];
      Enumeration nx=xi.elements();
      Enumeration ny=yi.elements();
      for(i=0;i<=n;i++)
      {
                //String ss1=(String)nx.nextElement();
                //String ss2=(String)ny.nextElement();
                Double ax=(Double)nx.nextElement();
                Double ay=(Double)ny.nextElement();
      aa[0][i]=ax.doubleValue();
      aa[1][i]=ay.doubleValue();
      }
  return aa;
 }


public static double[][] RKDP45(f_xy f,double x0,double y0,double xn,double h,double eps)
{
  double hmax=(xn-x0)/40;
  double hmin=hmax/100;
// Runge-Kutta Dormant-Price order 4-5
// eps relative error
int s=8;
double dydx=1.0;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
double b[]=new double[s];
c[1]=0.0e0;
     c[2]=0.2;
     c[3]=0.3;
     c[4]=0.8;
     c[5]=8.0/9.0;
     c[6]=1;
     c[7]=1;

     a[2][1]=0.2;
     a[3][1]=3.0/40.0;
     a[3][2]=9.0/40.0;
     a[4][1]=44.0/45.0;
     a[4][2]=-56.0/15.0;
```

```
    a[4][3]=32.0/9.0;
    a[5][1]=19372.0/6561.0;
    a[5][2]=-25360.0/2187.0;
    a[5][3]=64448.0/6561.0;
    a[5][4]=-212.0/729.0;
    a[6][1]=9017.0/3168.0;
    a[6][2]=-355.0/33.0;
    a[6][3]=46732.0/5247.0;
    a[6][4]=49.0/176.0;
    a[6][5]=-5103.0/18656.0;
    a[7][1]=35.0/384.0;
    a[7][3]=500.0/1113.0;
    a[7][4]=125.0/192.0;
    a[7][5]=-2187.0/6784.0;
    a[7][6]=11.0/84.0;

    b[1]=5179.0/57600.0;
    b[2]=0;
    b[3]=-7571.0/16695.0;
    b[4]=393.0/640.0;
    b[5]=-92097.0/339200.0;
    b[6]=187.0/2100.0;
    b[7]=1.0/40.0;

    bh[1]=35.0/384.0;
    bh[2]=0;
    bh[3]=500.0/1113.0;
    bh[4]=125.0/192.0;
    bh[5]=-2187.0/6784.0;
    bh[6]=11.0/84.0;
    bh[7]=0.0;

double alfa;
//double a[][]=new double[2][n+1];
int i=0;
//a[0][0]=x0;
//a[1][0]=y0;
double delta=h;
double deltayeni=eps*y0;
double k1,k2,k3,k4,k5,k6,k7;
Vector<Double> xi=new Vector<Double>();
Vector<Double> yi=new Vector<Double>();
double x,y,yp1_5,yp1_6,xeski,yeski;
x=x0;
y=y0;
xeski=x;
yeski=y;
Double Xi=new Double(x);
Double Yi=new Double(y);
xi.add(Xi);
yi.add(Yi);
while((x+h)<=xn)
 {
 xeski=x;
 yeski=y;
 k1=f.func(x,y);
 k2=f.func(x+c[2]*h,y+a[2][1]*k1*h);
 k3=f.func(x+c[3]*h,y+a[3][1]*k1*h+a[3][2]*k2*h);
 k4=f.func(x+c[4]*h,y+a[4][1]*k1*h+a[4][2]*k2*h+a[4][3]*k3*h);
 k5=f.func(x+c[5]*h,y+a[5][1]*k1*h+a[5][2]*k2*h+a[5][3]*k3*h+a[5][4]*k4*h);
 k6=f.func(x+c[6]*h,y+a[6][1]*k1*h+a[6][2]*k2*h+a[6][3]*k3*h+a[6][4]*k4*h+a[6][5]*k5*h);
 k7=f.func(x+c[7]*h,y+a[7][1]*k1*h+a[7][2]*k2*h+a[7][3]*k3*h+a[7][4]*k4*h+a[7][5]*k5*h+a[7][6]*k6*h);
 yp1_5=y+(b[1]*k1+b[2]*k2+b[3]*k3+b[4]*k4+b[5]*k5+b[6]*k6+b[7]*k7)*h;
 yp1_6=y+(bh[1]*k1+bh[2]*k2+bh[3]*k3+bh[4]*k4+bh[5]*k5+bh[6]*k6+bh[7]*k7)*h;
 y=yp1_6;
 x=x+h;
 Xi=new Double(x);
 Yi=new Double(y);
 xi.add(Xi);
 yi.add(Yi);
```

```
 delta=Math.abs(yp1_6-yp1_5);
 dydx=f.func(x,y);
 deltayeni=Math.abs(eps*h*dydx);
 if(delta<=deltayeni) alfa=0.2;
 else                 alfa=0.25;
 h=h*0.9437436*Math.pow((deltayeni/delta),alfa);
 if(h>hmax) h=hmax;
 if(h<hmin) h=hmin;
 i++;
 }
 if(x<xn)//son bir step daha
 {
 h=xn-x;
 xeski=x;
 yeski=y;
 k1=f.func(x,y);
 k2=f.func(x+c[2]*h,y+a[2][1]*k1*h);
 k3=f.func(x+c[3]*h,y+a[3][1]*k1*h+a[3][2]*k2*h);
 k4=f.func(x+c[4]*h,y+a[4][1]*k1*h+a[4][2]*k2*h+a[4][3]*k3*h);
 k5=f.func(x+c[5]*h,y+a[5][1]*k1*h+a[5][2]*k2*h+a[5][3]*k3*h+a[5][4]*k4*h);
 k6=f.func(x+c[6]*h,y+a[6][1]*k1*h+a[6][2]*k2*h+a[6][3]*k3*h+a[6][4]*k4*h+a[6][5]*k5*h);
 k7=f.func(x+c[7]*h,y+a[7][1]*k1*h+a[7][2]*k2*h+a[7][3]*k3*h+a[7][4]*k4*h+a[7][5]*k5*h+a[7][6]*k6*h);
 yp1_5=y+(b[1]*k1+b[2]*k2+b[3]*k3+b[4]*k4+b[5]*k5+b[6]*k6+b[7]*k7)*h;
 yp1_6=y+(bh[1]*k1+bh[2]*k2+bh[3]*k3+bh[4]*k4+bh[5]*k5+bh[6]*k6+bh[7]*k7)*h;
 y=yp1_6;
 //System.out.println("x="+x+"h="+h+"y="+yp1_6);
 x=xn;
 Xi=new Double(x);
 Yi=new Double(y);
 xi.add(Xi);
 yi.add(Yi);
 i++;
 }
 //vektöre aktarılan değerleri double matrise aktar
 int n=i;
     double aa[][]=new double[2][n+1];
     Enumeration nx=xi.elements();
     Enumeration ny=yi.elements();
     for(i=0;i<=n;i++)
     {
               //String ss1=(String)nx.nextElement();
               //String ss2=(String)ny.nextElement();
               Double ax=(Double)nx.nextElement();
               Double ay=(Double)ny.nextElement();
     aa[0][i]=ax.doubleValue();
     aa[1][i]=ay.doubleValue();
     }
 return aa;
}

public static double[][] RKF45(f_xy f,double x0,double y0,double xn,double h,double eps)
{
 //  Runge-Kutta Fehlberg order 4-5
 //  eps relative error
 double hmax=(xn-x0)/40;
 double hmin=hmax/1000;
double sh=1;
double dydx=1.0;
double alfa;
int i=0;
double delta=h;
double deltayeni=eps*y0;
double k1,k2,k3,k4,k5,k6;
Vector<Double> xi=new Vector<Double>();
Vector<Double> yi=new Vector<Double>();
double x,y,yp1_5,yp1_6,xeski,yeski;
x=x0;
y=y0;
xeski=x;
yeski=y;
```

```
Double Xi=new Double(x);
Double Yi=new Double(y);
xi.add(Xi);
yi.add(Yi);
while((x+h)<=xn)
 {
 xeski=x;
 yeski=y;
 k1=f.func(x,y);
 k2=f.func(x+1.0/4.0*h,y+1.0/4.0*k1*h);
 k3=f.func(x+3.0/8.0*h,y+3.0/32.0*k1*h+9.0/32.0*k2*h);
 k4=f.func(x+12/13*h,y+1932.0/2197.0*k1*h-7200.0/2197.0*k2*h+7296/2197*k3*h);
 k5=f.func(x+h,y+439.0/216.0*k1*h-8.0*k2*h+3680.0/513.0*k3*h-845.0/4104.0*k4*h);
 k6=f.func(x+1.0/2.0*h,y-8.0/27.0*k1*h+2.0*k2*h+-3544.0/2565.0*k3*h+1859.0/4104.0*k4*h-11.0/40.0*k5*h);

 yp1_5=y+(25.0/216.0*k1+1408.0/2565.0*k3+2197.0/4104.0*k4-1.0/5.0*k6)*h;
 yp1_6=y+(16.0/135.0*k1+6656.0/12825.0*k3+28561.0/56430.0*k4-9.0/50.0*k5+2.0/55.0*k6)*h;
 y=yp1_6;
 x=x+h;
 Xi=new Double(x);
 Yi=new Double(y);
 xi.add(Xi);
 yi.add(Yi);
 delta=Math.abs(yp1_6-yp1_5);
 dydx=f.func(x,y);
 deltayeni=Math.abs(eps*h*dydx);
 if(delta<=deltayeni) alfa=0.25;
 else                 alfa=1.0/6.0;
 h=h*0.98*Math.pow((deltayeni/delta),alfa);
 if(h>hmax) h=hmax;
 if(h<hmin) h=hmin;
 i++;
 }
 if(x<xn)//son bir step daha
 {
 h=xn-x;
 k1=f.func(x,y);
 k2=f.func(x+1.0/4.0*h,y+1.0/4.0*k1*h);
 k3=f.func(x+3.0/8.0*h,y+3.0/32.0*k1*h+9.0/32.0*k2*h);
 k4=f.func(x+12/13*h,y+1932.0/2197.0*k1*h-7200.0/2197.0*k2*h+7296/2197*k3*h);
 k5=f.func(x+h,y+439.0/216.0*k1*h-8.0*k2*h+3680.0/513.0*k3*h-845.0/4104.0*k4*h);
 k6=f.func(x+1.0/2.0*h,y-8.0/27.0*k1*h+2.0*k2*h+-3544.0/2565.0*k3*h+1859.0/4104.0*k4*h-11.0/40.0*k5*h);

 yp1_5=y+(25.0/216.0*k1+1408.0/2565.0*k3+2197.0/4104.0*k4-1.0/5.0*k6)*h;
 yp1_6=y+(16.0/135.0*k1+6656.0/12825.0*k3+28561.0/56430.0*k4-9.0/50.0*k5+2.0/55.0*k6)*h;
 y=yp1_6;
 //System.out.println("x="+x+"h="+h+"y="+yp1_6);
 x=xn;
 Xi=new Double(x);
 Yi=new Double(y);
 xi.add(Xi);
 yi.add(Yi);
 i++;
 }
 //vektöre aktarılan değerleri double matrise aktar
 int n=i;
     double aa[][]=new double[2][n+1];
     Enumeration nx=xi.elements();
     Enumeration ny=yi.elements();
     for(i=0;i<=n;i++)
     {
               //String ss1=(String)nx.nextElement();
               //String ss2=(String)ny.nextElement();
               Double ax=(Double)nx.nextElement();
               Double ay=(Double)ny.nextElement();
     aa[0][i]=ax.doubleValue();
     aa[1][i]=ay.doubleValue();
     }
 return aa;
}
```

```
public static double[][] RKV16(f_xy f,double x0,double y0,double xn,double h,double eps)
{  //Runge-Kutta-Verner 16th degree step constant step size version
double hmax=(xn-x0)/40;
double hmin=hmax/100;
double dydx=1.0;
double delta,deltayeni,alfa;
int s=17;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
double b[]=new double[s];
c[0]=0;
c[1] =  0.;
c[2] =  .6e-1;
c[3] =  .95933333333333333333333333333333333333333333e-1;
c[4] =  .1439;
c[5] =  .4973;
c[6] =  .9725;
c[7] =  .9995;
c[8] =  1.;
c[9] =  1.;
a[2][1] =  .6e-1;
a[3][1] =  .19239962962962962962962962962962962962962e-1;
a[3][2] =  .76693370370370370370370370370370370370370e-1;
a[4][1] =  .35975e-1;
a[4][2] =  0.;
a[4][3] =  .107925;
a[5][1] =  1.3186834152331482609197472764317356128 61;
a[5][2] =  0.;
a[5][3] = -5.0420580636285622254277616347156376933 44;
a[5][4] =  4.2206746483954139645080143582839020804 83;
a[6][1] = -41.8725916643275146180375778064434681290 5;
a[6][2] =  0.;
a[6][3] =  159.43256216313749177003656690703468304 53;
a[6][4] = -122.1192135650100309202516203389242140663;
a[6][5] =  5.5317430662000537682526312383329991500 76;
a[7][1] = -54.4301569353165043325064205129414246127 1;
a[7][2] =  0.;
a[7][3] =  207.0672513650184644273657173866509835987;
a[7][4] = -158.6108137845899991828742424365058599469;
a[7][5] =  6.9918165859502423219925972807917939070 96;
a[7][6] = -.1859723106220323397765171799549294623692e-1;

a[8][1] = -54.6637417872819768024121564805038695935 1;
a[8][2] =  0.;
a[8][3] =  207.9528062553893734515824816699834244238;
a[8][4] = -159.2889574744995071508959805871426654216;
a[8][5] =  7.0187437407969444346981707609642524908 17;
a[8][6] = -.1833878590504572306472782005141738268361e-1;
a[8][7] = -.5119484997882099077875432497245168395840e-3;

a[9][1] =  .3438957868357036009278820124728322386520e-1;
a[9][2] =  0.;
a[9][3] =  0.;
a[9][4] =  .2582624555633503404659558098586120858767;
a[9][5] =  .4209371189673537150642551514069801967032;
a[9][6] =  4.4053964696693101701488368161970956648 91;
a[9][7] = -176.4831190242986576151740942499002125029;
a[9][8] =  172.3641334014150730294022582711902413315;

b[1] =  .3438957868357036009278820124728322386520e-1;
b[2] =  0.;
b[3] =  0.;
b[4] =  .2582624555633503404659558098586120858767;
b[5] =  .4209371189673537150642551514069801967032;
b[6] =  4.4053964696693101701488368161970956648 91;
b[7] = -176.4831190242986576151740942499002125029;
b[8] =  172.3641334014150730294022582711902413315;
b[9] =  0.;
```

```
bh[ 1] =  .490996764838248973090685492797122583647 9e-1;
bh[ 2] =  0.;
bh[ 3] =  0.;
bh[ 4] =  .225111222951652415340139532053987532 9485;
bh[ 5] =  .469468225302956203943194852504738741 2553;
bh[ 6] =  .806579224998886770763416180899521798 1443;
bh[ 7] =  0.;
bh[ 8] = -.607119489177795979767295146525621712 2488;
bh[ 9] =  .568611394404756924114760317876613815 3594e-1;
 // Runge-Kutta Verner 16 variable step
 //eps relative error
 //double a[][]=new double[2][n+1];
 int i=0;
 double x1,y1;
 Vector<Double> xi=new Vector<Double>();
 Vector<Double> yi=new Vector<Double>();
 double x,y,yp1=0,yp2=0,xeski,yeski;

 x=x0;
 y=y0;
 xeski=x;
 yeski=y;
 Double Xi=new Double(x);
 Double Yi=new Double(y);
 double k[]=new double[s];
 xi.add(Xi);
 yi.add(Yi);
 int n=1;
 while((x+h)<=xn)
 {
  xeski=x;
  yeski=y;
  double total=0;
  //yp1=y;
  yp2=y;
  double w=1;
  for(int ii=1;ii<s;ii++)
  {  x1=x+c[ii]*h;
     y1=y;
               for(int j=1;j<ii;j++)
    {y1+=h*a[ii][j]*k[j];}
     k[ii]=f.func(x1,y1);
     yp2+=h*bh[ii]*k[ii];
     yp1+=h*b[ii]*k[ii];
  }
  delta=Math.abs(yp2-yp1);
  dydx=f.func(x,y);
  deltayeni=Math.abs(eps*h*dydx);
  if(delta<=deltayeni) alfa=0.2;
  else                 alfa=0.25;
  h=h*0.9437436*Math.pow((deltayeni/delta),alfa);
  if(h>hmax) h=hmax;
  if(h<hmin) h=hmin;
  y=yp2;
  x+=h;
  Xi=new Double(x);
  Yi=new Double(y);
  xi.add(Xi);
  yi.add(Yi);
  n++;
  } //end of while
  if(x<xn)//son bir step daha
  {
  h=xn-x;
  xeski=x;
  yeski=y;
  double total=0;
  yp1=y;
  yp2=y;
```

```
   for(int ii=1;ii<s;ii++)
   {  x1=x+c[ii]*h;
     y1=y;
                for(int j=1;j<ii;j++)
     {y1+=h*a[i][j]*k[j];}
     k[ii]=f.func(x1,y1);
     yp1+=h*b[ii]*k[ii];
     yp2+=h*bh[ii]*k[ii];
   }

   y=yp2;
   x+=h;
   Xi=new Double(x);
   Yi=new Double(y);
   xi.add(Xi);
   yi.add(Yi);
   }
       // vektöre aktarilan değerleri double matrise aktar
       double aa[][]=new double[2][n+1];
       Enumeration nx=xi.elements();
       Enumeration ny=yi.elements();
       for(i=0;i<=n;i++)
       {
                  Double ax=(Double)nx.nextElement();
                  Double ay=(Double)ny.nextElement();
       aa[0][i]=ax.doubleValue();
       aa[1][i]=ay.doubleValue();
       }
   return aa;
}

public static double[][] RKV13(f_xy f,double x0,double y0,double xn,double h,double eps)
{  //Runge-Kutta-Verner 13th degree step constant step size version
   double hmax=(xn-x0)/40;
   double hmin=hmax/1000;
   double dydx=1.0;
   double delta,deltayeni,alfa;
int s=14;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
double b[]=new double[s];
c[0]=0;
c[1] =  0.;
c[2] =  .5e-1;
a[2][1] =  .5e-1;
c[3] =  .1065625;
a[3][1] = -.69931640625e-2;
a[3][2] =  .1135556640625;
c[4] =  .15984375;
a[4][1] =  .399609375e-1;
a[4][2] =  0.;
a[4][3] =  .1198828125;
c[5] =  .39;
a[5][1] =  .361397562800457512405294072118402834512 9;
a[5][2] =  0.;
a[5][3] = -1.341524066700492771819987788202715834917;
a[5][4] =  1.370126503900035259414693716084313000404;
c[6] =  .465;
a[6][1] =  .490472027972027972027972027972027972028 0e-1;
a[6][2] =  0.;
a[6][3] =  0.;
a[6][4] =  .235097204221440473986298833549342714 3122;
a[6][5] =  .180855592981356728810903963653454488485 0;
//
c[7] =  .155;
a[7][1] =  .616928904428904428904428904428904428904 4e-1;
a[7][2] =  0.;
a[7][3] =  0.;
a[7][4] =  .112365683146402766226255703513001544230 3;
```

```
a[7][5] = -.38850460714513667670490481081112445674 56e-1;
a[7][6] =  .19791887125220458553791887125220458553 79e-1;
c[8] =  .943;
a[8][1] = -1.76763024022232687573559711957214558671 4;
a[8][2] =  0.;
a[8][3] =  0.;
a[8][4] = -62.5;
a[8][5] = -6.06188937737666910082136145965933319997 58;
a[8][6] =  5.65082319822276313856129803060084017420 1;
a[8][7] =  65.62169641937623283799566054863063741227;
c[9] =  .90180204173585695825970794067837214995 60;
a[9][1] = -1.18094506655497079982511628262829795788 2;
a[9][2] =  0.;
a[9][3] =  0.;
a[9][4] = -41.50473441114320841606641502701994225874;
a[9][5] = -4.43443831910372501122516922984610021177 6;
a[9][6] =  4.26040818858613302481219371074469324076 1;
a[9][7] =  43.75364022446171584987676829438379303004;
a[9][8] =  .78714254899123106874464750442263075508 60e-2;
c[10] =  .909;
a[10][1] = -1.28140599944148840545951029118205424626 6;
a[10][2] =  0.;
a[10][3] =  0.;
a[10][4] = -45.04713996013986630220754257136007322267;
a[10][5] = -4.73136206944957647731146426549128281094 3;
a[10][6] =  4.51496701659380784118585158459724099621 4;
a[10][7] =  47.44909557172985134869022392235929015114;
a[10][8] =  .10592282971116611356873939555165428752 28e-1;
a[10][9] = -.57468422638446162544323184782862962320 21e-2;
c[11] =  .94;
a[11][1] = -1.72447013426248519175670981748448186173 1;
a[11][2] =  0.;
a[11][3] =  0.;
a[11][4] = -60.92349008483054016518434619253765246063;
a[11][5] = -5.95151837622239245520283276706185486829 0;
a[11][6] =  5.55652373069845623597979165084359249683 9;
a[11][7] =  63.98301198033305336837536378635995939281;
a[11][8] =  .14642028250414961592759213917594526760 03e-1;
a[11][9] =  .64604087723582036036218651449776507148 92e-1;
a[11][10] = -.79303231690088789840244525486933732914 47e-1;
c[12] =  1.;
a[12][1] = -3.30162266774707901635399478979098362556 9;
a[12][2] =  0.;
a[12][3] =  0.;
a[12][4] = -118.01127235975250856669233039578988685 10;
a[12][5] = -10.14142238845611248642783916034510897595;
a[12][6] =  9.13931133223205792354401227355682700061 9;
a[12][7] =  123.37594282840426836848471809865018943 64;
a[12][8] =  4.62324437887458047483980762506763092479 2;
a[12][9] = -3.38327773806820192365255097153681124081 4;
a[12][10] =  4.52759210032461818945126533935112903532 5;
a[12][11] = -5.82849548581162296319308801916298570375 5;
c[13] =  1.;
a[13][1] = -3.03951503376630903004010285182120025105 6;
a[13][2] =  0.;
a[13][3] =  0.;
a[13][4] = -109.26086808941762546864441923221646233 52;
a[13][5] = -9.29064249740029344971766554265689754915 8;
a[13][6] =  8.43050498176491114213429925383616780345 4;
a[13][7] =  114.20100103783313135574240410955234274 76;
a[13][8] = -.96372713421454793581623756589879016527 62;
a[13][9] = -5.03488408880218979119868033618333232311 8;
a[13][10] =  5.95813082400292317754040216538817207279 4;
a[13][11] =  0.;
a[13][12] =  0.;

bh[1] =  .44312615229089792124864365102090297648 93e-1;
bh[2] =  0.;
bh[3] =  0.;
bh[4] =  0.;
```

```
bh[5] =  0.;
bh[6] =  .3546095642343226447863179350895055038855;
bh[7] =  .2478480431366653069619986721504458660016;
bh[8] =  4.448134732475784492725128317159648871312;
bh[9] =  19.846886366118733699309323992976879352 91;
bh[10] = -23.581623377465618419695179608703949650 85;
bh[11] =  0.;
bh[12] =  0.;
bh[13] = -.360167943728977516212453673774620240911 0;

 // Runge-Kutta Verner 16 variable step
 //eps relative error
 //double a[][]=new double[2][n+1];
 int i=0;
 double x1,y1;
 Vector<Double> xi=new Vector<Double>();
 Vector<Double> yi=new Vector<Double>();
 double x,y,yp1,yp2,xeski,yeski;
 x=x0;
 y=y0;
 xeski=x;
 yeski=y;
 Double Xi=new Double(x);
 Double Yi=new Double(y);
 double k[]=new double[s];
 xi.add(Xi);
 yi.add(Yi);
 int n=1;
 while((x+h)<=xn)
 {
  xeski=x;
  yeski=y;
  double total=0;
  yp1=y;
  yp2=y;
  double w=1;
  for(int ii=1;ii<s;ii++)
  {  x1=x+c[ii]*h;
     y1=y;
                for(int j=1;j<ii;j++)
    {y1+=h*a[ii][j]*k[j];}
     k[ii]=f.func(x1,y1);
     yp1+=h*b[ii]*k[ii];
     yp2+=h*bh[ii]*k[ii];
  }
  delta=Math.abs(yp2-yp1);
  dydx=f.func(x,y);
  deltayeni=Math.abs(eps*h*dydx);
  if(delta<=deltayeni) alfa=0.2;
  else                 alfa=0.25;
  h=h*0.9437436*Math.pow((deltayeni/delta),alfa);
  if(h>hmax) h=hmax;
  if(h<hmin) h=hmin;

  y=yp2;
  x+=h;
  Xi=new Double(x);
  Yi=new Double(y);
  xi.add(Xi);
  yi.add(Yi);
  n++;
  } //end of while
  if(x<xn)//son bir step daha
  {
  h=xn-x;
  xeski=x;
  yeski=y;
  double total=0;
  yp1=y;
  yp2=y;
```

```
   for(int ii=1;ii<s;ii++)
   {  x1=x+c[ii]*h;
     y1=y;
                for(int j=1;j<ii;j++)
     {y1+=h*a[i][j]*k[j];}
     k[ii]=f.func(x1,y1);
     yp1+=h*b[ii]*k[ii];
     yp2+=h*bh[ii]*k[ii];
   }
   y=yp2;
   x+=h;
   Xi=new Double(x);
   Yi=new Double(y);
   xi.add(Xi);
   yi.add(Yi);
   }
   //vektöre aktarilan değerleri double matrise aktar
       double aa[][]=new double[2][n+1];
       Enumeration nx=xi.elements();
       Enumeration ny=yi.elements();
       for(i=0;i<=n;i++)
       {
                Double ax=(Double)nx.nextElement();
                Double ay=(Double)ny.nextElement();
       aa[0][i]=ax.doubleValue();
       aa[1][i]=ay.doubleValue();
       }
   return aa;
}

public static double[][] RKV9(f_xy f,double x0,double y0,double xn,double h,double eps)
{  //Runge-Kutta-Verner 9th degree step constant step size version
   double hmax=(xn-x0)/40.0;
   double hmin=hmax/100.0;int s=10;
   double dydx=1.0;
double c[]=new double[s];
double a[][]=new double[s][s];
double bh[]=new double[s];
double b[]=new double[s];
c[0]=0;
c[1] =  0.;
c[2] =  .6e-1;
c[3] =  .95933333333333333333333333333333333333333333e-1;
c[4] =  .1439;
c[5] =  .4973;
c[6] =  .9725;
c[7] =  .9995;
c[8] =  1.;
c[9] =  1.;
a[2][1] =  .6e-1;
a[3][1] =  .19239962962962962962962962962962962962962296e-1;
a[3][2] =  .76693370370370370370370370370370370370370037e-1;
a[4][1] =  .35975e-1;
a[4][2] =  0.;
a[4][3] =  .107925;
a[5][1] =  1.31868341523314826091974727643173561286 1;
a[5][2] =  0.;
a[5][3] = -5.04205806362856222542776163471563769 3344;
a[5][4] =  4.22067464839541396450801435828390208048 3;
a[6][1] = -41.87259166432751461803757780644346812905;
a[6][2] =  0.;
a[6][3] =  159.43256216313749177003656690703468304 53;
a[6][4] = -122.11921356501003092025162033892421406 63;
a[6][5] =  5.53174306620005376825263123833299915007 6;
a[7][1] = -54.43015693531650433250642051294142461271;
a[7][2] =  0.;
a[7][3] =  207.06725136501846442736571738665098359 87;
a[7][4] = -158.61081378458999918287424243650585994 69;
a[7][5] =  6.99181658595024232199259728079179390709 6;
a[7][6] = -.18597231062203233977651717995492946236 92e-1;
```

```
a[8][1] = -54.663741787281976802412156480503869593 51;
a[8][2] =  0.;
a[8][3] =  207.952806255389373451582481669983424423 8;
a[8][4] = -159.288957474499507150895980587142665421 6;
a[8][5] =  7.018743740796944434698170760964252490817;
a[8][6] = -.183387859050457230647278200514173826836 1e-1;
a[8][7] = -.511948499788209907787543249724516839584 0e-3;

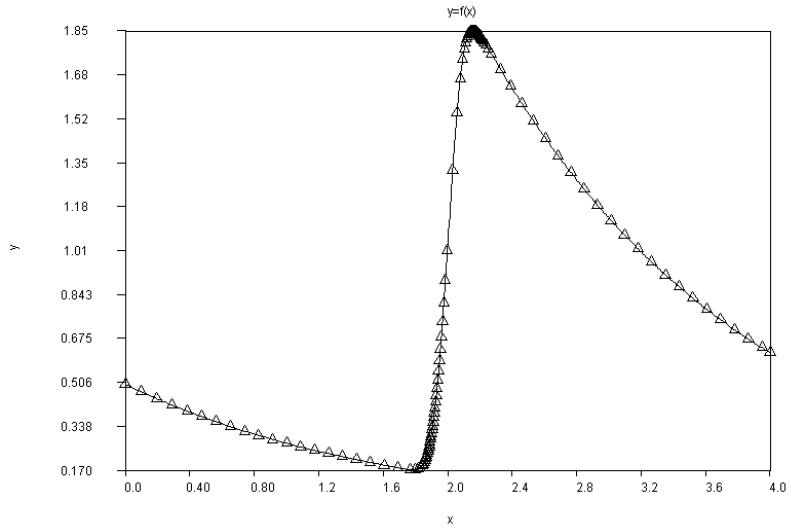
a[9][1] =  .343895786835703600927882012472832238652 0e-1;
a[9][2] =  0.;
a[9][3] =  0.;
a[9][4] =  .258262455563350340465955809858612085876 7;
a[9][5] =  .420937118967353715064255151406980196703 2;
a[9][6] =  4.405396469669310170148836816197095664891;
a[9][7] = -176.483119024298657615174094249900212502 9;
a[9][8] =  172.364133401415073029402258271190241331 5;

b[1] =  .343895786835703600927882012472832238652 0e-1;
b[2] =  0.;
b[3] =  0.;
b[4] =  .258262455563350340465955809858612085876 7;
b[5] =  .420937118967353715064255151406980196703 2;
b[6] =  4.405396469669310170148836816197095664891;
b[7] = -176.483119024298657615174094249900212502 9;
b[8] =  172.364133401415073029402258271190241331 5;
b[9] =  0.;

bh[ 1] =  .490996764838248973090685492797122583647 9e-1;
bh[ 2] =  0.;
bh[ 3] =  0.;
bh[ 4] =  .225111222951652415340139532053987532948 5;
bh[ 5] =  .469468225302956203943194852504738741255 3;
bh[ 6] =  .806579224998886770763416180899521798144 3;
bh[ 7] =  0.;
bh[ 8] = -.607119489177795979767295146525621712248 8;
bh[ 9] =  .568611394404756924114760317876613815359 4e-1;
  // Runge-Kutta Verner 16 variable step
  //eps relative error
  double alfa;
  //double a[][]=new double[2][n+1];
  int i=0;
  double x1,y1;
  Vector<Double> xi=new Vector<Double>();
  Vector<Double> yi=new Vector<Double>();
  double x,y,yp1,yp2,xeski,yeski;
  x=x0;
  y=y0;
  xeski=x;
  yeski=y;
  double delta,deltayeni;
  Double Xi=new Double(x);
  Double Yi=new Double(y);
  double k[]=new double[s];
  xi.add(Xi);
  yi.add(Yi);
  int n=1;
  while((x+h)<=xn)
  {
   xeski=x;
   yeski=y;
   double total=0;
   yp1=y;
   yp2=y;
   double w=1;
   for(int ii=1;ii<s;ii++)
   {  x1=x+c[ii]*h;
      y1=y;
                for(int j=1;j<ii;j++)
     {y1+=h*a[ii][j]*k[j];}
```

```
      k[ii]=f.func(x1,y1);
      yp2+=h*bh[ii]*k[ii];
   }
   delta=Math.abs(yp2-yp1);
   dydx=f.func(x,y);
   deltayeni=Math.abs(eps*h*dydx);
   if(delta<=deltayeni) alfa=0.2;
   else                 alfa=0.25;
   h=h*0.9437436*Math.pow((deltayeni/delta),alfa);
   if(h>hmax) h=hmax;
   if(h<hmin) h=hmin;
   y=yp2;
   x+=h;
   Xi=new Double(x);
   Yi=new Double(y);
   xi.add(Xi);
   yi.add(Yi);
   n++;
   } //end of while
   if(x<xn)//son bir step daha
   {
   h=xn-x;
   xeski=x;
   yeski=y;
   double total=0;
   yp1=y;
   yp2=y;
   for(int ii=1;ii<s;ii++)
   {  x1=x+c[ii]*h;
      y1=y;
                for(int j=1;j<ii;j++)
      {y1+=h*a[i][j]*k[j];}
      k[ii]=f.func(x1,y1);
      yp1+=h*b[ii]*k[ii];
      yp2+=h*bh[ii]*k[ii];
   }
   y=yp2;
   x+=h;
   Xi=new Double(x);
   Yi=new Double(y);
   xi.add(Xi);
   yi.add(Yi);
   }
   //vektöre aktarilan değerleri double matrise aktar
       double aa[][]=new double[2][n+1];
       Enumeration nx=xi.elements();
       Enumeration ny=yi.elements();
       for(i=0;i<=n;i++)
       {
                  Double ax=(Double)nx.nextElement();
                  Double ay=(Double)ny.nextElement();
       aa[0][i]=ax.doubleValue();
       aa[1][i]=ay.doubleValue();
       }
   return aa;
}
  public static void main(String args[]) throws IOException
  {
   double x0=0;
   double y0=0.5;
   fxy2 f1=new fxy2();
   double a[][]=RKF45(f1,x0,y0,4.0,0.1,1e-3);
    Text.print(Text.T(a));
    Plot p1=new Plot(a);
    //p1.setPlotType(0,24);
    //p1.addSpline(a,8);
    //p1.addData(a);
    p1.plot();
  }
}
```

Output 8.6-1 Runge-Kutta Fehlberg variable step methods with variable step size



DIFFERENTIAL EQUATIONS WITH MORE THEN ONE SET OF EQUATION

Consider the solution of differential equation set:

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_n)$$

….

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, ..., y_n)$$

This set requires n initial conditions to solve it. For example consider differential equation set

$$\frac{dy_1}{dx} = -0.5y_1$$

$$\frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1$$

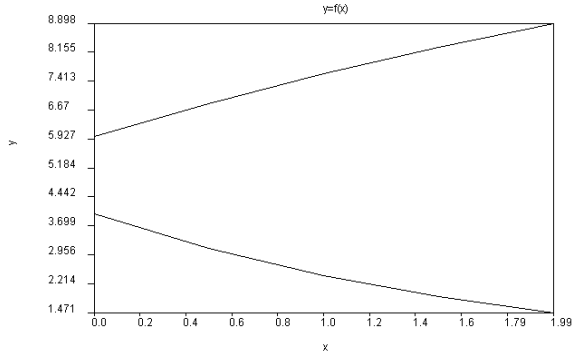
If for x=0, y_1=4 and y_2=6 is given. If this set is solved with 6th order Runge_Kutta method:

Program variable differential equation set with Runge-Kutta6 method

```
 import java.io.*;
import java.text.*;
import java.util.Locale;
class fm1 extends f_xi
{
//multivariable function
 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y
   if(x_ref==0) return -0.5*x[1];
   if(x_ref==1) return 4-0.3*x[2]-0.1*x[1];
   else return 0;
 }}
public class NA93A
{
public static void main(String arg[])
{ Buthcher bb=new Buthcher("RK6");
  Runge_Kutta rk=new Runge_Kutta(bb);
  fm1 f=new fm1();
```

```
  double y[]=new double[2];
  y[0]=4.0;
  y[1]=6.0;
  double a[][]=rk.hesapla(f,0.0,2.0,y,0.01);
  Text.printT(a);
  Plot p=new Plot(a);
  p.plot();
 }
 }
```

Output Solution of 2 equation set of differential equation by using RK6



Output Solution of 2 equation set of differential equation by using RK6

0.0	4.0	6.0
0.1	3.804917698014323	6.177326798708166
0.2	3.6193496721656535	6.351334649609415
0.30000000000000004	3.4428319057313583	6.522027912650391
0.4	3.274923012351407	6.689415390177402
0.5	3.1152031323325615	6.853509972700889
0.6000000000000001	2.9632728827804544	7.014328306002212
0.7000000000000001	2.818752358934318	7.171890478421663
0.8	2.681280184207202	7.326219727226854
0.9	2.5505126065562718	7.477342163017759
1.0	2.42612263892365	7.625286511178904
1.1	2.307799241598452	7.770083869440747
1.2000000000000002	2.195246544455496	7.911767480661146
1.3	2.0881831071258756	8.050372519984265
1.4000000000000001	1.9863412152494457	8.185935895578297
1.5	1.8894662110494735	8.318496062195234
1.6	1.7973158565555518	8.448092846835637
1.7000000000000002	1.709659727882498	8.574767285838973
1.8	1.626278639050617	8.698561472755916
1.9000000000000001	1.546964093906585	8.819518416392935
2.0	1.471517764774464	8.937681908451696

Set of differential equation solution concept can be used to solve nth degree differential equations as well.

$$\frac{d^n y}{dx^n} = g(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, ..., \frac{d^{n-1} y}{dx^{n-1}})$$

This equation can be cange to take form of

$x=x_0$

$y=x_1$

$$\frac{dy}{dx} = x_2$$

$$\frac{d^2y}{dx^2} = x_3$$

..

$$\frac{d^{n-1}y}{dx^{n-1}} = x_{n-1}$$

Our high order differential equation is converted to a set of single order differential equations

$$\frac{dx_k}{dx_0} = \frac{d^k y}{dx_0^k} = x_{k+1}\ ,\ 1 \le k < n$$

$$\frac{dx_n}{dx_0} = \frac{d^n y}{dx_0^n} = g(x_0, x_1, x_2, ..., x_n)$$

If the initial conditions are given, this type of set of equations can be solved easily. For example consider the diferential equation

$$\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 10y = 10u(t)$$

With u(t)=1,and the initial conditions at t=0 y(t)=0 and dy(t)/dt=0 . Exact solution of this differential equation is : y(t)=1-(10/9)exp(-t)+(1/9)exp(-10t) . To solve the same equation numerically, it can be divided into two equations as

$$\frac{dx_1}{dx_0} = x_2$$

$$\frac{dx_2}{dx_0} = 10u - 10x_1 - 11x_2$$

$x_1(0)=0$

$x_2(0)=0$

This form can be easily solved by using the previous RK6 program(NA93)

Program Solution of set of differential equation with Runge Kutta

```
 //=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta 6 method with multible differential equations
// Dr. Turhan Coban
// =====================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// ......
// dyn/dx=f(x,y1,y2,..,yn)
import java.io.*;


class fm1 extends f_xi
{
//multivariable function
 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y
   if(x_ref==0) return x[2];
   if(x_ref==1) return 10-10.0*x[1]-11*x[2];
   else return 0;
 }
}

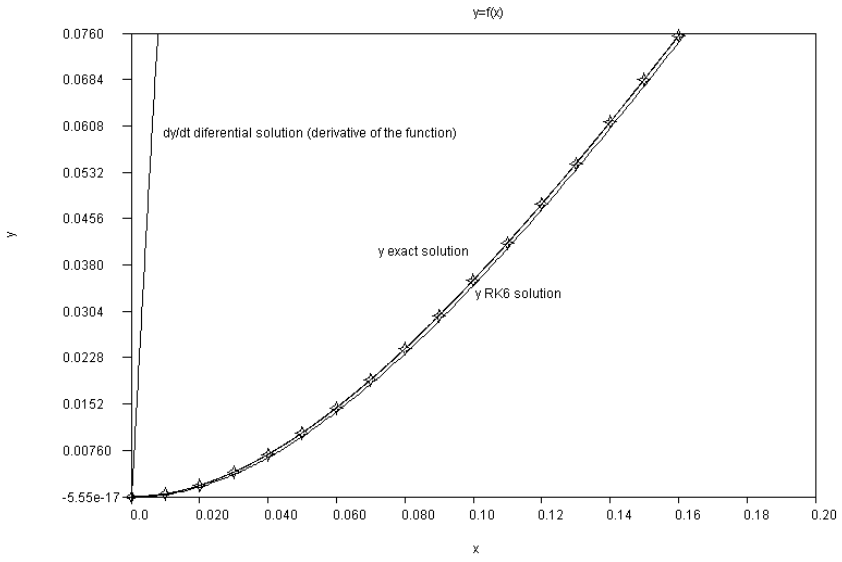
class fm2 extends f_x
{// gerçek çözüm
 public double func(double t)
 {return 1.0-10.0/9.0*Math.exp(-t)+1.0/9.0*Math.exp(-10.0*t);}
}

class NA94
```

```
{

public static void main(String args[]) throws IOException
  {
  fm1 b1=new fm1();
  fm2 b2=new fm2();
  double y[];
  y=new double[2];
  y[0]=0;
  y[1]=0;
  Buthcher bb=new Buthcher("RKV13");
  Runge_Kutta rk=new Runge_Kutta(bb);
  double a[][]=rk.hesapla(b1,0.0,0.2,y,0.01);
  Text.print(Text.T(a));
  Plot pp=new Plot(a);
  pp.addFunction(b2,0.0,0.2,20);
  pp.addSpline(2,100);
  pp.setPlotType(2,33);
  pp.plot();
  }
 }
```

Program Solution of set of differential equation with 6th order Runge Kutta



The same set of equation solved by using Python language:

Program f_xi abstract method for multidimension differential equation set in Python

```
from abc import ABC, abstractmethod
from math import *

class f_xi(ABC):
  # multifunction multi independent variable
  # vector of dependent variables are returned
  # example f[0]=x[0]+sin(x[1])
  #         f[1]=x[0]*x[1]-x[1]
  # func(x,1) returns the value of f[1]
  # func(x,0) returns the value of f[0]

  @abstractmethod
  def func(self,x,equation_ref):
     pass;
```

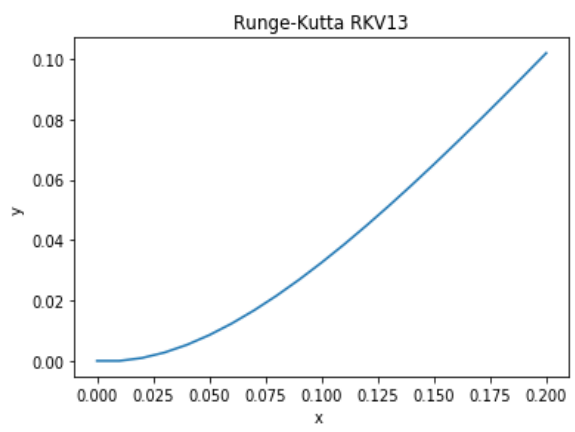
Program f_ Solution of set of differential equation with Runge Kutta with Python

```
# -*- coding: utf-8 -*-
"""
```

```
Created on Fri Aug 31 13:10:28 2018
@author: Mustafa Turhan Çoban
"""
from Runge_Kutta import *;
from  Butcher import *;
from f_xi import *;
import numpy as np;
import matplotlib.pyplot as plt;

class f1(f_xi):
    def func(self,x,x_ref):
        if x_ref==0: F=x[2];
        elif x_ref==1: F=10.0-10.0*x[1]-11.0*x[2];
        else: F=0.0;
        return F;

rk=Runge_Kutta("RKV13");
f=f1();
y=[0 for i in range(2)];
y[0]=0.0;
y[1]=0.0;
XY=rk.calc_n(f,0.0,0.2,y,0.01,0);
print(XY);
plt.title('Runge-Kutta RKV13')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(XY[0],XY[1]);
```



Few more example will be given in this chapter to understand nature of the real differential equation solutions. Fluid mechanics is a topic that mechanical engineering, chemical engineering and meterology is closely interested. Turbulent flow is one of the important sub topic. In the classical approach to the turbulance random movements are assumed. When it is investigated further it is shown that some diferential equations can be given chaotic results. This created a new science of chaotic systems. Edward Lorents, an athmospheric scientist developed Lorents set of diferential equations to investigate atmospheric turbulance effects. The equation set:

$$\frac{dx_1}{dt} = \sigma(x_2 - x_2)$$

$$\frac{dx_2}{dt} = (1 + \lambda - x_3)x_1 - x_2$$

$$\frac{dx_3}{dt} = x_1 x_2 - \gamma x_2$$

In this equations σ, λ, and γ are pozitif constants. Equation behave chaotically when
$\sigma > \gamma + 1$ and

$\lambda > \frac{(\sigma+1)(\sigma+\gamma+1)}{\sigma-\gamma-1}$. For example with $\sigma = 10$, $\lambda = 24$ ve $\gamma = 2$ system behaves chaotically. If we solve the Lorents equation set with $0\ s \le t \le 30 s$ time range, $x_1(0) = 1.0$; $x_2(0) = 0.0$; $x_3(0) = 20.0$ conditions

Program Solution of Lorenz equation with RKV13

```
 //=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta-Verner 16 constant step method with
// multible differential equations
// Dr. Turhan Coban
// =====================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// ......
// dyn/dx=f(x,y1,y2,..,yn)
// LORENTS equation solution
import java.io.*;

class fm2 extends f_xi
{
//multivariable function
double sigma=10.0;
 double lambda=28.0;
 double gamma=8.0/3.0;



 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y

   if(x_ref==0) return sigma*(x[2]-x[1]);
   else if(x_ref==1) return (1+lambda-x[3])*x[1]-x[2];
   else if(x_ref==2)       return x[1]*x[2]-gamma*x[3];
   else return 0;
 }
}

class NA95
{

public static void main(String args[]) throws IOException
  {

  fm2 b1=new fm2();
  double y[];
  y=new double[3];
  y[0]=2.0;
  y[1]=2.0;
  y[2]=2.0;
   Buthcher bb=new Buthcher("RKV13");
  Runge_Kutta rk=new Runge_Kutta(bb);
  double a[][]=rk.hesapla(b1,0.0,40.0,y,0.001);
  Text.printT(a);
  Plot p1=new Plot(a[1],a[3]);
  p1.setXlabel("x1");
  p1.setYlabel("x3");
  p1.setPlabel("Lorenz attractor");
  p1.plot();

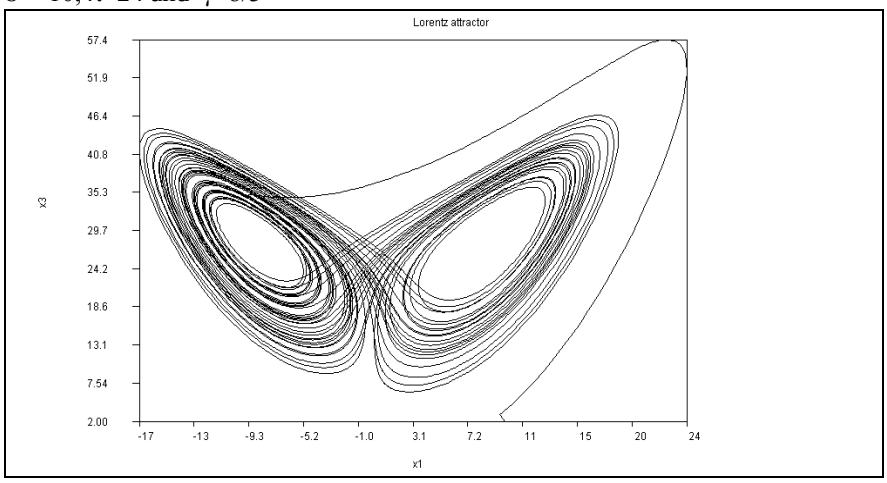
  Plot p2=new Plot(a[0],a[1]);
  p2.setXlabel("x0");
  p2.setYlabel("x3");
  p2.setPlabel("Lorentz equation");
  p2.plot();

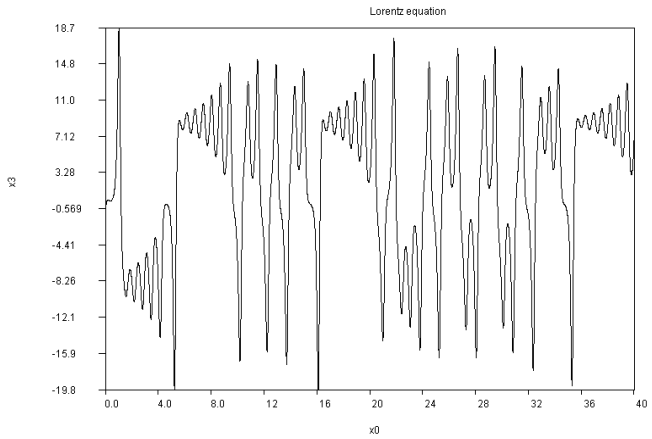
 }
```

```
}
```

Output 8.7-3 Solution of lorentz equation by using 6th order Runge-Kutta method

$\sigma = 10$, $\lambda=24$ and $\gamma=8/3$





Python equivalent of this solution:

Program Solution of Lorenz equation with RKV13 python version

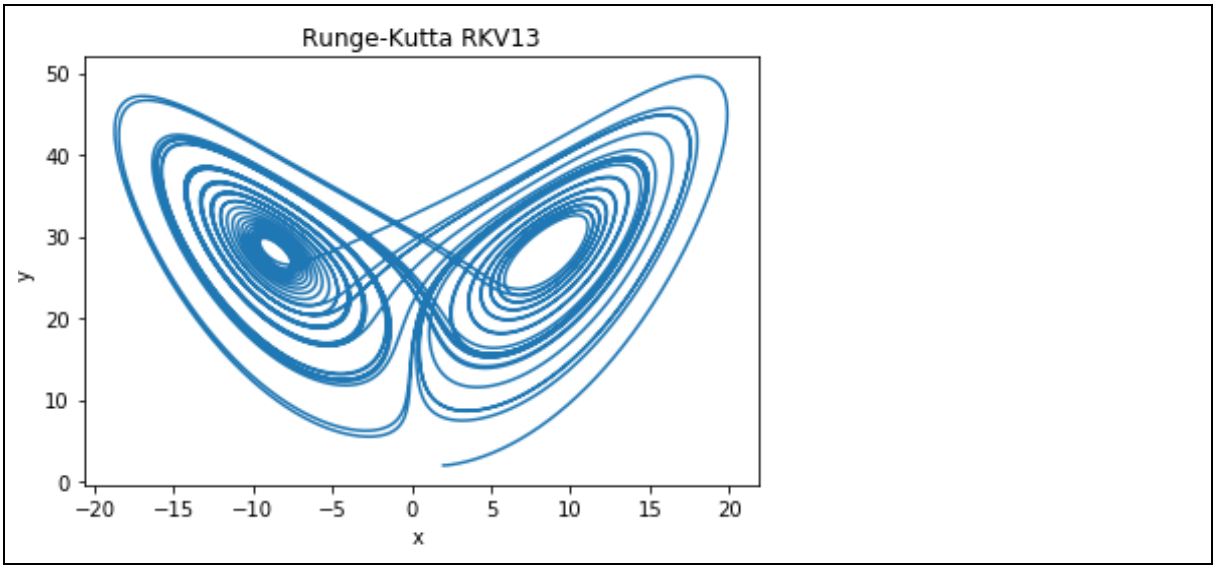
```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 13:10:28 2018

@author: Mustafa Turhan Çoban
"""
from Runge_Kutta import *;
from  Butcher import *;
from f_xi import *;
import numpy as np;
import matplotlib.pyplot as plt;

class f1(f_xi):
    def __init__(self):
      self.name="Lorentz equation";
      self.sigma=10.0;
      self.lambd=28.0;
      self.gamma=8.0/3.0;
    def func(self,x,x_ref):
        if x_ref==0: F=self.sigma*(x[2]-x[1]);
        elif x_ref==1: F=(1.0+self.lambd-x[3])*x[1]-x[2];
        elif x_ref==2: F=x[1]*x[2]-self.gamma*x[3];
        else: F=0.0;
        return F;

rk=Runge_Kutta("RKV13");
f=f1();
y=[0 for i in range(3)];
y[0]=2.0;
```

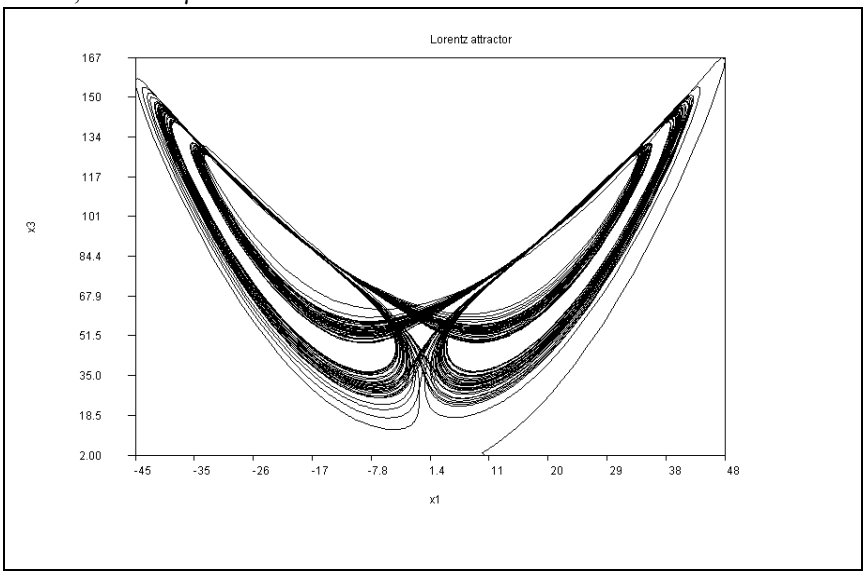
```
y[1]=2.0;
y[2]=2.0;
XY=rk.calc_n(f,0.0,40.0,y,0.001,0);
print(XY);
plt.title('Runge-Kutta RKV13')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(XY[1],XY[3]);
```

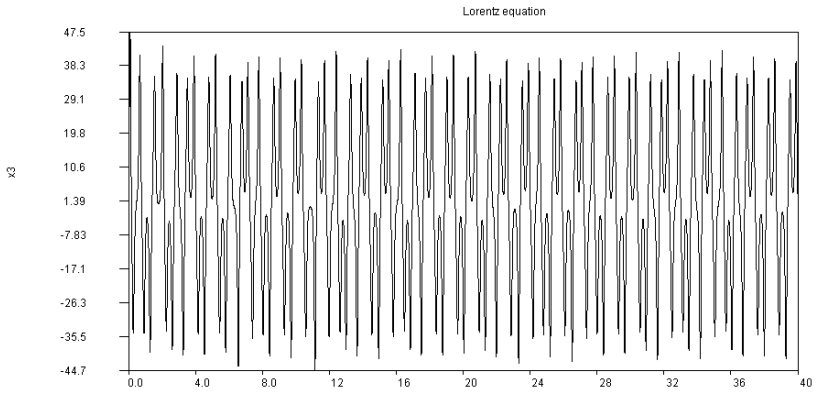


If parameters is changed a completely different kind of behavior can be observed.

Output Solution of Lorenz equation with 16 degree Runge-Kutta-Verner method

$\sigma = 10$, $\lambda=90$ ve $\gamma=5.5$





Another interesting example in chaotic behavior is Predator-Prey model. When the population of Predators are increased, the population of the Prey animal will decreased and it will caused population decrease of the Predator population as well. This relation is explained and put into a matematical model by italian mathematician Vito Voltera and American Biolog Alfret J. Lotka. Their equation is:

$$\frac{dy_1}{dt} = ay_1 - by_1y_2$$

$$\frac{dy_2}{dt} = -cy_2 + dy_1y_2$$

In this equation y_1 is Prey animal population, y_2 is Predator population, a is Prey animal birth rate, b is Predator dead rate, c is Predator-Prey relation ratio in Predator dead rate and d is Predator-Prey relation ratio in Predator population increase.

Program Predator-Prey model (Lotka-Volterra equation)

```
//=======================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta-Verner 16 method with multible differential equations
// Dr. Turhan Coban
// =======================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// ......
// dyn/dx=f(x,y1,y2,..,yn)
// Lotka - Volterra equation
// Preditor-Prey model
import java.io.*;

class fm1 extends f_xi
{
//Preditor-Prey model
// Lotka-Volterra equation
 double a=1.2;  //prey birth rate
 double b=0.6;  //preditor death rate
 double c=0.8;  //effect of preditor-prey relation on prey death rate
 double d=0.3;  //effect of preditor-prey relations on preditor birth rate
 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y

   if(x_ref==0) return a*x[1]-b*x[1]*x[2];
   else return -c*x[2]+d*x[1]*x[2];
 }
}

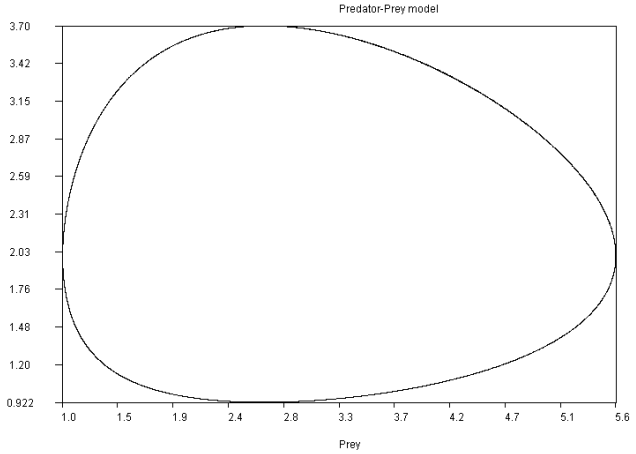
class NA96
{

public static void main(String args[]) throws IOException
{
  fm1 b1=new fm1();
```

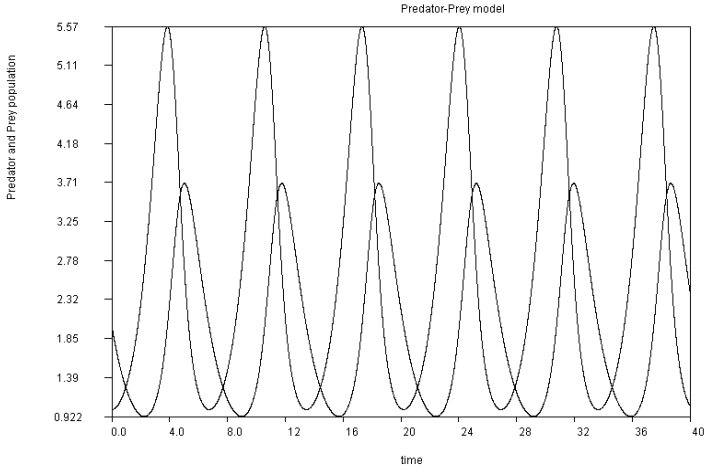
```
  double y[];
  y=new double[2];
  y[0]=1.0;
  y[1]=2.0;
  double a[][]=NA92F.RKV16(b1,0.0,40.0,y,10000);
  Plot p1=new Plot(a[0],a[1]);
  p1.addData(a[0],a[2]);
  p1.setXlabel("time");
  p1.setYlabel("Predator and Prey population");
  p1.setPlabel("Predator-Prey model");
  p1.plot();

  Plot p2=new Plot(a[1],a[2]);
  p2.setXlabel("Prey");
  p2.setYlabel("Predator");
  p2.setPlabel("Predator-Prey model");
  p2.plot();
 }
}
```

Output Predator-Prey model (Lotka-Volterra equation) a=1.2 b= 0.6 c=0.8 d=0.3, Predator-pray population change with respect to each other



Output Predator-Prey model (Lotka-Volterra equation) a=1.2 b= 0.6 c=0.8 d=0.3, Predator-pray population change with time



While investigating hard beats, Van der Pol developed the set of differential equation to represent hard beats. This equations are

$\frac{d^2y}{dt^2} + \mu(y^2 - 1)\frac{dy}{dt} + y = 0$ If this equation are converted into a system of equations with

$y=y_1$ and

$$\frac{dy}{dt} = y_2$$

$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2}{dt} = -y_1 - \mu(y_1^2 - 1)y_2 = 0$$

If this equation is solved for μ=1 and $y_1(0)$=0, $y_2(0)$=0.5 :

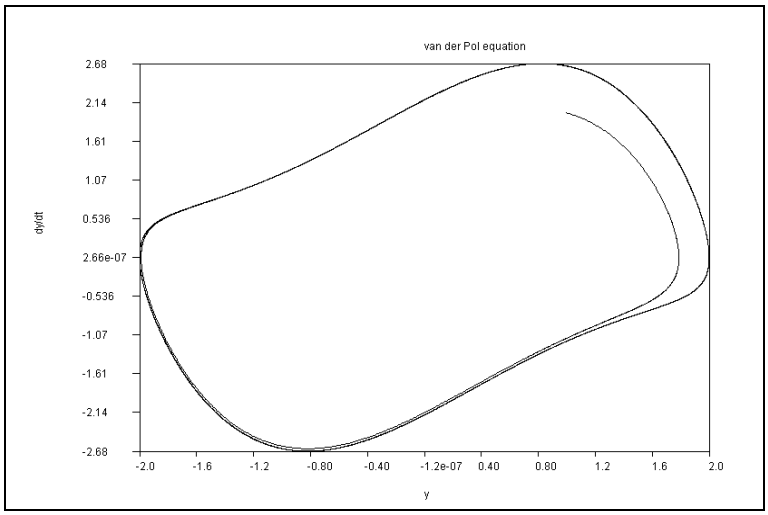
Program Van der Pol differential equation

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta_verner 16 method with multible differential equations
// Dr. Turhan Coban
// =====================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// .....
// dyn/dx=f(x,y1,y2,..,yn)
// Van der Pol equation
// hard bit differential equatin model

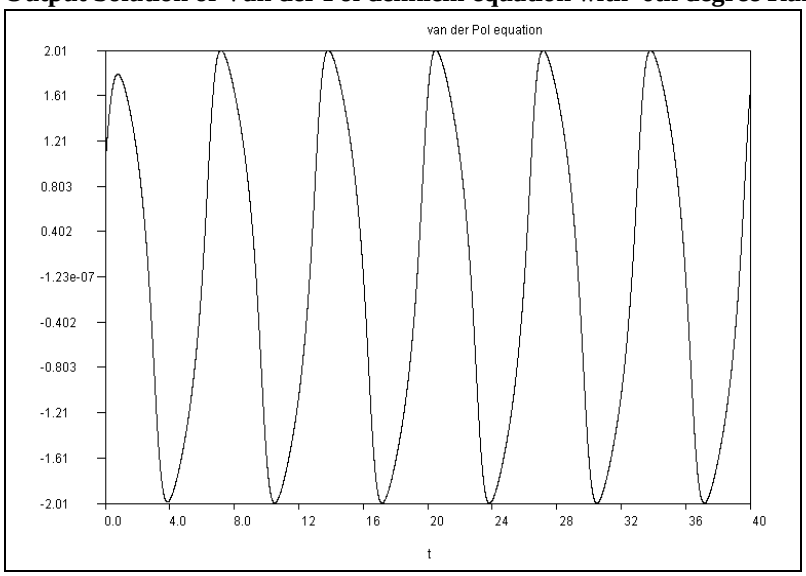
import java.io.*;


class fm1 extends f_xi
{
//Van der Pol equation
//simulates hard beats
 double mu=1.0;
 // d^2y/dt^2+mu(y^2-1)dy/dt=0
 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y
   if(x_ref==0) return x[2];
   else  return -x[1]-mu*(x[1]*x[1]-1.0)*x[2]; }}
class NA97
{
public static void main(String args[]) throws IOException
  {
  fm1 b1=new fm1();
  double y[];
  y=new double[3];
  y[0]=1;
  y[1]=2.0;
  double a[][]=NA92F.RKV16(b1,0.0,40.0,y,10000);
  Plot p1=new Plot(a[1],a[2]);
  p1.setXlabel("y");
  p1.setYlabel("dy/dt");
  p1.setPlabel("van der Pol equation");
  p1.plot();
  Plot p2=new Plot(a[0],a[1]);
  p2.setXlabel("t");
  p2.setYlabel("y");
  p2.setPlabel("van der Pol equation");
  p2.plot();
 }}
```

Output Van der Pol denklem setinin 6 derece Runge-Kutta metodu ile çözülmesi



Output Solution of Van der Pol denklem equation with 6th degree Runge-Kutta method



The next example is the harmonic motion with damping. The diferential equation is:

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + \omega^2 y = 0$$

The equation will be solved with initial conditions y(0)=1 $\frac{dy(0)}{dt} = 1$ and ω =1 ve b=0.1. When b=0, it is called simple harmonic motion. a periodic motion that is neither driven nor damped. When b=0.1, it is a damped motion. Oscilations will get smaller and stop after a certain time. Converted form of the equation:

$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2}{dt} = -by_2 - \omega^2 y$$

$y_1(0)=0$

$y_2(0)=1$

Program Harmonic motion with damping

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
```

```
// and differentiation (derivative) functions
// Dr. Turhan Coban
// ====================================================
import java.io.*;

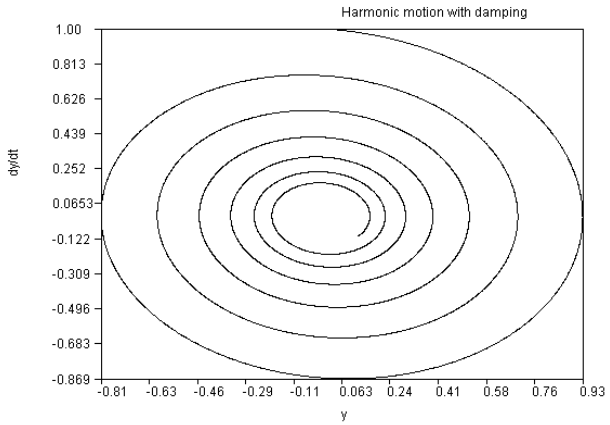
class fm1 extends f_xi
{
//Harmonic motion
 double b=0.1;
 double w=1.0;
 // d^2y/dt^2+b*dy/dt+w^2y=0
 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y
   if(x_ref==0) return x[2];
   else  return -b*x[2]-w*w*x[1];
 }
}

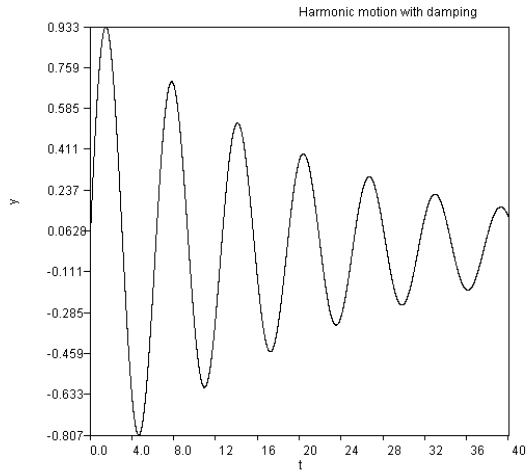
class NA98A
{

public static void main(String args[]) throws IOException
  {

  fm1 b1=new fm1();
  double y[];
  y=new double[3];
  y[0]=0.0;
  y[1]=1.0;
  double a[][]=NA92F.RKV16(b1,0.0,40.0,y,10000);
  Plot p1=new Plot(a[1],a[2]);
  p1.setXlabel("y");
  p1.setYlabel("dy/dt");
  p1.setPlabel("Harmonic motion with damping");
  p1.plot();
  Plot p2=new Plot(a[0],a[1]);
  p2.setXlabel("t");
  p2.setYlabel("y");
  p2.setPlabel("Harmonic motion with damping");
  p2.plot();
 }
}
```

Output Harmonic motion with damping





Python code of the Harmonic motion:

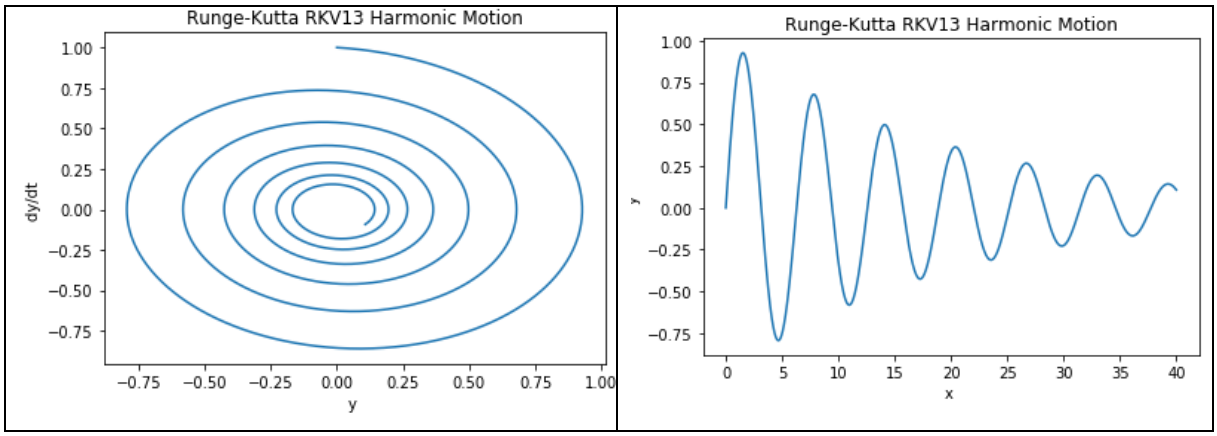
Program Harmonic motion with damping python code

```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 13:10:28 2018

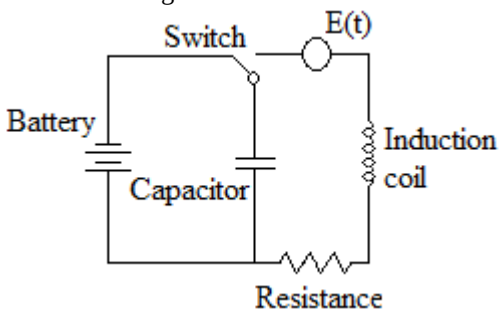
@author: Mustafa Turhan Çoban
"""
from Runge_Kutta import *;
from  Butcher import *;
from f_xi import *;
import numpy as np;
import matplotlib.pyplot as plt;

class f1(f_xi):
    #Harmonic motion
    #d^2y/dt^2+c*dy/dt+w^2y=0
     def __init__(self):
       self.name="Harmonic motion";
       self.b=0.1;
       self.w=1.0;
     def func(self,x,x_ref):
         if x_ref==0: F=x[2];
         elif x_ref==1: F=-self.b*x[2]-self.w*self.w*x[1];
         else: F=0.0;
         return F;

rk=Runge_Kutta("RKV13");
f=f1();
y=[0 for i in range(2)];
y[0]=0.0;
y[1]=1.0;
XY=rk.calc_n(f,0.0,40.0,y,0.001,0);
print(XY);
plt.title('Runge-Kutta RKV13 Harmonic Motion')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(XY[0],XY[1]);
```



The next example is the harmonic motion with damping in an electrical circuit. This kind of behavior can be seen frequently in circuits when current is not constant and changed as a function of time. Such a circuit is shown in the figure below.



The differential equation of the system:

$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}-E(t)=0$ In this equation q is capacitor energy storage capacity (C, Coulomb). L enductance (H, Henry), R res,stance (Ω, Ohm), C capacitance (F), and $E(t)=E_0\sin(wt)$ is time dependent voltage, w frequency and t is time (s) . Current density is the derivative of capacitor energy storage capacity.

$i=\frac{dq}{dt}$ (Ampere). An analytical solution of this equation for R=0 ve $E(t)=E_0\sin(wt)$:

$q(t)=\frac{-E_0}{L(p^2-w^2)}\frac{w}{p}\sin(pt)+\frac{E_0}{L(p^2-w^2)}\sin(wt)$. And the current density

$i=\frac{dq(t)}{dt}=\frac{-E_0w}{L(p^2-w^2)}\sin(pt)+\frac{E_0w}{L(p^2-w^2)}\sin(wt)$. In this equation

$p=\frac{1}{\sqrt{LC}}$ dir.

Now this equation will be solved numerically for initial conditions q(0)=0 ve $i(0)=\frac{dq(0)}{dt}=0$ and physical conditions $w^2=3.5\ 1/s^2$ p=2, L=1 H, E_0=1 V, C=0.25 F. Since analytical solution is known, two solutions can be compared. It is a good exampel to observe the numerical solution error generation due to very complex behaviour of the function.

Converted equation :

$\frac{dy_1}{dt}=y_2$

$$\frac{dy_2}{dt} = \frac{\left[-Ry_2 - \frac{y_1}{C} + E_0 \sin(wt) \right]}{L}$$

$y_1(0)=0$

$y_2(0)=0$

Program Harmonic motion electrical circuit (Changing voltage problem)

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta 6 method with multible differential equations
// Dr. Turhan Coban
// =====================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// ......
// dyn/dx=f(x,y1,y2,..,yn)
// Changing voltage problem

import java.io.*;

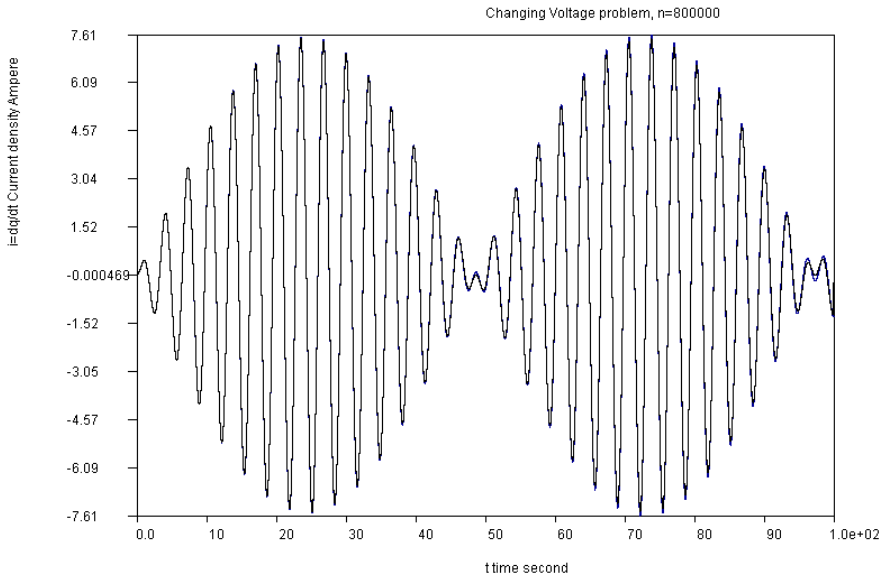

class fm1 extends f_xi
{
 double E0=1.0;  //Volt
 double L=1.0;   // Henry
 double C=0.25;  // Coulomb
 double w2=3.5;  // s^2
 double R=0.0;     // Ohm
 double w=Math.sqrt(w2);
 double p=1.0/Math.sqrt(L*C);
 // Ld^2q/dt^2+R*dq/dt+q/C-E(t)=0
 double func(double x[],int x_ref)
 {
   //x[0] is t, x[1] is q x[2] is i
   if(x_ref==0) return x[2];
   else  return (-R*x[2]-x[1]/C+E0*Math.sin(w*x[0]))/L;
 }
 double i(double t)
 {
 return -E0/L/(p*p-w2)*w*Math.cos(p*t)+E0/L/(p*p-w2)*w*Math.cos(w*t);
 }
 double q(double t)
 {
 return -E0/L/(p*p-w2)*w/p*Math.sin(p*t)+E0/L/(p*p-w2)*Math.sin(w*t);
 }
}

class NA98
{

public static void main(String args[]) throws IOException
  {

  fm1 b1=new fm1();
  double y[];
  y=new double[3];
  y[0]=0.0;
  y[1]=0.0;
  int n=800000;
  double a[][]=NA93.RK6(b1,0.0,100.0,y,n);
```

```
  Plot p1=new Plot(a[0],a[2]);
  double t=0;
  double dt=100.0/n;
  for(int i=0;i<(n-1);i++) {a[1][i]=b1.i(t);t+=dt;}
  p1.addData(a[0],a[1]);
  p1.setXlabel("t time second");
  p1.setYlabel("i=dq/dt Current density Ampere");
  p1.setPlabel("Changing Voltage problem");
  p1.plot();
 }
}
```



Python version of the code is given below, analyical solution is omitted:

Program Harmonic motion electrical circuit (Changing voltage problem) Python version

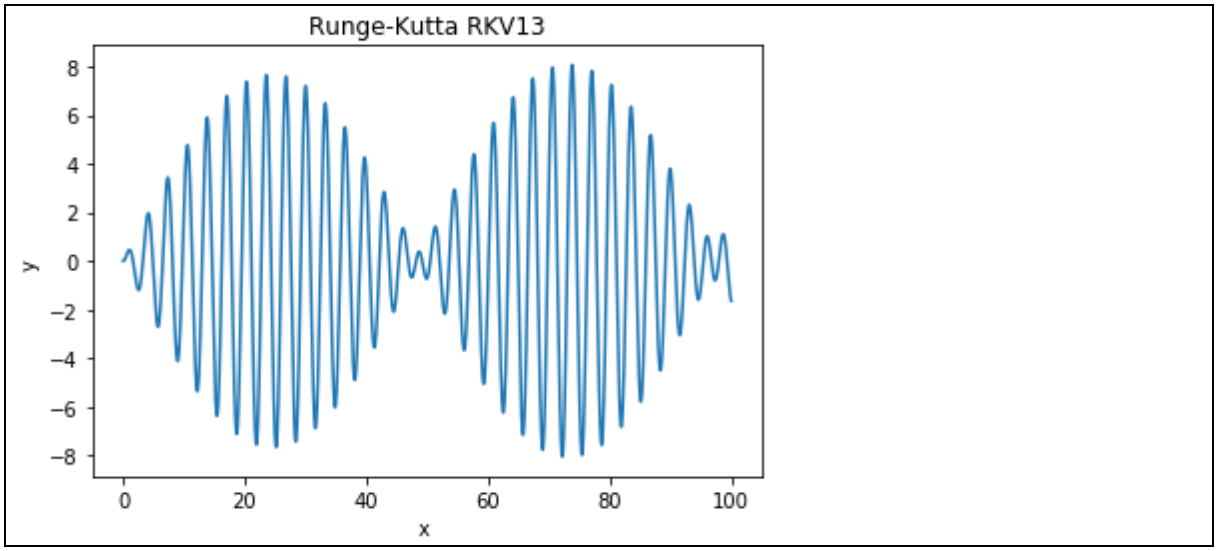
```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 13:10:28 2018

@author: Mustafa Turhan Çoban
"""
from Runge_Kutta import *;
from  Butcher import *;
from f_xi import *;
from math import *
import numpy as np;
import matplotlib.pyplot as plt;

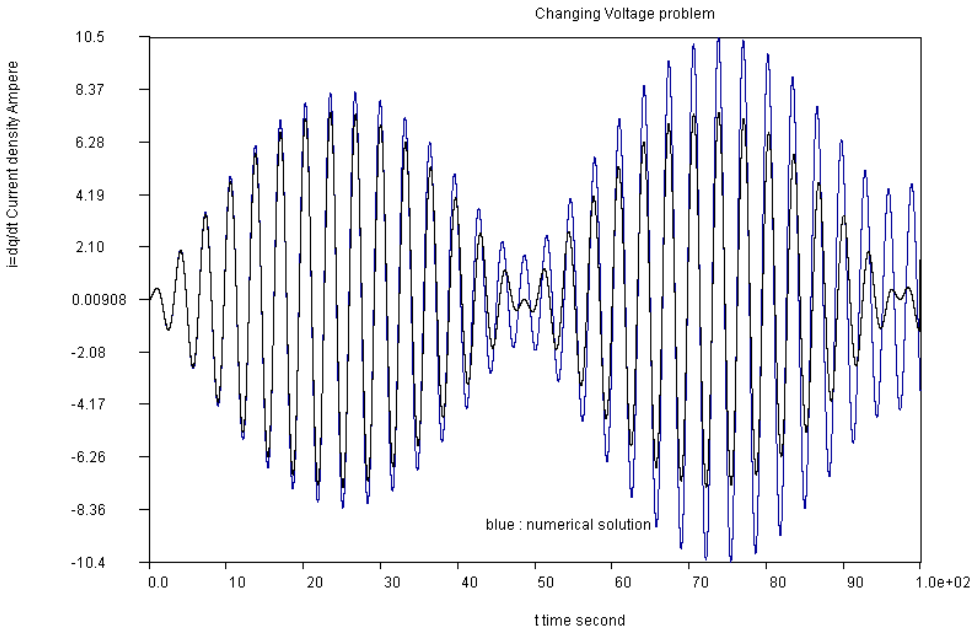
class f1(f_xi):
    def __init__(self):
      self.name="Harmonic electronic circuit";
      self.E0=1.0;#Volt
      self.L=1.0;#Henry
      self.C=0.25;#Coulomb
      self.w2=3.5;#s^2
      self.R=0.0;#Ohm
      self.w=sqrt(self.w2);
      self.p=1.0/sqrt(self.L*self.C);
    def func(self,x,x_ref):
        if x_ref==0: F=x[2];
        elif x_ref==1: F=(-self.R*x[2]-x[1]/self.C+self.E0*sin(self.w*x[0]))/self.L;
        else: F=0.0;
        return F;

rk=Runge_Kutta("RKV13");
f=f1();
y=[0 for i in range(2)];
y[0]=0.0;
y[1]=0.0;
```

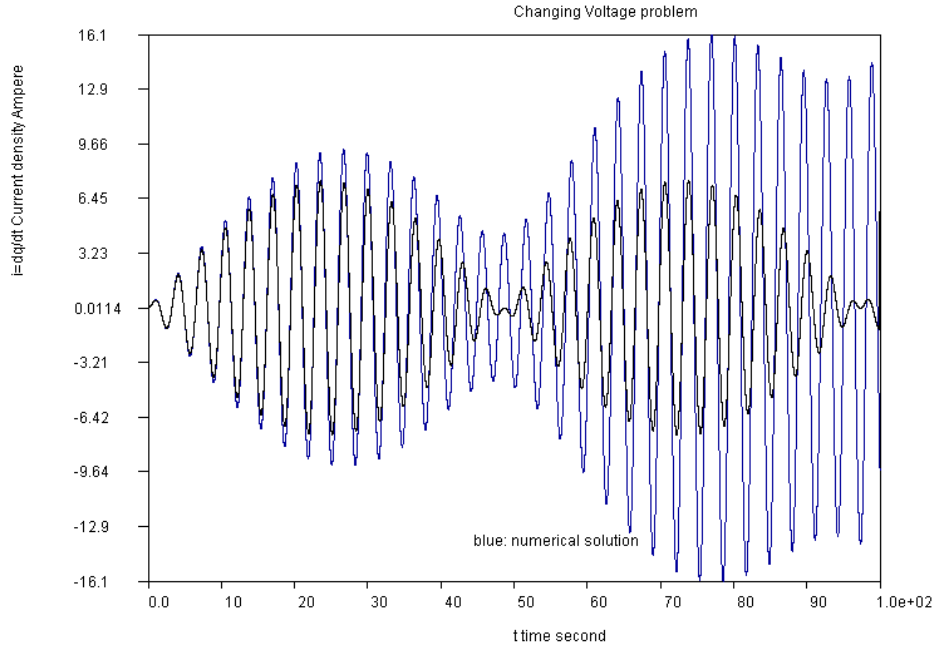
```
XY=rk.calc_n(f,0.0,100.0,y,0.001,0);
print(XY);
plt.title('Runge-Kutta RKV13')
plt.xlabel('x ')
plt.ylabel('y ');
plt.plot(XY[0],XY[2]);
```



To calculate current both numerical and analytical solutions are calculated and plotted. As number of points N=800000 points are used for the numerical calculations. If N=50000, error generation in numerical solution can be observed.



For N=25000 level of numerical error even gets bigger by time



If differential equation is changed to Runge-Kutta-Verner 16th degree constant step size with N=4000, Result is very accurate

Program Harmonic motion electrical circuit RKV16 N=4000

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta 6 method with multible differential equations
// Dr. Turhan Coban
// =====================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// ......
// dyn/dx=f(x,y1,y2,..,yn)
// Changing voltage problem

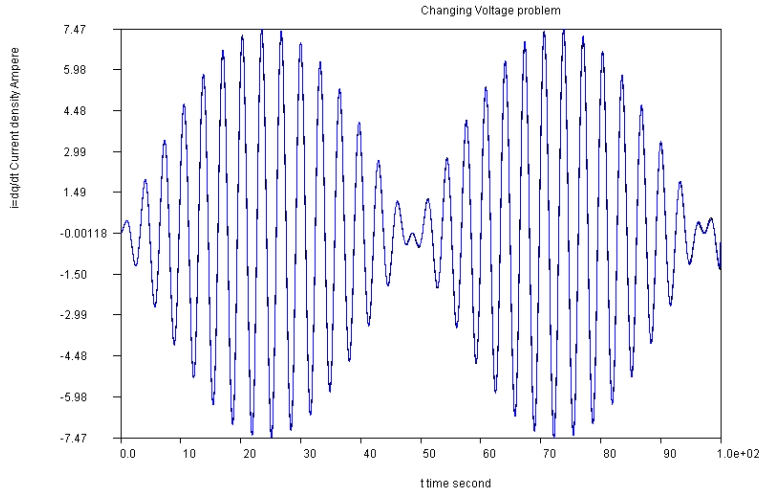
import java.io.*;


class fm1 extends f_xi
{
 double E0=1.0;  //Volt
 double L=1.0;   // Henry
 double C=0.25;  // Coulomb
 double w2=3.5;  // s^2
 double R=0.0;     // Ohm
 double w=Math.sqrt(w2);
 double p=1.0/Math.sqrt(L*C);
 // Ld^2q/dt^2+R*dq/dt+q/C-E(t)=0
 double func(double x[],int x_ref)
 {
   //x[0] is t, x[1] is q x[2] is i
   if(x_ref==0) return x[2];
   else  return (-R*x[2]-x[1]/C+E0*Math.sin(w*x[0]))/L;
 }
 double i(double t)
 {
 return -E0/L/(p*p-w2)*w*Math.cos(p*t)+E0/L/(p*p-w2)*w*Math.cos(w*t);
 }
 double q(double t)
 {
 return -E0/L/(p*p-w2)*w/p*Math.sin(p*t)+E0/L/(p*p-w2)*Math.sin(w*t);
 }
}
```

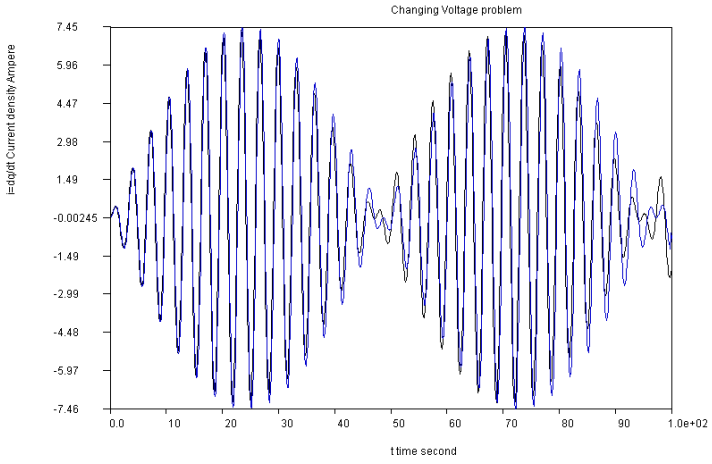
```
class NA98C
{

public static void main(String args[]) throws IOException
  {

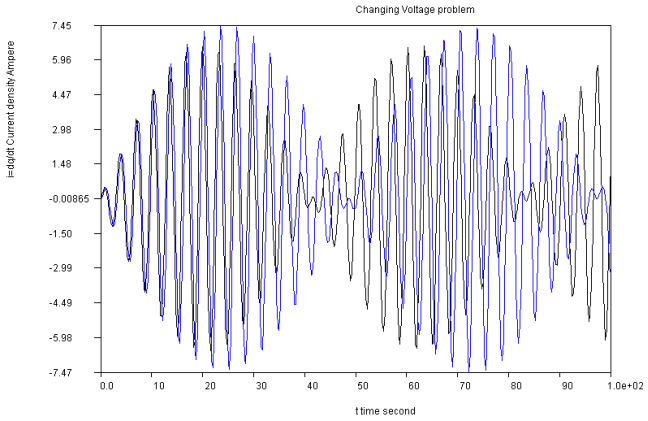
  fm1 b1=new fm1();
  double y[];
  y=new double[3];
  y[0]=0.0;
  y[1]=0.0;
  int n=4000;
  double a[][]=NA92F.RKV16(b1,0.0,100.0,y,n);
  Plot p1=new Plot(a[0],a[2]);
  double t=0;
  double dt=100.0/n;
  for(int i=0;i<(n-1);i++) {a[1][i]=b1.i(t);t+=dt;}
  p1.addData(a[0],a[1]);
  p1.setXlabel("t time second");
  p1.setYlabel("i=dq/dt Current density Ampere");
  p1.setPlabel("Changing Voltage problem");
  p1.plot();
 }
}
```



With RKV16 N=1000 system starts to diverge from exact solution



With RKV16 and N=400 divergence is quite clear.



Another example. Assume that a bullet is thrown from the gun into an atmosphere. Due to high velocity of the bullet, friction will effect the movement of the bullet. The diferential equation of the motion in cartesian coordinate system can be given as:

$$\frac{d^2x}{dt^2} = -\gamma(\sqrt{v_x^2 + v_y^2})v_x$$

$$\frac{d^2y}{dt^2} = -g - \gamma(\sqrt{v_x^2 + v_y^2})v_y$$

In this equation $\gamma = \frac{k}{m}$,where k friction coefficient between the bullet and air, v_x is x direction component of the velocity, v_y is y direction component of the velocity, m mass, g gravitational acceleration, and t is time. With t=x[0] , x=x[1], y=x[2], v_x=x[3], v_y=x[4]
variable convertion the diferential equation becomes

$$\frac{dx[1]}{dx[0]} = x[3]$$

$$\frac{dx[2]}{dx[0]} = x[4]$$

$$\frac{dx[3]}{dx[0]} = -\gamma(\sqrt{x^2[3] + x^2[4]})x[3]$$

$$\frac{dx[4]}{dx[0]} = -g - \gamma(\sqrt{x^2[3] + x^2[4]})x[4]^{z}$$

Initial conditions x[0]=0 da x[1]=0 x[2]=0,
V_0=250 m/s x[3]= V_0 sin(θ) m/s, x[4]= V_0 cos(θ) m/s, g=9.806 m/s^2, γ=0.01 , θ =30,45,60 degree

Program gun bullet trajectory with air friction

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// and differentiation (derivative) functions
// Dr. Turhan Coban
// =====================================================
import java.io.*;

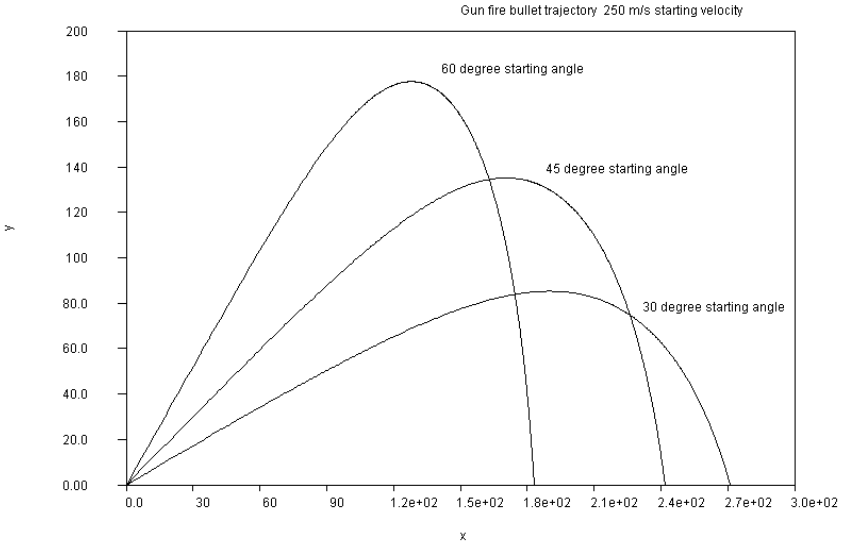

class fm1 extends f_xi
{
 double gamma;
            public fm1(double gammai)
   {gamma=gammai;}
 // sürtünmeli eğik atiş problemi
 double func(double x[],int x_ref)
 {
   //x[0] is x, x[1] is y
```

```
   double g=9.806;
   if(x_ref==0) return x[3];
   else if(x_ref==1) return x[4];
   else if(x_ref==2) return -gamma*Math.sqrt(x[3]*x[3]+x[4]*x[4])*x[3];
   else              return -g - gamma*Math.sqrt(x[3]*x[3]+x[4]*x[4])*x[4];
  }
}

class diferansiyel3F
{

public static void main(String args[]) throws IOException
  {
  fm1 b1=new fm1(0.01);
  double y[];
  y=new double[4];
  double V0=250.0;//  m/s
  double alfa=Math.PI/3; //60 derece
  y[0]=0.0;
  y[1]=0.0;
  y[2]=V0*Math.sin(alfa);
  y[3]=V0*Math.cos(alfa);
  double a[][]=diferansiyel3.RK6(b1,0.0,20.0,y,1000);
  Plot p1=new Plot(a[1],a[2]);
  alfa=Math.PI/4;  //45 derece
  y[2]=V0*Math.sin(alfa);
  y[3]=V0*Math.cos(alfa);
  double b[][]=diferansiyel3.RK6(b1,0.0,20.0,y,1000);
  p1.addData(b[1],b[2]);
  alfa=Math.PI/6;  //30 derece
  y[2]=V0*Math.sin(alfa);
  y[3]=V0*Math.cos(alfa);
  double c[][]=diferansiyel3.RK6(b1,0.0,20.0,y,1000);
  p1.addData(c[1],c[2]);
  double xa[]=new double[2];
  double ya[]=new double[2];
  xa[0]=0;xa[1]=12.0;
  ya[0]=0;ya[1]=0.0;
  p1.addData(xa,ya);
  p1.setXlabel("x");
  p1.setYlabel("y");
  p1.setPlabel("sürtünmeli top atişi 60,45 ve 30 derece açi ve 250 m/s mermi çikiş hizi");
  p1.setMinMax(0,300,0,200);
  p1.plot();
  }
}
```

Output 8.7-10 gun bullet trajectory with air friction
θ=30, 45, 60 derece, bullet gun exit speed 250 m/s, gravitational accaleration 8.806 m/s^2, Friction coefficient γ = (k / m)= 0.01



Not that a bigger y scale is used in order to see trajectory ore clearly. Python version of the problem is given below:

Program gun bullet trajectory with air friction , Python version

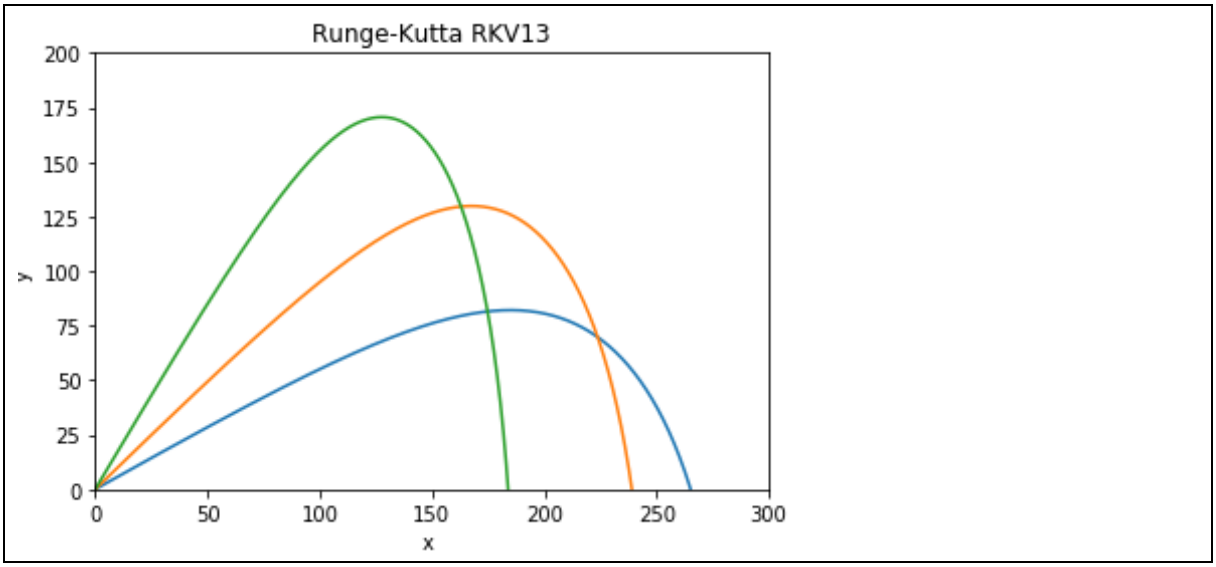
```
# -*- coding: utf-8 -*-
"""
Created on Fri Aug 31 13:10:28 2018

@author: Mustafa Turhan Çoban
"""
from Runge_Kutta import *;
from  Butcher import *;
from f_xi import *;
from math import *
import numpy as np;
import matplotlib.pyplot as plt;

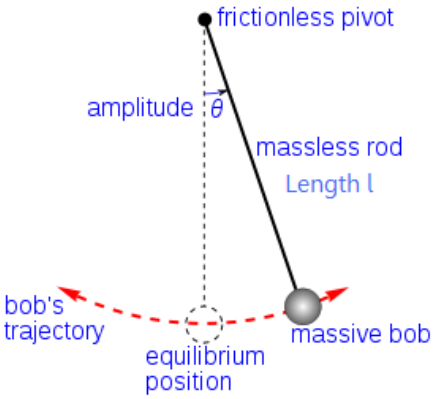
class f1(f_xi):
    def __init__(self,gammai):
      self.name="Bullet trajectory";
      self.gamma=gammai;
      self.g=9.806;#m/s^2
    def func(self,x,x_ref):
       if x_ref==0: F=x[3];
       elif x_ref==1: F=x[4];
       elif x_ref==2: F=-self.gamma*sqrt(x[3]*x[3]+x[4]*x[4])*x[3];
       elif x_ref==3: F=-self.g - self.gamma*sqrt(x[3]*x[3]+x[4]*x[4])*x[4];
       else: F=0.0;
       return F;

rk=Runge_Kutta("RKV13");
f=f1(0.01);
y=[0 for i in range(4)];
V0=250.0;#m/s
alpha=pi/3.0;
y[0]=0.0;
y[1]=0.0;
y[2]=V0*sin(alpha);
y[3]=V0*cos(alpha);
a=rk.calc_n(f,0.0,20.0,y,0.001,0);
alpha=pi/4.0;
y[2]=V0*sin(alpha);
y[3]=V0*cos(alpha);
b=rk.calc_n(f,0.0,20.0,y,0.001,0);
alpha=pi/6.0;
y[2]=V0*sin(alpha);
y[3]=V0*cos(alpha);
c=rk.calc_n(f,0.0,20.0,y,0.001,0);
plt.title('Runge-Kutta RKV13')
plt.xlabel('x ')
plt.ylabel('y ');
plt.ylim(ymin=0);
```

```
plt.ylim(ymax=200);
plt.xlim(xmin=0);
plt.xlim(xmax=300);
plt.plot(a[1],a[2],b[1],b[2],c[1],c[2]);
```



Pendulum is another interesting problem. Pendulum is an old mechanical problem. A **pendulum** is a weight suspended from a pivot, length l, so it can swing freely. When a pendulum is displaced from its resting equilibrium position with a displacement angle θ, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position (θ=0). When released, the restoring force combined with the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. From its discovery around 1602 by Galileo Galilei the regular motion of pendulums was used for timekeeping, and was the world's most accurate timekeeping technology until the 1930s.



The diferential equation of the pendulum motion is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta)$$

In this equation , θ swing angle, t time, and g is gravitational acceleration coefficient(m/s^2) . This is a non-linear differential equation. For very small swing angles, from sin(θ) Taylor series opening

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + ...$$

Diferential equation can be approximated as

$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$ This is a simple linear equation that can be solved by analytical methods easily. The solution is

$$\theta(t) = \theta_0 \cos(\sqrt{\frac{g}{l}}t)$$

An analytical solution of the non-linear equation is not existed, but it can be solved by using numerical methods. Both solution is calculated and plotted by using the following program. As it is seen from the plot outputs, for small angles (π/16 radian) both solutions are quite similar, but when the angle grows to bigger values(π/2 radian), solutions differs.

Program Pendulum problem

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta 6 method with multible differential equations
// Dr. Turhan Coban
// =====================================================
// dy1/dx=f(x,y1,y2,..,yn)
// dy2/dx=f(x,y1,y2,..,yn)
// .....
// dyn/dx=f(x,y1,y2,..,yn)
// Pendulum problem

import java.io.*;

class fm1 extends f_xi
{
// Pendulum equation
 double g=9.806;  //gravitational acceleration m/s2
 double l=0.6;   //pendulum length m

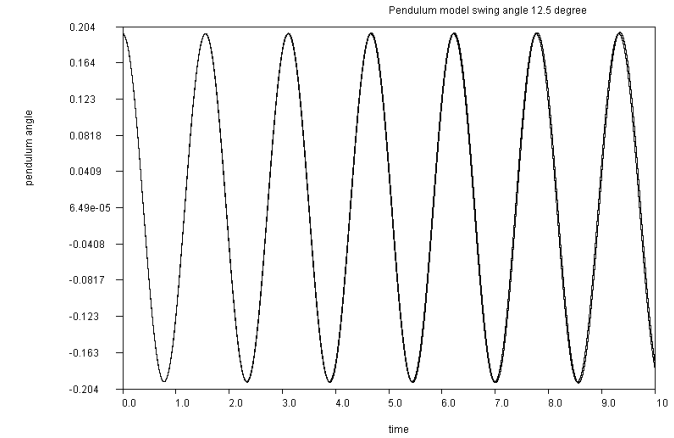
 double func(double x[],int x_ref)
 {
   //x[0] is t, x[1] is teta(pendulum swing angle) x[2] v velocity
   if(x_ref==0) return x[2];
   else return -g/l*Math.sin(x[1]);
 }
}

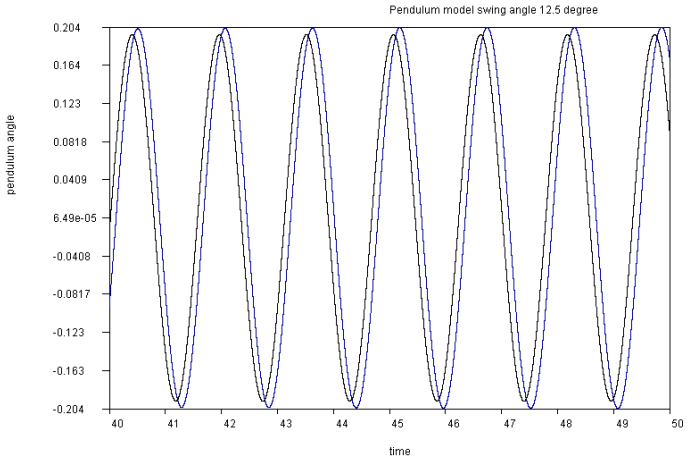
class NA101
{

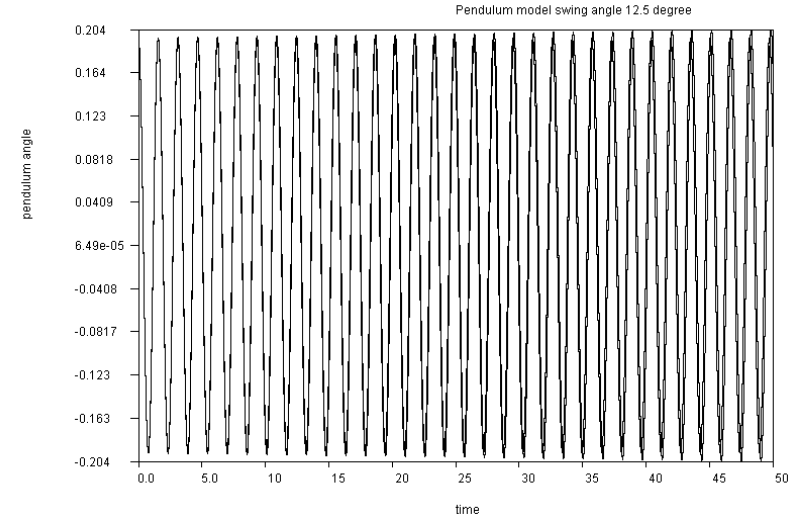
public static void main(String args[]) throws IOException
{
  fm1 b1=new fm1();
  double y[];
  y=new double[2];
  y[0]=Math.PI/16.0; //11.25 degree swing angle
  y[1]=0.0;
  double a[][]=NA93.RK6(b1,0.0,10.0,y,100000);
  int n=a[0].length;
  double teta[]=new double[n];
  for(int i=0;i<n;i++) {teta[i]=y[0]*Math.cos(Math.sqrt(b1.g/b1.l)*a[0][i]);}
  Plot p1=new Plot(a[0],a[1]);
```

```
  p1.addData(a[0],teta);
  p1.setXlabel("time");
  p1.setYlabel("pendulum angle");
  p1.setPlabel("Pendulum model");
  p1.plot();
 }
}
```

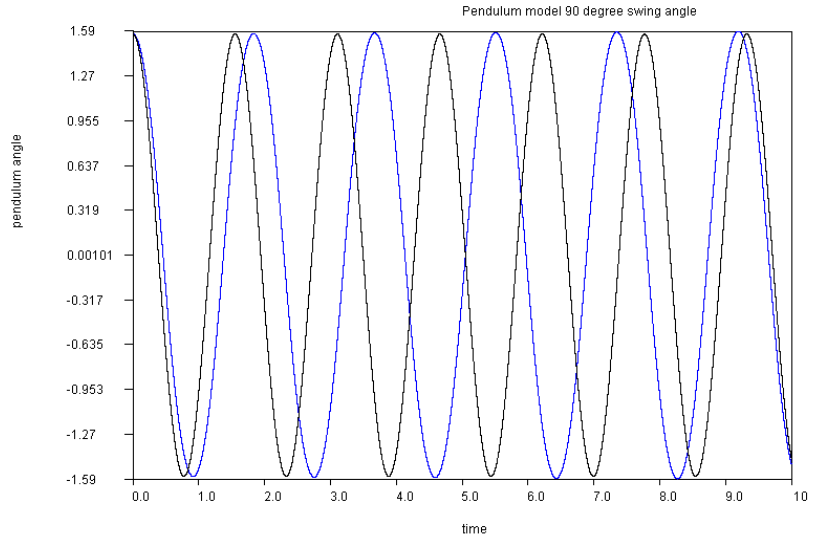
Output 8.7-11 Pendulum problem
$\theta=\pi/16$ radian, gravitational accaleration g=9.806 m/s^2, Pendulum length l= 0.6 m







Output Pendulum problem θ=π/16 radian, gravitational accaleration g=9.806 m/s^2, Pendulum length l= 0.6 m



Even with small angle two solutions differs by long period of time, but the wrong one could be the numerical solution here due to added error.

BOUNDARY PROBLEM AND SHOOTING METHODS IN DIFFERENTIAL EQUATIONS

In all differential equations solutions so far, it is asumed that initial condition (value of dependent value at satrting independent value). In the real system given boundary condition is not necassarily initial condition. Numerical differential equation solution methods requires an initial condition to be given. When a different type of boundary value is defined, following steps can be taken to solve the problem

Step 1 : Guess a value for the initial condition and solve the diferential equation for this initial condition

Step 2 : Repeat step 1 3 to 5 times. Find the value correspond to the given boundary condition (for example at the other end)

Step 3 : Make a curve fitting or interpolation or root finding between initial conditions and corresponding end result

Step 4 : By using the the curve fitting find initial condition corresponds to the given boundary condition

As an example let us investigate the following problem:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)\rightarrow\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}=0$$

Boundary conditions:

r=1 V=100

r=2 V=0

Program Boundary value problem solution

```
 //=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta methods with root finding function
// Dr. Turhan Coban
// =====================================================
// dy/dx=f(x,y)
// Multiequation method with error correction
import java.io.*;
import java.util.*;
import javax.swing.*;

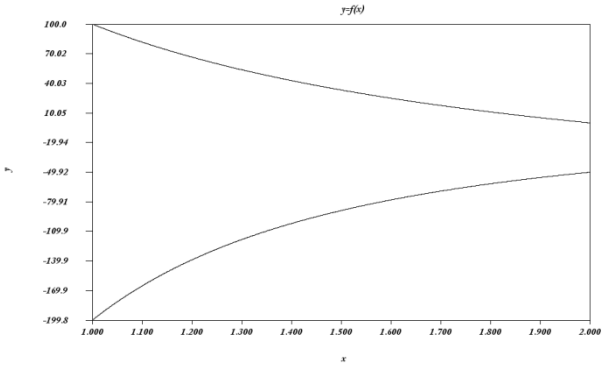
class fm1 extends f_xi
{
//multivariable function
```

```
 double func(double x[],int x_ref)
 { //x[0] is x, x[1] is y
   if(x_ref==0) return x[2];
   if(x_ref==1) return -2/x[0]*x[2];
   else return 0;
 }
}

class boundary
{   Runge_Kutta rk;
    int index_iy; //index of y value 0..number_of_equations for given b.c
    int index_ix; //index of x value which boundary condition is given
    int index_initial_y; //index of initial_y value to be calculated
    double y_value;
    double h;
    int n;
    double x0,xn;
    f_xi f;
                public boundary(f_xi fi,String eqn,int index_iyi,int index_ixi,int index_initial_yi,double y_valuei,double hi,double x0i,double
xni)
    {   f=fi;
                    rk=new Runge_Kutta(eqn);
                    h=hi;
                    n=(int)((xn-x0)/h+1e-30);
     index_iy=index_iyi;
     index_ix=index_ixi;
     index_initial_y=index_initial_yi;
     y_value=y_valuei;
     x0=x0i;
     xn=xni;
    }
     public static double[] doublearray(String s)
     {
     StringTokenizer token=new StringTokenizer(s);
     int n1=token.countTokens()-1;
     int m=n1+1;
     double a[]=new double[m];
     int j=0;
     while(token.hasMoreTokens())
     {
     Double ax=new Double(token.nextToken());
     a[j++]=ax.doubleValue();
     }
     return a;
     }
  public double func(double y[],double yi)
  {   y[index_initial_y]=yi;
                double a[][]=rk.hesapla(f,x0,xn,y,h);
      double yy=a[index_iy+1][index_ix]-y_value;
      return yy;
  }
  public double[][] func(double y[])
  {   double a[][]=rk.hesapla(f,x0,xn,y,h);
      return a;
  }
  public double[][] solve_bisection(double y[],double xl,double xu)
  {     y[index_initial_y]=bisection(y,xl,xu);
        double a[][]=func(y);
        return a;
  }

public  double bisection(double y[],double xl,double xu)
{ // bisection root finding method
double test;
double xr=0;
double es,ea;
double fxl,fxr;
int maxit=200,iter=0;
es=0.001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
{   xr=(xl+xu)/2.0;
    iter++;
    if((xl+xu)!=0)
        { ea=Math.abs((xu-xl)/(xu+xl))*100;}
    fxl= func(y,xl);
```

```
    fxr= func(y,xr);
    test= fxl*fxr;
    if(test==0.0)      ea=0;
    else if(test<0.0)  xu=xr;
    else               {xl=xr;}
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Maximum number of iteration is exceeded \n"+
   " result might not be valid","MAKSİMUM NUMBER OF ITERATION WARNING",JOptionPane.WARNING_MESSAGE);
return xr;
}
  public static void main(String args[]) throws IOException
  {
  fm1 b3=new fm1();
  double y[];
  y=new double[2];
  y[0]=100.0;
  y[1]=10;
  double x0=1.0;
  double xn=2.0;
  int index_iyi=0;
  double h=0.02;
  int index_ixi=(int)((xn-x0)/h+1e-30);
  int index_initial_yi=1;
  double y_value=0.0;
  boundary b=new boundary(b3,"RK6",index_iyi,index_ixi,index_initial_yi,y_value,h,x0,xn);
  double a[][]=b.solve_bisection(y,-300,300.0);
  Text.print(Text.T(a));
  Plot pp=new Plot(a);
  pp.plot();
  }
}
```



r	V1	V2
1	100	-197.0230103
1.02	96.05953979	-189.3723666
1.04	92.27209246	-182.1588482
1.06	88.6289155	-175.3497777
1.08	85.12191994	-168.9154752
1.1	81.74361044	-162.8289341
1.12	78.48703176	-157.0655375
1.14	75.34572101	-151.6028088
1.16	72.31366483	-146.4201919
1.18	69.38526099	-141.4988583
1.2	66.55528383	-136.8215349
1.22	63.81885313	-132.372353
1.24	61.17140607	-128.1367132
1.26	58.60867181	-124.1011654
1.28	56.1266485	-120.2533022
1.3	53.72158245	-116.5816629
1.32	51.3899492	-113.0756487
1.34	49.12843622	-109.7254457
1.36	46.93392731	-106.5219562
1.38	44.80348818	-103.4567372
1.4	42.73435344	-100.521944
1.42	40.72391456	-97.71028082
1.44	38.76970894	-95.01495479
1.46	36.86940985	-92.42963513
1.48	35.02081715	-89.94841593
1.5	33.22184883	-87.56578233
1.52	31.47053318	-85.27657992
1.54	29.76500158	-83.07598678
1.56	28.10348185	-80.9594881

1.58	26.48429208	-78.92285301
1.6	24.90583502	-76.96211338
1.62	23.36659276	-75.07354452
1.64	21.86512187	-73.25364748
1.66	20.40004892	-71.49913277
1.68	18.97006626	-69.80690556
1.7	17.57392815	-68.17405199
1.72	16.21044711	-66.59782661
1.74	14.87849058	-65.07564085
1.76	13.57697776	-63.60505238
1.78	12.30487671	-62.18375529
1.8	11.06120161	-60.80957106
1.82	9.845010187	-59.48044024
1.84	8.655401382	-58.19441465
1.86	7.491513089	-56.94965032
1.88	6.352520082	-55.74440082
1.9	5.237632066	-54.57701115
1.92	4.146091843	-53.44591207
1.94	3.077173602	-52.3496148
1.96	2.030181306	-51.28670613
1.98	1.004447183	-50.25584386
2	-6.70E-04	-49.25575256

Now let us try another one

$$\frac{\partial^2 y}{\partial x^2} = y^3 - y\frac{\partial y}{\partial x}$$

Boundary conditions:

x=1 y=0.5

x=2 y=1/3

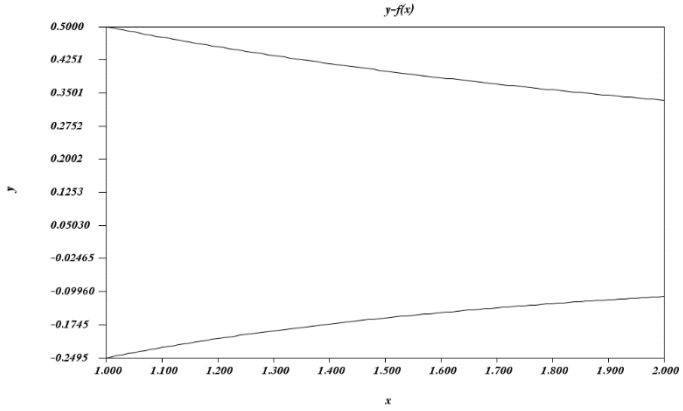
Program Boundary value problem solution

```
 //=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta methods with root finding function
// Dr. Turhan Coban
// =====================================================
// dy/dx=f(x,y)
// Multiequation method with error correction
import java.io.*;
import java.util.*;
import javax.swing.*;

class fm2 extends f_xi
{
//multivariable function
 double func(double x[],int x_ref)
 { //x[0] is x, x[1] is y
   if(x_ref==0) return x[2];
   if(x_ref==1) return x[1]*x[1]*x[1]-x[1]*x[2];
   else return 0;
 }
}

class boundarytest1
{      public static void main(String args[]) throws IOException
  {
  fm2 b3=new fm2();
  double y[];
  y=new double[2];
  y[0]=0.5;
  y[1]=-1.0;
  double x0=1.0;
  double xn=2.0;
  int index_iyi=0;
  double h=0.01;
  int index_ixi=(int)((xn-x0)/h+1e-30);
  int index_initial_yi=1;
  double y_value=1.0/3.0;
  boundary b=new boundary(b3,"RK6",index_iyi,index_ixi,index_initial_yi,y_value,h,x0,xn);
  double a[][]=b.solve_bisection(y,-4.0,4.0);
  //double a[][]=b.solve_cubic_spline(y,0.0,20.0);
  Text.print(Text.T(a));
  Plot pp=new Plot(a);
```

```
  pp.plot();
  }
}
```



1	0.5	-0.249497592
1.01	0.497505024	-0.247006338
1.02	0.495034961	-0.244552198
1.03	0.492589439	-0.242134437
1.04	0.490168094	-0.23975234
1.05	0.487770571	-0.237405207
1.06	0.485396519	-0.235092355
1.07	0.483045595	-0.232813119
1.08	0.480717464	-0.23056685
1.09	0.478411796	-0.228352914
1.1	0.476128266	-0.226170693
1.11	0.47386656	-0.224019581
1.12	0.471626364	-0.221898991
1.13	0.469407374	-0.219808345
1.14	0.46720929	-0.217747083
1.15	0.46503182	-0.215714654
1.16	0.462874673	-0.213710523
1.17	0.460737568	-0.211734166
1.18	0.458620226	-0.209785071
1.19	0.456522375	-0.207862738
1.2	0.454443748	-0.205966679
1.21	0.452384081	-0.204096416
1.22	0.450343117	-0.202251482
1.23	0.448320602	-0.200431422
1.24	0.446316288	-0.198635789
1.25	0.44432993	-0.196864148
1.26	0.442361289	-0.195116071
1.27	0.440410128	-0.193391143
1.28	0.438476217	-0.191688956
1.29	0.436559327	-0.190009111
1.3	0.434659236	-0.188351217
1.31	0.432775724	-0.186714894
1.32	0.430908575	-0.185099768
1.33	0.429057577	-0.183505474
1.34	0.427222522	-0.181931655
1.35	0.425403206	-0.180377961
1.36	0.423599426	-0.178844049
1.37	0.421810986	-0.177329585
1.38	0.42003769	-0.17583424
1.39	0.418279347	-0.174357693
1.4	0.416535771	-0.172899631
1.41	0.414806774	-0.171459745
1.42	0.413092177	-0.170037734
1.43	0.411391799	-0.168633302
1.44	0.409705466	-0.167246161
1.45	0.408033005	-0.165876028
1.46	0.406374245	-0.164522624
1.47	0.404729018	-0.163185678
1.48	0.403097161	-0.161864924
1.49	0.401478512	-0.160560101
1.5	0.399872911	-0.159270953
1.51	0.398280202	-0.15799723
1.52	0.396700229	-0.156738686
1.53	0.395132843	-0.15549508
1.54	0.393577892	-0.154266177
1.55	0.39203523	-0.153051744

1.56	0.390504713	-0.151851556
1.57	0.388986197	-0.15066539
1.58	0.387479543	-0.149493028
1.59	0.385984613	-0.148334257
1.6	0.38450127	-0.147188865
1.61	0.383029382	-0.146056649
1.62	0.381568815	-0.144937406
1.63	0.380119441	-0.143830938
1.64	0.378681132	-0.142737051
1.65	0.377253761	-0.141655556
1.66	0.375837206	-0.140586264
1.67	0.374431343	-0.139528993
1.68	0.373036053	-0.138483563
1.69	0.371651217	-0.137449797
1.7	0.370276719	-0.136427522
1.71	0.368912444	-0.135416568
1.72	0.367558279	-0.134416768
1.73	0.366214111	-0.133427959
1.74	0.364879831	-0.132449979
1.75	0.363555331	-0.13148267
1.76	0.362240505	-0.130525878
1.77	0.360935246	-0.12957945
1.78	0.359639451	-0.128643237
1.79	0.358353019	-0.127717092
1.8	0.357075848	-0.126800871
1.81	0.355807839	-0.125894432
1.82	0.354548895	-0.124997636
1.83	0.353298919	-0.124110347
1.84	0.352057815	-0.12323243
1.85	0.350825491	-0.122363754
1.86	0.349601853	-0.121504188
1.87	0.348386812	-0.120653606
1.88	0.347180276	-0.119811883
1.89	0.345982157	-0.118978896
1.9	0.344792368	-0.118154524
1.91	0.343610822	-0.117338648
1.92	0.342437436	-0.116531153
1.93	0.341272124	-0.115731922
1.94	0.340114805	-0.114940843
1.95	0.338965397	-0.114157807
1.96	0.337823819	-0.113382703
1.97	0.336689992	-0.112615424
1.98	0.335563837	-0.111855867
1.99	0.334445279	-0.111103926
2	0.33333424	-0.1103595

In the third problem

$$\frac{\partial^2 y}{\partial x^2} = -x^2 + x\frac{\partial y}{\partial x} - y$$

Boundary conditions:

x=0 y=-2

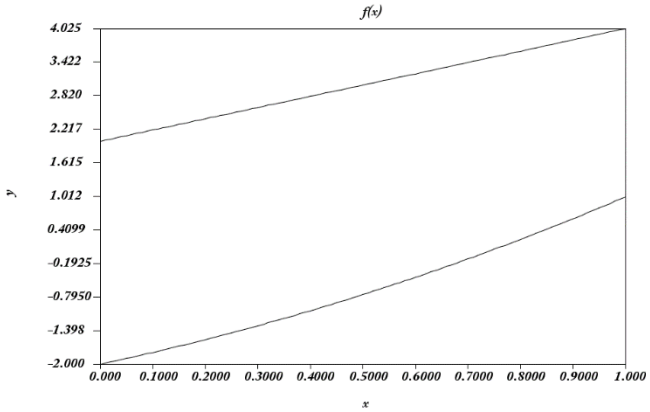
x=1 y=1

Program Boundary value problem solution

```
//=====================================================
// Numerical Analysis package in java
// example to show differential equation solution
// Runge-Kutta methods with root finding function
// Dr. Turhan Coban
// =====================================================
// dy/dx=f(x,y)
// Multiequation method with error correction
import java.io.*;
import java.util.*;
import javax.swing.*;

class fm2 extends f_xi
{
//multivariable function
 double func(double x[],int x_ref)
 { //x[0] is x, x[1] is y
   if(x_ref==0) return x[2];
   if(x_ref==1) return -x[0]*x[0]+x[0]*x[2]-x[1];
   else return 0;
 }
}
```

```
class boundarytest2
{      public static void main(String args[]) throws IOException
  {
  fm2 b3=new fm2();
  double y[];
  y=new double[2];
  y[0]=-2.0;
  y[1]=1.0;
  double x0=0.0;
  double xn=1.0;
  int index_iyi=0;
  double h=0.01;
  int index_ixi=(int)((xn-x0)/h+1e-30);
  int index_initial_yi=1;
  double y_value=1.0;
  boundary b=new boundary(b3,"RK6",index_iyi,index_ixi,index_initial_yi,y_value,h,x0,xn);
  double a[][]=b.solve_bisection(y,-4.0,4.0);
  //double a[][]=b.solve_cubic_spline(y,0.0,20.0);
  Text.print(Text.T(a));
  Plot pp=new Plot(a);
  pp.plot();
  }
}
```



0	-2	2.000961
0.01	-1.97999	2.021062
0.02	-1.95978	2.041164
0.03	-1.93937	2.061269
0.04	-1.91876	2.081375
0.05	-1.89794	2.101484
0.06	-1.87693	2.121594
0.07	-1.85571	2.141707
0.08	-1.83429	2.161822
0.09	-1.81268	2.181938
0.1	-1.79086	2.202057
0.11	-1.76884	2.222178
0.12	-1.74661	2.242302
0.13	-1.72419	2.262427
0.14	-1.70157	2.282555
0.15	-1.67874	2.302684
0.16	-1.65571	2.322816
0.17	-1.63249	2.342951
0.18	-1.60906	2.363087
0.19	-1.58543	2.383226
0.2	-1.56159	2.403367
0.21	-1.53756	2.42351
0.22	-1.51332	2.443656
0.23	-1.48889	2.463804
0.24	-1.46425	2.483954
0.25	-1.43941	2.504106
0.26	-1.41437	2.524262
0.27	-1.38913	2.544419
0.28	-1.36368	2.564579
0.29	-1.33804	2.584741
0.3	-1.31219	2.604906
0.31	-1.28614	2.625074
0.32	-1.25989	2.645243
0.33	-1.23344	2.665416
0.34	-1.20678	2.685591
0.35	-1.17993	2.705769
0.36	-1.15287	2.725949

0.37	-1.12561	2.746132
0.38	-1.09815	2.766318
0.39	-1.07049	2.786506
0.4	-1.04262	2.806697
0.41	-1.01455	2.826891
0.42	-0.98628	2.847088
0.43	-0.95781	2.867288
0.44	-0.92914	2.88749
0.45	-0.90027	2.907696
0.46	-0.87119	2.927904
0.47	-0.84191	2.948115
0.48	-0.81243	2.96833
0.49	-0.78275	2.988547
0.5	-0.75286	3.008768
0.51	-0.72277	3.028991
0.52	-0.69248	3.049218
0.53	-0.66199	3.069448
0.54	-0.6313	3.089681
0.55	-0.6004	3.109918
0.56	-0.5693	3.130158
0.57	-0.538	3.150401
0.58	-0.50649	3.170648
0.59	-0.47479	3.190898
0.6	-0.44288	3.211152
0.61	-0.41077	3.231409
0.62	-0.37845	3.25167
0.63	-0.34594	3.271934
0.64	-0.31322	3.292202
0.65	-0.28029	3.312474
0.66	-0.24717	3.33275
0.67	-0.21384	3.353029
0.68	-0.18031	3.373313
0.69	-0.14658	3.3936
0.7	-0.11264	3.413892
0.71	-0.0785	3.434187
0.72	-0.04416	3.454487
0.73	-0.00962	3.474791
0.74	0.02513	3.495099
0.75	0.060081	3.515412
0.76	0.095235	3.535728
0.77	0.130593	3.55605
0.78	0.166153	3.576376
0.79	0.201917	3.596706
0.8	0.237884	3.617041
0.81	0.274054	3.637381
0.82	0.310428	3.657725
0.83	0.347005	3.678075
0.84	0.383786	3.698429
0.85	0.42077	3.718789
0.86	0.457958	3.739153
0.87	0.49535	3.759523
0.88	0.532945	3.779898
0.89	0.570744	3.800279
0.9	0.608747	3.820664
0.91	0.646954	3.841056
0.92	0.685364	3.861453
0.93	0.723979	3.881855
0.94	0.762797	3.902264
0.95	0.80182	3.922678
0.96	0.841047	3.943099
0.97	0.880478	3.963525
0.98	0.920113	3.983958
0.99	0.959952	4.004397
1	0.999996	4.024842

As the fourth problem is the Blassius flat plate problem from fluid mechanics is considered. Blassius flat plate problem is one of the basic problem that shapes heat convection as we know today. Problem based on two differential eqaution the first one is called similarity monentum equation

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0 \text{ where } \eta=\frac{y}{\sqrt{\nu x/u_\infty}} \text{ and } f'(\eta)=\frac{u}{u_\infty}$$

Boundary conditions:

$$\eta=0 \quad f'(\eta)=0$$

$$\eta\to\infty \quad f'(\eta)=1$$

$$\eta = 0 \quad f''(\eta) = 0 \; f''(0) = 0$$

And energy equation

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{\Pr}{2} f \frac{\partial \theta}{\partial \eta} = 0$$

$$\theta(0) = 0$$

$$\theta(\infty) = 1$$

Not that Pr is Prandtl number, a parameter we can define as a constant.Equations can be solved directly as a set of differential equations. Equation set is as follows:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0$$

$$\theta'' + \frac{\Pr}{2} f\theta' = 0$$

Conversion of the equation:

$$x[0] = \eta$$

$$x[1] = f$$

$$x[2] = f'$$

$$x[3] = f''$$

$$x[4] = \theta$$

$$x[5] = \theta'$$ Differential equation set becomes:

$$\frac{dx[1]}{dx[0]} = x[2]$$

$$\frac{dx[2]}{dx[0]} = x[3]$$

$$\frac{dx[3]}{dx[0]} = -0.5 * x[1] * x[3]$$

$$\frac{dx[4]}{dx[0]} = x[5]$$

$$\frac{dx[4]}{dx[0]} = -\frac{\Pr}{2} x[1] * x[5]$$

Boundary conditions:

x[4]=0 at x[0]=0

x[4]=1 at x[0]= ∞

x[2]=0 at x[0]=0

x[2]=1 at x[0]= ∞

x[1]=0 at x[0]= 0

As it is seen from boundary conditions, two given conditions are not initial conditions. One way of solving boundary conditions is using non-linear system of equation solving methods. In order to solve differential

equation set 6th order Runge-Kutta Method is used. In order to solve boundary value nonlinear system of equation Nelder & Mead optimization method is used.

Program Blassius flat plate problem

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;

class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
double y[];
fm2 b3;
double a[][];
double b[][];
public fa(double Pr,double yi[])
{ b3=new fm2(Pr);
  y=new double[yi.length];
  for(int i=0;i<y.length;i++) y[i]=yi[i];
}

public double[] func(double x[])
{
// shooting method
y[2]=x[0];
y[4]=x[1];
a=RK6(b3,0.0,6.8,y,10000);
//System.out.println(a[0][10000]+" "+a[1][10000]+" "+a[2][10000]+" "+a[3][10000]+" "+a[4][10000]+" "+a[5][10000]+" ");
double ff[]=new double[2];
ff[0]=a[2][10000]-1.0;
ff[1]=a[4][10000]-1.0;
return ff;
}
 //differenetial equation solution
public static double[][] RK6(f_xi fp,double x0,double xn,double f0[],int N)
{
//6th order Runge Kutta Method
//fp  : given set of derivative functions dfj/dxi(fj,x)
// xo : initial value of the independent variable
// xn : final value of the independent variable
// f0 : initial value of the dependent variable
// N  : number of dependent variable to be calculated
// fi : dependent variable
double h=(xn-x0)/N;
int M=f0.length;
double fi[][];
fi=new double[M][N+1];
double xi[]=new double[M+1];
double k[]=new double[6];
int i,j;
double x;
for(j=0;j<M;j++)
  {
  fi[j][0]=f0[j];
  xi[j+1]=f0[j];
```

```
  }
for(x=x0,i=0;i<N;x+=h,i++)
  {
  for(j=1;j<=M;j++)
    {
    xi[0]=x;
    xi[j]=fi[j-1][i];
    k[0]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/2;
    k[1]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/4.0+k[1]/4.0;
    k[2]=h*fp.func(xi,j-1);
    xi[0]=x+h;
    xi[j]=fi[j-1][i]-k[1]+2.0*k[2];
    k[3]=h*fp.func(xi,j-1);
    xi[0]=x+2.0/3.0*h;
    xi[j]=fi[j-1][i]+7.0/27.0*k[0]+10.0/27.0*k[1]+1.0/27.0*k[3];
    k[4]=h*fp.func(xi,j-1);
    xi[0]=x+1.0/5.0*h;
    xi[j]=fi[j-1][i]+28.0/625.0*k[0]-1.0/5.0*k[1]+546.0/625.0*k[2]+54.0/625.0*k[3]-378/625.0*k[4];
    k[5]=h*fp.func(xi,j-1);
    fi[j-1][i+1]=fi[j-1][i]+k[0]/24.0+5.0*k[3]/48.0+27.0*k[4]/56.0+125.0*k[5]/336.0;
    xi[j]=fi[j-1][i];
    }
  }
double a[][]=new double[M+1][N+1];
for(x=x0,i=0;i<=N;x+=h,i++)
{
  a[0][i]=x;
  for(j=1;j<=M;j++)
  {
  a[j][i]=fi[j-1][i];
  }
}

return a;
}
}

//Blassius flat plate differential equation
class fm2 extends f_xi
{ double Pr;
  fm2(double Pri) {Pr=Pri;}
// multivariable function
 double func(double x[],int x_ref)
 {
   //Blasius flat plate diferential equation
   //f"'+0.5*ff" = 0
   //f"'=-0.5*x[1]*x[3] f"=x[3] f'=x[2] f=x[1]]
   double a=0;
   if(x_ref==4) a= -0.5*Pr*x[1]*x[5];
   if(x_ref==3) a= x[5];
   if(x_ref==2) a= -0.5*x[1]*x[3];
   if(x_ref==1) a= x[3];
   if(x_ref==0) a= x[2];;
   return a;
 }
}

public class blassius_flat_plate
{
                public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
```

```
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
public static String toString(double x[])
{ String s="";
int n=x.length;
  for(int k=0;k<n;k++)
  {s+=x[k]+" ";}
  return s;
}
public static double[] nelder(f_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
                int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
   {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
   }


   // Inlet variable definitions
   // fnelder : abstract multivariable function f(x)
   // x : independent variable set of n+1 simplex elements
   // maxiteration : maximum iteration number
   // tolerance :
       int NDIMS = x.length-1;
       int NPTS = x.length;
       int FUNC = NDIMS;
       int ncalls = 0;
         ////// construct the starting simplex //////////////////
         //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
         double z[]=new double[NDIMS];
         double best = 1E99;
         ///////////////// calculate the first function values for the simplex /////////////////

         int iter=0;

         for (iter=1; iter<maxiteration; iter++)
         {
                    ////////////  define lo, nhi, hi (low high next_to_high //////////////
           int  ilo=0, ihi=0, inhi = -1; // -1 means missing
           double flo = p[0][FUNC];
           double fhi = flo;
           double pavg,sterr;
           for (i=1; i<NPTS; i++)
           {
             if (p[i][FUNC] < flo)
               {flo=p[i][FUNC]; ilo=i;}
             if (p[i][FUNC] > fhi)
               {fhi=p[i][FUNC]; ihi=i;}
           }
           double fnhi = flo;
           inhi = ilo;
           for (i=0; i<NPTS; i++)
            if ((i != ihi) && (p[i][FUNC] > fnhi))
              {fnhi=p[i][FUNC]; inhi=i;}
           ////////// exit criteria //////////////
           if ((iter % 4*NDIMS) == 0)
           {
              // calculate the avarage (including maximum value)
              pavg=0;
```

```
    for(i=0;i<NPTS;i++)
      pavg+=p[i][FUNC];
    pavg/=NPTS;
    double tot=0;
    if(printlist!=0)
    {  System.out.print(iter);
       for (j=0; j<=NDIMS; j++)
          { System.out.print(p[ilo][j]+" ");}
       System.out.println("");
    }
    for(i=0;i<NPTS;i++)
    { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
    sterr=Math.sqrt(tot/NPTS);
    //if(sterr < tolerance)
      { for (j=0; j<NDIMS; j++)
          { z[j]=p[ilo][j];}
        //break;
      }
    best = p[ilo][FUNC];
}

///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);


///////// reflect ////////////////

double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
    for (j=0; j<NDIMS; j++)
      p[ihi][j] = r[j];
    p[ihi][FUNC] = fr;
    continue;
}

if (fr < flo)  //// expand
{
    double e[] = new double[NDIMS];
    for (j=0; j<NDIMS; j++)
      e[j] = 3*ave[j] - 2*p[ihi][j];
    double fe = fnelder.func(e);
    if (fe < fr)
    {
       for (j=0; j<NDIMS; j++)
         p[ihi][j] = e[j];
       p[ihi][FUNC] = fe;
       continue;
    }
    else
    {
       for (j=0; j<NDIMS; j++)
         p[ihi][j] = r[j];
       p[ihi][FUNC] = fr;
       continue;
    }
}
```

```
            //////////// shrink:

            if (fr < fhi)
            {
               double c[] = new double[NDIMS];
               for (j=0; j<NDIMS; j++)
                 c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
               double fc = fnelder.func(c);
               if (fc <= fr)
               {
                  for (j=0; j<NDIMS; j++)
                    p[ihi][j] = c[j];
                  p[ihi][FUNC] = fc;
                  continue;
               }
               else   /////// daralt
               {
                  for (i=0; i<NPTS; i++)
                    if (i != ilo)
                    {
                       for (j=0; j<NDIMS; j++)
                         p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                       p[i][FUNC] = fnelder.func(p[i]);
                    }
                  continue;
               }
            }

            if (fr >= fhi)   ///
            {
               double cc[] = new double[NDIMS];
               for (j=0; j<NDIMS; j++)
                 cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
               double fcc = fnelder.func(cc);
               if (fcc < fhi)
               {
                  for (j=0; j<NDIMS; j++)
                    p[ihi][j] = cc[j];
                  p[ihi][FUNC] = fcc;
                  continue;
               }
               else    /////////
               {
                  for (i=0; i<NPTS; i++)
                    if (i != ilo)
                    {
                       for (j=0; j<NDIMS; j++)
                         p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                       p[i][FUNC] = fnelder.func(p[i]);
                    }
               }
            }
         }
      return z;
}
     public static double[] nelder(f_xj fnelder,double a[],double da[],double tolerance)
     {return nelder(fnelder,a,da,500,tolerance,0);}

     public static double[] nelder(f_xj fnelder,double a[],double da[])
     {return nelder(fnelder,a,da,500,1.0e-10,0);}

     public static double[] nelder(f_xj fnelder,double a[])
     {
                        double [] da=new double[a.length];
           for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
           return nelder(fnelder,a,da);
     }
// Polynomial least square
  public static double[] gausswithpartialpivot(double a[][],double b[])
```

```
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                  if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
           b[i]   =b[i]   -carpan*b[k];
                }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] PolynomialLSQ(double xi[],double yi[],int n)
{ //Polynomial least square
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  for(j=0;j<n+1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= yi[k];
                   else      B[i] += Math.pow(xi[k],i)*yi[k];}
  }
X=gausswithpartialpivot(A,B);
double max=0;
for(i=0;i<n+1;i++)
```

```
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
return X;
}

public static double funcPolynomialLSQ(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcPolynomialLSQ(e,x[k]))*(y[k]-funcPolynomialLSQ(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcPolynomialLSQ(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
                //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                 int n=xi.length;
                int nn=(n-1)*(aradegersayisi+1)+1;
                double z[][]=new double[2][nn];
    double E[]=PolynomialLSQ(xi,yi,polinomkatsayisi);
    System.out.println("katsayilar :\n"+Matrix.toStringT(E));
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
       {z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;
       for(int j=0;j<aradegersayisi;j++)
       {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;}
       }
       z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);
    return z;
 }

public static void main(String arg[])
{
  double y[]={0,0,0.1,0,0.1};
 //String s="leave a space between initial root guess values: xmin0 xmin1.. ";
 double x[]={0.1,0.1};//inputdata(s);
 fa ff;
 f2 f;
 double r1[];
 double Pr[]={1.0};
 double xi[]=new double[Pr.length];
 double yi[]=new double[Pr.length];
   String s[]={""+'\u03B7',"f","f'","f''",""+'\u03B8',""+'\u03B8'+"''"};
```

```
 for(int i=0;i<xi.length;i++)
 {ff=new fa(Pr[i],y);
  f=new f2(ff);
  r1=nelder(f,x);
  xi[i]=Math.log(Pr[i]);
  yi[i]=Math.log(r1[1]);
  System.out.println("Pr = "+Pr[i]+" f'="+r1[0]+" teta'="+r1[1]);
   int nn=ff.a.length;
   int mm=ff.a[0].length;
   System.out.println("nn="+nn+"mm="+mm);
   double b[][]=new double[nn][201];
   for(int j=0;j<mm;j+=50)
   { for(int k=0;k<nn;k++)
                b[k][j/50]=ff.a[k][j];
   }
  Text.printT(b,s);
 }

 double[] b=PolynomialLSQ(xi,yi,1);
 b[0]=Math.exp(b[0]);
 System.out.println("Nux="+b[0]+"Rex^0.5Pr^"+b[1]);

}
}
```

data output(veri çıktısı) :

η	f	f'	f''	θ	θ'
0.0	0.0	0.0	0.33176751005680527	0.0	0.33176751005680527
0.033999999999999996	1.8040932058659557E-4	0.011280093052269017	0.3317671912349006	0.011280093052269017	0.3317671912349006
0.06799999999999999	7.441879823734188E-4	0.022560148177521513	0.3317647958415689	0.022560148177521513	0.3317647958415689
0.102	0.0016914866316839924	0.03384006020790119	0.33175816091239246	0.03384006020790119	0.33175816091239246
0.13599999999999976	0.0030222977553112187	0.04511965043852202	0.3317451237875651	0.04511965043852202	0.3317451237875651
0.16999999999999907	0.0047366065882229495	0.05639866664399907	0.3317235225097647	0.05639866664399907	0.3317235225097647
0.20399999999999838	0.006834388615884927	0.06767678311045437	0.33169119635968686	0.06767678311045437	0.33169119635968686
0.23799999999999769	0.009315607075648581	0.07895360068820427	0.3316459865463205	0.07895360068820427	0.3316459865463205
0.2719999999999979	0.012180210461265483	0.09022864687038464	0.3315857370516235	0.09022864687038464	0.3315857370516235
0.3059999999999986	0.015428130030402665	0.10150137590275614	0.3315082956290707	0.10150137590275614	0.3315082956290707
0.3399999999999993	0.019059277316458272	0.11277116892990928	0.33141151495535515	0.11277116892990928	0.33141151495535515
0.374	0.023073541645998348	0.12403733418306279	0.33129325393427445	0.12403733418306279	0.33129325393427445
0.4080000000000007	0.027470787663311115	0.13529910721460944	0.331151379151572	0.13529910721460944	0.331151379151572
0.4420000000000014	0.03225085286375044	0.14655565118452077	0.3309837664791938	0.14655565118452077	0.3309837664791938
0.4760000000000021	0.037413545137712015	0.1578060572036577	0.33078830282708815	0.1578060572036577	0.33078830282708815
0.5100000000000028	0.04295864032725748	0.16904934473897243	0.3305628880403074	0.16904934473897243	0.3305628880403074
0.5440000000000035	0.048885879797568824	0.18028446208550017	0.33030543693876885	0.18028446208550017	0.33030543693876885
0.5780000000000042	0.055194968025581125	0.19151028690994057	0.33001388149661964	0.19151028690994057	0.33001388149661964
0.6120000000000049	0.06188557020830324	0.20272562687052534	0.32968617315767645	0.20272562687052534	0.32968617315767645
0.6460000000000056	0.068957309893492 83	0.21392922031773196	0.3293202852829434	0.21392922031773196	0.3293202852829434
0.6800000000000063	0.07640976663550668	0.22511973708026517	0.3289142157257063	0.22511973708026517	0.3289142157257063
0.714000000000007	0.08424247367929254	0.23629577934056442	0.32846598952916756	0.23629577934056442	0.32846598952916756
0.7480000000000077	0.09245491567563323	0.2474558826039179	0.32797366174104037	0.2474558826039179	0.32797366174104037
0.7820000000000084	0.10104652643088778	0.2585985167650596	0.3274353203389618	0.2585985167650596	0.3274353203389618
0.816000000000009	0.1100166866946028	0.2697220872759172	0.32684908925999434	0.2697220872759172	0.32684908925999434
0.8500000000000097	0.1193647219884897	0.28082493641792416	0.32621313152690423	0.28082493641792416	0.32621313152690423
0.8840000000000104	0.1290899004803732	0.29190534468207696	0.3255256524632878	0.29190534468207696	0.3255256524632878
0.9180000000000111	0.13919143090682146	0.3029615322596094	0.32478490298903145	0.3029615322596094	0.32478490298903145
0.9520000000000118	0.14966846054826258	0.3139916606458778	0.32398918298696244	0.3139916606458778	0.32398918298696244
0.9860000000000125	0.16052007326047069	0.324993834359724	0.3231368447309542	0.324993834359724	0.3231368447309542
1.0200000000000131	0.17174528756638383	0.3359661027802382	0.32222629636514466	0.3359661027802382	0.32222629636514466
1.0540000000000138	0.18334305481226687	0.34690646210249265	0.32125600542333527	0.34690646210249265	0.32125600542333527
1.0880000000000145	0.19531225739228827	0.35781285741342334	0.3202245023770666	0.35781285741342334	0.3202245023770666
1.1220000000000152	0.2076517070456059	0.3686831848886484	0.3191303842003111	0.3686831848886484	0.3191303842003111
1.156000000000016	0.220360143230085	0.37951529411058466	0.31797231793818453	0.37951529411058466	0.31797231793818453
1.1900000000000166	0.23343623157676824	0.3903069905077934	0.3167490442665892	0.3903069905077934	0.3167490442665892
1.2240000000000173	0.24687856242921505	0.40105603791503397	0.3154593810292179	0.40105603791503397	0.3154593810292179

PROBLEMS

PROBLEM 1

dy/dx = y*sin(x)* sin(x) diferansiyel equation is given. Solution is desired in the x=0 to x=1 range for h=dx=0.2 with the initial conditions of (x=0) y=2. Use fourth order Runge-Kutte method.

PROBLEM 2

$d^2y/dx^2+6\ dy/dx= \sin(\pi x)$ diferansiyel equation at x=0 to x=1 range with the boundary conditions of
x=0 y=2 and
x=1 de y=-1
Solve diferential equation for h=0.02 step size.

PROBLEM 3

dy/dx= sin(πx) diferansiyel equation at x=0 to x=1 range with the boundary conditions of
x=0 y=2 and

Solve diferential equation for h=0.02 step size.

PROBLEM 4
Falkner-Scan flow equation is given as
f''' + f*f'' + β(1-f'^{2}) = 0
Boundary conditions of the equation
f(0)=0
f'(0)=1
f'(∝)=1
β = 0.3 ve β = 1.0
Instead of infinity(∝) , take value of 8.

PROBLEM 5
Solve diferential equation
y'' = x^2 - y^2 - 3y'
with initial conditions y(0)=0, y'(0)=0

PROBLEM 6
Solve diferential equation
y''' = y'^{2} - 4y
with initial conditions y(0)=0, y'(0)=1, y''(0)=0 .

PROBLEM 7
Solve diferential equation

x' = 3x-y
y'=x+y
with initial conditions x(0)=0.2, ve y(0)=0.5 .

PROBLEM 8
Solve diferential equation
2x''(t)-5x'(t)-3x(t)=45e^{2t} with initial conditions x(0)=0 ve x'(0)=1 .

PROBLEM 9
Solve diferential equation
x'(t)=x(t)-x(t)y(t)
y'(t)=-y(t)+x(t)y(t) with initial conditions x(0)=4 ve y(0)=1 at t=[0,8] range.

PROBLEM 10
Solve diferential equation
$x'(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$ with initial conditions x(0)=1.25 ve x(4)=-0.95 at range t=[0,4].

PROBLEM 11
Solve diferential equation
y'' = -(y')^2 – y +lnx
with initial conditions: y(1)=0 y(2)=ln2 at the range 1<=x<=2

PROBLEM 12
For diferential equation
$y' - 2(1+y^2) = 0$ with , $y(0)=0$ starting value and, h=0,25 step size find $y(1)$ değerini with Heun method

PROBLEM 13
Solve diferential equation
$\frac{dy}{dx} = yx^2 - y$ at the range x=0 ve x=2 , with initial condition y(0)=1 .

c) solve analytically from x=0 to x=2.
d) solve numerically by hand with h=0.5 step size with 4th order Runge_Kutte and calculate y(2)
e) solve numerically by computer with h=0.1 step size with 4th order Runge-Kutta
f) Plot to results and compare with analytical solution

PROBLEM 14
Solve diferential equation

$\frac{dy}{dx} = -5y + e^{-2x}$ with initial conditions **:** x_0=0 y_0=1 . $y(1)$, and with step size h=0.2 by using Simpson 1/3 rule.

PROBLEM 15
Solve diferential equation

$\frac{dy}{dx} = \frac{1}{x+y}$ with initial conditions x_0=0 , y_0=2 , at the interval 0-0.6 with step size h=0.2,

a-) with 4th order Runge-Kutta method
b-) with Euler method

PROBLEM 16
Cooling rate of a substance are given with the following equation

$$\frac{dT}{dt} = -k\left(T - T_a\right)$$

In the equation T indicates the temperature of the substance in (^{o}C), T_a is environmental temperature (^{o}C) ,k is a constant (minute^{-1}) dir. Therefore cooling rate is function of temperature difference between body and environment. If a metal ball initially at 90 ^{o}C drop into a water bath at T_a=20 ^{o}C and if k=0.1 minute^{-1}, Find the time period required ball to reach 30 ^{o}C temperature
a) By analytical methods
b) By a numerical method

PROBLEM 17
Solution of diferential equation

$\frac{dy}{dx} = yx^2 - y$ are investigated with range x=0 ve x=2. Initial condition is given as y(0)=1

a) Solve it anaytically Plot the solution.
b) Solve it with Heun method with step size h=0.5 adim
c) Plot the numerical results also and compare it with analytical results

PROBLEM 18
In a chemical reaction a chemical A and a chemical B reacts to give chemical C. Concentration of chemical C as a function of time, y(t) is given with the following equation

$y' = k(a - y)(b - y)$ $\quad y(0)=0$

In this equation k is a positive constant, a and b is the initial concentration of chemicals A and B.
Assume that
k=0.01 , a= 70 milimole/liter , b= 50 milimole/liter .
By using 4th order Runge-Kutta method with step size h=0.5 estimate concentration at t=1 second.

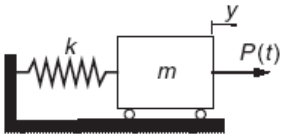
PROBLEM 19
Solve differential equation

$\frac{dy}{dx} = yx^2 - y$

With range of x=0 ve x=2. Initial value is given as y(0)=1 .
a) Solve the problem analytically
b) Solve it with Heun method, step size h=0.5

c) Plot both results and compare relative error.

PROBLEM 20



In the spring-mass system shown in the figure force P(t) is equal to

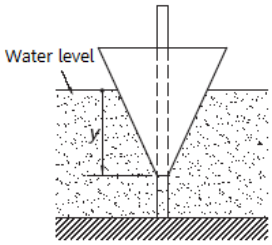
$$P(t)=\begin{cases}10t\ \ N & t<2s\\ 20\ N & t\geq 2s\end{cases}$$

Diferential equation of the movement

$\frac{d^2y}{dt^2}=\frac{P(t)}{m}-\frac{k}{m}y$ with m=2.5 kg, k=75 N/m. Determine amplitude (maximum amount of movement) of the movement.

PROBLEM 21 A cone buoy can slide freely on a bar is floating on the surface. If cone is disturbed by any effect, it will oscilate. The diferential equation of the oscilation is given as:

$\frac{d^2y}{dt^2}=g(1-ay^3)$ If a=16 m^{-3}, g=9.80665 m/s^2 and cone position initially moved upward to y=0.1 m and released, find the period(s) and amplitude(m) of the oscilation.



PROBLEM 23

Gun problem with air friction was given in the examples. Solve the same problem with 6th order Runge-Kutta method:

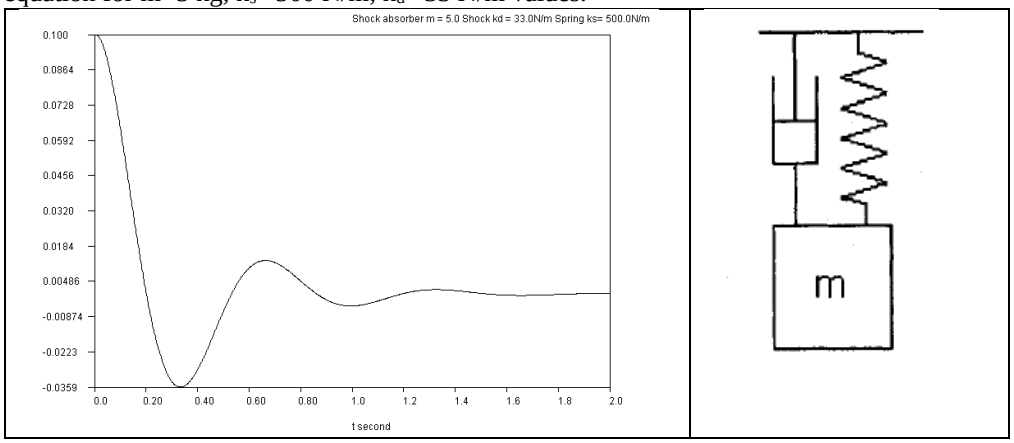
Nozle exit velocity 250 m/s, gravitational acceleration 9.806 m/s^2, friction coefficient γ = (k / m)= 0.01. A target at 750 m distance will be hit. Determine the angle θ.

PROBLEM 24

In cars and other machines for vibration control mechanical system with a spring and a shock absorber is used. The differential equation of such a system is given as:

$$m\frac{d^2x}{dt^2}+k_d\frac{dx}{dt}+k_sx=0$$

In this equation m is mass, k_d shock absorber coefficient, k_s is spring coefficient. Solve the diferential equation for m=5 kg, k_s =500 N/m, k_d =33 N/m values.



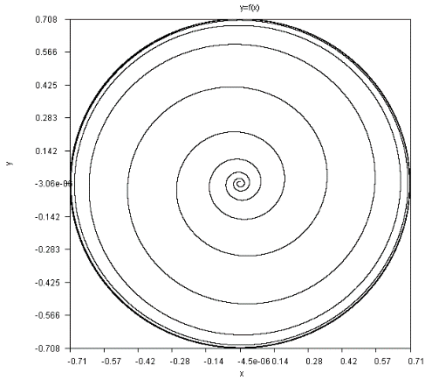
PROBLEM 25

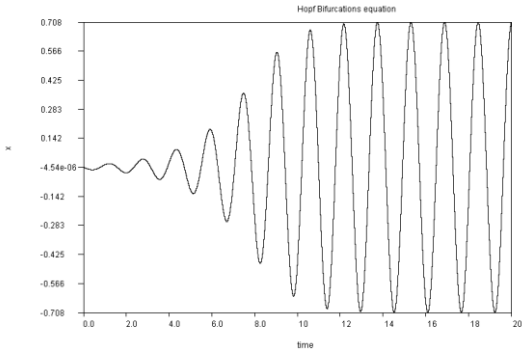
Hopf Bifurcations equation is given by

$$\frac{dx}{dt} = -(g + x^2 + y^2)x - wy$$

$$\frac{dy}{dt} = -(g + x^2 + y^2)y + wx$$

Solve the set of differential equations with g=-0.5,w=4, with the initial conditions x=0,y=0.01 at t=0. T_{final}=20 s





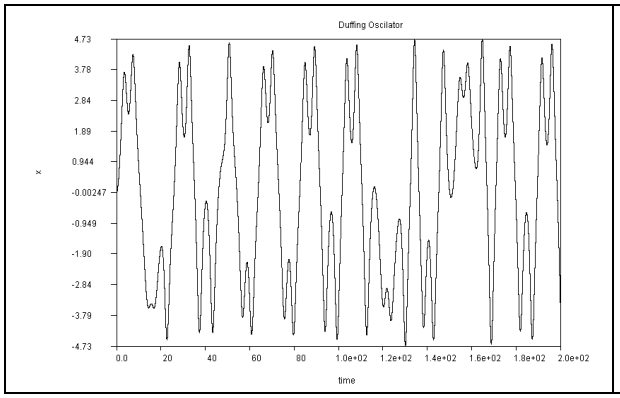
PROBLEM 26

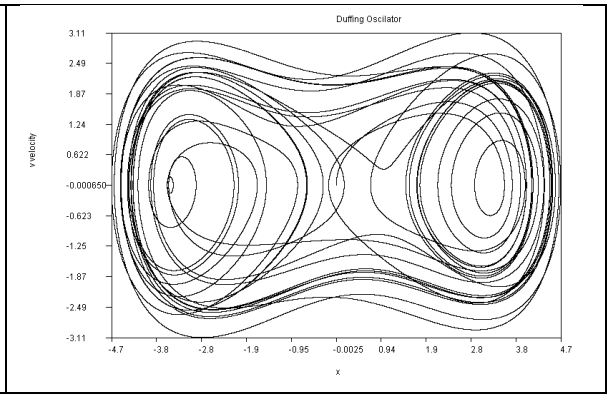
The Duffing Oscilator differential equations are given as

$$\frac{dx}{dt} = v$$

$$\frac{d^2x}{dt^2} = \frac{dv}{dx} = -0.2v + x - 0.1x^3 + \cos(t)$$

Solve the differential equation set with the initial conditions of x=v=0 at t=0 for 0 to t=200 s period.



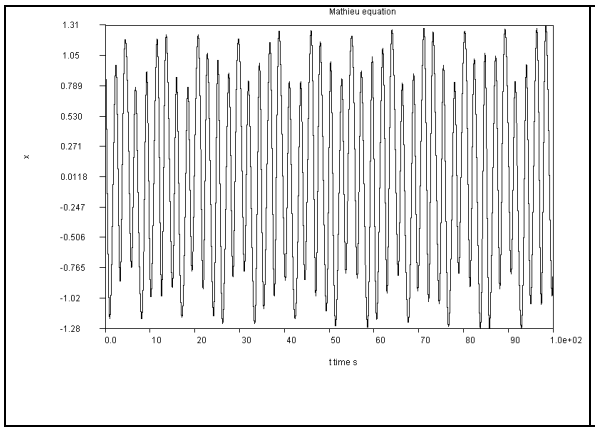


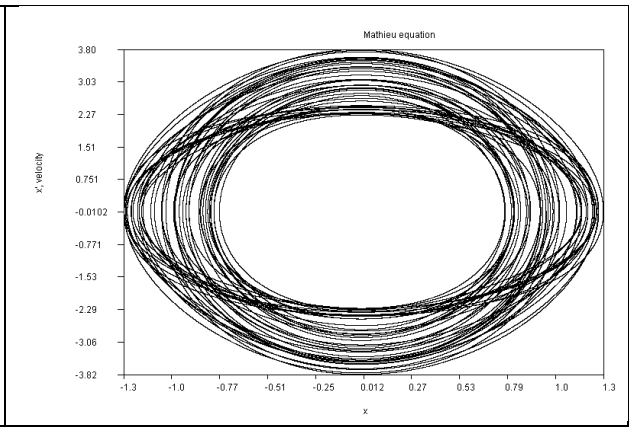
PROBLEM 27

The Mathieu equation is given as

$$\frac{d^2x}{dt^2} + (\delta + 2\varepsilon\cos(2t))x = 0$$

Solve the differential equation with initial conditions δ=8, ε=2.5, x=0,x'=0 for the time period of 0<=t<=100 s.





PROBLEM 28

H'enon-Heiles equation is given as

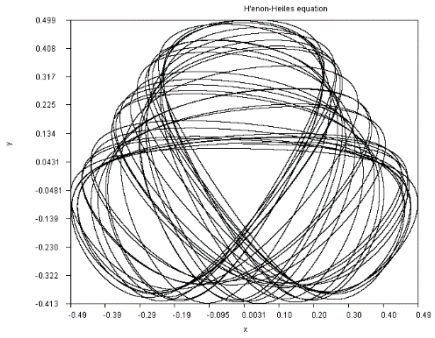
$$\frac{dx}{dt} = P_x$$

$$\frac{dy}{dt} = P_y$$

$$\frac{dP_x}{dt} = -x - 2xy$$

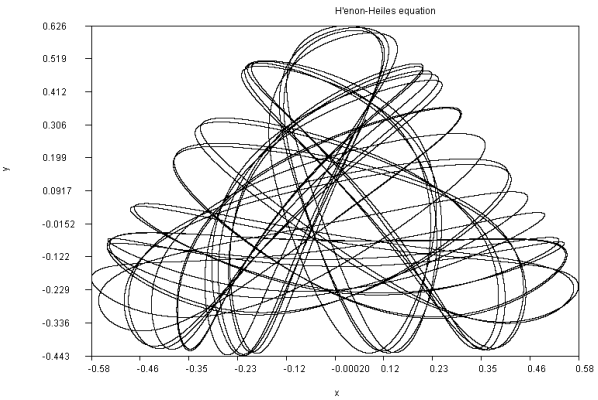
$$\frac{dP_y}{dt} = -y - x^2 + y^2$$

Solve the differential equation with initial conditions x=0,y=0.1 and P_x=0.49057, P_y=0 for the time period of 0<=t<=200 s.



PROBLEM 29

H'enon-Heiles equation is described in Problem 28. Solve the equation with the initial conditions of x=0,y=0 and P_x= 0.35355, P_y= 0.35355 for the time period of 0<=t<=200 s.



PROBLEM 30

Intermittency in Josephson Junctions: The study of the chaotic behavior of (quantum) Josephson junctions is of much fundamental and even practical interest. Written in dimensionless form, the differential equation for the quantum phase difference, φ, across the junction is given by

$$\frac{d^2\Phi}{dt^2} + \frac{1}{\sqrt{\beta_c}}\frac{d\Phi}{dt} + \sin\Phi = A\sin\Omega t$$

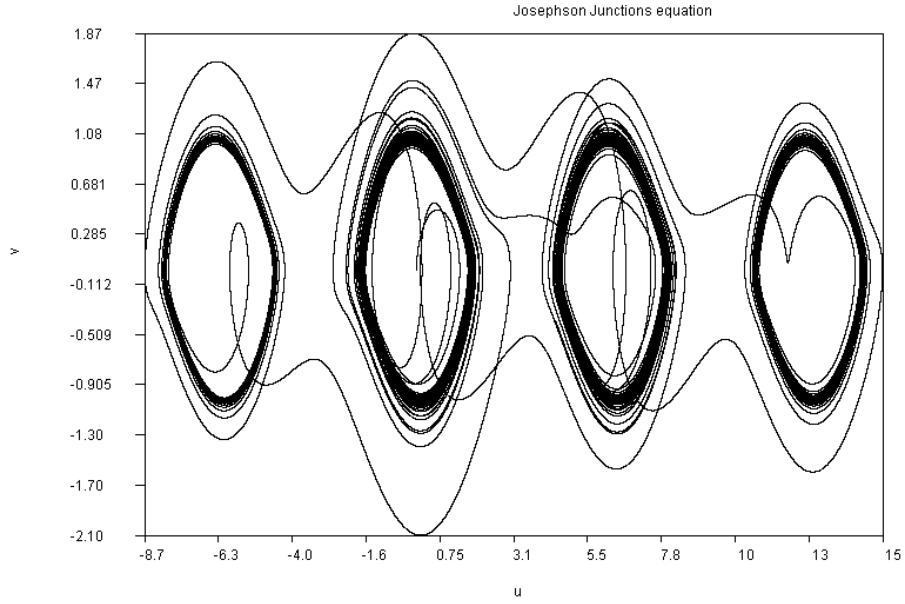
where β_c is the so-called McCumber parameter and Ω is the (normalized)

angular frequency of the driving current. $u = {}_{v=\Phi}$ and $v = \frac{d\Phi}{dt}$, this can be written as

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} = \mathrm{Asin}(\Omega t) - \mathrm{r}v - \sin(u)$$

For differential equation solution parametric values will be taken as A=0.9045, r=0.5, Ω=0.47. Solve the differential equation for the initial conditions u=v=0 for 0<=t<=1200 s



PROBLEM 31

Planar motion of a three body celestial system can be expressed by the following equations:

$$\frac{dx}{dt} = p + y$$

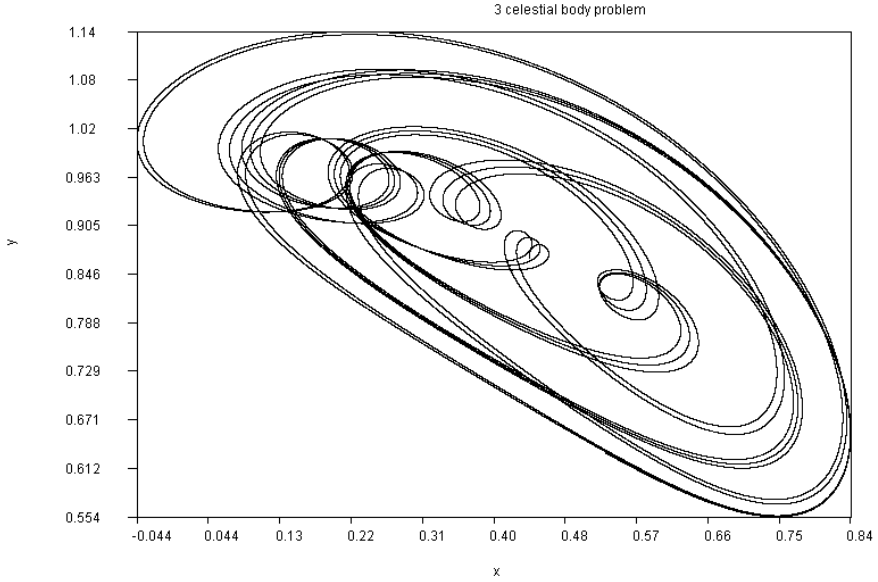
$$\frac{dy}{dt} = q - x$$

$$\frac{dp}{dt} = q - \frac{m(x+n)}{\left[(x+n)^2 + y^2\right]^{3/2}} - \frac{n(x-m)}{\left[(x+n)^2 + y^2\right]^{3/2}}$$

$$\frac{dq}{dt} = -p - \frac{my}{\left[(x+n)^2 + y^2\right]^{3/2}} - \frac{ny}{\left[(x+n)^2 + y^2\right]^{3/2}}$$

where the p and q are the canonical momenta of the coordinates x and y,

respectively. There are five equilibrium positions, called Lagrangian positions. One of the Lagrangian position is , L4, located at $(x_{average}, y_{average}) = (0.48, 0.866)$. In order to solve the diferential equation of the motion initial starting point will be taken as: (x,y,p,q)={0.44,0.866,-0.866,0.48} as initial conditions. Solve the differential equations for 0<t<200. Take m=0.98 n=0.02



PROBLEM 32 For Falkner–Skan solution transformation for fluid laminar free convection, we get the following variables for describing expressions for the stream function ψ and dimensionless coordinate variable η:

$$\eta = \left(\frac{1}{4}Gr\right)^{1/4}\frac{y}{x} \qquad \psi = 4\nu\left(\frac{1}{4}Gr\right)^{1/4} f(\eta) \quad \textit{where}$$

$$Gr = \frac{g(\rho_\infty / \rho_w - 1)x^3}{\nu^2}$$

Velocity profile for the free convection boundary layer:

$$w_x = \frac{\partial\psi}{dy} = \frac{4\nu}{x}\left(\frac{1}{4}Gr\right)^{1/2} f'(\eta) \quad w_y = -\frac{\partial\psi}{dx} = \frac{\nu}{x}\left(\frac{1}{4}Gr\right)^{1/4}\left[\eta f'(\eta) - 3f(\eta)\right]$$

Where x is vertical dimension, y is horizontal dimension, Gr is Grashoff number, w_x and w_y are x and y are dimension velocities. ψ is the stream function.

$\theta(\eta) = \dfrac{t - t_\infty}{t_w - t_\infty}$ is the dimensionless temperature**.** Under these condition falkner-skan solution is converted to the following differential equation set:

$$f'''(\eta) + 3f(\eta)f''(\eta) - 2(f'(\eta))^2 + \theta(\eta) = 0$$

$$\theta''(\eta) + 3\,\mathrm{Pr}\, f(\eta)\theta'(\eta) = 0$$

Where Pr is the Prandtl number. *The boundary conditions:*

$$\eta = 0: \quad f(\eta) = f'(\eta) = 0 \quad \theta(\eta) = 1$$

$$\eta \to \infty: \quad f'(\eta) = 0 \qquad \theta(\eta) = 0$$

Solve the differential equation set.

PROBLEM 33 A compound pendulum consisting of a shaft with a center of mass at point G is shown in the

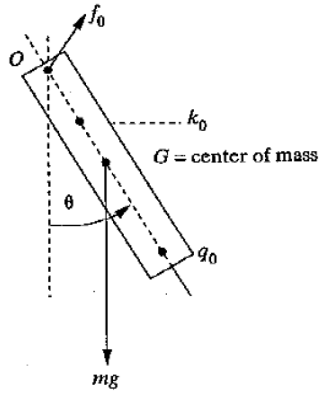


figure. For such a system diferential equation can be given as $J\frac{d^2\theta}{dt^2}+mgr\sin(\theta)=0$. In this equation J is mass momentum of inertia and for the rod given $J=\frac{1}{3}ml^2$ and r is the center of gravity so $r=\frac{l}{2}$ Differential equation becomes $\frac{d^2\theta}{dt^2}+\frac{3mg}{2l}\sin(\theta)=0$. Note that for small values of initial θ value $\sin(\theta)$ approximates to θ so differential equation can be solved analytically. Approximate analytical solution is: $\theta(t)=\theta_0\cos(\left[\sqrt{\frac{3mg}{2l}}\right]t)$ Solve differential equation numerically and compare the results for big and small initial value of θ

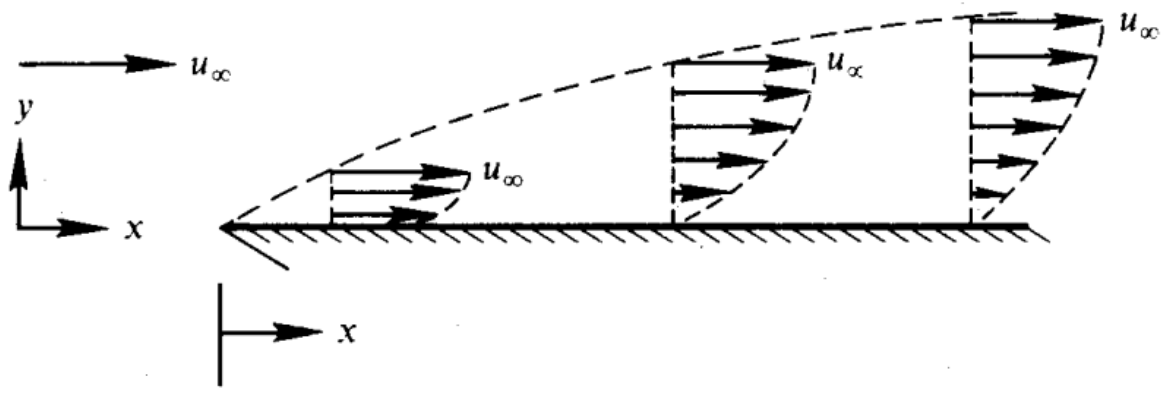
PROBLEM 34 : Solve boundary value equation :

$\frac{d^2y}{dt^2}=-\left(\frac{dy}{dt}\right)^2-y+\ln(x)$ x=1 y=0 and x=2 y=ln(2)

9 Partial differential equation solution methods: similarity theory solutions

EXAMPLE 1: BLASSIUS FLAT PLATE PROBLEM AND WEDGE FLOW PROBLEM SOLUTIONS

Blassius flat plate problem is one of the basic problem that shapes heat convection as we know today. Problem is given in all heat transfer text book, but details of the problem is mostly avoided. We would like to investigate problem here including necasary numeric methods to solve it.



Conservation of mass:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Conservation of momentum

$$\upsilon \frac{\partial^2 u}{\partial y^2} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

The boundary conditions are
u=0 v0 at y=0

$u \to u_\infty$ at $y \to \infty$

$u = u_\infty$ at x=0

The shape of velocity profile suggest that it might be similar , different only with a stretching factor on the y coordinate so Let us assume a velocity profile such as

$u = g(y.w(x))$ where

$\eta = y.w(x)$ so

$u = g(\eta)$

$$\frac{\partial u}{\partial x} = \frac{\partial g(\eta)}{\partial x} = \frac{\partial g(\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} = g'(\eta)yw'(x)$$

$$\frac{\partial u}{\partial y} = \frac{\partial g(\eta)}{\partial y} = \frac{\partial g(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} = g'(\eta)w(x)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 g(\eta)}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g(\eta)}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial g(\eta)}{\partial y}\right)\frac{\partial \eta}{\partial y} = f''(\eta)w^2(x)$$

Substituting these into the momentum equation

$$vg''(\eta)w^2(x) = g(\eta)g'(\eta)yw'(x) + vg'(\eta)w(x)$$

Conservation of mass

$$g'(\eta)yw'(x) + \frac{\partial v}{\partial y} = 0$$

Now, by substituting these equations into each other and solving them:

$$v = \frac{vg''(\eta)w^2(x) - g(\eta)g'(\eta)yw'(x)}{g'(\eta)w(x)}$$

$$\frac{\partial v}{\partial y} = -g'(\eta)yw'(x)$$

$$\frac{\partial}{\partial y}\left(\frac{vg''(\eta)w^2(x) - g(\eta)g'(\eta)yw'(x)}{g'(\eta)w(x)}\right) = -g'(\eta)yw'(x)$$

$$vw(x)\frac{\partial}{\partial y}\left(\frac{g''(\eta)}{g'(\eta)}\right) - \frac{w'(x)}{w(x)}\frac{\partial}{\partial y}\left(g(\eta)y\right) = -g'(\eta)yw'(x)$$

$$vw(x)\frac{\partial}{\partial y}\left(\frac{g''(\eta)}{g'(\eta)}\right) - \frac{w'(x)}{w(x)}\left(\frac{\partial g(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}y + g(\eta)\right) = -g'(\eta)yw'(x)$$

$$vw(x)\frac{\partial}{\partial y}\left(\frac{g''(\eta)}{g'(\eta)}\right) - \frac{w'(x)}{w(x)}\left(g'(\eta)w(x)y + g(\eta)\right) = -g'(\eta)yw'(x)$$

$$vw(x)\frac{\partial}{\partial y}\left(\frac{g''(\eta)}{g'(\eta)}\right) - \frac{w'(x)g'(\eta)w(x)y}{w(x)} - \frac{w'(x)}{w(x)}g(\eta) + g'(\eta)yw'(x) = 0$$

$$vw(x)\frac{\partial}{\partial y}\left(\frac{g''(\eta)}{g'(\eta)}\right) - \frac{w'(x)}{w(x)}g(\eta) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{g''(\eta)}{g'(\eta)}\right)=\frac{\partial}{\partial \eta}\left(\frac{g''(\eta)}{g'(\eta)}\right)\frac{\partial \eta}{\partial y}=\frac{\partial}{\partial \eta}\left(\frac{g''(\eta)}{g'(\eta)}\right)w(x)$$

$$\nu w^2(x)\frac{\partial}{\partial \eta}\left(\frac{g''(\eta)}{g'(\eta)}\right)-\frac{w'(x)}{w(x)}g(\eta)=0$$

$$\frac{1}{g(\eta)}\frac{\partial}{\partial \eta}\left(\frac{g''(\eta)}{g'(\eta)}\right)=\frac{w'(x)}{\nu w^3(x)}$$

In this form, the differential equation is separated so it can be solved as two different set of equation with the common arbitrary constant.

$$\frac{1}{f(\eta)}\frac{\partial}{\partial \eta}\left(\frac{f''(\eta)}{f'(\eta)}\right)=\frac{g'(x)}{\nu g^3(x)}=-k$$

$$\frac{dw(x)}{w^3(x)}=-k\nu dx$$

$$-\frac{1}{2w(x)}=-k\nu x+C$$

At $y\to\infty\ u\to u_\infty$; but x=0 a boundary condition is that $u=u_\infty$ even at y=0. Thus g(0) must be infinite. It thus follows that C=0. Then

$w(x)=\frac{1}{\sqrt{2k\nu x}}\qquad \eta=yw(x)=\frac{y}{\sqrt{2k\nu x}}$ So from this result, it is clear that velocity profile

$$u=f\left(\frac{y}{\sqrt{x}}\right)$$

The left hand side of the equation:

$$\frac{1}{g(\eta)}\frac{d}{d\eta}\left(\frac{g''(\eta)}{g'(\eta)}\right)=-k$$

$$d\left(\frac{g''(\eta)}{g'(\eta)}\right)=-kg(\eta)d\eta$$

$$\frac{g''(\eta)}{g'(\eta)}=-k\int g(\eta)d\eta+C$$

Boundary conditions $\eta=0$ u=0 v=0 g=0 at y=0

$\frac{\partial^2 u}{\partial y^2}=0=\frac{\partial^2 g}{\partial \eta^2}=g''$ and thus C=0

$$\frac{g''(\eta)}{g'(\eta)}=-k\int g(\eta)d\eta$$

Let us now define a non-dimensional velocity in terms of a derivative of a function of η , derivative being for the purpose of eliminating the integral of the above equation. Let

$f'(\eta)=\frac{u}{u_\infty}=\frac{g(\eta)}{u_\infty}$ then

$$g(\eta)=u_\infty f'(\eta)$$

$$g'(\eta)=u_\infty f''(\eta)$$

$$g''(\eta) = u_\infty f'''(\eta)$$

And

$$\frac{g''(\eta)}{g'(\eta)} = \frac{f'''(\eta)}{f''(\eta)}$$ Substituting,

$$\frac{f'''(\eta)}{f''(\eta)} = -k\int u_\infty \frac{df(\eta)}{d\eta} d\eta = -ku_\infty f(\eta)$$

k is an arbitrary constant but ku_∞ must be nondimensional, let $ku_\infty = \frac{1}{2}$ Thus,

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0$$ where $\eta = \frac{y}{\sqrt{\nu x / u_\infty}}$ and $f'(\eta) = \frac{u}{u_\infty}$

Boundary conditions:

u=0 at y=0

v=0 at y=0

$u \to u_\infty$ at $y \to \infty$

Then $f'(0) = 0 \quad f'(\infty) = 1$

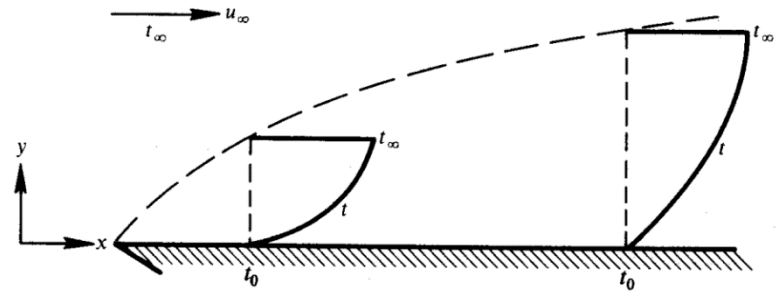
Since $f''(0) = 0 \quad f'''(0) = 0$ (provided that v(0)=0)

Then from the differential equation . $f(0) f''(0) = 0$ It is not possible that all the derivatives are zero, in fact $f''(0) = 0$ would correspond to zero shear stress at the wall, so we conclude that $f(0) = 0$

The same set of equations can also be developed by using stream function $\psi(x, y)$. Stream function is basically a single function to replace velociyies u and v. It is defined as:

$u = \frac{\partial \psi}{dy} \quad v = -\frac{\partial \psi}{dx}$ Due to the definition, conservation of mass is always satisfied. The relation between ψ and f is as follows: $\psi = \sqrt{\nu x u_\infty} f(\eta)$

Now that we have a single variable differential similarity equation for the momentum equation, a similarity solution for energy equation should also be developed.



For $\theta = \frac{t_0 - t}{t_0 - t_\infty}$ The energy equation for the boundary layer:

$$\alpha \frac{\partial^2 \theta}{\partial y^2} = u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y}$$

With the boundary conditions:

$\theta = 0$ at y=0

$\theta = 1$ at $y \to \infty$

$\theta = 1$ at x=0

Assume $\theta = \theta(\eta)$ where $\eta = \frac{y}{\sqrt{\nu x / u_\infty}}$

Energy equation becomes:

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{\Pr}{2} f \frac{\partial \theta}{\partial \eta} = 0$$

$$\theta(0) = 0$$

$$\theta(\infty) = 1$$

Equation can be solved directly as a set of differential equations. Equation set change as follows:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0$$

$$\theta'' + \frac{\Pr}{2} f\theta' = 0$$

Conversion of the equation:

$$x[0] = \eta$$

$$x[1] = f$$

$$x[2] = f'$$

$$x[3] = f''$$

$$x[4] = \theta$$

$x[5] = \theta'$ Differential equation set becomes:

$$\frac{dx[1]}{dx[0]} = x[2]$$

$$\frac{dx[2]}{dx[0]} = x[3]$$

$$\frac{dx[3]}{dx[0]} = -0.5 * x[1] * x[3]$$

$$\frac{dx[4]}{dx[0]} = x[5]$$

$$\frac{dx[4]}{dx[0]} = -\frac{\Pr}{2} x[1] * x[5]$$

Boundary conditions:

x[4]=0 at x[0]=0

x[4]=1 at x[0]= ∞

x[2]=0 at x[0]=0

x[2]=1 at x[0]= ∞

x[1]=0 at x[0]= 0

As it is seen from boundary conditions, two given conditions are not initial conditions. One way of solving boundary conditions is using non-linear system of equation solving methods. In order to solve differential equation set 6th order Runge-Kutta Method is used. In order to solve boundary value nonlinear system of equation Nelder & Mead optimization method is used.

Runge-Kutta method as in most general form can be given as: For differential form $\frac{dy}{dx} = f(x, y)$

$y_{n+1} = y_n + h*\phi(x_i,y_i,h)$

$$\phi(x_1, y_1, h) = \sum_{i=1}^{n} b_i k_i$$

In this equation a is constant values depends on polynamial degree of Runge-Kutta equation and coefficients k, can be defined as:

$k_1=f(x_i,y_i)$

$k_2=f(x_i+c_2h,y_i+a_{21}k_1h)$

$k_3=f(x_i+c_3h,y_i+a_{31}k_1h+ q_{32}k_2h)$

….

$k_n=f(x_i+c_nh,y_i+a_{n,1}k_1h+ \ldots+ a_{n,n-1}k_{n-1}h)$

In more generalized form, these equations can be written as:

$$x_i = x_n + c_i h \quad \text{i=1…s}$$

$$y_i = y_n + h\sum_{j=1}^{i-1} a_{ij} k_j$$

$$k_i = f(x_i, y_i)$$

$y_{n+1} = y_n + h\sum_{j=1}^{s} b_i k_i$ where s is the degree of coefficients of equations. In numerical analysis books, coefficients are usually shown as Buthcher tableu which puts the coefficients of the method in a table as follows:

$$\begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & a_{13} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \ldots & a_{2s} \\ c_3 & a_{31} & a_{32} & a_{33} & \ldots & a_{3s} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ c_s & a_{s1} & a_{s2} & a_{s3} & \ldots & a_{s4} \\ \hline & b_1 & b_2 & b_3 & \ldots & b_s \end{array}$$

As it is seen from the equation the definition is very general. Depends on polynamial degree of Runge-Kutta equation accuracy will improve. **Runge-Kutta with sixth degree polynomial solution RK6** is given as:

$y_{i+1} = y_i + (1/90)*(7k_1 + 32k_3 +12k_4+32k_5+7k_6)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+0.25h , y_i+0.25k_1h)$

$k_3=f(x_i+0.25h , y_i+0.125k_1h+0.125k_2h)$

$k_4=f(x_i+0.5h , y_i – 0.5k_2h+k_3h)$

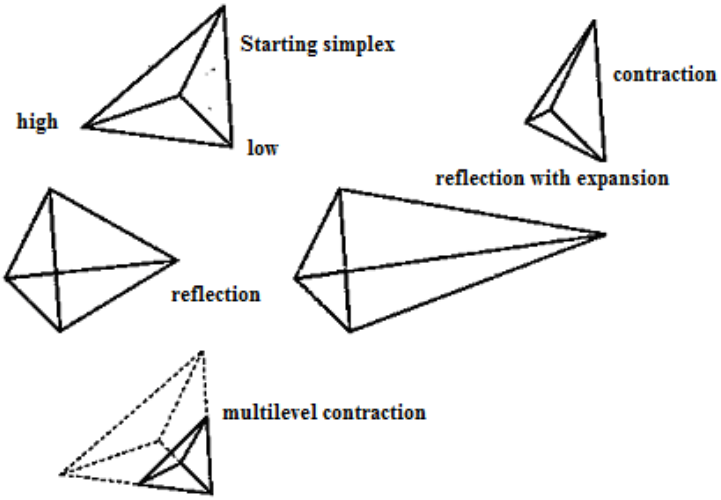
$k_5=f(x_i+0.75h , y_i + (3/16)k_1h+(9/16)k_4h)$

$k_6=f(x_i+h , y_i - (3/7)k_1h+(2/7)k_2h+(12/7)k_3h - (12/7)k_4h+(8/7)k_5h)$

This equation can be given as Buthcher tableu as:

0	0	0	0	0	0
1/4	1/4	0	0	0	0
1/4	1/8	1/8	0	0	0
2/4	1/4	−1/4	1	0	0
3/4	3/16	0	0	9/16	0
1	−3/7	2/7	12/7	−12/7	8/7
	1/90	7/90	32/90	12/90	7/90

Nelder & Mead optimization method is as follows:
The simplex method developed by Nelder and Mead method is a multidimensional search method . Simplex is an n dimensional geometric entity. It contains (n+1) points in n dimensional space. The method uses the concept of a simplex, which is a special polytop of $N + 1$ vertices in N dimensions. Examples of simplices include a line segment on a line, a triangle on a plane, a tetrahedron in three-dimensional space and so forth. Required processes for the simplex amobea's movements are shown in the figure



Process can be given as follows

- **1. Order** according to the values at the vertices:
  $f(\mathbf{x}_1) \leq f(\mathbf{x}_2) \leq \cdots \leq f(\mathbf{x}_{n+1})$

- **2.** Calculate x_o, the center of gravity of all points except x_{n+1}.

- **3. Reflection**
  Compute reflected point $\mathbf{x}_r = \mathbf{x}_o + \alpha(\mathbf{x}_o - \mathbf{x}_{n+1})$
  If the reflected point is better than the second worst, but not better than the best, i.e.:
  $f(\mathbf{x}_1) \leq f(\mathbf{x}_r) < f(\mathbf{x}_n)$,
  then obtain a new simplex by replacing the worst point x_{n+1} with the reflected point x_r, and go to step 1.

- **4. Expansion**
  If the reflected point is the best point so far, $f(\mathbf{x}_r) < f(\mathbf{x}_1)$,
  then compute the expanded point $\mathbf{x}_e = \mathbf{x}_o + \gamma(\mathbf{x}_o - \mathbf{x}_{n+1})$
  If the expanded point is better than the reflected point, $f(\mathbf{x}_e) < f(\mathbf{x}_r)$

then obtain a new simplex by replacing the worst point x_{n+1} with the expanded point x_e, and go to step 1.
Else obtain a new simplex by replacing the worst point x_{n+1} with the reflected point x_r, and go to step 1.
Else (i.e. reflected point is worse than second worst) continue at step 5.

- **5. Contraction**

  Here, it is certain that $f(\mathbf{x}_r) \geq f(\mathbf{x}_n)$
  Compute contracted point $\mathbf{x}_c = \mathbf{x}_{n+1} + \rho(\mathbf{x}_o - \mathbf{x}_{n+1})$
  If the contracted point is better than the worst point, i.e. $f(\mathbf{x}_c) \leq f(\mathbf{x}_{n+1})$
  then obtain a new simplex by replacing the worst point x_{n+1} with the contracted point x_c, and go to step 1.
  Else go to step 6.

- **6. Reduction**

  For all but the best point, replace the point with
  $x_i = x_1 + \sigma(x_i - x_1)$ for all i $\in \{2, \ldots, n+1\}$. go to step 1.

Note: α, γ, $\rho$and σ are respectively the reflection, the expansion, the contraction and the shrink coefficient. Standard values are $\alpha = 1$, $\gamma = 2$, $\rho = 1 / 2$ and $\sigma = 1 / 2$.

For the **reflection**, since x_{n+1} is the vertex with the higher associated value among the vertices, we can expect to find a lower value at the reflection of x_{n+1} in the opposite face formed by all vertices point x_i except x_{n+1}.

For the **expansion**, if the reflection point x_r is the new minimum along the vertices we can expect to find interesting values along the direction from x_o to x_r.

Concerning the **contraction**: If $f(x_r) > f(x_n)$ we can expect that a better value will be inside the simplex formed by all the vertices x_i.

The initial simplex is important, indeed, a too small initial simplex can lead to a local search, consequently the NM can get more easily stuck. So this simplex should depend on the nature of the problem. Instead of given n+1 point same process can be done by giving one point and a change vector by defining
$x_{i-1} = P_i + \lambda dx_j$ where if i=j λ=1 else if $i \neq j$ λ=0 . For example:

if $P = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$ and $dx = \begin{Bmatrix} da \\ db \\ dc \end{Bmatrix}$

$$x_0 = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \quad x_1 = \begin{Bmatrix} a+da \\ b \\ c \end{Bmatrix} \; x_2 = \begin{Bmatrix} a \\ b+db \\ c \end{Bmatrix} \; x_3 = \begin{Bmatrix} a \\ b \\ c+dc \end{Bmatrix}$$

Non-Linear system of equations can also be solved by using optimization methods through an adaptation function.
To solve function $f_i(x_j)=0$

$$g(x_j) = \sum_{i=0}^{n\_equation} f_i(x_j) * f_i(x_j)$$

The derivative of this equation is the root of non-linear system of equation again became $f_i(x_j)=0$, therefore optimum of $g(x_j)$ is the same problem as solution of $f_i(x_j)=0$.
Another program utilized in this set is generalized **least square linear curve fitting**. Assuming having a linear function $f(a_j,x)$ where a_j are m linear coefficient and x are independent variable and $\phi_j(x)$ are m sub-funtions linearly multiplied with the coefficients

$$f(a_j,x)=\sum_{j=0}^{m} a_j^{(m)}\phi_j(x)$$

It should be noted that linearity is only for the coefficients, $\phi_j(x)$ functions does not have to be linear. It is desired to fit data x_i, f_i, i=0..n into the equation so that the difference between data and fitted function dependent values for all points will be minimum. In order to establish this, the following function H will be minimized with respect to a_j

$$H(a_0^{(m)},....,a_m^{(m)})=\sum_{i=1}^{n} w(x_i)\left[\, f_i \,-\sum_{j=0}^{m} a_j^{(m)}\phi_j(x_i)\,\right]^2$$

Where $w(x_i)$ values in the equation are called weight function. In order to minimiz the function root of the derivative of the function can be calculated.

$$\frac{\partial H(a_0^{(m)},....,a_m^{(m)})}{\partial a_k^{(m)}}=2\sum_{i=1}^{n} w(x_i)\left[\, f_i \,-\sum_{j=0}^{m} a_j^{(m)}\phi_j(x_i)\,\right]\phi_k(x_i)=0 \quad k=0,.....,m$$

$$\left\{\sum_{j=0}^{m} w(x_i)\phi_j(x_i)\phi_k(x_i)\right\}\left[a_j^{(m)}\right]=\left[\sum_{i=1}^{n} w(x_i)\phi_k(x_i)f_i\right] \quad k=0,.....,m$$

For weight function to be taken equal to unity , $w(x_i)$=1, equation in the open form can be written in the following form.

$$\begin{Bmatrix} \sum_{i=1}^{n}\phi_0^2(x_i) & \sum_{i=1}^{n}\phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n}\phi_0(x_i)\phi_2(x_i) & .... & \sum_{i=1}^{n}\phi_0(x_i)\phi_m(x_i) \\ \sum_{i=1}^{n}\phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n}\phi_1^2(x_i) & \sum_{i=1}^{n}\phi_1(x_i)\phi_2(x_i) & .... & \sum_{i=1}^{n}\phi_1(x_i)\phi_m(x_i) \\ \sum_{i=1}^{n}\phi_0(x_i)\phi_2(x_i) & \sum_{i=1}^{n}\phi_1(x_i)\phi_2(x_i) & \sum_{i=1}^{n}\phi_2^2(x_i) & .... & \sum_{i=1}^{n}\phi_2(x_i)\phi_m(x_i) \\ .... & .... & .... & .... & .... \\ \sum_{i=1}^{n}\phi_0(x_i)\phi_m(x_i) & \sum_{i=1}^{n}\phi_1(x_i)\phi_m(x_i) & \sum_{i=1}^{n}\phi_2(x_i)\phi_m(x_i) & .... & \sum_{i=1}^{n}\phi_m^2(x_i) \end{Bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ .... \\ a_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\phi_0(x_i)f(x_i) \\ \sum_{i=1}^{n}\phi_1(x_i)f(x_i) \\ \sum_{i=1}^{n}\phi_2(x_i)f(x_i) \\ .... \\ \sum_{i=1}^{n}\phi_m(x_i)f(x_i) \end{bmatrix}$$

This equation is an m+1 linear system of equation. It can easily be solved by using a system of equation solving method.
Now that basic mathematical calculation methods are established, The equation can be solved by using differential equation boundary value solving method.

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;
class fb extends f_xr
{
            public double func(double x,int i)
            {
   double xx=1.0;
    if(i==0) xx=1.0;
    else if(i==1) xx=x;
     else if(i==2) xx=x*x;
      else if(i==3) xx=x*x*x;
       else if(i==4) xx=x*x*x*x;
        else if(i==5) xx=x*x*x*x*x;
         else if(i==6) xx=x*x*x*x*x*x;
          else if(i==7) xx=x*x*x*x*x*x*x;
           else if(i==8) xx=x*x*x*x*x*x*x*x;
```

```
            else if(i==9) xx=x*x*x*x*x*x*x*x*x;
             else if(i==10) xx=x*x*x*x*x*x*x*x*x*x;
  return xx;
              }
}
class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
double y[];
fm2 b3;
double w;
double a[][];
public fa(double Pr,double yi[],double wi)
{ b3=new fm2(Pr);
  y=new double[yi.length];
  for(int i=0;i<y.length;i++) y[i]=yi[i];
  w=wi;
}

public double[] func(double x[])
{
// shooting method
y[2]=x[0];
y[4]=x[1];
a=RK6(b3,0.0,w,y,10000);
double ff[]=new double[2];
ff[0]=a[2][10000]-1.0;
ff[1]=a[4][10000]-1.0;
return ff;
}
 //differenetial equation solution
public static double[][] RK6(f_xi fp,double x0,double xn,double f0[],int N)
{
//6th order Runge Kutta Method
//fp  : given set of derivative functions dfj/dxi(fj,x)
// xo : initial value of the independent variable
// xn : final value of the independent variable
// f0 : initial value of the dependent variable
// N  : number of dependent variable to be calculated
// fi : dependent variable
double h=(xn-x0)/N;
int M=f0.length;
double fi[][];
fi=new double[M][N+1];
double xi[]=new double[M+1];
double k[]=new double[6];
int i,j;
double x;
for(j=0;j<M;j++)
  {
  fi[j][0]=f0[j];
  xi[j+1]=f0[j];
  }
for(x=x0,i=0;i<N;x+=h,i++)
  {
  for(j=1;j<=M;j++)
    {
    xi[0]=x;
    xi[j]=fi[j-1][i];
    k[0]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/2;
    k[1]=h*fp.func(xi,j-1);
```

```
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/4.0+k[1]/4.0;
    k[2]=h*fp.func(xi,j-1);
    xi[0]=x+h;
    xi[j]=fi[j-1][i]-k[1]+2.0*k[2];
    k[3]=h*fp.func(xi,j-1);
    xi[0]=x+2.0/3.0*h;
    xi[j]=fi[j-1][i]+7.0/27.0*k[0]+10.0/27.0*k[1]+1.0/27.0*k[3];
    k[4]=h*fp.func(xi,j-1);
    xi[0]=x+1.0/5.0*h;
    xi[j]=fi[j-1][i]+28.0/625.0*k[0]-1.0/5.0*k[1]+546.0/625.0*k[2]+54.0/625.0*k[3]-378/625.0*k[4];
    k[5]=h*fp.func(xi,j-1);
    fi[j-1][i+1]=fi[j-1][i]+k[0]/24.0+5.0*k[3]/48.0+27.0*k[4]/56.0+125.0*k[5]/336.0;
    xi[j]=fi[j-1][i];
    }
  }
double a[][]=new double[M+1][N+1];
for(x=x0,i=0;i<=N;x+=h,i++)
{
  a[0][i]=x;
  for(j=1;j<=M;j++)
  {
  a[j][i]=fi[j-1][i];
  }
}

return a;
}
}

//Blassius flat plate differential equation
class fm2 extends f_xi
{ double Pr;
  fm2(double Pri) {Pr=Pri;}
// multivariable function
 double func(double x[],int x_ref)
 {
   //Blasius flat plate diferential equation
   //f'''+0.5*ff'' = 0
   //f'''=-0.5*x[1]*x[3] f''=x[3] f'=x[2] f=x[1]]
   double a=0;
   if(x_ref==4) a= -0.5*Pr*x[1]*x[5];
   if(x_ref==3) a= x[5];
   if(x_ref==2) a= -0.5*x[1]*x[3];
   if(x_ref==1) a= x[3];
   if(x_ref==0) a= x[2];;
   return a;
 }
}

public class blassius_flat_plate1
{
                public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
public static String toString(double x[])
{ String s="";
int n=x.length;
  for(int k=0;k<n;k++)
  {s+=x[k]+" ";}
  return s;
}
public static double[] nelder(f_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
```

```
                int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
  {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
  }


  // Inlet variable definitions
  // fnelder : abstract multivariable function f(x)
  // x : independent variable set of n+1 simplex elements
  // maxiteration : maximum iteration number
  // tolerance :
      int NDIMS = x.length-1;
      int NPTS = x.length;
      int FUNC = NDIMS;
      int ncalls = 0;
        ////// construct the starting simplex //////////////////
        //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
        double z[]=new double[NDIMS];
        double best = 1E99;
        /////////////// calculate the first function values for the simplex ////////////////

        int iter=0;

        for (iter=1; iter<maxiteration; iter++)
        {
                    ///////////  define lo, nhi, hi (low high next_to_high //////////////
          int  ilo=0, ihi=0, inhi = -1; // -1 means missing
          double flo = p[0][FUNC];
          double fhi = flo;
          double pavg,sterr;
          for (i=1; i<NPTS; i++)
          {
            if (p[i][FUNC] < flo)
              {flo=p[i][FUNC]; ilo=i;}
            if (p[i][FUNC] > fhi)
              {fhi=p[i][FUNC]; ihi=i;}
          }
          double fnhi = flo;
          inhi = ilo;
          for (i=0; i<NPTS; i++)
            if ((i != ihi) && (p[i][FUNC] > fnhi))
              {fnhi=p[i][FUNC]; inhi=i;}
          ////////// exit criteria //////////////
          if ((iter % 4*NDIMS) == 0)
          {
            // calculate the avarage (including maximum value)
            pavg=0;
            for(i=0;i<NPTS;i++)
              pavg+=p[i][FUNC];
            pavg/=NPTS;
            double tot=0;
            if(printlist!=0)
            {  System.out.print(iter);
              for (j=0; j<=NDIMS; j++)
                { System.out.print(p[ilo][j]+" ");}
              System.out.println("");
            }
            for(i=0;i<NPTS;i++)
            { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
            sterr=Math.sqrt(tot/NPTS);
            //if(sterr < tolerance)
              { for (j=0; j<NDIMS; j++)
                  { z[j]=p[ilo][j];}
                //break;
              }
            best = p[ilo][FUNC];
          }

          ///// calculate avarage without maximum value //////
```

```
double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);


///////// reflect ////////////////

double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
   for (j=0; j<NDIMS; j++)
     p[ihi][j] = r[j];
   p[ihi][FUNC] = fr;
   continue;
}

if (fr < flo)  //// expand
{
   double e[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     e[j] = 3*ave[j] - 2*p[ihi][j];
   double fe = fnelder.func(e);
   if (fe < fr)
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = e[j];
      p[ihi][FUNC] = fe;
      continue;
   }
   else
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = r[j];
      p[ihi][FUNC] = fr;
      continue;
   }
}

////////////// shrink:

if (fr < fhi)
{
   double c[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
   double fc = fnelder.func(c);
   if (fc <= fr)
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = c[j];
      p[ihi][FUNC] = fc;
      continue;
   }
   else   /////// daralt
   {
      for (i=0; i<NPTS; i++)
        if (i != ilo)
        {
           for (j=0; j<NDIMS; j++)
             p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
           p[i][FUNC] = fnelder.func(p[i]);
        }
      continue;
   }
}
```

```
            if (fr >= fhi)   ///
            {
               double cc[] = new double[NDIMS];
               for (j=0; j<NDIMS; j++)
                 cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
               double fcc = fnelder.func(cc);
               if (fcc < fhi)
               {
                  for (j=0; j<NDIMS; j++)
                    p[ihi][j] = cc[j];
                  p[ihi][FUNC] = fcc;
                  continue;
               }
               else    /////////
               {
                  for (i=0; i<NPTS; i++)
                    if (i != ilo)
                    {
                       for (j=0; j<NDIMS; j++)
                         p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                       p[i][FUNC] = fnelder.func(p[i]);
                    }
               }
            }
         }
      return z;
}
     public static double[] nelder(f_xj fnelder,double a[],double da[],double tolerance)
     {return nelder(fnelder,a,da,500,tolerance,0);}

     public static double[] nelder(f_xj fnelder,double a[],double da[])
     {return nelder(fnelder,a,da,500,1.0e-10,0);}

     public static double[] nelder(f_xj fnelder,double a[])
     {
                        double [] da=new double[a.length];
            for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
            return nelder(fnelder,a,da);
     }
// Polynomial least square
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
```

```
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] PolynomialLSQ(double xi[],double yi[],int n)
{ //Polynomial least square
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  for(j=0;j<n+1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= yi[k];
                  else      B[i] += Math.pow(xi[k],i)*yi[k];}
  }
X=gausswithpartialpivot(A,B);
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
return X;
}

public static double funcPolynomialLSQ(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcPolynomialLSQ(e,x[k]))*(y[k]-funcPolynomialLSQ(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcPolynomialLSQ(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
                //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                 int n=xi.length;
```

```
            int nn=(n-1)*(aradegersayisi+1)+1;
            double z[][]=new double[2][nn];
   double E[]=PolynomialLSQ(xi,yi,polinomkatsayisi);
   System.out.println("katsayilar :\n"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
     {z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;
     for(int j=0;j<aradegersayisi;j++)
     {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
     z[0][k]=z[0][k-1]+dx;z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;}
     }
     z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);
   return z;
 }
//general least square curve fitting

public static double[] GeneralLeastSquare(double xi[],double yi[],int n)
{
fb f=new fb();
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  B[i]=0;
            for(j=0;j<np1;j++)
    {
            A[i][j]=0.0;
    for(k=0;k<l;k++) A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);
    }
    for(k=0;k<l;k++) B[i]+= f.func(xi[k],i)*yi[k];
  }
//System.out.println("A = \n"+Matrix.toString(A));
//System.out.println("B = \n"+Matrix.toStringT(B));
X=gausswithpartialpivot(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
Text.printT(X);
return X;
}

public static double funcGeneralLeastSquare(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
fb f=new fb();
int n=e.length;
double ff=0;
if(n!=0.0)
  {
  for(int i=n-1;i>=0;i--)
   {ff+=e[i]*f.func(x,i);}
  }
return ff;
}

public static double hata(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcGeneralLeastSquare(e,x[k]))*(y[k]-funcGeneralLeastSquare(e,x[k]));
```

```
 }
total=Math.sqrt(total);
return total;
}
 public static double[][]  funcGeneralLeastSquare(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {  // aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                 int n=xi.length;
                int nn=(n-1)*(aradegersayisi+1)+1;
                double z[][]=new double[2][nn];
    double E[]=GeneralLeastSquare(xi,yi,polinomkatsayisi);
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
       {z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;
       for(int j=0;j<aradegersayisi;j++)
       {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;}
       }
       z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);
    return z;
 }
public static void main(String arg[]) throws IOException
{
  double y[]={0,0,0.1,0,0.1};
 double x[]={0.33,0.0173};//inputdata(s);
 fa ff;
 f2 f;
 double r1[];
 double
Pr[]={0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,
0.9,1.0,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9,2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0,10.0,11,12,13,14,15.0,30.0,40.0,50,60,70,80,90,100,110.0,120.0,1
30.0,140,150,160,170,180,190,200,300,400,500,600,700,800,900,1000};
 double
w[]={200,200,200,200,200,200,200,200,200,100.0,100,100,100,100,100,100,100,100,50,50,50,50,50,50,50,50,25,25,25,25,25,25,25,2
5,25,10,10,10,10,10,10,10,10,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3};
 double xi[]=new double[Pr.length];
 double yi[]=new double[Pr.length];
 double Xi[]=new double[Pr.length];
 double Yi[]=new double[Pr.length];
 //Plot pp=new Plot();
 //Plot pp1=new Plot();
 for(int i=0;i<xi.length;i++)
 {ff=new fa(Pr[i],y,w[i]);
  f=new f2(ff);
  r1=nelder(f,x);
  xi[i]=Pr[i];
  yi[i]=r1[1];
  Xi[i]=Math.log(Pr[i]);
  Yi[i]=Math.log(r1[1]);
  System.out.println("i="+i+"Pr="+xi[i]+"w="+w[i]+"logPr="+Xi[i]+"Nux/Rex^0.5="+Yi[i]);
}
  double a[][]={Xi,Yi};
  Text.printT(a);
  double b[]=GeneralLeastSquare(Xi,Yi,6);
  double gPr1[]=new double[Pr.length];
  double egPr1[]=new double[Pr.length];
  for(int i=0;i<Pr.length;i++)

{gPr1[i]=b[0]*Math.pow(Pr[i],0.25)+b[1]*Math.pow(Pr[i],0.5)+b[2]*Math.pow(Pr[i],0.75)+b[3]*Pr[i]+b[4]*Math.pow(Pr[i],1.25)+b[5]
*Math.pow(Pr[i],1.5)+b[6]*Math.pow(Pr[i],1.75);
   egPr1[i]=(gPr1[i]-yi[i])/yi[i]*100;
  }

  Text.printT(b);
  double d[][]={xi,yi};
  Text.printT(d,"d");
  //pp.plot();
  //pp1.plot();
  Plot pp5=new Plot(xi,yi);
  pp5.addData(xi,gPr1);
  pp5.setColor(1,0,0,255);
  pp5.setPlotType(1,23);
  pp5.setColor(2,255,0,0);
  pp5.setPlotType(2,28);
```

```
  pp5.plot();
  Plot pp6=new Plot(xi,egPr1);
  pp6.setColor(1,0,0,255);
  pp6.setColor(2,255,0,0);
  pp6.plot();
 }
 }
```

Program investigates a wide range of Prandtl numbers and as a results:

$$h_x = \frac{q''_s}{T_s - T\infty} = -\frac{T_\infty - T_s}{T_s - T_\infty} k \left.\frac{\partial\theta}{\partial y}\right|_{y=0}$$

$$h_x = k\left(\frac{u_\infty}{\nu x}\right)^{1/2} \left.\frac{\partial\theta(\eta)}{\partial\eta}\right|_{\eta=0} = \frac{k}{x}\left(\frac{u_\infty x}{\nu}\right)^{1/2} \left.\frac{\partial\theta(\eta)}{\partial\eta}\right|_{\eta=0}$$

$$Nu_x = \frac{h_x x}{k} = \mathrm{Re}_x^{1/2} \left.\frac{\partial\theta(\eta)}{\partial\eta}\right|_{\eta=0}$$

Partial continuous curve fitting values are as follows:

$$Nu_x = 0.4454717\ \mathrm{Re}^{0.5} * \mathrm{Pr}^{0.46914552} \qquad 10^{-3} \le \mathrm{Pr} \le 10^{-2}$$

$$Nu_x = 0.34666757\ \mathrm{Re}^{0.5} * \mathrm{Pr}^{0.40159677} \qquad 10^{-2} \le \mathrm{Pr} \le 0.5$$

$$Nu_x = 0.33206065\ \mathrm{Re}^{0.5} * \mathrm{Pr}^{0.33679327} \qquad 0.5 \le \mathrm{Pr} \le 1000$$

Energy equation can also be directly integrated as follows:

$\frac{\partial^2\theta}{\partial\eta^2} + \frac{\mathrm{Pr}}{2} f \frac{\partial\theta}{\partial\eta} = 0$ can be written as

$$\frac{\partial\theta'}{\partial\eta} + \frac{\mathrm{Pr}}{2} f\theta' = 0$$

$$\frac{\partial\theta'}{\theta'} + \frac{\mathrm{Pr}}{2} f d\eta = 0$$

$$\ln(\theta') = -\frac{\mathrm{Pr}}{2}\int_0^\eta f(\eta)d\eta + \mathrm{C}$$

$$\theta' = C_1 \exp\left(-\frac{\mathrm{Pr}}{2}\int_0^\eta f(\eta)d\eta\right)$$

$$\theta(\eta) = C_1 \int_0^\eta \left[\exp\left(-\frac{\mathrm{Pr}}{2}\int_0^\eta f(\eta)d\eta\right)\right] d\eta + C_2$$

By applying the boundary condition $\theta = 0$ at $\eta = 0$ $C_2 = 0$

For the boundary condition $\theta = 1$ at $\eta \to \infty$

$$C_1 = \frac{1}{\int_0^\infty \left[\exp\left(-\frac{\mathrm{Pr}}{2}\int_0^\eta f(\eta)d\eta\right)\right] d\eta}$$

Therefore

$$\theta(\eta)=\frac{\int_0^{\eta}\left[\exp\left(-\frac{\Pr}{2}\int_0^{\eta}f(\eta)d\eta\right)\right]d\eta}{\int_0^{\infty}\left[\exp\left(-\frac{\Pr}{2}\int_0^{\eta}f(\eta)d\eta\right)\right]d\eta}$$

Remember that

$$Nu_x=\frac{h_x x}{k}=\mathrm{Re}_x^{\,1/2}\theta'(\eta)\big|_{\eta=0}$$

and from the above equation

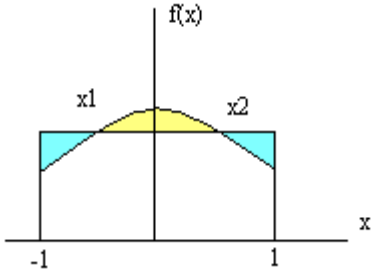
$$\theta'(\eta)=\frac{1}{\int_0^{\infty}\left[\exp\left(-\frac{\Pr}{2}\int_0^{\eta}f(\eta)d\eta\right)\right]d\eta}=C_1$$

The integration can still be solved by this equation an approximation of curve fit polynomial substitution will be used to find the integral. Of course momentum boundary layer still should be solved by using numerical differential equation solution methods.In order to solve integration the following polynomial equation is applied

$$f(\eta)=a_0+a_1\eta+a_2\eta^2+a_3\eta^3+a_4\eta^4+a_5\eta^5+a_6\eta^6+a_7\eta^7+a_7\eta^8+a_9\eta^9+a_{10}\eta^{10}$$

$$\int_0^{\eta}f(\eta)d\eta=a_0\eta+\frac{a_1}{2}\eta^2+\frac{a_2}{3}\eta^3+\frac{a_3}{4}\eta^4+\frac{a_4}{5}\eta^5+\frac{a_5}{6}\eta^6+\frac{a_6}{7}\eta^7+\frac{a_7}{8}\eta^8+\frac{a_8}{9}\eta^9+\frac{a_9}{10}\eta^{10}+\frac{a_{10}}{11}\eta^{11}$$

Outside integral is solved by using Gauss-Legendre Numerical integration method. A problem known from antique times, is to create a rectangle with the same area with a polynomial. It is called a quadrature problem.



a rectangle with the same area with a polynomial

If point x1 and x2 intersecting the polynomial and rectangle is known, area calculation of the polynomial can be interchange with the area calculation of the rectangle. Or in more general means instead of integration process of summation can be substituted. As a general definition:

$$I_w(-1,1)\cong\int_{-1}^{1}f(x)w(x)dx\cong\sum_{k=1}^{n}c_k f(x_k)$$

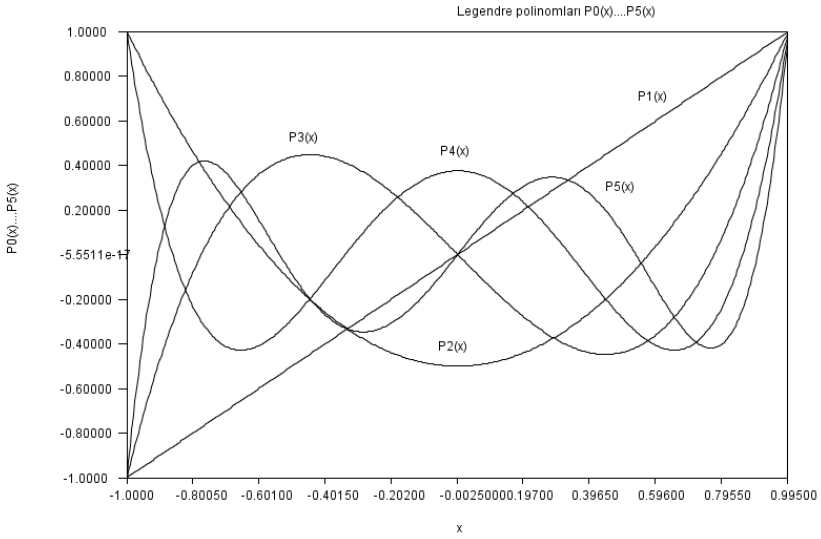
Can be written. In this equation w(x) is the weight factor. In the first of a such formulation, Gauss-Legendre integral formulation, weight factor can be taken as 1.

$$I_w(-1,1)\cong\int_{-1}^{1}f(x)dx\cong\sum_{k=1}^{n}c_k f(x_k)$$

The general solution of this problem can be defined with the Legendre polynomials. Legendre polynomials has a general definition as:

$(k+1)P_{k+1}(x)=(2k+1)xP_k(x)-kP_{k-1}(x),\quad k\geq 1$ and $P_0(x)=1$,$P_1(x)=x$

The first 6 Legendre polynomials are giving in graphic form.



The roots of Legendre Polynomials are the roots of the Gauss-Legendre integration Formula. For example if $P_2(x)$ value is calculated from the above general form:

$P_2(x)=(3x^2-1)/2$. The root of this is equal to $x_{1,2} = \pm 1/\sqrt{3}$. After finding the roots coeficients can be calculated as

$$c_1 x_1^{2j} + c_2 x_2^{2j} + ... + c_n x_n^{2j} = \frac{2}{2j+1}, \quad 0 \le j \le n \text{ or}$$

$$c_k = \frac{2(1-x_k^2)}{[nP_{n-1}(x_k)]^2}$$

Consider that Gauss-Legendre integration limits are -1 and 1
Region x=[-1,1] can be converted to z=[a,b] by changing the variables

$$z = \left(\frac{b-a}{2}\right)x + \left(\frac{b+a}{2}\right) = \alpha x + \beta$$

$$\alpha = \left(\frac{b-a}{2}\right) \quad \beta = \left(\frac{b+a}{2}\right)$$

In this case, integration Formula will became :

$$I_w(a,b) \cong \int_a^b f(z)dx \cong \alpha \sum_{k=1}^{n} c_k f(\alpha x_k + \beta)$$

Now the program using this approach to calculate Blassius flat plate problem is given below:

```
//=====================================================
// Blassius flat plate problem
// Dr. Turhan Coban
// =====================================================
//Blasius flat plate problem

import java.io.*;

class fm1 extends f_xi
{
//multivariable function
 double func(double x[],int x_ref)
 {double a=0;
   if(x_ref==2) a= -0.5*x[1]*x[3];
   if(x_ref==1) a= x[3];
```

```
   if(x_ref==0) a= x[2];
   return a;
 }
}

class f1x extends f_x
{  double Pr;
   double a[];
   double a1[][];
public f1x(double Pri,double w)
{ Pr=Pri;
  fm1 b2=new fm1();
  double y1[]=new double[3];
  y1[0]=0.0; //y0 first guess for f''
  y1[1]=0.0;
  y1[2]=0.3;
  //solution of the differential system :
  double xi1[]=new double[40];
  double fi1[]=new double[40];
  //shooting method for y[2]
  for(int i=1;i<40;i++)
  {
  a1=RK6(b2,0.0,w,y1,10000);
  xi1[i]=a1[2][10000];
  fi1[i]=y1[2];
  y1[2]+=0.02;  //change the boundary value
  }
  //curve fit to find actual boundary value
  double S1[][]=cubic_spline(xi1,fi1,0,0);
  y1[2]=funcSpline(S1,1.0);
  //System.out.println("y1[2]= "+y1[2]);
  a1=RK6(b2,0.0,w,y1,10000);
  // energy equation as integral equation solution
  double xx[]=a1[0];
  double yy[]=a1[1];
  a=PolynomialLSQ(xx,yy,10);
}
public double func(double x)
{   double x2=x*x,x3=x2*x,x4=x2*x2,x5=x4*x,x6=x5*x,x7=x6*x,x8=x7*x,x9=x8*x,x10=x9*x,x11=x10*x;
    double
y1=a[0]*x+a[1]*x2/2.0+a[2]*x3/3.0+a[3]*x4/4.0+a[4]*x5/5.0+a[5]*x6/6.0+a[6]*x7/7.0+a[7]*x8/8.0+a[8]*x9/9.0+a[9]*x10/10.0+a[10]*
x11/11.0;
    double y=Math.exp(-Pr/2.0*y1);
//return funcPolynomialLSQ(a,x);
return y;
}
public static double[][] RK6(f_xi fp,double x0,double xn,double f0[],int N)
{
//6th order Runge Kutta Method
//fp  : given set of derivative functions dfj/dxi(fj,x)
// xo : initial value of the independent variable
// xn : final value of the independent variable
// f0 : initial value of the dependent variable
// N  : number of dependent variable to be calculated
// fi : dependent variable
double h=(xn-x0)/N;
int M=f0.length;
double fi[][];
fi=new double[M][N+1];
double xi[]=new double[M+1];
double k[]=new double[6];
int i,j;
double x;
for(j=0;j<M;j++)
  {
  fi[j][0]=f0[j];
  xi[j+1]=f0[j];
  }
for(x=x0,i=0;i<N;x+=h,i++)
  {
  for(j=1;j<=M;j++)
    {
    xi[0]=x;
    xi[j]=fi[j-1][i];
    k[0]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
```

```
    xi[j]=fi[j-1][i]+k[0]/2;
    k[1]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/4.0+k[1]/4.0;
    k[2]=h*fp.func(xi,j-1);
    xi[0]=x+h;
    xi[j]=fi[j-1][i]-k[1]+2.0*k[2];
    k[3]=h*fp.func(xi,j-1);
    xi[0]=x+2.0/3.0*h;
    xi[j]=fi[j-1][i]+7.0/27.0*k[0]+10.0/27.0*k[1]+1.0/27.0*k[3];
    k[4]=h*fp.func(xi,j-1);
    xi[0]=x+1.0/5.0*h;
    xi[j]=fi[j-1][i]+28.0/625.0*k[0]-1.0/5.0*k[1]+546.0/625.0*k[2]+54.0/625.0*k[3]-378/625.0*k[4];
    k[5]=h*fp.func(xi,j-1);
    fi[j-1][i+1]=fi[j-1][i]+k[0]/24.0+5.0*k[3]/48.0+27.0*k[4]/56.0+125.0*k[5]/336.0;
    xi[j]=fi[j-1][i];
    }
  }
double a[][]=new double[M+1][N+1];
for(x=x0,i=0;i<=N;x+=h,i++)
{
  a[0][i]=x;
  for(j=1;j<=M;j++)
  {
  a[j][i]=fi[j-1][i];
  }
}

return a;
}
  public static double [] thomas(double a[][],double r[])
  {
  //
  int n=a.length;
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double x[]=new double[n];
  for(int i=0;i<n;i++)     {f[i]=a[i][i];}
  for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
  for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }

  public static double [] thomas(double f[],double e[],double g[],double r[])
  {
  int n=f.length;
  double x[]=new double[n];
  for(int k=1;k<n;k++)
     {e[k]=e[k]/f[k-1];
      f[k]=f[k]-e[k]*g[k-1];
     }
  for(int k=1;k<n;k++)
     {r[k]=r[k]-e[k]*r[k-1];
     }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
     {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }


  public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
  {
  int n=xi.length;
  double h[]=new double[n];
```

```
double w[]=new double[n];
double f[]=new double[n];
double e[]=new double[n];
double g[]=new double[n];
double d[]=new double[n];
double x[]=new double[n];
double S[][]=new double[4][n];
int k;
for(k=0;k<(n-1);k++)
   {h[k]=xi[k+1]-xi[k];
    w[k]=(yi[k+1]-yi[k])/h[k];
              }
d[0]=c0;
d[n-1]=cn;
for(k=1;k<(n-1);k++)
   {d[k]=6.0*(w[k]-w[k-1]);}
f[0]=1.0;
f[n-1]=1.0;
g[0]=0.0;
g[n-1]=0.0;
e[0]=0.0;
e[n-1]=0.0;
for(k=1;k<(n-1);k++)
{f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
S[2]=thomas(f,e,g,d);
S[3]=xi;
for(k=0;k<(n-1);k++)
  {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
   S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
  }
return S;
}

public static double funcSpline(double S[][],double x)
{
int n=S[0].length;
double xx1=0;
double xx2=0;
double y=0;
double hk=0;
for(int k=0;k<(n-1);k++)
   {if(S[3][k]<=x && x<=S[3][k+1])
    {hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     break;
     }
   }
     if(y==0 && S[3][n-2]<=x )
     {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     }
return y;
}

  public static double[][]  funcSpline(double xi[],double yi[],int aradegersayisi)
  {
              //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
              // ara değer sayısı 2 dir
              int n=xi.length;
              int nn=(n-1)*(aradegersayisi+1)+1;
              double z[][]=new double[2][nn];
  double S[][]=cubic_spline(xi,yi,0,0);
  double dx=0;
  int k=0;
  int i;
  for(i=0;i<(n-1);i++)
     {  z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k++;
        for(int j=0;j<aradegersayisi;j++)
           {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
        z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k++;}
```

```
      }
      z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);
    return z;
    }
// Polynomial least square
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
              //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
              }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] PolynomialLSQ(double xi[],double yi[],int n)
{ //Polynomial least square
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  for(j=0;j<n+1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= yi[k];
                  else      B[i] += Math.pow(xi[k],i)*yi[k];}
  }
X=gausswithpartialpivot(A,B);
double max=0;
for(i=0;i<n+1;i++)
```

```
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
return X;
}

public static double funcPolynomialLSQ(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcPolynomialLSQ(e,x[k]))*(y[k]-funcPolynomialLSQ(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcPolynomialLSQ(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
                //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                  int n=xi.length;
                int nn=(n-1)*(aradegersayisi+1)+1;
                double z[][]=new double[2][nn];
    double E[]=PolynomialLSQ(xi,yi,polinomkatsayisi);
    System.out.println("katsayilar :\n"+Matrix.toStringT(E));
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
       {z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;
       for(int j=0;j<aradegersayisi;j++)
       {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;}
       }
       z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);
    return z;
 }
}



class fm2 extends f_xi
{ double Pr;
  fm2(double Pri) {Pr=Pri;}
// multivariable function
 double func(double x[],int x_ref)
 {
   //Blasius flat plate diferential equation
   //f"'+0.5*ff" = 0
   //f"'=-0.5*x[1]*x[3] f"=x[3] f'=x[2] f=x[1]]
   double a=0;
   if(x_ref==4) a= -0.5*Pr*x[1]*x[5];
   if(x_ref==3) a= x[5];
   if(x_ref==2) a= -0.5*x[1]*x[3];
   if(x_ref==1) a= x[3];
   if(x_ref==0) a= x[2];;
```

```
   return a;
 }
}

class diferansiyel1B3
{
  public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
              //calculates legendre gauss-coefficients as coefficients of the integral
              //for n terms
              double EPS=3.0e-15;
              int m,j,i;
              double z1,z,xm,xl,pp,p3,p2,p1;
   //double x[]=new double[n];
   //double w[]=new double[n];
   double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
              m=(n+1)/2;
              xm=0.5*(x2+x1);
              xl=0.5*(x2-x1);
              for (i=1;i<=m;i++)  {
                        z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                        do {
                                  p1=1.0;
                                  p2=0.0;
                                  for (j=1;j<=n;j++) {
                                            p3=p2;
                                            p2=p1;
                                            p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                                  }
                                  pp=n*(z*p1-p2)/(z*z-1.0);
                                  z1=z;
                                  z=z1-p1/pp;
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
                                  }
return a;
}

public static double integral(f_x f_xnt,double x1,double x2,int n)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss legendre integral
double a[][]=new double[2][n];
a=gauss_legendre_coefficients(x1,x2,n);
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*f_xnt.func(a[0][i]);}
return z;
}
  public static void main(String args[]) throws IOException
  {  double
Pr[]={0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,
0.9,1.0,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9,2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0,10.0,11,12,13,14,15.0,30.0,40.0,50,60,70,80,90,100,110.0,120.0,1
30.0,140,150,160,170,180,190,200,300,400,500,600,700,800,900,1000};
     double
w[]={200,200,200,200,200,200,200,200,200,100.0,100,100,100,100,100,100,100,50,50,50,50,50,50,50,50,25,25,25,25,25,25,25,2
5,25,10,10,10,10,10,10,10,10,7,7,7,7,7,7,7,7,7,7,7,7,7,7,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3};
     double Xi[]=new double[Pr.length];
     double Yi[]=new double[Pr.length];
     for(int i=0;i<Pr.length;i++)
     {
         f1x ff=new f1x(Pr[i],w[i]);
         Xi[i]=Pr[i];
         Yi[i]=1.0/integral(ff,0,w[i],100);
         System.out.println("Pr = "+Xi[i]+"teta'="+Yi[i]);
     }
     double z[][]={Xi,Yi};
     Text.printT(z,"Blassius flat palete NuxRex^-0.5 integral method");
     double Pr1=1.0;
     f1x ff=new f1x(Pr1,6.8);
     Text.printT(ff.a,"coefficient of curve fitting f(eta)");
```

}}

Curve fitting coefficients for $f(\eta)$ as follows

a_0	-1.4214607426133100E-05
a_1	-2.7490470731684500E-05
a_2	1.6425932953124300E-01
a_3	3.6067764550437600E-03
a_4	-4.1973033042777800E-03
a_5	2.3935684285465600E-03
a_6	-1.1730013965284500E-03
a_7	2.9023032420905100E-04
a_8	-3.7647193571235300E-05
a_9	2.5045430633530600E-06
a_{10}	-6.8014637715603100E-08

Resulting $\theta'(\eta)$ values are giving in the following table

Pr	Nux/Rex^0.5
0.001	1.729413262306600E-02
0.002	2.415544166595620E-02
0.003	2.930890400324250E-02
0.004	3.358120336476310E-02
0.005	3.729221792552660E-02
0.006	4.060584965389220E-02
0.007	4.361937115572090E-02
0.008	4.639608880595110E-02
0.009	4.897991598433690E-02
0.01	5.149458266428290E-02
0.02	7.040630431060980E-02
0.03	8.414370225346290E-02
0.04	9.526381880762620E-02
0.05	1.047452799486890E-01
0.06	1.130837306523290E-01
0.07	1.205699518112150E-01
0.08	1.273911396645470E-01
0.09	1.336758511642230E-01
0.1	1.397049456770640E-01
0.2	1.835650337268280E-01
0.3	2.140695063892500E-01
0.4	2.381456439762600E-01
0.5	2.583263894319400E-01
0.6	2.758548026899100E-01
0.7	2.914431591789320E-01
0.8	3.055415336430820E-01
0.9	3.184543073705920E-01
1	3.312954754476690E-01
1.1	3.425243531610520E-01
1.2	3.530677925048750E-01
1.3	3.630204591015030E-01
1.4	3.724578178515520E-01
1.5	3.814410953788210E-01
1.6	3.900207199841780E-01
1.7	3.982387708167000E-01
1.8	4.061307623149510E-01
1.9	4.144877482111790E-01
2	4.218486966103280E-01
3	4.844673987151180E-01
4	5.341031722208660E-01
5	5.759124157275020E-01
6	6.123969275242920E-01
7	6.449824545098310E-01
8	6.745675885350030E-01
9	7.017594347981730E-01
10	7.273604360913650E-01
11	7.509684948102970E-01
12	7.731770279049560E-01
13	7.941765123191470E-01
14	8.141190622552570E-01
15	8.331283108236750E-01
30	1.050198952039010E+00
40	1.155975784381680E+00
50	1.245256993110980E+00
60	1.323271776036590E+00
70	1.393014856049930E+00
80	1.456380317117830E+00
90	1.514651201226540E+00
100	1.568741602664960E+00
110	1.619328478420450E+00
120	1.666928800405430E+00
130	1.716730579095950E+00

140	1.759627853164530E+00
150	1.800524439991490E+00
160	1.839638248832950E+00
170	1.877151533095050E+00
180	1.913218451133240E+00
190	1.947970697495480E+00
200	1.981521771720000E+00
300	2.267844938698240E+00
400	2.495681758564390E+00
500	2.688011500291470E+00
600	2.856078449312840E+00
700	3.006326418245540E+00
800	3.142832760682760E+00
900	3.268360331085450E+00
1000	3.384877975113260E+00

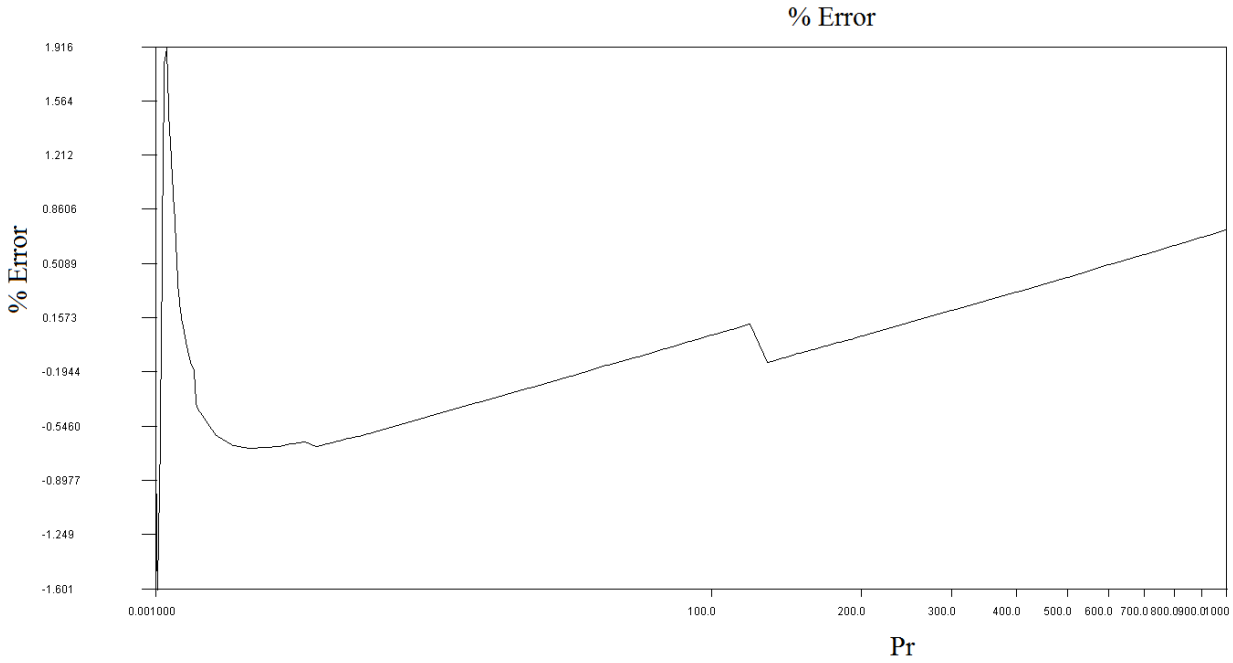
Partial continuous curve fitting values are as follows:

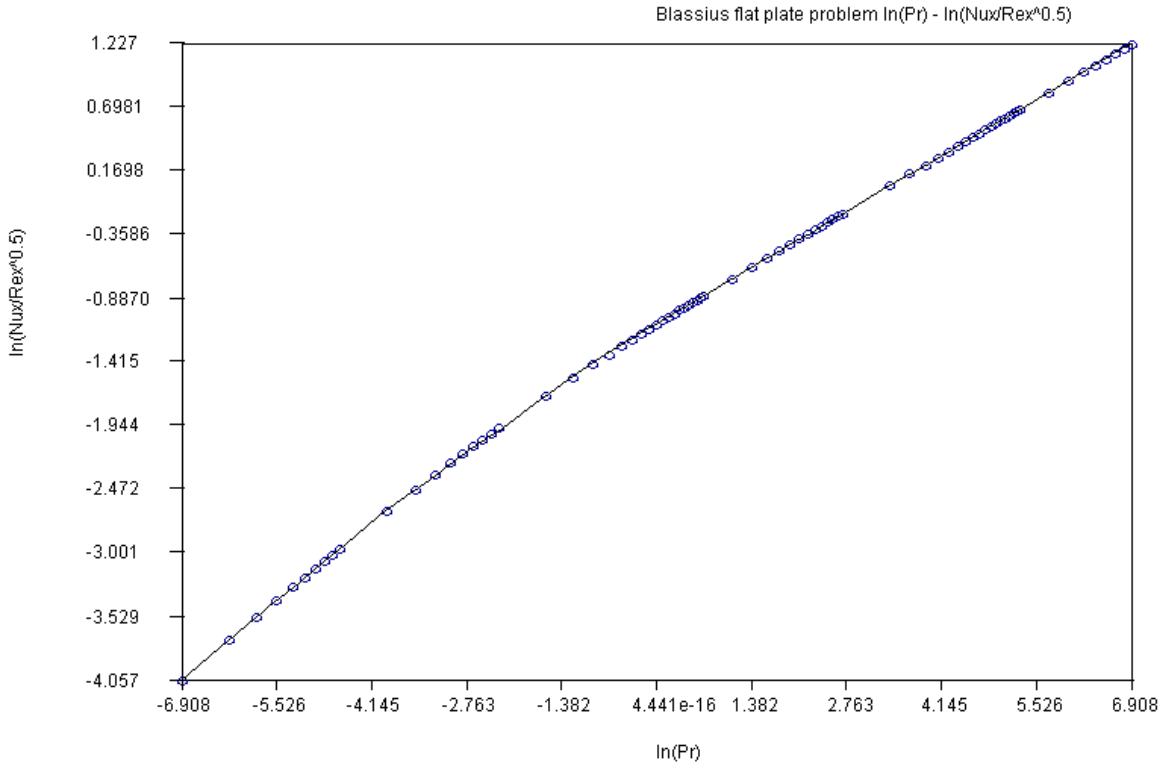
$$Nu_x = 0.45584275\,\mathrm{Re}^{0.5} * \mathrm{Pr}^{0.47294717} \qquad 10^{-3} \le \mathrm{Pr} \le 10^{-2}$$

$$Nu_x = 0.35271867\,\mathrm{Re}^{0.5} * \mathrm{Pr}^{0.40922589} \qquad 10^{-2} \le \mathrm{Pr} \le 0.5$$

$$Nu_x = 0.33253715\,\mathrm{Re}^{0.5} * \mathrm{Pr}^{0.33694685} \qquad 0.5 \le \mathrm{Pr} \le 1000$$

As it is seen from the results equations are slightly differs from the previous set, but not that much. Let us look at equation errors and fitted lines as plots as well.

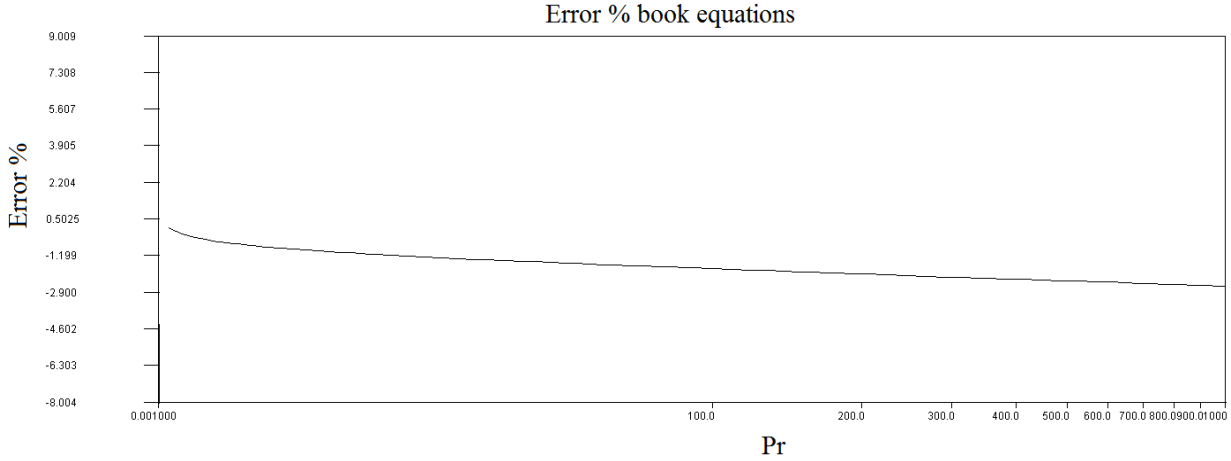


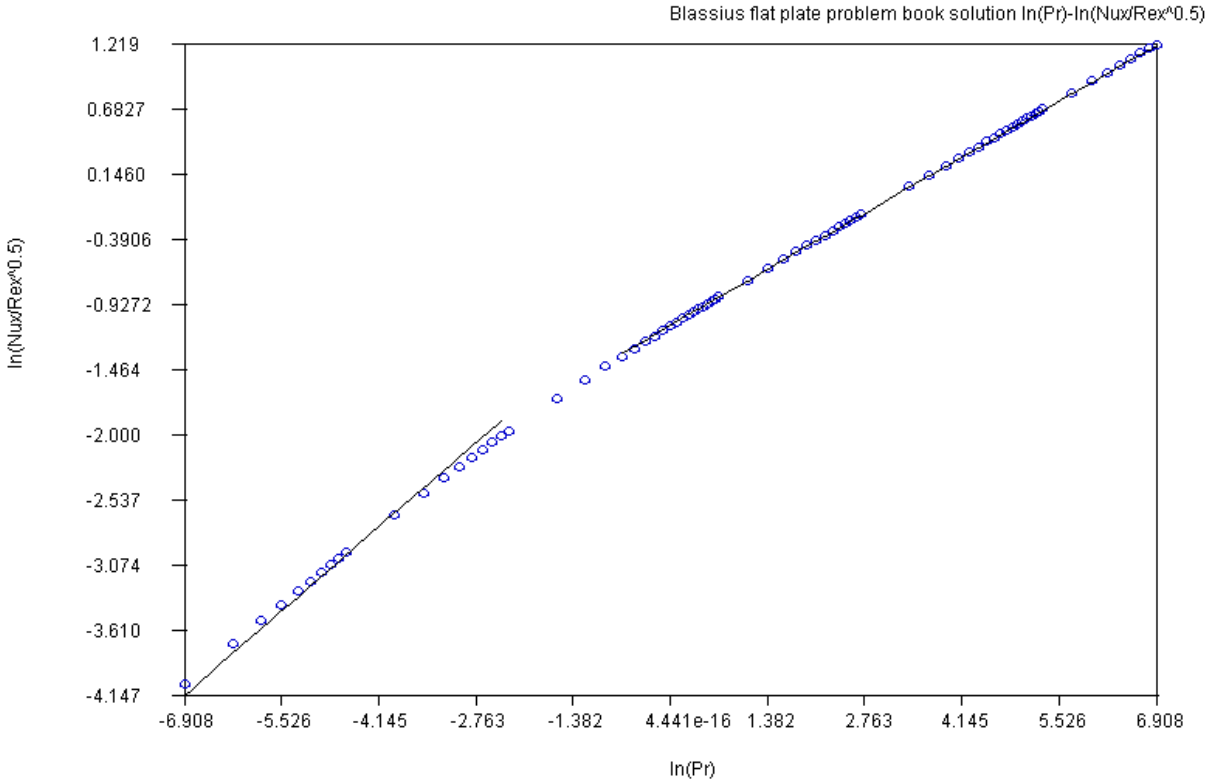


In most of the region plot errors are less than 1 %
As comparison solutions listed in books (a heat and mass transfer by Eckert and Drake) listed the following equations

$$Nu_x = 0.5\,\mathrm{Re}^{0.5} * \mathrm{Pr}^{0.5} \qquad 0.005 \le \mathrm{Pr} \le 0.05$$

$$Nu_x = 0.332\mathrm{Re}^{0.5} * \mathrm{Pr}^{0.33333333} \qquad 0.6 \le \mathrm{Pr}$$





This form of equation has an error less than 5% in most of the regions. As the last thing let us print out tables for η,f,f',f'',θ and θ' for **Pr=1.0**. In programs total of 10000 data points are used for accurate differential equation solution. In order to keep following table in manageable size one in every 50 data is printed out.

i	η	f	f'	f''	θ	θ'
0	0	0	0	0.3317675	0	0.33176751
50	0.034	1.80E-04	0.01128009	0.3317672	0.0112801	0.331767191
100	0.068	7.44E-04	0.02256015	0.3317648	0.0225601	0.331764796
150	0.102	0.0016915	0.03384006	0.3317582	0.0338401	0.331758161
200	0.136	0.0030223	0.04511965	0.3317451	0.0451197	0.331745124
250	0.17	0.0047366	0.05639867	0.3317235	0.0563987	0.331723523
300	0.204	0.0068344	0.06767678	0.3316912	0.0676768	0.331691196
350	0.238	0.0093156	0.0789536	0.331646	0.0789536	0.331645987
400	0.272	0.0121802	0.09022865	0.3315857	0.0902286	0.331585737
450	0.306	0.0154281	0.10150138	0.3315083	0.1015014	0.331508296
500	0.34	0.0190593	0.11277117	0.3314115	0.1127712	0.331411515
550	0.374	0.0230735	0.12403733	0.3312933	0.1240373	0.331293254
600	0.408	0.0274708	0.13529911	0.3311514	0.1352991	0.331151379
650	0.442	0.0322509	0.14655565	0.3309838	0.1465557	0.330983766
700	0.476	0.0374135	0.15780606	0.3307883	0.1578061	0.330788303
750	0.51	0.0429586	0.16904934	0.3305629	0.1690493	0.330562888
800	0.544	0.0488859	0.18028446	0.3303054	0.1802845	0.330305437
850	0.578	0.055195	0.19151029	0.3300139	0.1915103	0.330013881
900	0.612	0.0618856	0.20272563	0.3296862	0.2027256	0.329686173
950	0.646	0.0689573	0.21392922	0.3293203	0.2139292	0.329320285
1000	0.68	0.0764098	0.22511974	0.3289142	0.2251197	0.328914216
1050	0.714	0.0842425	0.23629578	0.328466	0.2362958	0.32846599
1100	0.748	0.0924549	0.24745588	0.3279737	0.2474559	0.327973662
1150	0.782	0.1010465	0.25859852	0.3274353	0.2585985	0.32743532
1200	0.816	0.1100167	0.26972209	0.3268491	0.2697221	0.326849089
1250	0.85	0.1193647	0.28082494	0.3262131	0.2808249	0.326213132
1300	0.884	0.1290899	0.29190534	0.3255257	0.2919053	0.325525652
1350	0.918	0.1391914	0.30296153	0.3247849	0.3029615	0.324784903
1400	0.952	0.1496685	0.31399166	0.3239892	0.3139917	0.323989183
1450	0.986	0.1605201	0.32499383	0.3231368	0.3249938	0.323136845
1500	1.02	0.1717453	0.3359661	0.3222263	0.3359661	0.322226296
1550	1.054	0.1833431	0.34690646	0.321256	0.3469065	0.321256005
1600	1.088	0.1953123	0.35781286	0.3202245	0.3578129	0.320224502
1650	1.122	0.2076517	0.36868318	0.3191304	0.3686832	0.319130384
1700	1.156	0.2203601	0.37951529	0.3179723	0.3795153	0.317972318

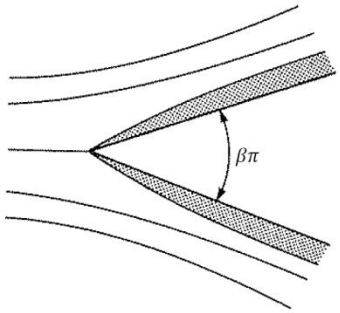
1750	1.19	0.2334362	0.39030699	0.316749	0.390307	0.316749044
1800	1.224	0.2468786	0.40105604	0.3154594	0.401056	0.315459381
1850	1.258	0.2606856	0.41176016	0.3141022	0.4117602	0.314102227
1900	1.292	0.2748559	0.42241705	0.3126766	0.422417	0.312676564
1950	1.326	0.2893878	0.43302436	0.3111815	0.4330244	0.311181463
2000	1.36	0.3042794	0.44357971	0.3096161	0.4435797	0.309616085
2050	1.394	0.3195291	0.45408071	0.3079797	0.4540807	0.307979684
2100	1.428	0.3351349	0.46452493	0.3062716	0.4645249	0.306271613
2150	1.462	0.3510949	0.47490992	0.3044913	0.4749099	0.304491323
2200	1.496	0.3674069	0.48523322	0.3026384	0.4852332	0.302638371
2250	1.53	0.384069	0.49549235	0.3007124	0.4954924	0.300712416
2300	1.564	0.4010788	0.50568483	0.2987132	0.5056848	0.298713226
2350	1.598	0.4184341	0.51580817	0.2966407	0.5158082	0.296640682
2400	1.632	0.4361324	0.52585987	0.2944948	0.5258599	0.294494775
2450	1.666	0.4541713	0.53583744	0.2922756	0.5358374	0.29227561
2500	1.7	0.4725482	0.54573838	0.2899834	0.5457384	0.28998341
2550	1.734	0.4912604	0.55556023	0.2876185	0.5555602	0.287618513
2600	1.768	0.5103054	0.56530052	0.2851814	0.5653005	0.285181378
2650	1.802	0.5296801	0.5749568	0.2826726	0.5749568	0.282672582
2700	1.836	0.5493818	0.58452664	0.2800928	0.5845266	0.280092822
2750	1.87	0.5694075	0.59400764	0.2774429	0.5940076	0.277442917
2800	1.904	0.5897541	0.60339745	0.2747238	0.6033974	0.274723804
2850	1.938	0.6104184	0.61269371	0.2719365	0.6126937	0.271936543
2900	1.972	0.6313973	0.62189412	0.2690823	0.6218941	0.269082309
2950	2.006	0.6526874	0.63099645	0.2661624	0.6309964	0.266162401
3000	2.04	0.6742855	0.63999846	0.2631782	0.6399985	0.263178232
3050	2.074	0.6961879	0.648898	0.2601313	0.648898	0.26013133
3100	2.108	0.7183913	0.65769297	0.2570233	0.657693	0.257023339
3150	2.142	0.740892	0.66638132	0.253856	0.6663813	0.253856013
3200	2.176	0.7636864	0.67496105	0.2506312	0.674961	0.250631216
3250	2.21	0.7867707	0.68343025	0.2473509	0.6834302	0.247350918
3300	2.244	0.8101412	0.69178705	0.2440172	0.6917871	0.24401719
3350	2.278	0.833794	0.70002968	0.2406322	0.7000297	0.240632204
3400	2.312	0.8577252	0.70815644	0.2371982	0.7081564	0.237198226
3450	2.346	0.8819308	0.71616569	0.2337176	0.7161657	0.233717615
3500	2.38	0.9064069	0.72405588	0.2301928	0.7240559	0.230192815
3550	2.414	0.9311493	0.73182556	0.2266264	0.7318256	0.226626353
3600	2.448	0.9561539	0.73947334	0.2230208	0.7394733	0.223020831
3650	2.482	0.9814166	0.74699795	0.2193789	0.746998	0.219378925
3700	2.516	1.0069331	0.75439819	0.2157034	0.7543982	0.215703377
3750	2.55	1.0326993	0.76167296	0.211997	0.761673	0.211996987
3800	2.584	1.0587107	0.76882126	0.2082626	0.7688213	0.208262613
3850	2.618	1.0849632	0.77584218	0.2045032	0.7758422	0.204503158
3900	2.652	1.1114524	0.78273491	0.2007216	0.7827349	0.200721569
3950	2.686	1.1381738	0.78949876	0.1969208	0.7894988	0.196920829
4000	2.72	1.1651231	0.79613312	0.1931039	0.7961331	0.19310395
4050	2.754	1.192296	0.80263748	0.189274	0.8026375	0.189273965
4100	2.788	1.2196879	0.80901146	0.1854339	0.8090115	0.185433924
4150	2.822	1.2472944	0.81525475	0.1815869	0.8152547	0.181586888
4200	2.856	1.2751111	0.82136717	0.1777359	0.8213672	0.177735917
4250	2.89	1.3031335	0.82734864	0.1738841	0.8273486	0.173884069
4300	2.924	1.3313573	0.83319917	0.1700344	0.8331992	0.170034391
4350	2.958	1.3597778	0.83891888	0.1661899	0.8389189	0.166189911
4400	2.992	1.3883907	0.84450801	0.1623536	0.844508	0.162353633
4450	3.026	1.4171916	0.84996687	0.1585285	0.8499669	0.158528532
4500	3.06	1.446176	0.8552959	0.1547175	0.8552959	0.154717542
4550	3.094	1.4753395	0.86049562	0.1509236	0.8604956	0.150923558
4600	3.128	1.5046778	0.86556665	0.1471494	0.8655666	0.147149424
4650	3.162	1.5341864	0.87050971	0.1433979	0.8705097	0.143397926
4700	3.196	1.563861	0.87532562	0.1396718	0.8753256	0.139671794
4750	3.23	1.5936974	0.88001528	0.1359737	0.8800153	0.135973688
4800	3.264	1.6236912	0.88457969	0.1323062	0.8845797	0.132306197
4850	3.298	1.6538382	0.88901993	0.1286718	0.8890199	0.128671835
4900	3.332	1.6841342	0.89333716	0.125073	0.8933372	0.125073034
4950	3.366	1.7145751	0.89753263	0.1215121	0.8975326	0.121512142
5000	3.4	1.7451566	0.90160766	0.1179914	0.9016077	0.117991415
5050	3.434	1.7758748	0.90556367	0.114513	0.9055637	0.114513018
5100	3.468	1.8067257	0.90940211	0.111079	0.9094021	0.11107902
5150	3.502	1.8377052	0.91312452	0.1076914	0.9131245	0.107691388
5200	3.536	1.8688094	0.91673253	0.104352	0.9167325	0.10435199
5250	3.57	1.9000344	0.92022779	0.1010626	0.9202278	0.101062588
5300	3.604	1.9313766	0.92361203	0.0978248	0.923612	0.097824838
5350	3.638	1.962832	0.92688703	0.0946403	0.926887	0.094640286
5400	3.672	1.9943971	0.93005463	0.0915104	0.9300546	0.091510371
5450	3.706	2.0260682	0.9331167	0.0884364	0.9331167	0.088436421
5500	3.74	2.0578417	0.93607517	0.0854197	0.9360752	0.085419653
5550	3.774	2.0897142	0.938932	0.0824612	0.938932	0.082461171
5600	3.808	2.1216822	0.94168918	0.079562	0.9416892	0.079561971
5650	3.842	2.1537424	0.94434875	0.0767229	0.9443487	0.076722936
5700	3.876	2.1858915	0.94691276	0.0739448	0.9469128	0.073944839
5750	3.91	2.2181263	0.9493833	0.0712283	0.9493833	0.071228345
5800	3.944	2.2504436	0.95176247	0.068574	0.9517625	0.068574011

5850	3.978	2.2828404	0.9540524	0.0659823	0.9540524	0.065982286
5900	4.012	2.3153136	0.95625521	0.0634535	0.9562552	0.063453518
5950	4.046	2.3478604	0.95837305	0.0609879	0.958373	0.060987949
6000	4.08	2.3804778	0.96040807	0.0585857	0.9604081	0.058585724
6050	4.114	2.4131631	0.96236243	0.0562469	0.9623624	0.05624689
6100	4.148	2.4459136	0.96423829	0.0539714	0.9642383	0.053971399
6150	4.182	2.4787267	0.96603778	0.0517591	0.9660378	0.051759112
6200	4.216	2.5115998	0.96776307	0.0496098	0.9677631	0.049609799
6250	4.25	2.5445303	0.96941628	0.0475231	0.9694163	0.04752315
6300	4.284	2.577516	0.97099954	0.0454988	0.9709995	0.045498768
6350	4.318	2.6105544	0.97251497	0.0435362	0.972515	0.043536181
6400	4.352	2.6436432	0.97396464	0.0416348	0.9739646	0.041634841
6450	4.386	2.6767804	0.97535064	0.0397941	0.9753506	0.039794131
6500	4.42	2.7099636	0.97667502	0.0380134	0.976675	0.038013366
6550	4.454	2.7431909	0.9779398	0.0362918	0.9779398	0.036291797
6600	4.488	2.7764603	0.97914698	0.0346286	0.979147	0.034628616
6650	4.522	2.8097699	0.98029853	0.033023	0.9802985	0.033022963
6700	4.556	2.8431177	0.9813964	0.0314739	0.9813964	0.031473922
6750	4.59	2.8765021	0.98244249	0.0299805	0.9824425	0.029980534
6800	4.624	2.9099212	0.98343869	0.0285418	0.9834387	0.028541793
6850	4.658	2.9433734	0.98438682	0.0271567	0.9843868	0.027156656
6900	4.692	2.9768571	0.98528871	0.025824	0.9852887	0.025824043
6950	4.726	3.0103707	0.98614611	0.0245428	0.9861461	0.024542842
7000	4.76	3.0439128	0.98696076	0.0233119	0.9869608	0.023311914
7050	4.794	3.077482	0.98773434	0.0221301	0.9877343	0.022130095
7100	4.828	3.1110768	0.98846851	0.0209962	0.9884685	0.0209962
7150	4.862	3.1446959	0.98916488	0.019909	0.9891649	0.019909025
7200	4.896	3.1783382	0.98982501	0.0188674	0.989825	0.018867354
7250	4.93	3.2120023	0.99045044	0.01787	0.9904504	0.017869958
7300	4.964	3.2456872	0.99104265	0.0169156	0.9910426	0.016915601
7350	4.998	3.2793917	0.99160308	0.016003	0.9916031	0.016003044
7400	5.032	3.3131147	0.99213314	0.015131	0.9921331	0.015131042
7450	5.066	3.3468554	0.99263418	0.0142984	0.9926342	0.014298354
7500	5.1	3.3806126	0.99310752	0.0135037	0.9931075	0.01350374
7550	5.134	3.4143854	0.99355443	0.012746	0.9935544	0.012745968
7600	5.168	3.4481731	0.99397616	0.0120238	0.9939762	0.012023812
7650	5.202	3.4819747	0.99437388	0.0113361	0.9943739	0.011336057
7700	5.236	3.5157895	0.99474875	0.0106815	0.9947488	0.0106815
7750	5.27	3.5496166	0.99510189	0.010059	0.9951019	0.010058953
7800	5.304	3.5834555	0.99543435	0.0094672	0.9954344	0.009467243
7850	5.338	3.6173053	0.99574718	0.0089052	0.9957472	0.008905215
7900	5.372	3.6511654	0.99604135	0.0083717	0.9960413	0.008371732
7950	5.406	3.6850353	0.99631782	0.0078657	0.9963178	0.007865679
8000	5.44	3.7189143	0.99657752	0.007386	0.9965775	0.007385961
8050	5.474	3.7528019	0.99682131	0.0069315	0.9968213	0.006931507
8100	5.508	3.7866975	0.99705004	0.0065013	0.99705	0.006501268
8150	5.542	3.8206007	0.99726451	0.0060942	0.9972645	0.006094222
8200	5.576	3.8545109	0.99746551	0.0057094	0.9974655	0.005709368
8250	5.61	3.8884278	0.99765376	0.0053457	0.9976538	0.005345736
8300	5.644	3.9223509	0.99782997	0.0050024	0.99783	0.005002378
8350	5.678	3.9562797	0.99799482	0.0046784	0.9979948	0.004678375
8400	5.712	3.9902141	0.99814895	0.0043728	0.9981489	0.004372834
8450	5.746	4.0241535	0.99829298	0.0040849	0.998293	0.004084891
8500	5.78	4.0580976	0.99842748	0.0038137	0.9984275	0.003813707
8550	5.814	4.0920462	0.99855303	0.0035585	0.998553	0.003558472
8600	5.848	4.1259989	0.99867014	0.0033184	0.9986701	0.003318403
8650	5.882	4.1599554	0.99877932	0.0030927	0.9987793	0.003092744
8700	5.916	4.1939155	0.99888104	0.0028808	0.998881	0.002880767
8750	5.95	4.227879	0.99897578	0.0026818	0.9989758	0.00268177
8800	5.984	4.2618456	0.99906394	0.0024951	0.9990639	0.002495079
8850	6.018	4.2958151	0.99914594	0.00232	0.9991459	0.002320043
8900	6.052	4.3297873	0.99922217	0.002156	0.9992222	0.002156042
8950	6.086	4.363762	0.999293	0.0020025	0.999293	0.002002477
9000	6.12	4.3977391	0.99935876	0.0018588	0.9993588	0.001858775
9050	6.154	4.4317182	0.99941978	0.0017244	0.9994198	0.00172439
9100	6.188	4.4656994	0.99947638	0.0015988	0.9994764	0.001598796
9150	6.222	4.4996825	0.99952884	0.0014815	0.9995288	0.001481494
9200	6.256	4.5336673	0.99957744	0.001372	0.9995774	0.001372005
9250	6.29	4.5676536	0.99962244	0.0012699	0.9996224	0.001269874
9300	6.324	4.6016414	0.99966407	0.0011747	0.9996641	0.001174667
9350	6.358	4.6356307	0.99970258	0.001086	0.9997026	0.00108597
9400	6.392	4.6696211	0.99973816	0.0010034	0.9997382	0.00100339
9450	6.426	4.7036127	0.99977103	9.27E-04	0.999771	9.27E-04
9500	6.46	4.7376055	0.99980138	8.55E-04	0.9998014	8.55E-04
9550	6.494	4.7715992	0.99982938	7.89E-04	0.9998294	7.89E-04
9600	6.528	4.8055938	0.9998552	7.27E-04	0.9998552	7.27E-04
9650	6.562	4.8395892	0.99987899	6.70E-04	0.999879	6.70E-04
9700	6.596	4.8735855	0.99990091	6.17E-04	0.9999009	6.17E-04
9750	6.63	4.9075824	0.99992108	5.68E-04	0.9999211	5.68E-04
9800	6.664	4.9415801	0.99993964	5.22E-04	0.9999396	5.22E-04
9850	6.698	4.9755783	0.9999567	4.80E-04	0.9999567	4.80E-04
9900	6.732	5.0095771	0.99997238	4.41E-04	0.9999724	4.41E-04

9950	6.766	5.0435764	0.99998678	4.05E-04	0.9999868	4.05E-04
10000	6.8	5.0775761	1	3.71E-04	1	3.71E-04

EXAMPLE 2: WEDGE FLOW AND FALKNER SMILARITY SOLUTION

In 1930 Falkner & Skan discovered a whole family of situations for which the same transformation is possible, and in 1939 Golstein[92] investigated in detail the actual transformations and solutions.



If a potential flow of constant property fluid at a surface of a wedge is given as:
$u_s = Cx^m$ In this equation exponent m is connected with the opening angle $\beta\pi$ of the wedge by relation

$$\beta = \frac{2m}{m+1}$$

Continuity equation can be eliminated by introducing of stream function φ satisfying the conditions

$$u = \frac{\partial\varphi}{\partial y} \qquad v = -\frac{\partial\varphi}{\partial x}$$

A similarity parameter $\eta(x,y)$ is introduced as a new variable.

$$\eta(x,y) = \sqrt{\frac{m+1}{2}}\frac{y}{x}\sqrt{Re_x} = \sqrt{\frac{m+1}{2}}y\sqrt{\frac{u_s}{\nu x}} = \sqrt{\frac{m+1}{2}}\sqrt{\frac{C}{\nu}}x^{(m-1)/2}y$$

And the new similarity function f which is function of this new η only

$$f(\eta) = \sqrt{\frac{m+1}{2}}\frac{\varphi}{u_s x}\sqrt{Re_x} = \sqrt{\frac{m+1}{2}}\frac{\varphi}{\sqrt{\nu\, u_s x}} = \sqrt{\frac{m+1}{2}}\frac{1}{\sqrt{\nu C}}x^{(m+1)/2}\varphi$$

the velocity component u can now be written as

$$u = \frac{\partial\varphi}{\partial y} = \sqrt{\frac{2C\nu}{m+1}}x^{(m+1)/2}\frac{df}{d\eta}\sqrt{\frac{m+1}{2}}\sqrt{\frac{C}{\nu}}x^{(m+1)/2} = Cx^m\frac{df}{d\eta} = u_s\frac{df}{d\eta}$$

the velocity component v can now be written as

$$u = -\frac{\partial\varphi}{\partial x} = -\sqrt{\frac{2C\nu}{m+1}}x^{(m-1)/2}\left(\frac{m+1}{2}f + \frac{m-1}{2}\frac{df}{d\eta}\eta\right)$$

Bernoulli equation

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = -u_s\frac{du_s}{dx} = -mC^2x^{2m-1}$$

Equation converts to

$$\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} - \beta\left[\left(\frac{df}{d\eta}\right)^2 - 1\right] = 0$$

Bounday conditions

$\eta=0$ $\frac{df}{d\eta} = 0$ $f = 0$

$\eta=\infty$ $\frac{df}{d\eta} = 1$

For heat transfer constant surface temperature

$$u\frac{dT}{dx} + v\frac{dT}{dy} = \alpha\frac{d^2T}{dy^2}$$

$T_w - T_s = C_1x^\gamma$ where T_w is Wall surface temperature and T_s is the stream temperature. $\gamma = 0$ means constant wall temperature.

$\theta=\frac{T-T_s}{T_w-T_s}$ assuming that θ is function of η only

$\frac{d^2\theta}{d\eta^2}+Prf\frac{d\theta}{d\eta}-(2-\beta)Pr\gamma\frac{df}{d\eta}\theta=0$

Boundary conditions

$\eta=0 \quad \theta=1$

$\eta=\infty \quad \theta=0$

Convertion of the equations:

$X[0]=\eta \quad X[1]=f \quad X[2]=f'=\frac{df}{d\eta} \quad X[3]=f''=\frac{d^2f}{d\eta^2} \quad X[4]=\theta \quad X[5]=\theta'=\frac{d\theta}{d\eta}$

Differential equation set becomes:

$\frac{dX[1]}{dX[0]}=X[2] \quad \frac{dX[2]}{dX[0]}=X[3] \quad \frac{dX[3]}{dX[0]}=-X[1]*X[3]+\beta*(X[2]*X[2]-1) \quad \frac{dX[4]}{dX[0]}=X[5]$

$\frac{dX[5]}{dX[0]}=-Pr*X[1]*X[5]+(2-\beta)*Pr*\gamma*X[2]*X[4]$

```
import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;
class fb extends f_xr
{
                public double func(double x,int i)
                { double y=Math.pow(x,0.25);
    double xx=1.0;
     if(i==0) xx=1;
     else if(i==1) xx=x;
      else if(i==2) xx=x*x;
       else if(i==3) xx=x*x*x;
        else if(i==4) xx=x*x*x*x;
         else if(i==5) xx=x*x*x*x*x;
          else if(i==6) xx=x*x*x*x*x*x;
           else if(i==7) xx=x*x*x*x*x*x*x;
            else if(i==8) xx=x*x*x*x*x*x*x*x;
             else if(i==9) xx=x*x*x*x*x*x*x*x*x;
              else if(i==10) xx=x*x*x*x*x*x*x*x*x*x*x;
  return xx;
                }
}
class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}

public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}}

class fa extends fi_xi
{ // Nonlinear system of equation
double y[];
fm2 b3;
double w;
double a[][];
public fa(double Pr,double beta,double gamma,double yi[],double wi)
{ b3=new fm2(Pr,beta,gamma);
  y=new double[yi.length];
  for(int i=0;i<y.length;i++) y[i]=yi[i];
  w=wi;
}

public double[] func(double x[])
{
// shooting method
y[2]=x[0];
y[4]=x[1];
```

```
a=RK6(b3,0.0,w,y,10000);
double ff[]=new double[2];
ff[0]=a[2][10000]-1.0;
ff[1]=a[4][10000]-1.0;
return ff;
}
 //differenetial equation solution
public static double[][] RK6(f_xi fp,double x0,double xn,double f0[],int N)
{
//6th order Runge Kutta Method
//fp  : given set of derivative functions dfj/dxi(fj,x)
// xo : initial value of the independent variable
// xn : final value of the independent variable
// f0 : initial value of the dependent variable
// N  : number of dependent variable to be calculated
// fi : dependent variable
double h=(xn-x0)/N;
int M=f0.length;
double fi[][];
fi=new double[M][N+1];
double xi[]=new double[M+1];
double k[]=new double[6];
int i,j;
double x;
for(j=0;j<M;j++)
  {
  fi[j][0]=f0[j];
  xi[j+1]=f0[j];
  }
for(x=x0,i=0;i<N;x+=h,i++)
  {
  for(j=1;j<=M;j++)
    {
    xi[0]=x;
    xi[j]=fi[j-1][i];
    k[0]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/2;
    k[1]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/4.0+k[1]/4.0;
    k[2]=h*fp.func(xi,j-1);
    xi[0]=x+h;
    xi[j]=fi[j-1][i]-k[1]+2.0*k[2];
    k[3]=h*fp.func(xi,j-1);
    xi[0]=x+2.0/3.0*h;
    xi[j]=fi[j-1][i]+7.0/27.0*k[0]+10.0/27.0*k[1]+1.0/27.0*k[3];
    k[4]=h*fp.func(xi,j-1);
    xi[0]=x+1.0/5.0*h;
    xi[j]=fi[j-1][i]+28.0/625.0*k[0]-1.0/5.0*k[1]+546.0/625.0*k[2]+54.0/625.0*k[3]-378/625.0*k[4];
    k[5]=h*fp.func(xi,j-1);
    fi[j-1][i+1]=fi[j-1][i]+k[0]/24.0+5.0*k[3]/48.0+27.0*k[4]/56.0+125.0*k[5]/336.0;
    xi[j]=fi[j-1][i];
    }
  }
double a[][]=new double[M+1][N+1];
for(x=x0,i=0;i<=N;x+=h,i++)
{
  a[0][i]=x;
  for(j=1;j<=M;j++)
  {
  a[j][i]=fi[j-1][i];
  }
}
return a;
}
}

//Blassius flat plate differential equation
class fm2 extends f_xi
{ double Pr;
  double beta,gamma;
  fm2(double Pri,double betai,double gammai) {Pr=Pri;beta=betai;gamma=gammai;}
// multivariable function
 double func(double x[],int x_ref)
 {
```

```
   //Falkner - Skan diferential equation
   //f"'+0.5*ff"-beta*(f'^2-1) = 0
   //teta"+Pr*f*teta'-(2-beta)Pr*gamma*teta=0
   double a=0;
   if(x_ref==0) a= x[2];
   else if(x_ref==1) a= x[3];
   else if(x_ref==2) a= -x[1]*x[3]+beta*(x[2]*x[2]-1.0);
   else if(x_ref==3) a= x[5];
   else if(x_ref==4) a= -Pr*x[1]*x[5]+(2.0-beta)*Pr*gamma*x[2]*x[4];
   return a;
 }
}

public class falkner_skan2
{
                public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
public static String toString(double x[])
{ String s="";
int n=x.length;
  for(int k=0;k<n;k++)
  {s+=x[k]+" ";}
  return s;
}
public static double[] nelder(f_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
                int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
   {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
   }


   // Inlet variable definitions
   // fnelder : abstract multivariable function f(x)
   // x : independent variable set of n+1 simplex elements
   // maxiteration : maximum iteration number
   // tolerance :
        int NDIMS = x.length-1;
        int NPTS = x.length;
        int FUNC = NDIMS;
        int ncalls = 0;
          ////// construct the starting simplex ///////////////////
          //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
          double z[]=new double[NDIMS];
          double best = 1E99;
          ///////////////// calculate the first function values for the simplex /////////////////

          int iter=0;

          for (iter=1; iter<maxiteration; iter++)
          {
                        ////////////  define lo, nhi, hi (low high next_to_high ///////////////
             int  ilo=0, ihi=0, inhi = -1; // -1 means missing
             double flo = p[0][FUNC];
             double fhi = flo;
             double pavg,sterr;
             for (i=1; i<NPTS; i++)
```

```
{
  if (p[i][FUNC] < flo)
    {flo=p[i][FUNC]; ilo=i;}
  if (p[i][FUNC] > fhi)
    {fhi=p[i][FUNC]; ihi=i;}
}
double fnhi = flo;
inhi = ilo;
for (i=0; i<NPTS; i++)
 if ((i != ihi) && (p[i][FUNC] > fnhi))
   {fnhi=p[i][FUNC]; inhi=i;}
////////// exit criteria //////////////
if ((iter % 4*NDIMS) == 0)
{
   // calculate the avarage (including maximum value)
   pavg=0;
   for(i=0;i<NPTS;i++)
     pavg+=p[i][FUNC];
   pavg/=NPTS;
   double tot=0;
   if(printlist!=0)
   {  System.out.print(iter);
      for (j=0; j<=NDIMS; j++)
         { System.out.print(p[ilo][j]+" ");}
      System.out.println("");
   }
   for(i=0;i<NPTS;i++)
   { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
   sterr=Math.sqrt(tot/NPTS);
   //if(sterr < tolerance)
     { for (j=0; j<NDIMS; j++)
          { z[j]=p[ilo][j];}
       //break;
     }
   best = p[ilo][FUNC];
}

///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
 ave[j] = 0;
for (i=0; i<NPTS; i++)
 if (i != ihi)
   for (j=0; j<NDIMS; j++)
     ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
  ave[j] /= (NPTS-1);


///////// reflect ////////////////

double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
 r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
   for (j=0; j<NDIMS; j++)
     p[ihi][j] = r[j];
   p[ihi][FUNC] = fr;
   continue;
}

if (fr < flo)  //// expand
{
   double e[] = new double[NDIMS];
   for (j=0; j<NDIMS; j++)
     e[j] = 3*ave[j] - 2*p[ihi][j];
   double fe = fnelder.func(e);
   if (fe < fr)
   {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = e[j];
      p[ihi][FUNC] = fe;
```

```
                continue;
              }
              else
              {
                for (j=0; j<NDIMS; j++)
                  p[ihi][j] = r[j];
                p[ihi][FUNC] = fr;
                continue;
              }
            }

            ////////////// shrink:

            if (fr < fhi)
            {
              double c[] = new double[NDIMS];
              for (j=0; j<NDIMS; j++)
                c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
              double fc = fnelder.func(c);
              if (fc <= fr)
              {
                for (j=0; j<NDIMS; j++)
                  p[ihi][j] = c[j];
                p[ihi][FUNC] = fc;
                continue;
              }
              else   /////// daralt
              {
                for (i=0; i<NPTS; i++)
                  if (i != ilo)
                  {
                    for (j=0; j<NDIMS; j++)
                      p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                    p[i][FUNC] = fnelder.func(p[i]);
                  }
                continue;
              }
            }

            if (fr >= fhi)   ///
            {
              double cc[] = new double[NDIMS];
              for (j=0; j<NDIMS; j++)
                cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
              double fcc = fnelder.func(cc);
              if (fcc < fhi)
              {
                for (j=0; j<NDIMS; j++)
                  p[ihi][j] = cc[j];
                p[ihi][FUNC] = fcc;
                continue;
              }
              else    /////////
              {
                for (i=0; i<NPTS; i++)
                  if (i != ilo)
                  {
                    for (j=0; j<NDIMS; j++)
                      p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                    p[i][FUNC] = fnelder.func(p[i]);
                  }
              }
            }
          }
       return z;
}
    public static double[] nelder(f_xj fnelder,double a[],double da[],double tolerance)
    {return nelder(fnelder,a,da,500,tolerance,0);}

    public static double[] nelder(f_xj fnelder,double a[],double da[])
    {return nelder(fnelder,a,da,500,1.0e-10,0);}

    public static double[] nelder(f_xj fnelder,double a[])
    {
                    double [] da=new double[a.length];
          for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
```

```
            return nelder(fnelder,a,da);
        }
// Polynomial least square
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[] PolynomialLSQ(double xi[],double yi[],int n)
{ //Polynomial least square
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  for(j=0;j<n+1;j++)
     {if(i==0 && j==0) A[i][j]=l;
     else  for(k=0;k<l;k++) A[i][j] += Math.pow(xi[k],(i+j));
     }
     for(k=0;k<l;k++) { if(i==0)  B[i]+= yi[k];
                   else      B[i] += Math.pow(xi[k],i)*yi[k];}
  }
X=gausswithpartialpivot(A,B);
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
```

```
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
return X;
}

public static double funcPolynomialLSQ(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcPolynomialLSQ(e,x[k]))*(y[k]-funcPolynomialLSQ(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcPolynomialLSQ(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
                //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                 int n=xi.length;
                int nn=(n-1)*(aradegersayisi+1)+1;
                double z[][]=new double[2][nn];
   double E[]=PolynomialLSQ(xi,yi,polinomkatsayisi);
   System.out.println("katsayilar :\n"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
     {z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;
     for(int j=0;j<aradegersayisi;j++)
     {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
     z[0][k]=z[0][k-1]+dx;z[1][k]=funcPolynomialLSQ(E,z[0][k]);k++;}
     }
     z[0][k]=xi[i];z[1][k]=funcPolynomialLSQ(E,z[0][k]);
   return z;
 }
//general least square curve fitting

public static double[] GeneralLeastSquare(double xi[],double yi[],int n)
{
fb f=new fb();
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  B[i]=0;
                for(j=0;j<np1;j++)
    {
                A[i][j]=0.0;
    for(k=0;k<l;k++) A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);
    }
```

```
    for(k=0;k<l;k++) B[i]+= f.func(xi[k],i)*yi[k];
  }
//System.out.println("A = \n"+Matrix.toString(A));
//System.out.println("B = \n"+Matrix.toStringT(B));
X=gausswithpartialpivot(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
Text.printT(X);
return X;
}

public static double funcGeneralLeastSquare(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
fb f=new fb();
int n=e.length;
double ff=0;
if(n!=0.0)
  {
  for(int i=n-1;i>=0;i--)
   {ff+=e[i]*f.func(x,i);}
  }
return ff;
}

public static double hata(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcGeneralLeastSquare(e,x[k]))*(y[k]-funcGeneralLeastSquare(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}
 public static double[][]  funcGeneralLeastSquare(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {  // aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
                // ara değer sayisi 2 dir
                 int n=xi.length;
                int nn=(n-1)*(aradegersayisi+1)+1;
                double z[][]=new double[2][nn];
    double E[]=GeneralLeastSquare(xi,yi,polinomkatsayisi);
    double dx=0;
    int k=0;
    int i;
    for(i=0;i<(n-1);i++)
       {z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;
       for(int j=0;j<aradegersayisi;j++)
       {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;}
       }
       z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);
    return z;
 }
public static void main(String arg[]) throws IOException
{
  double y[]={0,0,0.1,0,0.1};
 double x[]={0.33,0.0173};//inputdata(s);
 fa ff;
 f2 f;
 double r1[];
 double beta=0.1;
 double gamma=0.0;
 double Pr1[]={0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,0.01};
 double w1[]={200,200,200,200,200,200,200,200,200,100.0};
 double Pr2[]={0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0.2,0.3,0.4,0.5};
 double w2[]={100.0,100.0,100,100,100,100,100,100,100,100,50,50,50,50,50};
```

```
 double
Pr3[]={0.5,0.6,0.7,0.8,0.9,1.0,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9,2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0,10.0,11,12,13,14,15.0,30.0,40.0,50};
 double w3[]={50,50,50,50,25,25,25,25,25,25,25,25,10,10,10,10,10,10,10,10,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7};
 double Pr4[]={50,60,70,80,90,100,110.0,120.0,130.0,140,150,160,170,180,190,200,300,400,500,600,700,800,900,1000};
 double w4[]={5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3,5.3};
 double xi1[]=new double[Pr1.length];
 double yi1[]=new double[Pr1.length];
 double Xi1[]=new double[Pr1.length];
 double Yi1[]=new double[Pr1.length];
 double xi2[]=new double[Pr2.length];
 double yi2[]=new double[Pr2.length];
 double Xi2[]=new double[Pr2.length];
 double Yi2[]=new double[Pr2.length];
 double xi3[]=new double[Pr3.length];
 double yi3[]=new double[Pr3.length];
 double Xi3[]=new double[Pr3.length];
 double Yi3[]=new double[Pr3.length];
 double xi4[]=new double[Pr4.length];
 double yi4[]=new double[Pr4.length];
 double Xi4[]=new double[Pr4.length];
 double Yi4[]=new double[Pr4.length];
 double b1[];
 double b2[];
 double b3[];
 double b4[];
 for(int i=0;i<Pr1.length;i++)
 {ff=new fa(Pr1[i],beta,gamma,y,w1[i]*beta);
  f=new f2(ff);
  r1=nelder(f,x);
  xi1[i]=Pr1[i];
  yi1[i]=r1[1];
  Xi1[i]=Math.log(Pr1[i]);
  Yi1[i]=Math.log(r1[1]);
  System.out.println("i="+i+"Pr="+xi1[i]+" Nux/Rex^0.5="+yi1[i]);
  }
  for(int i=0;i<Pr2.length;i++)
  {ff=new fa(Pr2[i],beta,gamma,y,w2[i]*beta);
  f=new f2(ff);
  r1=nelder(f,x);
  xi2[i]=Pr2[i];
  yi2[i]=r1[1];
  Xi2[i]=Math.log(Pr2[i]);
  Yi2[i]=Math.log(r1[1]);
  System.out.println("i = "+i+" Pr= "+xi2[i]+" Nux/Rex^0.5="+yi2[i]);
  }
  for(int i=0;i<Pr3.length;i++)
  {ff=new fa(Pr3[i],beta,gamma,y,w3[i]*beta);
  f=new f2(ff);
  r1=nelder(f,x);
  xi3[i]=Pr3[i];
  yi3[i]=r1[1];
  Xi3[i]=Math.log(Pr3[i]);
  Yi3[i]=Math.log(r1[1]);
  System.out.println("i = "+i+"Pr = "+xi3[i]+" Nux/Rex^0.5="+yi3[i]);
  }
  for(int i=0;i<Pr4.length;i++)
  {ff=new fa(Pr4[i],beta,gamma,y,w4[i]*beta);
  f=new f2(ff);
  r1=nelder(f,x);
  xi4[i]=Pr4[i];
  yi4[i]=r1[1];
  Xi4[i]=Math.log(Pr4[i]);
  Yi4[i]=Math.log(r1[1]);
  System.out.println("i = "+i+" Pr="+xi4[i]+" Nux/Rex^0.5="+yi4[i]);
  }
  b1=GeneralLeastSquare(Xi1,Yi1,1);
  System.out.println("0.001<Pr<0.01 a="+Math.exp(b1[0])+" b="+b1[1]);
  b2=GeneralLeastSquare(Xi2,Yi2,1);
  System.out.println("0.01<Pr<0.5 a="+Math.exp(b2[0])+" b="+b2[1]);
  b3=GeneralLeastSquare(Xi3,Yi3,1);
  System.out.println("0.5<Pr<50 a="+Math.exp(b3[0])+" b="+b3[1]);
  b4=GeneralLeastSquare(Xi4,Yi4,1);
  System.out.println("50<Pr<1000 a="+Math.exp(b4[0])+" b="+b4[1]);
}
 }
```

```
---------- Capture Output ----------
> "D:\co\java\bin\java.exe" falkner_skan2
i=0Pr=0.001 Nux/Rex^0.5=0.052838585958506955
i=1Pr=0.002 Nux/Rex^0.5=0.05568439004805982
i=2Pr=0.003 Nux/Rex^0.5=0.058528451086160366
i=3Pr=0.004 Nux/Rex^0.5=0.06136255984188095
i=4Pr=0.005 Nux/Rex^0.5=0.06417932292947877
i=5Pr=0.006 Nux/Rex^0.5=0.0669721969228364
i=6Pr=0.007 Nux/Rex^0.5=0.0697354959029941
i=7Pr=0.008 Nux/Rex^0.5=0.07246437637025133
i=8Pr=0.009 Nux/Rex^0.5=0.07515480379431305
i=9Pr=0.01 Nux/Rex^0.5=0.11205588920743956
i = 0 Pr= 0.01 Nux/Rex^0.5=0.11205588920743956
i = 1 Pr= 0.02 Nux/Rex^0.5=0.12395151866250631
i = 2 Pr= 0.03 Nux/Rex^0.5=0.13553863140044367
i = 3 Pr= 0.04 Nux/Rex^0.5=0.14671316479801308
i = 4 Pr= 0.05 Nux/Rex^0.5=0.15741124821744934
i = 5 Pr= 0.06 Nux/Rex^0.5=0.16760215238090806
i = 6 Pr= 0.07 Nux/Rex^0.5=0.1772802936278064
i = 7 Pr= 0.08 Nux/Rex^0.5=0.18645770511575543
i = 8 Pr= 0.09 Nux/Rex^0.5=0.19515771000451182
i = 9 Pr= 0.1 Nux/Rex^0.5=0.20341002695454757
i = 10 Pr= 0.2 Nux/Rex^0.5=0.2841350963165023
i = 11 Pr= 0.3 Nux/Rex^0.5=0.32059954900699383
i = 12 Pr= 0.4 Nux/Rex^0.5=0.3530598178408308
i = 13 Pr= 0.5 Nux/Rex^0.5=0.3819538367853976
i = 0Pr = 0.5 Nux/Rex^0.5=0.3819538367853976
i = 1Pr = 0.6 Nux/Rex^0.5=0.4078354022748689
i = 2Pr = 0.7 Nux/Rex^0.5=0.43122731650764307
i = 3Pr = 0.8 Nux/Rex^0.5=0.45257066060822737
i = 4Pr = 0.9 Nux/Rex^0.5=0.5226234080781739
i = 5Pr = 1.0 Nux/Rex^0.5=0.5351747910351455
i = 6Pr = 1.1 Nux/Rex^0.5=0.5474862124668214
i = 7Pr = 1.2 Nux/Rex^0.5=0.5595561046378349
i = 8Pr = 1.3 Nux/Rex^0.5=0.5713842346231784
i = 9Pr = 1.4 Nux/Rex^0.5=0.5829715406219594
i = 10Pr = 1.5 Nux/Rex^0.5=0.5943199765440061
i = 11Pr = 1.6 Nux/Rex^0.5=0.605432366337465
i = 12Pr = 1.7 Nux/Rex^0.5=0.6163122690177371
i = 13Pr = 1.8 Nux/Rex^0.5=1.07843526057971
i = 14Pr = 1.9 Nux/Rex^0.5=1.0827301489808536
i = 15Pr = 2.0 Nux/Rex^0.5=1.087017850234281
i = 16Pr = 3.0 Nux/Rex^0.5=1.1294826213403983
i = 17Pr = 4.0 Nux/Rex^0.5=1.1711499568140122
i = 18Pr = 5.0 Nux/Rex^0.5=1.211963030834299
i = 19Pr = 6.0 Nux/Rex^0.5=1.2518790389776722
i = 20Pr = 7.0 Nux/Rex^0.5=1.2908679003183552
i = 21Pr = 8.0 Nux/Rex^0.5=1.3289108735798996
i = 22Pr = 9.0 Nux/Rex^0.5=1.6894698550922782
i = 23Pr = 10.0 Nux/Rex^0.5=1.7171634489663514
i = 24Pr = 11.0 Nux/Rex^0.5=1.744566745589914
i = 25Pr = 12.0 Nux/Rex^0.5=1.7716743069707244
i = 26Pr = 13.0 Nux/Rex^0.5=1.7984816070930691
i = 27Pr = 14.0 Nux/Rex^0.5=1.8249849823816064
i = 28Pr = 15.0 Nux/Rex^0.5=1.851181581603489
i = 29Pr = 30.0 Nux/Rex^0.5=2.2073023003762247
i = 30Pr = 40.0 Nux/Rex^0.5=2.409227460829409
i = 31Pr = 50.0 Nux/Rex^0.5=2.5876756067443543
i = 0 Pr=50.0 Nux/Rex^0.5=2.866196403058497
i = 1 Pr=60.0 Nux/Rex^0.5=3.0245877351786525
i = 2 Pr=70.0 Nux/Rex^0.5=3.1718665682913283
i = 3 Pr=80.0 Nux/Rex^0.5=3.3091378766342614
i = 4 Pr=90.0 Nux/Rex^0.5=3.437487478797009
i = 5 Pr=100.0 Nux/Rex^0.5=3.55792449115799
i = 6 Pr=110.0 Nux/Rex^0.5=3.6713556427982894
i = 7 Pr=120.0 Nux/Rex^0.5=3.7785789246436887
i = 8 Pr=130.0 Nux/Rex^0.5=3.8802879432061186
i = 9 Pr=140.0 Nux/Rex^0.5=3.977081466292545
i = 10 Pr=150.0 Nux/Rex^0.5=4.06947486723898
i = 11 Pr=160.0 Nux/Rex^0.5=4.157911638100899
i = 12 Pr=170.0 Nux/Rex^0.5=4.242774052422481
i = 13 Pr=180.0 Nux/Rex^0.5=4.324392594357212
i = 14 Pr=190.0 Nux/Rex^0.5=4.403054068457771
i = 15 Pr=200.0 Nux/Rex^0.5=4.479008456728227
```

```
i = 16 Pr=300.0 Nux/Rex^0.5=5.127398066520598
i = 17 Pr=400.0 Nux/Rex^0.5=5.643467206121851
i = 18 Pr=500.0 Nux/Rex^0.5=6.079180893347311
i = 19 Pr=600.0 Nux/Rex^0.5=6.459981724270481
i = 20 Pr=700.0 Nux/Rex^0.5=6.800451316074847
i = 21 Pr=800.0 Nux/Rex^0.5=7.109816763950922
i = 22 Pr=900.0 Nux/Rex^0.5=7.394330723521469
i = 23 Pr=1000.0 Nux/Rex^0.5=7.658449093385441
0.001<Pr<0.01 a=0.23533274210000105 b=0.23167892227279793
0.01<Pr<0.5 a=0.46410108309973847 b=0.34259990608566376
0.5<Pr<50 a=0.5649974547131759 b=0.43945107376090803
50<Pr<1000 a=0.7763291971873808 b=0.3310579556701941
```

For $\beta = 0.1$ opening angle =0.314159 radian = 18 degree $\gamma = 0$ (isothermal plate)

$Nu = \mathbf{0.23533274210000105}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.23167892227279793}} \quad 0.001 \leq Pr \leq 0.01$

$Nu = \mathbf{0.46410108309973847}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.34259990608566376}} \quad 0.01 \leq Pr \leq 0.5$

$Nu = \mathbf{0.5649974547131759}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.43945107376090803}} \quad 0.5 \leq Pr \leq 50$

$Nu = \mathbf{0.7763291971873808}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.3310579556701941}} \quad 0.5 \leq Pr \leq 50$

```
---------- Capture Output ----------
> "D:\co\java\bin\java.exe" falkner_skan2
i=0Pr=0.001 Nux/Rex^0.5=0.3335480441892363
i=1Pr=0.002 Nux/Rex^0.5=0.333762732229488
i=2Pr=0.003 Nux/Rex^0.5=0.3339773973373252
i=3Pr=0.004 Nux/Rex^0.5=0.3341920393961866
i=4Pr=0.005 Nux/Rex^0.5=0.3344066582896734
i=5Pr=0.006 Nux/Rex^0.5=0.33462125390162084
i=6Pr=0.007 Nux/Rex^0.5=0.3348358261160087
i=7Pr=0.008 Nux/Rex^0.5=0.33505037481704425
i=8Pr=0.009 Nux/Rex^0.5=0.3352648998890954
i=9Pr=0.01 Nux/Rex^0.5=0.33547940121673925
i = 0 Pr= 0.01 Nux/Rex^0.5=0.253421061894859
i = 1 Pr= 0.02 Nux/Rex^0.5=0.2568358547768611
i = 2 Pr= 0.03 Nux/Rex^0.5=0.2602435827426917
i = 3 Pr= 0.04 Nux/Rex^0.5=0.26364347566404794
i = 4 Pr= 0.05 Nux/Rex^0.5=0.2670347895158677
i = 5 Pr= 0.06 Nux/Rex^0.5=0.27041680663249856
i = 6 Pr= 0.07 Nux/Rex^0.5=0.27378883589400793
i = 7 Pr= 0.08 Nux/Rex^0.5=0.2771502128452698
i = 8 Pr= 0.09 Nux/Rex^0.5=0.28050029975043916
i = 9 Pr= 0.1 Nux/Rex^0.5=0.28383848558575353
i = 10 Pr= 0.2 Nux/Rex^0.5=0.316450169918763
i = 11 Pr= 0.3 Nux/Rex^0.5=0.34738929942134955
i = 12 Pr= 0.4 Nux/Rex^0.5=0.376435621274995
i = 13 Pr= 0.5 Nux/Rex^0.5=0.40353923080931065
i = 0Pr = 0.5 Nux/Rex^0.5=0.3967777602996435
i = 1Pr = 0.6 Nux/Rex^0.5=0.4245252151748051
i = 2Pr = 0.7 Nux/Rex^0.5=0.4495786853586725
i = 3Pr = 0.8 Nux/Rex^0.5=0.4724305505302953
i = 4Pr = 0.9 Nux/Rex^0.5=0.5357643115864267
i = 5Pr = 1.0 Nux/Rex^0.5=0.5495806291641702
i = 6Pr = 1.1 Nux/Rex^0.5=0.5631114536497109
i = 7Pr = 1.2 Nux/Rex^0.5=0.5763553971849682
i = 8Pr = 1.3 Nux/Rex^0.5=0.5893128347762455
i = 9Pr = 1.4 Nux/Rex^0.5=0.6019856624409455
i = 10Pr = 1.5 Nux/Rex^0.5=0.6143770703546839
```

i = 11Pr = 1.6 Nux/Rex^0.5=0.6264913330571373
i = 12Pr = 1.7 Nux/Rex^0.5=0.6383336178903841
i = 13Pr = 1.8 Nux/Rex^0.5=1.080860506314767
i = 14Pr = 1.9 Nux/Rex^0.5=1.085287745456006
i = 15Pr = 2.0 Nux/Rex^0.5=1.089707490793113
i = 16Pr = 3.0 Nux/Rex^0.5=1.1334742878878064
i = 17Pr = 4.0 Nux/Rex^0.5=1.1764058811902651
i = 18Pr = 5.0 Nux/Rex^0.5=1.2184405505728213
i = 19Pr = 6.0 Nux/Rex^0.5=1.2595320622144668
i = 20Pr = 7.0 Nux/Rex^0.5=1.2996482023212421
i = 21Pr = 8.0 Nux/Rex^0.5=1.338769212589115
i = 22Pr = 9.0 Nux/Rex^0.5=1.6935839761605147
i = 23Pr = 10.0 Nux/Rex^0.5=1.7217037719289932
i = 24Pr = 11.0 Nux/Rex^0.5=1.749525714515282
i = 25Pr = 12.0 Nux/Rex^0.5=1.777044154610579
i = 26Pr = 13.0 Nux/Rex^0.5=1.804254401475994
i = 27Pr = 14.0 Nux/Rex^0.5=1.8311526701643182
i = 28Pr = 15.0 Nux/Rex^0.5=1.8577360281534534
i = 29Pr = 30.0 Nux/Rex^0.5=2.218713821164285
i = 30Pr = 40.0 Nux/Rex^0.5=2.4230524425125015
i = 31Pr = 50.0 Nux/Rex^0.5=2.603460669031582
i = 0 Pr=50.0 Nux/Rex^0.5=2.8746451804241273
i = 1 Pr=60.0 Nux/Rex^0.5=3.0341867308060273
i = 2 Pr=70.0 Nux/Rex^0.5=3.182477947059634
i = 3 Pr=80.0 Nux/Rex^0.5=3.3206487477818905
i = 4 Pr=90.0 Nux/Rex^0.5=3.4498065393990984
i = 5 Pr=100.0 Nux/Rex^0.5=3.5709781453825187
i = 6 Pr=110.0 Nux/Rex^0.5=3.6850843511054094
i = 7 Pr=120.0 Nux/Rex^0.5=3.7929340944342878
i = 8 Pr=130.0 Nux/Rex^0.5=3.89522942779183
i = 9 Pr=140.0 Nux/Rex^0.5=3.9925756225433635
i = 10 Pr=150.0 Nux/Rex^0.5=4.0854930792072555
i = 11 Pr=160.0 Nux/Rex^0.5=4.174429208393844
i = 12 Pr=170.0 Nux/Rex^0.5=4.259769374018717
i = 13 Pr=180.0 Nux/Rex^0.5=4.341846531713771
i = 14 Pr=190.0 Nux/Rex^0.5=4.420949492777605
i = 15 Pr=200.0 Nux/Rex^0.5=4.497329894013379
i = 16 Pr=300.0 Nux/Rex^0.5=5.14935646315043
i = 17 Pr=400.0 Nux/Rex^0.5=5.668325369270466
i = 18 Pr=500.0 Nux/Rex^0.5=6.1064900529916315
i = 19 Pr=600.0 Nux/Rex^0.5=6.489434384478669
i = 20 Pr=700.0 Nux/Rex^0.5=6.831821245536183
i = 21 Pr=800.0 Nux/Rex^0.5=7.1429292829709565
i = 22 Pr=900.0 Nux/Rex^0.5=7.429046147303017
i = 23 Pr=1000.0 Nux/Rex^0.5=7.694652708450487
0.001<Pr<0.01 a=0.33909842208904095 b=0.002522323431681117
0.01<Pr<0.5 a=0.3970088423802754 b=0.12179317628513694
0.5<Pr<50 a=0.5808370023452061 b=0.4303534940489292
50<Pr<1000 a=0.7772410241736414 b=0.3315896030668632

For $\beta = 0.25$ opening angle =0785398 radian = 45 degree $\gamma = 0$ (isothermal plate)

$Nu = \mathbf{0.33909842208904095}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.002522323431681117}} \quad 0.001 \leq Pr \leq 0.01$

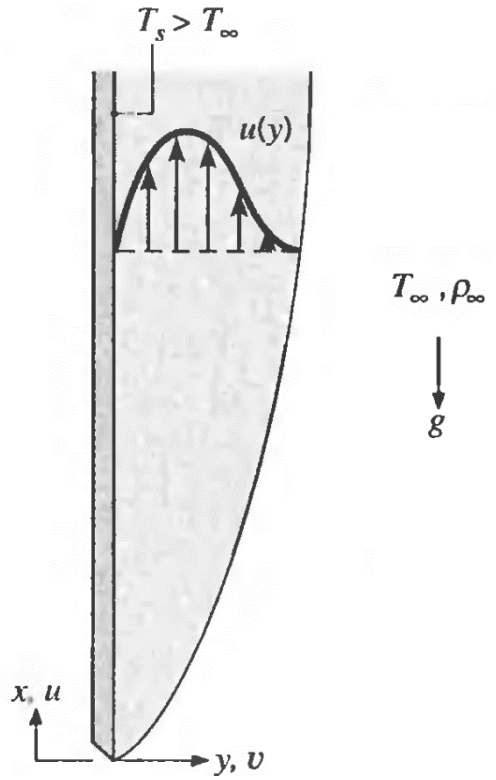
$Nu = \mathbf{0.3970088423802754}\mathrm{Re}^{0.5} Pr^{\mathbf{0.12179317628513694}} \quad 0.01 \leq Pr \leq 0.5$

$Nu = \mathbf{0.5808370023452061}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.4303534940489292}} \quad 0.5 \leq Pr \leq 50$

$Nu = \mathbf{0.7772410241736414}\ \mathrm{Re}^{0.5} Pr^{\mathbf{0.3315896030668632}} \quad 0.5 \leq Pr \leq 50$

EXAMPLE 3 : NATURAL CONVECTION SIMILARITY SOLUTION

The equations governing external free convection are essentially the same as those for forced convection. A boundary layer for free convection of a vertical flow is shown in the figure. Velocity in x direction is assumed to be zero at the wall and again is zero out of boundary layer.



Conservation of mass:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Conservation of momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) - \frac{dP}{dx} - \rho g$$

Conservation of energy

$$\rho u C_p \frac{\partial T}{\partial x} + \rho v C_p \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right)$$

The boundary conditions are

u=0, v=0, $T=T_s(x)$ at y=0

$u = 0$, $T = T_\infty$ at x=0

$u = 0$, $T = T_\infty$ as $y \to \infty$

Outside of the boundary layer, the only hydrostatic terms of the boundary layer will be left so:

$\frac{dP}{dx} = -\rho_\infty g$ if this substituted the previous momentum equation it becomes:

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+(\rho_\infty-\rho)g$$

Furthermore, if Definition of the thermodynamic property of thermal expansion coefficient is taken into consideration:

$$\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P\approx\frac{1}{\rho}\left(\frac{\rho_\infty-\rho}{T_\infty-T}\right)$$

Equation becomes:

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+g\beta(T-T_\infty)$$

Boundary layer equations can be converted by using stream function concept. Let

$$u=\frac{\partial\varphi}{\partial y}\quad v=-\frac{\partial\varphi}{\partial x}\quad\theta=\frac{T-T_\infty}{T_s-T_\infty}$$

Then equations become:

$$\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial y^2}=\nu\frac{\partial^3\varphi}{\partial y^3}+g\beta(T_s-T_\infty)\theta$$

$$\frac{\partial\varphi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\varphi}{\partial x}\frac{\partial\theta}{\partial y}=\alpha\frac{\partial^2\theta}{\partial y^2}+\theta\frac{\partial\varphi}{\partial y}\frac{dT_s}{dx}$$

A similarity transformation will be applied to this qauations in the form of

$\eta=yH(x)$ where $H(x)=\frac{1}{x}\left(\frac{1}{4}Gr_x\right)^{1/4}$ so $\eta=\frac{y}{x}\left(\frac{1}{4}Gr_x\right)^{1/4}$

And a separation of variable can be applied to the equations in the form of

$\varphi(x,\eta)=\upsilon F(\eta)G(x)$ where

$$G(x)=4\left(\frac{1}{4}Gr_x\right)^{1/4}$$

Where $Gr_x=\frac{g\beta(T_s(x)-T_\infty)}{\nu^2}$ is the Garshoff number. For the case of $T_s(x)$=constant, the equation becomes:

$$F'''(\eta)+3F(\eta)F'(\eta)-2\left(F'(\eta)\right)^2+\theta(\eta)=0$$

$$\theta'''(\eta)+3\Pr F(\eta)\theta'(\eta)=0$$

The boundary conditions on these equations

u=0 at y=0	F’(0)=0
v=0 ay y=0	F(0)=0
u=0 $y\to\infty$	$F'(\infty)=0$
$T=T_s$ y=0	$\theta(0)=1$
$T\to T_\infty\quad y\to\infty$	$\theta(\infty)=0$

This set of differential equation can only be solved by using numerical methods. Furthermore, it should be solved for different Prandtl number. In order to obtain thermal conductivity coefficient or Nussel number from the numerical solutions, following convertions can be carried out.

$$h_x = \frac{q_s''}{T_s - T_\infty} \qquad q_s'' = -k\left.\frac{dT}{dy}\right|_{y=0} = -k(T_s - T_\infty)\left(\frac{\partial\theta}{\partial\eta}\frac{\partial\eta}{\partial y}\right)_0$$

$q_s'' = -k(T_s - T_\infty)\theta'(0)H(x)$ Then the local Nusselt number will be defined as:

$Nu_x = \frac{xh_x}{k} = -\theta'(0)xH(x) = -\frac{\theta'(0)}{\sqrt{2}}Gr_x^{1/4}$ So we will seek solutions for $\theta'(0)$ for different Pr numbers. Since relation might not be linear curve fitting of the obtained solutions should be applied.

As it is seen from boundary conditions, two given conditions are not initial conditions. One way of solving boundary conditions is using non-linear system of equation solving methods. In order to solve differential equation set 6th order Runge-Kutta Method is used. In order to solve boundary value nonlinear system of equation Nelder & Mead optimization method is used.

Runge-Kutta method as in most general form can be given as: For differential form $\frac{dy}{dx} = f(x, y)$

$y_{n+1} = y_n + h*\phi(x_i,y_i,h)$

$$\phi(x_1, y_1, h) = \sum_{i=1}^{n} b_i k_i$$

In this equation a is constant values depends on polynamial degree of Runge-Kutta equation and coefficients k, can be defined as:

$k_1=f(x_i,y_i)$

$k_2=f(x_i+c_2h,y_i+a_{21}k_1h)$

$k_3=f(x_i+c_3h,y_i+a_{31}k_1h+ q_{32}k_2h)$

….

$k_n=f(x_i+c_nh,y_i+a_{n,1}k_1h+ \ldots+ a_{n,n-1}k_{n-1}h)$

In more generalized form, these equations can be written as:

$x_i = x_n + c_i h$ i=1…s

$$y_i = y_n + h\sum_{j=1}^{i-1} a_{ij}k_j$$

$$k_i = f(x_i, y_i)$$

$y_{n+1} = y_n + h\sum_{j=1}^{s} b_i k_i$ where s is the degree of coefficients of equations. In numerical analysis books, coefficients are usually shown as Buthcher tableu which puts the coefficients of the method in a table as follows:

$$\begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & a_{13} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \ldots & a_{2s} \\ c_3 & a_{31} & a_{32} & a_{33} & \ldots & a_{3s} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ c_s & a_{s1} & a_{s2} & a_{s3} & \ldots & a_{s4} \\ \hline & b_1 & b_2 & b_3 & \ldots & b_s \end{array}$$

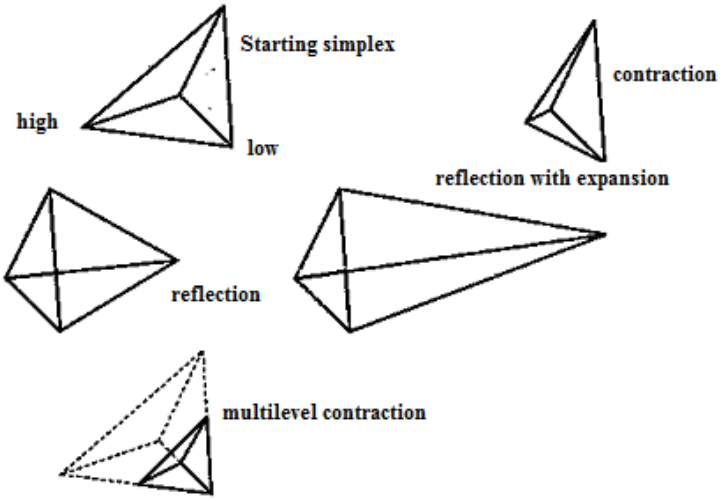
As it is seen from the equation the definition is very general. Depends on polynamial degree of Runge-Kutta equation accuracy will improve. **Runge-Kutta with sixth degree polynomial solution RK6** is given as:

$y_{i+1} = y_i + (1/90)*(7k_1 + 32k_3 +12k_4+32k_5+7k_6)h$
$k_1=f(x_i,y_i)$
$k_2=f(x_i+0.25h , y_i+0.25k_1h)$
$k_3=f(x_i+0.25h , y_i+0.125k_1h+0.125k_2h)$
$k_4=f(x_i+0.5h , y_i – 0.5k_2h+k_3h)$
$k_5=f(x_i+0.75h , y_i + (3/16)k_1h+(9/16)k_4h)$
$k_6=f(x_i+h , y_i - (3/7)k_1h+(2/7)k_2h+(12/7)k_3h - (12/7)k_4h+(8/7)k_5h)$

This equation can be given as Buthcher tableu as:

0	0	0	0	0	0
1/4	1/4	0	0	0	0
1/4	1/8	1/8	0	0	0
2/4	1/4	−1/4	1	0	0
3/4	3/16	0	0	9/16	0
1	−3/7	2/7	12/7	−12/7	8/7
	1/90	7/90	32/90	12/90	7/90

Nelder & Mead optimization method is as follows:
The simplex method developed by Nelder and Mead method is a multidimensional search method . Simplex is an n dimensional geometric entity. It contains (n+1) points in n dimensional space. The method uses the concept of a simplex, which is a special polytop of $N + 1$ vertices in N dimensions. Examples of simplices include a line segment on a line, a triangle on a plane, a tetrahedron in three-dimensional space and so forth. Required processes for the simplex amobea's movements are shown in the figure



Process can be given as follows

- **1. Order** according to the values at the vertices:
  $f(\mathbf{x}_1) \leq f(\mathbf{x}_2) \leq \cdots \leq f(\mathbf{x}_{n+1})$
- **2.** Calculate x_o, the center of gravity of all points except x_{n+1}.
- **3. Reflection**
  Compute reflected point $\mathbf{x}_r = \mathbf{x}_o + \alpha(\mathbf{x}_o - \mathbf{x}_{n+1})$
  If the reflected point is better than the second worst, but not better than the best, i.e.:
  $f(\mathbf{x}_1) \leq f(\mathbf{x}_r) < f(\mathbf{x}_n)$,

then obtain a new simplex by replacing the worst point x_{n+1} with the reflected point x_r, and go to step 1.

- **4. Expansion**

  If the reflected point is the best point so far, $f(\mathbf{x}_r) < f(\mathbf{x}_1),$
  then compute the expanded point $\mathbf{x}_e = \mathbf{x}_o + \gamma(\mathbf{x}_o - \mathbf{x}_{n+1})$
  If the expanded point is better than the reflected point, $f(\mathbf{x}_e) < f(\mathbf{x}_r)$
  then obtain a new simplex by replacing the worst point x_{n+1} with the expanded point x_e, and go to step 1.
  Else obtain a new simplex by replacing the worst point x_{n+1} with the reflected point x_r, and go to step 1.
  Else (i.e. reflected point is worse than second worst) continue at step 5.

- **5. Contraction**

  Here, it is certain that $f(\mathbf{x}_r) \geq f(\mathbf{x}_n)$
  Compute contracted point $\mathbf{x}_c = \mathbf{x}_{n+1} + \rho(\mathbf{x}_o - \mathbf{x}_{n+1})$
  If the contracted point is better than the worst point, i.e. $f(\mathbf{x}_c) \leq f(\mathbf{x}_{n+1})$
  then obtain a new simplex by replacing the worst point x_{n+1} with the contracted point x_c, and go to step 1.
  Else go to step 6.

- **6. Reduction**

  For all but the best point, replace the point with
  $x_i = x_1 + \sigma(x_i - x_1)$ for all i $\in \{2, \ldots, n+1\}$. go to step 1.

Note: $\alpha,\ \gamma,\ \rho$and σ are respectively the reflection, the expansion, the contraction and the shrink coefficient. Standard values are $\alpha = 1$, $\gamma = 2$, $\rho = 1 / 2$ and $\sigma = 1 / 2$.

For the **reflection**, since x_{n+1} is the vertex with the higher associated value among the vertices, we can expect to find a lower value at the reflection of x_{n+1} in the opposite face formed by all vertices point x_i except x_{n+1}.

For the **expansion**, if the reflection point x_r is the new minimum along the vertices we can expect to find interesting values along the direction from x_o to x_r.

Concerning the **contraction**: If $f(x_r) > f(x_n)$ we can expect that a better value will be inside the simplex formed by all the vertices x_i.

The initial simplex is important, indeed, a too small initial simplex can lead to a local search, consequently the NM can get more easily stuck. So this simplex should depend on the nature of the problem. Instead of given n+1 point same process can be done by giving one point and a change vector by defining
$x_{i-1} = P_i + \delta dx_j$ where if i=j δ=1 else if $i \neq j$ δ=0 . For example:
if $P = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$ and $dx = \begin{Bmatrix} da \\ db \\ dc \end{Bmatrix}$

$$x_0 = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \quad x_1 = \begin{Bmatrix} a+da \\ b \\ c \end{Bmatrix} \quad x_2 = \begin{Bmatrix} a \\ b+db \\ c \end{Bmatrix} \quad x_3 = \begin{Bmatrix} a \\ b \\ c+dc \end{Bmatrix}$$

Non-Linear system of equations can also be solved by using optimization methods through an adaptation function.
To solve function $f_i(x_j)=0$

$$g(x_j) = \sum_{i=0}^{n\_equation} f_i(x_j) * f_i(x_j)$$

The derivative of this equation is the root of non-linear system of equation again became $f_i(x_j)=0$, therefore optimum of $g(x_j)$ is the same problem as solution of $f_i(x_j)=0$.
Another program utilized in this set is generalized **least square linear curve fitting**. Assuming having a linear function $f(a_j,x)$ where a_j are m linear coefficient and x are independent variable and $\phi_j(x)$ are m sub-funtions linearly multiplied with the coefficients

$$f(a_j, x) = \sum_{j=0}^{m} a_j^{(m)} \phi_j(x)$$

It should be noted that linearity is only for the coefficients, $\phi_j(x)$ functions does not have to be linear. It is desired to fit data x_i, f_i, i=0..n into the equation so that the difference between data and fitted function dependent values for all points will be minimum. In order to establish this, the following function H will be minimized with respect to a_j

$$H(a_0^{(m)},....,a_m^{(m)}) = \sum_{i=1}^{n} w(x_i) \left[f_i - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x_i) \right]^2$$

Where $w(x_i)$ values in the equation are called weight function. In order to minimiz the function root of the derivative of the function can be calculated.

$$\frac{\partial H(a_0^{(m)},....,a_m^{(m)})}{\partial a_k^{(m)}} = 2\sum_{i=1}^{n} w(x_i) \left[f_i - \sum_{j=0}^{m} a_j^{(m)} \phi_j(x_i) \right] \phi_k(x_i) = 0 \quad k = 0,.....,m$$

$$\left\{ \sum_{j=0}^{m} w(x_i) \phi_j(x_i) \phi_k(x_i) \right\} \left[a_j^{(m)} \right] = \left[\sum_{i=1}^{n} w(x_i) \phi_k(x_i) f_i \right] \quad k = 0,.....,m$$

For weight function to be taken equal to unity , $w(x_i)$ =1, equation in the open form can be written in the following form.

$$\begin{Bmatrix} \sum_{i=1}^{n} \phi_0^2(x_i) & \sum_{i=1}^{n} \phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n} \phi_0(x_i)\phi_2(x_i) & .... & \sum_{i=1}^{n} \phi_0(x_i)\phi_m(x_i) \\ \sum_{i=1}^{n} \phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n} \phi_1^2(x_i) & \sum_{i=1}^{n} \phi_1(x_i)\phi_2(x_i) & .... & \sum_{i=1}^{n} \phi_1(x_i)\phi_m(x_i) \\ \sum_{i=1}^{n} \phi_0(x_i)\phi_2(x_i) & \sum_{i=1}^{n} \phi_1(x_i)\phi_2(x_i) & \sum_{i=1}^{n} \phi_2^2(x_i) & .... & \sum_{i=1}^{n} \phi_2(x_i)\phi_m(x_i) \\ .... & .... & .... & .... & .... \\ \sum_{i=1}^{n} \phi_0(x_i)\phi_m(x_i) & \sum_{i=1}^{n} \phi_1(x_i)\phi_m(x_i) & \sum_{i=1}^{n} \phi_2(x_i)\phi_m(x_i) & .... & \sum_{i=1}^{n} \phi_m^2(x_i) \end{Bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ .... \\ a_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \phi_0(x_i) f(x_i) \\ \sum_{i=1}^{n} \phi_1(x_i) f(x_i) \\ \sum_{i=1}^{n} \phi_2(x_i) f(x_i) \\ .... \\ \sum_{i=1}^{n} \phi_m(x_i) f(x_i) \end{bmatrix}$$

This equation is an m+1 linear system of equation. It can easily be solved by using a system of equation solving method.
Now that basic mathematical calculation methods are established, The equation can be solved by using differential equation boundary value solving method.

```
 import java.util.*;
import java.awt.*;
import java.applet.Applet;
import java.awt.event.*;
import javax.swing.*;
import java.io.*;

class f2 extends f_xj
{//adaptation function
fi_xi ff1;
public f2(fi_xi ffi) {ff1=ffi;}
```

```
public double func(double x[])
{
double ff=0.0;
double fa[]=ff1.func(x);
for(int i=0;i<fa.length;i++)  ff+=fa[i]*fa[i];
return ff;
}
}

class fa extends fi_xi
{ // Nonlinear system of equation
double y[];
double w;
fm2 b3;
double a[][];
public fa(double Pr,double yi[],double wi)
{ b3=new fm2(Pr);
  y=new double[yi.length];
  for(int i=0;i<y.length;i++) y[i]=yi[i];w=wi;
}

public double[] func(double x[])
{
// shooting method
y[2]=x[0];
y[4]=x[1];
a=RK6(b3,0.0,w,y,10000);
double ff[]=new double[2];
ff[0]=a[2][10000]-0.0;
ff[1]=a[4][10000]-0.0;
return ff;
}
 //differenetial equation solution
public static double[][] RK6(f_xi fp,double x0,double xn,double f0[],int N)
{
//6th order Runge Kutta Method
//fp  : given set of derivative functions dfj/dxi(fj,x)
// xo : initial value of the independent variable
// xn : final value of the independent variable
// f0 : initial value of the dependent variable
// N  : number of dependent variable to be calculated
// fi : dependent variable
double h=(xn-x0)/N;
int M=f0.length;
double fi[][];
fi=new double[M][N+1];
double xi[]=new double[M+1];
double k[]=new double[6];
int i,j;
double x;
for(j=0;j<M;j++)
  {
  fi[j][0]=f0[j];
  xi[j+1]=f0[j];
  }
for(x=x0,i=0;i<N;x+=h,i++)
  {
  for(j=1;j<=M;j++)
    {
    xi[0]=x;
    xi[j]=fi[j-1][i];
    k[0]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/2;
    k[1]=h*fp.func(xi,j-1);
    xi[0]=x+h/2.0;
    xi[j]=fi[j-1][i]+k[0]/4.0+k[1]/4.0;
    k[2]=h*fp.func(xi,j-1);
    xi[0]=x+h;
    xi[j]=fi[j-1][i]-k[1]+2.0*k[2];
    k[3]=h*fp.func(xi,j-1);
    xi[0]=x+2.0/3.0*h;
    xi[j]=fi[j-1][i]+7.0/27.0*k[0]+10.0/27.0*k[1]+1.0/27.0*k[3];
    k[4]=h*fp.func(xi,j-1);
    xi[0]=x+1.0/5.0*h;
```

```
    xi[j]=fi[j-1][i]+28.0/625.0*k[0]-1.0/5.0*k[1]+546.0/625.0*k[2]+54.0/625.0*k[3]-378/625.0*k[4];
    k[5]=h*fp.func(xi,j-1);
    fi[j-1][i+1]=fi[j-1][i]+k[0]/24.0+5.0*k[3]/48.0+27.0*k[4]/56.0+125.0*k[5]/336.0;
    xi[j]=fi[j-1][i];
    }
  }
double a[][]=new double[M+1][N+1];
for(x=x0,i=0;i<=N;x+=h,i++)
{
  a[0][i]=x;
  for(j=1;j<=M;j++)
  {
  a[j][i]=fi[j-1][i];
  }
}
return a;
}
}

//Blassius flat plate differential equation
class fm2 extends f_xi
{ double Pr;
  fm2(double Pri) {Pr=Pri;}
// multivariable function
 double func(double x[],int x_ref)
 {
   //Natural convection similarity solution
   double a=0;
   if(x_ref==4) a= -3.0*Pr*x[1]*x[5];
   if(x_ref==3) a= x[5];
   if(x_ref==2) a= -3.0*x[1]*x[3]+2.0*x[2]*x[2]-x[4];
   if(x_ref==1) a= x[3];
   if(x_ref==0) a= x[2];
   return a;
 }
}
class fb extends f_xr
{
                double func(double Pr,int i)
                {
     double xx=0.0;
      if(i==0) xx=Math.pow(Pr,0.25);
      else if(i==1) xx=Math.sqrt(Pr);
      else if(i==2) xx=Math.pow(Pr,0.75);
      else if(i==3) xx=Pr;
      else if(i==4) xx=Math.pow(Pr,1.25);
      else if(i==5) xx=Math.pow(Pr,1.5);
      else if(i==6) xx=Math.pow(Pr,1.75);
   return xx;
                }
}

public class natural_convection_similarity
{
                 public static double[] inputdata(String s)
    {
    String s1=JOptionPane.showInputDialog(s);
    StringTokenizer token=new StringTokenizer(s1);
    int n=token.countTokens()-1;
    int m=n+1;
    double a[]=new double[m];
    int j=0;
    while(token.hasMoreTokens())
    {
    Double ax=new Double(token.nextToken());
    a[j++]=ax.doubleValue();
    }
    return a;
    }
public static String toString(double x[])
{ String s="";
int n=x.length;
  for(int k=0;k<n;k++)
  {s+=x[k]+" ";}
  return s;
}
```

```
public static double[] nelder(f_xj fnelder,double a[],double da[],int maxiteration,double tolerance,int printlist)
{
                int i,j;
double x[][]=new double[a.length+1][a.length];
double p[][]=new double[a.length+1][a.length+1];
for(i=0;i<x.length;i++)
  {for(j=0;j<x[0].length;j++)
    {if(i==j){x[i][j]=a[j]+da[j];p[i][j]=x[i][j];}
     else    {x[i][j]=a[j];p[i][j]=x[i][j]; }
    }
    p[i][j] = fnelder.func(p[i]);
  }


  // Inlet variable definitions
  // fnelder : abstract multivariable function f(x)
  // x : independent variable set of n+1 simplex elements
  // maxiteration : maximum iteration number
  // tolerance :
       int NDIMS = x.length-1;
       int NPTS = x.length;
       int FUNC = NDIMS;
       int ncalls = 0;
         ////// construct the starting simplex //////////////////
         //double p[][]=new double[NPTS][NPTS]; // [row][col] = [whichvx][coord,FUNC]
         double z[]=new double[NDIMS];
         double best = 1E99;
         /////////////// calculate the first function values for the simplex /////////////////

         int iter=0;

         for (iter=1; iter<maxiteration; iter++)
         {
                        ///////////  define lo, nhi, hi (low high next_to_high //////////////
           int  ilo=0, ihi=0, inhi = -1; // -1 means missing
           double flo = p[0][FUNC];
           double fhi = flo;
           double pavg,sterr;
           for (i=1; i<NPTS; i++)
           {
             if (p[i][FUNC] < flo)
               {flo=p[i][FUNC]; ilo=i;}
             if (p[i][FUNC] > fhi)
               {fhi=p[i][FUNC]; ihi=i;}
           }
           double fnhi = flo;
           inhi = ilo;
           for (i=0; i<NPTS; i++)
            if ((i != ihi) && (p[i][FUNC] > fnhi))
              {fnhi=p[i][FUNC]; inhi=i;}
           ////////// exit criteria //////////////
           if ((iter % 4*NDIMS) == 0)
           {
              // calculate the avarage (including maximum value)
              pavg=0;
              for(i=0;i<NPTS;i++)
                pavg+=p[i][FUNC];
              pavg/=NPTS;
              double tot=0;
              if(printlist!=0)
              {  System.out.print(iter);
                 for (j=0; j<=NDIMS; j++)
                    { System.out.print(p[ilo][j]+" ");}
                 System.out.println("");
              }
              for(i=0;i<NPTS;i++)
              { tot=(p[i][FUNC]-pavg)*(p[i][FUNC]-pavg);}
              sterr=Math.sqrt(tot/NPTS);
              //if(sterr < tolerance)
                { for (j=0; j<NDIMS; j++)
                     { z[j]=p[ilo][j];}
                  //break;
                }
              best = p[ilo][FUNC];
           }
```

```
///// calculate avarage without maximum value //////

double ave[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  ave[j] = 0;
for (i=0; i<NPTS; i++)
  if (i != ihi)
    for (j=0; j<NDIMS; j++)
      ave[j] += p[i][j];
for (j=0; j<NDIMS; j++)
   ave[j] /= (NPTS-1);


///////// reflect ////////////////

double r[] = new double[NDIMS];
for (j=0; j<NDIMS; j++)
  r[j] = 2*ave[j] - p[ihi][j];
double fr = fnelder.func(r);

if ((flo <= fr) && (fr < fnhi))  // in zone: accept
{
    for (j=0; j<NDIMS; j++)
      p[ihi][j] = r[j];
    p[ihi][FUNC] = fr;
    continue;
}

if (fr < flo)  //// expand
{
    double e[] = new double[NDIMS];
    for (j=0; j<NDIMS; j++)
      e[j] = 3*ave[j] - 2*p[ihi][j];
    double fe = fnelder.func(e);
    if (fe < fr)
    {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = e[j];
      p[ihi][FUNC] = fe;
      continue;
    }
    else
    {
      for (j=0; j<NDIMS; j++)
        p[ihi][j] = r[j];
      p[ihi][FUNC] = fr;
      continue;
    }
}

////////////// shrink:

if (fr < fhi)
{
    double c[] = new double[NDIMS];
    for (j=0; j<NDIMS; j++)
      c[j] = 1.5*ave[j] - 0.5*p[ihi][j];
    double fc = fnelder.func(c);
    if (fc <= fr)
    {
        for (j=0; j<NDIMS; j++)
          p[ihi][j] = c[j];
        p[ihi][FUNC] = fc;
        continue;
    }
    else   /////// daralt
    {
        for (i=0; i<NPTS; i++)
          if (i != ilo)
          {
              for (j=0; j<NDIMS; j++)
                p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
              p[i][FUNC] = fnelder.func(p[i]);
          }
        continue;
    }
```

```
            }

            if (fr >= fhi)   ///
            {
               double cc[] = new double[NDIMS];
               for (j=0; j<NDIMS; j++)
                 cc[j] = 0.5*ave[j] + 0.5*p[ihi][j];
               double fcc = fnelder.func(cc);
               if (fcc < fhi)
               {
                  for (j=0; j<NDIMS; j++)
                    p[ihi][j] = cc[j];
                  p[ihi][FUNC] = fcc;
                  continue;
               }
               else    /////////
               {
                  for (i=0; i<NPTS; i++)
                    if (i != ilo)
                    {
                       for (j=0; j<NDIMS; j++)
                         p[i][j] = 0.5*p[ilo][j] + 0.5*p[i][j];
                       p[i][FUNC] = fnelder.func(p[i]);
                    }
               }
            }
         }
      return z;
}
     public static double[] nelder(f_xj fnelder,double a[],double da[],double tolerance)
     {return nelder(fnelder,a,da,500,tolerance,0);}

     public static double[] nelder(f_xj fnelder,double a[],double da[])
     {return nelder(fnelder,a,da,500,1.0e-10,0);}

     public static double[] nelder(f_xj fnelder,double a[])
     {
                       double [] da=new double[a.length];
            for(int i=0;i<a.length;i++) da[i]=0.1*a[i];
            return nelder(fnelder,a,da);
     }
//general least square curve fitting
  public static double[] gausswithpartialpivot(double a[][],double b[])
  { //Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
               p=k;
               buyuk=Math.abs(a[k][k]);
               for(ii=k+1;ii<n;ii++)
               {  dummy=Math.abs(a[ii][k]);
                  if(dummy > buyuk) {buyuk=dummy;p=ii;}
               }
               if(p!=k)
               { for(jj=k;jj<n;jj++)
                 { dummy=a[p][jj];
                   a[p][jj]=a[k][jj];
                   a[k][jj]=dummy;
                 }
                 dummy=b[p];
                 b[p]=b[k];
                 b[k]=dummy;
               }
               //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
```

```
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}


public static double[] GeneralLeastSquare(double xi[],double yi[],int n)
{
fb f=new fb();
int l=xi.length;
int i,j,k;
int np1=n+1;
double A[][];
A=new double[np1][np1];
double B[];
B=new double[np1];
double X[];
X=new double[np1];
for(i=0;i<n+1;i++)
  {  B[i]=0;
                for(j=0;j<np1;j++)
    {
                A[i][j]=0.0;
    for(k=0;k<l;k++) A[i][j]+=f.func(xi[k],i)*f.func(xi[k],j);
    }
    for(k=0;k<l;k++) B[i]+= f.func(xi[k],i)*yi[k];
  }
//System.out.println("A = \n"+Matrix.toString(A));
//System.out.println("B = \n"+Matrix.toStringT(B));
X=gausswithpartialpivot(A,B);
//X=B/A;
double max=0;
for(i=0;i<n+1;i++)
 if(Math.abs(X[i]) > max) max = Math.abs(X[i]);
for(i=0;i<n+1;i++)
 if((Math.abs(X[i]/max) > 0) && (Math.abs(X[i]/max) < 1.0e-100)) X[i]=0;
Text.printT(X);
return X;
}

public static double funcGeneralLeastSquare(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
fb f=new fb();
int n=e.length;
double ff=0;
if(n!=0.0)
  {
  for(int i=n-1;i>=0;i--)
   {ff+=e[i]*f.func(x,i);}
  }
return ff;
}

public static double hata(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-funcGeneralLeastSquare(e,x[k]))*(y[k]-funcGeneralLeastSquare(e,x[k]));
 }
total=Math.sqrt(total);
```

```
return total;
}

 public static double[][]  funcGeneralLeastSquare(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
               //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
               // ara değer sayisi 2 dir
                int n=xi.length;
               int nn=(n-1)*(aradegersayisi+1)+1;
               double z[][]=new double[2][nn];
   double E[]=GeneralLeastSquare(xi,yi,polinomkatsayisi);
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=funcGeneralLeastSquare(E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=funcGeneralLeastSquare(E,z[0][k]);
   return z;
 }


public static void main(String arg[]) throws IOException
{ //initial guess for differential equation set
  double y[]={0,0,0.7,1.0,-0.5};
 //String s="leave a space between initial root guess values: xmin0 xmin1.. ";
 //double x[]=inputdata(s);
 double x[]={0.7,-0.5};//inputdata(s);
 fa ff;
 f2 f;
 double r1[];
 //selected Prandtl number values
 double
Pr[]={0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0.2,0.3,0.4,0.5,0.6,0.8,1.0,1.2,1.4,1.6,1.8,2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0,10,11,12,13,14,15};
 //selected values for maximum eta(limit is not taken infinite)
 // corresponding to the Pr numbers
 double
w[]={16.0,15.0,15.0,15.0,15.0,15.0,15.0,15.0,15.0,10.0,10.0,9.0,8.0,8.0,9.0,7.0,7.0,7.0,6.0,5.0,5.0,4.5,4.5,4.5
```

```
  double egPr1[]=new double[Pr.length];
  double egPr2[]=new double[Pr.length];
  double b[]=GeneralLeastSquare(xi,yi,6);
  for(int i=0;i<Pr.length;i++)

{gPr1[i]=b[0]*Math.pow(Pr[i],0.25)+b[1]*Math.pow(Pr[i],0.5)+b[2]*Math.pow(Pr[i],0.75)+b[3]*Pr[i]+b[4]*Math.pow(Pr[i],1.25)+b[5]*Math.po
w(Pr[i],1.5)+b[6]*Math.pow(Pr[i],1.75);
   gPr2[i]=0.75*Math.sqrt(Pr[i])/Math.pow((0.609+1.221*Math.sqrt(Pr[i])+1.238*Pr[i]),0.25);
   egPr1[i]=(gPr1[i]-yi[i])/yi[i]*100;
   egPr2[i]=(gPr2[i]-yi[i])/yi[i]*100;
   }
  Text.printT(b);
  double d[][]={xi,yi};
  Text.printT(d,"d");
  System.out.println("f(Pr) ="+b[0]+b[1]+"*Pr^0.25+"+b[2]+"*Pr^0.5+"+b[3]+"*Pr^0.75+"+b[4]+"*Pr +"+b[5]+"*Pr^1.25+"+b[6]+"*Pr^1.5");
 pp.plot();
 pp1.plot();
 Plot pp5=new Plot(xi,yi);
 pp5.addData(xi,gPr1);
 pp5.setColor(1,0,0,255);
 pp5.setPlotType(1,23);
 pp5.addData(xi,gPr2);
  pp5.setColor(2,255,0,0);
 pp5.setPlotType(2,28);
 pp5.plot();
 Plot pp6=new Plot(xi,egPr1);
 pp6.setColor(1,0,0,255);

 pp6.addData(xi,egPr2);
  pp6.setColor(2,255,0,0);
 pp6.plot();
}
}
```

Program investigates a wide range of Prandtl numbers and as a results:

$$h_x = \frac{q''_s}{T_s - T\infty} = -\frac{T_\infty - T_s}{T_s - T_\infty} k \left.\frac{\partial \theta}{\partial y}\right|_{y=0}$$

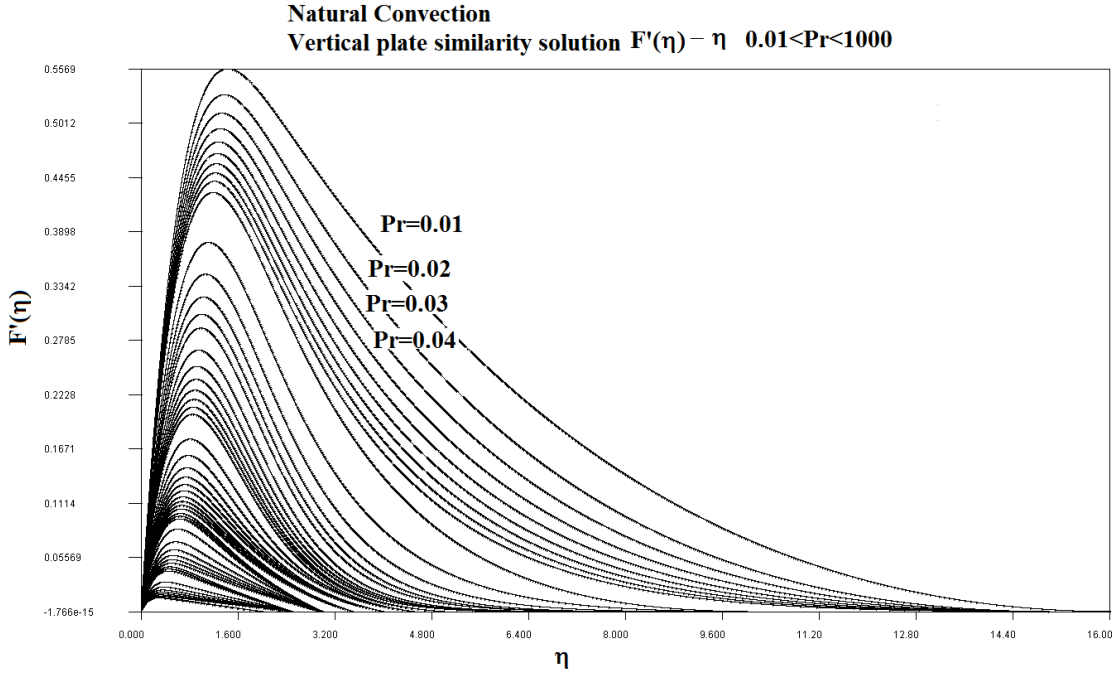
$$h_x = k\left(\frac{u_\infty}{\nu x}\right)^{1/2} \left.\frac{\partial \theta(\eta)}{\partial \eta}\right|_{\eta=0} = \frac{k}{x}\left(\frac{u_\infty x}{\nu}\right)^{1/2} \left.\frac{\partial \theta(\eta)}{\partial \eta}\right|_{\eta=0}$$

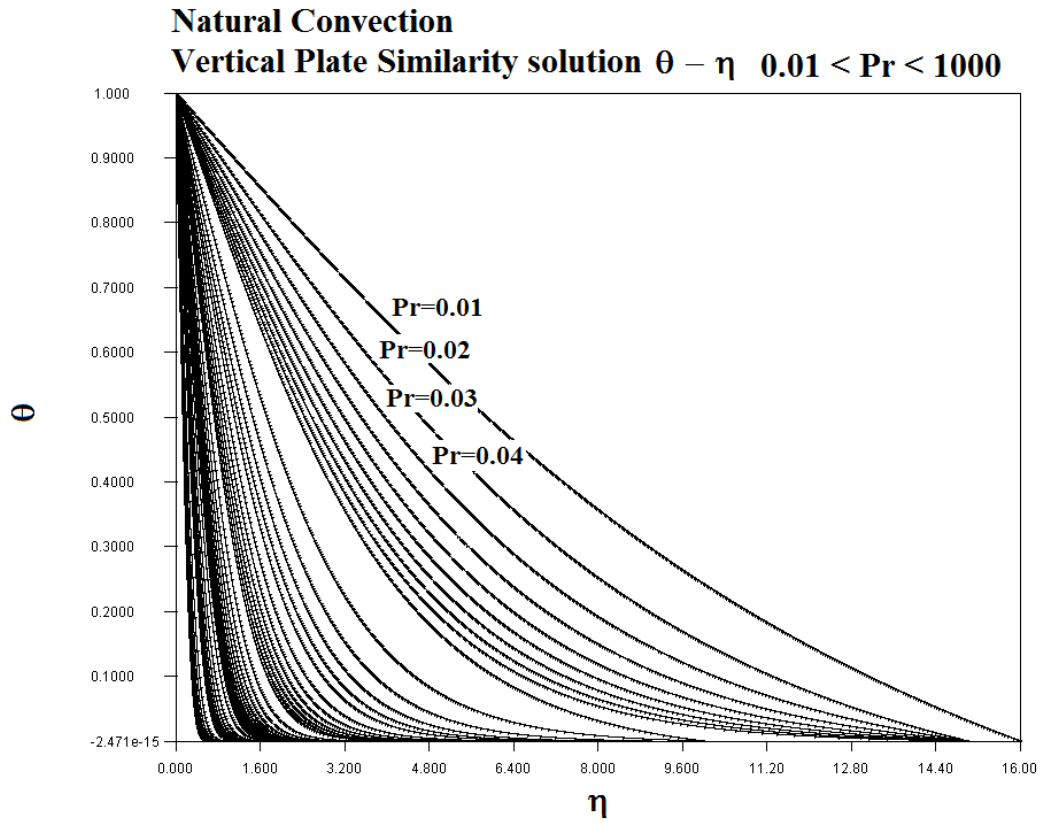
$$Nu_x = \frac{h_x x}{k}$$

Results for $\left.\frac{\partial \theta(\eta)}{\partial \eta}\right|_{\eta=0}$

Pr	□'(□□
0.01	0.091093502
0.02	0.116834579
0.03	0.136995175
0.04	0.154605985
0.05	0.170229134
0.06	0.184286962
0.07	0.19708927
0.08	0.208865088
0.09	0.219786662
0.1	0.231500963
0.2	0.307792789
0.3	0.362168135
0.4	0.4050892
0.5	0.440866301

0.6	0.471765381
0.8	0.523754747
1	0.56672913
1.2	0.603623472
1.4	0.636067035
1.6	0.664867361
1.8	0.691266588
2	0.715036357
3	0.813788218
4	0.889502603
5	0.950493977
6	1.003583771
7	1.050193207
8	1.09189393
9	1.127210608
10	1.161864748
11	1.193933662
12	1.223822704
13	1.246961238
14	1.273319569
15	1.298263279
20	1.406640617
30	1.572188344
40	1.699540604
50	1.804447867
60	1.894382556
70	1.973529018
80	2.034457087
90	2.098909425
100	2.158133722
200	2.587798708
300	2.874897742
400	3.096602009
500	3.279691561
600	3.436935238
700	3.575508896
800	3.699882055
900	3.796072907
1000	3.900184715





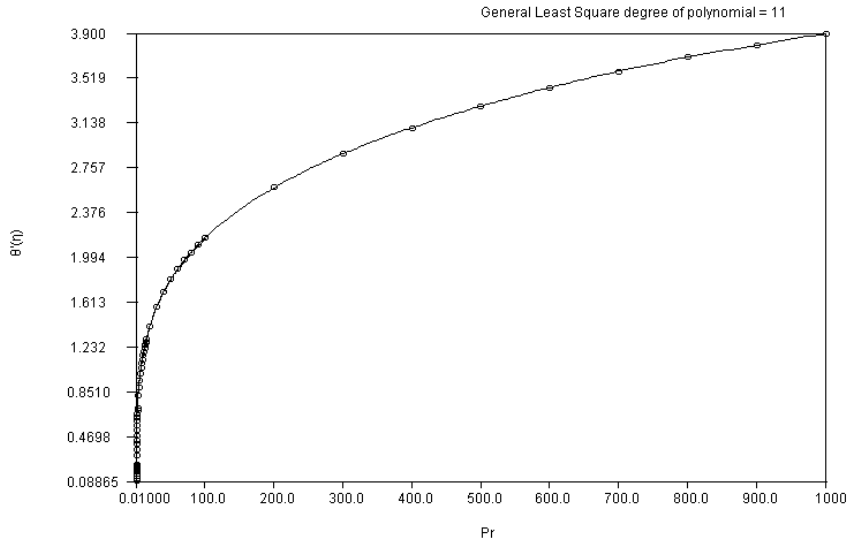
After establishing data, different curve fitting methods can be used to get Nusselt number correlations. The simplest correlation could be a linear polynomial such as an 11th degree polynomial

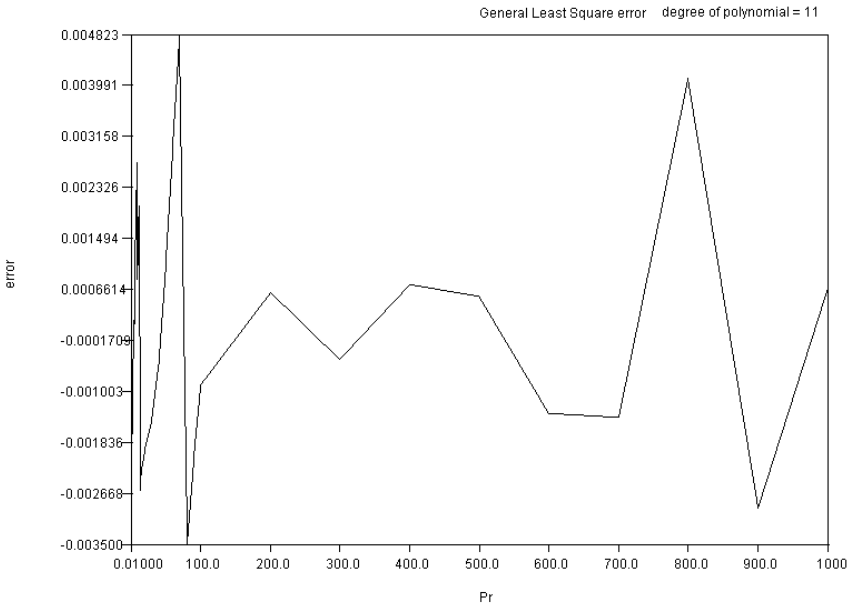
$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \theta' = a_0\,\mathrm{Pr}^{1/4} + a_1\,\mathrm{Pr}^{1/2} + a_2\,\mathrm{Pr}^{3/4} + a_3\,\mathrm{Pr} + a_4\,\mathrm{Pr}^{5/4} + a_5\,\mathrm{Pr}^{3/2} +$$

$$a_6\,\mathrm{Pr}^{2/4} + a_7\,\mathrm{Pr}^{2} + a_8\,\mathrm{Pr}^{9/4} + a_9\,\mathrm{Pr}^{5/2} + a_{10}\,\mathrm{Pr}^{11/4} + a_{11}\,\mathrm{Pr}^{3}$$

a_0	2.0989129231E-01
a_1	-3.0051209224E-01
a_2	2.6523436090E+00
a_3	-3.9991683936E+00
a_4	3.0819446934E+00
a_5	-1.4187784421E+00
a_6	4.0479301621E-01
a_7	-6.9334038631E-02
a_8	6.0379609656E-03
a_9	-3.3673993063E-05
a_{10}	-3.6909765186E-05
a_{11}	2.0994279162E-06

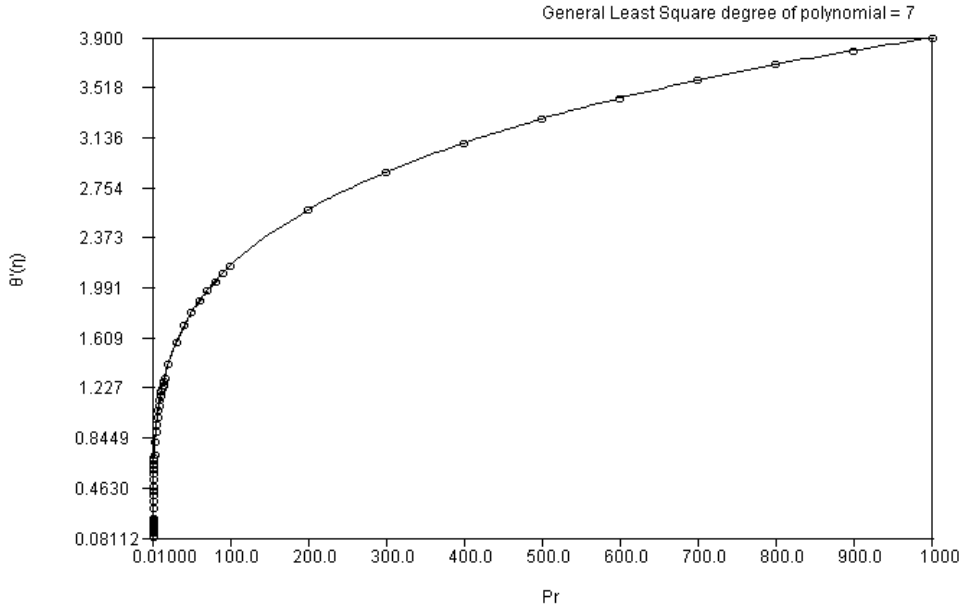
For this polynomial data and fitted line and error shown as follows

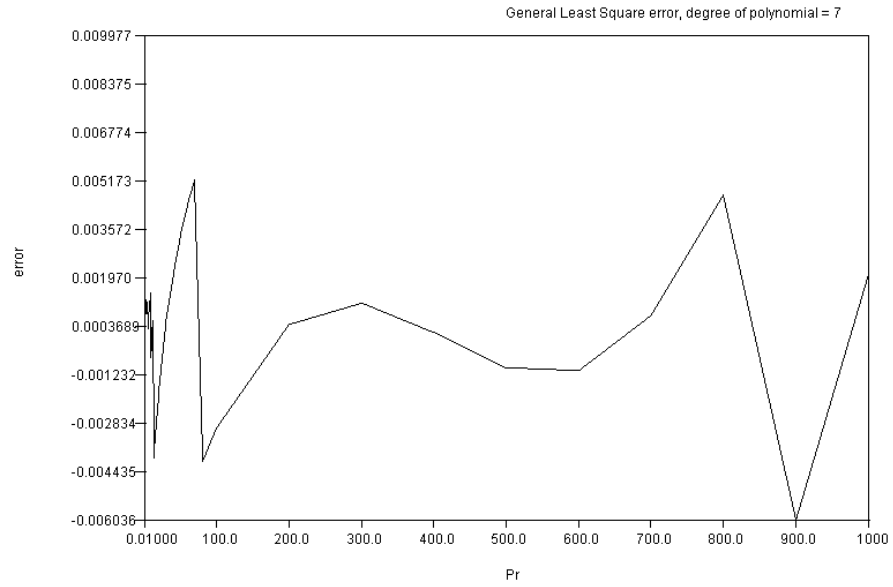




$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \theta' = a_0\,\mathrm{Pr}^{1/4} + a_1\,\mathrm{Pr}^{1/2} + a_2\,\mathrm{Pr}^{3/4} + a_3\,\mathrm{Pr} + a_4\,\mathrm{Pr}^{5/4} + a_5\,\mathrm{Pr}^{3/2} + a_6\,\mathrm{Pr}^{2/4} + a_7\,\mathrm{Pr}^{2}$$

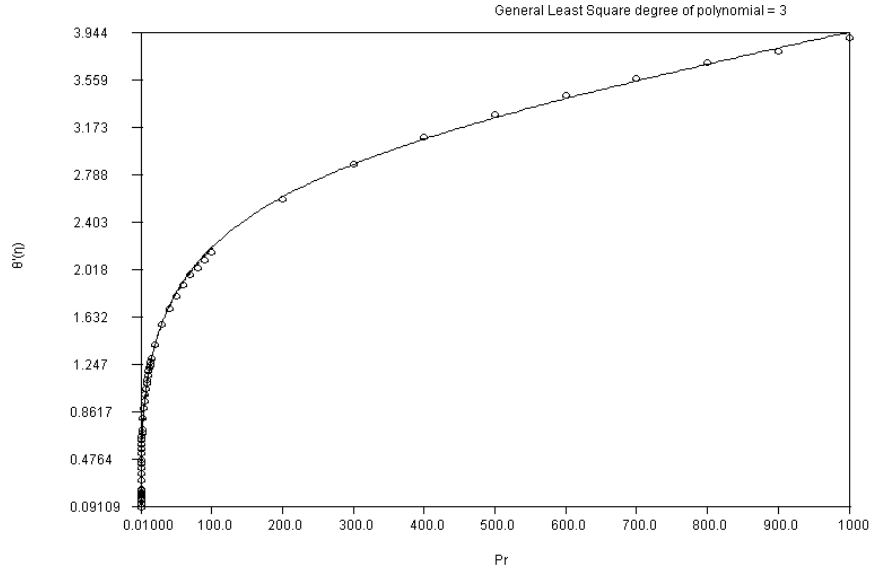
a0	-7.1616625047E-02
a1	1.3294314755E+00
a2	-1.0629736882E+00
a3	4.8311043084E-01
a4	-1.3183854844E-01
a5	2.1366843379E-02
a6	-1.8917609206E-03
a7	7.0288177613E-05

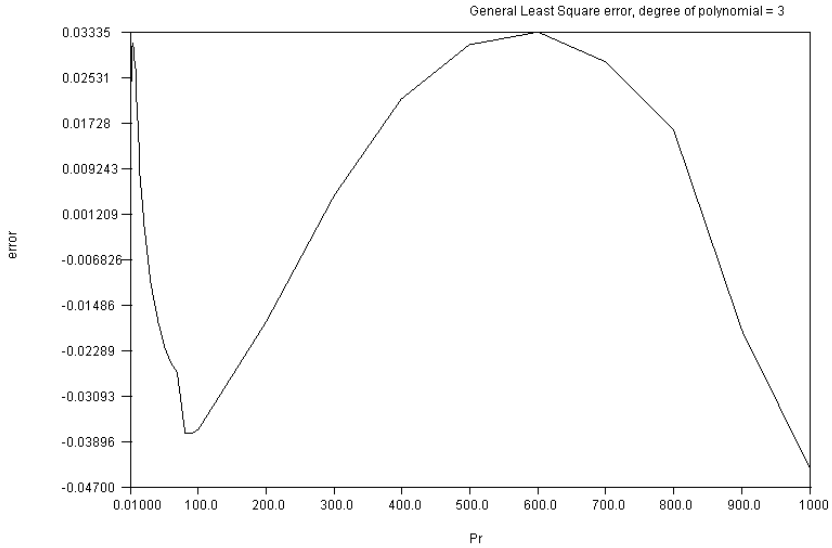




$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \theta' = a_0\,\mathrm{Pr}^{1/4} + a_1\,\mathrm{Pr}^{1/2} + a_2\,\mathrm{Pr}^{3/4} + a_3\,\mathrm{Pr}$$

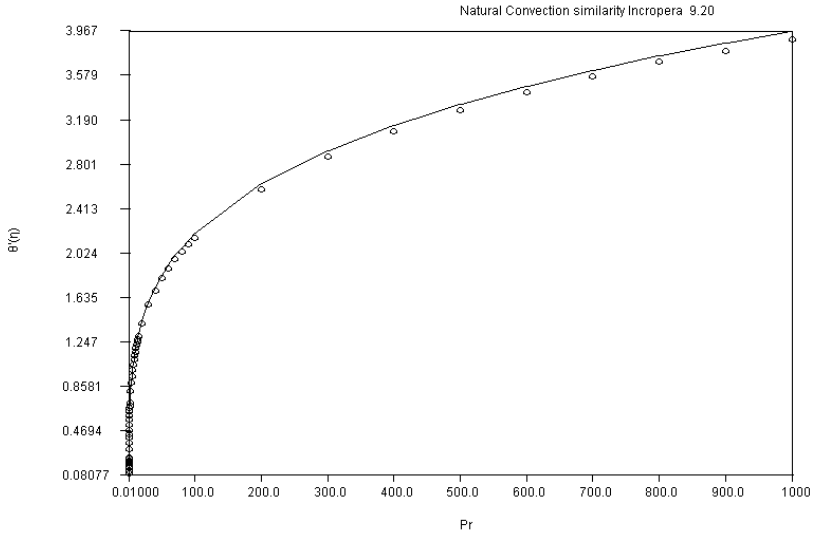
a0	0.337767076
a1	0.279380627
a2	-0.071379462
a3	0.005903166



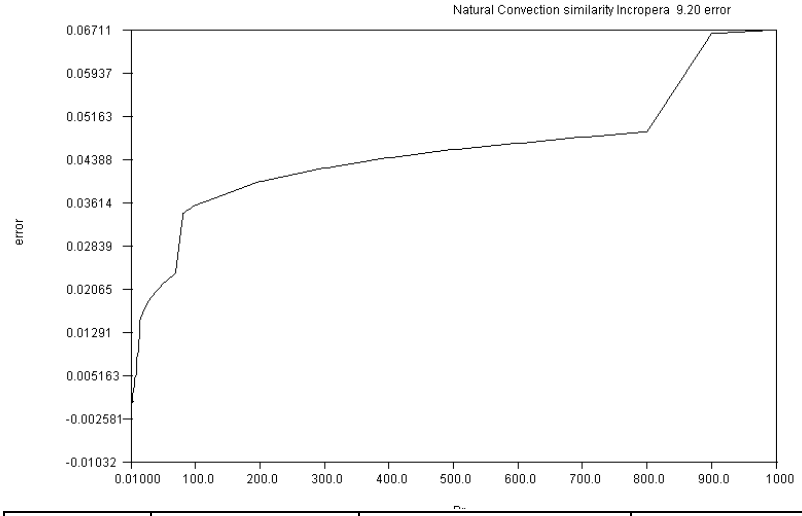


As a solution of this problem following equation is given in Fundamentals of Heat and Mass Transfer, Incropera/DeWitt/Bergman/Lavine, sixth edition as:

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \theta' = \frac{0.75\,\mathrm{Pr}^{1/2}}{(0.609+1.221\mathrm{Pr}^{1/2}+1.238\,\mathrm{Pr})^{1/4}}$$



Error level of the equation is as follow:



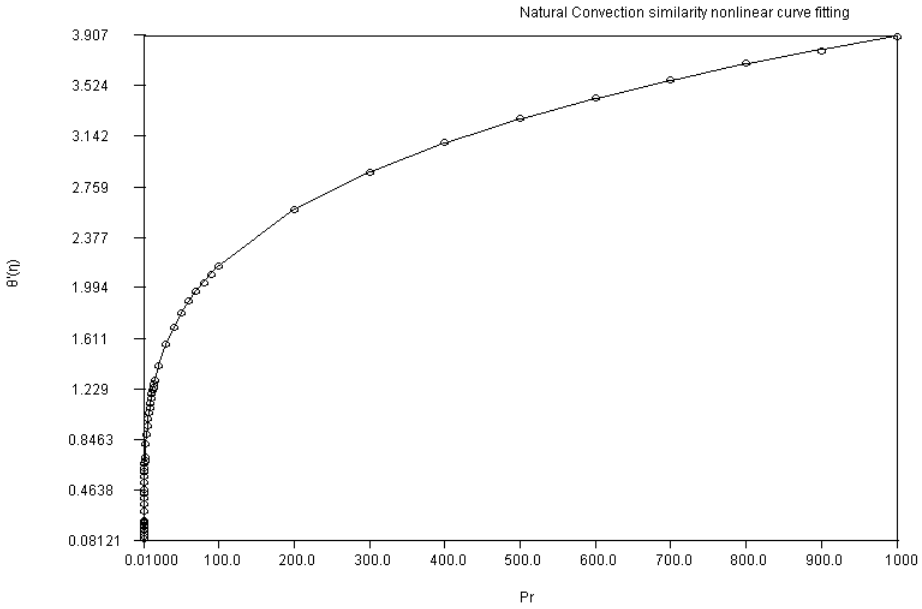
Pr	θ'(η) calculated	θ'(η) Incropera 9.20	Error
0.01	0.091093502	0.081214605	-0.009878897
0.02	0.116834579	0.112577416	-0.004257163
0.03	0.136995175	0.135789496	-0.001205679
0.04	0.154605985	0.154806134	2.00E-04
0.05	0.170229134	0.171157064	9.28E-04
0.06	0.184286962	0.185625468	0.001338506
0.07	0.19708927	0.198675703	0.001586433
0.08	0.208865088	0.210609701	0.001744613
0.09	0.219786662	0.221636726	0.001850064
0.1	0.231500963	0.231908792	4.08E-04
0.2	0.307792789	0.309682066	0.001889277
0.3	0.362168135	0.363931695	0.00176356
0.4	0.4050892	0.406583847	0.001494647
0.5	0.440866301	0.442124632	0.001258331
0.6	0.471765381	0.472793688	0.001028307
0.8	0.523754747	0.524235458	4.81E-04
1	0.56672913	0.566769318	4.02E-05
1.2	0.603623472	0.603273793	-3.50E-04
1.4	0.636067035	0.635395565	-6.71E-04
1.6	0.664867361	0.66417209	-6.95E-04
1.8	0.691266588	0.690302464	-9.64E-04
2	0.715036357	0.714281819	-7.55E-04

3	0.813788218	0.812086974	-0.001701244
4	0.889502603	0.887072196	-0.002430407
5	0.950493977	0.948637499	-0.001856478
6	1.003583771	1.001262098	-0.002321673
7	1.050193207	1.047456694	-0.002736513
8	1.09189393	1.088780187	-0.003113743
9	1.127210608	1.126271668	-9.39E-04
10	1.161864748	1.16066093	-0.001203818
11	1.193933662	1.192481565	-0.001452097
12	1.223822704	1.222136374	-0.00168633
13	1.246961238	1.249937427	0.002976189
14	1.273319569	1.276131774	0.002812205
15	1.298263279	1.300918579	0.0026553
20	1.406640617	1.408592852	0.001952235
30	1.572188344	1.572996939	8.09E-04
40	1.699540604	1.699415203	-1.25E-04
50	1.804447867	1.80352129	-9.27E-04
60	1.894382556	1.892748434	-0.001634122
70	1.973529018	1.971257678	-0.00227134
80	2.034457087	2.0416329	0.007175813
90	2.098909425	2.105597706	0.006688281
100	2.158133722	2.164363939	0.006230217
200	2.587798708	2.590451013	0.002652305
300	2.874897742	2.874952717	5.50E-05
400	3.096602009	3.094562562	-0.002039447
500	3.279691561	3.275874082	-0.003817479
600	3.436935238	3.431560807	-0.005374431
700	3.575508896	3.568742382	-0.006766514
800	3.699882055	3.691852017	-0.008030038
900	3.796072907	3.803853475	0.007780568
1000	3.900184715	3.906834359	0.006649644

The same accuracy equation is curve fitted by using Nelder_& Mead nonlinear optimization method.

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \theta' = \frac{a_0 \,\mathrm{Pr}^{1/2}}{(a_1 + a_2 \,\mathrm{Pr}^{1/2} + a_3 \,\mathrm{Pr})^{1/4}}$$

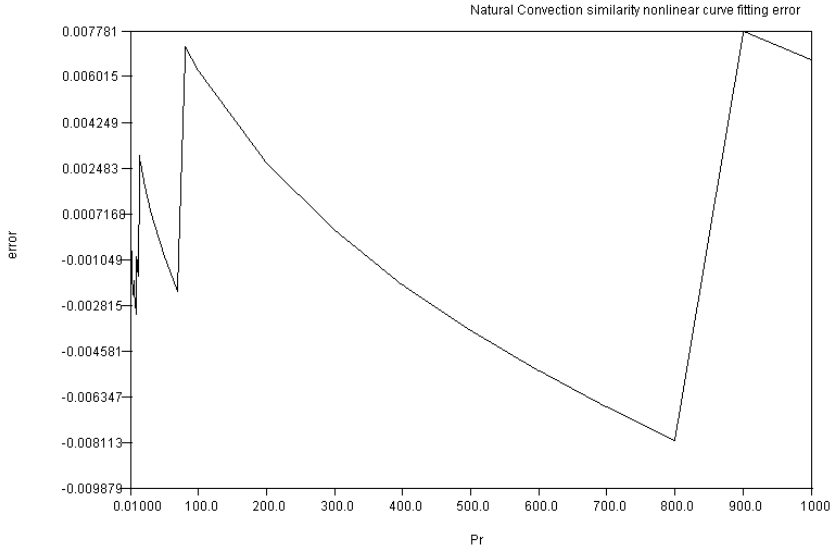
a0	0.756705949
a1	0.621261022
a2	1.187008971
a3	1.369213226



The error of these curve fitting value:

Pr	θ'(η) calculated	θ'(η) Non Lin. C.F.	Error
0.01	0.0910935	0.081214605	-0.009878897
0.02	0.11683458	0.112577416	-0.004257163
0.03	0.13699518	0.135789496	-0.001205679
0.04	0.15460599	0.154806134	2.00E-04
0.05	0.17022913	0.171157064	9.28E-04
0.06	0.18428696	0.185625468	0.001338506
0.07	0.19708927	0.198675703	0.001586433
0.08	0.20886509	0.210609701	0.001744613
0.09	0.21978666	0.221636726	0.001850064
0.1	0.23150096	0.231908792	4.08E-04
0.2	0.30779279	0.309682066	0.001889277
0.3	0.36216814	0.363931695	0.00176356
0.4	0.4050892	0.406583847	0.001494647
0.5	0.4408663	0.442124632	0.001258331
0.6	0.47176538	0.472793688	0.001028307
0.8	0.52375475	0.524235458	4.81E-04
1	0.56672913	0.566769318	4.02E-05
1.2	0.60362347	0.603273793	-3.50E-04
1.4	0.63606704	0.635395565	-6.71E-04
1.6	0.66486736	0.66417209	-6.95E-04
1.8	0.69126659	0.690302464	-9.64E-04
2	0.71503636	0.714281819	-7.55E-04
3	0.81378822	0.812086974	-0.001701244
4	0.8895026	0.887072196	-0.002430407
5	0.95049398	0.948637499	-0.001856478
6	1.00358377	1.001262098	-0.002321673
7	1.05019321	1.047456694	-0.002736513
8	1.09189393	1.088780187	-0.003113743
9	1.12721061	1.126271668	-9.39E-04
10	1.16186475	1.16066093	-0.001203818
11	1.19393366	1.192481565	-0.001452097
12	1.2238227	1.222136374	-0.00168633
13	1.24696124	1.249937427	0.002976189
14	1.27331957	1.276131774	0.002812205
15	1.29826328	1.300918579	0.0026553
20	1.40664062	1.408592852	0.001952235
30	1.57218834	1.572996939	8.09E-04

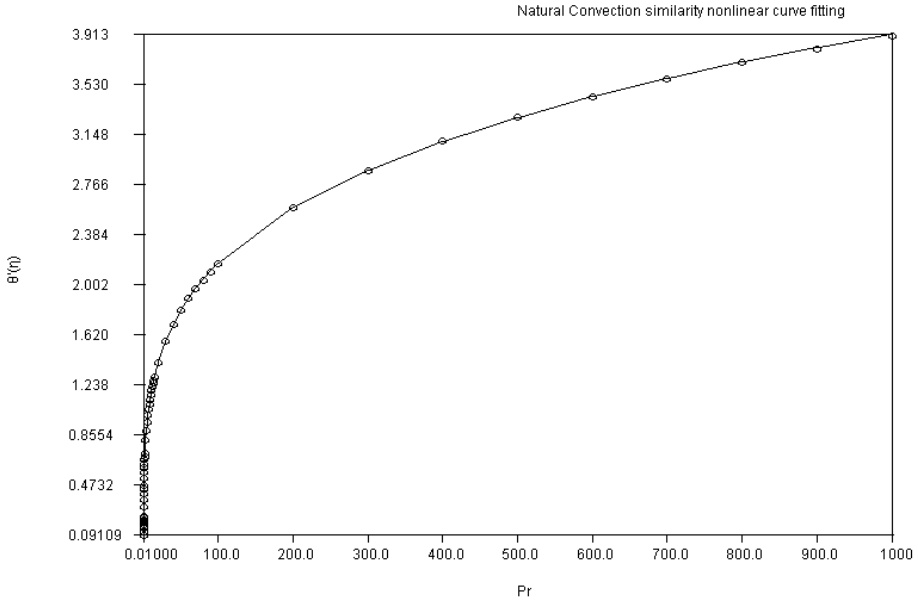
40	1.6995406	1.699415203	-1.25E-04
50	1.80444787	1.80352129	-9.27E-04
60	1.89438256	1.892748434	-0.001634122
70	1.97352902	1.971257678	-0.00227134
80	2.03445709	2.0416329	0.007175813
90	2.09890943	2.105597706	0.006688281
100	2.15813372	2.164363939	0.006230217
200	2.58779871	2.590451013	0.002652305
300	2.87489774	2.874952717	5.50E-05
400	3.09660201	3.094562562	-0.002039447
500	3.27969156	3.275874082	-0.003817479
600	3.43693524	3.431560807	-0.005374431
700	3.5755089	3.568742382	-0.006766514
800	3.69988206	3.691852017	-0.008030038
900	3.79607291	3.803853475	0.007780568
1000	3.90018472	3.906834359	0.006649644



As it is seen from the results, error level of this equation is one order better than the Incropera equation. Error level can be further reduced in nonlinear equations by increasing number of terms as well. Let us assume the equation in the form of

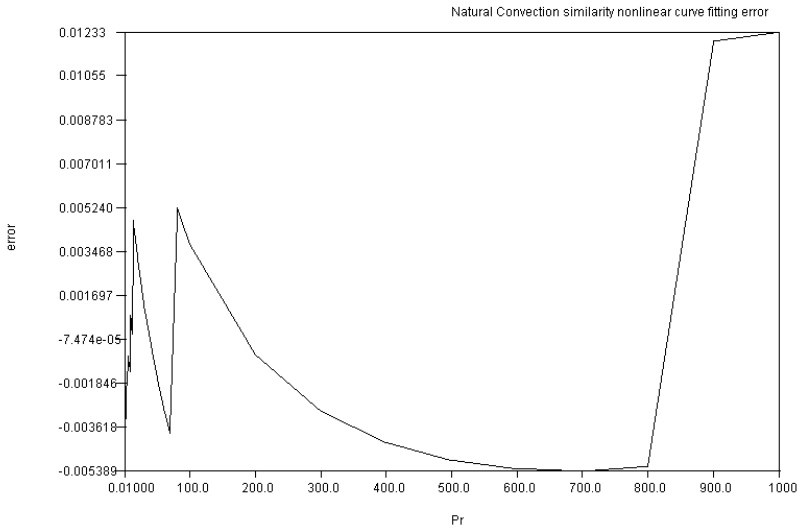
$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \theta' = \frac{a_0 + a_1 \,\mathrm{Pr}^{1/2} + a_2 \,\mathrm{Pr}}{(a_3 + a_4 \,\mathrm{Pr}^{1/2} + a_5 \,\mathrm{Pr})^{1/4}}$$

a0	0.022830145
a1	0.752624416
a2	2.08E-04
a3	1.222236219
a4	1.010421489
a5	1.389905434



Pr	θ'(η) calculated	θ'(η) Non Lin. C.F.	Error
0.01	0.091093502	0.091221691	1.28E-04
0.02	0.116834579	0.118992716	0.002158137
0.03	0.136995175	0.139872692	0.002877517
0.04	0.154605985	0.157180232	0.002574247
0.05	0.170229134	0.172202802	0.001973668
0.06	0.184286962	0.185601606	0.001314644
0.07	0.19708927	0.19776992	6.81E-04
0.08	0.208865088	0.208964269	9.92E-05
0.09	0.219786662	0.219363018	-4.24E-04
0.1	0.231500963	0.22909615	-0.002404813
0.2	0.307792789	0.304102599	-0.00369019
0.3	0.362168135	0.35757659	-0.004591545
0.4	0.4050892	0.400125376	-0.004963824
0.5	0.440866301	0.435841974	-0.005024327
0.6	0.471765381	0.466810409	-0.004954972
0.8	0.523754747	0.518973737	-0.00478101
1	0.56672913	0.562235954	-0.004493176
1.2	0.603623472	0.59941289	-0.004210582
1.4	0.636067035	0.632137656	-0.003929379
1.6	0.664867361	0.661449436	-0.003417925
1.8	0.691266588	0.688053664	-0.003212924
2	0.715036357	0.712452737	-0.00258362
3	0.813788218	0.811759058	-0.00202916
4	0.889502603	0.887621384	-0.001881219
5	0.950493977	0.949722148	-7.72E-04
6	1.003583771	1.002679508	-9.04E-04
7	1.050193207	1.04907898	-0.001114227
8	1.09189393	1.090522895	-0.001371035
9	1.127210608	1.128077086	8.66E-04
10	1.161864748	1.162488589	6.24E-04
11	1.193933662	1.194302537	3.69E-04
12	1.223822704	1.223929691	1.07E-04
13	1.246961238	1.25168775	0.004726512
14	1.273319569	1.277827826	0.004508257
15	1.298263279	1.302552067	0.004288788
20	1.406640617	1.409850676	0.003210059
30	1.572188344	1.57346755	0.001279206
40	1.699540604	1.699202238	-3.38E-04
50	1.804447867	1.802740682	-0.001707185

60	1.894382556	1.891497867	-0.002884689
70	1.973529018	1.969616401	-0.003912617
80	2.034457087	2.039665331	0.005208244
90	2.098909425	2.10335679	0.004447365
100	2.158133722	2.161893751	0.003760029
200	2.587798708	2.587090857	-7.08E-04
300	2.874897742	2.871901695	-0.002996047
400	3.096602009	3.092319163	-0.004282846
500	3.279691561	3.274700561	-0.004991
600	3.436935238	3.431612916	-0.005322322
700	3.575508896	3.570119596	-0.0053893
800	3.699882055	3.694620897	-0.005261158
900	3.796072907	3.808059921	0.011987014
1000	3.900184715	3.912510613	0.012325898



Non linear curve fitting program:

```
import java.io.*;
import javax.swing.*;

abstract class f_xj
{
  // single function  multi independent variable
  // a single value is returned  (
  // example f=x[0]+sin(x[1])
  // func(x) returns the value of f
  abstract public double func(double x[]);
}
     class yy extends f_xj
     {
               double xi[]; // independent variable data set
     double yi[]; //dependent variable data set
     double a[];  //fit function coefficient set
     int nn;
               public yy(String filename,double ia[])
     {
       //read the data to curvefit
       //get the data file and initial fit coefficient when class is defined
                  xi=new double[1001];
       yi=new double[1001];
       int n=ia.length;
       a=new double[n];
       seta(ia);
       int i=-1;
       try{
                  BufferedReader fin=new BufferedReader(new FileReader(filename));
       try {
       while(fin != null)
       {
       i++;
```

```
      xi[i]=Text.readDouble(fin);
      yi[i]=Text.readDouble(fin);
      }
      } catch(EOFException e_eof) {System.out.println("end of file"); }
      }catch(IOException e_io) {System.out.println("dosya bulunamadı"); }
    nn=i;
    a=ia;
    }

    public void seta(double ia[])
    {
                  //assign new fit coefficient set
                  for(int ii=0;ii<nn;ii++)
          a[ii]=ia[ii];
               }

               public double[] geta()
    {
                 // return fit coefficient set
                 return a;
               }

    double Ps(double Pr,double ai[])
    {a=ai;
    double pi=a[0]*Math.pow(Pr,0.5)/Math.pow((a[1]+a[2]*Math.pow(Pr,0.5)+a[3]*Pr),0.25);
    return pi;
    }

    public double func(double ai[])
    {
    double ff=0;
    double w;
    double yy;
     for(int i=0;i<nn;i++)
                 {w=Ps(xi[i],ai);
                 yy=yi[i];
      //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                 w-=yy;
      ff+=w*w;
      //System.out.println("x="+xi[i]+"w="+w+"yy="+yy+"ff="+ff);
                 }
    return ff;
               }
}

class zz extends yy
{    public zz(String filename,double ia[])
     {super(filename,ia);}
     public double func(double ai[])
     {
     return -super.func(ai);
               }
}

public class NA50
{
   public static void main(String args[]) throws IOException
   {
   //String in_name=JOptionPane.showInputDialog(" enter name of the input file : ");
   String in_name="Pr_t.txt";
   double p[];
   p=new double[4];
   p[0] =0.75;
   p[1] =0.609;
   p[2] =1.221;
   p[3] =1.238;

   double da[];
   da=new double[4];
   da[0] =0.1*p[0];
   da[1] =0.1*p[1];
   da[2] =0.1*p[2];
   da[3] =0.1*p[3];

   yy f=new yy(in_name,p);
   System.out.println("book function = ");
```

```
    double aa[][]=new double[4][53];
    for(int i=0;i<53;i++)
    {aa[0][i]=f.xi[i];aa[1][i]=f.yi[i];aa[2][i]=f.Ps(f.xi[i],p);aa[3][i]=aa[2][i]-aa[1][i];
    System.out.println("i="+i+"x="+aa[0][i]+"yi="+aa[1][i]+"y=f(xi)="+aa[2][i]+"error="+aa[3][i]);};
    Text.printT(aa,"incropera 9.20");
    Plot pp=new Plot(aa[0],aa[1]);
  pp.setPlabel("Natural Convection similarity Incropera  9.20 ");
  pp.setXlabel("Pr");
  pp.setYlabel(""+'\u03B8'+"'("+'\u03B7'+")");
  pp.addData(aa[0],aa[2]);
  pp.setPlotType(0,22);
  pp.plot();
  Plot pp1=new Plot(aa[0],aa[3]);
  pp1.setPlabel("Natural Convection similarity Incropera  9.20 error");
  pp1.setXlabel("Pr");
  pp1.setYlabel("error");
  pp1.plot();


    p=NA41.nelder(f,p,da,1e-15);

    for(int i=0;i<53;i++)
    {aa[0][i]=f.xi[i];aa[1][i]=f.yi[i];aa[2][i]=f.Ps(f.xi[i],p);aa[3][i]=aa[2][i]-aa[1][i];
    System.out.println("i="+i+"x="+aa[0][i]+"yi="+aa[1][i]+"y=f(xi)="+aa[2][i]+"error="+aa[3][i]);};
  Text.printT(aa,"nonlinear curve fitting");
  Text.printT(p,"coefficients of non-linear eqaution");
    pp=new Plot(aa[0],aa[1]);
  pp.setPlabel("Natural Convection similarity nonlinear curve fitting ");
  pp.setXlabel("Pr");
  pp.setYlabel(""+'\u03B8'+"'("+'\u03B7'+")");
  pp.addData(aa[0],aa[2]);
  pp.setPlotType(0,22);
  pp.plot();
  pp1=new Plot(aa[0],aa[3]);
  pp1.setPlabel("Natural Convection similarity nonlinear curve fitting error");
  pp1.setXlabel("Pr");
  pp1.setYlabel("error");
  pp1.plot();
       String s1=" coefficient of the curve fitting equation  : \n"+Matrix.toStringT(p)+"\n";
    String s2="Nelder-Mead nonlinear least square curve fitting : ";
    JOptionPane.showMessageDialog(null,s1,s2,JOptionPane.PLAIN_MESSAGE);
    }

}
```

It should also be note that in order to obtain average heat transfer coefficients, these local heat transfer coefficients should be integrated over the total distance.

$$\bar{h} = \frac{1}{L}\int_0^L h dx = \frac{k}{L}\left[\frac{g\beta(T_s - T_\infty)}{4\nu^2}\right]^{1/4}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}\int_0^L \frac{dx}{x^{1/4}}$$

And

$$\bar{N}u = \frac{\bar{h}L}{k} = \frac{4}{3}Nu_x$$

10. Partial diferential equation solutions, seperation of variable methods

SEPARATION OF VARIABLE METHODS FOR HOMOGENEOUS AND NON-HOMOGENEOUS HEAT CONDUCTION

Let us consider cartesian coordinate system heat conduction equation

$$\frac{\partial^2 T(x,y,z,t)}{\partial x^2} + \frac{\partial^2 T(x,y,z,t)}{\partial y^2} + \frac{\partial^2 T(x,y,z,t)}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T(x,y,z,t)}{\partial t}$$

Assume a separation of variables in the form:

$$T(x,y,z,t) = \Psi(x,y,z)\Gamma(t)$$

in this case above equation becomes:

$$\Gamma(t)\left[\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z)}{\partial z^2}\right] = \Psi(x,y,z)\left[\frac{1}{\alpha}\frac{\partial\Gamma(t)}{\partial t}\right]$$

$$\frac{\Gamma(t)}{\Gamma(t)\Psi(x,y,z)}\left[\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z)}{\partial z^2}\right] = \frac{\Psi(x,y,z)}{\Gamma(t)\Psi(x,y,z)}\left[\frac{1}{\alpha}\frac{\partial\Gamma(t)}{\partial t}\right]$$

$$\frac{1}{\Psi(x,y,z)}\left[\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z)}{\partial z^2}\right] = \left[\frac{1}{\alpha\Gamma(t)}\frac{\partial\Gamma(t)}{\partial t}\right] = -\lambda^2$$

Then separated functions becomes:

$$\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\lambda^2\Psi(x,y,z) = 0$$

$$\frac{\partial\Gamma(t)}{\partial t}+\alpha\Gamma(t)\lambda^2 = 0$$

The other side can also be separated the similar way such as:

$$\Psi(x,y,z) = X(x)Y(y)Z(z)$$

Substitution of this equation into above differential equation yields:

$$\frac{1}{X(x)}\frac{\partial^2\mathrm{X}(x)}{\partial x^2}+\frac{1}{Y(x)}\frac{\partial^2\mathrm{Y}(y)}{\partial y^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = -\lambda^2$$

$$\frac{1}{X(x)}\frac{\partial^2\mathrm{X}(x)}{\partial x^2} = -\beta^2$$

$$\frac{1}{Y(y)}\frac{\partial^2\mathrm{Y}(y)}{\partial y^2} = -\gamma^2$$

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = -\eta^2$$

$\frac{\partial^2\mathrm{X}(x)}{\partial x^2}+\beta^2\mathrm{X}(x) = 0$ $\quad A^2+\beta^2 = 0 \;\; A^2 = -\beta^2 \quad A = \pm\beta i$

$$\mathrm{X}(x) = D_1 exp(-\beta i x)+D_2 exp(+\beta i x)$$

$$\mathrm{X}(x) = C_1 sin(\beta x)+C_2 cos(\beta x)$$

$$\frac{\partial^2\mathrm{Y}(y)}{\partial y^2}+\gamma^2\mathrm{Y}(y) = 0$$

$$\frac{\partial^2 Z(z)}{\partial z^2}+\eta^2 Z(z) = 0$$

$\frac{\partial\Gamma(t)}{\partial t}+\alpha\Gamma(t)\lambda^2 = 0 \quad A+\alpha\lambda^2 = 0 \;\; A = -\alpha\lambda^2$

$$\Gamma(t) = C_7 exp(-\alpha\lambda^2 t)$$

$$\Gamma(t) = C_7 exp(-\alpha(\beta^2+\gamma^2+\eta^2)t)$$

Where

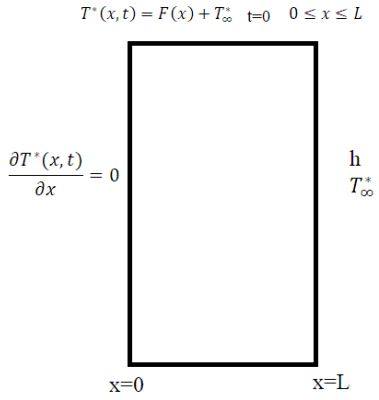
$$\lambda^2 = \beta^2+\gamma^2+\eta^2$$

General solution of the equation becomes:

$$T(x,y,z,t) = [C_1\sin(\beta x)+C_2\cos(\beta x)][C_3\sin(\gamma y)+C_4\cos(\gamma y)][C_5\sin(\eta z) + C_6\cos(\eta z)][\exp(-\alpha(\beta^2+\gamma^2+\eta^2)t]$$

EXAMPLE 1: SEPARATION OF VARIABLE HOMOGENEOUS SOLUTION, TIME DEPENDENT; CARTASIAN COORDINATES

In this problem one side of the wall (x=0) is insulated and other side is exposed to the convective heat transfer. At initial time, the wall has a temperature profile of F(x). It should be also note that by taking $T = T^* - T_\infty^*$ equation becomes homegeneous. [$q_{convection} = hT = h(T^* - T_\infty^*)$]



$\frac{\partial^2 T^*(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T^*(x,t)}{\partial t}$ Boundary conditions $\frac{\partial T^*(x,t)}{\partial x} = 0$ x=0 t >0 $-k\frac{\partial T^*(x,t)}{\partial x} = h(T^*(x,t) - T_\infty)$ by arranging $\frac{\partial T^*(x,t)}{\partial x} + H(T^*(x,t) - T_\infty)$ x=L t>0 $H = \frac{h}{k}$ $T(x,t) = F(x) - T_\infty$ t=0 $0 \le x \le L$	$T(x,t) = T^*(x,t) - T_\infty$ $\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ Boundary conditions $\frac{\partial T(x,t)}{\partial x} = 0$ x=0 t >0 $\frac{\partial T(x,t)}{\partial x} + HT(x,t) = 0$ x=L t>0 $H = \frac{h}{k}$ $T(x,t) = F(x)$ t=0 $0 \le x \le L$

Separation of variables:

$T(x,t) = X(x)\Gamma(t)$

$\Gamma(t)\frac{\partial^2 X(x)}{\partial x^2} = \frac{X(x)}{\alpha}\frac{\partial \Gamma(t)}{\partial t}$

$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha\Gamma(t)}\frac{\partial \Gamma(t)}{\partial t} = -\beta^2$

$\frac{\partial^2 X(x)}{\partial x^2} + \beta^2 X(x) = 0$

$\frac{\partial \Gamma(t)}{\partial t} + \beta^2\alpha\Gamma(t) = 0$

Boundary conditions

$\frac{\partial X(x)}{\partial x} = 0$ x=0 t >0

$\frac{\partial X(x)}{\partial x} + HX(x) = 0$ x=L t>0 $H = \frac{h}{k}$

$T(x,t) = F(x)$ t=0 $0 \le x \le L$

Solution set:

$X(x) = C_1\cos(\beta x) + C_2\sin(\beta x)$

$\Gamma(t) = C_3\exp(-\beta^2\alpha t)$

Application of boundary conditions:

$\frac{\partial X(x)}{\partial x} = -C_1\beta\sin(\beta x) + C_2\beta\cos(\beta x)$

$\frac{\partial X(x)}{\partial x} = 0$ x=0 t >0

$\frac{\partial X(x)}{\partial x} = -C_1\beta\sin(\beta 0) + C_2\beta\cos(\beta 0) = -C_1\beta * 0 + C_2\beta * 1 = 0 \xrightarrow{yields} C_2 = 0$

$\frac{\partial X(x)}{\partial x} + HX(x) = 0$ x=L t>0

$-C_1 \beta \sin(\beta L) + C_1 H \cos(\beta L) = 0 \quad \xrightarrow{because} C_1 \neq 0$

$\beta_m \sin(\beta_m L) - H \cos(\beta_m L) = 0$ or

$\beta_m \tan(\beta_m L) = H$

$\beta_m L \sin(\beta_m L) - HL \cos(\beta_m L) = 0$

$\beta_m \mathrm{L} \tan(\beta_m L) = HL \qquad \xi_m = \beta_m \mathrm{L} \quad Bi = HL = \frac{hL}{k} = Biot\ number$

$\xi_m \sin(\xi_m) + Bi \cos(\xi_m) = 0$ or $\xi_m \tan(\xi_m) = Bi$

These equations has more than one solution. It can be solved only by numerical root finding methods. If $\xi_m \tan(\xi_m) = Bi$ form is taken into account, a caution should be recmmended due to fact that $\tan(\xi_n)$ is nıt a continious function. Temperature profile becomes.

$$T^*(x,t) - T_\infty = T(x,t) = \sum_{m=1}^{\infty} C_m \cos(\beta_m x) \exp(-\beta_m^2 \alpha t)$$

We still do not know the C_n coefficients we will use the remaining boundary conditions to find it.

$T(x,t) = F(x)$ t=0 $0 \leq x \leq L$

$$F(x) = \sum_{m=1}^{\infty} C_m \cos(\beta_n x)$$

This equation has n coefficients. In order to solve these coefficients, we will use properties of orthogonal functions.

$$\int_0^L X(\beta_n x) X(\beta_m x) dx = \begin{cases} 0 & m \neq n \\ N(\beta_m) & m = n \end{cases}$$

$$\int_0^L \cos(\beta_n x) \cos(\beta_m x)\, dx = \begin{cases} 0 & m \neq n \\ N(\beta_m) & m = n \end{cases}$$

$$\int_0^L F(x) \cos(\beta_m x) dx = C_m \int_0^L cos^2(\beta_m x) dx$$

$$N(\beta_m) = \int_0^L cos^2(\beta_m x) dx = \int_0^L \frac{1 + cos(2\beta_m x)}{2} dx = \left[\frac{x}{2} + \frac{\sin(2\beta_m x)}{4\beta_m}\right]_{x=0}^{L} = \frac{L}{2} + \frac{\sin(2\beta_m L)}{4\beta_m}$$

To get an alternative formula for $N(\beta_m)$ eigenvalue function

$\beta_m \sin(\beta_n L) = H \cos(\beta_m L)$ can be utilised

$\sin(\beta_m L) = \frac{H}{\beta_m} \cos(\beta_m L) \qquad \sin(2\beta_m L) = 2 \sin(\beta_m L) \cos(\beta_m L) \qquad \sin(2\beta_m L) = 2 \frac{H}{\beta_m} \cos^2(\beta_m L)$

On the other hand taking square of eigenvalue function:

$\sin^2(\beta_m L) = \left[\frac{H}{\beta_m}\right]^2 \cos^2(\beta_m L) \qquad \sin^2(\beta_m L) + \cos^2(\beta_m L) = 1 \qquad \sin^2(\beta_m L) = 1 - \cos^2(\beta_m L)$

$1 - \cos^2(\beta_m L) = \frac{H^2}{\beta_m^2} \cos^2(\beta_m L) \qquad 1 = \left[\frac{H^2}{\beta_m^2} + 1\right] \cos^2(\beta_m L) \quad \cos^2(\beta_m L) = \frac{\beta_m^2}{H^2 + \beta_m^2}$

Substituting into previous equation

$$\sin(2\beta_m L) = 2 \frac{H}{\beta_m} \cos^2(\beta_m L) = 2 \frac{H}{\beta_m} \frac{\beta_m^2}{H^2 + \beta_m^2}$$

$$N(\beta_m) = \int_0^L cos^2(\beta_m x)dx = \frac{L}{2} + \frac{\sin(2\beta_m L)}{4\beta_m} = \frac{L}{2} + \frac{2\frac{H}{\beta_m}\frac{\beta_m^2}{H^2+\beta_m^2}}{4\beta_m} = \frac{L(\beta_m^2+H^2)+H}{2(\beta_m^2+H^2)}$$

$$C_m = \frac{1}{N(\beta_m)}\int_0^L F(x)\cos(\beta_m x)dx$$

Depends on value of F(x), this integration can be solved by using numerical or analytical methods.

$T(x,t) = T^*(x,t) - T_\infty^* = \sum_{m=1}^{\infty} C_m \cos(\beta_m x)\exp(-\beta_m^2 \alpha t) =$
$\sum_{m=1}^{\infty} \frac{1}{N(\beta_m)}\left[\int_0^L F(x)\cos(\beta_m x)dx\right]\cos(\beta_m x)\exp(-\beta_m^2 \alpha t)$

PROBLEM MATLAB SOLUTION

```
function  [T] = T1(x,t,L,H,alpha)
T=0;
for m=1:20
beta=beta1(m,L,H);
T=T+CN1(beta,L,H)*cos(beta*x)*exp(-beta*beta*alpha*t);
end
```

```
function [C] = CN1(beta,L,H)
y=@(x) F1(x).*cos(beta*x);
%matlab integration formula is used
C1=quad(y,0,L);
N=(L*(beta*beta+H*H)+H)/(2*(beta*beta+H*H));
C=C1/N;
end
```

```
function [beta] = beta1(m,L,H)
PI=4.0*atan(1.0);
x1=((m-1)*PI+0.0000001)/L;
x2= x1+(PI/2.0-0.0000001)/L;
beta=bisection(@(x) func1(x,L,H),x1,x2);
```

```
function xr=bisection(f,xl,xu)
% bisection root finding method
maxit=100;
iter=0;
es=0.0000001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
xr=(xl+xu)/2.0;
iter=iter+1;
if xr~=0 ea=abs((xu-xl)/(xu+xl))*100;end
fxl= f(xl);
fxr= f(xr);
test= fxl*fxr;
if test == 0.0      ea=0;
elseif test < 0.0  xu=xr;
else              xl=xr;
end
end
if(iter>=maxit) fprintf('Maximum number of iteration is exceeded result might not be valid',"MAKSİMUM NUMBER OF
ITERATION WARNING');
end
end
```

```
function [F] = F1(x)
F=1.0;
end
```

```
function [f] = func1(beta,L,H)
f=(beta*L)*tan(beta*L)-H*L;
end
```

```
function [N] = N1(m,L,H)
beta=beta1(m,L,H);
N=(L*(beta*beta+H*H)+H)/(2*(beta*beta+H*H));
end
```

PROBLEM JAVA SOLUTION

In order to simplify java codes a root class (interface created) to define functions. Some basic numerical processes such as derivatives, integral and root finding processes are embedded in this class so that it should not be repeated in every program. Interface if_x.java is listed first, and in the Wall problem this class is utilised as the root of all functions (defined as Lambda variable functions)

```
import static java.lang.Math.*;
import java.io.*;
import java.util.*;
import javax.swing.*;
import javax.swing.table.*;
import java.awt.*;
import static java.lang.Math.*;
  // single function single independent variable
  // example f=x*x
  // includes full set of derivatives
  // Reference :"Generation of Finite Difference Formulas on Arbitrary Spaced Grids",
  // Bength Fornberg, Mathematics of Computation, Volume 51, Number 184, October 1988
  // pages 699-706
  //interface (empty function) version
@FunctionalInterface
interface if_x
{  public double func(double x);
//first order derivative
default double dfunc(double x)
{double h=1.0e-3;
 int n=1;
 int M=10;
 return dfunc(x,n,M,h);
}
//second order derivative
default double dfunc2(double x)
{double h=1.0e-3;
 int n=2;
 int M=10;
 return dfunc(x,n,M,h);
}
default double dfunc(double x,int N,int Mi,double hi)
{// order of the maximum derivative
 // N order of derivative
 // M degree of difference formula
double c[][][];
double alpha[];
double h;
int M;
double a[]=new double[0];
h=0.01;
double x0=0;
M=20;
double alphai[]={0,1,-1,2,-2,3,-3,4,-4,5,-5,6,-6,7,-7,8,-8,9,-9,10,-10,11,-11,12,-12,13,-13,14,-14,15,
-15,16,-16,17,-17,18,-18,19,-19,20,-20,21,-21,22,-22,23,-23,24,-24,25,-25,26,-26,27,-27,28,-28,29,-29,30,-30,
31,-31,32,-32,33,-33,34,-34,35,-35,36,-36,37,-37,38,-38,39,-39,40,-40,41,-41,42,-42,43,-43,44,-44,45,-45,46,
-46,47,-47,48,-48,49,-49,50,-50,51,-51,52,-52,53,-53,54,-54,55,-55,56,-56,57,-57,58,-58,59,-59,60,-60,
-61,61,-62,62,-63,63,-64,64,-65,65,-66,66,-67,67,-68,68,-70,70,-71,71,-72,72,-73,73,-74,74,-75,75,
-76,76,-77,77,-78,78,-79,79,-80,80,-81,81,-82,82,-83,83,-84,84,-85,85,-86,86,-87,87,
-88,88,-89,89,-90,90,-91,91,-92,92,-93,93,-94,94,-95,95,-96,96,-97,97,-98,98,-99,99,-100,100};
alpha=alphai;
int N1=alpha.length-1;
// M   degree of highest derivative
// N+1 number of coefficients
double delta[][][]=new double[N1+1][N1+1][M+1];
double c1,c2,c3;
delta[0][0][0]=1.0;
c1=1.0;
for(int n=1;n<=N1;n++)
{  c2=1;
   for(int nu=0;nu<=(n-1);nu++)
   {c3=alpha[n]-alpha[nu];
    c2=c2*c3;
    if(n<=M) delta[n-1][nu][n]=0.0;
    for(int m=0;m<=Math.min(n,M);m++)
    {
       if(m==0)
       { delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m])/c3;}
```

```
        else
        {delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m]-m*delta[n-1][nu][m-1])/c3;}
     }//next m
    }//next nu
    for(int m=0;m<=Math.min(n,M);m++)
    {   if(m==0)
                {delta[n][n][m]=c1/c2*(-(alpha[n-1]-x0)*delta[n-1][n-1][m]);}
                else
                {delta[n][n][m]=c1/c2*(m*delta[n-1][n-1][m-1]-(alpha[n-1]-x0)*delta[n-1][n-1][m]);}
    }//next m
    c1=c2;
}//next n
c=delta;
  if(Mi<N) M=N;
  else M=Mi;
  h=hi;
  double deriv=0;
  double h1=1/h;
  double h2=1;
  for(int j=0;j<N;j++)
  {h2*=h1;}
  for(int i=0;i<c[0].length;i++)
  {   deriv+=c[M][i][N]*func(x+alpha[i]*h);}
  return deriv*h2;}
default double dfunc(double x,int N)
{int M=30;double h=0.05*N;return dfunc(x,N,M,h);}
default double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{
            //calculates legendre gauss-coefficients as coefficients of the integral
            //for n terms
            double EPS=3.0e-15;
            int m,j,i;
            double z1,z,xm,xl,pp,p3,p2,p1;
    //double x[]=new double[n];
    //double w[]=new double[n];
    double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
            m=(n+1)/2;
            xm=0.5*(x2+x1);
            xl=0.5*(x2-x1);
            for (i=1;i<=m;i++)
            { z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                    do {
                            p1=1.0;
                            p2=0.0;
                            for (j=1;j<=n;j++) {
                                    p3=p2;
                                    p2=p1;
                                    p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                            }
                            pp=n*(z*p1-p2)/(z*z-1.0);
                            z1=z;
                            z=z1-p1/pp;
                    } while (Math.abs(z-z1) > EPS);
                    a[0][i-1]=xm-xl*z;
                    a[0][n-i]=xm+xl*z;
                    a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                    a[1][n-i]=a[1][i-1];
            }
return a;
}

default double integral(double x1,double x2,int n)
{
//n : number of integral coefficients
// this routine first generates gauss legendre coefficients
// for [x1,x2] band
// then calculates gauss legendre integral
double a[][]=new double[2][n];
a=gauss_legendre_coefficients(x1,x2,n);
double z=0;
for(int i=0;i<n;i++)
{ z+=a[1][i]*func(a[0][i]);
}
return z;
}
default double bisection(double a,double b)
```

```
  {double b1=1.1*b;
   double r=(a+b)/2.0;
   double eps=1.0e-6;
   int nmax=100;
   int i=1;
   while(Math.abs(func(r))>eps && i<nmax)
   {if(func(a)*func(r)<0) b=r;
    else a=r;
    r=(a+b)/2.0;;
    i++;
   }
   if(i>=nmax) r=bisection(a,b1);
   return r;
  }
default double secant(double x)
{ double eps=1.0e-10;
  double y=1.0;
  double dy=0.0;
  int miter=100;
  int i=0;
  while(abs(y)>eps && i<miter)
  {y=func(x);
   dy=dfunc(x);
   x=x-y/dy;
   i++;
  }
  if(i>=miter)
    System.out.println("i="+i+"results may not be valid");
  return x;
}
default double illinois(double xa,double xu)
{
//Modified False position root finding: Illinois method
double test;
double p=0;
double es,ea;
double fa,fp,fu;
int maxit=100,iter=0;
es=0.000001;
ea=1.1*es;
double xold;
int ia=0,iu=0;
fa=func(xa);
fu=func(xu);
p=xu;
double xpold;
while(iter<maxit && ea>es)
{
xpold=p;
p=xu-fu*(xa-xu)/(fa-fu);
fp=func(p);
if(signum(fp)!=signum(fu))
{xa=xu;fa=fu;}
else
{fa=fa/2;}
xu=p;fu=fp;
iter++;
if(p!=0) ea=abs((p-xpold)/p)*100.0;
}
if(iter>=maxit) System.out.println("Maximum number of iteration is exceeded ");
return p;
}
default double newton_bisection(double x1, double x2)
{
double xacc=1.0e-10; //accuracy
double h=0.0001; //derivative function step
int MAXIT=100;  //Maximum number of iterations
int j;
double dfr,dx,dxold,fr,fh,fl;
double temp,b,a,r;
fl=func(x1);
fh=func(x2);
//Check if a root is existed in the given region
if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
System.out.println("There are no root in the given region or double root?");
if (fl == 0.0) return x1;
```

```
if (fh == 0.0) return x2;
if (fl < 0.0)   { a=x1;b=x2;}
else            { b=x1;a=x2;}
r=0.5*(x1+x2); //midpoint value
dxold=Math.abs(x2-x1);
dx=dxold;
fr=func(r);   //function value at midpoint
dfr=dfunc(r); //derivative of function value at midpoint
for (j=1;j<=MAXIT;j++)
{
if ((((r-b)*dfr-fr)*((r-a)*dfr-fr) > 0.0) || (Math.abs(2.0*fr) > Math.abs(dxold*dfr)))
   { //bisection step
                dxold=dx;
     dx=0.5*(b-a);
     r=a+dx;
     if (a == r) return r; //çözüm!!!!
   }
else{ //Newton-Raphson (Secant) step
                dxold=dx;
     dx=fr/dfr;
     temp=r;
     r -= dx;
     if (temp == r) return r;  //solution!!!!!
   }
if (Math.abs(dx) < xacc) return r; //solution!!!!!
fr=func(r);
dfr=dfunc(r);
if (fr < 0.0)
a=r;
else
b=r;
}
System.out.println("Maximum number of iterations are exceeded");
return 0.0; //program should never reach here
          //dummy return
}
default double brent(double a,double b)
{
double test;
double p=0;
double es,ea;
double f1,f2,f3,fp;
int maxit=500,iter=0;
double tol=1.0e-15;
es=0.0000001;
double x1=a;f1=func(x1);
double x2=b;f2=func(x2);
double x3=(x1+x2)/2.0;f3=func(x3);
if(f1==0) return x1;
else if(f2==0) return x2;
else if(f3==0) return x3;
if(f1*f2>0) System.out.println("No root is existed in the given region");
p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
fp=func(p);
ea=Math.abs(x3-p);
while((ea>es)&&(iter<maxit))
{               if(Math.abs(f3)<tol) return x3;
   if((p<x1) && (p>x2))
   {p=(x1+x2)/2.0;
    if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
    else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   }
   else
   {
   if(p>x3) {x1=x3;f1=f3;x3=p;f3=fp;}
   else if(p<x3) {x2=x3;f2=f3;x3=p;f3=fp;}
   p=-(f2*f3*x1*(f2-f3)+f3*f1*x2*(f3-f1)+f1*f2*x3*(f1-f2))/((f1-f2)*(f2-f3)*(f3-f1));
   fp=func(p);
   ea=Math.abs(x3-p);
   }
   iter++;
}
if(iter>=maxit) JOptionPane.showMessageDialog(null,"Warning : MAximum number of iteration is exceeded \n"+
  " result may not be valid","MAXIMUM ITERATION NUMBER WARNING",JOptionPane.WARNING_MESSAGE);
return p;
}
```

```
default double enlarge(double x0,double dx)
{  //enlarge region untill a root is existed
                double x1=x0;
                double x2=x1+dx;
                int NTRY=200;
                double a[]=new double[2];
                double FACTOR=1.001;
                int j;
                double f1,f2;
                if (x1 == x2) System.out.println("Giriş değerleri yanlış verildi");
                f1=func(x1);
                f2=func(x2);
                for (j=1;j<=NTRY;j++)
                { if (f1*f2 < 0.0) {break;}
                  else {x2+=dx;f2=func(x2);}
                }
                return x2;
}
}
```



```
import java.io.*;
import javax.swing.*;
import java.awt.*;
import static java.lang.Math.*;
public class HT_plane_wall
{
public if_x F;
public double h,L,k,alpha,Bi,Ti,Tinf,H;
public  HT_plane_wall(double hi,double Li,double ki,double alphai,double Tii,double Tinfi)
{h=hi;k=ki;L=Li;alpha=alphai;Bi=h*L/k;Ti=Tii;Tinf=Tinfi;H=h/k;
 //initial temperature profile is defined as constant change
 //if you desired to do so
 F=xx->Ti-Tinf;
}

  public static double ksi(int n,double Bi)
  {double x1=n*PI+0.0000001;
   double x2=x1+(PI/2-0.000001);
   if_x f1=x->x*tan(x)-Bi;
   double ksi=f1.bisection(x1,x2);
   return ksi;
  }
  public double teta(double x,double t)
  {double ksi=0;
   double beta=0;
   double C=0;
   double N=0,N1=0,C1=0,C2=0;
   double teta=0;
   //coefficient of equation
   for(int n=0;n<20;n++)
   {ksi=ksi(n,Bi);
    beta=ksi/L;
    final double beta1=beta;
    N=L/2.0+sin(2*ksi)/(4.0*beta);
    N1=(L*(beta*beta+H*H)+H)/(2*(beta*beta+H*H));
    if_x Cn=xx->F.func(xx)*cos(beta1*xx);
    C=1.0/N1*Cn.integral(0,L,20);
    teta+=C*cos(beta*x)*exp(-beta*beta*alpha*t);
    }
    return teta;
  }
  public double dteta(double x,double t)
  {double ksi=0;
   double beta=0;
   double C=0;
   double N=0,N1=0,C1=0,C2=0;
   double dteta=0;
   double x1,x2,x3;
   //coefficient of equation
   for(int n=0;n<20;n++)
   {ksi=ksi(n,Bi);
    beta=ksi/L;
    final double beta1=beta;
    x1=tan(ksi);
```

```
    x2=(H/beta);
    x3=Bi/ksi;
    N=L/2.0+sin(2*ksi)/(4.0*beta);
    N1=(L*(beta*beta+H*H)+H)/(2*(beta*beta+H*H));
    if_x Cn=xx->F.func(xx)*cos(beta1*xx);
    C=1.0/N*Cn.integral(0,L,20);
    dteta+=-C*beta*sin(beta*x)*exp(-beta*beta*alpha*t);
     }
    return dteta;
  }
  public double T(double x,double t)
  {double teta=teta(x,t);
   double T=teta+Tinf;
   return T;
  }
  public double q(double x,double t)
  {double q=-k*dteta(x,t);
   return q;
  }

}
```

When a numerical code is given it will be a good idea to check the code outputs if possible. The first three eigenvalues of this solution is listed in "Fundamentals of Heat and Mass Transfer" by Incropera, Dewitt, Bergman, Lavine as:

EXAMPLE 2: SEPARATION OF VARIABLE HOMOGENEOUS SOLUTION, TIME DEPENDENT; CARTASIAN COORDINATES

This examples has two different convective heat transfer coefficient in each side of the wall, but outside fluid temperatures are the same

$q_{convection\_0} = h_1(T(0,t) = h_1 T(0,t) = h_1(T^*(0,t) - T_\infty)$

$q_{convection\_1} = h_2(T(L,t) = h_2 T(L,t) = h_2(T^*(L,t) - T_\infty)$

By defining temperature T(x,t) differential equation becomes homogeneous.

$\frac{\partial^2 T^*(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T^*(x,t)}{\partial t}$ Boundary conditions $-\frac{\partial T^*(x,t)}{\partial x} + H_1(T^*(x,t) - T_\infty) = 0$ x=0 t >0 $H_1 = \frac{h_1}{k}$ $\frac{\partial T^*(x,t)}{\partial x} + H_2(T^*(x,t) - T_\infty) = 0$ x=L t>0 $H_2 = \frac{h_2}{k}$ $T^*(x,t) = F(x) + T_\infty$ t=0 $0 \le x \le L$	$T(x,t) = T^*(x,t) - T_\infty$ $\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ Boundary conditions $-\frac{\partial T(x,t)}{\partial x} + H_1 T(x,t) = 0$ x=0 t >0 $H_1 = \frac{h_1}{k}$ $\frac{\partial T(x,t)}{\partial x} + H_2 T(x,t) = 0$ x=L t>0 $H_2 = \frac{h_2}{k}$ $T(x,t) = F(x)$ t=0 $0 \le x \le L$

Separation of variables:

$T(x,t) = X(x)\Gamma(t)$

$\Gamma(t)\frac{\partial^2 X(x)}{\partial x^2} = \frac{X(x)}{\alpha}\frac{\partial \Gamma(t)}{\partial t}$

$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha\Gamma(t)}\frac{\partial \Gamma(t)}{\partial t} = -\beta^2$

$\frac{\partial^2 X(x)}{\partial x^2} + \beta^2 X(x) = 0$

$\frac{\partial \Gamma(t)}{\partial t} + \beta^2 \alpha \Gamma(t) = 0$

Boundary conditions

$-\frac{\partial X(x)}{\partial x} + H_1 X(x) = 0$ x=0 t >0 $H_1 = \frac{h_1}{k}$

$\frac{\partial T(x)}{\partial x} + H_2 X(x) = 0$ x=L t>0 $H_2 = \frac{h_2}{k}$

$T(x,t) = F(x)$ t=0 $0 \le x \le L$

Solution:

$X(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x)$

$\Gamma(t) = C_3 \exp(-\beta^2 \alpha t)$

$T(x,t) = [C_1 \cos(\beta x) + C_2 \sin(\beta x)] \exp(-\beta^2 \alpha t)$

Applications of boundary conditions:

$$\frac{\partial X(x)}{\partial x} = -C_1 \beta \sin(\beta x) + C_2 \beta \cos(\beta x)$$

$$-C_1 \beta \sin(\beta 0) + C_2 \beta \cos((\beta 0) + H_1 [C_1 \cos((\beta 0) + C_2 \sin((\beta 0)] = 0$$

$$-C_1 0 + C_2 \beta 1 + H_1 [C_1 1 + C_2 0] = 0$$

$$C_2 \beta + C_1 H_1 = 0$$

$$C_2 = \frac{C_1 H_1}{\beta}$$

$$X(x) = C_1 [\beta \cos(\beta x) + H_1 \sin(\beta x)]$$

$$\frac{\partial X(x)}{\partial x} = C_1 [-\beta^2 \sin(\beta x) + H_1 \beta \cos(\beta x)]$$

$\frac{\partial T(x)}{\partial x} + H_2 X(x) = 0$ x=L t>0

$C_1 [-\beta^2 \sin(\beta L) + H_1 \beta \cos(\beta L)] + H_2 C_1 [\beta \cos(\beta L) + H_1 \sin(\beta L)] = 0 \rightarrow C_1 \neq 0$

$[-\beta^2 \sin(\beta L) + H_1 \beta \cos(\beta L)] + H_2 [\beta \cos(\beta L) + H_1 \sin(\beta L)] = 0$

$\tan(\beta_m L) = \frac{\beta_m (H_1 + H_2)}{\beta_m^2 - H_1 H_2}$ $\xi_m = \beta_m L$ $Bi_1 = H_1 L = \frac{h_1 L}{k}$ $Bi_2 = H_2 L = \frac{h_2 L}{k}$

$\tan(\xi_m) = \frac{\xi_m (Bi_1 + Bi_2)}{\xi_m^2 - Bi_1 Bi_2}$ or

$\sin(\beta_m L) - \frac{\beta_m (H_1 + H_2)}{\beta_m^2 - H_1 H_2} \cos(\beta_m L) = 0$ or $\sin(\xi_m) - \frac{\xi_m (Bi_1 + Bi_2)}{\xi_m^2 - Bi_1 Bi_2} \cos(\xi_m) = 0$

This equation has multiple roots.

$T(x,t) = \sum_{m=1}^{\infty} C_m [\beta_m \cos(\beta_m x) + H_1 \sin(\beta x)] \exp(-\beta_m^2 \alpha t)$

$T(x,t) = F(x)$ t=0 $0 \le x \le L$

$F(x) = \sum_{m=1}^{\infty} C_m [\beta_m \cos(\beta_m x) + H_1 \sin(\beta x)]$

This equation has n coefficients. In order to solve these coefficients, we will use properties of orthogonal functions.

$$\int_0^L X(\beta_n x) X(\beta_m x) dx = \begin{cases} 0 & m \neq n \\ N(\beta_m) & m = n \end{cases}$$

$$\int_0^L F(x) [\beta_m \cos(\beta_m x) + H_1 \sin(\beta x)] dx = C_m \int_0^L [\beta_m \cos(\beta_m x) + H_1 \sin(\beta_m x)]^2 dx$$

$$\int_0^L [\beta_m \cos(\beta_m x) + H_1 \sin(\beta_m x)]^2 dx$$

$$= \int_0^L [\beta_m^2 \cos^2(\beta_m x) + 2\beta_m H_1 \sin(\beta_m x) \cos(\beta_m x) + H_1^2 \sin(\beta_m x)] dx$$

$$= \int_0^L \left[\beta_m^2 \frac{1+\cos(2\beta_m x)}{2} + \beta_m H_1 \sin(2\beta_m x) + H_1^2 \frac{1-\cos(2\beta_m x)}{2}\right] dx$$

$$= \int_0^L \left[\frac{\beta_m^2 + H_1^2}{2} + \frac{\beta_m^2 - H_1^2}{2} \cos(2\beta_m x) + \beta_m H_1 \sin(2\beta_m x)\right] dx$$

$$=\left[\frac{\beta_m^2+H_1^2}{2}x+\frac{\beta_m^2-H_1^2}{4\beta_m}\sin(2\beta_m x)-\beta_m H_1\frac{\cos(2\beta_m x)}{2\beta_m}\right]_{x=0}^{L}$$

$$N(\beta_m)=\frac{(\beta_m^2+H_1^2)}{2}+\frac{(\beta_m^2-H_1^2)}{4\beta_m}\sin(2\beta_m L)+\frac{H_1}{2}(1-\cos(2\beta_m L))$$

Substituting $\sin(\beta_m L)=\frac{\beta_m(H_1+H_2)}{\beta_m^2-H_1H_2}\cos(\beta_m L)$ eigenvalue function

$$N(\beta_m)=0.5\left[(\beta_m^2+H_1^2)\left(L+\frac{H_2}{(\beta_m^2+H_2^2)}\right)+H_1\right]$$

$$C_m=\frac{1}{N(\beta_m)}\int_0^L F(x)[\beta_m\cos(\beta_m x)+H_1\sin(\beta x)]dx$$

Depends on value of F(x), this integration can be solved by using numerical or analytical methods.

$$T(x,t)=\sum_{m=1}^{\infty}\left[\frac{1}{N(\beta_m)}\int_0^L F(x)[\beta_m\cos(\beta_m x)+H_1\sin(\beta x)]dx\right][\beta_m\cos(\beta_m x)+H_1\sin(\beta x)]\exp(-\beta_m^2\alpha t)$$

PROBLEM MATLAB SOLUTION

```
function  [T] = T2(x,t,L,H1,H2,alpha)
T=0;
for m=1:20
beta=beta2(m,L,H1,H2)
C2=CN2(beta,L,H1,H2)
T=T+C2*(beta*cos(beta*x)+H1*sin(beta*x))*exp(-beta*beta*alpha*t);
End
```

```
function [C] = CN2(beta,L,H1,H2)
y=@(x) F2(x).*(beta*cos(beta*x)+H1*sin(beta*x));
C1=quad(y,0,L);
N=0.5*(L*(beta*beta+H1*H1)*(L+H2/(beta*beta+H2*H2)+H1);
C=C1/N;
End
```

```
function [beta] = beta2(m,L,H1,H2)
PI=4*atan(1);
x1=((m-1)*PI+0.0000001)/L;
x2= x1+(PI/2-0.000001)/L;
beta=bisection(@(x) func2(x,L,H1,H2),x1,x2);
```

```
function [F] = F2(x)
F=1.0;
end
```

```
function [f] = func2(beta,L,H1,H2)
f=tan(beta*L)-beta*(H1+H2)/(beta*beta-H1*H2);
```

PROBLEM JAVA SOLUTION

```
import java.io.*;
import javax.swing.*;
import java.awt.*;
import static java.lang.Math.*;
public class HT_plane_wall2
{
public if_x F;
public double h1,h2,L,k,alpha,Bi1,Bi2,Ti,Tinf,H1,H2;
public  HT_plane_wall2(double h1i,double h2i,double Li,double ki,double alphai,double Tii,double Tinfi)
{h1=h1i;h2=h2i;k=ki;L=Li;alpha=alphai;Bi1=h1*L/k;Bi2=h2*L/k;Ti=Tii;Tinf=Tinfi;H1=h1/k;H2=h2/k;
//initial temperature profile is defined as constant change
//if you desired to do so
F=xx->Ti-Tinf;
}

public static double ksi(int n,double Bi1,double Bi2)
{double x1=n*PI+0.0000001;
double x2=x1+(PI/2-0.000001);
```

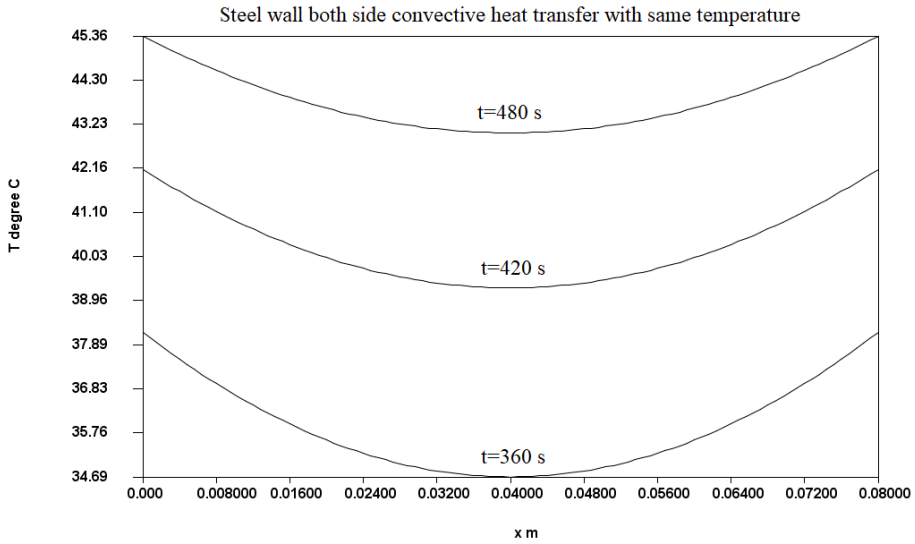
```
if_x f1=x->(x*x-Bi1*Bi2)*Math.tan(x)-x*(Bi1+Bi2);
double ksi=f1.illinois(x1,x2);
return ksi;
}
public double teta(double x,double t)
{double ksi=0;
double beta=0;
double C=0;
double N=0,N1=0,N2=0,C1=0,C2=0;
double teta=0;
//coefficient of equation
for(int n=0;n<10;n++)
{ksi=ksi(n,Bi1,Bi2);
beta=ksi/L;
final double beta1=beta;
//N=0.5*((beta*beta+H1*H1)*(L+H2/(beta*beta+H2*H2))+H1);
//if_x intN=(xx)->(beta1*cos(beta1*xx)+H1*sin(beta1*xx))*(beta1*cos(beta1*xx)+H1*sin(beta1*xx));
//N1=intN.integral(0,L,40);
N2=L/2.0*(beta*beta+H1*H1)+sin(2.0*beta*L)/(4*beta)*(beta*beta-H1*H1)+H1/2.0*(1.0-cos(2.0*beta*L));
if_x Cn=xx->F.func(xx)*(beta1*cos(beta1*xx)+H1*sin(beta1*xx));
C=1.0/N2*Cn.integral(0,L,20);
teta+=C*(beta*cos(beta*x)+H1*sin(beta*x))*exp(-beta*beta*alpha*t);
}
return teta;
}
public double dteta(double x,double t)
{double ksi=0;
double beta=0;
double C=0;
double N=0,N2=0,C1=0,C2=0;
double dteta=0;
//coefficient of equation
for(int n=0;n<10;n++)
{ksi=ksi(n,Bi1,Bi2);
beta=ksi/L;
final double beta1=beta;
N2=L/2.0*(beta*beta+H1*H1)+sin(2.0*beta*L)/(4*beta)*(beta*beta-H1*H1)+H1/2.0*(1.0-cos(2.0*beta*L));    if_x Cn=xx-
>F.func(xx)*(beta1*cos(beta1*xx)+H1*sin(beta1*xx));
C=1.0/N2*Cn.integral(0,L,20);
dteta+=C*(-beta*beta*sin(beta*x)+H1*beta*cos(beta*x))*exp(-beta*beta*alpha*t);
}
return dteta;
}
public double T(double x,double t)
{double T=teta(x,t)+Tinf;
return T;
}
public double q(double x,double t)
{double q=-k*dteta(x,t);
return q;
}

public static void main (String args[])
{ double h1=500; //W/m^2K
double h2=500; //W/m^2K
double k=63.9;
double alpha=18.8e-6;
double L=2.0*40e-3;
double t[]={8*60};  //s
double Tinf=60;
double Ti=-20;
int n1=t.length;
double x[]=new double[101];
double T[][]=new double[n1][101];
double q[][]=new double[n1][101];
HT_plane_wall2 htpw=new HT_plane_wall2(h1,h2,L,k,alpha,Ti,Tinf);
//initial temperature variation t=0
htpw.F=xx->(Ti-Tinf);
for(int j=0;j<n1;j++)
{
for(int i=0;i<=100;i++)
{   x[i]=i*L/100;
            T[j][i]=htpw.T(x[i],t[j]);
            q[j][i]=htpw.q(x[i],t[j]);
}
}
```

```
Plot pp=new Plot(x,T[0]);
for(int j=1;j<n1;j++)
{pp.addData(x,T[j]);}
pp.setPlabel("Steel wall one side insulated other side convective heat transfer");
pp.setXlabel(" x m");
pp.setYlabel(" T degree C");
pp.plot();
Plot pp1=new Plot(x,q[0]);
for(int j=1;j<n1;j++)
{pp1.addData(x,q[j]);}
pp1.setPlabel("Steel wall one side insulated other side convective heat transfer");
pp1.setXlabel(" x m");
pp1.setYlabel(" q W/m^2");
pp1.plot();
}}
```

Numerical example:

A steel wall (AISI 1010, k=63.9 W7mK) has Wall thickness of 0.04 m. Initially the pipe temperature is -20 ℃. (constant) The pipe is insulated on one surface. The other wall is exposed to 60 ℃ fluid with thermal convectivity coefficient of h=500 W/m^{2}K. Find the temperature profile of the Wall at a) t=360 s, b)t=420 s and c) t=480 s F(x)=-20-60=-80 ℃,

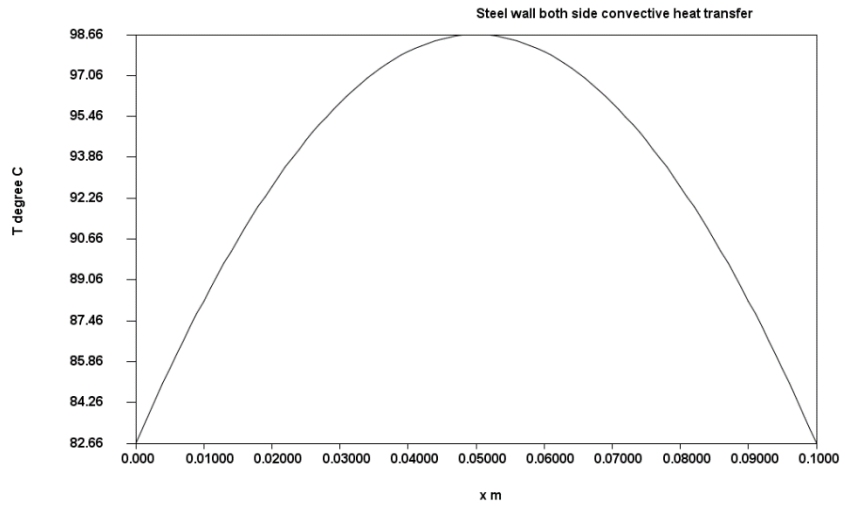


Numerical example:

A steel plate [α=1.2e-5 and k=43 W/(mK)] of thickness L=0.1 m, initially at a uniform temperature of 240 ℃, is suddenly immersed in an oil bath at $T_\infty = 40$ ℃. The convective heat transfer coefficients are h_1=600 W/(m^{2}K), h_2=600 W/(m^{2}K) calculate temperature profile for t=478.8 seconds.
F(x)=240-40=200 K

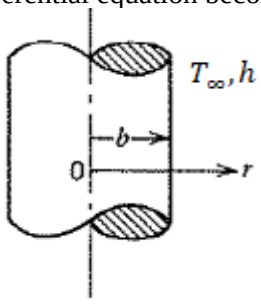
```
public class HT_plane_wall2_test1
{
public static void main (String args[])
{ double h1=600; //W/m^2K
double h2=600; //W/m^2K
double k=42;
double alpha=1.2e-5;
double L=0.1;
double t[]={478.8};  //s
double Tinf=40;
double Ti=240;
int n1=t.length;
double x[]=new double[101];
double T[][]=new double[n1][101];
double q[][]=new double[n1][101];
HT_plane_wall2 htpw=new HT_plane_wall2(h1,h2,L,k,alpha,Ti,Tinf);
//initial temperature variation t=0
htpw.F=xx->(Ti-Tinf);
for(int j=0;j<n1;j++)
{
```

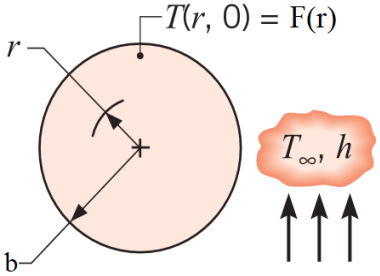
```
for(int i=0;i<=100;i++)
{   x[i]=i*L/100;
            T[j][i]=htpw.T(x[i],t[j]);
            q[j][i]=htpw.q(x[i],t[j]);
}
}
Plot pp=new Plot(x,T[0]);
for(int j=1;j<n1;j++)
{pp.addData(x,T[j]);}
pp.setPlabel("Steel wall both side convective heat transfer");
pp.setXlabel(" x m");
pp.setYlabel(" T degree C");
pp.plot();
Plot pp1=new Plot(x,q[0]);
for(int j=1;j<n1;j++)
{pp1.addData(x,q[j]);}
pp1.setPlabel("Steel wall both side convective heat transfer");
pp1.setXlabel(" x m");
pp1.setYlabel(" q W/m^2");
pp1.plot();
}
}
```



EXAMPLE 3: SEPARATION OF VARIABLES METHOD,HOMOGENEOUS SOLUTION, TIME DEPENDENT; CYLINDRICAL COORDINATES

Problems defines a solid rod of radius r, outside exposed to convective heat transfer of $q_{convection\_b} = h_1 T(b,t) = h_1(T^*(b,t) - T_\infty^*)$. By defining temperature $T(r,t) = T^*(r,t) - T_\infty^*$ differential equation becomes homogeneous.





$\frac{\partial^2 T^*(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T^*(r,t)}{\partial r} = \frac{1}{\alpha}\frac{\partial T^*(r,t)}{\partial t}$ Boundary conditions:	$\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t)}{\partial r} = \frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t}$ $T(r,t) = T^*(r,t) - T_\infty$ Boundary conditions:

$\frac{\partial T^*(r,t)}{\partial r} = 0$ r=0 t >0 $H = \frac{h}{k}$ $\frac{\partial T^*(r,t)}{\partial r} + H(T^*(r,t) - T_\infty) = 0$ r = b t > 0 $T^*(r,t) = F(r) + T_\infty$ t=0 $0 \le x \le b$	$\frac{\partial T(r,t)}{\partial r} = 0$ r=0 t >0 $H = \frac{h}{k}$ $\frac{\partial T(r,t)}{\partial r} + HT(r,t) = 0$ r = b t > 0 $T(r,t) = F(r)$ t=0 $0 \le x \le b$

Separation of variables:

$T(r,t) = R(r)\Gamma(t)$

$\Gamma(t)\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r}\right] = \frac{R(r)}{\alpha}\frac{\partial \Gamma(t)}{\partial t}$

$\frac{1}{R(r)}\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r}\right] = \frac{1}{\alpha\Gamma(t)}\frac{\partial \Gamma(t)}{\partial t} = -\beta^2$

$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \beta^2 R(r) = 0$

$\frac{\partial \Gamma(t)}{\partial t} + \beta^2 \alpha \Gamma(t) = 0$

Boundary conditions:

$\frac{\partial R(r)}{\partial r} = 0$ r=0 t >0 $H = \frac{h}{k}$

$\frac{\partial R(r)}{\partial r} + HR(r) = 0$ r = b t > 0

$T(r,t) = F(r)$ t=0 $0 \le r \le b$

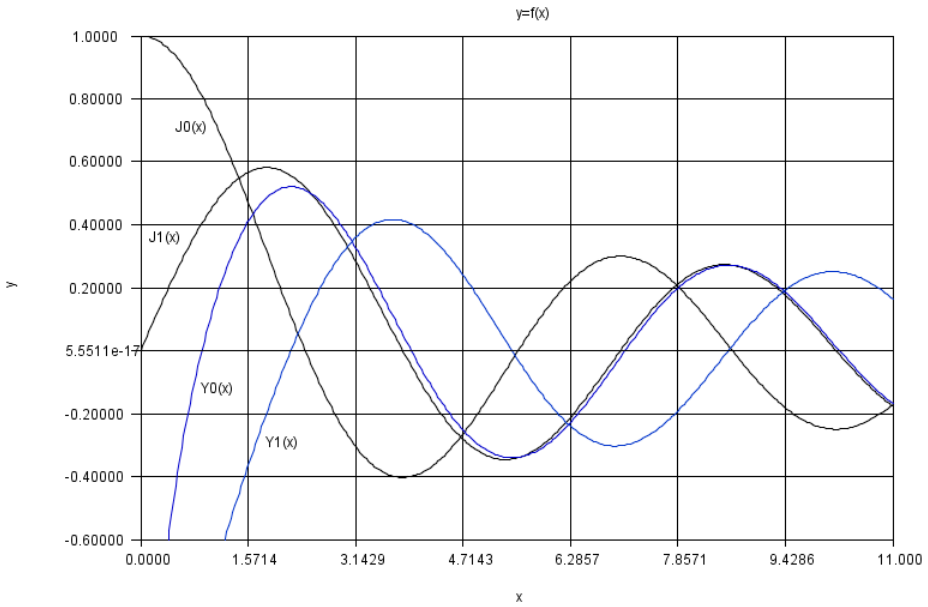
Solution:

$R(r) = C_1 J_0(\beta r) + C_2 Y_0(\beta r)$

$\Gamma(t) = C_3 \exp(-\beta^2 \alpha t)$

$T(r,t) = [C_1 J_0(\beta r) + C_2 Y_0(\beta r)] \exp(-\beta^2 \alpha t)$

J_0 and Y_0 in this equation called Bessel functions. In the figure, behavior of bessel functions are given as a plot.



As it is seen from the plot, $J_0(0) = 1$, $J_1(0) = 0$, $Y_0(0) = -\infty$, $Y_1(0) = -\infty$

Before going any further, let us look at some properties of bessel functions. Bessel functions can be defined as series. Bessel function $J_\nu(z)$ can be defined by using the following series:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{2\kappa}}{k!\,\Gamma(\nu+k+1)}$$

Where gamma function defined as

$\Gamma(z)=\int_{z=0}^{\infty} t^{z-1}e^{-t}dt=\frac{e^{-\gamma t}}{t}\prod_{n=1}^{\infty}\left(1+\frac{t}{n}\right)^{-1}e^{t/n}$

Where γ=0.57721566490153286... is the Euler-Mascheroni constant. The Bessel function $Y_\nu(z)$ is given by:

$Y_\nu(z)=\frac{J_\nu(z)\cos(\nu\pi)-J_{-\nu}(z)}{\sin(\nu\pi)}$

When ν=n is a positive integer, the solution $J_n(z)$ and $J_{-n}(z)$ are not independent: they are related by:

$J_n(z)=(-1)^n J_{-n}(z)$

Derivatives of the J or Y bessel functions can be given as:

$\frac{d}{dz}[z^\nu J_\nu(\beta z)]=\beta z^\nu J_{\nu-1}(\beta z)$

$\frac{d}{dz}[J_0(\beta z)]=-\beta J_1(\beta z)$

Integrals of the J or Y bessel functions can be given as:

$\int[z^\nu J_{\nu-1}(\beta z)]\,dz=\frac{1}{\beta}z^\nu J_\nu(\beta z)$

$\int[zJ_0(\beta z)]\,dz=\frac{1}{\beta}zJ_1(\beta z)$

$\int[zJ_0^2(z)]dz=\frac{z^2}{2}z\left(J_0^2(z)+J_1^2(z)\right)$ After this introduction to bessel functions, boundary conditions can be applied now:

$\frac{\partial R(r)}{\partial r}=-C_1\beta J_1(\beta r)-C_2\beta Y_1(\beta r)$

$\frac{\partial R(r)}{\partial r}=0$ r=0 t>0

$\frac{\partial R(r)}{\partial r}=-C_1\beta J_1(0)-C_2\beta Y_1(0)=0$

$\frac{\partial R(r)}{\partial r}=-C_1\beta 0-C_2\beta Y_1(\infty)=0 \quad \rightarrow C_2=0$

$\frac{\partial R(r)}{\partial r}+HR(r)=0$ r=b t>0

$-C_1\beta J_1(\beta b)+HC_1J_0(\beta b)=0 \quad \rightarrow C_1\neq 0$

$-\beta_m J_1(\beta_m b)+HJ_0(\beta_m b)=0$ This equation has multiple roots.

Equation can also be written as:

$-\beta_m bJ_1(\beta_m b)+HbJ_0(\beta_m b)=0$ or $-\xi_m J_1(\xi_m)+BiJ_0(\xi_m)$

where $\xi_m=\beta_m b$ and $Bi=Hb=\frac{hb}{k}$ is Biot number

$T(r,t)=\sum_{m=1}^{\infty}C_mJ_0(\beta_m r)\exp(-\beta_m^2\alpha t)$

$T(r,t)=F(r)$ t=0 $0\le r\le b$

$F(r)=\sum_{m=1}^{\infty}C_mJ_0(\beta_m r)$

This equation has n coefficients. In order to solve these coefficients, we will use properties of orthogonal functions.

$\int_0^b rJ_0(\beta_m r)J_0(\beta_n r)dr=\begin{cases}0 & m\neq n\\ N(\beta_m) & m=n\end{cases}$

$\int_0^b rF(r)J_0(\beta_m r)dr=C_m\int_0^b rJ_0^2(\beta_m r)dr$

$N(\beta_m)=\int_{r=0}^b rJ_0^2(\beta_m r)dr \qquad z=\beta_m r \qquad dz=\beta_m dr$

$N(\beta_m)=\frac{1}{\beta_m^2}\int_{z=0}^{b\beta_m}zJ_0^2(z)dz=\frac{1}{\beta_m^2}\frac{z^2}{2}[J_0^2(z)+J_1^2(z)]\Big|_{z=0}^{\beta_m b}=\frac{b^2}{2}[J_0^2(\beta_m b)+J_1^2(\beta_m b)]$

Or alternatively due to boundary condition $-\beta_m b J_1(\beta_m b) + H b J_0(\beta_m b) = 0 \;\rightarrow\; J_1(\beta_m b) = \frac{H b J_0(\beta_m b)}{\beta_m b}$

$$N(\beta_m) = \frac{b^2}{2}\left[J_0^2(\beta_m b) + \left(\frac{H b J_0(\beta_m b)}{\beta_m b}\right)^2\right] = \frac{b^2}{2}\left[J_0^2(\beta_m b) + \frac{H^2 J_0^2(\beta_m b)}{\beta_m^2}\right] = \frac{J_0^2(\beta_m b) b^2}{2}\left[\frac{\beta_m^2 + H^2}{\beta_m^2}\right]$$

$$\frac{1}{N(\beta_m)} = \frac{2}{J_0^2(\beta_m b)} \frac{\beta_m^2}{b^2(\beta_m^2 + H^2)}$$

$$C_m = \frac{1}{N(\beta_m)} \int_0^b r F(r) J_0(\beta_m r) dr$$

If $F(r) = T_i = const$

$$\int_0^b r T_i J_0(\beta_m r) dr = \frac{T_i}{\beta_m} r J_1(\beta_m r)\Big|_{r=0}^{b} = \frac{T_i b}{\beta_m} J_1(\beta_m b)$$

$$\frac{T_i b}{\beta_m} J_1(\beta_m b) = C_m N(\beta_m) = C_m \frac{b^2}{2}[J_0^2(\beta_m b) + J_1^2(\beta_m b)]$$

$$C_m = \frac{2T_i}{\beta_m b} \frac{J_1(\beta_m b)}{J_0^2(\beta_m b) + J_1^2(\beta_m b)} \qquad \xi_m = \beta_m b \quad C_m = \frac{2T_i}{\xi_m} \frac{J_1(\xi_m)}{J_0^2(\xi_m) + J_1^2(\xi_m)}$$

Depends on value of F(r), this integration can be solved by using numerical or analytical methods.

$$T^*(r,t) - T_\infty = T(r,t) = \sum_{m=1}^{\infty} C_m J_0(\beta_m r) \exp(-\beta_m^2 \alpha t)$$

$$= \sum_{m=1}^{\infty} \left[\frac{1}{N(\beta_m)} \int_0^b r F(r) J_0(\beta_m r) dr\right] J_0(\beta_m r) \exp(-\beta_m^2 \alpha t)$$

In this case heat transfer will be:

$$q = -k\frac{dT(r,t)}{dr} = \sum_{m=1}^{\infty} k C_m \beta_m J_1(\beta_m r) \exp(-\beta_m^2 \alpha t)$$

Before solving the entire problem eigenvalues will be solved by a small program and listed.

```
import java.io.*;
import javax.swing.*;
import java.awt.*;
import static java.lang.Math.*;
class bisection
{
//J Bessel function of integer order 0
public static double J0(double x)
{   double ax,z;
double xx,y,ans,ans1,ans2;
if ((ax=Math.abs(x)) < 8.0) {
                        y=x*x;
                        ans1=57568490574.0+y*(-13362590354.0+y*(651619640.7
                                +y*(-11214424.18+y*(77392.33017+y*(-184.9052456)))));
                        ans2=57568490411.0+y*(1029532985.0+y*(9494680.718
                                +y*(59272.64853+y*(267.8532712+y*1.0))));
                        ans=ans1/ans2;
} else {
                        z=8.0/ax;
                        y=z*z;
                        xx=ax-0.785398164;
                        ans1=1.0+y*(-0.1098628627e-2+y*(0.2734510407e-4
                                +y*(-0.2073370639e-5+y*0.2093887211e-6)));
                        ans2 = -0.1562499995e-1+y*(0.1430488765e-3
                                +y*(-0.6911147651e-5+y*(0.7621095161e-6
                                -y*0.934935152e-7)));
                        ans=sqrt(0.636619772/ax)*(cos(xx)*ans1-z*sin(xx)*ans2);
}
return ans;
}
//J Bessel function of integer order 1
public static double J1(double x)
{
double ax,z;
double xx,y,ans,ans1,ans2;
```

```
if ((ax=Math.abs(x)) < 8.0) {
                        y=x*x;
                        ans1=x*(72362614232.0+y*(-7895059235.0+y*(242396853.1
                                +y*(-2972611.439+y*(15704.48260+y*(-30.16036606))))));
                        ans2=144725228442.0+y*(2300535178.0+y*(18583304.74
                                +y*(99447.43394+y*(376.9991397+y*1.0))));
                        ans=ans1/ans2;
} else {
                        z=8.0/ax;
                        y=z*z;
                        xx=ax-2.356194491;
                        ans1=1.0+y*(0.183105e-2+y*(-0.3516396496e-4
                                +y*(0.2457520174e-5+y*(-0.240337019e-6))));
                        ans2=0.04687499995+y*(-0.2002690873e-3
                                +y*(0.8449199096e-5+y*(-0.88228987e-6
                                +y*0.105787412e-6)));
                        ans=sqrt(0.636619772/ax)*(cos(xx)*ans1-z*sin(xx)*ans2);
                        if (x < 0.0) ans = -ans;
}
return ans;
}
public static void main (String args[]) throws java.io.IOException
{    double r;
char beta='\u03B2';
char ksi='\u03B6';
double Bi[]={0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,2,3,4,5,6,7,8,9,10,20,30,40,50,100};
String name="Eigenvalues "+ksi+"n * J1("+ksi+"n)=Bi* J0("+ksi+"n)  or "+beta+"nb * J1("+beta+"nb)=Bi* J0("+beta+"nb)";
String sBi[]=new String[Bi.length];
String sC[]=new String[11];
double A[][]=new double[Bi.length][11];
for(int i=0;i<Bi.length;i++)
{A[i][0]=Bi[i];}
sC[0]="Bi=Hb=hb/k";
for(int i=1;i<11;i++)
{sC[i]=""+ksi+i+" = "+beta+i+"*b";}
for(int i=0;i<Bi.length;i++)
{ final double bi=Bi[i];
if_x f1=x->x*J1(x)-bi*J0(x);
double x0=0;
double dx=0.0001;
double x1=0;
for(int n=1;n<11;n++)
{
x1=f1.enlarge(x0,dx);
r= f1.bisection(x0,x1);
A[i][n]=r;
x0=r+dx;
}
}
Text.print(A,sC,name);
}}
```

Eigenvalues ζn * J1(ζn)=Bi* J0(ζn) or βnb * J1(βnb)=Bi* J0(βnb)

Bi=Hb=hb/k	ζ1 = β1*b	ζ2 = β2*b	ζ3 = β3*b	ζ4 = β4*b	ζ5 = β5*b	ζ6 = β6*b	ζ7 = β7*b	ζ8 = β8*b	ζ9 = β9*b	ζ10 = β10*b
0.01	0.1412443870220242 4	3.834314632249809	7.017011746616954	10.174451080675095	13.324442509503786	16.471237133801893	19.616368435731 69	22.76052392161146	25.904058346767 14	29.047173004683046
0.02	0.1994973416960884 6	3.836921558951536 4	7.018436719724432	10.17543382511955	13.325192819188775	16.47184446081137	19.616878043619 37	22.760963191495527	25.904444325067637	29.0475170919872
0.03	0.2440353141524760 3	3.839527182895581	7.019861355482003	10.1764165392711	13.32594357463093	16.472451532114572	19.617387764218 15	22.761402584161385	25.904830438604883	29.0478613281863
0.04	0.281434255337335	3.8421311322554885	7.021286299591365	10.177399263286501	13.326693651736012	16.473058500324736	19.617897779111 38	22.76184179504383	25.90521634105921	29.048205665204684
0.05	0.3142624617593016	3.8447315601894303	7.022709522378595	10.178381745704364	13.32744389858538	16.473665642199645	19.618407119830138	22.7622812467134	25.905602214184498	29.048549966740115
0.06	0.3438282413256903 4	3.8473322057748 82	7.024133921668673	10.17936370559209	13.328194536767967	16.474272461040684	19.6189169258261	22.762720199722125	25.905988113407666	29.048893954925077
0.07	0.3709155453475383 4	3.84992938895705	7.02555675007628	10.18034653773546	13.328944446537767	16.474879500276536	19.619426978365887	22.763159945666878	25.906374312377626	29.04923827632775
0.08	0.396035425390744	3.8525254516938 27	7.026979667131505	10.181328169971463	13.329695039887277	16.475486577102366	19.619936194780095	22.763599229730698	25.906760280527394	29.04958267645783
0.09	0.4195360767333189 4	3.8551178865234057	7.02840272435999	10.182310465566292	13.330445220256248	16.476093493513392	19.620446069997868	22.76403823934689	25.907146231004894	29.049926714179698
0.1	0.4416820262004045 3	3.857709635251578	7.029825389657756	10.183292656409002	13.331195161066617	16.476700078451117	19.620955553537181	22.764477796276317	25.907532201894497	29.050271112131163
0.2	0.6169732894317739	3.8835055007230928	7.044029547609666	10.19310581382953	13.338693403125603	16.482767914022116	19.626051260913655	22.768869539903505	25.911391801908593	29.053713125252173
0.3	0.7464605051130706	3.9090647380738073	7.058193606434175	10.202905281081176	13.346185368251872	16.488832142282234	19.631145033802245	22.773260812013948	25.915250129835183	29.057154470776887
0.4	0.8515795607756881	3.9343610269278377	7.072312220369719	10.212690366622734	13.353670828488404	16.494893271160542	19.636236597390 25	22.777650286641197	25.919108069656975	29.060595459357657
0.5	0.9407713612644059	3.9593715887451 04	7.086380652101916	10.222458626579524	13.361149046452297	16.500949638182263	19.641326146180262	22.782038706512918	25.92296497219719	29.064035363391538
0.6	1.0184417444598162	3.9840747999877 87	7.100394187771341	10.232207703370841	13.368618819567672	16.50700209628063	19.646412967852573	22.786424732598903	25.926820212925 41	29.067474505801 01
0.7	1.0872539108246193	4.0084518367043 31	7.114347144726919	10.241936617052357	13.376078975928529	16.513049996609258	19.651496730092937	22.790809529454116	25.930674817760 38	29.070913014170667
0.8	1.1489720079369725	4.0324873463818 05	7.128235449432799	10.251644050442756	13.383529683891506	16.519092024798432	19.656577178107007	22.795192288962 12	25.934527569800 43	29.074350206763604
0.9	1.2048409334855388	4.0561667457570 99	7.142053980375918	10.261327230585215	13.390969224814475	16.525128406637 55	19.661654154235 8	22.799572315070876	25.938379174870 05	29.077786534464 35
1.0	1.2557842296241377	4.0794778404572 06	7.155799268134112	10.270985344249315	13.398397255818805	16.531158746854643	19.666727630255487	22.803950610100742	25.942228678948 19	29.081221950405133
2.0	1.5994493658393016	4.2909581916677855	7.288388727533503	10.365831031180004	13.471882165561022	16.59103272510179	19.717206438221552	22.847565608317 44	25.980614675016536	29.11549357459593
3.0	1.7886574563090574	4.4633714063926 98	7.410269783972261	10.456598743179503	13.543436480953545	16.64985359459543	19.767050899543033	22.890769855315 39	26.018719861406836	29.14956541642165
4.0	1.9080785113777559	4.6018454637532 48	7.520070369540813	10.542287494003851	13.612476461498979	16.707268619496688	19.816036638734104	22.933413959068154	26.056440344388374	29.18336142238403
5.0	1.9898144198944885	4.7131419498053 99	7.617707511018007	10.622300258039939	13.678558027742419	16.76298356143578	19.863966429489 2	22.975361123498956	26.093678193179006	29.216810846555575
6.0	2.049011194318341	4.8033119494383918	7.703886379154738	10.696387724334514	13.741377962936724	16.816763785434148	19.910670967521696	23.016489171618133	26.130344192026975	29.249846149496207
7.0	2.093731248705719	4.8771678347049 68	7.779697711800363	10.764570974144624	13.800759377195828	16.868434856637148	19.956011362654 56	23.056693528453238	26.166358959573 57	29.282406577069004
8.0	2.1286387336464525	4.9383789654977 94	7.846358034587544	10.827059626473815	13.856637010900307	16.917880358105823	19.999879844039064	23.0958847624916	26.201651816285683	29.314437177859705
9.0	2.1566066766378382	4.9897029447778 96	7.905056641637977	10.884181571247845	13.909031211994224	16.965035156303212	20.042197105920 06	23.133991179458704	26.235162676281175	29.345888452837144
10.0	2.1794966093614576	5.0332120886896 43	7.956883498494168	10.936330089852158	13.958030525050916	17.009878339800 04	20.082910423361 22	23.170956891689592	26.269841545434 46	29.376717447808787
20.0	2.2880484750155157	5.2568106855303 88	8.25341301893258	11.267663966214656	14.298318054966355	17.344204004844848	20.403686243663863	23.474927587898666	26.556135968262748	29.645700949015996
30.0	2.3261434363779805	5.3409843167894 7	8.377067428121602	11.422149192175402	14.474795683703014	17.534811013737116	20.602043126426743	23.676198980927374	26.75684480367 79	29.843460942561123
40.0	2.3455180724465268	5.3846012687826 19	8.44315668909114	11.508052408618068	14.577392198857684	17.6508459188906	20.728390988965764	23.810028323893206	26.895707183897 77	29.985315912288 68
50.0	2.3572420695716 94	5.4111970267556 91	8.483989094100505	11.562078409347 19	14.643305104074038	17.727179736751552	20.813614898802815	23.902625085614404	26.994235641036063	30.088450638808816
100.0	2.380901658741439	5.4652069771618 54	8.567831649098185	11.674735397910997	14.783420900872102	17.893136601237245	21.003600975493104	24.114699347668132	27.226387203083604	30.338652546149298

PROBLEM MATLAB SOLUTION

```
function [f] = func3(beta,H,b)
f=(beta*b)*besselj(1,beta*b)-H*b*besselj(0,beta*b);
end
```

```
function [beta] = genislet(x0,H,b)
x1=x0+0.01;
x2= x1+0.00001;
f1=func3(x1,H,b);
f2=func3(x2,H,b);
j=0;
NTRY=200;
FACTOR=1.01;
while ((f1*f2>0) && (j<NTRY))
x2=x2+FACTOR*(x2-x1);
f2=func3(x2,H,b);
j=j+1;
end
beta=bisection(@(x) func3(x,H,b),x1,x2);
end
```

```
function xr=bisection(f,xl,xu)
% bisection root finding method
maxit=100;
iter=0;
es=0.000000001;
ea=1.1*es;
while((ea>es)&&(iter<maxit))
xr=(xl+xu)/2.0;
iter=iter+1;
if xr~=0 ea=abs((xu-xl)/(xu+xl))*100;end
fxl= f(xl);
fxr= f(xr);
test= fxl*fxr;
if test == 0.0      ea=0;
elseif test < 0.0  xu=xr;
else               xl=xr;
```

```
end
end
if(iter>=maxit) fprintf('Maximum number of iteration is exceeded result might not be valid","MAKSİMUM NUMBER OF
ITERATION WARNING');
end
end
```

```
function [beta] = beta3(m,H,b)
x0=0;
for i=1:m
x0=genislet(x0,H,b);
beta(i)=x0;
end
end
```

```
function [C] = CN3(beta,H,b)
y=@(r) r.*F3(r,b).*besselj(0,beta*r);
C1=quad(y,0,b);
J0=besselj(0,beta*b);
N=J0*J0/2*b*b*(H*H+beta*beta)/(beta*beta);
C=C1/N;
end
```

```
function [F] = F3(r,b)
F=1.0;
end
```

```
function  [T] = T3(r,t,H,b,alpha)
T=0;
m=2;
beta=beta3(m,H,b);
for i=1:m
C1=CN3(beta(i),H,b)
T=T+C1*besselj(0,(beta(i)*r))*exp(-beta(i)*beta(i)*alpha*t);
end
end
```

PROBLEM JAVA SOLUTION

```
import java.io.*;
import javax.swing.*;
import java.awt.*;
import static java.lang.Math.*;
public class HT_cylinder
{public if_x F;
public double h,b,k,alpha,Bi,Ti,Tinf,H;
public  HT_cylinder(double hi,double bi,double ki,double alphai,double Tii,double Tinfi)
{h=hi;k=ki;b=bi;alpha=alphai;Bi=h*b/k;Ti=Tii;Tinf=Tinfi;H=h/k;
F=xx->Ti-Tinf;}
//J Bessel function of integer order 0
public static double J0(double x)
{   double ax,z;
double xx,y,ans,ans1,ans2;
if ((ax=Math.abs(x)) < 8.0) {
                        y=x*x;
                        ans1=57568490574.0+y*(-13362590354.0+y*(651619640.7
                                +y*(-11214424.18+y*(77392.33017+y*(-184.9052456)))));
                        ans2=57568490411.0+y*(1029532985.0+y*(9494680.718
                                +y*(59272.64853+y*(267.8532712+y*1.0))));
                        ans=ans1/ans2;
} else {
                        z=8.0/ax;
                        y=z*z;
                        xx=ax-0.785398164;
                        ans1=1.0+y*(-0.1098628627e-2+y*(0.2734510407e-4
                                +y*(-0.2073370639e-5+y*0.2093887211e-6)));
                        ans2 = -0.1562499995e-1+y*(0.1430488765e-3
                                +y*(-0.6911147651e-5+y*(0.7621095161e-6
                                -y*0.934935152e-7)));
                        ans=sqrt(0.636619772/ax)*(cos(xx)*ans1-z*sin(xx)*ans2);
}
return ans;
}
```

```
//J Bessel function of integer order 1
public static double J1(double x)
{   double ax,z;
double xx,y,ans,ans1,ans2;
if ((ax=Math.abs(x)) < 8.0) {
                        y=x*x;
                        ans1=x*(72362614232.0+y*(-7895059235.0+y*(242396853.1
                                +y*(-2972611.439+y*(15704.48260+y*(-30.16036606))))));
                        ans2=144725228442.0+y*(2300535178.0+y*(18583304.74
                                +y*(99447.43394+y*(376.9991397+y*1.0))));
                        ans=ans1/ans2;
} else {
                        z=8.0/ax;
                        y=z*z;
                        xx=ax-2.356194491;
                        ans1=1.0+y*(0.183105e-2+y*(-0.3516396496e-4
                                +y*(0.2457520174e-5+y*(-0.240337019e-6))));
                        ans2=0.04687499995+y*(-0.2002690873e-3
                                +y*(0.8449199096e-5+y*(-0.88228987e-6
                                +y*0.105787412e-6)));
                        ans=sqrt(0.636619772/ax)*(cos(xx)*ans1-z*sin(xx)*ans2);
                        if (x < 0.0) ans = -ans;
}
return ans;
}

public double teta(double r,double t)
{double ksi=0;
double beta=0;
double C=0;
double teta=0;
if_x f1=xx->xx*J1(xx)-Bi*J0(xx);
//search for eigenvalues starting from 0
double x0=0;
double dx=0.0001;
double x1=0;
for(int n=1;n<11;n++)
{x1=f1.enlarge(x0,dx);
ksi= f1.bisection(x0,x1);
beta=ksi/b;
double j0=J0(ksi);
double j1=J1(ksi);
double nn=2.0/(j0*j0)/(b*b)*(beta*beta)/(beta*beta+H*H);
double nn1=b*b/2*(j0*j0+j1*j1);
double NN=1.0/nn1;
final double beta1=beta;
if_x y=rr->rr*Ti*J0(beta1*rr);
C=y.integral(0.0,b,20)/nn1;
teta+=C*J0(beta*r)*Math.exp(-beta*beta*alpha*t);
x0=ksi+dx;
}
return teta;
}
//derivative of tetha
public double dteta(double r,double t)
{double ksi=0;
double beta=0;
double C=0;
double teta=0;
if_x f1=xx->xx*J1(xx)-Bi*J0(xx);
//search for eigenvalues starting from 0
double x0=0;
double dx=0.0001;
double x1=0;
for(int n=1;n<11;n++)
{x1=f1.enlarge(x0,dx);
ksi= f1.bisection(x0,x1);
beta=ksi/b;
double j0=J0(ksi);
double j1=J1(ksi);
double nn=2.0/(j0*j0)/(b*b)*(beta*beta)/(beta*beta+H*H);
double nn1=b*b/2*(j0*j0+j1*j1);
double NN=1.0/nn1;
final double beta1=beta;
if_x y=rr->rr*Ti*J0(beta1*rr);
C=y.integral(0.0,b,20)*nn1;
```

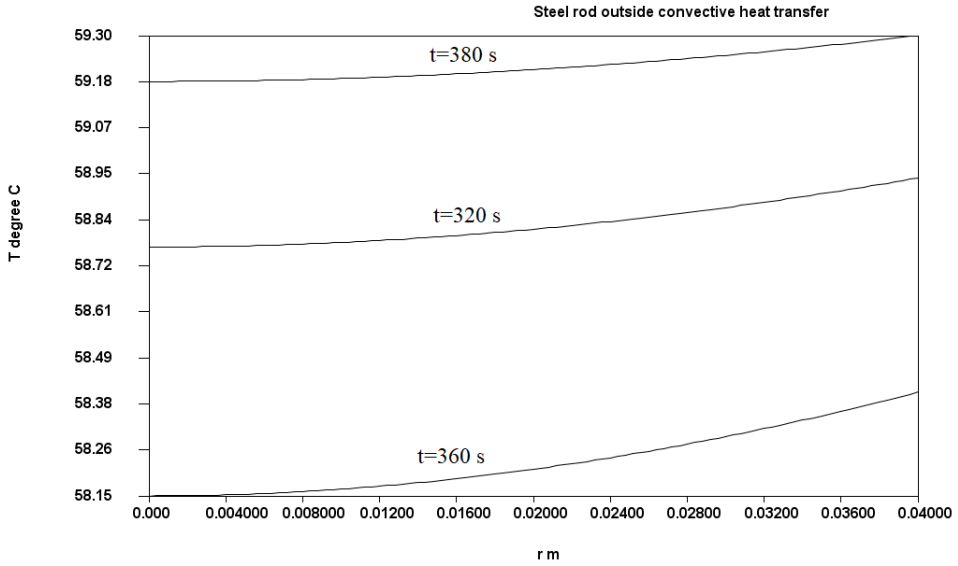
```
teta+=-C*beta*J1(beta*r)*Math.exp(-beta*beta*alpha*t);
x0=ksi+dx;
}
return teta;
}
public double T(double r,double t)
{double teta=teta(r,t);
double T=teta+Tinf;
return T;
}
public double q(double x,double t)
{double q=-k*dteta(x,t);
return q;
}
}}
```

Numerical example:

A steel rod (AISI 1010, k=63.9 W7mK) has Radius of 0.04 m. Initially the rod temperature is -20 ℃. (constant) The outside of the rod is exposed to 60 ℃ fluid with thermal convectivity coefficient of h=500 W/m^2K. Find the temperature profile of the rod at

a) t=360 s, b)t=420 s and c) t=480 s F(x)=-20-60=-80 ℃,

```
public class HT_cylinder_test
{
public static void main (String args[])
{ double h=500; //W/m^2K
double k=63.9;
double alpha=18.8e-6;
double b=40e-3;
double t[]={360,420,480};  //s
double Tinf=60;
double Ti=-20;
int n1=t.length;
double r[]=new double[101];
double T[][]=new double[n1][101];
double q[][]=new double[n1][101];
HT_cylinder htc=new HT_cylinder(h,b,k,alpha,Ti,Tinf);
//initial temperature variation t=0
htc.F=xx->Ti-Tinf;
for(int j=0;j<n1;j++)
{
for(int i=0;i<=100;i++)
{   r[i]=i*b/100;
            T[j][i]=htc.T(r[i],t[j]);
            q[j][i]=htc.q(r[i],t[j]);
}
}
Plot pp=new Plot(r,T[0]);
for(int j=1;j<n1;j++)
{pp.addData(r,T[j]);}
pp.setPlabel("Steel rod outside convective heat transfer");
pp.setXlabel(" r m");
pp.setYlabel(" T degree C");
pp.plot();
Plot pp1=new Plot(r,q[0]);
for(int j=1;j<n1;j++)
{pp1.addData(r,q[j]);}
pp1.setPlabel("Steel rod outside convective heat transfer");
pp1.setXlabel(" r m");
pp1.setYlabel(" q W/m^2");
pp1.plot();
}
}
```


Steel rod outside convective heat transfer
59.30
59.18
59.07
58.95
58.84
58.72
58.61
58.49
58.38
58.26
58.15
T degree C
t=380 s
t=320 s
t=360 s
0.000
0.004000
0.008000
0.01200
0.01600
0.02000
0.02400
0.02800
0.03200
0.03600
0.04000
r m

$Bi = \frac{hL}{k}$	ξ_1	ξ_2	ξ_3	ξ_4
0	0	3.1416	6.2832	9.4248
0.001	0.0316	3.1419	6.2833	9.4249
0.002	0.0447	3.1422	6.2835	9.4250
0.004	0.0632	3.1429	6.2838	9.4252
0.006	0.0774	3.1435	6.2841	9.4254
0.008	0.0893	3.1441	6.2845	9.4256
0.01	0.0998	3.1448	6.2848	9.4258
0.02	0.1410	3.1479	6.2864	9.4269
0.04	0.1987	3.1543	6.2895	9.4290
0.06	0.2425	3.1606	6.2927	9.4311
0.08	0.2791	3.1668	6.2959	9.4333
0.1	0.3111	3.1731	6.2991	9.4354
0.2	0.4328	3.2039	6.3148	9.4459
0.3	0.5218	3.2341	6.3305	9.4565
0.4	0.5932	3.2636	6.3461	9.4670
0.5	0.6533	3.2923	6.3616	9.4775
0.6	0.7051	3.3204	6.3770	9.4879
0.7	0.7506	3.3477	6.3923	9.4983
0.8	0.7910	3.3744	6.4074	9.5087
0.9	0.8274	3.4003	6.4224	9.5190
1.0	0.8603	3.4256	6.4373	9.5293
1.5	0.9882	3.5422	6.5097	9.5801
2.0	1.0769	3.6436	6.5783	9.6296
3.0	1.1925	3.8088	6.7040	9.7240
4.0	1.2646	3.9352	6.8140	9.8119
5.0	1.3138	4.0336	6.9096	9.8928
6.0	1.3496	4.1116	6.9924	9.9667
7.0	1.3766	4.1746	7.0640	10.0339
8.0	1.3978	4.2264	7.1263	10.0949
9.0	1.4149	4.2694	7.1806	10.1502
10.0	1.4289	4.3058	7.2281	10.2003
15.0	1.4729	4.4255	7.3959	10.3898
20.0	1.4961	4.4915	7.4954	10.5117
30.0	1.5202	4.5615	7.6057	10.6543
40.0	1.5325	4.5979	7.6647	10.7334
50.0	1.5400	4.6202	7.7012	10.7832
60.0	1.5451	4.6353	7.7259	10.8172
80.0	1.5514	4.6543	7.7573	10.8606
100.0	1.5552	4.6658	7.7764	10.8871
∞	1.5708	4.7124	7.8540	10.9956

Now let us check the eigenvalues by using our own root finding routine. The actual root finding is located at if_x class (bisection) . Eigenvalues are calculated at HT_plane_wall class, public static double ksi(int n,double Bi) method.

Java program results for eigenvalues:

```
public class Bi_eps
{public static void main(String arg[])
{
double
Bi[]={0.001,0.002,0.004,0.006,0.008,0.01,0.02,0.04,0.06,0.08,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,1.5,2,3,4,5,6,7,8,9,10,15,20,30,40,50,60,80,1
00,1000};
int n=Bi.length;
int n1=8;
double ksi[][]=new double[n][n1];
String
s[]={"Bi=hl/k",""+'\u03BE'+"1",""+'\u03BE'+"2",""+'\u03BE'+"3",""+'\u03BE'+"4",""+'\u03BE'+"5",""+'\u03BE'+"6",""+'\u03BE'+"7"};
double A1[]=new double[n1];
for(int i=0;i<n;i++)
{ ksi[i][0]=Bi[i];
for(int j=0;j<n1-1;j++){ksi[i][j+1]=HT_plane_wall.ksi(j,Bi[i]);}
}
Text.print(ksi,s,"eigenvalues");
```

```
}}
```

eigenvalues

Bi=hl/k	ξ1	ξ2	ξ3	ξ4	ξ5	ξ6	ξ7
0.001	0.03161446517398786	3.141911084361186	6.283344572565282	9.424884046190915	12.566450203425324	15.708026940476545	18.849609014249527
0.002	0.04470123043933842	3.1422294151325785	6.28350373795098	9.424990218865844	12.566529692491475	15.708090606630824	18.849662006960294
0.004	0.0632048619134238	3.1428653276617844	6.2838218814689775	9.425202376962304	12.566688857877168	15.708217938939383	18.849768086008524
0.006	0.07738219095088689	3.1435012401909903	6.284140024986975	9.42541453505877	12.566848023262864	15.70834517762124	18.849874258683457
0.008	0.0893184713577451	3.1441371527201953	6.2844583557583675	9.425626693155234	12.567007188648564	15.708472509929798	18.849980337731687
0.01	0.0998346220174018	3.144772316235822	6.28477631202297	9.425838851251694	12.56716635403426	15.708599842238357	18.850086416779916
0.02	0.14095247149122755	3.147945886773212	6.286366842359566	9.42689964173401	12.567961993709343	15.709236410154444	18.850616905647776
0.04	0.19867445400089873	3.154273178988 13	6.289544906978435	9.429020099178278	12.569552898552729	15.710509358733225	18.85167778975679
0.06	0.24252470501364384	3.1605742557277576	6.292719975542983	9.431139807608968	12.571143428889325	15.711782026431912	18.85273848661241
0.08	0.2791275006287509	3.1668487424853033	6.295891299039635	9.433258392519281	12.572733491092434	15.713054506877208	18.85379908984133
0.1	0.31105345545191376	3.1730970137675585	6.2990592519751765	9.435376041162623	12.574323178788749	15.714326800069111	18.85485950581686
0.2	0.43284006746455905	3.203935026366138	6.314846211195519	9.445947806080575	12.582264595268022	15.7206846145874	18.86015990041394
0.3	0.5217914221682278	3.234089938579987	6.330539169211603	9.456491482989286	12.590194214783413	15.727036436997054	18.865456643569832
0.4	0.5932413255675135	3.263550140698621	6.3461325084215785	9.467004824848022	12.59811072656116	15.733381424657795	18.870749548031117
0.5	0.6532710189163873	3.29231001512019	6.361620423970205	9.477485771869429	12.606013381587683	15.739719296689522	18.876038332917712
0.6	0.7050653079559356	3.320366565790377	6.376997298255633	9.487932077012776	12.613901430849403	15.746049584958762	18.881322623722834
0.7	0.7505581457490673	3.3477205417227602	6.392258262689596	9.498342054997504	12.62177356357255	15.75237182133202	18.88660223319308
0.8	0.7910333415714695	3.374375687985239	6.407398074177035	9.50871364603627	12.629629217996943	15.758685350422406	18.891876880448358
0.9	0.8274009469225359	3.400338745700031	6.422412800396655	9.519044977595122	12.637467457855607	15.764989891349842	18.89714619098188
1.0	0.8603335759968795	3.4256187030300937	6.4372983217737705	9.529334364393511	12.645287159628172	15.771284882354133	18.902409977540255
1.5	0.9882407524764324	3.542166339559194	6.5096590256860285	9.580091644131901	12.684082317991205	15.802600859243292	18.928635658141445
2.0	1.0768739636621034	3.6435970095150374	6.57833383377235	9.629560246006331	12.722298769737177	15.833611425845303	18.954681762736893
3.0	1.1924587432344387	3.808762181238582	6.703955769820094	9.724027523964654	12.796648385583758	15.894513064627699	19.00610828610878
4.0	1.264591559765059	3.9351615933821034	6.814010394879755	9.811878203212885	12.867755708897006	15.953625778101568	19.05645482552538
5.0	1.313837704606306	4.033567746502927	6.909595762859249	9.892752569979343	12.93522132818552	16.010658574491167	19.105519896333718
6.0	1.3495528255027147	4.111617770313552	6.99235183899143	9.966671409390873	12.998813142866279	16.065394988871972	19.15313721751911
7.0	1.3766150606457863	4.174637806752871	7.064030753292723	10.033914290766113	13.058441926696595	16.117689244475997	19.199176835225458
8.0	1.39781560914045	4.226362250276107	7.126280991025457	10.094915548354615	13.114131741737713	16.16745923068958	19.24354343747474
9.0	1.4148652882228738	4.269444387862823	7.180559196348842	10.150186852058049	13.165990820451912	16.214677421263666	19.286173919872873
10.0	1.4288700164435535	4.3058014133970755	7.228109761698937	10.200262623065564	13.214185724733982	16.25936123076397	19.327034295928648
15.0	1.4729157365939387	4.425495258061233	7.395901349539864	10.389796202592619	13.407759670763587	16.44736878748985	19.505124539611025
20.0	1.496128954797408	4.491480054095912	7.495412087030749	10.511669981060606	13.541976795112355	16.5863946973131	19.643935387651055
30.0	1.5201673840831922	4.561494917696423 5	7.605689964950525	10.654324087998347	13.708547139718178	16.76905682625955	19.83614055817702
40.0	1.5325024937404261	4.597942714313872	7.6646599737824 44	10.733414632111291	13.804845147305453	16.879444504888784	19.957553927360202
50.0	1.540005942083949	4.620245731800399	7.701159370413534	10.783164231111495	13.866633360692852	16.95188394885361	20.0391706953209
60.0	1.5450511637002098	4.635287337659696	7.725920951829726	10.817203633079954	13.909368026198585	17.002622122894046	20.097145224143656
80.0	1.551406180023728	4.654276035887111	7.757317371607012	10.860641186387 76	13.964353645589702	17.068554432838 96	20.17333538871908
100.0	1.5552451291675182	4.665765142020605	7.7763740779268	10.887130101374716	13.998089735094208	17.109307260626032	20.220834187282986
1000.0	1.569227100980503 2	4.707681333828946	7.846135659321186	10.984590139208333	14.123044835195774	17.261499809046036	20.39995512238452

If the same values are checked in mathlab(octave):

```
>> L=1;H=0.1;alfa=1e-5;x=0.05;t=1000;
>> x=[beta1(1,L,H),beta1(2,L,H),beta1(3,L,H),beta1(4,L,H),beta1(5,L,H)]
x =0.311052848268378   3.173097175184144   6.299059360548651   9.435375971163087  12.574323171641831
```

```
>> L=1;H=0.5;alfa=1e-5;x=0.05;t=1000;
>> x=[beta1(1,L,H),beta1(2,L,H),beta1(3,L,H),beta1(4,L,H),beta1(5,L,H)]
x =0.653271187307584   3.292310022477304   6.361620392838449   9.477485702590318 12.606013439411305
```

```
>> L=1;H=1;alfa=1e-5;x=0.05;t=1000;
>> x=[beta1(1,L,H),beta1(2,L,H),beta1(3,L,H),beta1(4,L,H),beta1(5,L,H)]
x =0.860333588363406   3.425618459073878   6.437298181858815   9.529334406744880 12.645287216547336
```

```
>> L=1;H=2;alfa=1e-5;x=0.05;t=1000;
>> x=[beta1(1,L,H),beta1(2,L,H),beta1(3,L,H),beta1(4,L,H),beta1(5,L,H)]
x =1.076873985989610   3.643597165479853   6.578333727851468   9.629560345782938 12.722298777154021
```

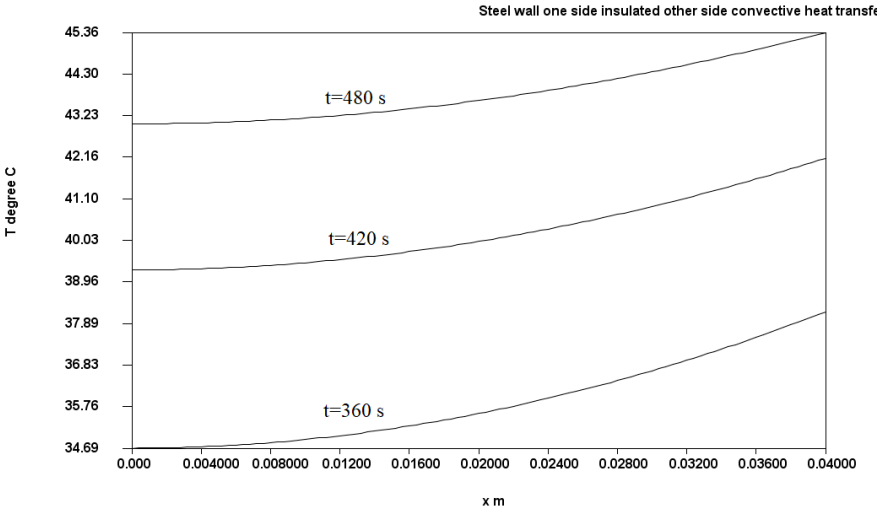
As it is seen from the examples, as long as algorithms and codes entered correctly the results are independent of the language used for the code.

Numerical example:

A steel wall (AISI 1010, k=63.9 W7mK) has Wall thickness of 0.04 m. Initially the pipe temperature is -20 ℃. (constant) The pipe is insulated on one surface. The other wall is exposed to 60 ℃ fluid with thermal convectivity coefficient of h=500 W/m^{2}K. Find the temperature profile of the Wall at a) t=360 s, b)t=420 s and c) t=480 s F(x)=-20-60=-80 ℃,

```
public class HT_plane_wall_test
{
public static void main (String args[])
{ double h=500; //W/m^2K
double k=63.9;
double alpha=18.8e-6;
double L=40e-3;
double t[]={6*60,7*80,8*60};  //s
double Tinf=60;
double Ti=-20;
int n1=t.length;
double x[]=new double[101];
double T[][]=new double[n1][101];
double q[][]=new double[n1][101];
HT_plane_wall htpw=new HT_plane_wall(h,L,k,alpha,Ti,Tinf);
```

```
//initial temperature variation t=0
htpw.F=xx->(Ti-Tinf);
for(int j=0;j<n1;j++)
{
for(int i=0;i<=100;i++)
{   x[i]=i*L/100;
                T[j][i]=htpw.T(x[i],t[j]);
                q[j][i]=htpw.q(x[i],t[j]);
}
}
Plot pp=new Plot(x,T[0]);
for(int j=1;j<n1;j++)
{pp.addData(x,T[j]);}
pp.setPlabel("Steel wall one side insulated other side convective heat transfer");
pp.setXlabel(" x m");
pp.setYlabel(" T degree C");
pp.plot();
Plot pp1=new Plot(x,q[0]);
for(int j=1;j<n1;j++)
{pp1.addData(x,q[j]);}
pp1.setPlabel("Steel wall one side insulated other side convective heat transfer");
pp1.setXlabel(" x m");
pp1.setYlabel(" q W/m^2");
pp1.plot();
}
}
```

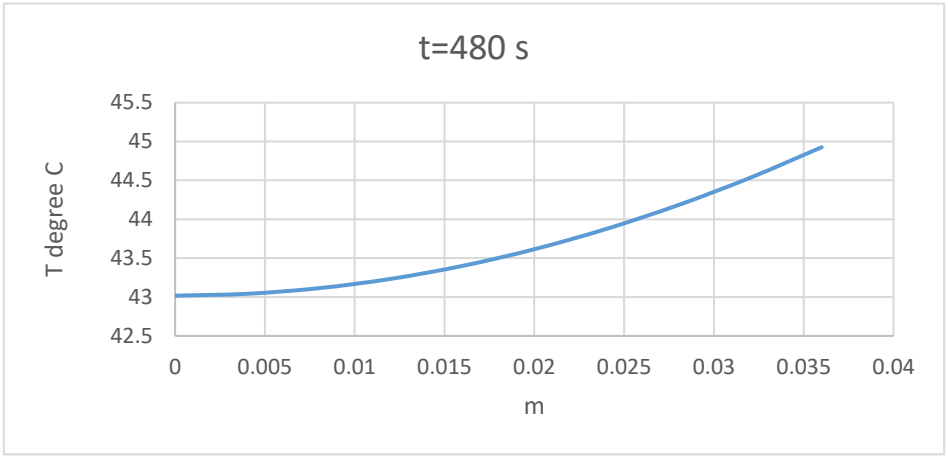


t=480 s , x=0-0.04 solution in excel:

	H	7	m										
	L	0	m										
h		5	W										
k		6	W										
B	H	0											
F	a	5											
α		1	m										
t		4	s										
x	m	0	0	0	0	0	0	0	0	0	0	0	0
T		6	d										
T		-	d										

m	β	β	C	Σ									
1	1	0.	1	0.	0.	0.	0.	0.	0.	0.	0.	0	0.
2	8	3.	-	-	-	-	-	-	5.	4.	7.	1	1.
3	1	6.	0	9.	7.	2.	-	-	-	-	-	3	7.
4	2	9.	-	-	-	1.	5.	4.	-	-	-	-	3.
5	3	1	0	0	0	0	0	0	0	0	0	0	0

	0.	0.	0.	0.	0.	0.	0.	0.	0	0.
T	4	4	4	4	4	4	4	4	4	4
	d									



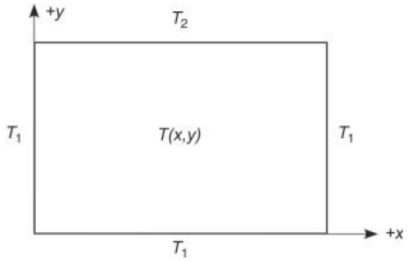
Root finding process (Bisection method):

H	7.825
L	0.04
Bi=H*L	0.313
	root_findin
m	1

xl	xu	xr	fxl	fxu	fxr	fxr*fxl	
3E-06	39.27	19.635	-0.313	3E+16	0.47241	-0.1479	0.7854
3E-06	19.635	9.81748	-0.313	0.472	-0.1503	0.04705	0.3927
9.8175	19.635	14.7262	-0.15	0.472	0.0806	-0.0121	0.58905
9.8175	14.726	12.2718	-0.15	0.081	-0.0506	0.00761	0.49087
12.272	14.726	13.499	-0.051	0.081	0.01065	-0.0005	0.53996
12.272	13.499	12.8854	-0.051	0.011	-0.021	0.00106	0.51542
12.885	13.499	13.1922	-0.021	0.011	-0.0054	0.00011	0.52769
13.192	13.499	13.3456	-0.005	0.011	0.00254	-1E-05	0.53383
13.192	13.346	13.2689	-0.005	0.003	-0.0015	8E-06	0.53076
13.269	13.346	13.3073	-0.001	0.003	0.00053	-8E-07	0.53229
13.269	13.307	13.2881	-0.001	5E-04	-0.0005	6.9E-07	0.53152
13.288	13.307	13.2977	-5E-04	5E-04	3E-05	-1E-08	0.53191
13.288	13.298	13.2929	-5E-04	3E-05	-0.0002	1E-07	0.53172
13.293	13.298	13.2953	-2E-04	3E-05	-1E-04	2.1E-08	0.53181
13.295	13.298	13.2965	-1E-04	3E-05	-3E-05	3.1E-09	0.53186
13.296	13.298	13.2971	-3E-05	3E-05	-2E-06	5.5E-11	0.53188
13.297	13.298	13.2974	-2E-06	3E-05	1.4E-05	-2E-11	0.5319
13.297	13.297	13.2972	-2E-06	1E-05	6.1E-06	-1E-11	0.53189

13.297	13.297	13.2972	-2E-06	6E-06	2.2E-06	-4E-12	0.53189	
13.297	13.297	13.2971	-2E-06	2E-06	2.8E-07	-5E-13	0.53189	

EXAMPLE 4: SEPARATION OF VARIABLES,HOMOGENEOUS, STEADY STATE SOLUTION; CARTESIAN COORDINATES



$$\frac{\partial^2 T^*(x,y)}{\partial x^2}+\frac{\partial^2 T^*(x,y)}{\partial y^2}=0$$

$$T^*(x,y)=T_1 \quad x=0 \quad 0\le y\le W$$

$$T^*(x,y)=T_1 \quad x=L \quad 0\le y\le W$$

$$T^*(x,y)=T_1 \quad y=0 \quad 0\le x\le L$$

$$T^*(x,y)=T_2 \quad y=W \quad 0\le x\le L$$

$$T(x,y)=T^*(x,y)-T_1$$

$$\frac{\partial^2 T(x,y)}{\partial x^2}+\frac{\partial^2 T(x,y)}{\partial y^2}=0$$

$$T(x,y)=0 \quad x=0 \quad 0\le y\le W$$

$$T(x,y)=0 \quad x=L \quad 0\le y\le W$$

$$T(x,y)=0 \quad y=0 \quad 0\le x\le L$$

$$T(x,y)=T_3=\left(T_2-T_1\right) \quad y=W \quad 0\le x\le L$$

Separation of variables:

$$T(x,y)=X(x)*Y(y)$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}=-\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=-\lambda^2$$

$$\frac{\partial^2 X(x)}{\partial x^2}+\lambda^2 X(x)=0$$

$$\frac{\partial^2 Y(x)}{\partial y^2}-\lambda^2 Y(y)=0 \quad A^2-\lambda^2=0 \quad A=\pm\lambda \quad Y(y)=D_3 exp(\lambda y)+D_4 exp(-\lambda y)$$

$$X(x)=0 \quad x=0$$

$$X(x)=0 \quad x=L$$

$$Y(y)=0 \quad y=0$$

$$Y(y)=\left(T_2-T_1\right) \quad y=W$$

$$X(x)=C_1\sin(\lambda x)+C_2\cos(\lambda x)$$

$$Y(y) = C_3 \sinh(\lambda x) + C_4 \cosh(\lambda x)$$

Boundary values:

$$X(x) = C_1 \sin(\lambda 0) + C_2 \cos(\lambda 0) = 0 \rightarrow C_2=0$$

$$X(x) = C_1 \sin(\lambda L) = 0 \qquad \lambda_n = \frac{n\pi}{L}$$

$$X(x) = C_1 \sin(\lambda 0) + C_2 \cos(\lambda 0) = 0$$

$$X(x) = C_3 \sinh(\lambda 0) + C_4 \cosh(\lambda 0) = 0 \rightarrow C_4=0$$

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) \sinh(\frac{n\pi y}{L})$$

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) \sinh(\frac{n\pi W}{L}) = T_3 = (T_2 - T_1)$$

This equation has n coefficients. In order to solve these coefficients, we will use properties of orthogonal functions.

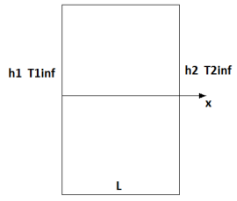
$$\int_0^L X(\lambda_n x) * X(\lambda_m x) dx = \begin{cases} 0 & m \neq n \\ N(\lambda_m) & m = n \end{cases}$$

$$\int_0^L \frac{T_3}{\sinh(\frac{n\pi W}{L})} * \sin(\frac{n\pi}{L} x) dx = C_n \int_0^L \sin^2(\frac{n\pi}{L} x) dx$$

$$C_n = \frac{T_3 \frac{L}{\pi}}{\sinh(\frac{n\pi W}{L})} \left[\frac{(-1)^{n+1} + 1}{n} \right]$$

$$T^*(x,y) = T(x,y) + T_1$$

EXAMPLE 5: SEPARATION OF VARIABLES, NON-HOMOGENEOUS, TIME DEPENDENT SOLUTION; CARTESIAN COORDINATES



$$\frac{\partial^2 T^*(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T^*(x,t)}{\partial t}$$

Boundary conditions

$$-k \frac{\partial T(x,t)}{\partial x} = h_1 (T_{1\infty} - T^*(x,t)) \quad \text{x=0} \quad \text{t >0}$$

$$-k \frac{\partial T(x,t)}{\partial x} = h_2 (T^*(x,t) - T_{2\infty}) \quad \text{x=L} \quad \text{t >0}$$

$$T^*(x,t) = F^*(x) \quad \text{t=0} \quad \text{0<=x<=L}$$

$$T(x,t) = T^*(x,t) - T_{2\infty}$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Boundary conditions

$$-k\frac{\partial T(x,t)}{\partial x} = h_1(T_{1\infty} - T(x,t) + T_{2\infty}) = -T(x,t) + h_1(T_{1\infty} + T_{2\infty}) \quad \text{x=0} \quad \text{t >0}$$

$$H_1 = \frac{h_1}{k} \qquad H_2 = \frac{h_2}{k}$$

$$-\frac{\partial T(x,t)}{\partial x} + H_1 * T(x,t) = h_1(T_{1\infty} + T_{2\infty})$$

$$-k\frac{\partial T(x,t)}{\partial x} = h_2 T(x,t) \quad \text{x=L} \quad \text{t >0}$$

$$\frac{\partial T(x,t)}{\partial x} + H_2 * T(x,t) = 0 \quad \text{x=L} \quad \text{t >0}$$

$$T(x,t) = F(x) = F^*(x) + T_{2\infty} \quad \text{t=0} \quad \text{0<=x<=L}$$

$$T(x,t) = T_H(x,t) + T_{ss}(x)$$

$$\frac{\partial^2 T_H(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_H(x,t)}{\partial t}$$

Boundary conditions

$$-\frac{\partial T_H(x,t)}{\partial x} + H_1 * T_H(x,t) = 0 \quad \text{x=0} \quad \text{t >0}$$

$$\frac{\partial T_H(x,t)}{\partial x} + H_2 * T_H(x,t) = 0 \quad \text{x=L} \quad \text{t >0}$$

$$T_H(x,t) = F(x) - T_{ss}(x) = F^*(x) + T_{2\infty} - T_{ss}(x) \quad \text{t=0} \quad \text{0<=x<=L}$$

$$\frac{\partial^2 T_{ss}(x)}{\partial x^2} = 0$$

Boundary conditions

$$-\frac{\partial T_{ss}(x)}{\partial x} + H_1 * T_{ss}(x) = H_1 * (T_{1\infty} + T_{2\infty}) \quad \text{x=0}$$

$$\frac{\partial T_{ss}(x)}{\partial x} + H_2 * T_{ss}(x) = 0 \quad \text{x=L} \quad \text{t >0}$$

Solution:

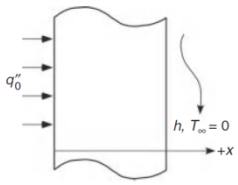
$$\frac{\partial T_{ss}(x)}{\partial x} = C_1 \qquad T_{ss}(x) = C_1 x + C_2$$

$$C_1 = \frac{H_2}{1 + H_2 L} C_2$$

$$C_2 = \frac{H_1(T_{1\infty} + T_{1\infty})}{\left(H_1 - \frac{H_2}{1 + H_2 L}\right)}$$

The first part of the problem is solved as problem 2.

EXAMPLE 6: SEPARATION OF VARIABLES, NON-HOMOGENEOUS, TIME DEPENDENT SOLUTION; CARTESIAN COORDINATES



$$\frac{\partial^2 T^*(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T^*(x,t)}{\partial t}$$

Boundary conditions

$$-k\frac{\partial T^*(x,t)}{\partial x} = q''_0 \quad \text{x=0} \quad \text{t >0}$$

$$-k\frac{\partial T^*(x,t)}{\partial x} = h(T^*(x,t) - T_\infty) \quad \text{x=L} \quad \text{t >0}$$

$$T^*(x,t) = F^*(x) \quad \text{t=0} \quad \text{0<=x<=L}$$

$$T(x,t) = T^*(x,t) - T_\infty$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Boundary conditions

$$-k\frac{\partial T(x,t)}{\partial x} = q''_0 \quad \text{x=0} \quad \text{t >0}$$

$$\frac{\partial T(x,t)}{\partial x} + HT(x,t) = 0 \quad \text{x=L} \quad \text{t >0}$$

$$T(x,t) = F^*(x) - T_\infty = F(x) \quad \text{t=0} \quad \text{0<=x<=L}$$

$$T(x,t) = T_H(x,t) + T_{ss}(x)$$

$$\frac{\partial^2 T_H(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_H(x,t)}{\partial t}$$

Boundary conditions

$$\frac{\partial T_H(x,t)}{\partial x} = 0 \quad \text{x=0} \quad \text{t >0}$$

$$\frac{\partial T_H(x,t)}{\partial x} + H * T_H(x,t) = 0 \quad \text{x=L} \quad \text{t >0}$$

$$T_H(x,t) = F(x) - T_{ss}(x) = F^*(x) + T_\infty - T_{ss}(x) \quad \text{t=0} \quad \text{0<=x<=L}$$

$$\frac{\partial^2 T_{ss}(x)}{\partial x^2} = 0$$

Boundary conditions

$$-\frac{\partial T(x)}{\partial x} = \frac{q''_0}{k} \quad \text{x=0}$$

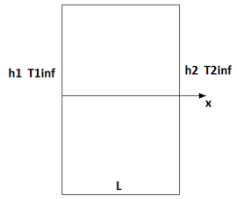
$$\frac{\partial T_{ss}(x)}{\partial x} + H * T_{ss}(x) = 0 \quad \text{x=L} \quad \text{t >0}$$

Solution:

$$\frac{\partial T_{ss}(x)}{\partial x} = C_1 \qquad T_{ss}(x) = C_1 x + C_2$$

$$T_{ss}(x) = \frac{q''_0}{k}(L - x) + \frac{q''_0}{h}$$

EXAMPLE 7: SEPARATION OF VARIABLES, NON-HOMOGENEOUS, TIME DEPENDENT SOLUTION; CARTESIAN COORDINATES



$$\frac{\partial^2 T^*(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T^*(x,t)}{\partial t}$$

Boundary conditions

$$-k\frac{\partial T(x,t)}{\partial x} = h_1(T_{1\infty} - T^*(x,t)) \quad \text{x=0} \quad \text{t >0}$$

$$-k\frac{\partial T(x,t)}{\partial x} = h_2(T^*(x,t) - T_{2\infty}) \quad \text{x=L} \quad \text{t >0}$$

$$T^*(x,t) = F^*(x) \quad \text{t=0} \quad \text{0<=x<=L}$$

$$T(x,t) = T^*(x,t) - T_{2\infty}$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Boundary conditions

$$-k\frac{\partial T(x,t)}{\partial x} = h_1(T_{1\infty} - T(x,t) + T_{2\infty}) = -T(x,t) + h_1(T_{1\infty} + T_{2\infty}) \quad \text{x=0} \quad \text{t >0}$$

$$H_1 = \frac{h_1}{k} \qquad H_2 = \frac{h_2}{k}$$

$$-\frac{\partial T(x,t)}{\partial x} + H_1 * T(x,t) = h_1(T_{1\infty} + T_{2\infty})$$

$$-k\frac{\partial T(x,t)}{\partial x} = h_2 T(x,t) \quad \text{x=L} \quad \text{t >0}$$

$$\frac{\partial T(x,t)}{\partial x} + H_2 * T(x,t) = 0 \quad \text{x=L} \quad \text{t >0}$$

$$T(x,t) = F(x) = F^*(x) + T_{2\infty} \quad \text{t=0} \quad \text{0<=x<=L}$$

$$T(x,t) = T_H(x,t) + T_{ss}(x)$$

$$\frac{\partial^2 T_H(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_H(x,t)}{\partial t}$$

Boundary conditions

$$-\frac{\partial T_H(x,t)}{\partial x} + H_1 * T_H(x,t) = 0 \quad \text{x=0} \quad \text{t >0}$$

$$\frac{\partial T_H(x,t)}{\partial x} + H_2 * T_H(x,t) = 0 \quad x=L \quad t>0$$

$$T_H(x,t) = F(x) - T_{ss}(x) = F^*(x) + T_{2\infty} - T_{ss}(x) \quad t=0 \quad 0<=x<=L$$

$$\frac{\partial^2 T_{ss}(x)}{\partial x^2} = 0$$

Boundary conditions

$$-\frac{\partial T_{ss}(x)}{\partial x} + H_1 * T_{ss}(x) = H_1 * (T_{1\infty} + T_{2\infty}) \quad x=0$$

$$\frac{\partial T_{ss}(x)}{\partial x} + H_2 * T_{ss}(x) = 0 \quad x=L \quad t>0$$

Solution:

$$\frac{\partial T_{ss}(x)}{\partial x} = C_1 \qquad T_{ss}(x) = C_1 x + C_2$$

$$C_1 = \frac{H_2}{1 + H_2 L} C_2$$

$$C_2 = \frac{H_1 (T_{1\infty} + T_{1\infty})}{\left(H_1 - \frac{H_2}{1 + H_2 L} \right)}$$

EXAMPLE 8: SEPARATION OF VARIABLES, NON-HOMOGENEOUS, STEADY STATE SOLUTION; CARTESIAN COORDINATES, HEAT GENERATION

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{g}{k} = 0$$

$$T(x,y) = 0 \quad x = L \quad 0 \le y \le W$$

$$T(x,y) = 0 \quad y = 0 \quad 0 \le x \le L$$

$$T(x,y) = T_1 \quad y = W \quad 0 \le x \le L$$

This equation can be divided into two equation to define a homogeneous problem

$T(x,y) = \psi(x,y) + \phi(x)$ where

$$\frac{\partial^2 \psi(x,y)}{\partial x^2} + \frac{\partial^2 \psi(x,y)}{\partial y^2} = 0$$

$$T(x,y) = 0 \quad x = L \quad 0 \le y \le W$$

$$T(x,y) = 0 \quad y = 0 \quad 0 \le x \le L$$

$$T(x,y) = T_1 - \phi(x) \quad y = W \quad 0 \le x \le L$$

and

$$\frac{\partial^2 \phi(x)}{\partial x^2} + \frac{g}{k} = 0$$

$$\phi(x) = 0 \quad x = 0$$

$$\phi(x) = 0 \quad x = L$$

Solution:

$$\frac{\partial \phi(x)}{\partial x} = -\frac{g}{k} x + C_1$$

$$\phi(x) = -\frac{g}{k}\frac{x^2}{2} + C_1 x + C_2$$

$$\phi(x) = 0 \quad x = 0 \rightarrow C_2 = 0$$

$$\phi(x) = 0 \quad x = L \rightarrow C_1 = \frac{g}{k}\frac{L}{2}$$

$$\phi(x) = -\frac{g}{k}\frac{x^2}{2} + \frac{g}{k}\frac{L}{2}x$$

$$\psi(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L})\sinh(\frac{n\pi y}{L})$$

$$\psi(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L})\sinh(\frac{n\pi W}{L}) = T_1 - \phi(x) = T_1 + \frac{g}{k}\frac{x^2}{2} - \frac{g}{k}\frac{L}{2}x$$

Orthogonality conditions:

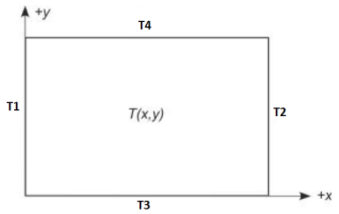
$$C_n\left[\int_{x=0}^{L} \sin^2(\frac{n\pi x}{L})dx\right]\sinh(\frac{n\pi W}{L}) = \int_{x=0}^{L}\left(T_1 + \frac{g}{k}\frac{x^2}{2} - \frac{g}{k}\frac{L}{2}x\right)\sin(\frac{n\pi x}{L})dx$$

$$N = \left[\int_{x=0}^{L} \sin^2(\frac{n\pi x}{L})dx\right] = \frac{L}{2}$$

$$C_n = \frac{2}{L\sinh(\frac{n\pi W}{L})}\int_{x=0}^{L}\left(T_1 + \frac{g}{k}\frac{x^2}{2} - \frac{g}{k}\frac{L}{2}x\right)\sin(\frac{n\pi x}{L})dx$$

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L})\sinh(\frac{n\pi y}{L}) + \frac{g}{k}\frac{x^2}{2} - \frac{g}{k}\frac{L}{2}x$$

EXAMPLE 9: SEPARATION OF VARIABLES, NON-HOMOGENEOUS, STEADY STATE SOLUTION; CARTESIAN COORDINATES , SUPERPOSITION PRINCIPLE



$$\frac{\partial^2 T^*(x,y)}{\partial x^2} + \frac{\partial^2 T^*(x,y)}{\partial y^2} = 0$$

$$T^*(x,y) = T_1 \quad x = 0 \quad 0 \le y \le W$$

$$T^*(x,y) = T_2 \quad x = L \quad 0 \le y \le W$$

$$T^*(x,y) = T_3 \quad y = 0 \quad 0 \le x \le L$$

$$T^*(x,y) = T_4 \quad y = W \quad 0 \le x \le L$$

$$T(x,y) = T^*(x,y) - T_1$$

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$

$$T(x,y) = 0 \quad x = 0 \quad 0 \le y \le W$$

$$T(x,y) = (T_2 - T_1) \quad x = L \quad 0 \le y \le W$$

$$T(x,y)=(T_3-T_1) \quad y=0 \quad 0\le x\le L$$
$$T(x,y)=(T_4-T_1) \quad y=W \quad 0\le x\le L$$

New problem can be constitute as superposition of three homegenous problem

$$T(x,y)=T^*(x,y)-T_1=\psi_1(x,y)+\psi_2(x,y)+\psi_3(x,y)$$

$$\frac{\partial^2\psi_1(x,y)}{\partial x^2}+\frac{\partial^2\psi_1(x,y)}{\partial y^2}=0$$
$$\psi_1(x,y)=0 \quad x=0 \quad 0\le y\le W$$
$$\psi_1(x,y)=(T_2-T_1) \quad x=L \quad 0\le y\le W$$
$$\psi_1(x,y)=0 \quad y=0 \quad 0\le x\le L$$
$$\psi_1(x,y)=0 \quad y=W \quad 0\le x\le L$$

$$\frac{\partial^2\psi_2(x,y)}{\partial x^2}+\frac{\partial^2\psi_2(x,y)}{\partial y^2}=0$$
$$\psi_2(x,y)=0 \quad x=0 \quad 0\le y\le W$$
$$\psi_2(x,y)=0 \quad x=L \quad 0\le y\le W$$
$$\psi_2(x,y)=(T_3-T_1) \quad y=0 \quad 0\le x\le L$$
$$\psi_2(x,y)=0 \quad y=W \quad 0\le x\le L$$

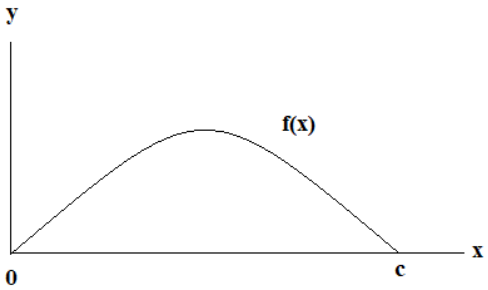
$$\frac{\partial^2\psi_3(x,y)}{\partial x^2}+\frac{\partial^2\psi_3(x,y)}{\partial y^2}=0$$
$$\psi_3(x,y)=0 \quad x=0 \quad 0\le y\le W$$
$$\psi_3(x,y)=0 \quad x=L \quad 0\le y\le W$$
$$\psi_3(x,y)=0 \quad y=0 \quad 0\le x\le L$$
$$\psi_3(x,y)=(T_3-T_1) \quad y=W \quad 0\le x\le L$$

Solution of each homegenous problem can be carried out similar to the previous problem

$$\psi_1(x,y)=\sum_{n=1}^{\infty}C_{n1}\sinh(\frac{n\pi}{W}x)\sin(\frac{n\pi}{W}y)$$
$$\psi_2(x,y)=\sum_{n=1}^{\infty}C_{n2}\sin(\frac{n\pi}{L}x)\sinh(\frac{n\pi}{L}(W-y))$$
$$\psi_3(x,y)=\sum_{n=1}^{\infty}C_{n3}\sin(\frac{n\pi}{L}x)\sinh(\frac{n\pi}{L}y)$$

EXAMPLE 10:

We are looking to find an expression for the transverse displacement y(x,t) of a string streched at t=0 time between the points y(0,0) and y(c,0). The string is initially displaced into position y(x,0) =f(x), and released at rest from that position. It is assumed that no external forces act along the string. String is frictionless.



Then equation of motion of frictionless string will be

$$\frac{\partial^2 y(x,t)}{\partial t^2} = a^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

Where $a^2 = \frac{F_0}{\rho_0}$ F_0 is tensile force ρ_0 is linear density of string (assumed to be constant)

So boundary conditions of the equation:

$$y(x,t) = 0 \quad at\ x = 0 \quad t > 0$$

$$y(x,t) = 0 \quad at\ x = c \quad t > 0$$

$$\frac{\partial y(x,t)}{\partial t} = 0 \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$y(x,t) = f(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

We will apply seperation of variables method to solve this equation

$$y(x,t) = X(x)\tau(t)$$

$$X(x)\frac{\partial^2 \tau(t)}{\partial t^2} = a^2 \tau(t) \frac{\partial^2 X(x)}{\partial x^2}$$

$$\frac{1}{a^2 \tau(t)} \frac{\partial^2 \tau(t)}{\partial t^2} = \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = -\alpha^2$$

$$\frac{\partial^2 X(x)}{\partial x^2} + \alpha^2 X(x) = 0$$

$$\frac{\partial^2 \tau(t)}{\partial t^2} + a^2 \alpha^2 \tau(t) = 0$$

So boundary conditions of the equation:

$$X(x) = 0 \quad at\ x = 0$$

$$X(x) = 0 \quad at\ x = c$$

$$\frac{\partial \tau(t)}{\partial t} = 0 \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$y(x,t) = f(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

Solution of differential equations:

$$A^2 + \alpha^2 = 0$$

$$A = \mp \alpha i$$

$$A^2 + a^2\alpha^2 = 0$$

$$A = \mp \alpha a i$$

$$X(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x)$$

$$\tau(t) = C_3 \sin(\alpha a t) + C_4 \cos(\alpha a t)$$

Applying boundary conditions to solve coefficients :

$$X(x) = 0 \;\; at\; x = 0 \;\; \rightarrow \;\; X(x) = C_1 \sin(\alpha 0) + C_2 \cos(\alpha 0) = 0 \;\; \rightarrow \;\; C_2 = 0$$

$$X(x) = 0 \;\; at\; x = c \;\; \rightarrow \;\; X(x) = C_1 \sin(\alpha c) = 0 \;\; \rightarrow \;\; \alpha c = n\pi \;\; \rightarrow \;\; \alpha = \frac{n\pi}{c}$$

$$\frac{\partial \tau(t)}{\partial t} = C_3 \alpha a \cos(\alpha a t) - C_4 \alpha a \sin(\alpha a t)$$

$$\frac{\partial \tau(t)}{\partial t} = 0 \;\; at\; t = 0 \quad 0 \le x \le c \;\; \rightarrow \;\; C_3 \alpha a \cos(\alpha a 0) - C_4 \alpha a \sin(\alpha a 0) = 0 \;\; \rightarrow \;\; C_3 = 0$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c} x\right) \cos\left(\frac{n\pi}{c} a t\right)$$

Applying last remaining boundary condition

$$y(x,t) = f(x) \;\; at\; t = 0 \quad 0 \le x \le c$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c} x\right) \cos\left(\frac{n\pi}{c} a 0\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c} x\right)$$

This equation has n coefficients. In order to solve these coefficients, we will use properties of orthogonal functions.

$$\int_0^c X(\alpha_n x) X(\alpha_m x) dx = \begin{cases} 0 & m \neq n \\ N(\alpha_m) & m = n \end{cases}$$

$$\int_0^c f(x) \sin\left(\frac{n\pi}{c} x\right) = C_n \int_0^c sin^2\left(\frac{n\pi}{c} x\right) dx$$

$$2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B)$$

$$2\sin^2(A) = 1 - \cos(2A)$$

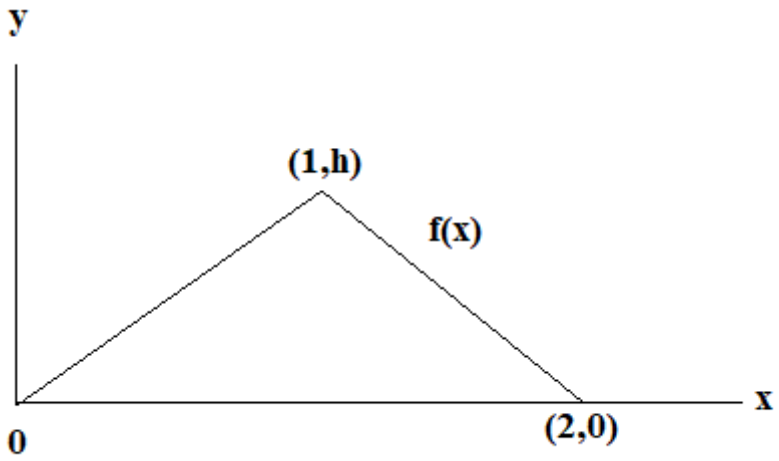
$$\sin^2(A) = \frac{1 - \cos(2A)}{2}$$

$$\int_0^c sin^2\left(\frac{n\pi}{c} x\right) dx = \int_0^c \frac{1 - \cos\left(2\frac{n\pi}{c} x\right)}{2}) dx = \frac{1}{2}\left[x - \frac{c}{2n\pi} \cos\left(2\frac{n\pi}{c} x\right)\right]_0^c = \frac{c}{2}$$

$$C_n = \frac{2}{c} \int_0^c f(x) \sin\left(\frac{n\pi}{c} x\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right)\cos\left(\frac{n\pi}{c}at\right)$$

Let us apply solution to a specific function:



Let the string streched between points (0,0) and (2,0) and suppose that its midpoint is raised to a height above the x axis. The string is then releasedfrom rest in thta positionconsisting of two line segments. In this case function f(x) can be described as:

$$f(x) = \begin{Bmatrix} hx & 0 \le x \le 1 \\ -hx + 2h & 1 \le x \le 2 \end{Bmatrix}$$

$$C_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi}{c}x\right) = h\int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx + h\int_1^2 (-x+2)\sin\left(\frac{n\pi}{2}x\right)dx$$

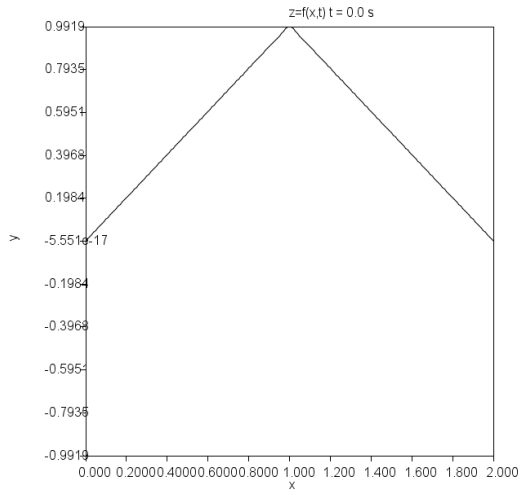
$$C_n = \frac{8h}{n^2\pi^2}sin\left(\frac{n\pi}{2}\right)$$

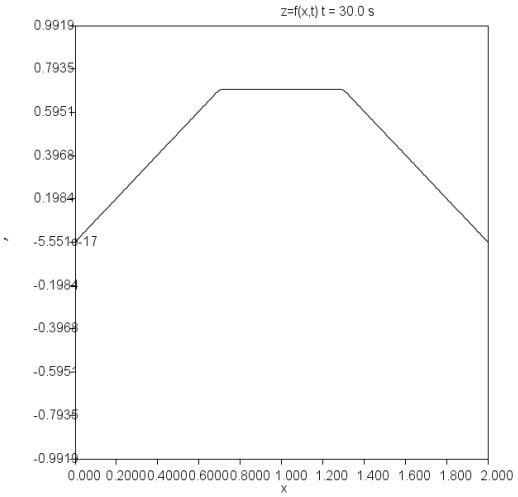
$$y(x,t) = \frac{8h}{\pi^2}sin\left(\frac{n\pi}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^2}sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}at\right)$$

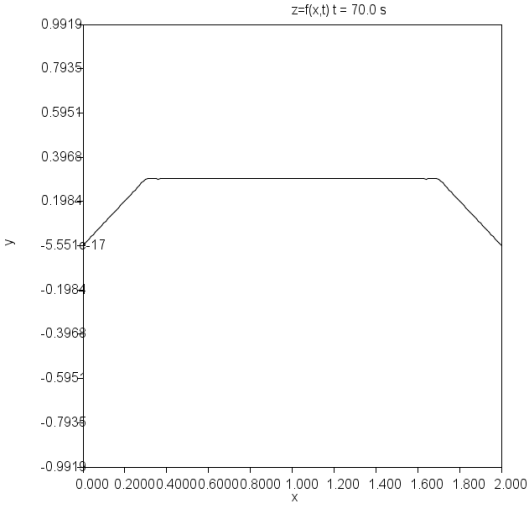
```
class i_wall extends fi_xt
{
double a;
double h;
public i_wall(double ai,double hi)
{a=ai;h=hi;}
public double[] func(double x,double t)
{ double b[]=new double[1];
  double y=0.0;
  double n1=1.0;
  for(int n=1;n<50;n++)
  {   n1=n*Math.PI/2.0;
                y+=1.0/(n*n)*Math.sin(n1)*Math.sin(n1*x)*Math.cos(n1*a*t);
  }
  y*=8.0*h/(Math.PI*Math.PI);
  b[0]=y;
  return b;
}             }
public class timeplot_ntest4
{
public static void main(String args[])
  {
 i_wall ff=new i_wall(0.01,1.0);
  timeplot_n p=new timeplot_n(ff,0.0,2.0,0.01,0.0,500.0,1.0,10.0);
```

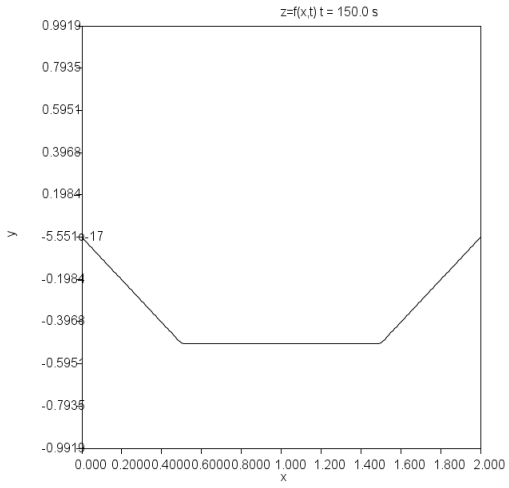
```
  p.start();
  }
}
```

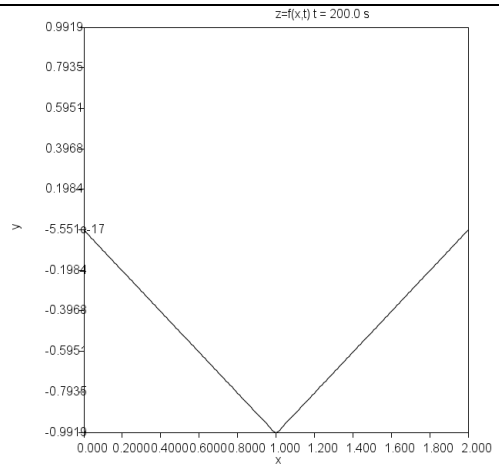
```
abstract class fi_xt implements iMathd
{
  // n function function  two independent variable : x,t
  // n value is returned
  abstract double[] func(double x,double t);
}
```











Let us try anoher f(x) function

$$f(x) = sin\left(\frac{\pi}{2}x\right) \quad 0 \le x \le 2 \quad c = 2$$

$$C_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi}{c}x\right)dx$$

$$C_n = \frac{2}{2}\int_0^2 sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{n\pi}{2}x\right)dx$$

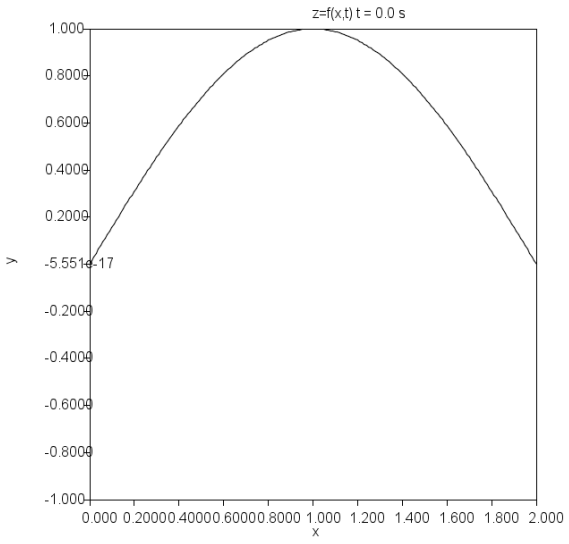
Because of orthogonality n=1 so

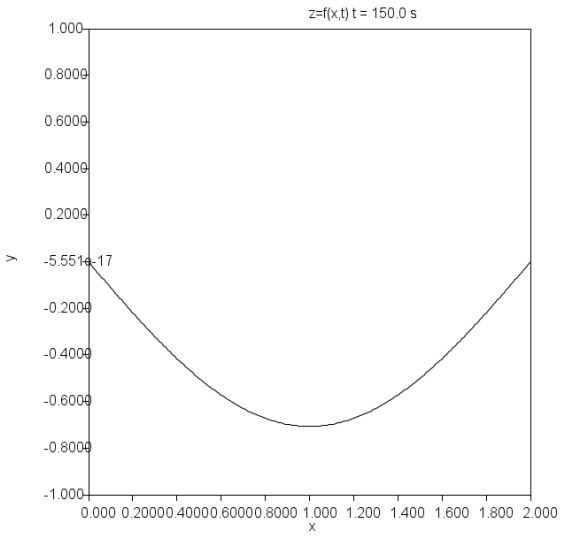
$$C_1 = \int_0^2 sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right)dx = 1$$

$$y(x,t) = sin\left(\frac{\pi}{2}x\right)cos\left(\frac{\pi}{2}at\right)$$

```
class i_wall extends fi_xt
{
double a;
double h;
public i_wall(double ai,double hi)
{a=ai;h=hi;}
public double[] func(double x,double t)
```

```
{ double b[]=new double[1];
  double y=Math.sin(Math.PI/2.0*x)*Math.cos(Math.PI/2.0*a*t);
  b[0]=y;
  return b;
}               }
public class timeplot_ntest5
{
public static void main(String args[])
  {
 i_wall ff=new i_wall(0.01,1.0);
  timeplot_n p=new timeplot_n(ff,0.0,2.0,0.01,0.0,500.0,1.0,10.0);
  p.start();
  }
}
```





EXAMPLE 11:

We are looking to find an expression for the transverse displacement y(x,t) of a string streched at t=0 time between the points y(0,0) and y(c,0). The string is initially displaced into position y(x,0) =f(x), and released with a velocity of g(x) from that position. String is frictionless.

Then equation of motion of frictionless string will be

$$\frac{\partial^2 y(x,t)}{\partial t^2} = a^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

Where $a^2 = \frac{F_0}{\rho_0}$ F_0 is tensile force ρ_0 is linear density of string (assumed to be constant)

So boundary conditions of the equation:

$$y(x,t) = 0 \quad at\ x = 0 \quad t > 0$$

$$y(x,t) = 0 \quad at\ x = c \quad t > 0$$

$$\frac{\partial y(x,t)}{\partial t} = g(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$y(x,t) = f(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

For this problem superposition principle can be applied

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

Where

$$\frac{\partial^2 y_1(x,t)}{\partial t^2} = a^2 \frac{\partial^2 y_1(x,t)}{\partial x^2}$$

So boundary conditions of the equation:

$$y_1(x,t) = 0 \quad at\ x = 0 \quad t > 0$$

$$y_1(x,t) = 0 \quad at\ x = c \quad t > 0$$

$$\frac{\partial y_1(x,t)}{\partial t} = 0 \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$y_1(x,t) = f(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$\frac{\partial^2 y_2(x,t)}{\partial t^2} = a^2 \frac{\partial^2 y_2(x,t)}{\partial x^2}$$

So boundary conditions of the equation:

$$y_2(x,t) = 0 \quad at\ x = 0 \quad t > 0$$

$$y_2(x,t) = 0 \quad at\ x = c \quad t > 0$$

$$\frac{\partial y_2(x,t)}{\partial t} = g(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$y_2(x,t) = 0 \ \ at\ t = 0 \qquad 0 \le x \le c$$

The solution of y_1 was already obtained in previous problem as

$$y_1(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right)\cos\left(\frac{n\pi}{c}at\right)$$

$$C_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi}{c}x\right)$$

$$y_1(x,t) = \sum_{n=1}^{\infty} \frac{2}{c} \int_0^c f(x) \sin\left(\frac{n\pi}{c}x\right) \sin\left(\frac{n\pi}{c}x\right) \cos\left(\frac{n\pi}{c}at\right)$$

Now let us look at the second problem:

We will apply seperation of variables method to solve this equation

$$y_2(x,t) = X(x)\tau(t)$$

$$X(x)\frac{\partial^2\tau(t)}{\partial t^2} = a^2\tau(t)\frac{\partial^2 X(x)}{\partial x^2}$$

$$\frac{1}{a^2\tau(t)}\frac{\partial^2\tau(t)}{\partial t^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\alpha^2$$

$$\frac{\partial^2 X(x)}{\partial x^2} + \alpha^2 X(x) = 0$$

$$\frac{\partial^2\tau(t)}{\partial t^2} + a^2\alpha^2\tau(t) = 0$$

So boundary conditions of the equation:

$$X(x) = 0 \quad at\ x = 0$$

$$X(x) = 0 \quad at\ x = c$$

$$\frac{\partial\tau(t)}{\partial t} = g(x) \ \ at\ t = 0 \qquad 0 \le x \le c$$

$$y_2(x,t) = 0 \ \ at\ t = 0 \qquad 0 \le x \le c$$

Solution of differential equations:

$$A^2 + \alpha^2 = 0$$

$$A = \mp\alpha i$$

$$A^2 + a^2\alpha^2 = 0$$

$$A = \mp\alpha a i$$

$$X(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x)$$

$$\tau(t) = C_3 \sin(\alpha a t) + C_4 \cos(\alpha a t)$$

Applying boundary conditions to solve coefficients :

$$X(x) = 0 \quad at\ x = 0 \ \rightarrow \ X(x) = C_1 \sin(\alpha 0) + C_2 \cos(\alpha 0) = 0 \ \rightarrow \ C_2 = 0$$

$$X(x) = 0 \quad at\ x = c \ \rightarrow \ X(x) = C_1 \sin(\alpha c) = 0 \ \rightarrow \ \alpha c = n\pi \ \rightarrow \ \alpha = \frac{n\pi}{c}$$

$$\tau(t) = C_3 \sin(\alpha a t) + C_4 \cos(\alpha a t)$$

$$\tau(t) = 0 \ at\ t = 0 \qquad 0 \le x \le c \ \rightarrow \ C_3 sin(\alpha a 0) + C_4 cos(\alpha a 0) = 0 \ \rightarrow \ C_4 = 0$$

$$\frac{\partial\tau(t)}{\partial t} = C_3 \alpha a \cos(\alpha a t)$$

$$\frac{\partial \tau(t)}{\partial t} = 0 \;\; at\; t = 0 \quad 0 \le x \le c \;\; \rightarrow \;\; C_3 \alpha a \cos(\alpha a 0) - C_4 \alpha a \sin(\alpha a 0) = 0 \;\rightarrow\; C_3 = 0$$

$$y_2(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right)\sin\left(\frac{n\pi}{c}at\right)$$

Applying last remaining boundary condition

$$\frac{\partial y_2(x,t)}{\partial t} = g(x) \;\; at\; t = 0 \quad 0 \le x \le c$$

$$\frac{\partial y_2(x,t)}{\partial t} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right)\frac{n\pi}{c}a\cos\left(\frac{n\pi}{c}a0\right) = g(x)$$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right) = \frac{2g(x)}{na\pi}$$

This equation has n coefficients. In order to solve these coefficients, we will use properties of orthogonal functions.

$$\int_0^c X(\alpha_n x)X(\alpha_m x)dx = \begin{cases} 0 & m \neq n \\ N(\alpha_m) & m = n \end{cases}$$

$$\int_0^c \frac{g(x)}{\alpha a}\sin\left(\frac{n\pi}{c}x\right) = C_n \int_0^c sin^2\left(\frac{n\pi}{c}x\right)dx$$

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

$$2\sin^2(A) = 1 - \cos(2A)$$

$$\sin^2(A) = \frac{1-\cos(2A)}{2}$$

$$\int_0^c sin^2\left(\frac{n\pi}{c}x\right)dx = \int_0^c \frac{1-\cos\left(2\frac{n\pi}{c}x\right)}{2})dx = \frac{1}{2}\left[x - \frac{c}{2n\pi}\cos\left(2\frac{n\pi}{c}x\right)\right]_0^c = \frac{c}{2}$$

$$C_n = \frac{2}{na\pi}\int_0^c g(x)\sin\left(\frac{n\pi}{c}x\right)$$

$$y_2(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right)\sin\left(\frac{n\pi}{c}at\right)$$

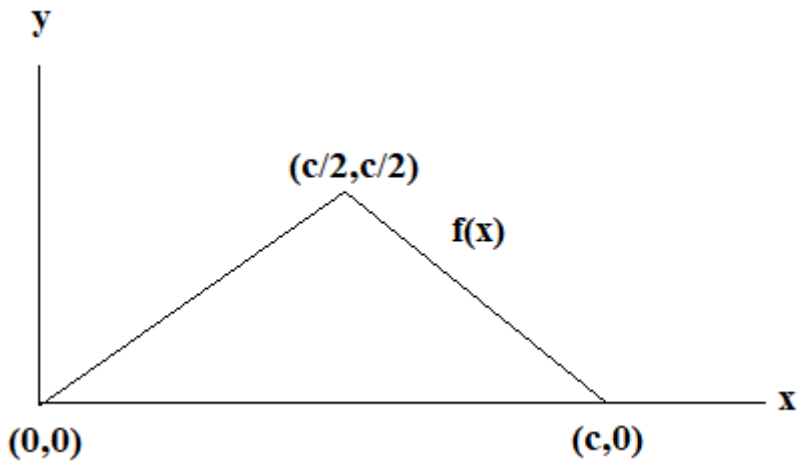
So total solution will be:

$$y_1(x,t) = \frac{2}{c}\sum_{n=1}^{\infty}\left[\int_0^c f(x)\sin\left(\frac{n\pi}{c}x\right)\right]\sin\left(\frac{n\pi}{c}x\right)\cos\left(\frac{n\pi}{c}at\right)$$

$$y_2(x,t) = \frac{2}{a\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\int_0^c \frac{g(x)}{\alpha a}\sin\left(\frac{n\pi}{c}x\right)dx\right]\sin\left(\frac{n\pi}{c}x\right)\sin\left(\frac{n\pi}{c}at\right)$$

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

As a numerical example suppose that



Displacement at t=0

$$f(x) = \begin{cases} x & 0 \le x \le c/2 \\ c - x & c/2 \le x \le c \end{cases}$$

Velocity at t=0

$$g(x) = 1$$

In this case:

EXAMPLE 12:

Considered vibration of the streched string released from rest from its natural strechd position, but an external force F(x)=Ax acting on it.

$$\frac{\partial^2 y(x,t)}{\partial t^2} = a^2 \frac{\partial^2 y(x,t)}{\partial x^2} + Ax \quad 0 \le x \le c \quad t \ge 0$$

Where $a^2 = \frac{F_0}{\rho_0}$ F_0 is tensile force ρ_0 is linear density of string (assumed to be constant)

So boundary conditions of the equation:

$$y(x,t) = 0 \;\; at\; x = 0 \;\; t > 0$$

$$y(x,t) = 0 \;\; at\; x = c \;\; t > 0$$

$$\frac{\partial y(x,t)}{\partial t} = 0 \;\; at\; t = 0 \quad 0 \le x \le c$$

$$y(x,t) = 0 \;\; at\; t = 0 \quad 0 \le x \le c$$

Equation will be divided into two unique solution such as:

$$y(x,t) = Y(x,t) + \varphi(x)$$

As it is seen from the formula, $\varphi(x)$ is onlyfunction of x, therefore $\frac{\partial^2 \varphi(x)}{\partial t^2} = 0$ so this part of equation reduced to:

$$a^2 \frac{\partial^2 \varphi(x)}{\partial x^2} + Ax \quad 0 \le x \le c$$

Boundary conditions:

$$x = 0 \quad \varphi(x) = 0$$

$$x = c \quad \varphi(x) = 0$$

$$\frac{\partial^2 \varphi(x)}{\partial x^2} = -\frac{A}{a^2} x$$

$$K = \frac{A}{a^2}$$

$$\frac{\partial^2 \varphi(x)}{\partial x^2} = -Kx$$

$$\frac{\partial \varphi(x)}{\partial x} = -K\frac{x^2}{2} + C_1$$

$$\varphi(x) = -K\frac{x^3}{6} + C_1 x + C_2$$

B.C

$$x = 0 \quad \varphi(x) = 0$$

$$C_2 = 0$$

B.C

$$x = c \quad \varphi(x) = 0$$

$$-K\frac{c^3}{6} + C_1 c = 0$$

$$C_1 = K\frac{c^2}{6}$$

$$\varphi(x) = -K\frac{x^3}{6} + K\frac{c^2}{6} x$$

$$\varphi(x) = K\frac{1}{6} x \ (c^2 - x^2)$$

Second part of the differential equation:

$$\frac{\partial^2 Y(x,t)}{\partial t^2} = a^2 \frac{\partial^2 Y(x,t)}{\partial x^2} + Ax \quad 0 \le x \le c \quad t \ge 0$$

So boundary conditions of the equation:

$$Y(x,t) = 0 \ \ at\ x = 0 \ \ t > 0$$

$$Y(x,t) = 0 \ \ at\ x = c \ \ t > 0$$

$$\frac{\partial Y(x,t)}{\partial t} = 0 \ \ at\ t = 0 \quad 0 \le x \le c$$

$$Y(x,t) = -\varphi(x) = \tfrac{K}{6} x \ (x^2 - c^2) \ \ at\ t = 0 \quad 0 \le x \le c$$

This problem previously solved and the result was:

$$C_n = -\frac{2}{c}\int_0^c \varphi(x)\sin\left(\frac{n\pi}{c}x\right)dx = \frac{2c}{6}K\int_0^c (x^3 - c^2 x)\sin\left(\frac{n\pi}{c}x\right)dx$$

$$C_n = \frac{2Kc^3}{n^3\pi^3}\cos(n\pi) = \frac{2Kc^3}{n^3\pi^3}(-1)^n$$

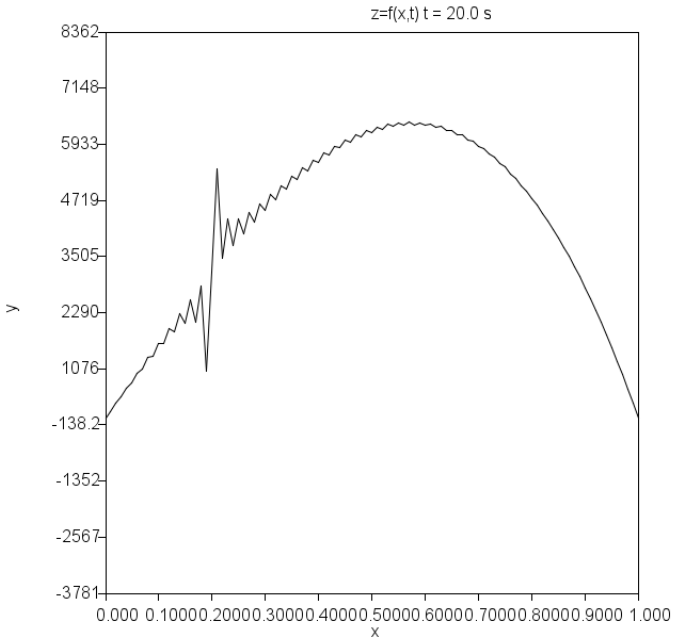
$$Y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{c}x\right)\cos\left(\frac{n\pi}{c}at\right)$$

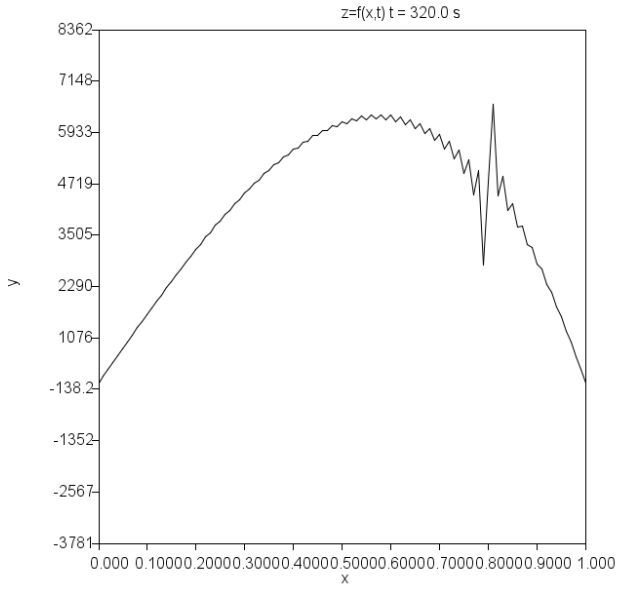
$$Y(x,t) = \sum_{n=1}^{\infty}\left[\frac{2Kc^3}{n^3\pi^3}(-1)^n\right]\sin\left(\frac{n\pi}{c}x\right)\cos\left(\frac{n\pi}{c}at\right)$$

$$y(x,t) = Y(x,y) + \varphi(x)$$

$$y(x,t) = \sum_{n=1}^{\infty}\left[\frac{2Kc^3}{n^3\pi^3}(-1)^n\right]\sin\left(\frac{n\pi}{c}x\right)\cos\left(\frac{n\pi}{c}at\right) + \frac{K}{6}x\ \ (c^2 - x^2)$$

```
class i_wall extends fi_xt
{
double a;
double A;
double K;
public i_wall(double ai,double Ai)
{a=ai;K=Ai/(a*a);}
public double[] func(double x,double t)
{ double b[]=new double[1];
  double y=0.0;
  double n1=1.0;
  double c=1.0;
  double PI=Math.PI;
  for(int n=1;n<100;n++)
  {   n1=n*Math.PI/c;
                y+=2.0*K*c*c*c/(PI*PI*PI)*Math.sin(n1*x/c)*Math.cos(n1*a*t/c)+1.0/6.0*K*x*(c*c-x*x);
  }
  b[0]=y;
  return b;
}              }
public class timeplot_ntest6
{
public static void main(String args[])
  {
 i_wall ff=new i_wall(0.01,0.1);
  timeplot_n p=new timeplot_n(ff,0.0,1.0,0.01,0.0,500.0,1.0,10.0);
  p.start();
  }
}
```





11. Partial differential equation solution methods: Duhamel problem & Green Functions

ANALYTIC SOLUTION METHODS: DUHAMEL'S THEOREM

Consider the three dimensional, nonhomogeneous heat conduction problem in a region R with time dependent boundary condition function and heat generation in the form

$$\nabla_2 T(r,t) + \frac{1}{k} g(r,t) = \frac{1}{\alpha} \frac{\partial T(r,t)}{\partial t} \quad \text{in region R t>0}$$

$$k_i \frac{\partial T(r,t)}{\partial n_i} + h_i T = f_i(r,t) \quad \text{on boundary } S_i$$

$$T(r,t) = F(r) \quad \text{for t=0}$$

This problem can not be solved with seperation of variables methods described in previous problems because of the non-homegeneous terms g(r,t) and f_i(r,t). Therefore instead of solving this problem directly, problem will be expressed in terms of solution of a simpler auxilary problem as follows: Let $\Phi(r,t,\tau)$ be solution of the above problemon the assumption that the nonhomogeneous terms $g(r,\tau)$ and $f_i(r,\tau)$ do not depend on time; namely , the variable τ is merely a parameter but not a time variable. Then $\Phi(r,t,\tau)$ is the solution of the following auxilary problem

$$\nabla_2\Phi(r,t,\tau)+\frac{1}{k}g(r,\tau)=\frac{1}{\alpha}\frac{\partial T(r,t,\tau)}{\partial t} \text{ in region R t>0}$$

$$k_i\frac{\partial\Phi(r,t,\tau)}{\partial n_i}+h_i\Phi(r,t,\tau)=f_i(r,\tau) \text{ on boundary } S_i$$

$$\Phi(r,t,\tau)=F(r) \text{ for t=0}$$

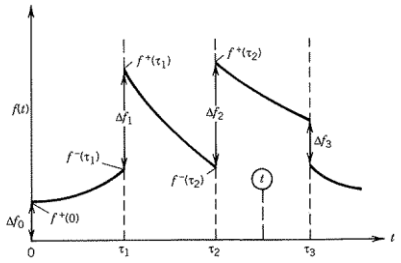
This probem can be solved by using methods such as seperation of variables. Then Duhamel's theorem relates the solution $\Phi(r,t,\tau)$ and $T(r,t)$ by the following intgral equation

$$T(r,t)=\frac{\partial}{\partial t}\int_{\tau=0}^{t}\Phi(r,t-\tau,\tau)d\tau$$

This result can also be expressed in the alternative form by performing the differential

$$T(r,t)=F(r)+\int_{\tau=0}^{t}\frac{\partial}{\partial t}\Phi(r,t-\tau,\tau)d\tau$$

If the time function has an F(r) function with discontinuities such as:



Equation becomes:

$$T(r,t)=\int_{\tau=0}^{t}\frac{\partial}{\partial t}\Phi(r,t-\tau)d\tau+\sum_{j=0}^{N-1}\Phi(r,t-\tau_j)\Delta f_j$$

After these initial definitions, now A heat transfer problem can be investigated

A slab of thickness L is initially at zero temperature. For times t>0 boundary surface at x=0 kept at zero temperature, while the surface at x=L is subjected to a time varying temperature f(t) defined by

$$f(t)=\begin{Bmatrix} F(t) & for & 0<t<\tau_1 \\ 0 & for & t>\tau_1 \end{Bmatrix}$$

equation

$$\frac{\partial^2 T(x,t)}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$ with boundary condition:

T(x,t) = 0 x=0 t > 0
T(x,t) = f(t) x=L t > 0
T(x,t) = 0 at t = 0

Becomes

$$\frac{\partial^2\Phi(x,t)}{\partial x^2}=\frac{1}{\alpha}\frac{\partial\Phi(x,t)}{\partial t}$$

Φ (x,t) = 0 x=0 t > 0
Φ (x,t) = 1 x=L t > 0
Φ (x,t) = 0 at t = 0

Problem divided into two part

$$\Phi(x,t) = \Phi_H(x,t) + \Phi_{ss}(x)$$

Where

$$\frac{\partial^2 \Phi_{ss}(x)}{\partial x^2} = 0$$

$\Phi_{ss}(x) = 0$ x=0

$\Phi_{ss}(x) = 1$ x=L

$$\frac{\partial \Phi_{ss}(x)}{\partial x} = C_1 \qquad \Phi_{ss}(x) = C_1 x + C_2$$

$\Phi_{ss}(x) = 0$ x=0 ➔ $C_2=0$

$\Phi_{ss}(x) = 1$ x=L ➔ $C_1 = \frac{1}{L}$

$$\Phi_{ss}(x) = \frac{x}{L}$$

$$\frac{\partial^2 \Phi_H(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Phi_H(x,t)}{\partial t}$$

$\Phi_H(x,t) = 0$ x=0 t > 0

$\Phi_H(x,t) = 0$ x=L t > 0

$\Phi_H(x,t) = -\Phi_{ss}(x)$ at t = 0

Seperation of variables:

$$\Phi_H(x,t) = X(x) * \Gamma(t)$$

$$\Gamma(t)\frac{\partial^2 X(x)}{\partial x^2} = \frac{X(x)}{\alpha}\frac{\partial \Gamma(x,t)}{\partial t}$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha * \Gamma(t)}\frac{\partial \Gamma(x,t)}{\partial t} = -\beta^2$$

$$\frac{\partial^2 X(x)}{\partial x^2} + \beta^2 X(x) = 0$$

$$\frac{\partial \Gamma(x,t)}{\partial t} + \beta^2 * \alpha * \Gamma(t) = 0$$

Boundary conditions

$X(x) = 0$ x=0 t >0

$X(x) = 0$ x=L t >0

$$X(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x)$$

$$\Gamma(t) = C_3 \exp(-\beta^2 \alpha t)$$

$$\Phi_H(x,t) = (C_1 \cos(\beta x) + C_2 \sin(\beta x) * \exp(-\beta^2 \alpha t)$$

Applications of boundary conditions:

$X(x) = C_1 \cos(\beta 0) + C_2 \sin(\beta 0) = 0$ $C_1=0$

$$X(x) = C_2 \sin(\beta L) = 0$$

$$\Phi_H(x,t) = \sum_{n=0}^{\infty} C_m \sin(\beta_m x) * \exp(-\beta_m^2 \alpha t)$$

t=0 $\Phi_H(x,t) = 0$ $\sin(\beta_m x) = 0$

$$\beta_m = \frac{m\pi}{L}$$

Φ_H (x,t) = - Φ_{ss} (x) at t = 0

$$-\Phi_s(x) = \sum_{n=0}^{\infty} C_n \sin(\beta_m x)$$

$$-\frac{x}{L} = \sum_{n=0}^{\infty} C_n \sin(\beta_m x)$$ from ortogonality condition

$$-\int_{x=0}^{L} \frac{x}{L}\sin(\beta_m x)dx = C_m \int_{x=0}^{L} \sin^2(\beta_m x)dx$$

$$\cos^2(\beta_m x) + \sin^2(\beta_m x) = 1$$

$$\cos^2(\beta_m x) - \sin^2(\beta_m x) = \cos(2\beta_m x) \rightarrow \sin^2(\beta_m x) = \frac{1-\cos(2\beta_m x)}{2}$$

$$u = \frac{x}{L} \quad dv = \sin(\beta_m x)dx \quad du = \frac{dx}{L} \quad dv = \sin(\beta_m x)dx \quad v = -\frac{\sin(\beta_m x)dx}{\beta_m}$$

$$\int u dv = uv - \int v du$$

$$\frac{x}{L}\left(\frac{1}{\beta_m}\cos(\beta_m x)\right)\Bigg|_{x=0}^{L} + \int_{x=0}^{L} \frac{1}{L\beta_m}\cos(\beta_m x)dx = C_m \int_{x=0}^{L} \frac{1-\cos(2\beta_m x)}{2}dx$$

$$\frac{L}{L}\left(\frac{1}{\beta_m}\cos(m\pi)\right) + \frac{L}{\beta_m^2}\sin(\beta_m x)\Bigg|_{x=0}^{L} = C_m\left(\frac{L}{2} - \frac{\sin(2\beta_m x)}{4\beta_m}\right)$$

$$C_m = \frac{2}{L\beta_m}(-1)^m$$

$$\Phi_H(x,t) = \frac{2}{L}\sum_{n=0}^{\infty} \frac{(-1)^m}{\beta_m}\sin(\beta_m x) * \exp(-\beta_m^2 \alpha t)$$

$$\Phi(x,t) = \Phi_H(x,t) + \Phi_{ss}(x) = \frac{x}{L} + \frac{2}{L}\sum_{n=0}^{\infty} \frac{(-1)^m}{\beta_m}\sin(\beta_m x) * \exp(-\beta_m^2 \alpha t)$$

The function $\Phi(x, t-\tau)$ is obtained by replacing t by $t-\tau$ in equation

$$\Phi(x,t) = \frac{x}{L} + \frac{2}{L}\sum_{n=0}^{\infty} \frac{(-1)^m}{\beta_m}\sin(\beta_m x) * \exp(-\beta_m^2 \alpha (t-\tau))$$

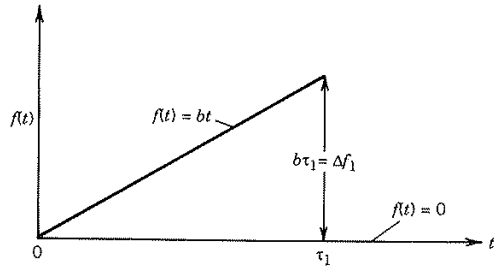
Duhamel theorem now can be applied to solve the problem. Remember that function to be considered was

$$f(t) = \begin{Bmatrix} F(t) & for & 0 < t < \tau_1 \\ 0 & for & t > \tau_1 \end{Bmatrix}$$

For boundary condition T(x,t) = f(t) x=L t > 0

The solution for times $t < \tau_1$

$$T(x,t) = \int_{\tau=0}^{t} \phi(x, t-\tau)\frac{\partial f(\tau)}{\partial \tau}d\tau$$ for a specific function $f(t) = \begin{Bmatrix} F(t) = bt & for & 0 < t < \tau_1 \\ 0 & for & t > \tau_1 \end{Bmatrix}$



$$T(x,t)=\int_{\tau=0}^{t}\left(\frac{x}{L}+\frac{2}{L}\sum_{n=0}^{\infty}\frac{(-1)^m}{\beta_m}\sin(\beta_m x)*\exp(-\beta_m^2\alpha(t-\tau))\right)bd\tau$$

The integration is performed to give

$$T(x,t)=\left(b\frac{x}{L}t+b\frac{2}{L}\sum_{n=0}^{\infty}\frac{(-1)^m}{\alpha\beta_m^3}\sin(\beta_m x)*(1-\exp(-\beta_m^2\alpha t))\right)$$

The solution for times $t>\tau_1$

$$T(x,t)=\int_{\tau=0}^{\tau_1}\phi(x,t-\tau)\frac{\partial f(\tau)}{\partial\tau}d\tau+\int_{\tau=\tau_1}^{t}\phi(x,t-\tau)\frac{\partial f(\tau)}{\partial\tau}d\tau+\phi(x,t-\tau_1)\Delta f_1$$ for a specific function

Java solution for this problem is given as follows:

```
 import java.util.*;
class fc extends f_xxi
{
public double func(double t,int i)
{
  double ff=0;
  if(i==0)
  {
              if(t<=10)ff=2.0*t;
              df=20.0;
  }
  else     ff=0.0;
  return ff;
}
}

public class HT_duhamel1
{

//integration function
public static double intfr(double to,double t,double x,double L,double alpha)
{ double fi=x/L;
  double bm=0;
  int plusminus=-1;
  for(int m=1;m<20;m++)
  {  bm=m*Math.PI/L;
               fi+=2.0/L*Math.exp(-alpha*bm*bm*(t-to))*plusminus/bm*Math.sin(bm*x);
               plusminus*=-1;
  }
              //double T=x/Math.sqrt(4.0*Math.PI*alpha)*f(to)/Math.pow((t-to),1.5)*Math.exp(-x*x/(4.0*alpha*(t-to)));
              return fi;
}
// analytic solution function
public static double intfr1(double t,double x,double L,double alpha)
{ double fi=2*x/L*t;
  double bm=0;
  int plusminus=-1;
  for(int m=1;m<20;m++)
  {  bm=m*Math.PI/L;
               fi+=4.0/L*plusminus/(bm*bm*bm*alpha)*(1.0-Math.exp(-alpha*bm*bm*t))*Math.sin(bm*x);
               plusminus*=-1;
  }
              //double T=x/Math.sqrt(4.0*Math.PI*alpha)*f(to)/Math.pow((t-to),1.5)*Math.exp(-x*x/(4.0*alpha*(t-to)));
              return fi;
}
//integral
```

```
//Gauss Kronrod integration formula
public static double integral(f_xxi f,int ii,double a,double b,double t,double x,double L,double alpha)
   {
                //eps hata miktari
                int n=50;
                double eps=1e-10;
      int Aused=0;
                boolean sonuc;
      double integral=0;
      double A[][];
      int Aw = 0;
      double toplamhata = 0;
      double[] c;
      double[] xg;
      double[] xk;
      int nn = 0;
      int ng = 0;
      int i = 0;
      int j = 0;
      int h = 0;
      double v = 0;
      double k1 = 0;
      double k2 = 0;
      double intg = 0;
      double intk = 0;
      double ta = 0;
      double tb = 0;
      Aw = 4;
      A = new double[n-1+1][Aw-1+1];
      nn = 61;
      ng = 15;
      c = new double[nn];
      xk = new double[nn];
      xg = new double[nn];
      //Gauss katsayilari
      xg[0] = 0.00796819249616660561546588347467 4;
      xg[1] = 0.01846646831109095914230213191204 7;
      xg[2] = 0.02878470788332336934971917961129 2;
      xg[3] = 0.03879919256962704959680193644634 8;
      xg[4] = 0.04840267283059405290293814042280 8;
      xg[5] = 0.05749315621761906648172168940205 6;
      xg[6] = 0.06597422988218049512812851511596 2;
      xg[7] = 0.07375597473770520626824385002219 1;
      xg[8] = 0.08075589522942021535469493846053 0;
      xg[9] = 0.08689978720108297980238753071512 6;
      xg[10] = 0.09212252223778612871763270708761 9;
      xg[11] = 0.09636873717464425963946862635181 0;
      xg[12] = 0.09959342058679526706278028210356 9;
      xg[13] = 0.10176238974840550459642895216855 4;
      xg[14] = 0.10285265289355884034128563670541 5;
      c[0] = 0.99948441005049063757132589570581 1;
      c[1] = 0.99689348407464954027163005091869 5;
      c[2] = 0.99163099687040459485862836610948 6;
      c[3] = 0.98366812327974720997003258160566 3;
      c[4] = 0.97311632250112626837469386842370 7;
      c[5] = 0.96002186496830751221687102558179 8;
      c[6] = 0.94437444474855997941583132403743 9;
      c[7] = 0.92620004742927432587932427708047 4;
      c[8] = 0.90557330769990779854652255892595 8;
      c[9] = 0.88256053579205268154311646253022 6;
      c[10] = 0.85720523354606109895865851065894 4;
      c[11] = 0.82956576238276839744289811973250 2;
      c[12] = 0.79972783582183908301366894232268 3;
      c[13] = 0.76777743210482619491797734097450 3;
      c[14] = 0.73379006245322680472617113136952 8;
      c[15] = 0.69785049479331579693229238802664 0;
      c[16] = 0.66006106412662696137005366814927 1;
      c[17] = 0.62052618298924286114047755643118 9;
      c[18] = 0.57934523582636169175602493217254 0;
      c[19] = 0.53662414814201989926416979331107 3;
      c[20] = 0.49248046786177857499369306120770 9;
      c[21] = 0.44703376953808917678060990032285 4;
      c[22] = 0.40040125483039439253547621154266 1;
      c[23] = 0.35270472553087811347103720708937 4;
      c[24] = 0.30407320227362507737267710719925 7;
      c[25] = 0.25463692616788984643980512981780 5;
```

```
c[26] = 0.20452511668230989143895767100202 5;
c[27] = 0.15386991360858354696379467274325 6;
c[28] = 0.10280693796673703014709675131800 1;
c[29] = 0.05147184255531769583302521316672 3;
c[30] = 0.00000000000000000000000000000000 0;
//Kronrod katsayilari
xk[0] = 0.00138901369867700762455159122676 0;
xk[1] = 0.00389046112709988405126720184451 6;
xk[2] = 0.00663070391593129217331982636975 0;
xk[3] = 0.00927327965951776342844114689202 4;
xk[4] = 0.01182301525349634174223289885325 1;
xk[5] = 0.01436972950704580481245143244358 0;
xk[6] = 0.01692088918905327262757228942032 2;
xk[7] = 0.01941414119394238117340895105012 8;
xk[8] = 0.02182803582160919229716748573833 9;
xk[9] = 0.02419116207808060136568637072523 2;
xk[10] = 0.02650995488233310161060170933507 5;
xk[11] = 0.02875404876504129284397878535433 4;
xk[12] = 0.03090725756238776247288425294309 2;
xk[13] = 0.03298144705748372603181419101685 4;
xk[14] = 0.03497933802806002413749967073146 8;
xk[15] = 0.03688236465182122922391106561713 6;
xk[16] = 0.03867894562472759295034865153228 1;
xk[17] = 0.04037453895153595911199527975246 8;
xk[18] = 0.04196981021516424614714754128597 0;
xk[19] = 0.04345253970135606931683172811707 3;
xk[20] = 0.04481480013316266319235555161672 3;
xk[21] = 0.04605923827100698811627173555937 4;
xk[22] = 0.04718554656929915394526147818109 9;
xk[23] = 0.04818586175708712914077949229830 5;
xk[24] = 0.04905543455502977888752816536723 8;
xk[25] = 0.04979568342707420635781156937994 2;
xk[26] = 0.05040592140278234684089308565358 5;
xk[27] = 0.05088179589874960649229747304980 5;
xk[28] = 0.05122154784925877217065628260494 4;
xk[29] = 0.05142612853745902593386287921578 1;
xk[30] = 0.05149472942945156755834043364709 9;
for(i=nn-1; i>=nn/2; i--)
{c[i] = -c[nn-1-i];}
for(i=nn-1; i>=nn/2; i--)
{ xk[i] = xk[nn-1-i]; }
for(i=ng-1; i>=0; i--)
{ xg[nn-2-2*i] = xg[i];
  xg[1+2*i] = xg[i];
}
for(i=0; i<=nn/2; i++)
{ xg[2*i] = 0; }
k1 = 0.5*(b-a);
k2 = 0.5*(b+a);
intg = 0;
intk = 0;
for(i=0; i<=nn-1; i++)
{  double tt=(k1*c[i]+k2);
                v = intfr(tt,t,x,L,alpha)*f.dfunc(tt,ii);
  intk = intk+v*xk[i];
  if( i%2==1 )
  {intg = intg+v*xg[i];}
}
intk = intk*(b-a)*0.5;
intg = intg*(b-a)*0.5;
A[0][0] = Math.abs(intg-intk);
A[0][1] = intk;
A[0][2] = a;
A[0][3] = b;
toplamhata = A[0][0];
if( toplamhata<eps )
{
   sonuc = true;
   integral = intk;
   Aused = 1;
   return integral;
}
Aused = 1;
for(h=1; h<=n-1; h++)
{
   Aused = h+1;
```

```
        gir(A, h, Aw);
        toplamhata = toplamhata-A[h-1][0];
        ta = A[h-1][2];
        tb = A[h-1][3];
        A[h-1][2] = ta;
        A[h-1][3] = 0.5*(ta+tb);
        A[h][2] = 0.5*(ta+tb);
        A[h][3] = tb;
        for(j=h-1; j<=h; j++)
        {
           k1 = 0.5*(A[j][3]-A[j][2]);
           k2 = 0.5*(A[j][3]+A[j][2]);
           intg = 0;
           intk = 0;
           for(i=0; i<=nn-1; i++)
           {   double tt=(k1*c[i]+k2);
                       v = intfr(tt,t,x,L,alpha)*f.dfunc(tt,ii);
              intk = intk+v*xk[i];
              if( i%2==1 )
              { intg = intg+v*xg[i];}
           }
           intk = intk*(A[j][3]-A[j][2])*0.5;
           intg = intg*(A[j][3]-A[j][2])*0.5;
           A[j][0] = Math.abs(intg-intk);
           A[j][1] = intk;
           toplamhata = toplamhata+A[j][0];
        }
        cik(A, h-1, Aw);
        cik(A, h, Aw);
        if( toplamhata<eps )
        {break;}
     }
     sonuc = toplamhata<eps;
     integral = 0;
     for(j=0; j<=Aused-1; j++)
     { integral = integral+A[j][1];}
     return integral;
  }
  public static void gir(double A[][],int n,int Awidth)
  {
     int i = 0;
     int p = 0;
     double t = 0;
     int maxcp = 0;
     if( n==1 )
     { return; }
     for(i=0; i<=Awidth-1; i++)
     {   t = A[n-1][i];
        A[n-1][i] = A[0][i];
        A[0][i] = t;
     }//end of for
     p = 0;
     while( 2*p+1<n-1 )
     {   maxcp = 2*p+1;
        if( 2*p+2<n-1 )
        {   if( A[2*p+2][0]>A[2*p+1][0] )
           { maxcp = 2*p+2; }
        }
        if( A[p][0]<A[maxcp][0] )
        {   for(i=0; i<=Awidth-1; i++)
           {   t = A[p][i];
              A[p][i] = A[maxcp][i];
              A[maxcp][i] = t;
           }
           p = maxcp;
        }
        else
        { break; }
     } //end of while
  }

  public static void cik(double A[][],int n,int Awidth)
  {
     int i = 0;
     int p = 0;
     double t = 0;
```

```
        int kk = 0;
        if( n==0 )
        {return;}
        p = n;
        while( p!=0 )
        {   kk = (p-1)/2;
            if( A[p][0]>A[kk][0] )
            { for(i=0; i<=Awidth-1; i++)
                { t = A[p][i];
                  A[p][i] = A[kk][i];
                  A[kk][i] = t;
                }
                p = kk;
            }
            else
            {break;}
        }
    }
    public static double Tplus(double T1,double x,double t,double L,double alpha)
    { double to1=9.0;
      double to2=20.0;
      ////to>to1
      fc f=new fc();
                  double TT=T1+integral(f,0,0.0,to1,t,x,L,alpha)+
                  integral(f,0,to1,to2,t,x,L,alpha)+
                  intfr(to1,t,x,L,alpha)*f.df;
                  return TT;
                }
                public static double T(double T1,double x,double to,double t,double L,double alpha)
    { double to1=9.0;
      double to2=20.0;
      //to<to1
      fc f=new fc();
                  double TT=T1+integral(f,0,0.0,to,t,x,L,alpha);
                  return TT;
                }
      public static void main (String args[])
      {
        double alpha=4.0e-5;
        double x=0.4; //m
        double t=9.0;//s
        double L=0.5; //m
        double T1=20.0;// degree C;
        double T=T(T1,x,t,t,L,alpha);
        double T2=T1+intfr1(t,x,L,alpha);
        double T3=Tplus(T1,x,t,L,alpha);
        System.out.println("T="+T+"T2="+T2);
        System.out.println("T3="+T3);
      }
}
```

```
import static java.lang.Math.*;

abstract  class f_xxi extends Mathd
{
  // single function single independent variable
  // example f=x*x
  // Reference :"Generation of Finite Difference Formulas on Arbitrary Spaced Grids",
  // Bength Fornberg, Mathematics of Computation, Volume 51, Number 184, October 1988
  // pages 699-706
double df;
double c[][][];
double alpha[];
double h;
int M;
public double a[];
//constructors
   public f_xxi(double ai[])
   {setA(ai);
   derivative_coef();
   df=0;
   }

   public f_xxi()
   { double ai[]=new double[0];
```

```
                setA(ai);
                derivative_coef();
                df=0;
   }

  public void setA(double ai[])
  {int n=ai.length;
   a=new double[n];
   for(int i=0;i<n;i++)
   {a[i]=ai[i];}
  }
// abstract function func
  abstract public double func(double x,int i);

// 1st order derivative

public double dfunc(double x,int i)
{double h=1.0e-3;
 int n=1;
 return dfunc(x,i,n,h);
}
// nth order derivative
public double dfunc(double x,int i,int n)
{double h=1.0e-2;
 return dfunc(x,i,n,h);
}

double dfunc(double x,int k,int N,double hi)
{// order of the maximum derivative
 // N order of derivative
  setH(hi);
  double deriv=0;
  double h1=1/h;
  double h2=1;
  for(int j=0;j<N;j++)
  {h2*=h1;}
  for(int i=0;i<c[0].length;i++)
  {deriv+=c[M][i][N]*func(x+alpha[i]*h,k);}
  return deriv*h2;
}


public void setH(double hi)
{h=hi;}
public void coef_out(int N)
{  double a[][]=new double[alpha.length][2];
   for(int i=0;i<alpha.length;i++)
   {a[i][0]=alpha[i];
    a[i][1]=c[M][i][N];
   }
   String s1="Derivative coefficients for order of the highest derivative M = "+M;
   String s[]=new String[2];
   s[0]=""+'\u0394'+"x";
   if(N==1){s[1]="df(x)/dx";}
   else if(N==2) {s[1]="d"+'\u00B2'+"f(x)/dx"+'\u00B2';}
   else if(N==3) {s[1]="d"+'\u00B3'+"f(x)/dx"+'\u00B3';}
   else  {s[1]="d("+N+")f(x)/dx"+N;}
                Text.print(a,s,s1);
}
public void coef_out()
{  double a[][]=new double[alpha.length][M+1];
   String s[]=new String[M+1];
   s[0]=""+'\u0394'+"x";
   for(int j=1;j<=M;j++)
   {
   if(j==1){s[1]="df(x)/dx";}
   else if(j==2) {s[j]="d"+'\u00B2'+"f(x)/dx"+'\u00B2';}
   else if(j==3) {s[j]="d"+'\u00B3'+"f(x)/dx"+'\u00B3';}
   else  {s[j]="d("+j+")f(x)/dx"+j;}
   }
   for(int i=0;i<alpha.length;i++)
   {a[i][0]=alpha[i];
    for(int j=1;j<=M;j++)
    a[i][j]=c[M][i][j];
   }
    String s1="Derivative coefficients for order of the highest derivative M = "+M;
```

```
                Text.print(a,s,s1);
}
public void setOrder(int Mi)
{M=Mi;}

public  void derivative_coef()
{
h=0.0001;
double x0=0;
M=60;
double alphai[]={0,1,-1,2,-2,3,-3,4,-4,5,-5,6,-6,7,-7,8,-8,9,-9,10,-10,11,-11,12,-12,13,-13,14,-14,15,
-15,16,-16,17,-17,18,-18,19,-19,20,-20,21,-21,22,-22,23,-23,24,-24,25,-25,26,-26,27,-27,28,-28,29,-29,30,-30,31,-31,32,-32,33,-33,34,-
34,35,-35,36,-36,37,-37,38,-38,39,-39,30,-40,41,-41};
alpha=alphai;
int N=alpha.length-1;
// M   degree of highest derivative
// N+1 number of coefficients
double delta[][][]=new double[N+1][N+1][M+1];
double c1,c2,c3;
delta[0][0][0]=1.0;
c1=1.0;
//for(int i=0;i<alpha.length;i++)
//{System.out.println(i+" "+alpha[i]);}
for(int n=1;n<=N;n++)
{  c2=1;
   for(int nu=0;nu<=(n-1);nu++)
   {c3=alpha[n]-alpha[nu];
    c2=c2*c3;
    if(n<=M) delta[n-1][nu][n]=0.0;
    for(int m=0;m<=Math.min(n,M);m++)
    {   //System.out.println("n="+n+"m="+m+"nu="+nu);
       if(m==0)
       { delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m])/c3;}
       else
       {delta[n][nu][m]=((alpha[n]-x0)*delta[n-1][nu][m]-m*delta[n-1][nu][m-1])/c3;}
    }//next m
   }//next nu
   for(int m=0;m<=Math.min(n,M);m++)
   {   if(m==0)
                {delta[n][n][m]=c1/c2*(-(alpha[n-1]-x0)*delta[n-1][n-1][m]);}
                else
                {delta[n][n][m]=c1/c2*(m*delta[n-1][n-1][m-1]-(alpha[n-1]-x0)*delta[n-1][n-1][m]);}
   }//next m
   c1=c2;
}//next n
c=delta;
}

public double dfunc_n(double x,int i,int n,double h)
{
double hh=1/h;
double df=0;
if(n==0) df=func(x,i);
else if(n==1)
{df=(3.0*func(x-4.0*h,i)-32.0*func(x-3.0*h,i)+168.0*func(x-2.0*h,i)-672.0*func(x-h,i)+672.0*func(x+h,i)-
168.0*func(x+2.0*h,i)+32.0*func(x+3.0*h,i)-3.0*func(x+4.0*h,i))/840.0*hh;}
else if(n==2)
{df=(-14350.0*func(x,i)-9.0*func(x-4*h,i)+128*func(x-3*h,i)-1008*func(x-2*h,i)+8064*func(x-h,i)+8064.0*func(x+h,i)-
1008.0*func(x+2.0*h,i)+128.0*func(x+3.0*h,i)-9.0*func(x+4.0*h,i))/5040.0*hh*hh;}
else if(n==3)
{df=(-7.0*func(x-4.0*h,i)+72.0*func(x-3.0*h,i)-338.0*func(x-2.0*h,i)+488.0*func(x-h,i)-488.0*func(x+h,i)+338.0*func(x+2.0*h,i)-
72.0*func(x+3.0*h,i)+7.0*func(x+4.0*h,i))/240.0*hh*hh*hh;}
else if(n==4)
{df=(2730.0*func(x,i)+7.0*func(x-4.0*h,i)-96.0*func(x-3.0*h,i)+676.0*func(x-2*h,i)-1952.0*func(x-h,i)-
1952.0*func(x+h,i)+676.0*func(x+2.0*h,i)-96.0*func(x+3.0*h,i)+7.0*func(x+4.0*h,i))/240.0*hh*hh*hh*hh;}
else if(n==5)
{df=(func(x-4.0*h,i)-9.0*func(x-3.0*h,i)+26.0*func(x-2.0*h,i)-29.0*func(x-h,i)+29.0*func(x+h,i)-
26.0*func(x+2.0*h,i)+9.0*func(x+3.0*h,i)-func(x+4.0*h,i))/6.0*hh*hh*hh*hh*hh;}
else if(n==6)
{df=(-150.0*func(x,i)-func(x-4.0*h,i)+12.0*func(x-3.0*h,i)-52.0*func(x-2.0*h,i)+116.0*func(x-h,i)+116.0*func(x+h,i)-
52.0*func(x+2.0*h,i)+12.0*func(x+3.0*h,i)-func(x+4.0*h,i))/4.0*hh*hh*hh*hh*hh*hh;}
else if(n==7)
{df=(-func(x-4.0*h,i)+6.0*func(x-3.0*h,i)-14.0*func(x-2.0*h,i)+14.0*func(x-h,i)-14.0*func(x+h,i)+14.0*func(x+2.0*h,i)-
6.0*func(x+3.0*h,i)+func(x+4.0*h,i))/2.0*hh*hh*hh*hh*hh*hh*hh;}
else if(n==8)
```

```
{df=(70.0*func(x,i)+func(x-4.0*h,i)-8.0*func(x-3.0*h,i)+28.0*func(x-2.0*h,i)-56.0*func(x-h,i)-56.0*func(x+h,i)+28.0*func(x+2.0*h,i)-
8.0*func(x+3.0*h,i)+func(x+4.0*h,i))*hh*hh*hh*hh*hh*hh*hh*hh;}
else df=0;
return df;
}

}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" HT_duhamel1
T=20.144848295310858T2=20.14484829531157
T3=19.428225405496892

> Terminated with exit code 0.
```

ANALYTIC SOLUTION METHOD: GREEN FUNCTIONS

Consider the following three-dimensional nonhomogeneous boundary value problem of heat conduction:

$$\nabla^2 T(r,t)+\frac{1}{k}g(r,t)=\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t}$$ in region R t>0

With boundary conditions

$$k_i\frac{\partial T}{\partial n_i}+h_iT=h_iT_\infty i=f_i(r,t)$$ on S_i t>0

T(r,t)=F(r) for t=0

Where $\frac{\partial T}{\partial n_i}$ denotes differentiation along the outward-drawn normal to the boundary surface S_i, i=1,2,3..N

N is the continous boundary surfaces in the region. For generality it is assumed that the generation term g(r,t) and boundary condition function $f_i(r,t)$ vary both position and time. In this equation k_i and h_i considered to be constant. To solve the problem we will considered the following auxilary problem for the same region R

$$\nabla^2 G(r,t|r',\tau)+\frac{1}{k}\delta(r-r')\delta(t-\tau)=\frac{1}{\alpha}\frac{\partial G(r,t|r',\tau)}{\partial t}$$ in region R t>τ

With boundary conditions

$$k_i\frac{\partial G}{\partial n_i}+h_iG=o$$ on S_i t>0

In this equation G is called Green's function. Obeying the casuality requirements the Green's function G be zero for $t<\tau$. The source in equation is a unit impulsive source fort he three-dimensional problem considered here. The delta function ,

δ (r-r') represent a point heat source located at r', while the δ (t-τ) indicates that its an intantaneous heat source releasing its energy spontaneously at time t=τ. In the case of three-dimensional transient non-homogeneous heat conduction problem given by the equation above T(r,t) is expressed in terms of three-dimensional Green's function $G(r,t|r',\tau)$ as

$$T(r,t)=\int_R G(r,t|r',\tau)\Big|_{\tau=0}F(r')dv'+\frac{\alpha}{k}\int_{\tau=0}^{t}d\tau\int_R G(r,t|r',\tau)g(r',\tau)dv'+\alpha\int_{\tau=0}^{t}d\tau\sum_{i=1}^{N}G(r,t|r',\tau)\Big|_{r'=r_i}\frac{1}{k_i}f_i(r',\tau)ds_i'$$

Where R refers to entire volüme of the region considered where S_i refers boundary surface of the region R, i=1,2,3..N. dv' and ds' refers differential volume and surface elements. If the same equation is written for two dimensional problem:

$$T(r,t)=\int_A G(r,t|r',\tau)\Big|_{\tau=0}F(r')dA'+\frac{\alpha}{k}\int_{\tau=0}^{t}d\tau\int_R G(r,t|r',\tau)g(r',\tau)dA'+\alpha\int_{\tau=0}^{t}d\tau\sum_{i=1}^{N}G(r,t|r',\tau)\Big|_{r'=r_i}\frac{1}{k_i}f_i(r',\tau)dl_i'$$

And for one dimensional problem:

$$T(x,t)=\int_L x'^P\,G(x,t|x',\tau)\Big|_{\tau=0}F(x')dx'+\frac{\alpha}{k}\int_{\tau=0}^{t}d\tau\int_L x'^P\,G(x,t|x',\tau)g(x',\tau)dx'+\alpha\int_{\tau=0}^{t}d\tau\sum_{i=1}^{N}x'^P\,G(x,t|x',\tau)\Big|_{r'=r_i}\frac{1}{k_i}f_i$$

Where x'^{P} is the Sturm-Liouville weight function such that

$P = \begin{Bmatrix} 0 & slab \\ 1 & cylinder \\ 2 & sphere \end{Bmatrix}$ Here L refers to the thickness or radius of one dimensional region.

Let us consider an example now:

A slab, $0 \le x \le L$, is initially at temperature F(x). For times t>0 the boundaries at x=0 and x=l are maintained at temperatures $f_1(t)$ and $f_2(t)$ respectively, whereas heat is generated in the medium at a rate of g(x,t) W/m^3. Obtain an expression and solve the equation

$$\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{1}{k} g(x,t) = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \quad \text{in } 0 \le x \le L \quad t>0$$

$T(x,t) = f_1(t)$ at x=0 t>0

$T(x,t) = f_2(t)$ at x=L t>0

$T(x,t) = F(x)$ in $0 \le x \le L$ for t=0

To determine appropriate Green's function, we consider the homogeneous version of this problem as

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{k} g(x,t) = \frac{1}{\alpha} \frac{\partial \psi(x,t)}{\partial t} \quad \text{in } 0 \le x \le L \quad t>0$$

$\psi(x,t) = 0$ at x=0 t>0

$\psi(x,t) = 0$ at x=L t>0

$\psi(x,t) = F(x)$ in $0 \le x \le L$ for t=0

Seperation of variables:

$\psi(x,t) = X(x) * \Gamma(t)$

$$\Gamma(t) \frac{\partial^2 X(x)}{\partial x^2} = \frac{X(x)}{\alpha} \frac{\partial \Gamma(x,t)}{\partial t}$$

$$\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha * \Gamma(t)} \frac{\partial \Gamma(x,t)}{\partial t} = -\beta^2$$

$$\frac{\partial^2 X(x)}{\partial x^2} + \beta^2 X(x) = 0$$

$$\frac{\partial \Gamma(x,t)}{\partial t} + \beta^2 * \alpha * \Gamma(t) = 0$$

Boundary conditions

$X(x) = 0$ x=0 t >0

$X(x) = 0$ x=L t >0

$$X(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x)$$

$$\Gamma(t) = C_3 \exp(-\beta^2 \alpha t)$$

$$\psi(x,t) = (C_1 \cos(\beta x) + C_2 \sin(\beta x) * \exp(-\beta^2 \alpha t)$$

Applications of boundary conditions:

$X(x) = C_1 \cos(\beta 0) + C_2 \sin(\beta 0) = 0$ $C_1=0$

$$X(x) = C_2 \sin(\beta L) = 0$$

$$\Phi_H(x,t) = \sum_{n=0}^{\infty} C_m \sin(\beta_m x) * \exp(-\beta_m^2 \alpha t)$$

t=0 $\Phi_H(x,t)=0 \qquad \sin(\beta_m x)=0$

$$\beta_m=\frac{m\pi}{L}$$

$\psi(x,t)=F(x)$ in $0\le x\le L$ for t=0

$$F(x)=\sum_{n=0}^{\infty}C_n\sin(\beta_m x)$$

from ortogonality condition

$$\int_{x=0}^{L}F(x)\sin(\beta_m x)dx=C_m\int_{x=0}^{L}\sin^2(\beta_m x)dx$$

$$\cos^2(\beta_m x)+\sin^2(\beta_m x)=1$$

$$\cos^2(\beta_m x)-\sin^2(\beta_m x)=\cos(2\beta_m x) \rightarrow \sin^2(\beta_m x)=\frac{1-\cos(2\beta_m x)}{2}$$

$$\int_{x=0}^{L}F(x)\sin(\beta_m x)dx=C_m\int_{x=0}^{L}\frac{1-\cos(2\beta_m x)}{2}dx$$

$$\int_{x=0}^{L}F(x)\sin(\beta_m x)dx=C_m\left(\frac{L}{2}-\frac{\sin(2\beta_m x)}{4\beta_m}\right)=C_m\left(\frac{L}{2}-\frac{\sin(2m\pi)}{4\beta_m}\right)=C_m\frac{L}{2}$$

$$C_m=\frac{2}{L}\int_{x=0}^{L}F(x)\sin(\beta_m x)dx$$

$$\psi(x,t)=\left[\frac{2}{L}\sum_{m=1}^{\infty}\sin(\beta_m x)*\exp(-\beta_m^2\alpha t)\right]\int_{x'=0}^{L}\sin(\beta_m x')F(x')dx'$$

$$\psi(x,t)=\int_{x'=0}^{L}\left[\frac{2}{L}\sum_{m=1}^{\infty}\sin(\beta_m x)*\exp(-\beta_m^2\alpha t)\right]\sin(\beta_m x')F(x')dx' \text{ where } \beta_m=\frac{m\pi}{L} \text{ m=1,2,3,..}$$

the solution of problem in terms of Green's function is given as:

$$T(x,t)=\int_L G(x,t|x',\tau)\Big|_{\tau=0}F(x')dx' \text{ A comparison of the equations yields}$$

$$G(x,t|x',\tau)\Big|_{\tau=0}=\frac{2}{L}\sum_{n=0}^{\infty}\sin(\beta_m x)*\sin(\beta_m x')\exp(-\beta_m^2\alpha t)$$

The desired Green function is obtained by replacing t with $t-\tau$ in the equation so that

$$G(x,t|x',\tau)=\frac{2}{L}\sum_{n=0}^{\infty}\sin(\beta_m x)*\sin(\beta_m x')\exp\left[-\beta_m^2\alpha(t-\tau)\right]$$

Then the solution of the actual problem is given in terms of Green function will be

$$T(x,t)=\int_L G(x,t|x',\tau)\Big|_{\tau=0}F(x')dx'+$$

$$\frac{\alpha}{k}\int_{\tau=0}^{t}d\tau\int_L x'^{P}G(x,t|x',\tau)g(x',\tau)dx'+\alpha\int_{\tau=0}^{t}\frac{\partial G(x,t|x',\tau)}{\partial x'}\Big|_{x'=0}f_1(\tau)d\tau-\int_{\tau=0}^{t}\frac{\partial G(x,t|x',\tau)}{\partial x'}\Big|_{x'=L}f_1(\tau)d\tau$$

It should be note that in the original problem given boundary condition was second(derivative) kind. In this equation it is both the first kind(direct equation), therefore $G(x,t|x',\tau)\Big|_{x'=0}$ replaced with $\frac{\partial G(x,t|x',\tau)}{\partial x'}\Big|_{x'=0}$

for terms involving $f_1(\tau)$ and $G(x,t|x',\tau)\big|_{x'=L}$ replaced with $\left.\frac{\partial G(x,t|x',\tau)}{\partial x'}\right|_{x'=L}$ for terms involving $f_2(\tau)$.

Introducing Green's function solution for these equation will become:

$$T(x,t)=\left[\frac{2}{L}\sum_{m=1}^{\infty}\sin(\beta_m x)*\exp(-\beta_m^2\alpha t)\right]\int_{x'=0}^{L}\sin(\beta_m x')F(x')dx'$$

$$+\frac{\alpha}{k}\left[\frac{2}{L}\sum_{m=1}^{\infty}\sin(\beta_m x)*\exp(-\beta_m^2\alpha t)\right]\int_{\tau=0}^{t}\exp(-\beta_m^2\alpha\tau)d\tau\int_{x'=0}^{L}\sin(\beta_m x')F(x')dx'$$

$$+\alpha\left[\frac{2}{L}\sum_{m=1}^{\infty}\beta_m*\sin(\beta_m x)*\exp(-\beta_m^2\alpha t)\right]\int_{\tau=0}^{t}\exp(-\beta_m^2\alpha\tau)f_1(\tau)d\tau$$

$$-\alpha\left[\frac{2}{L}\sum_{m=1}^{\infty}(-1)^m\beta_m\sin(\beta_m x)*\exp(-\beta_m^2\alpha t)\right]\int_{\tau=0}^{t}\exp(-\beta_m^2\alpha\tau)f_2(\tau)d\tau$$

12. Partial differential equation solution methods: Laplace transform solutions

LAPLACE TRANSFORMS

The Laplace transform and inversion formula of function F(t) is defined by:

$$L[F(t)]=F(s)=\int_{t=0}^{\infty}e^{-st}F(t)dt$$

$F(t)=\frac{1}{2\pi i}\int_{s=\gamma-i\infty}^{\gamma+i\infty}e^{st}F(s)ds$ where s is the Laplace transform variable, $i=\sqrt{-1}$ is the complex number.

Function could be continuos or piecewise continuous in any interval. For example if

F(t)=1 $F(s)=\int_{t=0}^{\infty}1.e^{-st}dt=-\frac{1}{s}e^{-st}\,|_0^{\infty}=-\frac{1}{s}e^{-\infty}+\frac{1}{s}e^{0}=\frac{1}{s}$

F(t)=t $F(s)=\int_{t=0}^{\infty}t.e^{-st}dt=\frac{1}{s^2}$

Laplace transforms have the following relational properties:

$L[c_1F(t)+c_2G(t)]=c_1F(s)+c_2G(s)$

Derivatives:

$$L[F'(t)]=\int_{t=0}^{\infty}F'(t).e^{-st}dt=[F(t)e^{-st}]_0^{\infty}+s\int_0^{\infty}F(t)e^{-st}dt$$

$$L[F'(t)]=sF(s)-F(0)$$

Second deriatives:

$$L[F''(t)]=sL[F'(t)]-F'(0)=s[sF(s)-F(0)]-F'[0]=s^2F(s)-sF(0)-F'(0)$$

In general nth derivative can be written as:

$$L[F^{(n)}(t)]=s^nF(s)-s^{n-1}F(0)-s^{n-2}F^{(1)}(0)-s^{n-3}F^{(2)}(0)..-F^{(n-1)}(0)$$

Laplace transform of the integral:

$$g(t)=\int_0^t F(t)dt$$

$$g(s)=\frac{1}{s}F(s)$$

Change of scale:

$$L[F(at)]=\int_0^\infty F(at)e^{-st}dt=\frac{1}{a}\int_0^\infty F(u)e^{-(s/a)u}du=\frac{1}{a}F\left(\frac{s}{a}\right)$$

Similarly

$$L[F(\frac{t}{a})]=\int_0^\infty F(\frac{t}{a})e^{-st}dt=a\int_0^\infty F(u)e^{-asu}du=aF(as)$$

Shift property:

$$L[e^{\pm at}F(t)]=\int_0^\infty e^{-st}e^{\pm at}F(t)dt=\int_0^\infty e^{-(st\mp a)t}F(t)dt=F(s\pm a)$$

Some of the Laplace Transform functions

No	F(s)	F(t)
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^n}$ (n=1,2,3,…)	$\frac{t^{n-1}}{(n-1)!}$
4	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
5	$\frac{1}{s^{-3/2}}$	$2\sqrt{t/\pi}$
6	$s^{(-n+1/2)}$ (n=1,2,3,….)	$\frac{2^n}{[1.3.5...(2n-1)]\sqrt{\pi}}t^{n-1/2}$
7	$\frac{1}{s^n}$ (n>0 real number)	$\frac{t^{n-1}}{\Gamma(n)}$
8	$\frac{1}{s+a}$	e^{-at}
9	$\frac{1}{(s+a)^n}$ (n=1,2,3,….)	$\frac{t^{n-1}e^{-at}}{(n-1)!}$
10	$\frac{\Gamma(k)}{(s+a)^k}$ (k>0)	$t^{k-1}e^{-at}$
11	$\frac{1}{(s+a)(s+b)}$ $(a\neq b)$	$\frac{e^{-at}-e^{-bt}}{b-a}$
12	$\frac{s}{(s+a)(s+b)}$ $(a\neq b)$	$\frac{ae^{-at}-be^{-bt}}{a-b}$
13	$\frac{1}{s^2+a^2}$	$\frac{\sin(at)}{a}$
14	$\frac{s}{s^2+a^2}$	$\cos(at)$

15	$\frac{1}{s^2-a^2}$	$\frac{\sinh(at)}{a}$
16	$\frac{s}{s^2-a^2}$	$\cosh(at)$
17	$\frac{1}{s\left(s^2+a^2\right)}$	$\frac{1}{a^2}\left(1-\cos(at)\right)$
18	$\frac{s}{s^2\left(s^2+a^2\right)}$	$\frac{1}{a^3}\left(at-\sin(at)\right)$
19	$\frac{1}{\left(s^2+a^2\right)^2}$	$\frac{1}{2a^3}\left(\sin(at)+(at)\cos(at)\right)$
20	$\frac{s}{\left(s^2+a^2\right)^2}$	$\frac{t}{2a}\sin(at)$
21	$\frac{s^2}{\left(s^2+a^2\right)^2}$	$\frac{1}{2a}\left(\sin(at)+(at)\cos(at)\right)$
22	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$	t*cos(at)
23	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}erfc(a\sqrt{t})$
24	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}erfc(a\sqrt{t})$
25	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$
26	$\frac{1}{\sqrt{s}\left(s-a^2\right)}$	$\frac{1}{a}e^{a^2t}erf(a\sqrt{t})$
27	$\frac{1}{\sqrt{s}\left(s+a^2\right)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$
28	$\frac{b^2-a^2}{\left(s-a^2\right)\left(b+\sqrt{s}\right)}$	$e^{a^2t}[b-aerf(a\sqrt{t})]-be^{b^2t}erf(b\sqrt{t})$
29	$\frac{1}{\sqrt{s}\left(s+a\right)}$	$e^{a^2t}erfc(a\sqrt{t})$
30	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}e^{-at}erf(\sqrt{b-a}\sqrt{t})$
31	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$	$ae^{-at}\left[I_1(at)+I_0(at)\right]$
32	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t}I_0(\frac{a-b}{2}t)$

33	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
34	$\frac{\left(\sqrt{s^2+a^2}-s\right)^{\nu}}{\sqrt{s^2+a^2}}$ *where* $(\nu>-1)$	$a^{\nu}J_{\nu}(at)$
35	$\frac{\left(s-\sqrt{s^2-a^2}\right)^{\nu}}{\sqrt{s^2-a^2}}$ *where* $(\nu>-1)$	$a^{\nu}I_{\nu}(at)$
36	$\frac{1}{s}\,e^{-ks}$	$u(t-k)$
37	$\frac{1}{s^2}\,e^{-ks}$	$(t-k)u(t-k)$
38	$\frac{1}{s}\,e^{-k/s}$	$J_0(2\sqrt{kt})$
39	$\frac{1}{s^{\mu}}\,e^{-k/s}$ $\mu>0$	$\left(\frac{t}{k}\right)^{(\mu-1)/2} J_{\mu-1}(2\sqrt{kt})$
40	$\frac{1}{s^{\mu}}\,e^{-k/s}$ $\mu<0$	$\left(\frac{t}{k}\right)^{(\mu-1)/2} I_{\mu-1}(2\sqrt{kt})$
41	$e^{-k\sqrt{s}}$ $k\geq 0$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$
42	$\frac{e^{-k\sqrt{s}}}{s}$ $k\geq 0$	$erfc\left(\frac{k}{2\sqrt{t}}\right)$
43	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}}$ $k\geq 0$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$
44	$\frac{e^{-k\sqrt{s}}}{s^{3/2}}$ $k\geq 0$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k*erfc\left(\frac{k}{2\sqrt{t}}\right)$
45	$\frac{e^{-k\sqrt{s}}}{s^{1+n/2}}$ $k\geq 0$ n=0,1,2,…	$(4t)^{n/2}i^n erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$
46	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}}$ $k\geq 0$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}*e^{a^2t}erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$
47	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}\left(a+\sqrt{s}\right)}$ $k\geq 0$	$e^{ak}*a^{a^2t}erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$
48	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}$ $k\geq 0$	$e^{-at/2}*I_0\left(\frac{1}{2}a\sqrt{t^2-k^2}\right)u(t-k)$
49	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s^2+a^2)}}$ $k\geq 0$	$J_0\left(a\sqrt{t^2-k^2}\right)u(t-k)$
50	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s^2-a^2)}}$ $k\geq 0$	$I_0\left(a\sqrt{t^2-k^2}\right)u(t-k)$

51	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \qquad k \geq 0$	$-e^{ak} * a^{a^2 t} erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+erfc\left(\frac{k}{2\sqrt{t}}\right)$
52	$\frac{e^{-k\sqrt{s}}}{s^2}$	$\left(t+\frac{k^2}{2}\right)erfc\left(\frac{k}{2\sqrt{t}}\right)-k\left(\frac{t}{\pi}\right)\exp\left(-\frac{k^2}{4t}\right)$
53	$\frac{\ln(s)}{s}$	$-\gamma-\ln(t)$ $\gamma = 0.5772156649..$ Euler's constant
54	$\ln\frac{s+a}{s+b}$	$\frac{1}{t}\left(e^{-bt}-e^{-at}\right)$
55	$\ln\frac{s^2+a^2}{s^2}$	$\frac{2}{t}(1-\cos(at))$
56	$\ln\frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1-\cosh(at))$
57	$K_0(ks) \qquad (k>0)$	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$
58	$K_0(k\sqrt{s}) \qquad (k>0)$	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$
59	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s}) \qquad (k>0)$	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$

Note that u(x) is unit step function where

$$u(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$

After these initial definitions, now A heat transfer problem can be investigated

EXAMPLE 1: SEMI-INFINITE WALL PROBLEM LAPLACE TRANSFORM SOLUTION TEMPERATURE PROFILE BOUNDARY

$\frac{\partial^2 T^*(x,t)}{\partial T^{*2}} = \frac{1}{\alpha}\frac{\partial T^*}{\partial t}$
$T^*(x,t) = F^*(x)$ x=0 t > 0
$T^*(x,t) = T_i$ $x \to \infty$ t > 0
$T^*(x,t) = T_i$ at t=0
$\frac{\partial^2 T(x,t)}{\partial T^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
$T(x,t) = F^*(x) - T_i = F^*(x)$ x=0 t > 0
$T(x,t) = 0$ $x \to \infty$ t > 0
$T(x,t) = T_i$ at t=0

$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ with boundary condition:

T(x,t)=f(t) x=0 t > 0
T(x,t)= 0 $X \to \infty$ t > 0
T(x,t)= 0 at t=0

Laplace transform of the differential equation:

$$\frac{\partial^2 T(x,s)}{\partial x^2} - \frac{s}{\alpha} T(x,s) = 0 \quad 0 < x < \infty$$

Boundary conditions:

$T(x,s) = f(s) \quad at\ x = 0$

$T(x,s) = 0 \quad at\ x \to \infty$

Solution of this ordinary differential equation is:

$$T(x,s) = C_1 e^{-x\sqrt{\frac{s}{\alpha}}} + C_2 e^{x\sqrt{\frac{s}{\alpha}}}$$

$$T(x,s) = 0 \quad at\ x \to \infty \ C_2 = 0$$

$$T(x,s) = f(s) \quad at\ x = 0 \ C_1 = f(s)$$

So the solution is:

$$T(x,s) = f(s)g(x,s) = f(s)e^{-x\sqrt{\frac{s}{\alpha}}}$$

Inverse function is:

$$T(x,t) = \int_0^t f(\tau) g(x, t-\tau) d\tau$$

Inverse Laplace transform of $g(x,s) = e^{-x\sqrt{\frac{s}{\alpha}}}$ is $g(x,t) = \frac{x}{2\sqrt{\pi\alpha t^3}} e^{-x^2/4\alpha t}$ so

$$T(x,t) = \frac{x}{\sqrt{4\pi\alpha}} \int_{\tau=0}^{t} \frac{f(\tau)}{(t-\tau)^{3/2}} \exp\left[\frac{-x^2}{4\alpha(t-\tau)}\right] d\tau$$

Problem Java Solution

```
 import java.util.*;
public class HT_infinite_wall1
{
public static double f(double t)
{  double ff=0;
            if(t<100)ff=30+0.5*t;
            else     ff=70.0;
   return ff;
}
//integration function
public static double intfr(double to,double t,double x,double alpha)
{ double T=x/Math.sqrt(4.0*Math.PI*alpha)*f(to)/Math.pow((t-to),1.5)*Math.exp(-x*x/(4.0*alpha*(t-to)));
            return T;
}

//integral
//Gauss Kronrod integration formula
public static double integral(double a,double b,double x,double alpha)
   {
               //eps hata miktari
               int n=50;
               double eps=1e-10;
      int Aused=0;
               boolean sonuc;
      double integral=0;
      double A[][];
      int Aw = 0;
      double toplamhata = 0;
      double[] c;
      double[] xg;
      double[] xk;
      int nn = 0;
      int ng = 0;
      int i = 0;
      int j = 0;
      int h = 0;
      double v = 0;
      double k1 = 0;
```

```
        double k2 = 0;
        double intg = 0;
        double intk = 0;
        double ta = 0;
        double tb = 0;
        Aw = 4;
        A = new double[n-1+1][Aw-1+1];
        nn = 61;
        ng = 15;
        c = new double[nn];
        xk = new double[nn];
        xg = new double[nn];
        //Gauss katsayilari
        xg[0] = 0.00796819249616660561546588347467 4;
        xg[1] = 0.01846646831109095914230213191204 7;
        xg[2] = 0.02878470788332336934971917961129 2;
        xg[3] = 0.03879919256962704959680193644634 8;
        xg[4] = 0.04840267283059405290293814042280 8;
        xg[5] = 0.05749315621761906648172168940205 6;
        xg[6] = 0.06597422988218049512812851511596 2;
        xg[7] = 0.07375597473770520626824385002219 1;
        xg[8] = 0.08075589522942021535469493846053 0;
        xg[9] = 0.08689978720108297980238753071512 6;
        xg[10] = 0.09212252223778612871763270708761 9;
        xg[11] = 0.09636873717464425963946862635181 0;
        xg[12] = 0.09959342058679526706278028210356 9;
        xg[13] = 0.10176238974840550459642895216855 4;
        xg[14] = 0.10285265289355884034128563670541 5;
        c[0] = 0.99948441005049063757132589570581 1;
        c[1] = 0.99689348407464954027163005091869 5;
        c[2] = 0.99163099687040459485862836610948 6;
        c[3] = 0.98366812327974720997003258160566 3;
        c[4] = 0.97311632250112626837469386842370 7;
        c[5] = 0.96002186496830751221687102558179 8;
        c[6] = 0.94437444474855997941583132403743 9;
        c[7] = 0.92620004742927432587932427708047 4;
        c[8] = 0.90557330769990779854652255892595 8;
        c[9] = 0.88256053579205268154311646253022 6;
        c[10] = 0.85720523354606109895865851065894 4;
        c[11] = 0.82956576238276839744289811973250 2;
        c[12] = 0.79972783582183908301366894232268 3;
        c[13] = 0.76777743210482619491797734097450 3;
        c[14] = 0.73379006245322680472617113136952 8;
        c[15] = 0.69785049479331579693229238802664 0;
        c[16] = 0.66006106412662696137005366814927 1;
        c[17] = 0.62052618298924286114047755643118 9;
        c[18] = 0.57934523582636169175602493217254 0;
        c[19] = 0.53662414814201989926416979331107 3;
        c[20] = 0.49248046786177857499369306120770 9;
        c[21] = 0.44703376953808917678060990032285 4;
        c[22] = 0.40040125483039439253547621154266 1;
        c[23] = 0.35270472553087811347103720708937 4;
        c[24] = 0.30407320227362507737267710719925 7;
        c[25] = 0.25463692616788984643980512981780 5;
        c[26] = 0.20452511668230989143895767100202 5;
        c[27] = 0.15386991360858354696379467274325 6;
        c[28] = 0.10280693796673703014709675131800 1;
        c[29] = 0.05147184255531769583302521316672 3;
        c[30] = 0.00000000000000000000000000000000 0;
        //Kronrod katsayilari
        xk[0] = 0.00138901369867700762455159122676 0;
        xk[1] = 0.00389046112709988405126720184451 6;
        xk[2] = 0.00663070391593129217331982636975 0;
        xk[3] = 0.00927327965951776342844114689202 4;
        xk[4] = 0.01182301525349634174223289885325 1;
        xk[5] = 0.01436972950704580481245143244358 0;
        xk[6] = 0.01692088918905327262757228942032 2;
        xk[7] = 0.01941414119394238117340895105012 8;
        xk[8] = 0.02182803582160919229716748573833 9;
        xk[9] = 0.02419116207808060136568637072523 2;
        xk[10] = 0.02650995488233310161060170933507 5;
        xk[11] = 0.02875404876504129284397878535433 4;
        xk[12] = 0.03090725756238776247288425294309 2;
        xk[13] = 0.03298144705748372603181419101685 4;
        xk[14] = 0.03497933802806002413749967073146 8;
        xk[15] = 0.03688236465182122922391106561713 6;
```

```
xk[16] = 0.038678945624727592950348651532281;
xk[17] = 0.040374538951535959111995279752468;
xk[18] = 0.041969810215164246147147541285970;
xk[19] = 0.043452539701356069316831728117073;
xk[20] = 0.044814800133162663192355551616723;
xk[21] = 0.046059238271006988116271735559374;
xk[22] = 0.047185546569299153945261478181099;
xk[23] = 0.048185861757087129140779492298305;
xk[24] = 0.049055434555029778887528165367238;
xk[25] = 0.049795683427074206357811569379942;
xk[26] = 0.050405921402782346840893085653585;
xk[27] = 0.050881795898749606492297473049805;
xk[28] = 0.051221547849258772170656282604944;
xk[29] = 0.051426128537459025933862879215781;
xk[30] = 0.051494729429451567558340433647099;
for(i=nn-1; i>=nn/2; i--)
{c[i] = -c[nn-1-i];}
for(i=nn-1; i>=nn/2; i--)
{ xk[i] = xk[nn-1-i]; }
for(i=ng-1; i>=0; i--)
{ xg[nn-2-2*i] = xg[i];
  xg[1+2*i] = xg[i];
}
for(i=0; i<=nn/2; i++)
{ xg[2*i] = 0; }
k1 = 0.5*(b-a);
k2 = 0.5*(b+a);
intg = 0;
intk = 0;
for(i=0; i<=nn-1; i++)
{ v = intfr((k1*c[i]+k2),b,x,alpha);
  intk = intk+v*xk[i];
  if( i%2==1 )
  {intg = intg+v*xg[i];}
}
intk = intk*(b-a)*0.5;
intg = intg*(b-a)*0.5;
A[0][0] = Math.abs(intg-intk);
A[0][1] = intk;
A[0][2] = a;
A[0][3] = b;
toplamhata = A[0][0];
if( toplamhata<eps )
{
   sonuc = true;
   integral = intk;
   Aused = 1;
   return integral;
}
Aused = 1;
for(h=1; h<=n-1; h++)
{
   Aused = h+1;
   gir(A, h, Aw);
   toplamhata = toplamhata-A[h-1][0];
   ta = A[h-1][2];
   tb = A[h-1][3];
   A[h-1][2] = ta;
   A[h-1][3] = 0.5*(ta+tb);
   A[h][2] = 0.5*(ta+tb);
   A[h][3] = tb;
   for(j=h-1; j<=h; j++)
   {
      k1 = 0.5*(A[j][3]-A[j][2]);
      k2 = 0.5*(A[j][3]+A[j][2]);
      intg = 0;
      intk = 0;
      for(i=0; i<=nn-1; i++)
      {   v = intfr((k1*c[i]+k2),b,x,alpha);
         intk = intk+v*xk[i];
         if( i%2==1 )
         { intg = intg+v*xg[i];}
      }
      intk = intk*(A[j][3]-A[j][2])*0.5;
      intg = intg*(A[j][3]-A[j][2])*0.5;
      A[j][0] = Math.abs(intg-intk);
```

```
            A[j][1] = intk;
            toplamhata = toplamhata+A[j][0];
        }
        cik(A, h-1, Aw);
        cik(A, h, Aw);
        if( toplamhata<eps )
        {break;}
    }
    sonuc = toplamhata<eps;
    integral = 0;
    for(j=0; j<=Aused-1; j++)
    { integral = integral+A[j][1];}
    return integral;
}
public static void gir(double A[][],int n,int Awidth)
{
    int i = 0;
    int p = 0;
    double t = 0;
    int maxcp = 0;
    if( n==1 )
    { return; }
    for(i=0; i<=Awidth-1; i++)
    {   t = A[n-1][i];
        A[n-1][i] = A[0][i];
        A[0][i] = t;
    }//end of for
    p = 0;
    while( 2*p+1<n-1 )
    {   maxcp = 2*p+1;
        if( 2*p+2<n-1 )
        {   if( A[2*p+2][0]>A[2*p+1][0] )
            { maxcp = 2*p+2; }
        }
        if( A[p][0]<A[maxcp][0] )
        {   for(i=0; i<=Awidth-1; i++)
            {   t = A[p][i];
                A[p][i] = A[maxcp][i];
                A[maxcp][i] = t;
            }
            p = maxcp;
        }
        else
        { break; }
    } //end of while
}

public static void cik(double A[][],int n,int Awidth)
{
    int i = 0;
    int p = 0;
    double t = 0;
    int kk = 0;
    if( n==0 )
    {return;}
    p = n;
    while( p!=0 )
    {   kk = (p-1)/2;
        if( A[p][0]>A[kk][0] )
        { for(i=0; i<=Awidth-1; i++)
            { t = A[p][i];
              A[p][i] = A[kk][i];
              A[kk][i] = t;
            }
            p = kk;
        }
        else
        {break;}
    }
}


 public static void main (String args[])
 {
   double alpha=4.0e-5;
   double x=0.1; //m
```

```
      double t=300.0;//s
      double T1=20.0;// degree C;
      double T=integral(0.0,t,x,alpha)+T1;
      System.out.println("T="+T);
    }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\java.exe" HT_infinite_wall1
T=55.14859399275895

> Terminated with exit code 0.
```

Special case of the previous problem: when $T(x,t)=f(x)=T_0$ at x=0, solution converges:

$T(x,s)=\frac{T_0}{s}e^{-x\sqrt{\frac{s}{\alpha}}}$ so solution can be directly given as:

$$T(x,t)=T_0 erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Numerical example: A water pipe will be burried underground in a cold climate. When meterological data is examined it is detected that after a long stable period of 20 ℃ temperatures, which it can be assumed that upper layers of the earth will be reached to this temperatures, air cools according to the table given below:

day	temperature	day	temperature	day	temperature	day	temperature
1	20	21	-12	41	-12	61	-8
2	16	22	-15	42	-12	62	-4
3	12	23	-17	43	-11	63	0
4	10	24	-14	44	-11	64	4
5	7	25	-13	45	-11	65	6
6	3	26	-15	46	-12	66	8
7	2	27	-14	47	-11	67	10
8	0	28	-13	48	-14	68	12
9	-2	29	-12	49	-11	69	12
10	-6	30	-11	50	-15	70	16
11	-10	31	-14	51	-12	71	16
12	-11	32	-15	52	-11	72	17
13	-13	33	-14	53	-14	73	18
14	-14	34	-13	54	-15	74	19
15	-15	35	-14	55	-15	75	20
16	-16	36	-14	56	-14	76	20
17	-15	37	-12	57	-15	77	20
18	-17	38	-12	58	-15	78	20
19	-18	39	-15	59	-14	79	20
20	-14	40	-12	60	-15	80	20

When observed the temperature profile, it is seen that temperature starts to warm up again after 60 days. Determine the required depth for the pipe to avoid freezing. Take thermal diffusivity coefficient $\alpha = 0.138 \times 10^{-6}$ m^2/s

```
 import java.util.*;


public class HT_infinite_walla
{
double a[][;
public HT_infinite_walla(String s)
{double d[][]=Text.readDoubleT(s);
 a=cubic_spline(d[0],d[1],0.0,0.0);
}
public  double f(double t,double Ti)
{   double t1=t/86400.0;
            double ff=funcSpline(a,t1)-Ti;
   return ff;
}

//integration function
public double  intfr(double to,double t,double x,double alpha,double Ti)
{ double T=x/Math.sqrt(4.0*Math.PI*alpha)*f(to,Ti)/Math.pow((t-to),1.5)*Math.exp(-x*x/(4.0*alpha*(t-to)));
  //System.out.println("TT="+T);
            return T;
```

```
}

//integral
//Gauss Kronrod integration formula
public  double integral(double a,double b,double x,double alpha,double Ti)
   {
                //eps hata miktari
                int n=50;
                double eps=1e-10;
      int Aused=0;
                boolean sonuc;
      double integral=0;
      double A[][];
      int Aw = 0;
      double toplamhata = 0;
      double[] c;
      double[] xg;
      double[] xk;
      int nn = 0;
      int ng = 0;
      int i = 0;
      int j = 0;
      int h = 0;
      double v = 0;
      double k1 = 0;
      double k2 = 0;
      double intg = 0;
      double intk = 0;
      double ta = 0;
      double tb = 0;
      Aw = 4;
      A = new double[n-1+1][Aw-1+1];
      nn = 61;
      ng = 15;
      c = new double[nn];
      xk = new double[nn];
      xg = new double[nn];
      //Gauss katsayilari
      xg[0] = 0.00796819249616660561546588347467 4;
      xg[1] = 0.01846646831109095914230213191204 7;
      xg[2] = 0.02878470788332336934971917961129 2;
      xg[3] = 0.03879919256962704959680193644634 8;
      xg[4] = 0.04840267283059405290293814042280 8;
      xg[5] = 0.05749315621761906648172168940205 6;
      xg[6] = 0.06597422988218049512812851511596 2;
      xg[7] = 0.07375597473770520626824385002219 1;
      xg[8] = 0.08075589522942021535469493846053 0;
      xg[9] = 0.08689978720108297980238753071512 6;
      xg[10] = 0.09212252223778612871763270708761 9;
      xg[11] = 0.09636873717464425963946862635181 0;
      xg[12] = 0.09959342058679526706278028210356 9;
      xg[13] = 0.10176238974840550459642895216855 4;
      xg[14] = 0.10285265289355884034128563670541 5;
      c[0] = 0.99948441005049063757132589570581 1;
      c[1] = 0.99689348407464954027163005091869 5;
      c[2] = 0.99163099687040459485862836610948 6;
      c[3] = 0.98366812327974720997003258160566 3;
      c[4] = 0.97311632250112626837469386842370 7;
      c[5] = 0.96002186496830751221687102558179 8;
      c[6] = 0.94437444474855997941583132403743 9;
      c[7] = 0.92620004742927432587932427708047 4;
      c[8] = 0.90557330769990779854652255892595 8;
      c[9] = 0.88256053579205268154311646253022 6;
      c[10] = 0.85720523354606109895865851065894 4;
      c[11] = 0.82956576238276839744289811973250 2;
      c[12] = 0.79972783582183908301366894232268 3;
      c[13] = 0.76777743210482619491797734097450 3;
      c[14] = 0.73379006245322680472617113136952 8;
      c[15] = 0.69785049479331579693229238802664 0;
      c[16] = 0.66006106412662696137005366814927 1;
      c[17] = 0.62052618298924286114047755643118 9;
      c[18] = 0.57934523582636169175602493217254 0;
      c[19] = 0.53662414814201989926416979331107 3;
      c[20] = 0.49248046786177857499369306120770 9;
      c[21] = 0.44703376953808917678060990032285 4;
      c[22] = 0.40040125483039439253547621154266 1;
```

```
c[23] = 0.35270472553087811347103720708937 4;
c[24] = 0.30407320227362507737267710719925 7;
c[25] = 0.25463692616788984643980512981780 5;
c[26] = 0.20452511668230989143895767100202 5;
c[27] = 0.15386991360858354696379467274325 6;
c[28] = 0.10280693796673703014709675131800 1;
c[29] = 0.05147184255531769583302521316672 3;
c[30] = 0.00000000000000000000000000000000 0;
//Kronrod katsayilari
xk[0] = 0.00138901369867700762455159122676 0;
xk[1] = 0.00389046112709988405126720184451 6;
xk[2] = 0.00663070391593129217331982636975 0;
xk[3] = 0.00927327965951776342844114689202 4;
xk[4] = 0.01182301525349634174223289885325 1;
xk[5] = 0.01436972950704580481245143244358 0;
xk[6] = 0.01692088918905327262757228942032 2;
xk[7] = 0.01941414119394238117340895105012 8;
xk[8] = 0.02182803582160919229716748573833 9;
xk[9] = 0.02419116207808060136568637072523 2;
xk[10] = 0.02650995488233310161060170933507 5;
xk[11] = 0.02875404876504129284397878535433 4;
xk[12] = 0.03090725756238776247288425294309 2;
xk[13] = 0.03298144705748372603181419101685 4;
xk[14] = 0.03497933802806002413749967073146 8;
xk[15] = 0.03688236465182122922391106561713 6;
xk[16] = 0.03867894562472759295034865153228 1;
xk[17] = 0.04037453895153595911199527975246 8;
xk[18] = 0.04196981021516424614714754128597 0;
xk[19] = 0.04345253970135606931683172811707 3;
xk[20] = 0.04481480013316266319235555161672 3;
xk[21] = 0.04605923827100698811627173555937 4;
xk[22] = 0.04718554656929915394526147818109 9;
xk[23] = 0.04818586175708712914077949229830 5;
xk[24] = 0.04905543455502977888752816536723 8;
xk[25] = 0.04979568342707420635781156937994 2;
xk[26] = 0.05040592140278234684089308565358 5;
xk[27] = 0.05088179589874960649229747304980 5;
xk[28] = 0.05122154784925877217065628260494 4;
xk[29] = 0.05142612853745902593386287921578 1;
xk[30] = 0.05149472942945156755834043364709 9;
for(i=nn-1; i>=nn/2; i--)
{c[i] = -c[nn-1-i];}
for(i=nn-1; i>=nn/2; i--)
{ xk[i] = xk[nn-1-i]; }
for(i=ng-1; i>=0; i--)
{ xg[nn-2-2*i] = xg[i];
  xg[1+2*i] = xg[i];
}
for(i=0; i<=nn/2; i++)
{ xg[2*i] = 0; }
k1 = 0.5*(b-a);
k2 = 0.5*(b+a);
intg = 0;
intk = 0;
for(i=0; i<=nn-1; i++)
{ v = intfr((k1*c[i]+k2),b,x,alpha,Ti);
  intk = intk+v*xk[i];
  if( i%2==1 )
  {intg = intg+v*xg[i];}
}
intk = intk*(b-a)*0.5;
intg = intg*(b-a)*0.5;
A[0][0] = Math.abs(intg-intk);
A[0][1] = intk;
A[0][2] = a;
A[0][3] = b;
toplamhata = A[0][0];
if( toplamhata<eps )
{
   sonuc = true;
   integral = intk;
   Aused = 1;
   return integral;
}
Aused = 1;
for(h=1; h<=n-1; h++)
```

```
      {
         Aused = h+1;
         gir(A, h, Aw);
         toplamhata = toplamhata-A[h-1][0];
         ta = A[h-1][2];
         tb = A[h-1][3];
         A[h-1][2] = ta;
         A[h-1][3] = 0.5*(ta+tb);
         A[h][2] = 0.5*(ta+tb);
         A[h][3] = tb;
         for(j=h-1; j<=h; j++)
         {
            k1 = 0.5*(A[j][3]-A[j][2]);
            k2 = 0.5*(A[j][3]+A[j][2]);
            intg = 0;
            intk = 0;
            for(i=0; i<=nn-1; i++)
            {   v = intfr((k1*c[i]+k2),b,x,alpha,Ti);
               intk = intk+v*xk[i];
               if( i%2==1 )
               { intg = intg+v*xg[i];}
            }
            intk = intk*(A[j][3]-A[j][2])*0.5;
            intg = intg*(A[j][3]-A[j][2])*0.5;
            A[j][0] = Math.abs(intg-intk);
            A[j][1] = intk;
            toplamhata = toplamhata+A[j][0];
         }
         cik(A, h-1, Aw);
         cik(A, h, Aw);
         if( toplamhata<eps )
         {break;}
      }
      sonuc = toplamhata<eps;
      integral = 0;
      for(j=0; j<=Aused-1; j++)
      { integral = integral+A[j][1];}
      return integral;
   }
   public  void gir(double A[][],int n,int Awidth)
   {
      int i = 0;
      int p = 0;
      double t = 0;
      int maxcp = 0;
      if( n==1 )
      { return; }
      for(i=0; i<=Awidth-1; i++)
      {   t = A[n-1][i];
         A[n-1][i] = A[0][i];
         A[0][i] = t;
      }//end of for
      p = 0;
      while( 2*p+1<n-1 )
      {   maxcp = 2*p+1;
         if( 2*p+2<n-1 )
         {   if( A[2*p+2][0]>A[2*p+1][0] )
            { maxcp = 2*p+2; }
         }
         if( A[p][0]<A[maxcp][0] )
         {   for(i=0; i<=Awidth-1; i++)
            {   t = A[p][i];
               A[p][i] = A[maxcp][i];
               A[maxcp][i] = t;
            }
            p = maxcp;
         }
         else
         { break; }
      } //end of while
   }

   public  void cik(double A[][],int n,int Awidth)
   {
      int i = 0;
      int p = 0;
```

```
        double t = 0;
        int kk = 0;
        if( n==0 )
        {return;}
        p = n;
        while( p!=0 )
        {   kk = (p-1)/2;
            if( A[p][0]>A[kk][0] )
            { for(i=0; i<=Awidth-1; i++)
                { t = A[p][i];
                  A[p][i] = A[kk][i];
                  A[kk][i] = t;
                }
                p = kk;
            }
            else
            {break;}
        }
    }
//Cubic spline interpolation
  public static double [] thomas(double f[],double e[],double g[],double r[])
  {
  // 3 band matrix system of equation solving algorithm
  int n=f.length;
  double x[]=new double[n];
  for(int k=1;k<n;k++)
    {e[k]=e[k]/f[k-1];
     f[k]=f[k]-e[k]*g[k-1];
    }
  for(int k=1;k<n;k++)
    {r[k]=r[k]-e[k]*r[k-1];
    }
  x[n-1]=r[n-1]/f[n-1];
  for(int k=(n-2);k>=0;k--)
    {x[k]=(r[k]-g[k]*x[k+1])/f[k];}
  return x;
  }


  public static double [][] cubic_spline(double xi[],double yi[],double c0,double cn)
  {
  int n=xi.length;
  double h[]=new double[n];
  double w[]=new double[n];
  double f[]=new double[n];
  double e[]=new double[n];
  double g[]=new double[n];
  double d[]=new double[n];
  double x[]=new double[n];
  double S[][]=new double[4][n];
  int k;
  for(k=0;k<(n-1);k++)
    {h[k]=xi[k+1]-xi[k];
     w[k]=(yi[k+1]-yi[k])/h[k];
                }
  d[0]=c0;
  d[n-1]=cn;
  for(k=1;k<(n-1);k++)
    {d[k]=6.0*(w[k]-w[k-1]);}
  f[0]=1.0;
  f[n-1]=1.0;
  g[0]=0.0;
  g[n-1]=0.0;
  e[0]=0.0;
  e[n-1]=0.0;
  for(k=1;k<(n-1);k++)
  {f[k]=2.0*(h[k]+h[k-1]);e[k]=h[k-1];g[k]=h[k];}
  S[2]=thomas(f,e,g,d);
  S[3]=xi;
  for(k=0;k<(n-1);k++)
    {S[0][k]=(6.*yi[k+1]-h[k]*h[k]*S[2][k+1])/(6.0*h[k]);
     S[1][k]=(6.*yi[k]-h[k]*h[k]*S[2][k])/(6.0*h[k]);
    }
  return S;
  }
```

```
 public static double funcSpline(double S[][],double x)
 {
 int n=S[0].length;
 double xx1=0;
 double xx2=0;
 double y=0;
 double hk=0;
 for(int k=0;k<(n-1);k++)
   {if(S[3][k]<=x && x<=S[3][k+1])
    {hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     break;
     }
   }
     if(y==0 && S[3][n-2]<=x )
     {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
     xx1=(x-S[3][k]);
     xx2=(S[3][k+1]-x);
     y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
     }
 return y;
 }
public static double[][]  funcSpline(double xi[],double yi[],int numberofmidpoints,double x0,double xn)
 {
              //numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
              // numberofmidpoints is 2
              int n=xi.length;
              int nn=(n-1)*(numberofmidpoints+1)+1;
              double z[][]=new double[2][nn];
  double S[][]=cubic_spline(xi,yi,x0,xn);
  double dx=0;
  int k=0;
  int i;
  for(i=0;i<(n-1);i++)
     {  z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k++;
       for(int j=0;j<numberofmidpoints;j++)
          {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k++;}
     }
     z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);
  return z;
}
public static double[][]  funcSpline(double xi[],double yi[],int numberofmidpoints)
{
              //numberofmidpoints :  in x--o--o--x--o--o--x chain if x's are esxperimental points
              // numberofmidpoints is 2
              int n=xi.length;
              int nn=(n-1)*(numberofmidpoints+1)+1;
              double z[][]=new double[2][nn];
  double S[][]=cubic_spline(xi,yi,0,0);
  double dx=0;
  int k=0;
  int i;
  for(i=0;i<(n-1);i++)
     {  z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);k++;
       for(int j=0;j<numberofmidpoints;j++)
          {dx=(xi[i+1]-xi[i])/((double)numberofmidpoints+1.0);
       z[0][k]=z[0][k-1]+dx;z[1][k]=funcSpline(S,z[0][k]);k++;}
     }
     z[0][k]=xi[i];z[1][k]=funcSpline(S,z[0][k]);
  return z;
}
 // Türev formülleri
// =================
 public static double dfSpline(double xi[],double yi[],double c0,double cn,double x)
 { //kübik şerit türev formülü
              double S[][]=cubic_spline(xi,yi,c0,cn);
  return dfSpline(S,x);
 }

 public static double dfSpline(double S[][],double x)
 {
```

```
   //kübik şerit türev formülü
 int n=S[0].length;
 double xx1=0;
 double xx2=0;
 double y=0;
 double hk=0;
 for(int k=0;k<(n-1);k++)
    {if(S[3][k]<=x && x<=S[3][k+1])
     {hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]-S[1][k]+(xx1*xx1*S[2][k+1]-xx2*xx2*S[2][k])/(2.0*hk);
      break;
      }
    }
      if(y==0 && S[3][n-2]<=x )
      {
                int k=n-2;
                hk=(S[3][k+1]-S[3][k]);
      xx1=(x-S[3][k]);
      xx2=(S[3][k+1]-x);
      y=S[0][k]*xx1+S[1][k]*xx2+(xx1*xx1*xx1*S[2][k+1]+xx2*xx2*xx2*S[2][k])/(6.0*hk);
      }
 return y;
 }

public static double  dfSpline(double xi[],double yi[],double x)
{
   //doğal kübik şerit türev formülü
              double S[][]=cubic_spline(xi,yi,0,0);
   return dfSpline(S,x);
}

 public static double intSpline(double S[][],double a,double b)
 {
   //kübik şerit integral[S(x)]dx  formülü
 int n=S[0].length;
 double xx1=0;
 double xx2=0;
 double y1,y2;
 double hk=0;
 double toplam=0;
 y1=0;y2=0;
 for(int k=0;k<(n-1);k++)
    { hk=(S[3][k+1]-S[3][k]);
     if(a>S[3][k+1])
     {toplam=0;
     }
               else if(S[3][k]<=a && a<=S[3][k+1] && b>S[3][k+1])//şart 2
      {
      xx1=(a-S[3][k]);
      xx2=(S[3][k+1]-a);
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=hk;
      xx2=0;
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      }
      else if(S[3][k]<=a && a<=S[3][k+1] && S[3][k]<=b && b<=S[3][k+1])//şart 3
      {
      xx1=(a-S[3][k]);
      xx2=(S[3][k+1]-a);
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=(b-S[3][k]);
      xx2=(S[3][k+1]-b);
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      toplam+=(y2-y1);
      }
      else if(a<S[3][k] && b>=S[3][k+1]) //şart 4
      {
      xx1=0;
      xx2=hk;
      y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
      xx1=hk;
      xx2=0;
      y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
```

```
       toplam+=(y2-y1);
       }
       else if(S[3][k]<=b && b<=S[3][k+1])//şart 5
       {xx1=0;
        xx2=hk;
        y1=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
        xx1=(b-S[3][k]);
        xx2=(S[3][k+1]-b);
        y2=S[0][k]*xx1*xx1/2.0-S[1][k]*xx2*xx2/2.0+(xx1*xx1*xx1*xx1*S[2][k+1]-xx2*xx2*xx2*xx2*S[2][k])/(24.0*hk);
        toplam+=(y2-y1);
        }
        else break;
     }
  return toplam;
  }


public static double golden(double S[][],double a,double b)
{
//  find the minimum of the function
//  note  maximum f(x) = minimum (-f(x))
    double epsilon=1.0e-10;
    double delta=1.0e-8;
    int print=0;
    double r1 = (Math.sqrt(5.0)-1.0)/2.0; // golden ratio
    double r2 = r1*r1;
    double h = b - a;
    double ya = -funcSpline(S,a);
    double yb = -funcSpline(S,b);
    double c = a + r2*h;
    double d = a + r1*h;
    double yc = -funcSpline(S,c);
    double yd = -funcSpline(S,d);
    int k = 1;
    double dp,dy,p,yp;
    while ((Math.abs(yb-ya)>epsilon) || (h>delta))
    {
      k++;
      if (yc<yd)
        {
        b = d;
        yb = yd;
        d = c;
        yd = yc;
        h = b - a;
        c = a + r2 * h;
        yc = -funcSpline(S,c);
        }
      else
        {
         a = c;
         ya = yc;
         c = d;
         yc = yd;
         h = b - a;
         d = a + r1 * h;
         yd = -funcSpline(S,d);
        }//end of if
    }//end of while
    dp = Math.abs(b-a);
    dy = Math.abs(yb-ya);
    p = a;
    yp = ya;
    if (yb<ya)
    {
      p = b;
      yp = yb;
    }
    if(print==1)
    {System.out.println("x min = "+p+"ymin = "+yp+"error of x ="+dp+"error of y"+dy); }
    return p;
}
public static void  printData(double c[][])
  {       String s3[]={"x[i]","y[i]"};
       Text.print(Text.T(c),s3,"Inpput data");
  }
```

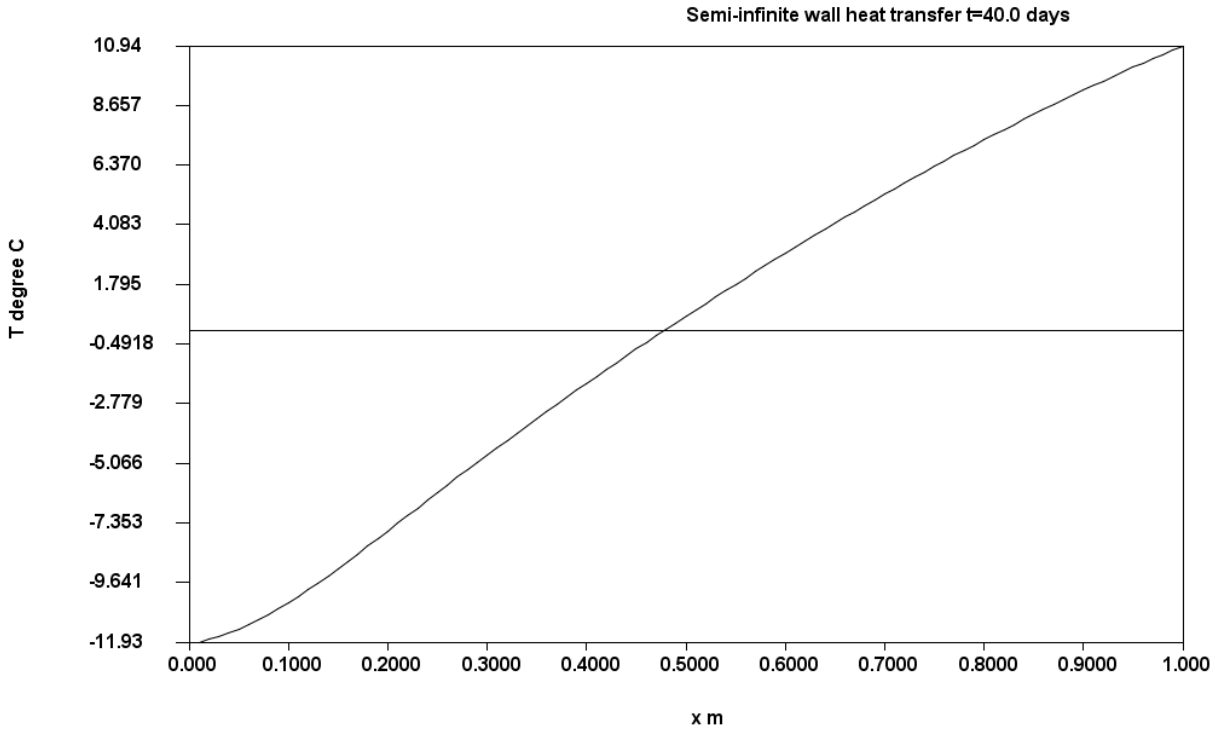
```
public static void  printOutputData(double c[][],int xarray[],int yarray[],int ngroup)
{
            int n=c[0].length;
            int m=xarray.length;
            double z[][];
 double E[][];
 double xi[]=new double[n];
 double yi[]=new double[n];
 double dx=0;
 int k=0;
 int i;
 String s="";
 String s1[]={"x","y"};
 for(i=0;i<m;i++)
 {
            s="data set no : "+i+" x ref : "+xarray[i]+" y ref : "+yarray[i];
            xi=c[xarray[i]];
            yi=c[yarray[i]];
 z=funcSpline(xi,yi,ngroup);
 Text.printT(z,s1,s);
 }

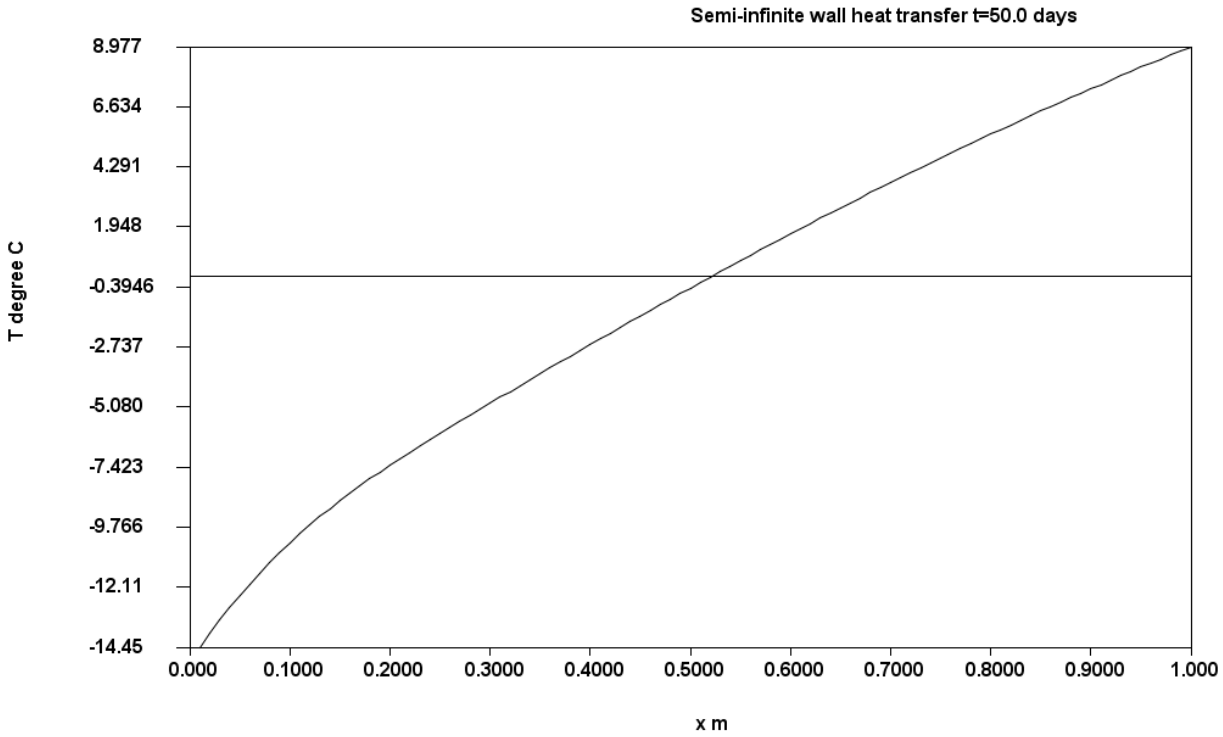
}

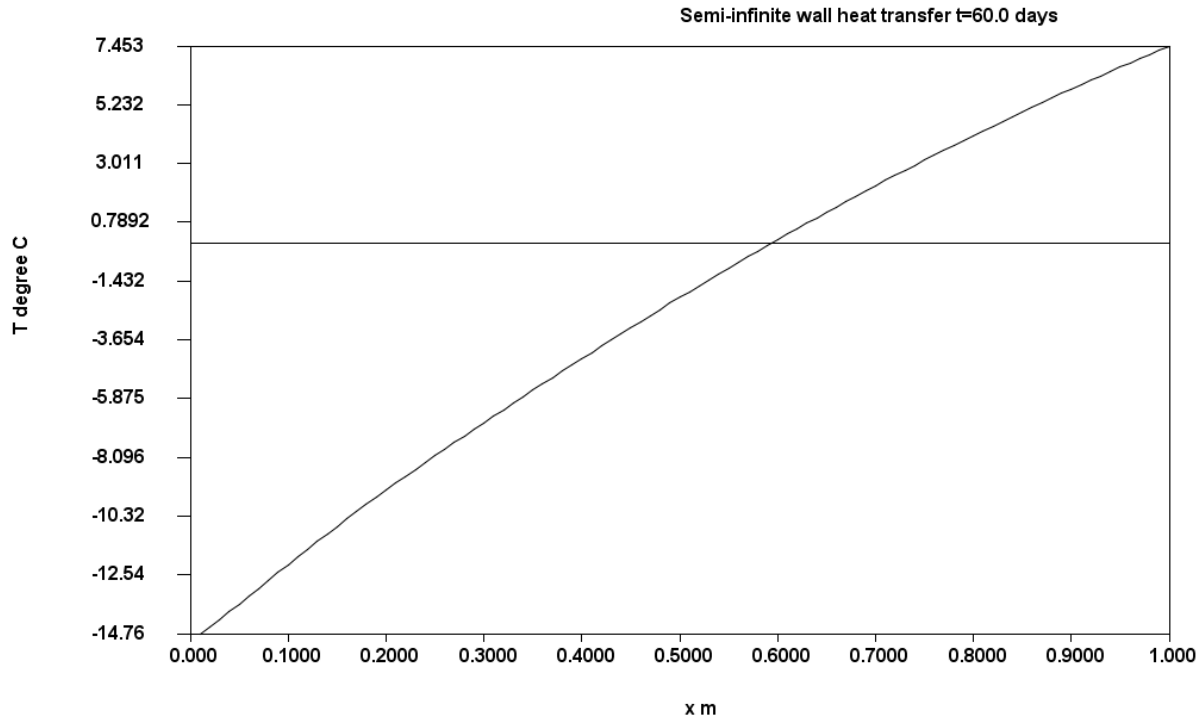
public static void printCoefficients(int datano,double c[][],int x1array[],int y1array[],String s)
{    int n=c[0].length;
    String s2[]={"dataset number : k","a[k]","b[k]","c[k]","x[k]"};
    double d[][]=Text.T(cubic_spline(c[x1array[datano]],c[y1array[datano]],0,0));
    double e[][]=new double[d.length][d[0].length+1];
    for(int i=0;i<e.length;i++)
    {for(int j=0;j<e[0].length;j++)
     {if(j==0) e[i][j]=i;
      else e[i][j]=d[i][j-1];
     }
    }
    int i=1;
    Text.print(e,s2,s);
}
public static void  funcEKKPlot(double c[][],int xarray[],int yarray[],int ngroup)
{ int sp[]={20,22,24,25,26,27,32,33,34,35,21,23};
 int sp1[]={0,1,2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9};
            int n=c[0].length;
            int m=xarray.length;
            double z[][];
 double E[][];
 double xi[]=new double[n];
 double yi[]=new double[n];
 double dx=0;
 int k=0;
 int i;
 String s="";
 String s1[]={"x","y"};
 xi=c[xarray[0]];
            yi=c[yarray[0]];
 Plot pp=new Plot(xi,yi);
 pp.setPlotType(0,sp[0]);
 for(i=0;i<m;i++)
 {
            s="data set no : "+i+" x ref : "+xarray[i]+" y ref : "+yarray[i];
            xi=c[xarray[i]];
            yi=c[yarray[i]];
 z=funcSpline(xi,yi,ngroup);
 if(i!=0) {pp.addData(xi,yi,sp[i]);}
 pp.addData(z[0],z[1],sp1[i]);
 }
 pp.setPlabel("y=f(x)");
 pp.setXlabel("x");
 pp.setYlabel("y");
 pp.plot();
}
public static void  errorEKKPlot(double c[][],int xarray[],int yarray[],int ngroup)
{ int sp[]={1,2,3,4,5,6,7,8,9,1,2,3,4,5,6,7,8,9};
            int n=c[0].length;
            int m=xarray.length;
            double z[][];
 double E[][];
```

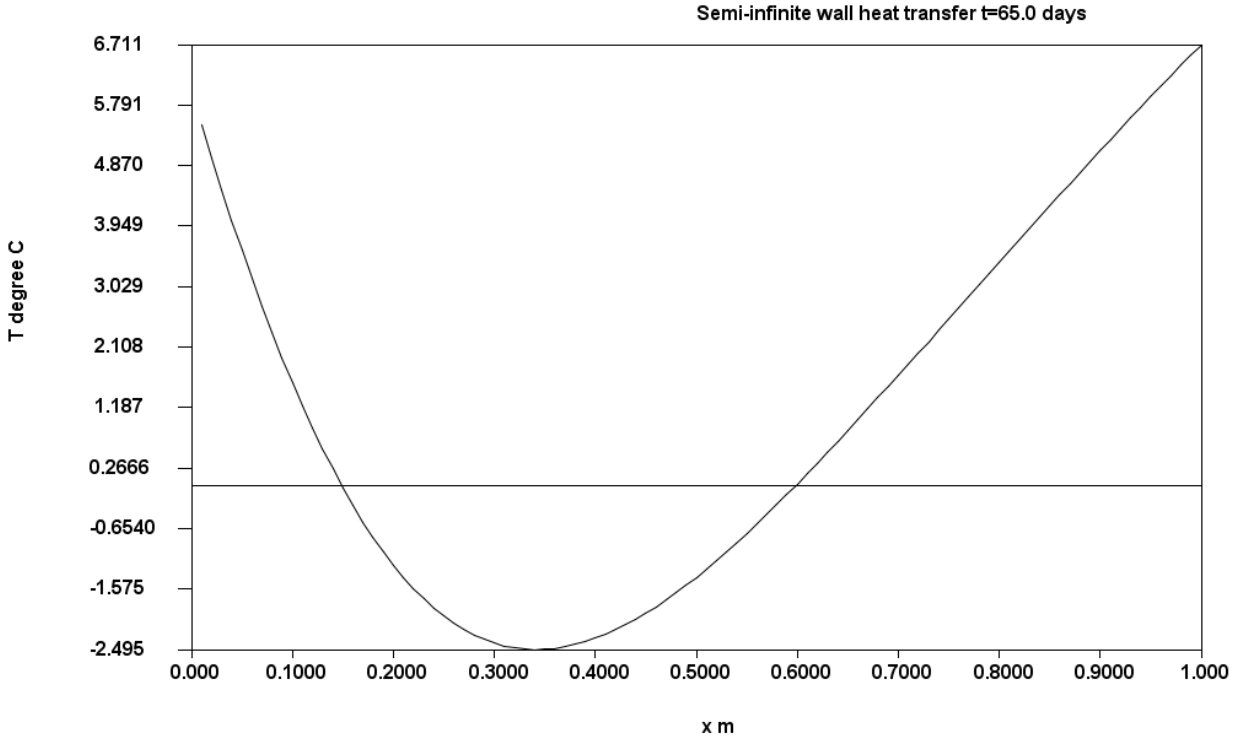
```
    double xi[]=new double[n];
    double yi[]=new double[n];
    double dx=0;
    int k=0;
    int i;
    String s="";
    String s1[]={"x","y"};
    xi=c[xarray[0]];
              yi=c[yarray[0]];
    Plot pp=new Plot(xi,yi);
    i=0;
    do
    {
              s="data set no : "+i+" x ref : "+xarray[i]+" y ref : "+yarray[i];
              xi=c[xarray[i]];
              yi=c[yarray[i]];
    z=funcSpline(xi,yi,ngroup);
    if(i==0) {pp=new Plot(z[0],z[1]);pp.setPlotType(i,sp[i]);}
    else     {pp.addData(z[0],z[1],sp[i]);}
    i++;
    } while(i<m);
    pp.setPlabel("Error=y-yi");
    pp.setXlabel("x");
    pp.setYlabel("dy");
    pp.plot();
  }
 // end of cubbi spline interpolation programs

     public static void main (String args[])
     { HT_infinite_walla iw=new HT_infinite_walla("a.txt");
       double alpha=0.138e-6;
       double x0=1.0; //m
       double x[]=new double[100];
       double T[]=new double[100];
       double t=60*86400;//s
       double Ti=20.0;// degree C;
       for(int i=0;i<100;i++)
       {x[i]=x0/100*(i+1);
        T[i]=iw.integral(0.0,t,x[i],alpha,Ti)+Ti;
        System.out.println(i+" "+x[i]+" "+T[i]);
       }
       Plot pp=new Plot(x,T);
       double xx[]={0,x0};
       double yy[]={0,0};
       pp.setPlabel("Semi-infinite wall heat transfer t="+t/86400+" days");
       pp.setXlabel(" x m");
       pp.setYlabel(" T degree C");
       pp.addData(xx,yy);
       pp.plot();
     }
}
```


Semi-infinite wall heat transfer t=40.0 days
T degree C
10.94
8.657
6.370
4.083
1.795
-0.4918
-2.779
-5.066
-7.353
-9.641
-11.93
0.000
0.1000
0.2000
0.3000
0.4000
0.5000
0.6000
0.7000
0.8000
0.9000
1.000
x m


Semi-infinite wall heat transfer t=50.0 days
T degree C
8.977
6.634
4.291
1.948
-0.3946
-2.737
-5.080
-7.423
-9.766
-12.11
-14.45
0.000
0.1000
0.2000
0.3000
0.4000
0.5000
0.6000
0.7000
0.8000
0.9000
1.000
x m





65 day plot clearly shows reheating effect of the surface. So the distance should be determined from 60 day case as 0.6 m.

EXAMPLE 2: SEMI-INFINITE WALL PROBLEM LAPLACE TRANSFORM SOLUTION, CONVECTIVE BOUNDARY

A second example: Semi-infinite medium at initially zero temperature. At t>0 boundary surface at x=0 is subjected to **convective heat transfer**

$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ with boundary condition:

$-k\frac{\partial T(x,t)}{\partial x} + h(T - T_\infty) = 0$ x=0 t > 0

T(x,t)= 0 $X \to \infty$ t > 0

T(x,t)= 0 at t=0

Laplace transform of the differential equation:

$\frac{\partial^2 T(x,s)}{\partial x^2} - \frac{s}{\alpha}T(x,s) = 0 \quad 0 < x < \infty$

Boundary conditions:

$-k\frac{\partial T(x,s)}{\partial x} + hT(x,s) = \frac{hT_\infty}{s}$ at x=0 (define H=h/k)

$T(x,s) = 0 \quad as\, x \to \infty$

Solution of this ordinary differential equation is:

$T(x,s) = C_1 e^{-x\sqrt{\frac{s}{\alpha}}} + C_2 e^{x\sqrt{\frac{s}{\alpha}}}$

$T(x,s) = 0 \quad at\, x \to \infty \;\; C_2 = 0$

$T(x,s) = C_1 e^{-x\sqrt{\frac{s}{\alpha}}}$

$\frac{dT(x,s)}{dx} = -C_1\sqrt{\frac{s}{\alpha}}e^{-x\sqrt{\frac{s}{\alpha}}}$

$-k\frac{\partial T(x,s)}{\partial x} + hT(x,s) = \frac{hT_\infty}{s}$ at x=0 (define H=h/k)

$-C_1\sqrt{\frac{s}{\alpha}}e^{-x\sqrt{\frac{s}{\alpha}}} + HC_1 e^{-x\sqrt{\frac{s}{\alpha}}} = H\frac{T\infty}{s}$ for x=0

$C_1\left(H - \sqrt{\frac{s}{\alpha}}\right) = H\frac{T\infty}{s}$

$C_1 = \frac{HT_\infty\sqrt{\alpha}}{s\left(H\sqrt{\alpha} + \sqrt{s}\right)}$

So the solution is:

$T(x,s) = T_\infty H\sqrt{\alpha}\,\frac{e^{-x\sqrt{\frac{s}{\alpha}}}}{s(H\sqrt{\alpha} + \sqrt{s})}$

The inversion of Laplace transform results:

$$T(x,t) = T_\infty\left[erfc\left(\frac{x}{\sqrt{4\alpha t}}\right)\right] - e^{Hx + H^2\alpha t}erfc\left(H\sqrt{\alpha t} + \frac{x}{\sqrt{4\alpha t}}\right)$$

Problem Java solution:

```
 import java.util.*;
public class HT_infinite_wall2
{
        //incomplete gamma function P(a,x)
public static double gammp(double a,double x)
{
        double gamser,gammcf;
        if (x < 0.0 || a <= 0.0) System.out.println("Invalid arguments in routine GAMMP");
```

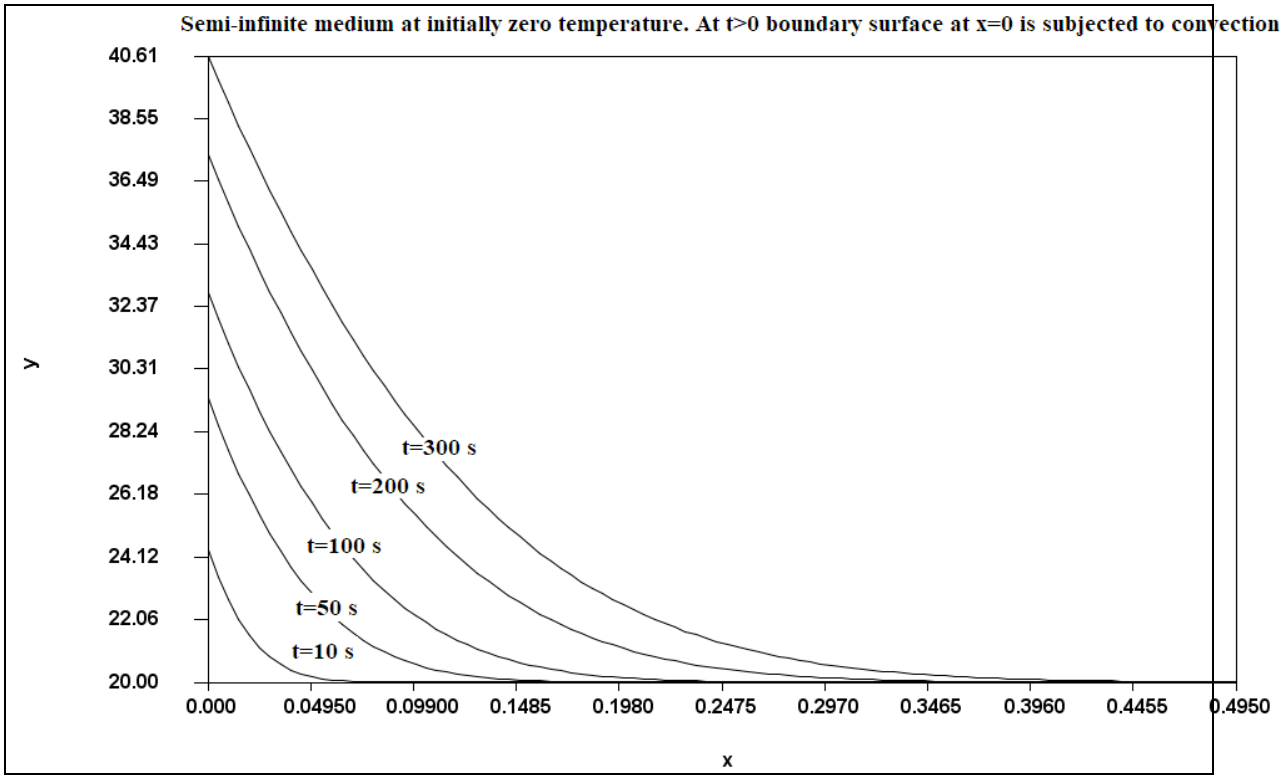
```
		if (x < (a+1.0)) {
			gamser=gser(a,x);
			return gamser;
		} else {
			gammcf=gcf(a,x);
			return 1.0-gammcf;
		}
}

//incomplete gamma function Q(a,x)=1-P(a,x);
public static double gammq(double a,double x)
{
		double gamser,gammcf;
		if (x < 0.0 || a <= 0.0) System.out.println("Invalid arguments in routine GAMMQ");
		if (x < (a+1.0)) {
			gamser=gser(a,x);
			return 1.0-gamser;
		} else {
			gammcf=gcf(a,x);
			return gammcf;
		}
}
public static double gser(double a,double x)
{		int ITMAX=200;
		double EPS=3.0e-7;
		int n;
		double sum,del,ap;
		double gln=lngamma(a)[0];
		double gamser=0;
		if (x <= 0.0) {
			if (x < 0.0) System.out.println("x less than 0 in routine GSER");
			gamser=0.0;
			return gamser;
		} else {
			ap=a;
			del=sum=1.0/a;
			for (n=1;n<=ITMAX;n++) {
				ap += 1.0;
				del *= x/ap;
				sum += del;
				if (Math.abs(del) < Math.abs(sum)*EPS) {
					gamser=sum*Math.exp(-x+a*Math.log(x)-(gln));
					return gamser;
				}
			}
			//System.out.println("a too large, ITMAX too small in routine GSER");
			return gamser;
		}
}
public static double gcf(double a,double x)
{
		int ITMAX=200;
		double EPS=3.0e-7;
		int n;
		double gln;
		double gammcf=0;
		double gold=0.0,g,fac=1.0,b1=1.0;
		double b0=0.0,anf,ana,an,a1,a0=1.0;
		gln=lngamma(a)[0];
		a1=x;
		for (n=1;n<=ITMAX;n++) {
			an=(double) n;
			ana=an-a;
			a0=(a1+a0*ana)*fac;
			b0=(b1+b0*ana)*fac;
			anf=an*fac;
			a1=x*a0+anf*a1;
			b1=x*b0+anf*b1;
			if (a1!=0) {
				fac=1.0/a1;
				g=b1*fac;
				if (Math.abs((g-gold)/g) < EPS) {
					gammcf=Math.exp(-x+a*Math.log(x)-gln)*g;
					return gammcf;
				}
				gold=g;
```

```
                        }
                }
                //System.out.println("a too large, ITMAX too small in routine GCF");
    return gammcf;
}
//error function
public static double  erf(double x)
{
   //error function
   return x < 0.0 ? -gammp(0.5,x*x) : gammp(0.5,x*x);
}
//natural logarithm of gamma function
                public static double [] lngamma(double x)
    {//logaithm of gamma function
    //xx[0] value of logarithm of gamma function
    //xx[1] sign of gamma functon
                double xx[]=new double[2];
    double result;
    double a;
    double b;
    double c;
    double p;
    double q;
    double u;
    double w;
    double z;
    int i;
    double logpi;
    double ls2pi;
    double tmp;
    double sgngam = 1;
    logpi = 1.14472988584940017414;
    ls2pi = 0.91893853320467274178;
    if( x<-34.0 )
    {
       q = -x;
       double yy[]=lngamma(q);
       w = yy[0];
       tmp=yy[1];
       p = Math.floor(q);
       i = (int)Math.round(p);
       if( i%2==0 )
       {
          sgngam = -1;
       }
       else
       {
          sgngam = 1;
       }
       z = q-p;
       if( z>0.5 )
       {
          p = p+1;
          z = p-q;
       }
       z = q*Math.sin(Math.PI*z);
       result = logpi-Math.log(z)-w;
       xx[0]=result;
       xx[1]=sgngam;
       return xx;
    }
    if( x<13 )
    {
       z = 1;
       p = 0;
       u = x;
       while(u>=3)
       {
          p = p-1;
          u = x+p;
          z = z*u;
       }
       while(u<2)
       {
          z = z/u;
          p = p+1;
```

```
            u = x+p;
        }
        if( z<0 )
        {
            sgngam = -1;
            z = -z;
        }
        else
        {
            sgngam = 1;
        }
        if( u==2 )
        {
            result = Math.log(z);
            //return result;
            xx[0]=result;
            xx[1]=sgngam;
            return xx;
        }
        p = p-2;
        x = x+p;
        b = -1378.25152569120859100;
        b = -38801.6315134637840924+x*b;
        b = -331612.992738871184744+x*b;
        b = -1162370.97492762307383+x*b;
        b = -1721737.00820839662146+x*b;
        b = -853555.664245765465627+x*b;
        c = 1;
        c = -351.81570143652347O549+x*c;
        c = -17064.2106651881159223+x*c;
        c = -220528.590553854454839+x*c;
        c = -1139334.44367982507207+x*c;
        c = -2532523.07177582951285+x*c;
        c = -2018891.41433532773231+x*c;
        p = x*b/c;
        result = Math.log(z)+p;
        //return result;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
    }
    q = (x-0.5)*Math.log(x)-x+ls2pi;
    if( x>100000000 )
    {
        result = q;
        //return result;
        xx[0]=result;
        xx[1]=sgngam;
        return xx;
    }
    p = 1/(x*x);
    if( x>=1000.0 )
    {
        q = q+((7.936507936507936507
```

```
double x1=x/Math.sqrt(4.0*alpha*t);
double TT=Tinf*(erfc(x1)-Math.exp(H*x+H*H*alpha*t)*erfc(H*Math.sqrt(alpha*t)+x1));
return TT;
}
    public static void main (String args[])
    {
     double alpha=4.0e-5;
     double H=2.0;
     double x[]=new double[100]; //m
     double t[]={10.0,50.0,100.0,200.0,300.0};//s
     double T1=20.0;// degree C;
     double Tinf=100.0;
     double T[][]=new double[t.length][100];
     for(int i=0;i<t.length;i++)
     {for(int j=0;j<x.length;j++)
      {
                    x[j]=0.5*j/100.0;
                    T[i][j]=T(x[j],t[i],Tinf,alpha,H)+T1;}
     }
     Plot pp=new Plot(x,T[0]);
     for(int i=1;i<t.length;i++)
     {pp.addData(x,T[i]); }
      pp.plot();
    }
}
```



EXAMPLE 3: SEMI-INFINITE WALL PROBLEM LAPLACE TRANSFORM SOLUTION, CONSTANT SURFACE HEAT FLUX BOUNDARY

Semi-infinite medium: **Constant surface heat flux**

$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ with boundary condition:

$-k\frac{\partial T(x,t)}{\partial x} = q_0$ x=0 t > 0

T(x,t)= 0 $x \to \infty$ t > 0

T(x,t)= 0 at t=0

Laplace transform of the differential equation:

$$\frac{\partial^2 T(x,s)}{\partial x^2} - \frac{s}{\alpha} T(x,s) = 0 \quad 0 < x < \infty$$

Boundary conditions:

$$-k\frac{\partial T(x,s)}{\partial x} = \frac{q_0}{s} \quad \text{at x=0 (define H=h/k)}$$

$$T(x,s) = 0 \quad as\, x \to \infty$$

Solution of this ordinary differential equation is:

$$T(x,s) = C_1 e^{-x\sqrt{\frac{s}{\alpha}}} + C_2 e^{x\sqrt{\frac{s}{\alpha}}}$$

$$T(x,s) = 0 \quad at\, x \to \infty \;\; C_2 = 0$$

$$T(x,s) = C_1 e^{-x\sqrt{\frac{s}{\alpha}}}$$

$$\frac{dT(x,s)}{dx} = -C_1\sqrt{\frac{s}{\alpha}} e^{-x\sqrt{\frac{s}{\alpha}}}$$

$$-k\frac{\partial T(x,s)}{\partial x} = \frac{q_0}{s} \quad \text{at x=0}$$

$$kC_1\sqrt{\frac{s}{\alpha}} = \frac{q_0}{s} \quad \text{for x=0}$$

$$C_1 = \frac{q_0}{ks\sqrt{s}} = \frac{q_0}{ks^{3/2}}$$

So the solution is:

$$T(x,s) = \frac{q_0}{k}\frac{1}{s^{3/2}} e^{-\frac{x}{\sqrt{\alpha}}\sqrt{s}}$$

Case 44 in the table:

$$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right) - k * erfc\left(\frac{k}{2\sqrt{t}}\right) \quad \text{so}$$

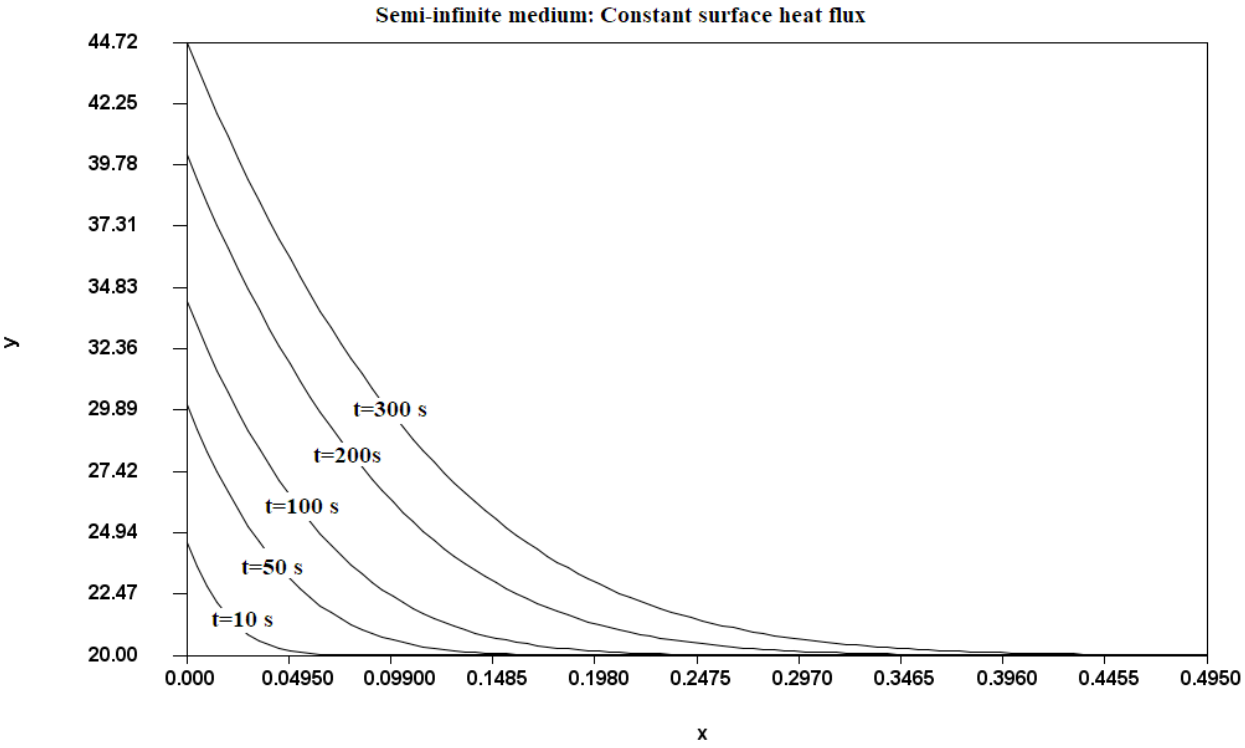
The inversion of Laplace transform results:

$$T(x,t) = \frac{q_0\sqrt{\alpha t/\pi}}{k}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_0 x}{k} erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$T(x,t) = \frac{q_0\sqrt{\alpha t/\pi}}{k}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_0 x}{k} erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

```
class HT_infinite_wall3
{   //Function  is written by using lambda variables
    public static void main (String args[])
    {
     double alpha=4.0e-5;
     double q0=10000.0;  //W/m^2
     double k=50.0; //W/mK;
     double t0=20;
     if_xj f1=(double x[])->2.0*q0*Math.sqrt(alpha*x[0]/Math.PI)/k*Math.exp(-x[1]*x[1]/(4.0*alpha*x[0]))-
q0*x[1]/k*Mathd.erfc(x[1]/(2.0*Math.sqrt(alpha*x[0])))+t0;
     double xx[]=new double[100]; //m
     double t[]={10.0,50.0,100.0,200.0,300.0};//s
     double T1=20.0;// degree C;
     double T[][]=new double[t.length][100];
     for(int i=0;i<t.length;i++)
     {for(int j=0;j<xx.length;j++)
      {   xx[j]=0.5*j/100.0;
```

```
                double y[]={t[i],xx[j]};
                T[i][j]=f1.func(y);}
        }
        Plot pp=new Plot(xx,T[0]);
        for(int i=1;i<t.length;i++)
        {pp.addData(xx,T[i]); }
         pp.plot();
      }
}
```



13. Partial differential equation solution methods: Finite difference solutions

Introduction to finite difference method

Finite difference concept based on the difference concept introduced in the derivative section. We actually used finite differences to approximate to derivative of the function. In order to approximate derivation difference equations can be used. Diference equations can basically be derived from Taylor formulations

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f"(x_i)}{2}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \frac{f^{(4)}(x_i)}{4!}h^4 + \cdots..$$

If the first derivative is solved from this equation it is found as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \left[\frac{f"(x_i)}{2}h + \frac{f^{(3)}(x_i)}{3!}h^2 + \frac{f^{(4)}(x_i)}{4!}h^3 + O(h^4)..\right]$$

Or

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

Term O(h) is the order of error. Error is proportional to the difference parameter h. When h is getting smaller, error will also shrinked. The second derivative can be written the same way:

$$f"(x_i) = \frac{f(x_{i+2}) - 2(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

In order to decrease the error, if the second derivative difference equation is substituted into the first derivative Taylor equation a first derivative equation with a second order error can be obtained.

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \left[\frac{f(x_{i+2}) - 2(x_{i+1}) + f(x_i)}{2h^2}h + O(h^2)\right]$$

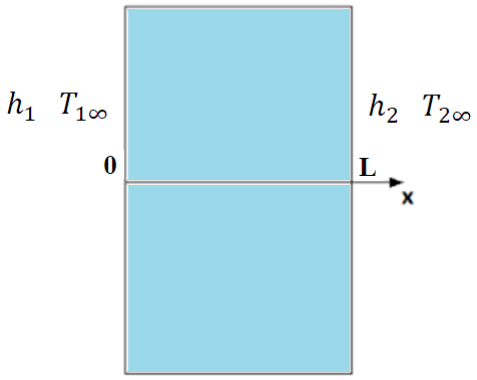
$$= \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

Difference equation can be written as forward or bakward way. The backward difference equation: $f'(x_i) = \frac{f(x_i)-f(x_{i-1})}{h}$. By taken avarage of forward and backward formulas can create a central formula

$$f'(x_i) = \frac{f'(x_i)_{forward} + f'(x_i)_{backward}}{2} = \frac{\frac{f(x_{i+1})-f(x_i)}{h} - \frac{f(x_i)-f(x_{i-1})}{h}}{2}$$
$$= \frac{f(x_{i+1})-f(x_{i-1})}{2h} + O(h^2)$$

Now let us consider a differential equation such as heat conduction equation in cartesian coordinates. It should be note that when a differential equation is given, in order to solve the differential equations boundary conditions should also be specified.

EXAMPLE 1: FINITE DIFFERENCE SOLUTION, NON-HOMOGENEOUS, TIME DEPENDENT , CARTESIAN COORDINATES



$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Boundary conditions

$$-\frac{\partial T(x,t)}{\partial x} + H_1(T_{1\infty} - T(x,t)) = 0 \quad \text{x=0} \quad \text{t >0} \quad H_1 = \frac{h_1}{k}$$

$$\frac{\partial T^*(x,t)}{\partial x} + H_2(T(x,t) - T_{2\infty}) = 0 \quad \text{x=L} \quad \text{t>0} \quad H_2 = \frac{h_2}{k}$$

$$T^*(x,t) = F^*(x) \quad \text{t=0} \quad 0 \le x \le L$$

$$\phi\frac{T^p_{m+1}-2T^p_m+T^p_{m-1}}{\Delta x^2} + (1-\phi)\frac{T^{p+1}_{m+1}-2T^{p+1}_m+T^{p+1}_{m-1}}{\Delta x^2} = \frac{1}{\alpha}\frac{T^{p+1}_m - T^p_m}{\Delta t}$$

$Fo = \frac{\alpha\Delta t}{\Delta x^2}$ Fourier number $Bi = \frac{h\Delta x}{k} = H\Delta x$ Biot number $\alpha = \frac{k}{\rho C_p}$ thermal diffusivity

$t = p\Delta t \quad x = m\Delta x$

By rearranging the equation

$$\phi\text{Fo}T^p_{m+1} + (1-2\phi\text{Fo})T^p_m + \phi\text{Fo}T^p_{m-1} = -(1-\phi)\text{Fo}T^{p+1}_{m+1} + (1-2(1-\phi)\text{Fo})T^{p+1}_m - (1-\phi)T^{p+1}_{m-1}$$

=1 explicit solution: error level: $O[\Delta t, (\Delta x)^2]$

=0 implicit solution: error level: $O[\Delta t, (\Delta x)^2]$

=0.5 Crank-Nicolson solution: error level: $O[\Delta t, (\Delta x)^2]$

ϕ =0.5-$\frac{(\Delta x)^2}{12\alpha\Delta t}$: error level: $O[(\Delta t)^2, (\Delta x)^4]$

Any other ϕmixed solution

The stability criterion depends on the value of the ϕ value

$0 \le \phi \le 0.5$ unconditionally stable

$0.5 \le \phi \le 1$ stable only if $0 \le Fo \le \frac{1}{4\phi\text{-}2}$

Boundary conditions:

$$\phi h_1 A\left(T_{1\infty}-T_0^p\right)+\phi\frac{kA}{\Delta x}\left(T_1^p-T_0^p\right)+(1-\phi)h_1A\left(T_{1\infty}-T_0^{p+1}\right)+(1-\phi)\frac{kA}{\Delta x}\left(T_1^{p+1}-T_0^{p+1}\right)$$
$$=\rho C_p\left(A\frac{\Delta x}{2}\right)\frac{T_0^{p+1}-T_0^p}{\Delta t}$$
$$\phi h_2 A\left(T_{2\infty}-T_{k-1}^p\right)+\phi\frac{kA}{\Delta x}\left(T_{k-2}^p-T_{k-1}^p\right)+(1-\phi)h_2A\left(T_{2\infty}-T_{k-1}^{p+1}\right)+(1-\phi)\frac{kA}{\Delta x}\left(T_{k-2}^{p+1}-T_{k-1}^{p+1}\right)$$
$$=\rho C_p\left(A\frac{\Delta x}{2}\right)\frac{T_{k-1}^{p+1}-T_{k-1}^p}{\Delta t}$$

By rearranging the equations

$$2\phi Bi_1Fo\left(T_{1\infty}-T_0^p\right)+2\phi\left(T_1^p-T_0^p\right)+2(1-\phi)Bi_1Fo\left(T_{1\infty}-T_0^{p+1}\right)+2(1-\phi)\left(T_1^{p+1}-T_0^{p+1}\right)=T_0^{p+1}-T_0^p$$
$$2\phi Bi_2Fo\left(T_{2\infty}-T_{k-1}^p\right)+2\phi\left(T_{k-2}^p-T_{k-1}^p\right)+2(1-\phi)Bi_2Fo\left(T_{2\infty}-T_{k-1}^{p+1}\right)+2(1-\phi)\left(T_{k-2}^{p+1}-T_{k-1}^{p+1}\right)$$
$$=T_{k-1}^{p+1}-T_{k-1}^p$$

Problem Java solution:

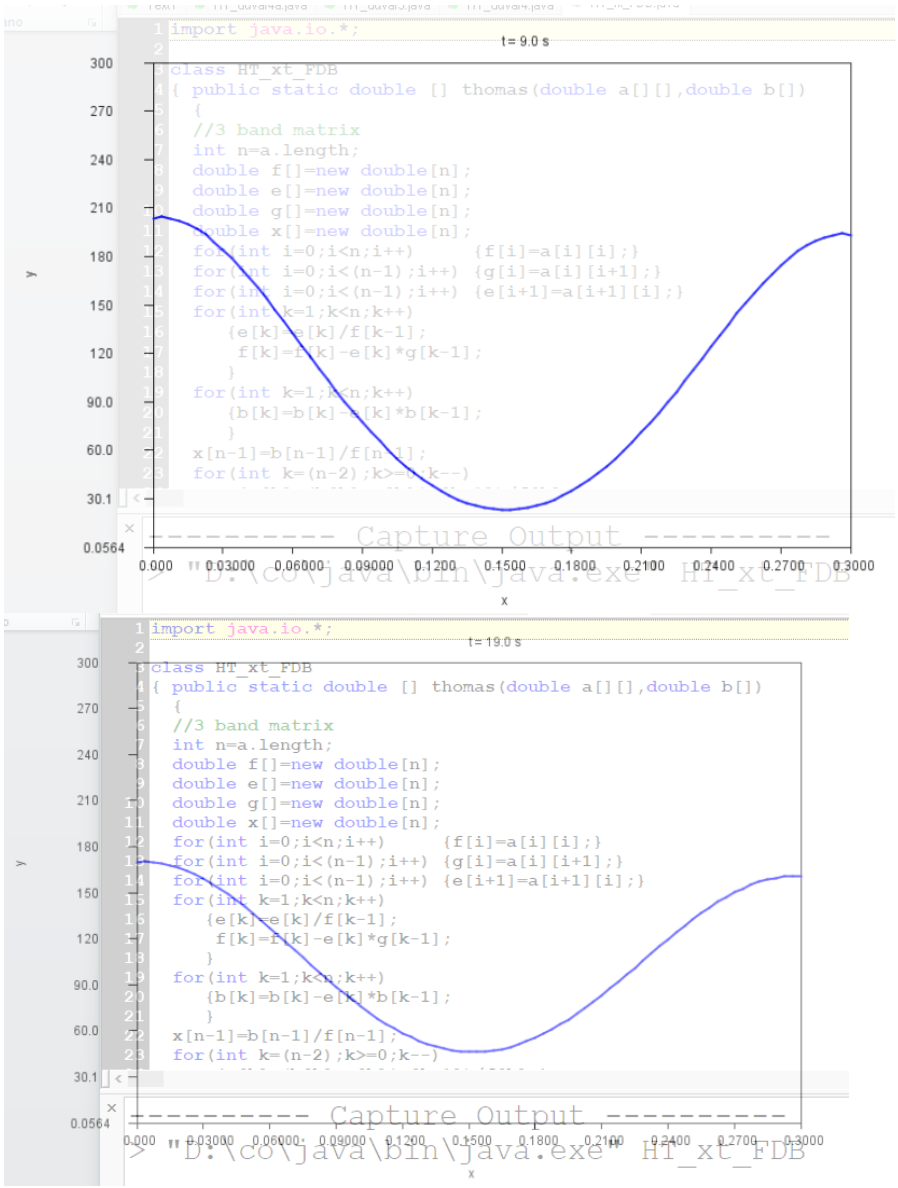
```
import java.io.*;

class HT_xt_FDB
{ public static double [] thomas(double a[][],double b[])
{
//3 band matrix
int n=a.length;
double f[]=new double[n];
double e[]=new double[n];
double g[]=new double[n];
double x[]=new double[n];
for(int i=0;i<n;i++)    {f[i]=a[i][i];}
for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
for(int k=1;k<n;k++)
{e[k]=e[k]/f[k-1];
f[k]=f[k]-e[k]*g[k-1];
}
for(int k=1;k<n;k++)
{b[k]=b[k]-e[k]*b[k-1];
}
x[n-1]=b[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
{x[k]=(b[k]-g[k]*x[k+1])/f[k];}
return x;
}
public static double [][] q(double k, double dx,double T[][])
{  //heat transfer
int nx=T[0].length;
int nt=T.length;
double qi[][]=new double[nt][nx-1];
for(int p=0;p<nt;p++)
{  for(int m=1;m<nx;m++)
{ qi[p][m-1]=-k*(T[p][m]-T[p][m-1])/dx;}
}
return qi;
}

public static double [][] HT_xtcoz(double fi, double L,double t, double dt,double T0[],double alpha,double Bi1,double Bi2,double
T1inf,double T2inf)
{
//  L duvar kalınlığı
//  t maksimum zaman
//  dt zaman stebi
//  T1 x=0 t>=0 boundary temperature
//  T2 x=L t>=0 boundary temperature
int nx=T0.length; //x step sayısı
//System.out.println(nx="+nx)
double A[][]=new double[nx][nx];
double B[]=new double[nx];
double dx=L/(nx-1); //x steb uzunluğu
int nt=(int)(t/dt+0.001);
double Fo=alpha*dt/(dx*dx);
double T[][]=new double[nt+1][nx];
for(int m=0;m<nx;m++) {T[0][m]=T0[m];}
for(int p=0;p<nt;p++)
```

```
{ //t>0 wall temperatures
for(int m=0;m<nx;m++)
{
if(m==0) B[m]=(1.0-2.0*Fo*fi-2*Fo*Bi1*fi)*T[p][m]+2.0*fi*Fo*T[p][m+1]+2.0*Bi1*Fo*T1inf;
else if(m==(nx-1)) B[m]=(1.0-2.0*Fo*fi-2*Fo*Bi2*fi)*T[p][m]+2.0*fi*Fo*T[p][m-1]+2.0*Bi2*Fo*T2inf;
else  B[m]=fi*Fo*T[p][m-1]+(1.0-2.0*Fo*fi)*T[p][m]+fi*Fo*T[p][m+1];
if(m==0)
{A[m][m]=(1+2.0*Fo*Bi1*(1.0-fi)+2.0*Fo*(1.0-fi));
A[m][m+1]=-2.0*(1.0-fi)*Fo;
}
else if(m==(nx-1))
{A[m][m]=(1+2.0*Fo*Bi2*(1.0-fi)+2.0*Fo*(1.0-fi));
A[m][m-1]=-2.0*(1.0-fi)*Fo;
}
else
{A[m][m]=(2.0*Fo*(1.0-fi)+1);
A[m][m-1]=-(1.0-fi)*Fo;
A[m][m+1]=-(1.0-fi)*Fo;
}
}
T[p+1]=thomas(A,B);
}
return T;
}

public static void main(String args[])
{
double T0[]=new double[81];
double L=0.8;
// t=0 daki T0(x) sıcaklık profili
int N=T0.length;
for(int m=0;m<N;m++) {T0[m]=300-300.0*Math.sin(m*Math.PI/N);}
double T1inf=100.0;  //x=0 da t>0 zamanı için duvar sıcaklığı sınır şartı
double T2inf=25.0;  //x=L de t>0 zamanı için duvar sıcaklığı sınır şartı
double h1=50.0;  //x=0 da t>0 zamanı için duvar sıcaklığı sınır şartı
double h2=50.0;  //x=L de t>0 zamanı için duvar sıcaklığı sınır şartı
double t=600.0; //sn toplam zaman
double dt=1.0; //sn gerçek zaman adımı
double dtplot=0.3; //sn her bir plotta bekleme zamanı
double alpha=8.4e-5;
double x0=0.0;
double xn=L;
double fi=0.5;
double k=200.0;
double dx=L/T0.length;
double h=250.0;
double Bi1=h*dx/k;
double Bi2=h*dx/k;
double T[][]=HT_xtcoz(fi,L,t,dt,T0,alpha,Bi1,Bi2,T1inf,T2inf);
double b[][]=Matrix.T(T);
double q[][]=q(k,dx,T);
timeplot p=new timeplot(T,x0,xn,dtplot,dt);
//timeplot p=new timeplot(q,x0,xn,dtplot,dt);
p.start();
}
}
```



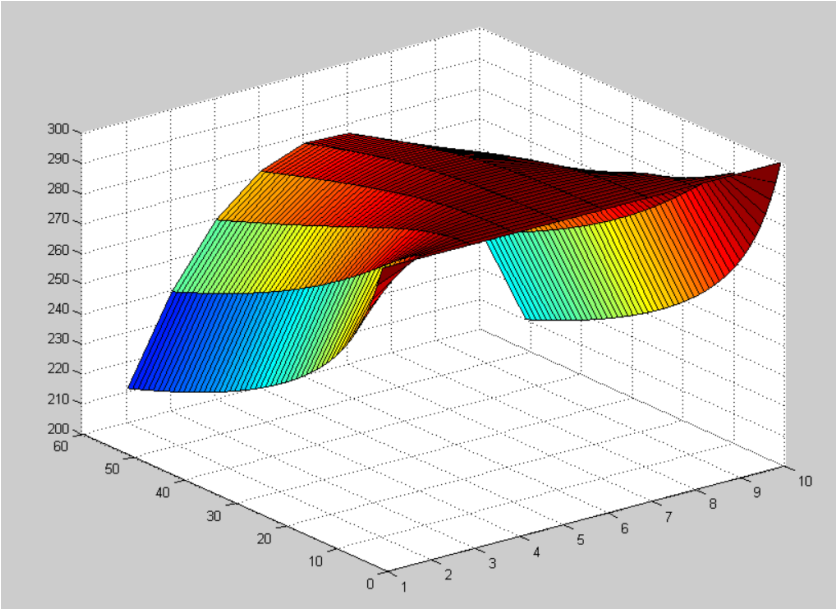
Note: time plot is time dependent real time plot program, and scene above captured from the real time plot .

Problem Matlab solution

```
function x=gauss1(a,b)
% Gauss elimination
n=length(b);
carpan=0;
toplam=0;
for k=1:n
%pivoting
p=k;
buyuk=abs(a(k,k));
for ii=k+1:n
dummy=abs(a(ii,k));
if dummy > buyuk buyuk=dummy;p=ii;end
end
if p~=k
for jj=k:n
dummy=a(p,jj);
a(p,jj)=a(k,jj);
a(k,jj)=dummy;
end
dummy=b(p);
b(p)=b(k);
b(k)=dummy;
end
%gauss eleme
```

```
for i=k+1:n
carpan=a(i,k)/a(k,k);
a(i,k)=0;
for j=k+1:n
a(i,j)=a(i,j)-carpan*a(k,j);
end
b(i)   =b(i)   -carpan*b(k);
end
end
%back substituting
x(n)=b(n)/a(n,n);
for i=n-1:-1:1
toplam=0;
for j=i+1:n
toplam=toplam+a(i,j)*x(j);end
x(i)=(b(i)-toplam)/a(i,i);
end
end
```

```
function [T] = HT_xtcoz(fi,L,t,dt,TT,alpha,Bi1,Bi2,T1inf,T2inf)
%  L wall thickness
%  t maksimum time
%  dt time step
%  T1 x=0 t>=0 boundary temperature
%  T2 x=L t>=0 boundary temperature
nx=length(TT); %x number of steps
dx=L/(nx-1);
nt=t/dt;
Fo=alpha*dt/(dx*dx);
for p=1:nt
for m=1:nx
if p==1 T(p,m)=TT(m);
else T(p,m)=0.0;
end
end
end
A=zeros(nx,nx);
B=zeros(nx);
for p=1:nt
for m=1:nx
if(m==1)
A(m,m)=(1+2.0*Fo*Bi1*(1.0-fi)+2.0*Fo*(1.0-fi));
A(m,m+1)=-2.0*(1.0-fi)*Fo;
elseif(m==nx)
A(m,m)=(1+2.0*Fo*Bi2*(1.0-fi)+2.0*Fo*(1.0-fi));
A(m,m-1)=-2.0*(1.0-fi)*Fo;
else
A(m,m)=(2.0*Fo*(1.0-fi)+1);
A(m,m-1)=-(1.0-fi)*Fo;
A(m,m+1)=-(1.0-fi)*Fo;
end
if m==1
B(m)=(1.0-2.0*Fo*fi)*T(p,m)+fi*Fo*T(p,m+1)+Fo*T1inf;
elseif m==nx
B(m)=(1.0-2.0*Fo*fi-2*Fo*Bi2*fi)*T(p,m)+2.0*fi*Fo*T(p,m-1)+2.0*Bi2*Fo*T2inf;
else
B(m)=fi*Fo*T(p,m-1)+(1.0-2.0*Fo*fi)*T(p,m)+fi*Fo*T(p,m+1);
end
end
W=gauss1(A,B);
for m=1:nx
T(p+1,m)=W(m);
end
end
end
```



EXAMPLE 2: FINITE DIFFERENCE SOLUTION, NON-HOMOGENEOUS, TIME INDEPENDENT SOLUTION; CARTESIAN COORDINATES, TWO DIMENSION



$$\frac{\partial^2 T(x,y)}{\partial x^2}+\frac{T(x,y)}{\partial y^2}=0$$

$$-k\frac{\partial T(x,y)}{\partial x}=h_1\left(T_{1\infty}-T(x,y)\right) \qquad x=0, \qquad 0\le y\le W$$

$$-k\frac{\partial T(x,y)}{\partial x}=h_2\left(T_{2\infty}-T(x,y)\right) \qquad x=L, \qquad 0\le y\le W$$

$$-k\frac{\partial T(x,y)}{\partial y}=h_3\left(T_{3\infty}-T(x,y)\right) \qquad y=0, \qquad 0\le x\le L$$

$$-k\frac{\partial T(x,y)}{\partial y}=h_4\left(T_{4\infty}-T(x,y)\right) \qquad y=W, \qquad 0\le x\le L$$

$$\frac{T_{m+1,n}-2T_{m,n}+T_{m-1,n}}{\Delta x^2}+\frac{T_{m,n+1}-2T_{m,n}+T_{m,n-1}}{\Delta y^2}=0$$

If $\Delta x=\Delta y$

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

Boundary conditions:

m =0 $n\neq 0$

$$-k(\Delta y.1)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}-k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}-k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}=h_1(\Delta y.1)\left(T_{m,n}-T_{1\infty}\right)$$

m=0 n=0

$$-k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m+1,n}-T_{m+1,n}}{\Delta x}-k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}=h_1\left(\frac{\Delta y}{2}.1\right)\left(T_{m,n}-T_{1\infty}\right)+h_3\left(\frac{\Delta x}{2}.1\right)\left(T_{m,n}-T_{3\infty}\right)$$

m=M (x=L) $n\neq 0$

$$-k(\Delta y.1)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}-k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}-k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}=h_2(\Delta y.1)\left(T_{m,n}-T_{2\infty}\right)$$

m=M (x=L) n=0

$$-k(\frac{\Delta y}{2}.1)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}-k(\frac{\Delta x}{2}.1)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}=h_2(\frac{\Delta y}{2}.1)(T_{m,n}-T_{1\infty})+h_3(\frac{\Delta x}{2}.1)(T_{m,n}-T_{3\infty})$$

$m\neq 0$ n=0

$$-k(\Delta x.1)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}-k(\frac{\Delta y}{2}.1)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}-k(\frac{\Delta y}{2}.1)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}=h_3(\Delta y.1)(T_{m,n}-T_{3\infty})$$

$m\neq 0$ n=N (y=W)

$$-k(\Delta x.1)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}-k(\frac{\Delta y}{2}.1)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}-k(\frac{\Delta y}{2}.1)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}=h_4(\Delta y.1)(T_{m,n}-T_{4\infty})$$

m=0 n=N (y=W)

$$-k(\frac{\Delta x}{2}.1)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}-k(\frac{\Delta y}{2}.1)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}=h_4(\frac{\Delta y}{2}.1)(T_{m,n}-T_{4\infty})+h_1(\frac{\Delta x}{2}.1)(T_{m,n}-T_{1\infty})$$

m=M(x=L) n=N (y=W)

$$-k(\frac{\Delta x}{2}.1)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}-k(\frac{\Delta y}{2}.1)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}=h_4(\frac{\Delta y}{2}.1)(T_{m,n}-T_{4\infty})+h_2(\frac{\Delta x}{2}.1)(T_{m,n}-T_{1\infty})$$

Java solution:

```
import java.io.*;
import visad.*;
import visad.java3d.DisplayImplJ3D;
import visad.java2d.DisplayImplJ2D;
import java.rmi.RemoteException;

class HT_plane_FD2
{
public static double [] gauss_seidel(double a[][],double b[],double x[],double lambda)
{
//denklem sistemi çözümü b iteratif gauss_seidel metodu metodu
//yakınsama faktörü lambda ile birlikte verilmiş formül
int n=a.length;
int imax=1000;//maksimum iterasyon sayısı
double es=1.0e-10;
double dummy=0;
double ea=1e-1;
double sum=0;
double old=0;
int iter=0;
int sentinel=0;
//System.out.print("iter = "+(iter+1)+" ");
for(int i=0;i<n;i++)
{  dummy=a[i][i];
for(int j=0;j<n;j++)
{a[i][j]/=dummy;}
b[i]/=dummy;
}
for(int i=0;i<n;i++)
{  sum=b[i];
for(int j=0;j<n;j++)
{if(i!=j)sum-=a[i][j]*x[j];}
x[i]=sum;
//System.out.print(x[i]+ "  ");
}
//System.out.println();
iter++;
do
{
sentinel=1;
//System.out.print("iter = "+(iter+1)+" ");
for(int i=0;i<n;i++)
{
old=x[i];
```

```
sum=b[i];
for(int j=0;j<n;j++)
{if(i!=j) sum-=a[i][j]*x[j];}
x[i]=lambda*sum+(1.0-lambda)*old;
if(sentinel==1 && x[i]!=0.0 ) {ea=Math.abs((x[i]-old)/x[i])*100.0;}
if(ea<es) sentinel=0;
//System.out.print(x[i]+ "  ");
}
//System.out.println();
iter++;
}while(sentinel==1 && iter<imax);
if(iter>=imax) System.out.println("Maksimum iterasyon sayısı aşıldı sonuç geçerli olmayabilir iter = "+iter);
return x;
}

public static double[] pivotlugauss(double a[][],double b[])
{ //kısmi pivotlu gauss eleme yöntemi
int n=b.length;
double x[]=new double[n];
double carpan=0;
double toplam=0;
double buyuk;
double dummy=0;
//gauss eleme
int i,j,k,p,ii,jj;
//System.out.println("Orijinal Matris :\n"+Matrix.toString(a));
for(k=0;k<(n-1);k++)
{ //pivotlama
p=k;
buyuk=Math.abs(a[k][k]);
for(ii=k+1;ii<n;ii++)
{  dummy=Math.abs(a[ii][k]);
if(dummy > buyuk) {buyuk=dummy;p=ii;}
}
if(p!=k)
{ for(jj=k;jj<n;jj++)
{ dummy=a[p][jj];
a[p][jj]=a[k][jj];
a[k][jj]=dummy;
}
dummy=b[p];
b[p]=b[k];
b[k]=dummy;
}
//System.out.println("pivotlu Matris : k="+k+"\n"+Matrix.toString(a));
//gauss elemeyi çözme
for(i=k+1;i<n;i++)
{  carpan=a[i][k]/a[k][k];
a[i][k]=0;
for(j=k+1;j<n;j++)
{   a[i][j]-=carpan*a[k][j]; }
b[i]   =b[i]   -carpan*b[k];
}
}
//geriye doğru yerine koyma
x[n-1]=b[n-1]/a[n-1][n-1];
for(i=n-2;i>=0;i--)
{
toplam=0;
for(j=i+1;j<n;j++)
{  toplam+=a[i][j]*x[j];}
x[i]=(b[i]-toplam)/a[i][i];
}
return x;
}

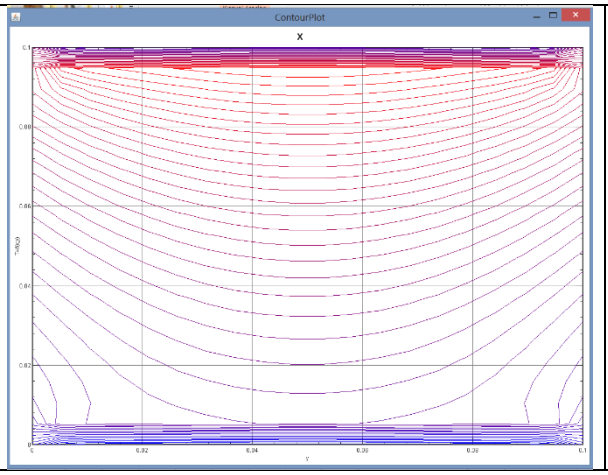
public static double [][] HT_planecoz( double L,double W,double dx, double Tx0,double Txn,double Ty0,double Tyn,double
hx0,double hxn,double hy0,double hyn,double kc)
{
//  Assume dx=dy=const;
//  Temperature of 4 sides are known
int nx=(int)(L/dx+0.01);
int ny=(int)(W/dx+0.01);
System.out.println("nx="+nx+"ny="+ny);

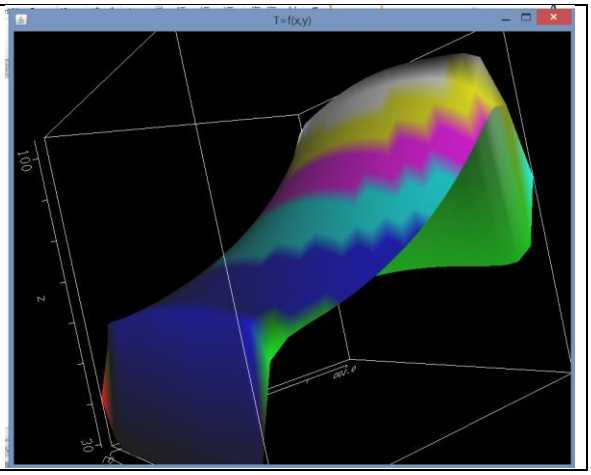
int mx=nx;
```

```
int my=ny;
int mt=mx*my;
double A[][]=new double[mt][mt];
double b[]=new double[mt];
double x[]=new double[mt];
double x1,x2,y1,y2;
// boundary conditions
int i=0,j=0;
for(i=0;i<my;i++)
{
for(j=0;j<mx;j++)
{
x1=dx*j;
x2=(L-x1);
y1=dx*i;
y2=(W-y1);
int k=i*my+j;
x[k]=((x1*Tx0+x2*Txn)/L+(y1*Ty0+y2*Tyn)/W)/2.0;
int kp1=(i+1)*my+j;
int km1=(i-1)*my+j;
// 1
if((i==0)&&(j==0)) {A[k][k]+=-(hy0*dx/kc+hx0*dx/kc+2.0);A[kp1][k]+=1.0;A[k][k+1]+=1.0;b[k]+=-hy0*dx/kc*Tx0-
hx0*dx/kc*Ty0;}
// 4
else if((i==0)&&(j==(mx-1))) {A[k][k]+=-(hxn*dx/kc+hy0*dx/kc+2.0);A[kp1][k]+=1.0;A[k][k-1]+=1.0;b[k]+=-hxn*dx/kc*Txn-
hy0*dx/kc*Ty0;}
// 2
else if((i==(my-1))&&(j==0)) {A[k][k]+=-(hx0*dx/kc+hyn*dx/kc+2.0);A[km1][k]+=1.0;A[k][k+1]+=1.0;b[k]+=-hyn*dx/kc*Tyn-
hx0*dx/kc*Tx0;}
// 3
else if((i==(my-1))&&(j==(mx-1))) {A[k][k]+=-(hxn*dx/kc+hyn*dx/kc+2.0);A[km1][k]+=1.0;A[k][k-1]+=1.0;b[k]+=-
hyn*dx/kc*Tyn-hxn*dx/kc*Txn;}
// 7
else if(j==0) {A[k][k]+=-(2.0*hx0*dx/kc+4.0);A[kp1][k]+=1.0;A[km1][k]+=1.0;A[k][k+1]+=2.0;b[k]+=-2.0*hx0*dx/kc*Tx0;}
// 5
else if(i==0) {A[k][k]+=-(2.0*hy0*dx/kc+4.0);A[kp1][k]+=2.0;A[k][k-1]+=1.0;A[k][k+1]+=1.0;b[k]+=-2.0*hy0*dx/kc*Ty0;}
// 8
else if(j==(mx-1)) {A[k][k]+=-(2.0*hxn*dx/kc+4.0);A[kp1][k]+=1.0;A[km1][k]+=1.0;A[k][k-1]+=2.0;b[k]+=-
2.0*hxn*dx/kc*Txn;}
// 6
else if(i==(my-1)) {A[k][k]+=-(2.0*hyn*dx/kc+4.0);A[k][k-1]+=1.0;A[k][k+1]+=1.0;A[km1][k]+=2.0;b[k]+=-
2.0*hyn*dx/kc*Tyn;}
// iç bölge
else {A[k][k]+=-4.0;A[km1][k]+=1.0;A[kp1][k]+=1.0;A[k][k-1]+=1.0;A[k][k+1]+=1.0;}
}
}
//Text.print(A);
double T[][]=new double[ny][nx];
//double a[]=conjugate_gradient1.conjugate_gradient(x,A,b);
//double a[]=gauss_seidel(A,b,x,1.2);
double a[]=pivotlugauss(A,b);
for(i=0;i<ny;i++)
{ for(j=0;j<nx;j++)
{
{int k=i*my+j;
T[i][j]=a[k];
}
}
}

return T;
}

public static void main(String args[]) throws IOException,RemoteException, VisADException
{
double a[][]=HT_planecoz(1.0,1.0,0.1,20.0,20.0,20.0,100.0,200.0,200.0,200.0,200.0,200.0);
Text.print(a);
ContourPlot pp=new ContourPlot(a,0.0,0.1,0.0,0.1,40, false,"x","y","T=f(x,y)");
pp.plot();
Plot3D.data(a,0.0,0.1,0.0,0.1,400,"x","y","T=f(x,y)");
}
}
```



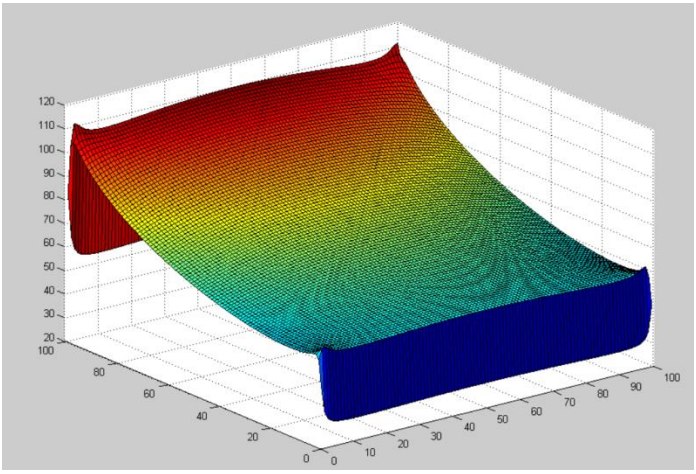


Matlab solution:

```
function [T] = HT_plane_xy(L,W,dx,Tx0,Txn,Ty0,Tyn,hx0,hxn,hy0,hyn,kc)
nx=L/dx;
ny=W/dx;
mx=nx;
my=ny;
mt=mx*my;
A=zeros(mt,mt);
B=zeros(mt);
T=zeros(mt,mt);
% boundary conditions
for i=1:my
for j=1:mx
k=(i-1)*my+j;
kp1=i*my+j;
km1=(i-2)*my+j;
% 1
if( i==1 && j==1 )
A(k,k)=A(k,k)-(hy0*dx/kc+hx0*dx/kc+2.0);
A(kp1,k)=A(kp1,k)+1.0;
A(k,k+1)=A(k,k+1)+1.0;
B(k)=B(k)-hy0*dx/kc*Tx0-hx0*dx/kc*Ty0;
% 4
elseif( i==1&& j==mx )
A(k,k)=A(k,k)-(hxn*dx/kc+hy0*dx/kc+2.0);
A(kp1,k)=A(kp1,k)+1.0;
A(k,k-1)=A(k,k-1)+1.0;
B(k)=B(k)-hxn*dx/kc*Txn-hy0*dx/kc*Ty0;
% 2
elseif( i==my && j==1)
A(k,k)=A(k,k)-(hx0*dx/kc+hyn*dx/kc+2.0);
A(km1,k)=A(km1,k)+1.0;
A(k,k+1)=A(k,k+1)+1.0;
B(k)=B(k)-hyn*dx/kc*Tyn-hx0*dx/kc*Tx0;
```

```
% 3
elseif( i==my && j==mx)
A(k,k)=A(k,k)-(hxn*dx/kc+hyn*dx/kc+2.0);
A(km1,k)=A(km1,k)+1.0;
A(k,k-1)=A(k,k-1)+1.0;
B(k)=B(k)-hyn*dx/kc*Tyn-hxn*dx/kc*Txn;
% 7
elseif(j==1)
A(k,k)=A(k,k)-(2.0*hx0*dx/kc+4.0);
A(kp1,k)=A(kp1,k)+1.0;
A(km1,k)=A(km1,k)+1.0;
A(k,k+1)=A(k,k+1)+2.0;
B(k)=B(k)-2.0*hx0*dx/kc*Tx0;
% 5
elseif(i==1)
A(k,k)=A(k,k)-(2.0*hy0*dx/kc+4.0);
A(kp1,k)=A(kp1,k)+2.0;
A(k,k-1)=A(k,k-1)+1.0;
A(k,k+1)=A(k,k+1)+1.0;
B(k)=B(k)-2.0*hy0*dx/kc*Ty0;
% 8
elseif(j==mx)
A(k,k)=A(k,k)-(2.0*hxn*dx/kc+4.0);
A(kp1,k)=A(kp1,k)+1.0;
A(km1,k)=A(km1,k)+1.0;
A(k,k-1)=A(k,k-1)+2.0;
B(k)=B(k)-2.0*hxn*dx/kc*Txn;
% 6
elseif(i==my) A(k,k)=A(k,k)-(2.0*hyn*dx/kc+4.0);
A(k,k-1)=A(k,k-1)+1.0;
A(k,k+1)=A(k,k+1)+1.0;
A(km1,k)=A(km1,k)+2.0;
B(k)=B(k)-2.0*hyn*dx/kc*Tyn;
% internal region
else
A(k,k)=A(k,k)-4.0;
A(km1,k)=A(km1,k)+1.0;
A(kp1,k)=A(kp1,k)+1.0;
A(k,k-1)=A(k,k-1)+1.0;
A(k,k+1)=A(k,k+1)+1.0;
end
end
end
T=zeros(ny,nx);
B=B(:,1);
aa=linsolve(A,B);
for i=1:ny
for j=1:nx
k=(i-1)*my+j;
T(i,j)=aa(k);
end
end
end
```

```
>> [T]=HT_plane_xy(1.0,1.0,0.01,20.0,20.0,20.0,100.0,200.0,200.0,200.0,200.0,200.0);
>> surf(T)
```



EXAMPLE 3: FINITE DIFFERENCE SOLUTION, NON-HOMOGENEOUS, TIME INDEPENDENT SOLUTION; CARTESIAN COORDINATES, TWO DIMENSION



$$\frac{\partial^2 T(x,y)}{\partial x^2}+\frac{\partial^2 T(x,y)}{\partial y^2}=0$$

$T(x,y)=T_1$ x=0, $0\le y\le W$

$T(x,y)=T_2$ x=L, $0\le y\le W$

$T(x,y)=T_3$ y=0, $0\le x\le L$

$T(x,y)=T_4$ y=W, $0\le x\le L$

$$\frac{T_{m+1,n}-2T_{m,n}+T_{m-1,n}}{\Delta x^2}+\frac{T_{m,n+1}-2T_{m,n}+T_{m,n-1}}{\Delta y^2}=0$$

If $\Delta x=\Delta y$

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

Java program:

```
import java.io.*;
import visad.*;
import visad.java3d.DisplayImplJ3D;
import visad.java2d.DisplayImplJ2D;
import java.rmi.RemoteException;

class HT_plane_FD_T
{
public static double [] gauss_seidel(double a[][],double b[],double x[],double lambda)
{
//denklem sistemi çözümü b iteratif gauss_seidel metodu metodu
//yakınsama faktörü lambda ile birlikte verilmiş formül
int n=a.length;
int imax=1000;//maksimum iterasyon sayısı
```

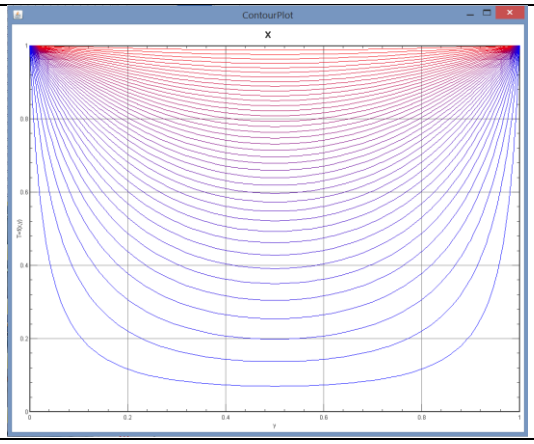
```
double es=1.0e-10;
double dummy=0;
double ea=1e-1;
double sum=0;
double old=0;
int iter=0;
int sentinel=0;
//System.out.print("iter = "+(iter+1)+" ");
for(int i=0;i<n;i++)
{  dummy=a[i][i];
for(int j=0;j<n;j++)
{a[i][j]/=dummy;}
b[i]/=dummy;
}
for(int i=0;i<n;i++)
{  sum=b[i];
for(int j=0;j<n;j++)
{if(i!=j)sum-=a[i][j]*x[j];}
x[i]=sum;
//System.out.print(x[i]+ "  ");
}
//System.out.println();
iter++;
do
{
sentinel=1;
//System.out.print("iter = "+(iter+1)+" ");
for(int i=0;i<n;i++)
{
old=x[i];
sum=b[i];
for(int j=0;j<n;j++)
{if(i!=j) sum-=a[i][j]*x[j];}
x[i]=lambda*sum+(1.0-lambda)*old;
if(sentinel==1 && x[i]!=0.0 ) {ea=Math.abs((x[i]-old)/x[i])*100.0;}
if(ea<es) sentinel=0;
//System.out.print(x[i]+ "  ");
}
//System.out.println();
iter++;
}while(sentinel==1 && iter<imax);
if(iter>=imax) System.out.println("Maksimum iterasyon sayısı aşıldı sonuç geçerli olmayabilir iter = "+iter);
return x;
}

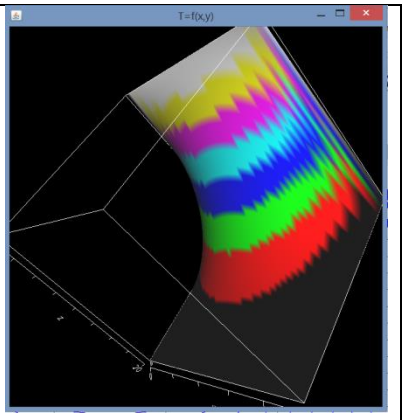
public static double[] pivotlugauss(double a[][],double b[])
{ //kısmi pivotlu gauss eleme yöntemi
int n=b.length;
double x[]=new double[n];
double carpan=0;
double toplam=0;
double buyuk;
double dummy=0;
//gauss eleme
int i,j,k,p,ii,jj;
//System.out.println("Orijinal Matris :\n"+Matrix.toString(a));
for(k=0;k<(n-1);k++)
{ //pivotlama
p=k;
buyuk=Math.abs(a[k][k]);
for(ii=k+1;ii<n;ii++)
{  dummy=Math.abs(a[ii][k]);
if(dummy > buyuk) {buyuk=dummy;p=ii;}
}
if(p!=k)
{ for(jj=k;jj<n;jj++)
{ dummy=a[p][jj];
a[p][jj]=a[k][jj];
a[k][jj]=dummy;
}
dummy=b[p];
b[p]=b[k];
b[k]=dummy;
}
//System.out.println("pivotlu Matris : k="+k+"\n"+Matrix.toString(a));
//gauss elemeyi çözme
```

```
for(i=k+1;i<n;i++)
{  carpan=a[i][k]/a[k][k];
a[i][k]=0;
for(j=k+1;j<n;j++)
{   a[i][j]-=carpan*a[k][j]; }
b[i]   =b[i]   -carpan*b[k];
}
}
//back substitution
x[n-1]=b[n-1]/a[n-1][n-1];
for(i=n-2;i>=0;i--)
{
toplam=0;
for(j=i+1;j<n;j++)
{  toplam+=a[i][j]*x[j];}
x[i]=(b[i]-toplam)/a[i][i];
}
return x;
}


public static double [][] HT_planecoz( double L,double W,double dx, double Tx0,double Txn,double Ty0,double Tyn)
{
int nx=(int)(L/dx+0.01);
int ny=(int)(W/dx+0.01);
int mx=(nx-1);
int my=(ny-1);
int mt=mx*my;
double A[][]=new double[mt][mt];
double b[]=new double[mt];
double x[]=new double[mt];
double x1,x2,y1,y2;
// boundary conditions
int i=0,j=0;
for(i=0;i<my;i++)
{
for(j=0;j<mx;j++)
{
x1=dx*j;
x2=(L-x1);
y1=dx*i;
y2=(W-y1);
int k=i*my+j;
x[k]=((x1*Tx0+x2*Txn)/L+(y1*Ty0+y2*Tyn)/W)/2.0;
int kp1=(i+1)*my+j;
int km1=(i-1)*my+j;
if(j==0)           {b[k]+=-Tx0;}
if(j== (mx-1))     {b[k]+=-Txn;}
if(i==0)           {b[k]+=-Ty0;}
else if(i==(my-1))  {b[k]+=-Tyn;}
A[k][k]+=-4.0;
if((i==0)&&(j==0)) {A[kp1][k]+=1.0;A[k][k+1]+=1.0;}
else if((i==0)&&(j==(mx-1))) {A[kp1][k]+=1.0;A[k][k-1]+=1.0;}
else if((i==(my-1))&&(j==0)) {A[km1][k]+=1.0;A[k][k+1]+=1.0;}
else if((i==(my-1))&&(j==(mx-1))) {A[km1][k]+=1.0;A[k][k-1]+=1.0;}
else if(j==0) {A[kp1][k]+=1.0;A[km1][k]+=1.0;A[k][k+1]+=1.0;}
else if(i==0) {A[k][k+1]+=1.0;A[k][k-1]+=1.0;A[kp1][k]+=1.0;}
else if(j==(mx-1)) {A[kp1][k]+=1.0;A[km1][k]+=1.0;A[k][k-1]+=1.0;}
else if(i==(my-1)) {A[km1][k]+=1.0;A[k][k-1]+=1.0;A[k][k+1]+=1.0;}
else {A[km1][k]+=1.0;A[kp1][k]+=1.0;A[k][k-1]+=1.0;A[k][k+1]+=1.0;}
}
}
double T[][]=new double[ny+1][nx+1];
double a[]=conjugate_gradient1.conjugate_gradient(A,b);
for(i=0;i<=ny;i++)
{ for(j=0;j<=nx;j++)
{if(j==0) {T[i][j]=Tx0;}
else if(i==0) {T[i][j]=Ty0;}
else if(j==nx) {T[i][j]=Txn;}
else if(i==ny) {T[i][j]=Tyn;}
else
{int k=(i-1)*my+(j-1);
T[i][j]=a[k];
}
}
```

```
}
return T;
}
public static void main(String args[]) throws IOException,RemoteException, VisADException
{  double a[][]=HT_planecoz(1.0,1.0,0.05,20.0,20.0,20.0,100.0);
Text.print(a);
ContourPlot pp=new ContourPlot(a,0.0,1.0,0.0,1.0,40, false,"x","y","T=f(x,y)");
pp.plot();
Plot3D.data(a,0.0,1.0,0.0,1.0,400,"x","y","T=f(x,y)");
}
}
```

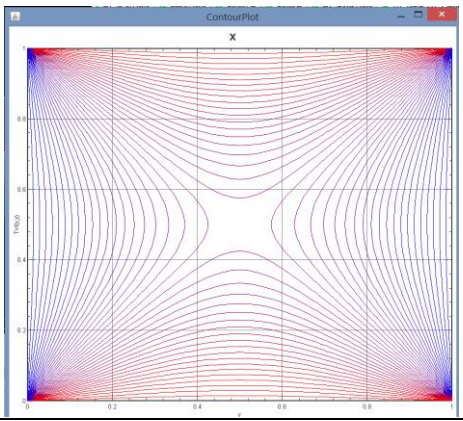




data output(veri çıktısı) :

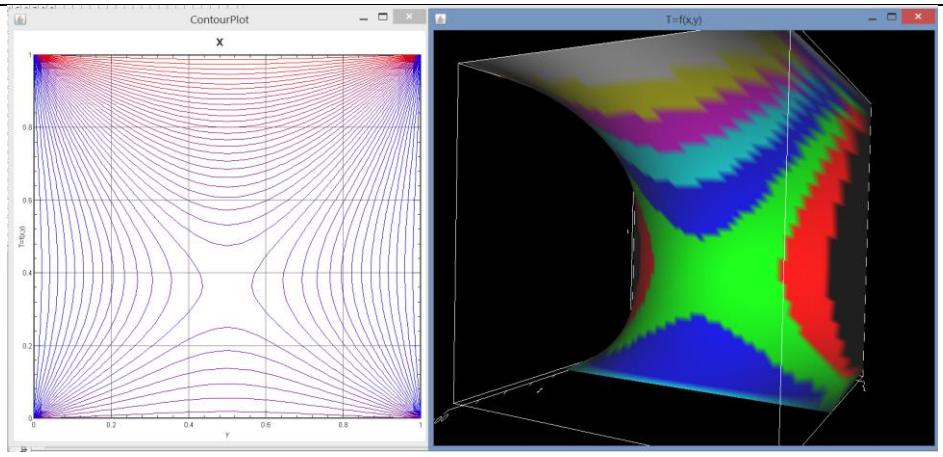
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0
20.0	20.219525...	20.433372...	20.636065...	20.822508...	20.988136...	21.129019...	21.241926...	21.324348...	21.374507...	21.391345...	21.374507...	21.324348...	21.241926...	21.129019...	20.988136...	20.822508...	20.636065...	20.433372...	20.219525...	20.0
20.0	20.444728...	20.877900...	21.288380...	21.665831...	22.001018...	22.286017...	22.514336...	22.680959...	22.782338...	22.816364...	22.782338...	22.680959...	22.514336...	22.286017...	22.001018...	21.665831...	21.288380...	20.877900...	20.444728...	20.0
20.0	20.681487...	21.345119...	21.973724...	22.551418...	23.064087...	23.499693...	23.848442...	24.102814...	24.257521...	24.309436...	24.257521...	24.102814...	23.848442...	23.499693...	23.064087...	22.551418...	21.973724...	21.345119...	20.681487...	20.0
20.0	20.936101...	21.847366...	22.709977...	23.502032...	24.204219...	24.800228...	25.276925...	25.624333...	25.835498...	25.906336...	25.835498...	25.624333...	25.276925...	24.800228...	24.204219...	23.502032...	22.709977...	21.847366...	20.936101...	20.0
20.0	21.215552...	22.398266...	23.516786...	24.542515...	25.450528...	26.220074...	26.834698...	27.282093...	27.553804...	27.644911...	27.553804...	27.282093...	26.834698...	26.220074...	25.450528...	24.542515...	23.516786...	22.398266...	21.215552...	20.0
20.0	21.527842...	23.013361...	24.416387...	25.700714...	26.835304...	27.794841...	28.559699...	29.115539...	29.452712...	29.565701...	29.452712...	29.115539...	28.559699...	27.794841...	26.835304...	25.700714...	24.416387...	23.013361...	21.527842...	20.0
20.0	21.882457...	23.710947...	25.434687...	27.008649...	28.395135...	29.564286...	30.493720...	31.167650...	31.575806...	31.712468...	31.575806...	31.167650...	30.493720...	29.564286...	28.395135...	27.008649...	25.434687...	23.710947...	21.882457...	20.0
20.0	22.291040...	24.513284...	26.602763...	28.504059...	30.172303...	31.573447...	32.683245...	33.485535...	33.970394...	34.132558...	33.970394...	33.485535...	32.683245...	31.573447...	30.172303...	28.504059...	26.602763...	24.513284...	22.291040...	20.0
20.0	22.768418...	25.448387...	27.959024...	30.232521...	32.216569...	33.873955...	35.180277...	36.120853...	36.687676...	36.876978...	36.687676...	36.120853...	35.180277...	33.873955...	32.216569...	30.232521...	27.959024...	25.448387...	22.768418...	20.0
20.0	23.334248...	26.552819...	29.552425...	32.250433...	34.587498...	36.525526...	38.043056...	39.129923...	39.782480...	40.0	39.782480...	39.129923...	38.043056...	36.525526...	34.587498...	32.250433...	29.552425...	26.552819...	23.334248...	20.0
20.0	24.015755...	27.876218...	31.447422...	34.629289...	37.357462...	39.597596...	41.336497...	42.573301...	43.312323...	43.558060...	43.312323...	42.573301...	41.336497...	39.597596...	37.357462...	34.629289...	31.447422...	27.876218...	24.015755...	20.0
20.0	24.852553...	29.488877...	33.731756...	37.461840...	40.615465...	43.170899...	45.132035...	46.514464...	47.335450...	47.607595...	47.335450...	46.514464...	45.132035...	43.170899...	40.615465...	37.461840...	33.731756...	29.488877...	24.852553...	20.0
20.0	25.905582...	31.494981...	36.528884...	40.870851...	44.471660...	47.338499...	49.506279...	51.017069...	51.907418...	52.201419...	51.907418...	51.017069...	49.506279...	47.338499...	44.471660...	40.870851...	36.528884...	31.494981...	25.905582...	20.0
20.0	27.274793...	34.056581...	40.017949...	45.021017...	49.061827...	52.205158...	54.537514...	56.140113...	57.075735...	57.383245...	57.075735...	56.140113...	54.537514...	52.205158...	49.061827...	45.021017...	40.017949...	34.056581...	27.274793...	20.0
20.0	29.137010...	37.438601...	44.465314...	50.133442...	54.549471...	57.882793...	60.298504...	61.930137...	62.872163...	63.180092...	62.872163...	61.930137...	60.298504...	57.882793...	54.549471...	50.133442...	44.465314...	37.438601...	29.137010...	20.0
20.0	31.834645...	42.095499...	50.271264...	56.497967...	61.119822...	64.478039...	66.843574...	68.409766...	69.302689...	69.592796...	69.302689...	68.409766...	66.843574...	64.478039...	61.119822...	56.497967...	50.271264...	42.095499...	31.834645...	20.0
20.0	36.106073...	48.837487...	58.026275...	64.467339...	68.953811...	72.065969...	74.187985...	75.562665...	76.336030...	76.585713...	76.336030...	75.562665...	74.187985...	72.065969...	68.953811...	64.467339...	58.026275...	48.837487...	36.106073...	20.0
20.0	43.752162...	59.122099...	68.529012...	74.391302...	78.162113...	80.644040...	82.279734...	83.316878...	83.893055...	84.077995...	83.893055...	83.316878...	82.279734...	80.644040...	78.162113...	74.391302...	68.529012...	59.122099...	43.752162...	20.0
20.0	59.780474...	75.369737...	82.576373...	86.406744...	88.659300...	90.068343...	90.970033...	91.532057...	91.841317...	91.940157...	91.841317...	91.532057...	90.970033...	90.068343...	88.659300...	86.406744...	82.576373...	75.369737...	59.780474...	20.0
20.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	20.0

```
public static void main(String args[]) throws IOException,RemoteException, VisADException
{  double a[][]=HT_planecoz(1.0,1.0,0.02,20.0,20.0,100.0,100.0);
Text.print(a);
ContourPlot pp=new ContourPlot(a,0.0,1.0,0.0,1.0,40, false,"x","y","T=f(x,y)");
pp.plot();
Plot3D.data(a,0.0,1.0,0.0,1.0,400,"x","y","T=f(x,y)");
}
```

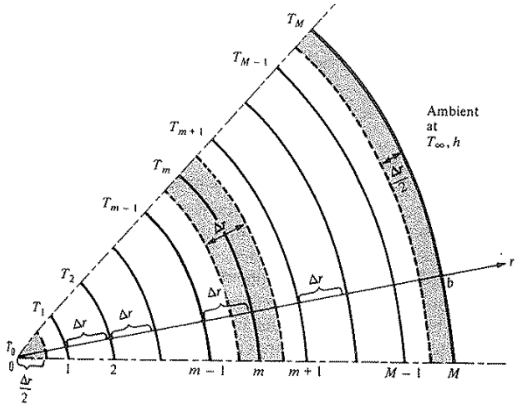



```
public static void main(String args[]) throws IOException,RemoteException, VisADException
{  double a[][]=HT_planecoz(1.0,1.0,0.02,20.0,20.0,60.0,100.0);
```

```
ContourPlot pp=new ContourPlot(a,0.0,1.0,0.0,1.0,40, false,"x","y","T=f(x,y)");
pp.plot();
Plot3D.data(a,0.0,1.0,0.0,1.0,400,"x","y","T=f(x,y)");
}
```



EXAMPLE 4: FINITE DIFFERENCE SOLUTION, NON-HOMOGENEOUS, TIME INDEPENDENT SOLUTION; CYLINDRICAL COORDINATES, ONE DIMENSION WITH HEAT GENERATION



$$\Delta r = \frac{b}{M}$$

$\frac{T_{m-1}-T_m}{R_{m-1,m}} + \frac{T_{m+1}-T_m}{R_{m+1,m}} + \Delta V_m g_m = 0$ where g_m is heat generation, ΔV_m is volume of element m.

$$R_{m-1,m} = \frac{\Delta r}{2\pi(m\Delta r - \frac{1}{2}\Delta r)Hk} = \frac{1}{2\pi m[1-1/(2m)]Hk}$$

$$R_{m+1,m} = \frac{\Delta r}{2\pi(m\Delta r + \frac{1}{2}\Delta r)Hk} = \frac{1}{2\pi m[1+1/(2m)]Hk}$$

$$\Delta V_m = 2\pi m \Delta r \Delta r H$$

H=the length of the cylinder. Equation can be arranged as:

$$\left(1-\frac{1}{2m}\right)T_{m-1} - 2T_m + \left(1+\frac{1}{2m}\right)T_{m+1} + \frac{(\Delta r)^2 g_m}{k} = 0$$

Boundary conditions:

At the center of the cylinder:

$$\frac{T_1 - T_0}{R_{1,0}} + \Delta V_0 g_0 = 0$$

$$R_{1,0} = \frac{\Delta r}{2\pi(\Delta r/2)Hk} = \frac{1}{\pi Hk}$$

$$\Delta V_0 = \pi\left(\frac{\Delta r}{2}\right)^2 H$$

After the arrangement

$$4(T_1 - T_0) + \frac{(\Delta r)^2 g_0}{k} = 0$$

Second boundary condition (convective boundary condition)

$$-kA\frac{T_{M+1} - T_{M-1}}{2\Delta r} + Ah(T_M - T_\infty) = 0$$

Combining this equation with previous finite difference formula :

$$\left(1 - \frac{1}{2M}\right)T_{m-1} - 2T_m + \left(1 + \frac{1}{2M}\right)T_{m+1} + \frac{(\Delta r)^2 g_M}{k} = 0$$

gives

$$2T_{M-1} - 2\beta_M T_M + \gamma_M + \frac{(\Delta r)^2 g_m}{k} = 0$$

$$\beta_M = 1 + \left(1 + \frac{1}{2M}\right)\frac{(\Delta r)h}{k}$$

$$\gamma_M = \left(1 + \frac{1}{2M}\right)\frac{(\Delta r)h}{k}T_\infty$$

Problem 4.4 Java Solution:

```
import java.io.*;

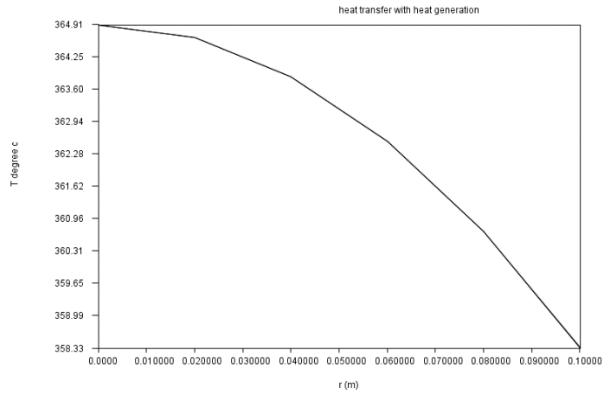
class HT_cylinder_FD_g
{
public static double [] thomas(double a[][],double b[])
{
//3 band matrix
int n=a.length;
double f[]=new double[n];
double e[]=new double[n];
double g[]=new double[n];
double x[]=new double[n];
for(int i=0;i<n;i++)     {f[i]=a[i][i];}
for(int i=0;i<(n-1);i++) {g[i]=a[i][i+1];}
for(int i=0;i<(n-1);i++) {e[i+1]=a[i+1][i];}
for(int k=1;k<n;k++)
{e[k]=e[k]/f[k-1];
f[k]=f[k]-e[k]*g[k-1];
}
for(int k=1;k<n;k++)
{b[k]=b[k]-e[k]*b[k-1];
}
x[n-1]=b[n-1]/f[n-1];
for(int k=(n-2);k>=0;k--)
{x[k]=(b[k]-g[k]*x[k+1])/f[k];}
return x;
}

public static double [] HT_cylinder( double b,double gm[],double dr, double Tr,double hr,double kr)
{
//b : cylinder radius
//gm : heat generatşon per unit volume
int M=(int)(b/dr);
int mt=M+1;
double A[][]=new double[mt][mt];
double B[]=new double[mt];
```

```
double x[]=new double[mt];
double x1,x2,y1,y2;

// boundary conditions
int m=0;
double GM=0,betaM=0,gammaM=0;
for(m=0;m<=M;m++)
{  GM=dr*dr*gm[m]/kr;
betaM=1+(1+1.0/(2.0*m))*dr*hr/kr;
gammaM=(1+1.0/(2.0*m))*dr*hr/kr*Tr;
if(m==0)          {B[m]+=-dr*dr*gm[0]/kr;}
else if(m==M)     {B[m]+=-2.0*gammaM-GM;}
else              {B[m]+=-dr*dr*gm[m]/kr;}
if(m==0)     {A[m][m]+=-4.0;A[m][m+1]+=4.0;}
else if(m==M) {A[m][m-1]=2.0;A[m][m]+=-2*betaM;}
else         {A[m][m-1]+=1.0-1/(2.0*m);A[m][m]+=-2.0;A[m][m+1]+=1.0+1/(2.0*m);}
}
Text.print(A,"A");
Text.printT(B,"B");
//double T[]=conjugate_gradient1.conjugate_gradient(A,B);
double T[]=thomas(A,B);
return T;
}
public static void main(String args[])
{  int M=5;
double b=10.0e-2;
double dr=b/M;
double gm[]=new double[M+1];
double r[]=new double[M+1];
for(int m=0;m<M+1;m++)
{gm[m]=1.0e6;//W/m3
r[m]=dr*m;
}
double Tr=25.0;//degree C
double hr=150.0;//W/(m2K)
double k=380.0;//W/(mK)
double T[]=HT_cylinder( b,gm,dr,Tr,hr,k);
//compare result with analytical case with qm=const.
double T1[]=new double[T.length];
for(int m=0;m<M+1;m++)
{T1[m]=gm[0]*b*b/(4.0*k)*(1.0-(m*dr/b)*(m*dr/b))+gm[0]*b/(2.0*hr)+Tr;
}
double TT[][]={T,T1};
Text.print(TT);
Plot pp=new Plot(r,T);
pp.addData(r,T1);
pp.setPlabel("heat transfer with heat generation");
pp.setXYlabel("r (m)","T degree c");
pp.plot();
}
}
```

Graphic output:



Solution and analytical solution to compare the results

0	1	2	3	4	5
364.9122807017613	364.64912280702447	363.8596491228139	362.5438596491297	360.70175438597175	358.333333333340
364.91228070175436	364.64912280701753	363.859649122807	362.5438596491228	360.7017543859649	358.333333333333

The matrix equations: (M=5)

A

0	1	2	3	4	5
-4.0	4.0	0.0	0.0	0.0	0.0
0.5	-2.0	1.5	0.0	0.0	0.0
0.0	0.75	-2.0	1.25	0.0	0.0
0.0	0.0	0.8333333...	-2.0	1.1666666...	0.0
0.0	0.0	0.0	0.875	-2.0	1.125
0.0	0.0	0.0	0.0	2.0	-2.0173684...

B

0
-1.0526315789473684
-1.0526315789473684
-1.0526315789473684
-1.0526315789473684
-1.0526315789473684
-1.486842105263158

If the same problem solved by M=10, matrices will become :

A

0	1	2	3	4	5	6	7	8	9	10
-4.0	4.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	-2.0	1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.75	-2.0	1.25	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.83...	-2.0	1.16...	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.875	-2.0	1.125	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.9	-2.0	1.1	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.91...	-2.0	1.08...	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.92...	-2.0	1.07...	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.9375	-2.0	1.0625	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.94...	-2.0	1.05...
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.0	-2.00...

B

0
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.2631578947368421
-0.47039473684210525

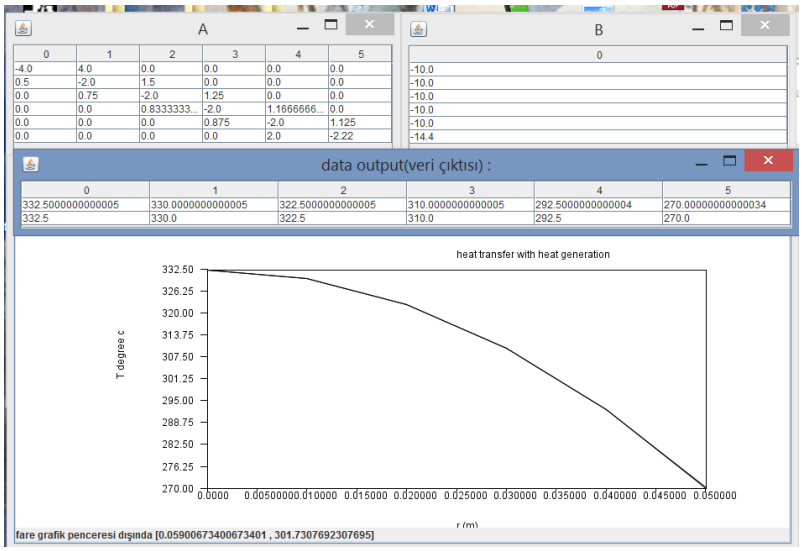
data output(veri çıktısı) :

0	1	2	3	4	5	6	7	8	9	
364.912280...	364.846491...	364.649122...	364.320175...	363.859649...	363.267543...	362.543859...	361.688596...	360.701754...	359.583333...	358.
364.912280...	364.846491...	364.649122...	364.320175...	363.859649...	363.267543...	362.543859...	361.688596...	360.701754...	359.583333...	358.

A $10x10^{-2}$ m diameter solid steel bar of thermal conductivity k=40 W/(mK) is heated electrically by the passage of electrical current which henerates energy within the rod at a rate of $g=4x10^6$ W/m^3. Heat is dissipated from the surface of the rod by convection with a heat transfer coefficient h=400 W/(m^{2}K) into an ambient temperature T_∞ =20 C. By dividing the Radius into 5 and 10 equal parts develop and solve the finite difference equation.

M=5 division

```
public static void main(String args[])
{  int M=5;
double b=5.0e-2;
double dr=b/M;
double gm[]=new double[M+1];
double r[]=new double[M+1];
for(int m=0;m<M+1;m++)
{gm[m]=4.0e6;//W/m3
r[m]=dr*m;
}
double Tr=20.0;//degree C
double hr=400.0;//W/(m2K)
double k=40.0;//W/(mK)
double T[]=HT_cylinder( b,gm,dr,Tr,hr,k);
//compare result with analytical case with qm=const.
double T1[]=new double[T.length];
for(int m=0;m<M+1;m++)
{T1[m]=gm[0]*b*b/(4.0*k)*(1.0-(m*dr/b)*(m*dr/b))+gm[0]*b/(2.0*hr)+Tr;
}
double TT[][]={T,T1};
Text.print(TT);
Plot pp=new Plot(r,T);
pp.addData(r,T1);
pp.setPlabel("heat transfer with heat generation");
pp.setXYlabel("r (m)","T degree c");
pp.plot();
}
```



M=10 division

14. Partial differential equation solution methods: Finite element solutions

FINITE ELEMENT

The simplest finite element will be the linear one dimensional element. The linear temperature variation in the element is represented by:

$T(x) = \alpha_1 + \alpha_2 x$ where T is independent variable (tempearture if heat transfer problems are considered). Since there are two arbitrary constants in the linear representation, it requires only two nodes to determine the values of α_1, and α_2, namely,

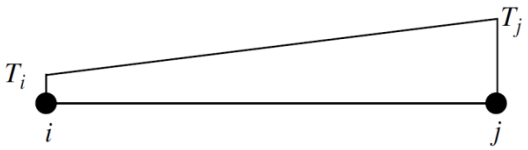


Figure 5.1 linear one dimensional finite element

$T_i = \alpha_1 + \alpha_2 x_i$

$T_j = \alpha_1 + \alpha_2 x_j$

From the above equation we get

$\alpha_1 = \frac{T_i x_j - T_j x_i}{x_j - x_i}$

$\alpha_2 = \frac{T_j - T_i}{x_j - x_i}$

On substituting the values α_1, and α_2, into the main equation, we get

$T(x) = \alpha_1 + \alpha_2 x = \frac{T_i x_j - T_j x_i}{x_j - x_i} + \frac{T_j - T_i}{x_j - x_i} x$ by rearranging the equation

$T(x) = T_i \left[\frac{x_j - x}{x_j - x_i}\right] + T_j \left[\frac{x - x_i}{x_j - x_i}\right]$ or

$T(x) = N_i T_i + N_j T_j = [N_i \quad N_j] \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$

Where N_i and N_j are called shape functions or interpolation functions. Equation can be rewritten as

$T = [N]\{T\}$ where $[N] = [N_i \quad N_j]$ and $\{T\} = \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$. This is basically linear interpolation formula that formulated in the shape function form. The derivative of the function can be easily formulated from this function

$$\frac{dT}{dx} = \frac{dN_i}{dx}T_i + \frac{dN_j}{dx}T_j = \left[-\frac{1}{x_j - x_i}\right]T_i + \left[\frac{1}{x_j - x_i}\right]T_j$$

$$l = x_j - x_i$$

$$g = \frac{dT}{dx} = \left[-\frac{1}{l}\right]T_i + \left[\frac{1}{l}\right]T_j = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} = [B]\{T\}$$

If linear curve fitting representation will not be sufficient for the accuracy, degree of shape function can be increased. Instead of linear approximation, a quadratic approximation can be used:

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

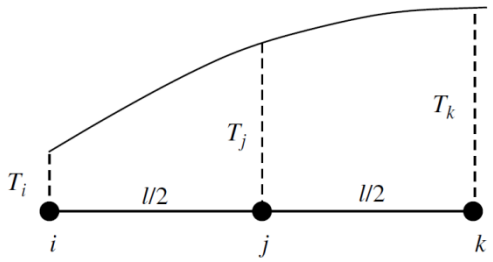


Figure 5.2 quadratic one dimensional finite element

$$T_i = \alpha_1$$

$$T_j = \alpha_1 + \alpha_2\left(\frac{l}{2}\right) + \alpha_3\left(\frac{l}{2}\right)^2$$
$$T_k = \alpha_1 + \alpha_2 l + \alpha_3 l^2$$

substituting these values back into the equation gives:

$$T = N_i T_i + N_j T_j + + N_k T_k = [N_i \quad N_j \quad N_k]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

$$N_i = \left[1 - \frac{3x}{l} + \frac{2x^2}{l^2}\right]$$
$$N_j = \left[\frac{4x}{l} - \frac{4x^2}{l^2}\right]$$
$$N_k = \left[-\frac{x}{l} + \frac{2x^2}{l^2}\right]$$

The derivative of the function will be

$$g = \frac{dT}{dx} = \left[\left(\frac{4x}{l^2} - \frac{3}{l}\right) \quad \left(\frac{4}{l} - \frac{8x}{l^2}\right) \quad \left(-\frac{1}{l} + \frac{4x}{l^2}\right)\right]\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} = [B]\{T\}$$

By using a linear or non-linear simplex elements a surface curve fitting can be achieved. The simplest geometric shape that can be employed to approximate irregular surfaces is the triangle and it is one of the popular elements currently used in finite element calculations. The two-dimensional linear triangular element, also known as a *simplex element*, is represented by

$$T(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

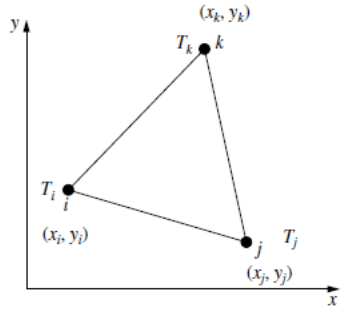


Figure 5.3 triangular finite element

where the polynomial is linear in *x* and *y* and contains three coefficients. Since a linear triangle has three nodes, the values of α_1, α_2 and α_3 are determined from

$T_i=\alpha_1+\alpha_2 x_i+\alpha_3 y_i$

$T_j=\alpha_1+\alpha_2 x_j+\alpha_3 y_j$

$T_k=\alpha_1+\alpha_2 x_k+\alpha_3 y_k$

which results in the following:

$$\alpha_1 = \frac{1}{2A}\left[(x_j y_k - x_k y_j)T_i + (x_k y_i - x_i y_k)T_j + (x_i y_j - x_j y_i)T_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[(y_j - y_k)T_i + (y_k - y_i)T_j + (y_i - y_j)T_k\right]$$

$$\alpha_3 = \frac{1}{2A}\left[(x_k - x_j)T_i + (x_i - x_k)T_j + (x_j - x_i)T_k\right]$$

where '*A*' is the area of the triangle given by

$$2A = det\begin{bmatrix}1 & x_i & y_i\\ 1 & x_j & y_j\\ 1 & x_k & y_k\end{bmatrix} = (x_i y_j - x_j y_i) + (x_k y_j - x_i y_k) + (x_j y_k - x_k y_j)$$

Substituting the values of α_1, α_2 and α_3 into the equation and collating the coefficients of T_i, T_j and T_k, we get

$$T = N_i T_i + N_j T_j + N_k T_k = [N_i \quad N_j \quad N_k]\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix}$$

$$N_i = \frac{1}{2A}[a_i + b_i x + c_i y]$$

$$N_j = \frac{1}{2A}[a_j + b_j x + c_j y]$$

$$N_k = \frac{1}{2A}[a_k + b_k x + c_k y]$$

$$a_i = (x_j y_k - x_k y_j) \qquad a_j = (x_k y_i - x_i y_k) \qquad a_k = (x_i y_j - x_j y_i)$$
$$b_i = (y_j - y_k) \qquad b_j = (y_k - y_i) \qquad b_k = (y_i - y_j)$$
$$c_i = (x_k - x_j) \qquad c_j = (x_i - x_k) \qquad c_k = (x_j - x_i)$$

If we evaluate N_i at node i, where the coordinates are (x_i,y_i), then we obtain $N_i(x_i,y_i)=1$ and similarly $N_i(x_j,y_j)=0$, $N_i(x_k,y_k)=0$ Thus, we see that the shape functions have a value of unity at the designated vertex and zero at all other vertices. It is possible to show that $N_i+N_j+N_k=1$ everywhere in the element, including the boundaries. Linear simplex is given us a relatively simple way of interpolating any point in the surface (a linear interpolation). So the temperature profile of The simplex element can be calculated as:

$$T = N_i T_i + N_j T_j + + N_k T_k = [N_i \quad N_j \quad N_k]\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix} = [N]\{T\}$$

The derivative of the temperature field can be given as:

$$\{g\} = \begin{Bmatrix}\frac{\partial T}{\partial x}\\ \frac{\partial T}{\partial y}\end{Bmatrix} = \frac{1}{2A}\begin{bmatrix}b_i & b_j & b_k\\ c_i & c_j & c_k\end{bmatrix}\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix} = [B]\{T\}$$

Therefore heat transfer can be given as:

$$\begin{Bmatrix}q_x\\ q_y\end{Bmatrix} = -k[B]\{T\} = -\frac{k}{2A}\begin{bmatrix}b_i & b_j & b_k\\ c_i & c_j & c_k\end{bmatrix}\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix}$$

Integration of the finite element can be carried out as:

One dimensional integral:

$$\int_l N_i^a N_j^b N_k^c dl = \frac{a!\,b!\,c!}{(a+b+c+1)!}l$$

Two dimensional integral

$$\int_A N_i^a N_j^b N_k^c dA = \frac{a!\,b!\,c!}{(a+b+c+2)!}2A$$

Different Finite elements can be defined the most common two dimensional element after tridigonal element is quadrilateral elements. Quadrilateral element is described by

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

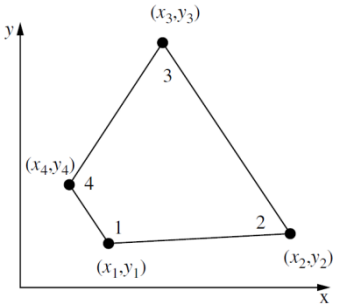


Figure 5.4 quadrilateral finite element

A special case of this element, rectangular element is used commonly.

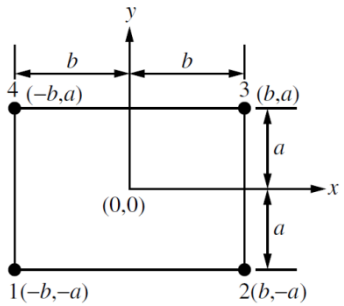


Figure 5.5 rectangular finite element

For this element shape function can be given as:

$$T(x,y) = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4$$
$$N_1 = \frac{1}{4ab}(b-x)(a-y)$$
$$N_2 = \frac{1}{4ab}(b+x)(a-y)$$
$$N_3 = \frac{1}{4ab}(b+x)(a+y)$$
$$N_4 = \frac{1}{4ab}(b-x)(a+y)$$

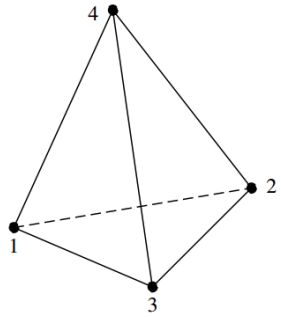
The derivatives can be expressed as follows:

$$\{g\} = \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix} = \begin{bmatrix} -(a-y) & (a-y) & (a+y) & -(a+y) \\ -(b-x) & -(b+x) & (b+x) & (b-x) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = [B]\{T\}$$

As the last thing in this part, let us consider a three dimensional simplex element as well. The linear temperature representation of a tree-dimensional linear element (tetrahedron) is given by:

$T(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$ The constatnt of equation can be solved in a simialar manner of 2D simplex

$$T(x,y,z) = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4$$



Where

$$N_i = \frac{1}{6V}[a_i + b_i x + c_i y + d_i z]$$

The volume of the simplex element is:

$$6V = det\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}$$

$$\{g\} = \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix} = \frac{1}{6V}\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = [B]\{T\}$$

Therefore heat transfer can be given as:

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -\frac{1}{6V}\begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = -[D][B]\{T\}$$

Integration of the finite element can be carried out as:

Where:

$$b_1 = -det\begin{bmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{bmatrix} = (y_2 - y_4)(z_3 - z_4) - (y_3 - y_4)(z_2 - z_4)$$

$$b_2 = -det\begin{bmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{bmatrix} = (y_3 - y_4)(z_1 - z_4) - (y_1 - y_4)(z_3 - z_4)$$

$$b_3 = -det\begin{bmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{bmatrix} = (y_1 - y_4)(z_2 - z_4) - (y_2 - y_4)(z_1 - z_4)$$

$$b_4 = b_1 + b_2 + b_3$$

$$c_1 = -det\begin{bmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{bmatrix} = (x_3 - x_4)(z_2 - z_4) - (x_2 - x_4)(z_3 - z_4)$$

$$c_2 = -det\begin{bmatrix} x_1 & 1 & z_1 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{bmatrix} = (x_1 - x_4)(z_3 - z_4) - (x_3 - x_4)(z_3 - z_4)$$

$$c_3 = -det\begin{bmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_4 & 1 & z_4 \end{bmatrix} = (x_2 - x_4)(z_1 - z_4) - (x_1 - x_4)(z_2 - z_4)$$

$$c_4 = -(c_1 + c_2 + c_3)$$

$$d_1 = -det\begin{bmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{bmatrix} = (x_2 - x_4)(y_3 - y_4) - (x_3 - x_4)(y_3 - y_4)$$

$$d_2 = -det\begin{bmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{bmatrix} = (x_3 - x_4)(y_1 - y_4) - (x_1 - x_4)(y_3 - y_4)$$

$$d_3 = -det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{bmatrix} = (x_1 - x_4)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_4)$$

$$d_4 = -(d_1 + d_2 + d_3)$$

Three dimensional integral

$\int_A N_1^a N_2^b N_3^c N_4^d dV = \frac{a!b!c!d!}{(a+b+c+d+3)!} 6V$

SOLUTION OF HEAT CONDUCTION EQUATION BY USING FINITE ELEMENT METHOD

Now general steady state general heat conduction equation can be investigated and converted into an integral equation

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + g = 0$$

Boundary conditions

$T=T_b$ on surface S_1

$k_x \frac{\partial T}{\partial x}\vec{l} + k_y \frac{\partial T}{\partial y}\vec{m} + q = 0$ on surface S_2

$k_x \frac{\partial T}{\partial x}\vec{l} + k_y \frac{\partial T}{\partial y}\vec{m} + h(T - T_a) = 0$ on surface S_3

Now that galarkin integration method will be applied to our differention equation . The method requires that the following expression will be satisfied

$\int_\Omega w_k L(T) d\Omega = 0$ where Ω is the volume of the element. Substituring steady state heat conduction equation and finite element node as weight factor into the galarkin integration equation gives:

$$\int_\Omega N_k \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + g\right] d\Omega = 0$$

Note that for finite element $T = [N]\{T\}$ and integral can be converted by partial integration rule

$$\int udv = uv - \int vdu$$

$$\int_\Omega N_k \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right)\right] d\Omega = \int_S N_k \left(k_x \frac{\partial T}{\partial x}\right) dS - \int_\Omega \frac{\partial N_k}{\partial x}(k_x) \frac{\partial N_m}{\partial x} T d\Omega$$

By applying these equation becomes

$$-\int_\Omega \left[k_x \frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x} + k_y \frac{\partial N_k}{\partial y}\frac{\partial N_m}{\partial y}\right]\{T_m\} d\Omega + \int_\Omega g N_k d\Omega + \int_S N_k \left(k_x \frac{\partial T}{\partial x} + k_y \frac{\partial T}{\partial y}\right) dS = 0$$

From boundary conditions:

$$\int_S N_k \left(k_x \frac{\partial T}{\partial x} + k_y \frac{\partial T}{\partial y}\right) dS = -\int_S N_k (q + h(T - T_a) dS = -\int_S N_k q dS - \int_S N_k h N_m \{T_m\} dS + \int_S N_k h T_a dS$$

$$-\int_\Omega \left[k_x \frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x} + k_y \frac{\partial N_k}{\partial y}\frac{\partial N_m}{\partial y}\right]\{T_m\} d\Omega + \int_\Omega g N_k d\Omega - \int_S N_k q dS - \int_S N_k h N_m \{T_m\} dS + \int_S N_k h T_a dS = 0$$

If this equation is written in matrix form:

$[K]\{T\} = \{f\}$ where

$$[K] = -\int_\Omega \left[k_x \frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x} + k_y \frac{\partial N_k}{\partial y}\frac{\partial N_m}{\partial y}\right] d\Omega + \int_S N_k h N_m dS$$

$$\{f\} = \int_\Omega g N_k d\Omega - \int_S N_k q dS + \int_S N_k h T_a dS$$

If equation put into the matrix form:

$$[K] = \int_\Omega [B]^T [D][B] d\Omega + \int_S h [N]^T [N] dS$$

Where

$$[D] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$$

$$\{f\} = \int_{\Omega} g[N]^T d\Omega - \int_{S} [N]^T q dS + \int_{S} h[N]^T T_a dS$$

$$\int_l N_i^a N_j^k dl = \frac{a!b!}{(a+b+1)!} l \quad \int_l N_i^2 N_j^0 dl = \frac{2!0!}{(2+0+1)!} l = \frac{2l}{6} \quad \int_l N_i^1 N_j^1 dl = \frac{1!1!}{(1+1+1)!} l = \frac{l}{6}$$

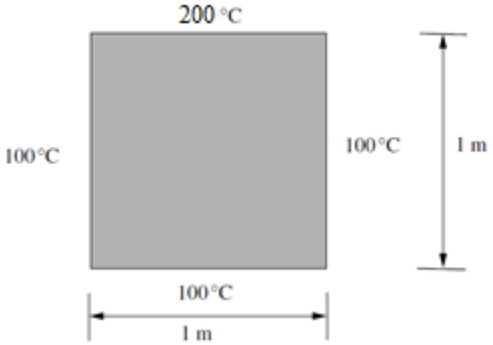
Matrix terms can be opened for two dimensional triangular element as:

$$[K] = \frac{t}{4A}\left\{ k_x \begin{bmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_i b_j & b_j^2 & b_j b_k \\ b_j b_k & b_j b_k & b_k^2 \end{bmatrix} + k_y \begin{bmatrix} c_i^2 & c_i c_j & c_i c_k \\ c_i c_j & c_j^2 & c_j c_k \\ c_j c_k & c_j c_k & c_k^2 \end{bmatrix} \right\} + \frac{htl_{ij}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{htl_{jk}}{6} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{htl_{ki}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\{f\} = \frac{gAt}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} - \frac{qtl_{ij}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} - \frac{qtl_{jk}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} - \frac{qtl_{ik}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} + + \frac{hT_a tl_{jk}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + + \frac{hT_a tl_{ik}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

EXAMPLE 1: FINITE ELEMENT SOLUTION, NON-HOMOGENEOUS, STEADY STATE SOLUTION; CARTESIAN COORDINATES

A square plate with temperatures of 100 C in three sides and 200 C in the fourth side is given. Plate sides are given as 1 m. Calculate the temperature profile



Java program:

```
public class element_number
{int number;
 int ijk;
 public element_number(int numberi,int ijki)
 {
 number=numberi;
 if(ijki<=2) ijk=ijki;
 else {ijk=0;System.out.println("wrong value");}
 }
 public boolean equals(element_number en)
 {if(en.number==number && en.ijk==ijk) return true;
 else return false;
 }
 public int getNumber() {return number;}
 public int getIjk() {return ijk;}
 public int[] output()
 {int x[]={number,ijk};
  return x;
 }
}
```

```
import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;

public class setElement
{   ArrayList<element_number> en;
    int no[];
    int ij[];
                public setElement()
                {en=new ArrayList<element_number>();}
```

```
            public void add(int n1,int ij1)
            {element_number en=new element_number(n1,ij1);
             add(en);
   }
            public void add(element_number s1)
            { //int n=en.size();
              boolean b=false;
              element_number s2;
              Iterator n2=en.iterator();
    while(n2.hasNext()&& !b)
    { s2=(element_number)n2.next();
      b=s2.equals(s1);
    }
            if(!b) en.add(s1);
            }

 public int[][] output()
 {
   int n=en.size();
   int a[][]=new int[n][2];
   Iterator i=en.iterator();
   int j=0;
   while(i.hasNext()) {element_number el=(element_number)i.next();a[j]=el.output();j++;}
   return a;
 }
 public static void main(String arg[])
 {setElement se=new setElement();
  element_number en1=new element_number(11,0);
  element_number en2=new element_number(11,1);
  element_number en3=new element_number(13,2);
  element_number en4=new element_number(13,0);
  se.add(en1);
  se.add(en2);
  se.add(en3);
  se.add(en4);
  int a[][]=se.output();
  Text.print(a);
 }
}
```

```
 import javax.swing.*;
import static java.lang.Math.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;
public class triangular_grid
{
public triangular_element te[];
public setElement se[];
public int N,M,K,K1,K2,Nm1,Mm1;
public double x[][];
public double y[][];
public double T[][];
public double Ti[];
public double xi[];
public double yi[];
public boolean bi[];
public double dx,dy;
public int Bx0[],BxM[],By0[],ByN[];
public double L,W;
public double min(double xi[][])
{ double min=1e200;
  for(int i=0;i<xi.length;i++)
   {for(int j=0;j<xi[0].length;j++)
     {if(xi[i][j]<min) min=xi[i][j];}
 }
 return min;
}
public double max(double xi[][])
{double max=-1e200;
 for(int i=0;i<xi.length;i++)
 {for(int j=0;j<xi[0].length;j++)
   {if(xi[i][j]<max) max=xi[i][j];}
 }
```

```
 return max;
}
public triangular_grid(double xi[][],double yi[][])
{x=xi;
 y=yi;
 N=x.length-1;
 M=x[0].length-1;
 T=new double[N+1][M+1];
 L=max(xi);
 W=max(yi);
 setElements();
}

public void generateTriangularGrid(double Li, double Wi,int Ni, int Mi)
{int i,j;
 N=Ni;M=Mi;
 L=Li;
 W=Wi;
 x=new double[N+1][M+1];
 y=new double[N+1][M+1];
 T=new double[N+1][M+1];
 setElements();
}

public void setElements()
{ int i,j;
 dx=L/M;
 dy=W/N;
 K=M*N;
 K1=K*2;
 K2=K1;
 //grid points
 xi=new double[K2];
 yi=new double[K2];
 Ti=new double[K2];
 bi=new boolean[K2];
 se=new setElement[K2];
 for(i=0;i<K2;i++) {se[i]=new setElement();}
 te=new triangular_element[K1];
 int k1=-1,k=0,n=0,m=0;
for(n=0;n<=N;n++)
{ for(m=0;m<=M;m++)
  {k=(N+1)*n+m;
   x[n][m]=m*dx;y[n][m]=n*dy;
   xi[k]=x[n][m];
   yi[k]=y[n][m];
   bi[k]=false;
   Ti[k]=0.0;
  }
}

for(n=0;n<N;n++)
{for(m=0;m<M;m++)
  { k=(N+1)*n+m;
                int kp1=(N+1)*(n+1)+m;
                int kp1p1= kp1+1;
                double xi1[]={x[n][m],x[n][m+1],x[n+1][m]};
                double yi1[]={y[n][m],y[n][m+1],y[n+1][m]};

                k1++;
                int n_point1[]={k,(k+1),kp1};
                se[k].add(k1,0);
                se[k+1].add(k1,1);
                se[kp1].add(k1,2);
                te[k1]=new triangular_element(k1,n_point1,xi1,yi1);
                double xi2[]={x[n][m+1],x[n+1][m+1],x[n+1][m]};
                double yi2[]={y[n][m+1],y[n+1][m+1],y[n+1][m]};
                k1++;
                int n_point2[]={k+1,kp1p1,kp1};
   se[k+1].add(k1,0);
                se[kp1p1].add(k1,1);
                se[kp1].add(k1,2);
                te[k1]=new triangular_element(k1,n_point2,xi2,yi2);
  }
}
}
```

```
public triangular_grid(double Li, double Wi,int Ni, int Mi)
{generateTriangularGrid(Li,Wi,Ni,Mi);}
public int find_element(double z[])
{ return find_element(z[0],z[1]);}
public int find_element(double x,double y)
{double R1=Math.sqrt(x*x+y*y);
 int kmin=0;
 double min=1.0e50;
 for(int k=0;k<K1;k++)
 {   te[k].setN(x,y);
    double r0=Math.sqrt((te[k].x[0]-x)*(te[k].x[0]-x)+(te[k].y[0]-y)*(te[k].y[0]-y));
    double r1=Math.sqrt((te[k].x[1]-x)*(te[k].x[1]-x)+(te[k].y[1]-y)*(te[k].y[1]-y));
    double r2=Math.sqrt((te[k].x[2]-x)*(te[k].x[2]-x)+(te[k].y[2]-y)*(te[k].y[2]-y));
    double dr=Math.sqrt(r0*r0+r1*r1+r2*r2);
                boolean b1=(x>=te[k].x[0]) && (x<=te[k].x[2]);
                boolean b2=(y>=te[k].y[0]) && (y<=te[k].y[1]);
                boolean b= b1 && b2;
    if(dr<min) {min=dr;kmin=k;}
 }
 return kmin;
}
public void setConvection(int cond,double L0,double L,double W0,double W, double hi, double tinfi)
{//cond 0: x==L0
 //cond 1: x==L
 //cond 2: y==W0
 //cond 3: y==W
 int j=0;
 for(int i=0;i<K1;i++)
 {  j=te[i].setI(cond,L0,L,W0,W);
              te[i].setConvection(j,hi,tinfi);
 }
}
public void setHeatFlux(int cond,double L0,double L,double W0,double W, double qi)
{//cond 0: x==L0
 //cond 1: x==L
 //cond 2: y==W0
 //cond 3: y==W
 int j=-1;
 for(int i=0;i<K1;i++)
 {  j=te[i].setI(cond,L0,L,W0,W);
              te[i].setHeatFlux(j,qi);
 }
}
public void setTemperature(int cond,double L0,double L,double W0,double W, double TTi)
{//cond 0: x==L0
 //cond 1: x==L
 //cond 2: y==W0
 //cond 3: y==W
 for(int i=0;i<K2;i++)
 {  if(cond==0)
    {if(xi[i]==L0) {Ti[i]=TTi;bi[i]=true;}}
    else if(cond==1)
    {if(xi[i]==L) {Ti[i]=TTi;bi[i]=true;}}
    else if(cond==2)
    {if(yi[i]==W0) {Ti[i]=TTi;bi[i]=true;}}
    else if(cond==3)
    {if(yi[i]==W) {Ti[i]=TTi;bi[i]=true;}}
 }
}
public void setHeatGeneration(int node[],double qgeni)
{  int n=node.length;
   for(int i=0;i<n;i++)
   {te[node[i]].setHeatGeneration(qgeni);}
}

public void solve(double ti,double k[])
{
double K[][]=new double[K2][K2];
double KK[][]=new double[K2][K2];
double ff[]=new double[K2];
double KN[][]=new double[3][3];
double f[]=new double[K2];
double fN[]=new double[3];
for(int i=0;i<K1;i++)
{KN=te[i].K(ti,k);
 fN=te[i].f(ti,k);
```

```
 int n1=te[i].n_element;
 int m[]=te[i].n_point;
 for(int j=0;j<3;j++)
   { for(int k1=0;k1<3;k1++)
     {K[m[j]][m[k1]]+=KN[j][k1];KK[m[j]][m[k1]]=K[m[j]][m[k1]];}
      f[m[j]]+=fN[j];
      ff[m[j]]=f[m[j]];
   }
}
//redesign K matrix and f vector if any known
//temperature boundary is given
//int j=0;
for(int j=0;j<K2;j++)
{
            for(int i=0;i<K2;i++)
    { if(bi[i])
      {
              if(i==j)
              {f[i]=Ti[i];K[i][j]=1.0;}
              else
              {
              K[i][j]=0.0;}
              //System.out.println("i="+i+"j="+j+"bi="+bi[i]+"Ti="+Ti[i]+"fi="+f[i]);
      }
      }//end of for
} //end of for
//System.out.println("K =\n"+Matrix.toString(K));
//System.out.println("f =\n"+Matrix.toStringT(f));
double TT[]=AXB(K,f);
int k1=0;
for(int n=0;n<=N;n++)
{ for(int m=0;m<=M;m++)
  {k1=(N+1)*n+m;
  T[n][m]=TT[k1];
  }
}
}

//LU decomposition method
public static double[][] LU(double c[][],int indx[],int d[])
{
//returns LU decomposition of matrix c and index indx
double a[][];
int n=c.length;
a=new double[n][n];
double vv[];
vv=new double[n];
double sum,dum,big,temp;
int i,j,k;
int imax;
int nmax=100;
double tiny=1.0e-40;
imax=0;
for(i=1;i<=n;i++)
{
  for(j=1;j<=n;j++)
     a[i-1][j-1]=c[i-1][j-1];
}
d[0]=1;
for(i=1;i<=n;i++)
  {
  big=0.0;
  for(j=1;j<=n;j++)
    {
    if(Math.abs(a[i-1][j-1])>big) big=Math.abs(a[i-1][j-1]);
    }
  if(big==0) {System.out.println("singular matrix");return a;}
  vv[i-1]=1.0/big;
  }
for(j=1;j<=n;j++)
{
  for(i=1;i<j;i++)
    {
      sum=a[i-1][j-1];
      for(k=1;k<i;k++)
              {
```

```
                sum-=a[i-1][k-1]*a[k-1][j-1];
               }
    a[i-1][j-1]=sum;
    }
  big=0;
  for(i=j;i<=n;i++)
    {
    sum=a[i-1][j-1];
    for(k=1;k<j;k++)
      {
      sum-=a[i-1][k-1]*a[k-1][j-1];
      }
    a[i-1][j-1]=sum;
    dum=vv[i-1]*Math.abs(sum);
    if(dum>=big)
      {
      imax=i;
      big=dum;
      }
    } //end of i=0
if(j != imax)
    {
    for(k=1;k<=n;k++)
      {
      dum=a[imax-1][k-1];
      a[imax-1][k-1]=a[j-1][k-1];
      a[j-1][k-1]=dum;
      }
    d[0]=-d[0];
    vv[imax-1]=vv[j-1];
    } //end of if
indx[j-1]=imax;
if(a[j-1][j-1]==0) a[j-1][j-1]=tiny;
if(j!=n)
  {
  dum=1.0/a[j-1][j-1];
  for(i=j+1;i<=n;i++)
    a[i-1][j-1]*=dum;
  }//endif
} //end for j=
return a;
}
public static double[] LUaxb(double a[][],double x[],int indx[])
{
//solves AX=B system of linear equation of LU decomposed matrix a
//(calculated by method LU)
int ii=0;
int i,j,ll=0;
double sum=0;
int n=a.length;
double b[];
b=new double[n];
for(i=1;i<=n;i++)
{
    b[i-1]=x[i-1];
}
for(i=1;i<=n;i++)
  {
  ll=indx[i-1];
  sum=b[ll-1];
  b[ll-1]=b[i-1];
  if(ii!=0)
     {
     for(j=ii;j<=(i-1);j++)
       {
       sum-=a[i-1][j-1]*b[j-1];
       }
     }
  else if(sum!=0) ii=i;
  b[i-1]=sum;
  }
for(i=n;i>=1;i--)
  {
  sum=b[i-1];
  if(i<n)
    {
```

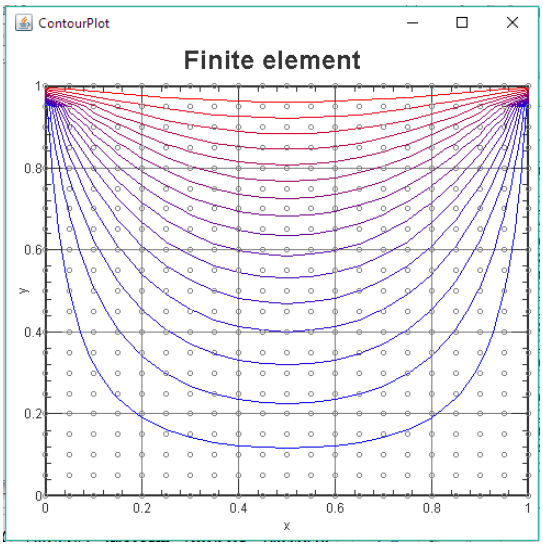
```
    for(j=(i+1);j<=n;j++)
      {
      sum-=a[i-1][j-1]*b[j-1];
      }
    }
  b[i-1]=sum/a[i-1][i-1];
  }
return b;
}
public void ContourPlot()
{ ContourPlot pp=new ContourPlot( x,y,T,16,true,"Finite element","x","y");
  pp.plot();
}

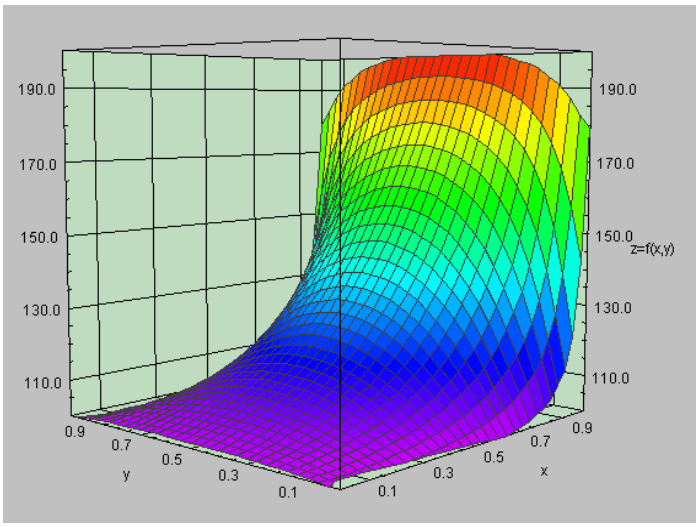
public void surfacePlot()
{    double b[][]=surfacePlot.dataformat(0,L,0,W,T);
     surfacePlot p1;
     p1=new surfacePlot(b);
     p1.plot();
}
public static double[] AXB(double a[][],double b[])
{
//Solution of system of linear equations by LU method
// note that the same calculation can be done by divide method.

Text.print(a,"K");
Text.print(b,"f");
int n=a.length;
double c[]=new double[n];
int d[]={1};
int indx[]=new int[n];
double e[][]=new double[n][n];
e=LU(a,indx,d);
c=LUaxb(e,b,indx);
return c;
} //end of AXB
public static void main(String arg[]) //throws RemoteException, VisADException
{double L=1.0;
 double W=1.0;
 int N=20,M=20;
 double hi=20.0;
 triangular_grid tg=new triangular_grid(L,W,M,N);
 int cond0=0;//x==0
 int cond1=1;//x==L
 int cond2=2;//y==0
 int cond3=3;//y==W
 double Ti0=100.0;
 double Ti1=100.0;
 double Ti2=100.0;
 double Ti3=200.0;
 tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
 tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
 tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
 tg.setTemperature(cond3,0.0,L,0.0,W,Ti3);
 //tg.setConvection(cond3,0.0,L,0.0,W,hi,Ti3);
 double k[]={50,50};
 tg.solve(1,k);
 Text.print(tg.T);
 tg.ContourPlot();
}}
```

data output(veri çıktısı) :

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	100.2...	100.5...	100.79...	101.0...	101.23...	101.4...	101.55...	101.6...	101.71...	101.7...	101.71...	101.6...	101.55...	101.4...	101.23...	101.0...	100.79...	100.5...	100.27...	100.0
100.0	100.5...	101.0...	101.61...	102.0...	102.50...	102.8...	103.14...	103.3...	103.47...	103.5...	103.47...	103.3...	103.14...	102.8...	102.50...	102.0...	101.61...	101.0...	100.55...	100.0
100.0	100.8...	101.6...	102.46...	103.1...	103.83...	104.3...	104.81...	105.1...	105.32...	105.3...	105.32...	105.1...	104.81...	104.3...	103.83...	103.1...	102.46...	101.6...	100.85...	100.0
100.0	101.1...	102.3...	103.38...	104.3...	105.25...	106.0...	106.59...	107.0...	107.29...	107.3...	107.29...	107.0...	106.59...	106.0...	105.25...	104.3...	103.38...	102.3...	101.17...	100.0
100.0	101.5...	102.9...	104.39...	105.6...	106.81...	107.7...	108.54...	109.1...	109.44...	109.5...	109.44...	109.1...	108.54...	107.7...	106.81...	105.6...	104.39...	102.9...	101.51...	100.0
100.0	101.9...	103.7...	105.52...	107.1...	108.54...	109.7...	110.69...	111.3...	111.81...	111.9...	111.81...	111.3...	110.69...	109.7...	108.54...	107.1...	105.52...	103.7...	101.90...	100.0
100.0	102.3...	104.6...	106.79...	108.7...	110.49...	111.9...	113.11...	113.9...	114.46...	114.6...	114.46...	113.9...	113.11...	111.9...	110.49...	108.7...	106.79...	104.6...	102.35...	100.0
100.0	102.8...	105.6...	108.25...	110.6...	112.71...	114.4...	115.85...	116.8...	117.46...	117.6...	117.46...	116.8...	115.85...	114.4...	112.71...	110.6...	108.25...	105.6...	102.86...	100.0
100.0	103.4...	106.8...	109.94...	112.7...	115.27...	117.3...	118.97...	120.1...	120.85...	121.0...	120.85...	120.1...	118.97...	117.3...	115.27...	112.7...	109.94...	106.8...	103.46...	100.0
100.0	104.1...	108.1...	111.94...	115.3...	118.23...	120.6...	122.55...	123.9...	124.72...	124.9...	124.72...	123.9...	122.55...	120.6...	118.23...	115.3...	111.94...	108.1...	104.16...	100.0
100.0	105.0...	109.8...	114.30...	118.2...	121.69...	124.4...	126.67...	128.2...	129.14...	129.4...	129.14...	128.2...	126.67...	124.4...	121.69...	118.2...	114.30...	109.8...	105.01...	100.0
100.0	106.0...	111.8...	117.16...	121.8...	125.76...	128.9...	131.41...	133.1...	134.16...	134.5...	134.16...	133.1...	131.41...	128.9...	125.76...	121.8...	117.16...	111.8...	106.06...	100.0
100.0	107.3...	114.3...	120.66...	126.0...	130.58...	134.1...	136.88...	138.7...	139.88...	140.2...	139.88...	138.7...	136.88...	134.1...	130.58...	126.0...	120.66...	114.3...	107.38...	100.0
100.0	109.0...	117.5...	125.02...	131.2...	136.32...	140.2...	143.17...	145.1...	146.34...	146.7...	146.34...	145.1...	143.17...	140.2...	136.32...	131.2...	125.02...	117.5...	109.09...	100.0
100.0	111.4...	121.7...	130.58...	137.6...	143.18...	147.3...	150.37...	152.4...	153.59...	153.9...	153.59...	152.4...	150.37...	147.3...	143.18...	137.6...	130.58...	121.7...	111.42...	100.0
100.0	114.7...	127.6...	137.83...	145.6...	151.39...	155.5...	158.55...	160.5...	161.62...	161.9...	161.62...	160.5...	158.55...	155.5...	151.39...	145.6...	137.83...	127.6...	114.79...	100.0
100.0	120.1...	136.0...	147.53...	155.5...	161.19...	165.0...	167.73...	169.4...	170.42...	170.7...	170.42...	169.4...	167.73...	165.0...	161.19...	155.5...	147.53...	136.0...	120.13...	100.0
100.0	129.6...	148.9...	160.66...	167.9...	172.70...	175.8...	177.84...	179.1...	179.86...	180.0...	179.86...	179.1...	177.84...	175.8...	172.70...	167.9...	160.66...	148.9...	129.69...	100.0
100.0	149.7...	169.2...	178.22...	183.0...	185.82...	187.5...	188.71...	189.4...	189.80...	189.9...	189.80...	189.4...	188.71...	187.5...	185.82...	183.0...	178.22...	169.2...	149.72...	100.0
200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0

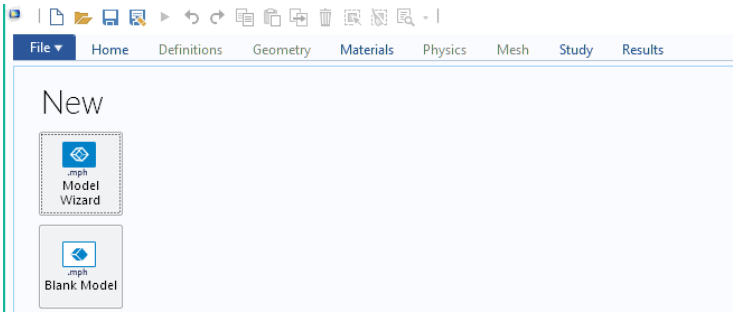




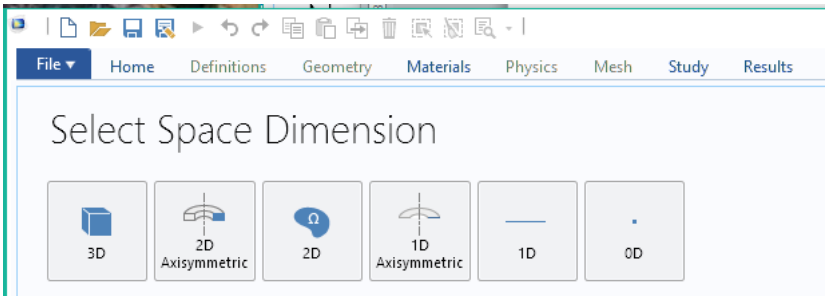
One of the advantages of finite element methods is integration of boundary values into the basic differential equation. It will allow to develop a general diferential equation solution packages, some commercial and public Access packages are available to solve heat transfer problems. Some of the package programs are:

Package	Web adres	Commercial/public licence	Company
COMSOL	https://www.comsol.com	Commercial licence	COMSOL.ltd
ABACUS	https://www.3ds.com/products-services/simulia/products/abaqus/	Commercial licence	
NASTRAN	https://www.autodesk.com/education/free-software/nastran	Commercial licence /public licence available for students and educators	AUTODESK
Open Source Field Operation and Manipulation (OpenFOAM)	https://openfoam.org/version/6/	Public licence	OpenCFD Ltd
ANSYS FLUENT	https://www.ansys.com/products/fluids/ansys-fluent	Comercial licence	ANSYS

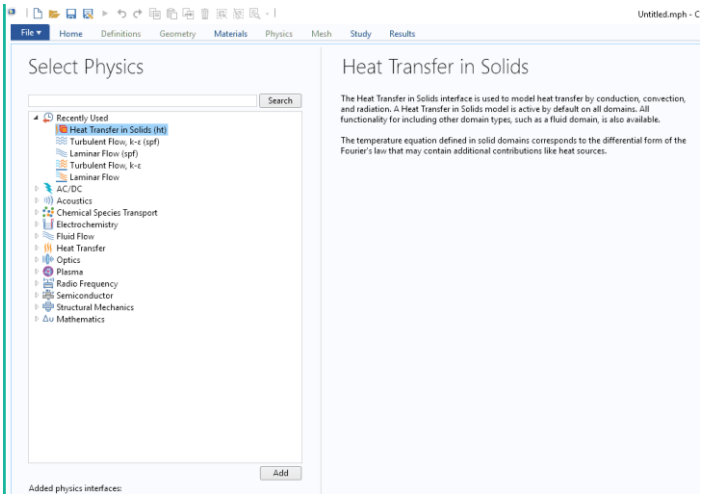
As an example, we will run comsol package to solve the previous problem

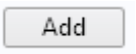


Model Wisard will be selected



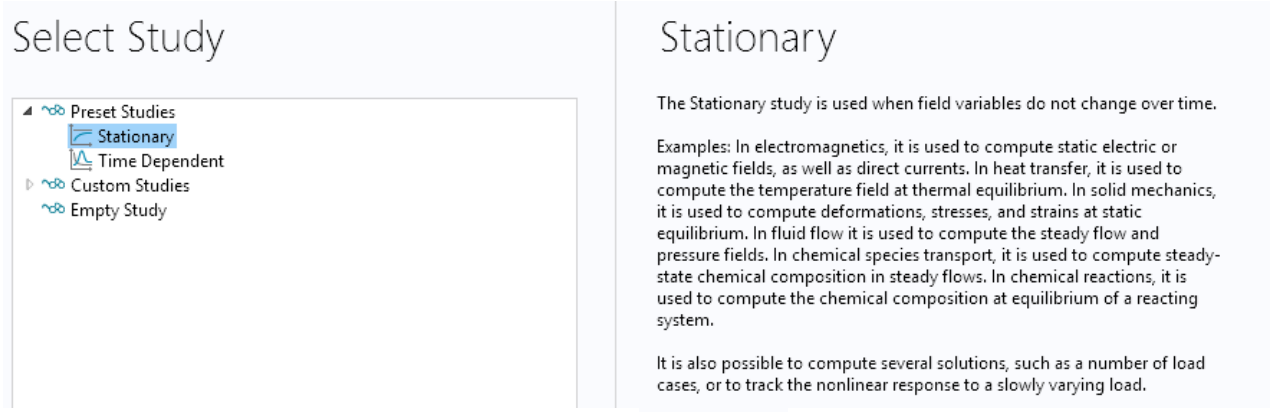
2D will be selected

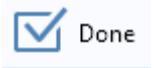


Heat transfer in solids will be selected and then push 

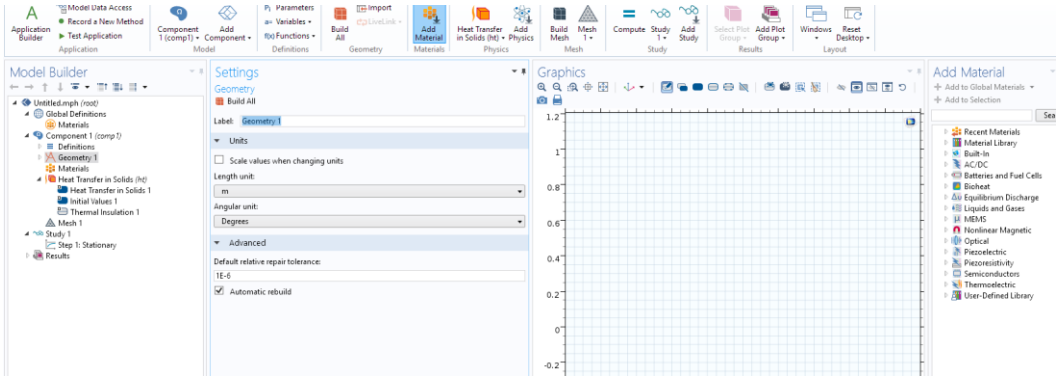
button then push study 

button

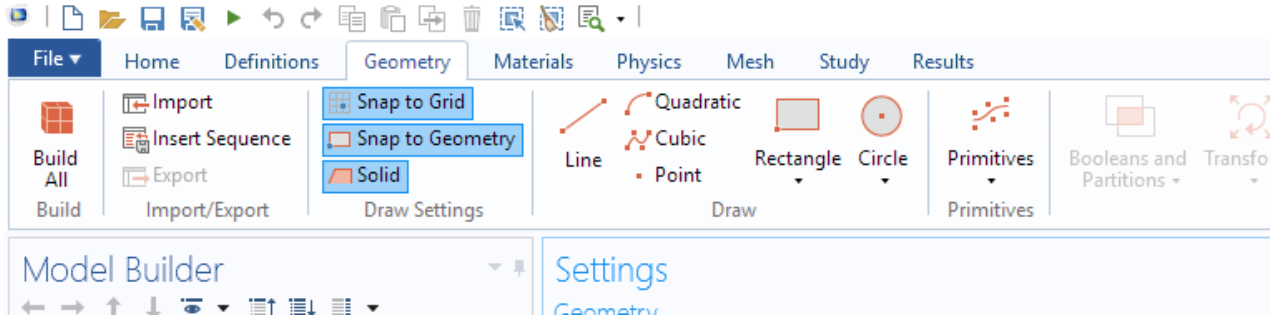


Select Stationary (steady state heat transfer) and push 

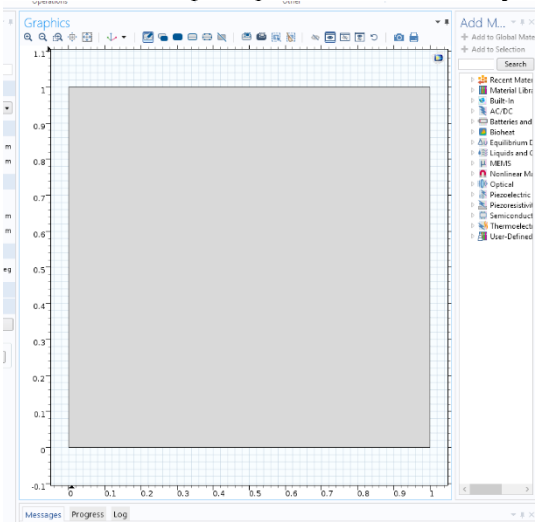
button



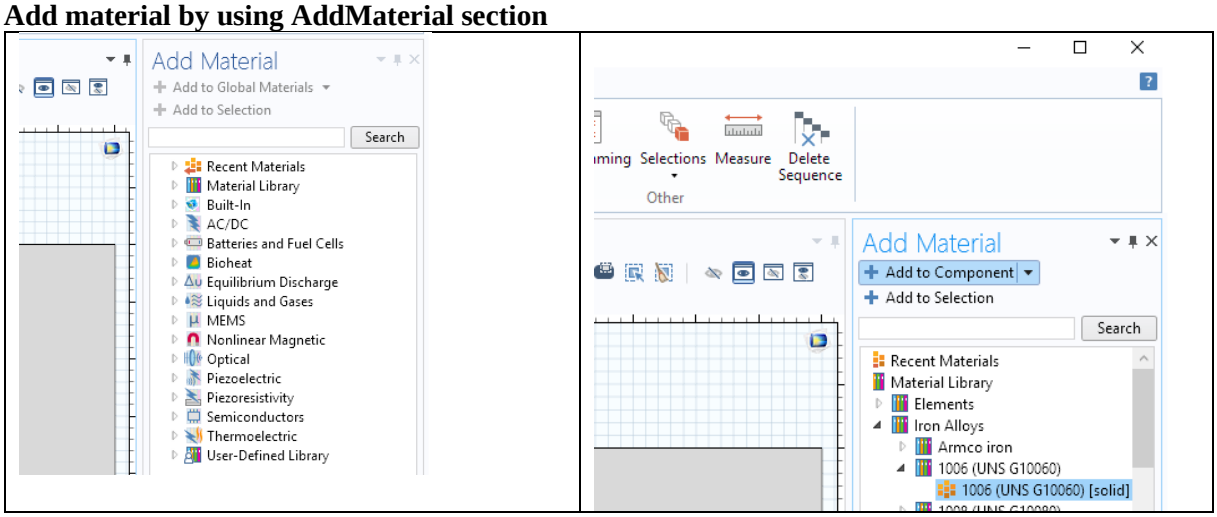
Page will open to draw the shape. By using Geometry and Rectangle draw a square 1 m x 1 m dimensions



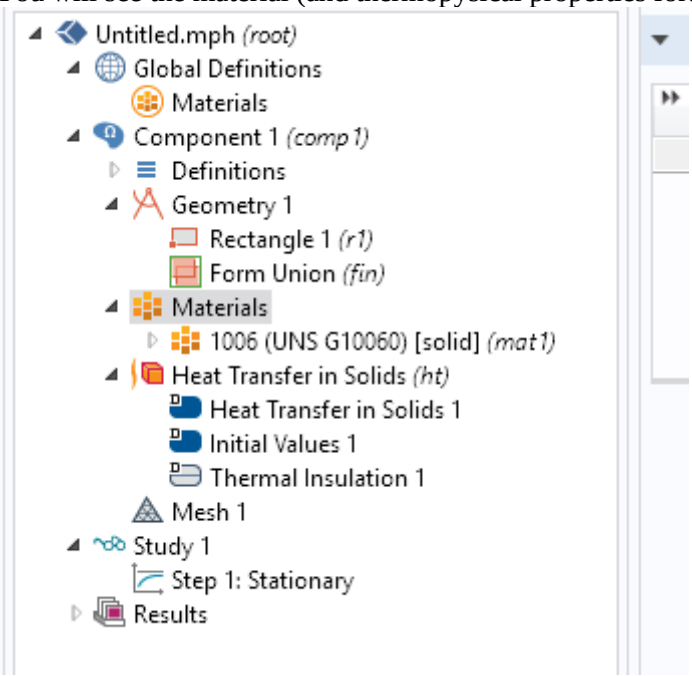
Zoom extent to put square at the center of your drawing screen



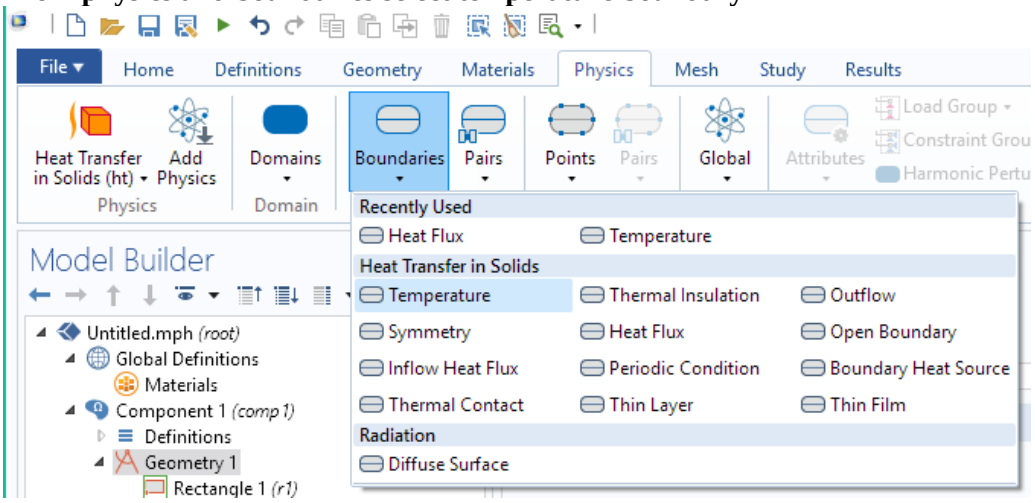
Add material by using AddMaterial section



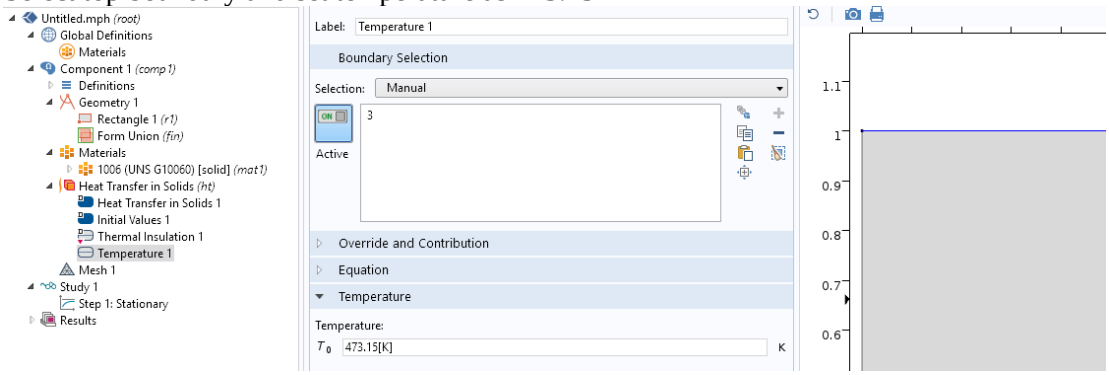
You will see the material (and thermopysical properties fort hat material) added into input section



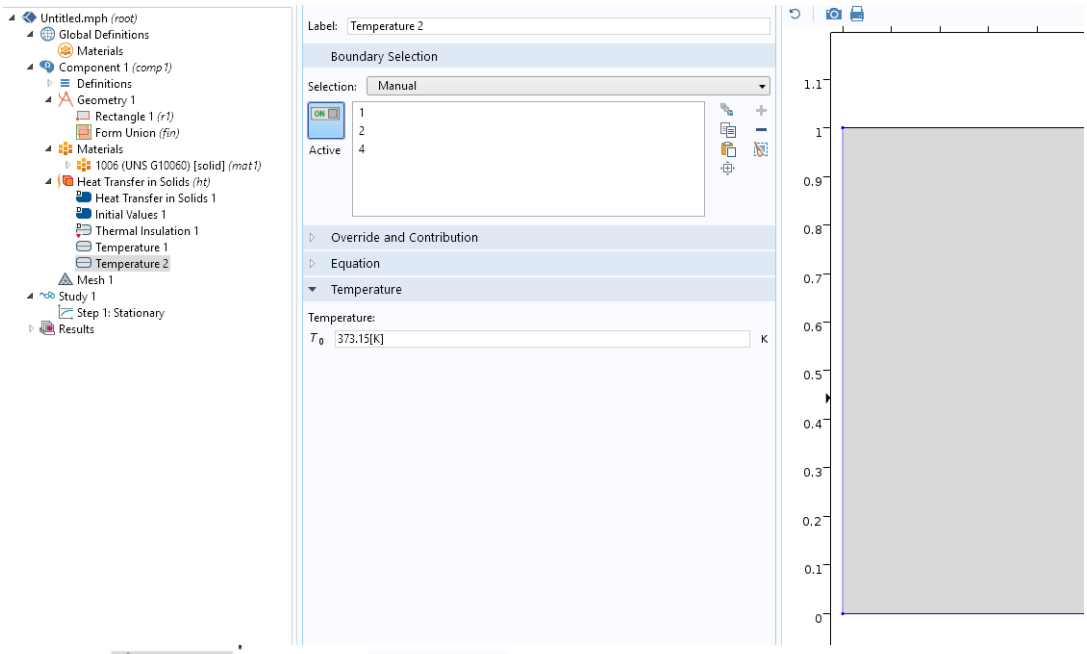
From physics and boundaries select temperature boundary



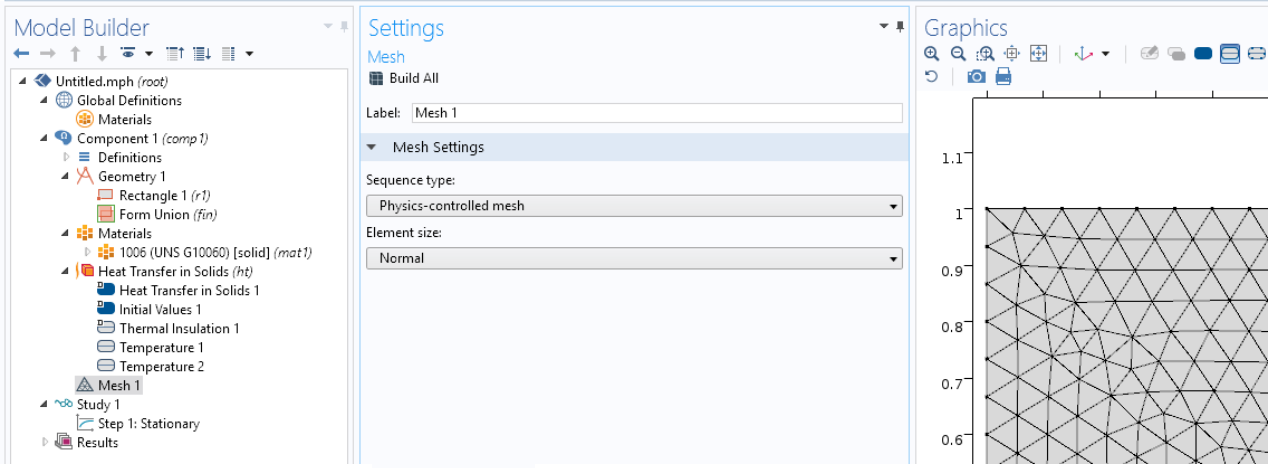
Select top boundary and set temperature as 473.15 K



Repeat the same process (add a new temperature profile select the other surfaces) and select temperature of other surfaces as 373.15 K

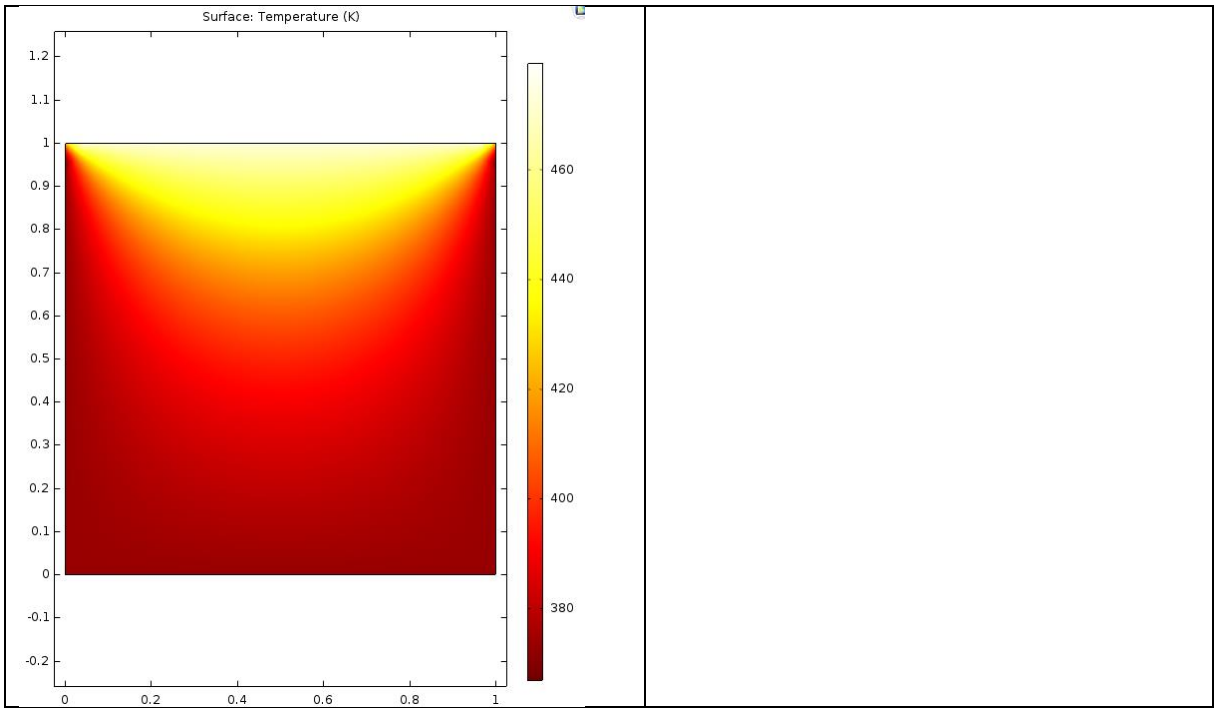


Select Mesh 1 and push Build All



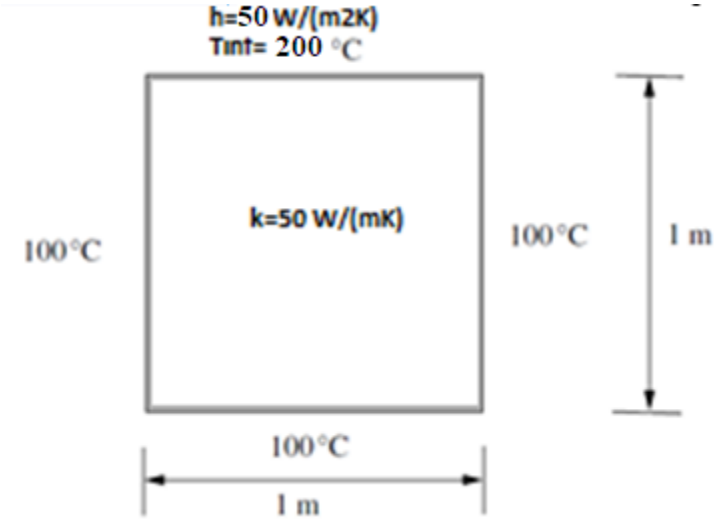
Select Study 1 and push Compute

Problem solved and temperature profile is given



EXAMPLE 2: NON-HOMOGENEOUS, TIME INDEPENDENT SOLUTION; CARTESIAN COORDINATES, FINITE ELEMENT SOLUTION

A second problem: In this one, three side temperatures specified as 100 C, and in the third side convective heat transfer is existed.

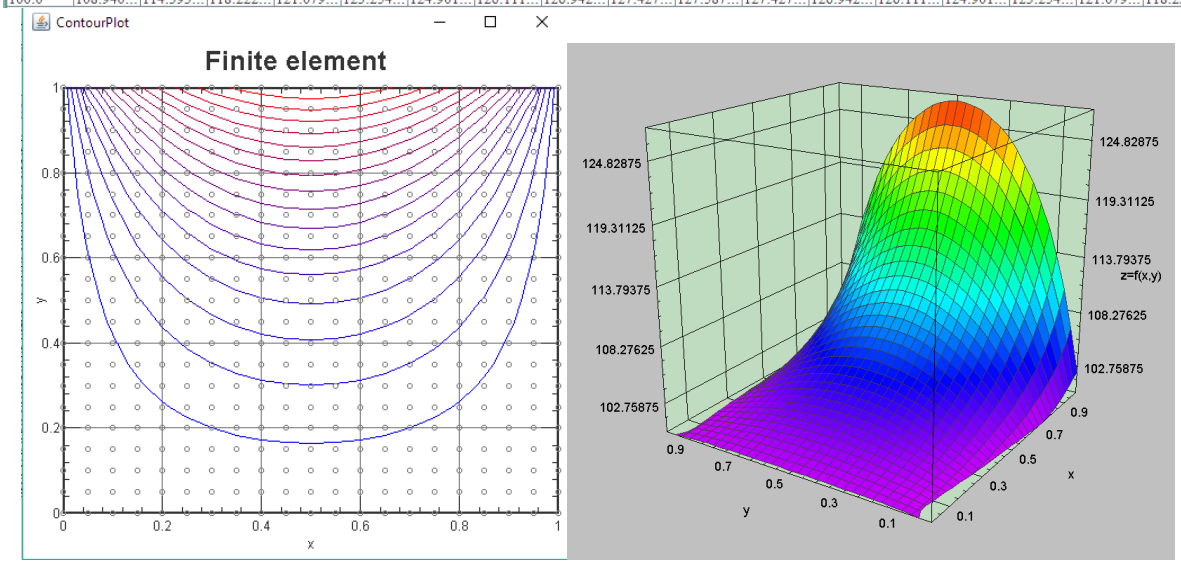



```
 public static void main(String arg[]) //throws RemoteException, VisADException
{double L=1.0;
 double W=1.0;
 int N=20,M=20; //Number of division in x and y side
 double hi=50.0;
 triangular_grid tg=new triangular_grid(L,W,M,N);
 int cond0=0;//x==0
 int cond1=1;//x==L
 int cond2=2;//y==0
 int cond3=3;//y==W
 double Ti0=100.0;
 double Ti1=100.0;
 double Ti2=100.0;
 double Ti3=200.0;
 tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
```

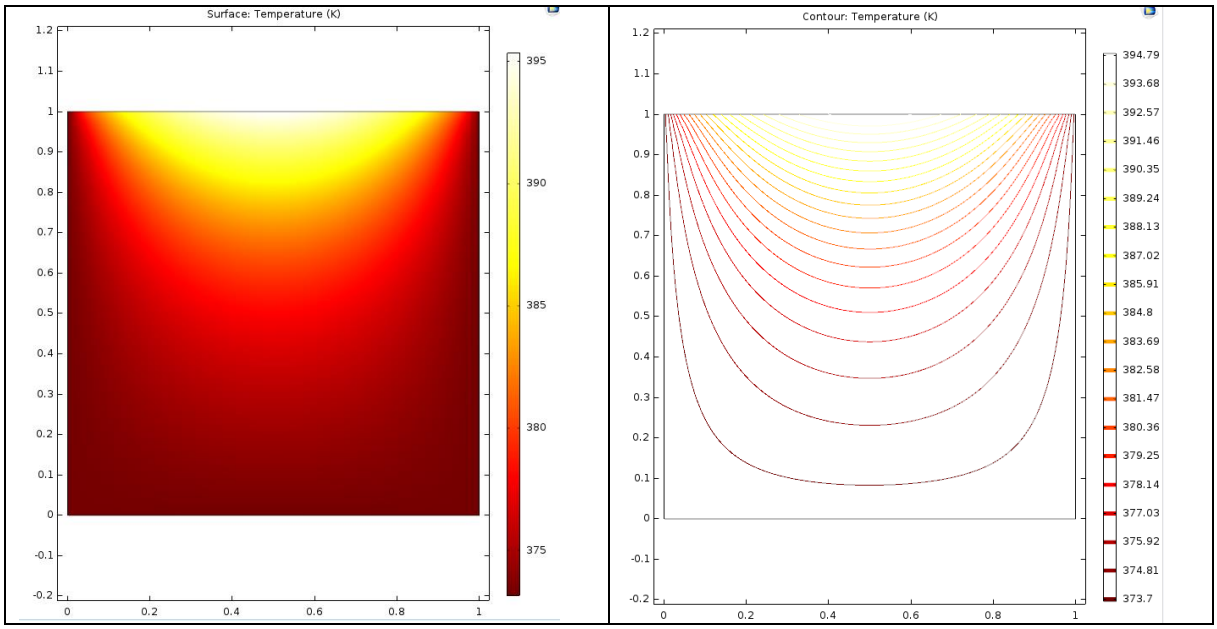
```
 tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
 tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
tg.setConvection(cond3,0.0,L,0.0,W,hi,Ti3);
 double k[]={50,50};
 tg.solve(1,k);
 Text.print(tg.T);
 tg.ContourPlot();
 tg.surfacePlot();
 }
 }
```

data output(veri çıktısı) :

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	100.065...	100.129...	100.190...	100.246...	100.296...	100.339...	100.373...	100.398...	100.413...	100.419...	100.413...	100.398...	100.373...	100.339...	100.296...	100.246...	100.190...	100.129...	100.065...	100.0
100.0	100.133...	100.263...	100.386...	100.499...	100.601...	100.687...	100.756...	100.807...	100.837...	100.848...	100.837...	100.807...	100.756...	100.687...	100.601...	100.499...	100.386...	100.263...	100.133...	100.0
100.0	100.203...	100.402...	100.591...	100.765...	100.920...	101.051...	101.157...	101.235...	101.282...	101.298...	101.282...	101.235...	101.157...	101.051...	100.920...	100.765...	100.591...	100.402...	100.203...	100.0
100.0	100.279...	100.552...	100.811...	101.049...	101.262...	101.442...	101.587...	101.693...	101.758...	101.780...	101.758...	101.693...	101.587...	101.442...	101.262...	101.049...	100.811...	100.552...	100.279...	100.0
100.0	100.362...	100.716...	101.051...	101.360...	101.635...	101.869...	102.056...	102.193...	102.277...	102.305...	102.277...	102.193...	102.056...	101.869...	101.635...	101.360...	101.051...	100.716...	100.362...	100.0
100.0	100.455...	100.898...	101.319...	101.705...	102.049...	102.341...	102.576...	102.747...	102.851...	102.886...	102.851...	102.747...	102.576...	102.341...	102.049...	101.705...	101.319...	100.898...	100.455...	100.0
100.0	100.559...	101.103...	101.619...	102.094...	102.514...	102.872...	103.158...	103.367...	103.494...	103.537...	103.494...	103.367...	103.158...	102.872...	102.514...	102.094...	101.619...	101.103...	100.559...	100.0
100.0	100.678...	101.337...	101.962...	102.535...	103.043...	103.474...	103.818...	104.070...	104.222...	104.273...	104.222...	104.070...	103.818...	103.474...	103.043...	102.535...	101.962...	101.337...	100.678...	100.0
100.0	100.814...	101.607...	102.356...	103.042...	103.648...	104.162...	104.572...	104.870...	105.051...	105.112...	105.051...	104.870...	104.572...	104.162...	103.648...	103.042...	102.356...	101.607...	100.814...	100.0
100.0	100.974...	101.920...	102.812...	103.628...	104.347...	104.954...	105.437...	105.788...	106.001...	106.072...	106.001...	105.788...	105.437...	104.954...	104.347...	103.628...	102.812...	101.920...	100.974...	100.0
100.0	101.160...	102.287...	103.346...	104.310...	105.158...	105.870...	106.435...	106.844...	107.092...	107.175...	107.092...	106.844...	106.435...	105.870...	105.158...	104.310...	103.346...	102.287...	101.160...	100.0
100.0	101.382...	102.720...	103.974...	105.110...	106.103...	106.933...	107.589...	108.062...	108.348...	108.443...	108.348...	108.062...	107.589...	106.933...	106.103...	105.110...	103.974...	102.720...	101.382...	100.0
100.0	101.648...	103.238...	104.720...	106.054...	107.210...	108.171...	108.926...	109.467...	109.793...	109.902...	109.793...	109.467...	108.926...	108.171...	107.210...	106.054...	104.720...	103.238...	101.648...	100.0
100.0	101.972...	103.865...	105.615...	107.174...	108.513...	109.616...	110.475...	111.089...	111.456...	111.579...	111.456...	111.089...	110.475...	109.616...	108.513...	107.174...	105.615...	103.865...	101.972...	100.0
100.0	102.374...	104.636...	106.700...	108.514...	110.052...	111.303...	112.270...	112.956...	113.365...	113.501...	113.365...	112.956...	112.270...	111.303...	110.052...	108.514...	106.700...	104.636...	102.374...	100.0
100.0	102.888...	105.604...	108.036...	110.131...	111.876...	113.277...	114.346...	115.099...	115.546...	115.694...	115.546...	115.099...	114.346...	113.277...	111.876...	110.131...	108.036...	105.604...	102.888...	100.0
100.0	103.575...	106.857...	109.708...	112.098...	114.044...	115.581...	116.740...	117.549...	118.026...	118.184...	118.026...	117.549...	116.740...	115.581...	114.044...	112.098...	109.708...	106.857...	103.575...	100.0
100.0	104.554...	108.541...	111.842...	114.508...	116.623...	118.262...	119.483...	120.329...	120.826...	120.990...	120.826...	120.329...	119.483...	118.262...	116.623...	114.508...	111.842...	108.541...	104.554...	100.0
100.0	106.101...	110.912...	114.611...	117.469...	119.677...	121.361...	122.603...	123.458...	123.959...	124.124...	123.959...	123.458...	122.603...	121.361...	119.677...	117.469...	114.611...	110.912...	106.101...	100.0
100.0	108.940...	114.393...	118.222...	121.079...	123.254...	124.901...	126.111...	126.942...	127.427...	127.587...	127.427...	126.942...	126.111...	124.901...	123.254...	121.079...	118.222...	114.393...	108.940...	100.0

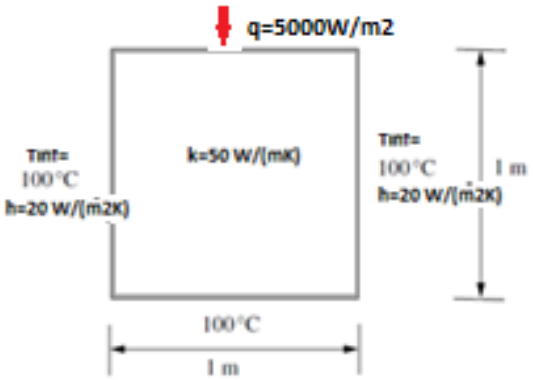


The same problem solved by comsol multiphysics finite element package:



EXAMPLE 3: NON-HOMOGENEOUS, TIME INDEPENDENT SOLUTION; CARTESIAN COORDINATES, FINITE ELEMENT SOLUTION

A third problem: In this,one side temperatures specified as 100 C, and in second and third side convective heat transfer is existed. In the fourth side heat flux is given

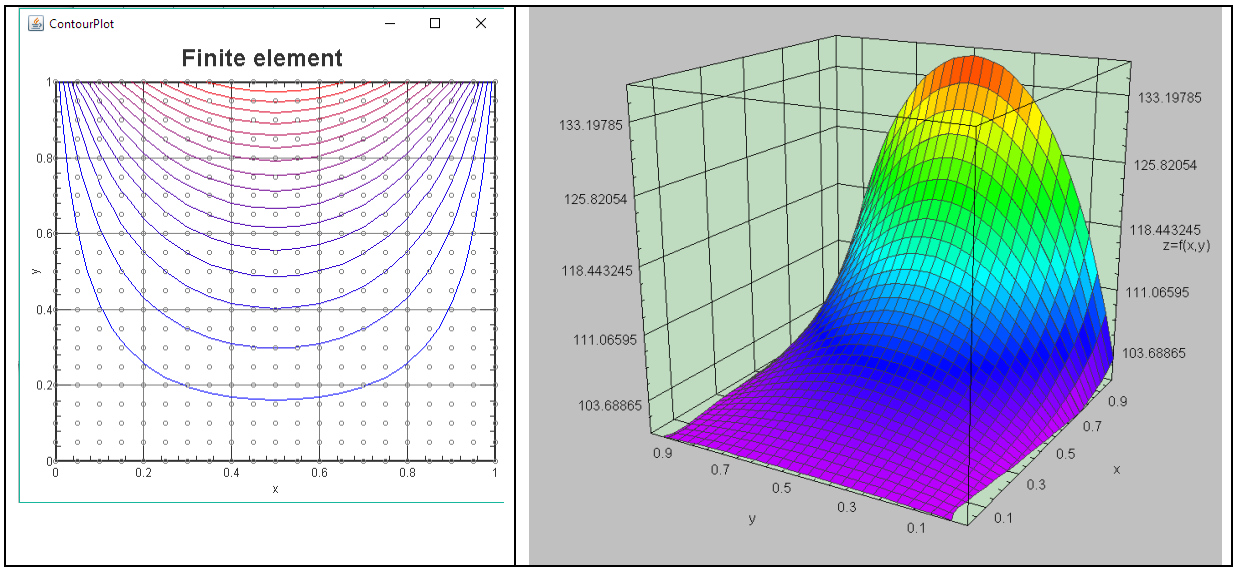



```
 public static void main(String arg[]) //throws RemoteException, VisADException
{double L=1.0;
 double W=1.0;
 int N=20,M=20;
 double hi=50.0;
 triangular_grid tg=new triangular_grid(L,W,M,N);
 int cond0=0;//x==0
 int cond1=1;//x==L
 int cond2=2;//y==0
 int cond3=3;//y==W
 double Ti0=100.0;
 double Ti1=100.0;
 double Ti2=100.0;
  double qi=-5000.0; //W/m2
 tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
 tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
 tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
 tg.setHeatFlux(cond3,0.0,L,0.0,W,qi);
```

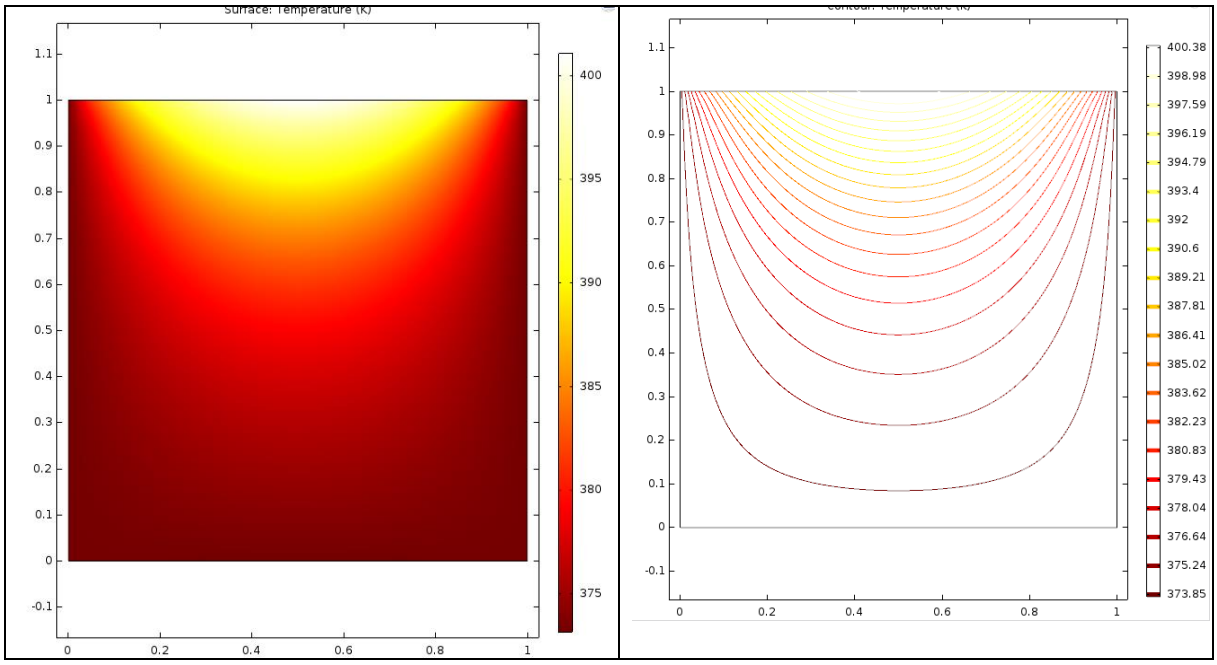
```
 double k[]={50,50};
 tg.solve(1,k);
 Text.print(tg.T);
 tg.ContourPlot();
 tg.surfacePlot();
}
```

data output(veri çıktısı) :

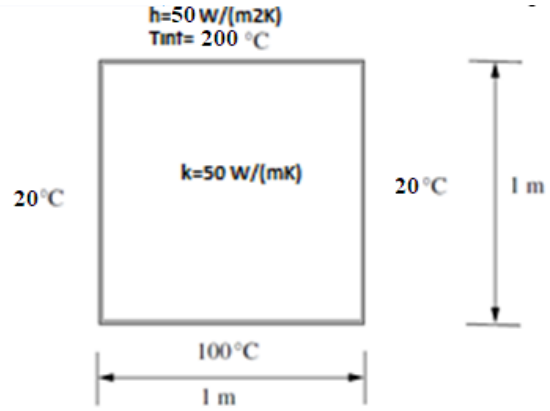
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	100.0864...	100.1707...	100.2508...	100.3245...	100.3903...	100.4463...	100.4914...	100.5243...	100.5444...	100.5511...	100.5444...	100.5243...	100.4914...	100.4463...	100.3903...	100.3245...	100.2508...	100.1707...	100.0864...	100.0
100.0	100.1751...	100.3458...	100.5078...	100.6572...	100.7902...	100.9037...	100.9949...	101.0615...	101.1022...	101.1158...	101.1022...	101.0615...	100.9949...	100.9037...	100.7902...	100.6572...	100.5078...	100.3458...	100.1751...	100.0
100.0	100.2681...	100.5295...	100.7776...	101.0062...	101.2098...	101.3834...	101.5228...	101.6248...	101.6869...	101.7078...	101.6869...	101.6248...	101.5228...	101.3834...	101.2098...	101.0062...	100.7776...	100.5295...	100.2681...	100.0
100.0	100.3678...	100.7264...	101.0667...	101.3803...	101.6594...	101.8972...	102.0882...	102.2279...	102.3130...	102.3416...	102.3130...	102.2279...	102.0882...	101.8972...	101.6594...	101.3803...	101.0667...	100.7264...	100.3678...	100.0
100.0	100.4769...	100.9417...	101.3827...	101.7888...	102.1501...	102.4579...	102.7050...	102.8856...	102.9955...	103.0325...	102.9955...	102.8856...	102.7050...	102.4579...	102.1501...	101.7888...	101.3827...	100.9417...	100.4769...	100.0
100.0	100.5981...	101.1808...	101.7335...	102.2422...	102.6944...	103.0794...	103.3882...	103.6138...	103.7512...	103.7973...	103.7512...	103.6138...	103.3882...	103.0794...	102.6944...	102.2422...	101.7335...	101.1808...	100.5981...	100.0
100.0	100.7346...	101.4501...	102.1283...	102.7519...	103.3059...	103.7771...	104.1546...	104.4303...	104.5980...	104.6543...	104.5980...	104.4303...	104.1546...	103.7771...	103.3059...	102.7519...	102.1283...	101.4501...	100.7346...	100.0
100.0	100.8901...	101.7568...	102.5775...	103.3314...	104.0002...	104.5683...	105.0230...	105.3546...	105.5563...	105.6239...	105.5563...	105.3546...	105.0230...	104.5683...	104.0002...	103.3314...	102.5775...	101.7568...	100.8901...	100.0
100.0	101.0691...	102.1095...	103.0935...	103.9960...	104.7953...	105.4730...	106.0145...	106.4089...	106.6485...	106.7289...	106.6485...	106.4089...	106.0145...	105.4730...	104.7953...	103.9960...	103.0935...	102.1095...	101.0691...	100.0
100.0	101.2770...	102.5185...	103.6909...	104.7639...	105.7119...	106.5138...	107.1532...	107.6180...	107.9001...	107.9946...	107.9001...	107.6180...	107.1532...	106.5138...	105.7119...	104.7639...	103.6909...	102.5185...	101.2770...	100.0
100.0	101.5204...	102.9967...	104.3876...	105.6569...	106.7746...	107.7170...	108.4664...	109.0098...	109.3391...	109.4494...	109.3391...	109.0098...	108.4664...	107.7170...	106.7746...	105.6569...	104.3876...	102.9967...	101.5204...	100.0
100.0	101.8080...	103.5603...	105.2060...	106.7016...	108.0127...	109.1134...	109.9854...	110.6159...	110.9972...	111.1247...	110.9972...	110.6159...	109.9854...	109.1134...	108.0127...	106.7016...	105.2060...	103.5603...	101.8080...	100.0
100.0	102.1512...	104.2307...	106.1744...	107.9306...	109.4610...	110.7386...	111.7457...	112.4713...	112.9089...	113.0551...	112.9089...	112.4713...	111.7457...	110.7386...	109.4610...	107.9306...	106.1744...	104.2307...	102.1512...	100.0
100.0	102.5662...	105.0368...	107.3302...	109.3856...	111.1621...	112.6343...	113.7877...	114.6148...	115.1120...	115.2778...	115.1120...	114.6148...	113.7877...	112.6343...	111.1621...	109.3856...	107.3302...	105.0368...	102.5662...	100.0
100.0	103.0769...	106.0200...	108.7240...	111.1196...	113.1675...	114.8487...	116.1559...	117.0881...	117.6464...	117.8323...	117.6464...	117.0881...	116.1559...	114.8487...	113.1675...	111.1196...	108.7240...	106.0200...	103.0769...	100.0
100.0	103.7214...	107.2423...	110.4262...	113.2013...	115.5397...	117.4370...	118.8992...	119.9353...	120.5533...	120.7587...	120.5533...	119.9353...	118.8992...	117.4370...	115.5397...	113.2013...	110.4262...	107.2423...	103.7214...	100.0
100.0	104.5663...	108.8018...	112.5371...	115.7197...	118.3530...	120.4605...	122.0687...	123.2005...	123.8728...	124.0958...	123.8728...	123.2005...	122.0687...	120.4605...	118.3530...	115.7197...	112.5371...	108.8018...	104.5663...	100.0
100.0	105.7419...	110.8615...	115.2008...	118.7873...	121.6921...	123.9832...	125.7145...	126.9252...	127.6416...	127.8788...	127.6416...	126.9252...	125.7145...	123.9832...	121.6921...	118.7873...	115.2008...	110.8615...	105.7419...	100.0
100.0	107.5401...	113.7015...	118.6171...	122.5366...	125.6449...	128.0658...	129.8809...	131.1442...	131.8897...	132.1361...	131.8897...	131.1442...	129.8809...	128.0658...	125.6449...	122.5366...	118.6171...	113.7015...	107.5401...	100.0
100.0	110.7169...	117.7874...	123.0297...	127.0970...	130.2852...	132.7540...	134.5992...	135.8811...	136.6367...	136.8864...	136.6367...	135.8811...	134.5992...	132.7540...	130.2852...	127.0970...	123.0297...	117.7874...	110.7169...	100.0



The same problem solved by comsol multiphysics finite element package:



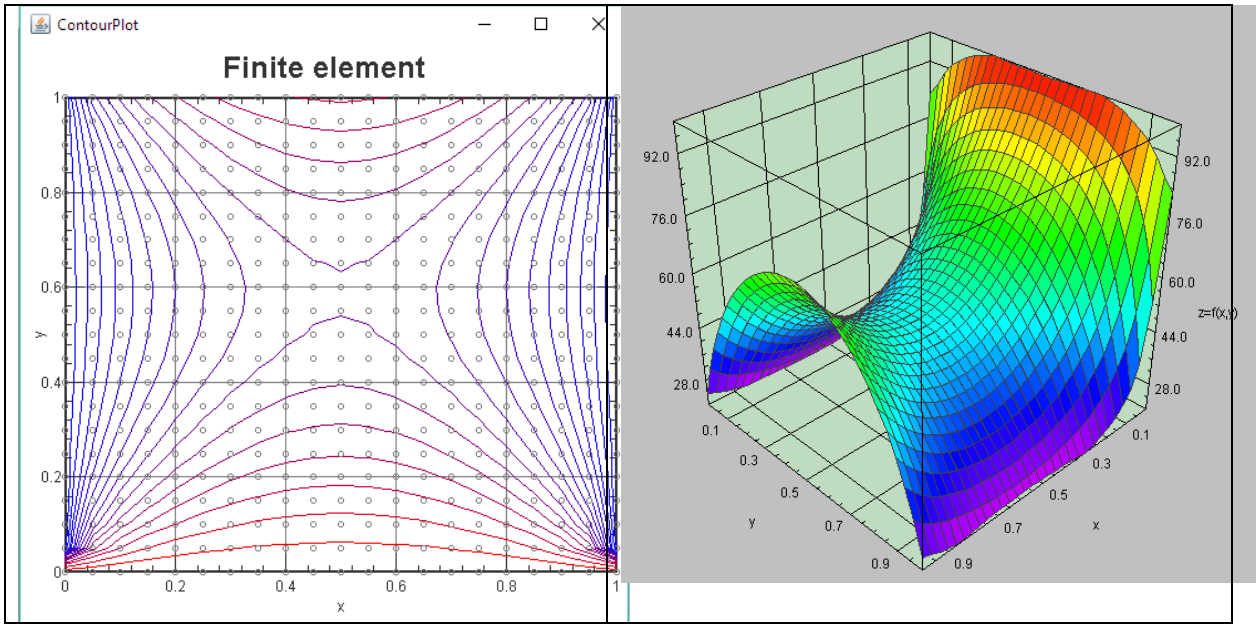
EXAMPLE 4: NON-HOMOGENEOUS, TIME INDEPENDENT SOLUTION; CARTESIAN COORDINATES, FINITE ELEMENT SOLUTION



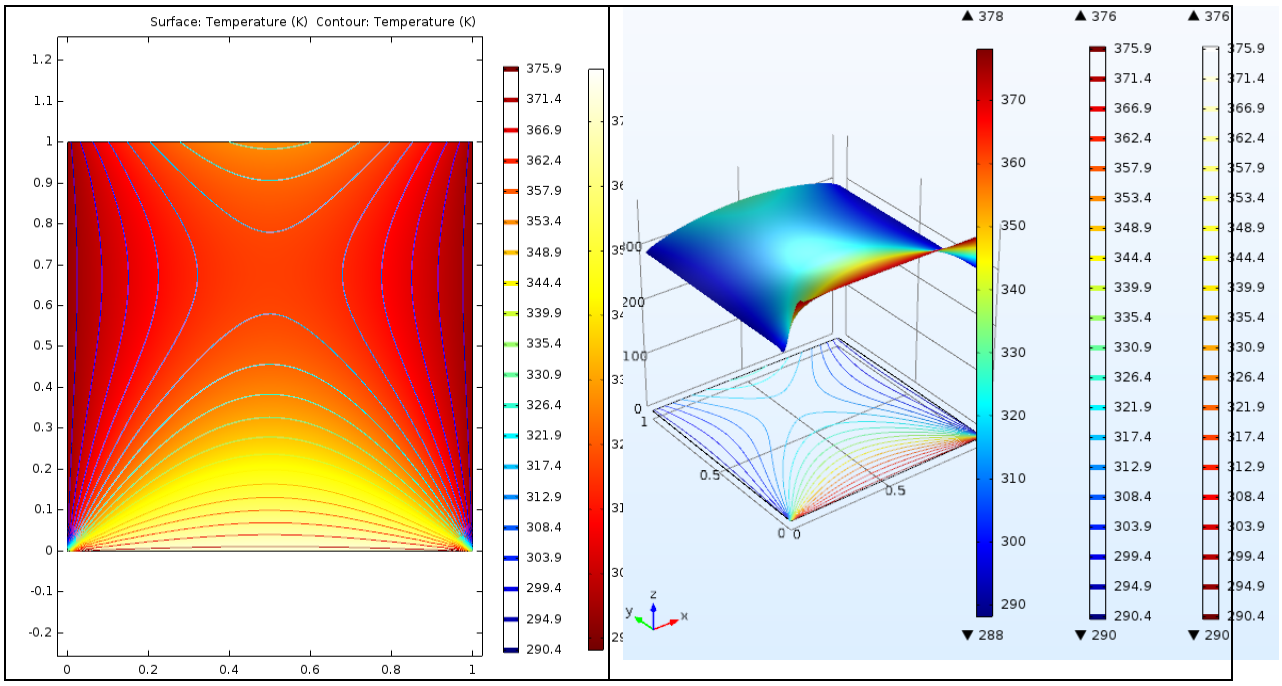

```
public static void main(String arg[]) //throws RemoteException, VisADException
{double L=1.0;
 double W=1.0;
 int N=20,M=20;
 double hi=50.0;
 triangular_grid tg=new triangular_grid(L,W,M,N);
 int cond0=0;//x==0
 int cond1=1;//x==L
 int cond2=2;//y==0
 int cond3=3;//y==W
 double Ti0=20.0;
 double Ti1=20.0;
 double Ti2=100.0;
 double Ti3=200.0;
 tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
 tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
 tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
 tg.setConvection(cond3,0.0,L,0.0,W,hi,Ti3);
 double k[]={50,50};
 tg.solve(1,k);
 Text.print(tg.T);
 tg.ContourPlot();
 tg.surfacePlot();
 }
 }
```

data output(veri çıktısı) :

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
20.0	59.9132...	75.6319...	82.9615...	86.9052...	89.2586...	90.7537...	91.7246...	92.3372...	92.6772...	92.7864...	92.6772...	92.3372...	91.7246...	90.7537...	89.2586...	86.9052...	82.9615...	75.6319...	59.9132...	20.0
20.0	44.0210...	59.6531...	69.3089...	75.4006...	79.3756...	82.0318...	83.8074...	84.9469...	85.5854...	85.7913...	85.5854...	84.9469...	83.8074...	82.0318...	79.3756...	75.4006...	69.3089...	59.6531...	44.0210...	20.0
20.0	36.5178...	49.6506...	59.2204...	66.0126...	70.8116...	74.1903...	76.5263...	78.0575...	78.9263...	79.2080...	78.9263...	78.0575...	76.5263...	74.1903...	70.8116...	66.0126...	59.2204...	49.6506...	36.5178...	20.0
20.0	32.3996...	43.2112...	51.9096...	58.6177...	63.6680...	67.3914...	70.0501...	71.8306...	72.8542...	73.1881...	72.8542...	71.8306...	70.0501...	67.3914...	63.6680...	58.6177...	51.9096...	43.2112...	32.3996...	20.0
20.0	29.8695...	38.8850...	46.5889...	52.8807...	57.8514...	61.6572...	64.4521...	66.3608...	67.4716...	67.8362...	67.4716...	66.3608...	64.4521...	61.6572...	57.8514...	52.8807...	46.5889...	38.8850...	29.8695...	20.0
20.0	28.1935...	35.8704...	42.6805...	48.4646...	53.1997...	56.9339...	59.7401...	61.6889...	62.8352...	63.2135...	62.8352...	61.6889...	59.7401...	56.9339...	53.1997...	48.4646...	42.6805...	35.8704...	28.1935...	20.0
20.0	27.0341...	33.7226...	39.7980...	45.0976...	49.5488...	53.1387...	55.8857...	57.8193...	58.9669...	59.3472...	58.9669...	57.8193...	55.8857...	53.1387...	49.5488...	45.0976...	39.7980...	33.7226...	27.0341...	20.0
20.0	26.2202...	32.1880...	37.6913...	42.5790...	46.7592...	50.1863...	52.8448...	54.7357...	55.8659...	56.2417...	55.8659...	54.7357...	52.8448...	50.1863...	46.7592...	42.5790...	37.6913...	32.1880...	26.2202...	20.0
20.0	25.6588...	31.1177...	36.2003...	40.7680...	44.7227...	48.0024...	50.5716...	52.4129...	53.5191...	53.8879...	53.5191...	52.4129...	50.5716...	48.0024...	44.7227...	40.7680...	36.2003...	31.1177...	25.6588...	20.0
20.0	25.2972...	30.4238...	35.2243...	39.5698...	43.3613...	46.5289...	49.0263...	50.8251...	51.9096...	52.2719...	51.9096...	50.8251...	49.0263...	46.5289...	43.3613...	39.5698...	35.2243...	30.4238...	25.2972...	20.0
20.0	25.1064...	30.0559...	34.7034...	38.9255...	42.6239...	45.7256...	48.1794...	49.9518...	51.0224...	51.3803...	51.0224...	49.9518...	48.1794...	45.7256...	42.6239...	38.9255...	34.7034...	30.0559...	25.1064...	20.0
20.0	25.0727...	29.9898...	34.6079...	38.8049...	42.4833...	45.5702...	48.0141...	49.7803...	50.8477...	51.2047...	50.8477...	49.7803...	48.0141...	45.5702...	42.4833...	38.8049...	34.6079...	29.9898...	25.0727...	20.0
20.0	25.1946...	30.2227...	34.9336...	39.2028...	42.9341...	46.0579...	48.5262...	50.3078...	51.3834...	51.7430...	51.3834...	50.3078...	48.5262...	46.0579...	42.9341...	39.2028...	34.9336...	30.2227...	25.1946...	20.0
20.0	25.4832...	30.7728...	35.7012...	40.1385...	43.9924...	47.2011...	49.7252...	51.5411...	52.6352...	53.0005...	52.6352...	51.5411...	49.7252...	47.2011...	43.9924...	40.1385...	35.7012...	30.7728...	25.4832...	20.0
20.0	25.9653...	31.6842...	36.9597...	41.6576...	45.6959...	49.0287...	51.6325...	53.4963...	54.6156...	54.9888...	54.6156...	53.4963...	51.6325...	49.0287...	45.6959...	41.6576...	36.9597...	31.6842...	25.9653...	20.0
20.0	26.6939...	33.0390...	38.7959...	43.8363...	48.1048...	51.5854...	54.2799...	56.1961...	57.3421...	57.7234...	57.3421...	56.1961...	54.2799...	51.5854...	48.1048...	43.8363...	38.7959...	33.0390...	26.6939...	20.0
20.0	27.7715...	34.9818...	41.3487...	46.7869...	51.3016...	54.9281...	57.7055...	59.6661...	60.8332...	61.2208...	60.8332...	59.6661...	57.7055...	54.9281...	51.3016...	46.7869...	41.3487...	34.9818...	27.7715...	20.0
20.0	29.4102...	37.7681...	44.8303...	50.6610...	55.3864...	59.1201...	61.9478...	63.9295...	65.1040...	65.4931...	65.1040...	63.9295...	61.9478...	59.1201...	55.3864...	50.6610...	44.8303...	37.7681...	29.4102...	20.0
20.0	32.1013...	41.8501...	49.5435...	55.6403...	60.4631...	64.2182...	67.0362...	69.0000...	70.1601...	70.5439...	70.1601...	69.0000...	67.0362...	64.2182...	60.4631...	55.6403...	49.5435...	41.8501...	32.1013...	20.0
20.0	37.1451...	47.9873...	55.8532...	61.8935...	66.6075...	70.2532...	72.9790...	74.8742...	75.9924...	76.3622...	75.9924...	74.8742...	72.9790...	70.2532...	66.6075...	61.8935...	55.8532...	47.9873...	37.1451...	20.0



The same problem solved by comsol multiphysics finite element package:



Now **time dependent** general heat conduction equation can be investigated as a finite element equation

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+g-\rho C_p\frac{\partial T}{\partial t}=0$$

Boundary conditions

T=T_b on surface S_1

$k_x\frac{\partial T}{\partial x}\vec{l}+k_y\frac{\partial T}{\partial y}\vec{m}+q=0$ on surface S_2

$k_x\frac{\partial T}{\partial x}\vec{l}+k_y\frac{\partial T}{\partial y}\vec{m}+h(T-T_a)=0$ on surface S_3

T=T_0 on x=0

Galarkin integration method will be applied to our differential equation . The method requires that the following expression will be satisfied

$\int_\Omega w_k L(T)d\Omega=0$ where Ω is the volume of the element.

$\int_\Omega N_k\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+g-\rho C_p\frac{\partial T}{\partial t}\right]d\Omega=0$

Note that for finite element T=[N]{T} $\frac{\partial T}{\partial t}=[N]\frac{\partial\{T\}}{\partial t}$ and integral can be converted by partial integration rule

$$\int udv=uv-\int vdu$$

$$\int_\Omega \alpha_1\left[\frac{\partial\alpha_2}{\partial x}\right]d\Omega=\int_S \alpha_1\alpha_2 dS-\int_\Omega\left[\frac{\partial\alpha_1}{\partial x}\right]\alpha_2 d\Omega$$

$\int_\Omega N_k\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)\right]d\Omega=\int_S N_k\left(k_x\frac{\partial T}{\partial x}\right)dS-\int_\Omega\frac{\partial N_k}{\partial x}(k_x)\frac{\partial N_m}{\partial x}Td\Omega$

By applying these equation becomes

$$-\int_\Omega\left[k_x\frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x}+k_y\frac{\partial N_k}{\partial y}\frac{\partial N_m}{\partial y}\right]\{T_m\}d\Omega-\int_\Omega\rho C_p N_k N_m\left\{\frac{\partial T_m}{\partial t}\right\}d\Omega+\int_\Omega gN_k d\Omega+\int_S N_k\left(k_x\frac{\partial T}{\partial x}+k_y\frac{\partial T}{\partial y}\right)dS=0$$

From boundary conditions:

$$\int_S N_k\left(k_x\frac{\partial T}{\partial x}+k_y\frac{\partial T}{\partial y}\right)dS=-\int_S N_k\,(q+h(T-T_a)dS=-\int_S N_k\,qdS-\int_S N_k h\,N_m\{T_m\}dS+\int_S N_k h\,T_a dS$$

$$-\int_\Omega\left[k_x\frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x}+k_y\frac{\partial N_k}{\partial y}\frac{\partial N_m}{\partial y}\right]\{T_m\}d\Omega-\int_\Omega\rho C_p N_k N_m\left\{\frac{\partial T_m}{\partial t}\right\}d\Omega+\int_\Omega gN_k d\Omega-\int_S N_k\,qdS-\int_S N_k h\,N_m\{T_m\}dS+\int_S N_k h\,T_a dS=0$$

If this equation is written in matrix form :

$[C]\left\{\frac{dT}{dt}\right\}+[K]\{T\}=\{f\}$ where

$[C]=\int_\Omega\rho C_p N_k N_m d\Omega$

$[K]=-\int_\Omega\left[k_x\frac{\partial N_k}{\partial x}\frac{\partial N_m}{\partial x}+k_y\frac{\partial N_k}{\partial y}\frac{\partial N_m}{\partial y}\right]d\Omega+\int_S N_k h\,N_m dS$

$\{f\}=\int_\Omega gN_k d\Omega-\int_S N_k\,qdS+\int_S N_k h\,T_a dS$

$[C]=\int_\Omega\rho C_p[N]^T[N]d\Omega$

$[K]=-\int_\Omega[B]^T[D][B]d\Omega+\int_S h[N]^T[N]\,dS$ where

$[D]=\begin{bmatrix}k_x & 0\\ 0 & k_y\end{bmatrix}$

$\{f\}=\int_\Omega g[N]^T d\Omega-\int_S[N]^T qdS+\int_S[N]^T h\,T_a dS$

Line integrals

$\int_l N_i^a N_j^b dl=\frac{a!b!}{(a+b+1)!}l$ $\int_l N_i^2 N_j^0 dl=\frac{2!0!}{(2+o+1)!}l=\frac{2l}{6}$ $\int_l N_i N_j dl=\frac{1!1!}{(1+1+1)!}l=\frac{l}{6}$

Area integrals

$\int_A N_i^a N_j^b N_k^c dA=\frac{a!b!c!}{(a+b+c+2)!}2A$

$\int_A N_i^2 N_j^0 N_k^0 dA=\frac{2!0!0!}{(2+0+0+2)!}2A=\frac{2A}{12}=\frac{A}{6}$ $\int_A N_i^1 N_j^1 N_k^0 dA=\frac{1!1!0!}{(1+1+0+2)!}2A=\frac{A}{12}$

The basic equation is still a differential equation in time, therefore should be solved by using finite difference method.

$[C]\left\{\frac{\{T\}^{p+1}-\{T\}^p}{\Delta t}\right\}+\phi[K]\{T\}^p+(1-\phi)[K]\{T\}^{p+1}=\{f\}$

where $t = p\Delta t$ and ϕ is implicit explicit-scheme. If ϕ=1 It is fully explicit scheme, If ϕ=0, it is fully implicit scheme. ϕ=0.5 is called Crank-Nicolson method. Note that properties k,h,g, ρ , C_p and q can be changed in each step as a function of temperature, but at each single step calculations, they can be considered constant. When equation is arranged for p+1 time step (unknown) temperatures, it becomes

$$([C] + (1-\phi)\Delta t[K])\{T\}^{p+1} = ([C] - \phi\Delta t[K])\{T\}^{p} + \Delta t\{f\}$$

Matrix terms can be opened for **two dimensional triangular element** as:

$$[C] = \frac{\rho C_p A t}{12}\begin{bmatrix}4 & 1 & 1\\ 1 & 4 & 1\\ 1 & 1 & 4\end{bmatrix}$$

$$[K] = \frac{t}{4A}\left\{k_x\begin{bmatrix}b_i^2 & b_ib_j & b_ib_k\\ b_ib_j & b_j^2 & b_jb_k\\ b_jb_k & b_jb_k & b_k^2\end{bmatrix} + k_y\begin{bmatrix}c_i^2 & c_ic_j & c_ic_k\\ c_ic_j & c_j^2 & c_jc_k\\ c_jc_k & c_jc_k & c_k^2\end{bmatrix}\right\} + \frac{htl_{ij}}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix} + \frac{htl_{jk}}{6}\begin{bmatrix}0 & 1 & 0\\ 0 & 2 & 1\\ 0 & 1 & 0\end{bmatrix} + \frac{htl_{ki}}{6}\begin{bmatrix}2 & 1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}$$

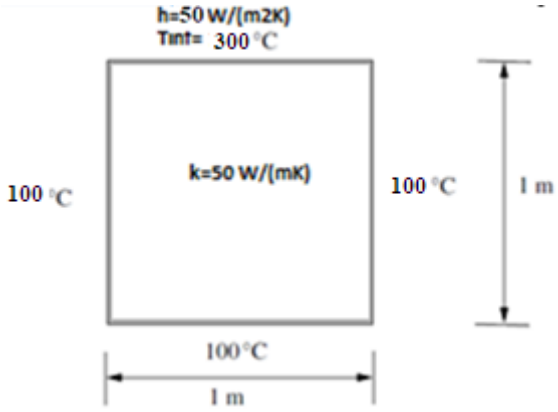
$$\{f\} = \frac{gAt}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix} - \frac{qtl_{ij}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} - \frac{qtl_{jk}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} - \frac{qtl_{ik}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix} + + \frac{hT_atl_{jk}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} + + \frac{hT_atl_{ik}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$$

Initial temperature profile will be defined and loaded into the program, and then Matrix will be solved and temperature profiles for each time step will be calculated.

EXAMPLE 5: NON-HOMOGENEOUS, TIME DEPENDENT SOLUTION; CARTESIAN COORDINATES, FINITE ELEMENT SOLUTION

Plate initial temperature is T_i=50 ℃. (t=0). The plate exposed to 100 C temperature at 3 surfaces as shown in the figure and convectve heat flowwith temperature T_∞=300 ℃ and thermal conductivity h=50 W/(m^{2}K) (t>0). Thermal conductivty of plate is k=51 W/(mK), specific heat Cp=490 J/(kgK) and density ρ=7800 kg/m^3

Calculate temperature profile as a function of time.




```
public class triangular_element2
{
public double t; //Thickness of two dimensional element
public double a[],b[],c[];
public double x[],y[];//coordinates of data points
public double xc,yc; //center point
public double A; //Area
public double N[],B[][]; //Shape factor and derivative
public double Ti[];      //Temperature nodes
public int n_point[]; //point numbers
public int n_element; //element number
public double l[];
public boolean isconv[];
public boolean isheatflux[];
public boolean isheatgen;
public double h[];
public double Tinf[];
public double qflux[];
public double qgen;
public triangular_element2(int n_elementi,int n_pointi[],double xi[],double yi[])
```

```
{x=new double[3];
y=new double[3];
Ti=new double[3];
l=new double[3];//0:lij,1:ljk,2:lik
isconv=new boolean[3];
isheatflux=new boolean[3];
n_point=new int[3];
xc=0;yc=0;
isheatgen=false;
h=new double[3];
Tinf=new double[3];
qflux=new double[3];
n_element=n_elementi;
for(int i=0;i<3;i++)
{n_point[i]=n_pointi[i];
x[i]=xi[i];y[i]=yi[i];
xc+=x[i];yc+=y[i];
isconv[i]=false;
isheatflux[i]=false;
h[i]=0.0;
Tinf[i]=0.0;
qflux[i]=0.0;
qgen=0.0;
}
xc/=3.0;
yc/=3.0;
a=new double[3];
b=new double[3];
c=new double[3];
N=new double[3];
B=new double[2][3];
a[0]=x[1]*y[2]-x[2]*y[1];
a[1]=x[2]*y[0]-x[0]*y[2];
a[2]=x[0]*y[1]-x[1]*y[0];
b[0]=y[1]-y[2];
b[1]=y[2]-y[0];
b[2]=y[0]-y[1];
c[0]=x[2]-x[1];
c[1]=x[0]-x[2];
c[2]=x[1]-x[0];
A=0.5*((x[0]*y[1]-x[1]*y[0])+(x[2]*y[0]-x[0]*y[2])+(x[1]*y[2]-x[2]*y[1]));
for(int i=0;i<3;i++)
{ B[0][i]=1.0/(2.0*A)*b[i];
B[1][i]=1.0/(2.0*A)*c[i];
}
l[0]=Math.sqrt(b[2]*b[2]+c[2]*c[2]);
l[1]=Math.sqrt(b[0]*b[0]+c[0]*c[0]);
l[2]=Math.sqrt(b[1]*b[1]+c[1]*c[1]);
}

public double[][] K(double ti,double k[])
{t=ti;
double a0=t/(4.0*A)*k[0];
double a1=t/(4.0*A)*k[1];
double b0=t*h[0]*l[0]/6.0;
double b1=t*h[1]*l[1]/6.0;
double b2=t*h[2]*l[2]/6.0;
double K1[][]={{(a0*b[0]*b[0]+a1*c[0]*c[0]+2*b0+2.0*b2),
(a0*b[0]*b[1]+a1*c[0]*c[1]+b0),
(a0*b[0]*b[2]+a1*c[0]*c[2]+b2)},
{(a0*b[1]*b[0]+a1*c[1]*c[0]+b0),
(a0*b[1]*b[1]+a1*c[1]*c[1]+2*b0+2*b1),
(a0*b[1]*b[2]+a1*c[1]*c[2]+b1)},
{(a0*b[2]*b[0]+a1*c[2]*c[0]+b2),
(a0*b[2]*b[1]+a1*c[2]*c[1]+b1),
(a0*b[2]*b[2]+a1*c[2]*c[2]+2*b1+2*b2)}
};
return K1;
}
public double[] f(double ti,double k[])
{
t=ti;
double a0=qgen*A*t/3.0;
double b0=-qflux[0]*t*l[0]/2.0;
double b1=-qflux[1]*t*l[1]/2.0;
double b2=-qflux[2]*t*l[2]/2.0;
```

```
double c0= h[0]* Tinf[0]*t*l[0]/2.0;
double c1= h[1]* Tinf[1]*t*l[1]/2.0;
double c2= h[2]* Tinf[2]*t*l[2]/2.0;
double f1[]={(a0+b0+b2+c0+c2),(a0+b0+b1+c0+c1),(a0+b1+b2+c1+c2)};
return f1;
}
public double[][] C(double ti,double Cp,double ro)
{   t=ti;
            double x=ro*Cp*A*t/12.0;
            double cc[][]={{4.0*x,x,x},{x,4.0*x,x},{x,x,4.0*x}};
   return cc;
}
public int setI(int cond,double L0,double L,double W0,double W)
{
int i=100;
if(cond==0) //x=0
{if(x[0]==L0 && x[1]==L0) i=0;
else if(x[1]==L0 && x[2]==L0) i=1;
else if(x[0]==L0 && x[2]==L0) i=2;
else i=-1;
}
else if(cond==1) //x=L
{if(x[0]==L && x[1]==L) i=0;
else if(x[1]==L && x[2]==L) i=1;
else if(x[0]==L && x[2]==L) i=2;
else i=-1;
}
else if(cond==2) //y=0
{if(y[0]==W0 && y[1]==W0) i=0;
else if(y[1]==W0 && y[2]==W0) i=1;
else if(y[0]==W0 && y[2]==W0) i=2;
else i=-1;
}
else if(cond==3) //y=W
{if(y[0]==W && y[1]==W) i=0;
else if(y[1]==W && y[2]==W) i=1;
else if(y[0]==W && y[2]==W) i=2;
else i=-1;
}
return i;
}

public void setConvection(int i, double hi, double tinfi)
{if(i>=0 && i<=2)
{
h[i]=hi;Tinf[i]=tinfi;isconv[i]=true;
}
}

public void setHeatFlux(int i, double qi)
{if(i>=0 && i<=2 ){qflux[i]=qi;isheatflux[i]=true;}     }

public void setHeatGeneration(double qgeni)
{qgen=qgeni;isheatgen=true;}
public void setN(double x,double y)
{for(int i=0;i<3;i++)
{N[i]=1.0/(2.0*A)*(a[i]+b[i]*x+c[i]*y);}
}
public double[]  q(double k)
{ double qi[]=new double[2];
for(int i=0;i<3;i++)
{qi[0]+=-k*B[0][i]*Ti[i];
qi[1]+=-k*B[1][i]*Ti[i];
}
return qi;
}
public double[]  q(double k,double Ti[])
{  //heat transfer
setT(Ti);
double qi[]=new double[2];
for(int i=0;i<3;i++)
{qi[0]+=-k*B[0][i]*Ti[i];
qi[1]+=-k*B[1][i]*Ti[i];
}
return qi;
}
```

```
public void setN(double z[])
{setN(z[0],z[1]);}
public void setT(double TT[])
{for(int i=0;i<3;i++)
{Ti[i]=TT[i];}
}
public double T(double x,double y)
{ //interpolation
double TT=0;
setN(x,y);
for(int i=0;i<3;i++)
{TT+=N[i]*Ti[i];}
return TT;
}

public double T(double T[],double x,double y)
{ //set grid point temperatures and get interpolated value
setT(T);
setN(x,y);
double TT=0;
for(int i=0;i<3;i++)
{TT+=N[i]*Ti[i];}
return TT;
}
public double T(double T[],double z[])
{ return T(T,z[0],z[1]);}
public static void main(String arg[])
{
                /*double L=1.0;
double W=1.0;
int N=16,M=16;
double hi=20.0;
triangular_grid tg=new triangular_grid(L,W,M,N);
int cond0=0;//x==0
int cond1=1;//x==L
int cond2=2;//y==0
int cond3=3;//y==W
double Ti0=100.0;
double Ti1=100.0;
double Ti2=100.0;
double Ti3=500.0;
double qi=-5000.0;;
tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
tg.setConvection(cond3,0.0,L,0.0,W,hi,Ti3);
double k[]={50,50};
tg.solve(1,k);
Text.print(tg.T);
tg.ContourPlot();
//tg.Plot3D(0.0,L,0.0,W);
//tg.GridPlot(0.0,L,0.0,W);
tg.surfacePlot();
*/
}}
```

```
 import javax.swing.*;
import static java.lang.Math.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;
public class triangular_grid2
{
public triangular_element2 te[];
public setElement se[];
public int N,M,K,K1,K2,Nm1,Mm1;
public double x[][];
public double y[][];
public double T[][];
public double TT[][][];
public double Ti[];
public double xi[];
public double yi[];
public boolean bi[];
```

```
public double dx,dy,dt,time;
int Np;
//In this version it is assumed that Cp,k, and ro is constant
public int Bx0[],BxM[],By0[],ByN[];
public double L,W;
public double min(double xi[][])
{ double min=1e200;
  for(int i=0;i<xi.length;i++)
   {for(int j=0;j<xi[0].length;j++)
     {if(xi[i][j]<min) min=xi[i][j];}
 }
 return min;
}
public double max(double xi[][])
{double max=-1e200;
 for(int i=0;i<xi.length;i++)
 {for(int j=0;j<xi[0].length;j++)
   {if(xi[i][j]<max) max=xi[i][j];}
 }
 return max;
}
public triangular_grid2(double xi[][],double yi[][],double TTi[][],double dti,double timei)
{x=xi;
 y=yi;
 T=TTi;
 N=x.length-1;
 M=x[0].length-1;
 T=new double[N+1][M+1];
 L=max(xi);
 W=max(yi);
 setElements();
 time=timei;
 dt=dti;
 Np=(int)((timei+0.00001)/dti);
 TT=new double[Np+1][N+1][N+1];
}

public void generateTriangularGrid(double Li, double Wi,int Ni, int Mi)
{int i,j;
 N=Ni;M=Mi;
 L=Li;
 W=Wi;
 x=new double[N+1][M+1];
 y=new double[N+1][M+1];
 T=new double[N+1][M+1];
 setElements();
}

public void setElements()
{ int i,j;
 dx=L/M;
 dy=W/N;
 K=M*N;
 K1=K*2;
 K2=K1;
 //grid points
 xi=new double[K2];
 yi=new double[K2];
 Ti=new double[K2];
 bi=new boolean[K2];
 se=new setElement[K2];
 for(i=0;i<K2;i++) {se[i]=new setElement();}
 te=new triangular_element2[K1];
 int k1=-1,k=0,n=0,m=0;
for(n=0;n<=N;n++)
{ for(m=0;m<=M;m++)
  {k=(N+1)*n+m;
   x[n][m]=m*dx;y[n][m]=n*dy;
   xi[k]=x[n][m];
   yi[k]=y[n][m];
   bi[k]=false;
   Ti[k]=0.0;
  }
}

 for(n=0;n<N;n++)
```

```
{for(m=0;m<M;m++)
  { k=(N+1)*n+m;
                int kp1=(N+1)*(n+1)+m;
                int kp1p1= kp1+1;
                double xi1[]={x[n][m],x[n][m+1],x[n+1][m]};
                double yi1[]={y[n][m],y[n][m+1],y[n+1][m]};

                k1++;
                int n_point1[]={k,(k+1),kp1};
                se[k].add(k1,0);
                se[k+1].add(k1,1);
                se[kp1].add(k1,2);
                te[k1]=new triangular_element2(k1,n_point1,xi1,yi1);
                double xi2[]={x[n][m+1],x[n+1][m+1],x[n+1][m]};
                double yi2[]={y[n][m+1],y[n+1][m+1],y[n+1][m]};
                k1++;
                int n_point2[]={k+1,kp1p1,kp1};
    se[k+1].add(k1,0);
                se[kp1p1].add(k1,1);
                se[kp1].add(k1,2);
                te[k1]=new triangular_element2(k1,n_point2,xi2,yi2);
  }
}
}
public triangular_grid2(double Li, double Wi,int Ni, int Mi,double TTi[][],double dti,double timei)
{generateTriangularGrid(Li,Wi,Ni,Mi);
 T=TTi;
 time=timei;
 dt=dti;
  Np=(int)((timei+0.00001)/dti);
 TT=new double[Np+1][N+1][N+1];
}
public int find_element(double z[])
{ return find_element(z[0],z[1]);}
public int find_element(double x,double y)
{double R1=Math.sqrt(x*x+y*y);
 int kmin=0;
 double min=1.0e50;
 for(int k=0;k<K1;k++)
 {   te[k].setN(x,y);
     double r0=Math.sqrt((te[k].x[0]-x)*(te[k].x[0]-x)+(te[k].y[0]-y)*(te[k].y[0]-y));
     double r1=Math.sqrt((te[k].x[1]-x)*(te[k].x[1]-x)+(te[k].y[1]-y)*(te[k].y[1]-y));
     double r2=Math.sqrt((te[k].x[2]-x)*(te[k].x[2]-x)+(te[k].y[2]-y)*(te[k].y[2]-y));
     double dr=Math.sqrt(r0*r0+r1*r1+r2*r2);
                 boolean b1=(x>=te[k].x[0]) && (x<=te[k].x[2]);
                 boolean b2=(y>=te[k].y[0]) && (y<=te[k].y[1]);
                 boolean b= b1 && b2;
     if(dr<min) {min=dr;kmin=k;}
 }
 return kmin;
}
public void setConvection(int cond,double L0,double L,double W0,double W, double hi, double tinfi)
{//cond 0: x==L0
 //cond 1: x==L
 //cond 2: y==W0
 //cond 3: y==W
 int j=0;
 for(int i=0;i<K1;i++)
 {  j=te[i].setI(cond,L0,L,W0,W);
                te[i].setConvection(j,hi,tinfi);
 }
}
public void setHeatFlux(int cond,double L0,double L,double W0,double W, double qi)
{//cond 0: x==L0
 //cond 1: x==L
 //cond 2: y==W0
 //cond 3: y==W
 int j=-1;
 for(int i=0;i<K1;i++)
 {  j=te[i].setI(cond,L0,L,W0,W);
                te[i].setHeatFlux(j,qi);
 }
}
public void setTemperature(int cond,double L0,double L,double W0,double W, double TTi)
{//cond 0: x==L0
 //cond 1: x==L
```

```
 //cond 2: y==W0
 //cond 3: y==W
 for(int i=0;i<K2;i++)
 {  if(cond==0)
    {if(xi[i]==L0) {Ti[i]=TTi;bi[i]=true;}}
    else if(cond==1)
    {if(xi[i]==L) {Ti[i]=TTi;bi[i]=true;}}
    else if(cond==2)
    {if(yi[i]==W0) {Ti[i]=TTi;bi[i]=true;}}
    else if(cond==3)
    {if(yi[i]==W) {Ti[i]=TTi;bi[i]=true;}}
 }
}
public void setHeatGeneration(int node[],double qgeni)
{  int n=node.length;
   for(int i=0;i<n;i++)
   {te[node[i]].setHeatGeneration(qgeni);}
}

public void solve(double ti,double k[],double Cp,double ro,double fi)
{
double TT1[]=new double[K2];
double C[][]=new double[K2][K2];
double K[][]=new double[K2][K2];
double AA[][]=new double[K2][K2];
double AA1[][]=new double[K2][K2];
double BB[]=new double[K2];
double CK[][]=new double[K2][K2];
double KK[][]=new double[K2][K2];
double ff[]=new double[K2];
double CN[][]=new double[3][3];
double KN[][]=new double[3][3];
double f[]=new double[K2];
double fN[]=new double[3];
int k1=0;
for(int n=0;n<=N;n++)
{ for(int m=0;m<=M;m++)
  {k1=(N+1)*n+m;
   TT1[k1]=T[n][m];
   TT[0][n][m]=T[n][m];
  }
}
for(int ip=0;ip<Np;ip++)
{  //time steps
for(int i=0;i<K1;i++)
{KN=te[i].K(ti,k);
 CN=te[i].C(ti,Cp,ro);
 fN=te[i].f(ti,k);
 int n1=te[i].n_element;
 int m[]=te[i].n_point;
 for(int j=0;j<3;j++)
   { for(k1=0;k1<3;k1++)
      {K[m[j]][m[k1]]+=KN[j][k1];KK[m[j]][m[k1]]=K[m[j]][m[k1]];
       C[m[j]][m[k1]]+=CN[j][k1];CK[m[j]][m[k1]]=C[m[j]][m[k1]];
      }
       f[m[j]]+=fN[j];
       ff[m[j]]=f[m[j]];
   }
}
for(int i=0;i<K2;i++)
{for(int j=0;j<K2;j++)
 {AA[i][j]=C[i][j]+(1.0-fi)*dt*K[i][j];AA1[i][j]=C[i][j]-fi*dt*K[i][j];}
  BB[i]=f[i]*dt;
}
BB=add(multiply(AA1,TT1),BB);
//redesign AA matrix and BB vector if any known
//temperature boundary is given
//int j=0;
for(int j=0;j<K2;j++)
{
                for(int i=0;i<K2;i++)
    { if(bi[i])
      {
                  if(i==j)
                  {BB[i]=Ti[i];AA[i][j]=1.0;}
                  else
```

```
                {
                AA[i][j]=0.0;}
                //System.out.println("i="+i+"j="+j+"bi="+bi[i]+"Ti="+Ti[i]+"fi="+f[i]);
      }
      }//end of for
} //end of for
//Text.print(K,"K");
//Text.print(C,"C");
//Text.print(f,"f");
TT1=AXB(AA,BB);
for(int n=0;n<=N;n++)
{ for(int m=0;m<=M;m++)
  {k1=(N+1)*n+m;
  T[n][m]=TT1[k1];
  TT[ip+1][n][m]=TT1[k1];
  }
}

} //end of time steps
}
public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
//System.out.println(n1+"x"+m1+"  "+n2+"x"+m2);
double b[][];
b=new double[n1][m2];
 if(m1 != n2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
   for(jj=0;jj<m2;jj++)
    b[ii][jj]=0;
   }
   return b;
 }
 for(i=0;i<n1;i++)
   {
   for(j=0;j<m2;j++)
    {
    for(k=0;k<m1;k++){
      //System.out.println(i+"  "+j+"   "+k);
                b[i][j]+=left[i][k]*right[k][j];
                //System.out.println(b[i][j]);
   }
    }
   }
return b;
//end of multiply of two matrices
}
public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
    b[ii]=0;
   }
   return b;
 }
 for(i=0;i<m1;i++)
   {
    b[i]=0;
```

```
     for(k=0;k<n1;k++)
                b[i]+=left[i][k]*right[k];
   }
return b;
//end of multiply of a matrix and a vector
}
public static double[] add(double[] left,double[] right)
{
//addition of two vectors
int n1=left.length;
int n2=right.length;
int nMax;
int i;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[];
b=new double[nMax];
for(i=0;i<n1;i++)
  {
    b[i]=b[i]+left[i];
  }
for(i=0;i<n2;i++)
  {
    b[i]=b[i]+right[i];
  }
return b;
//end of vector addition method
}
//LU decomposition method
public static double[][] LU(double c[][],int indx[],int d[])
{
//returns LU decomposition of matrix c and index indx
double a[][];
int n=c.length;
a=new double[n][n];
double vv[];
vv=new double[n];
double sum,dum,big,temp;
int i,j,k;
int imax;
int nmax=100;
double tiny=1.0e-40;
imax=0;
for(i=1;i<=n;i++)
{
  for(j=1;j<=n;j++)
     a[i-1][j-1]=c[i-1][j-1];
}
d[0]=1;
for(i=1;i<=n;i++)
  {
  big=0.0;
  for(j=1;j<=n;j++)
    {
    if(Math.abs(a[i-1][j-1])>big) big=Math.abs(a[i-1][j-1]);
    }
  if(big==0) {System.out.println("singular matrix");return a;}
  vv[i-1]=1.0/big;
  }
for(j=1;j<=n;j++)
{
  for(i=1;i<j;i++)
    {
      sum=a[i-1][j-1];
      for(k=1;k<i;k++)
                {
                  sum-=a[i-1][k-1]*a[k-1][j-1];
                }
    a[i-1][j-1]=sum;
    }
  big=0;
  for(i=j;i<=n;i++)
    {
    sum=a[i-1][j-1];
    for(k=1;k<j;k++)
      {
```

```
      sum-=a[i-1][k-1]*a[k-1][j-1];
      }
    a[i-1][j-1]=sum;
    dum=vv[i-1]*Math.abs(sum);
    if(dum>=big)
      {
      imax=i;
      big=dum;
      }
    } //end of i=0
if(j != imax)
    {
    for(k=1;k<=n;k++)
      {
      dum=a[imax-1][k-1];
      a[imax-1][k-1]=a[j-1][k-1];
      a[j-1][k-1]=dum;
      }
    d[0]=-d[0];
    vv[imax-1]=vv[j-1];
    } //end of if
indx[j-1]=imax;
if(a[j-1][j-1]==0) a[j-1][j-1]=tiny;
if(j!=n)
  {
  dum=1.0/a[j-1][j-1];
  for(i=j+1;i<=n;i++)
    a[i-1][j-1]*=dum;
  }//endif
} //end for j=
return a;
}
public static double[] LUaxb(double a[][],double x[],int indx[])
{
//solves AX=B system of linear equation of LU decomposed matrix a
//(calculated by method LU)
int ii=0;
int i,j,ll=0;
double sum=0;
int n=a.length;
double b[];
b=new double[n];
for(i=1;i<=n;i++)
{
    b[i-1]=x[i-1];
}
for(i=1;i<=n;i++)
  {
  ll=indx[i-1];
  sum=b[ll-1];
  b[ll-1]=b[i-1];
  if(ii!=0)
     {
     for(j=ii;j<=(i-1);j++)
       {
       sum-=a[i-1][j-1]*b[j-1];
       }
     }
  else if(sum!=0) ii=i;
  b[i-1]=sum;
  }
for(i=n;i>=1;i--)
  {
  sum=b[i-1];
  if(i<n)
    {
    for(j=(i+1);j<=n;j++)
      {
      sum-=a[i-1][j-1]*b[j-1];
      }
    }
  b[i-1]=sum/a[i-1][i-1];
  }
return b;
}
public void ContourPlot(double Tx[][], String ss)
```

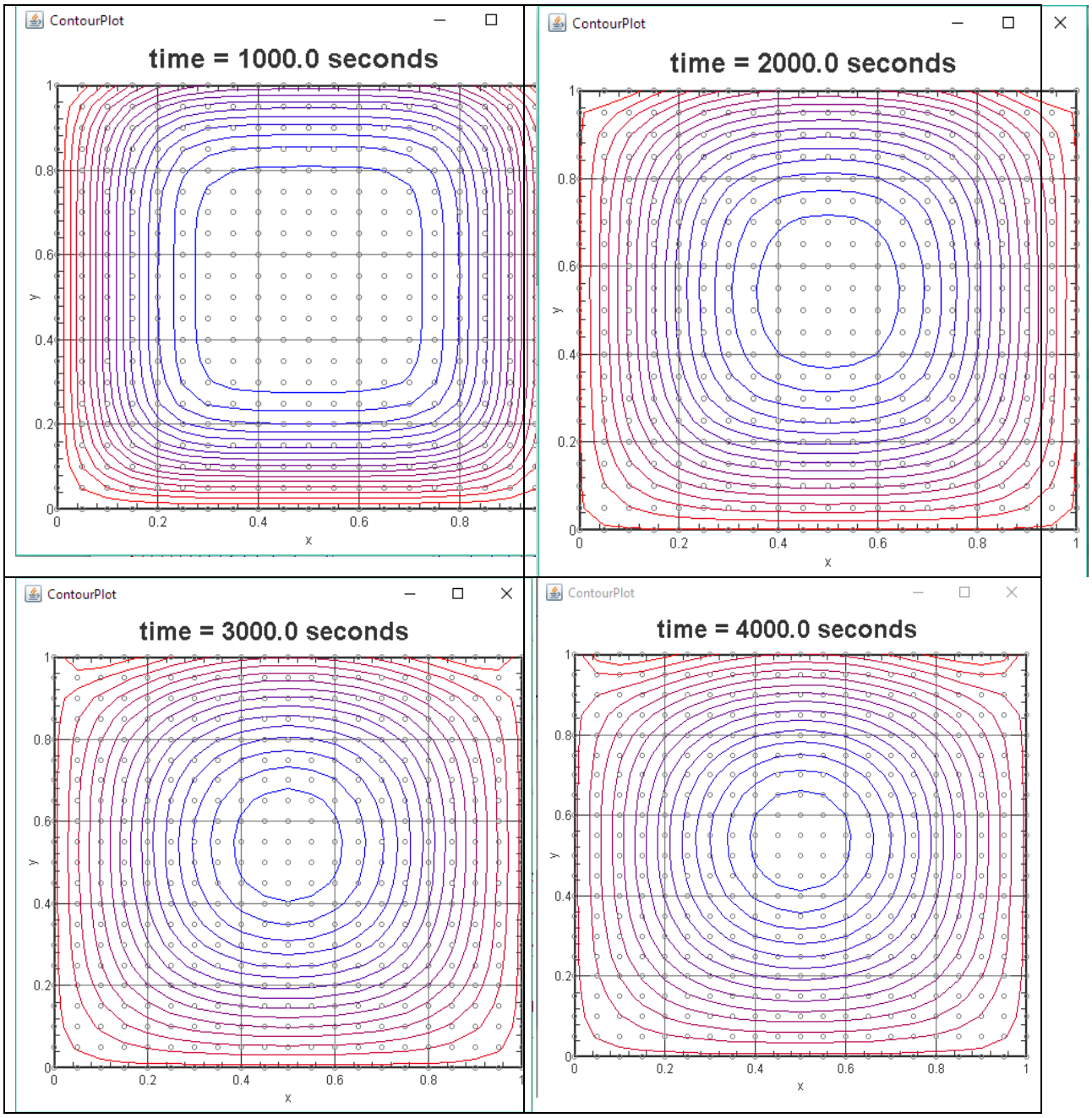
```
{ ContourPlot pp=new ContourPlot( x,y,Tx,16,true,ss,"x","y");
  pp.plot();
}

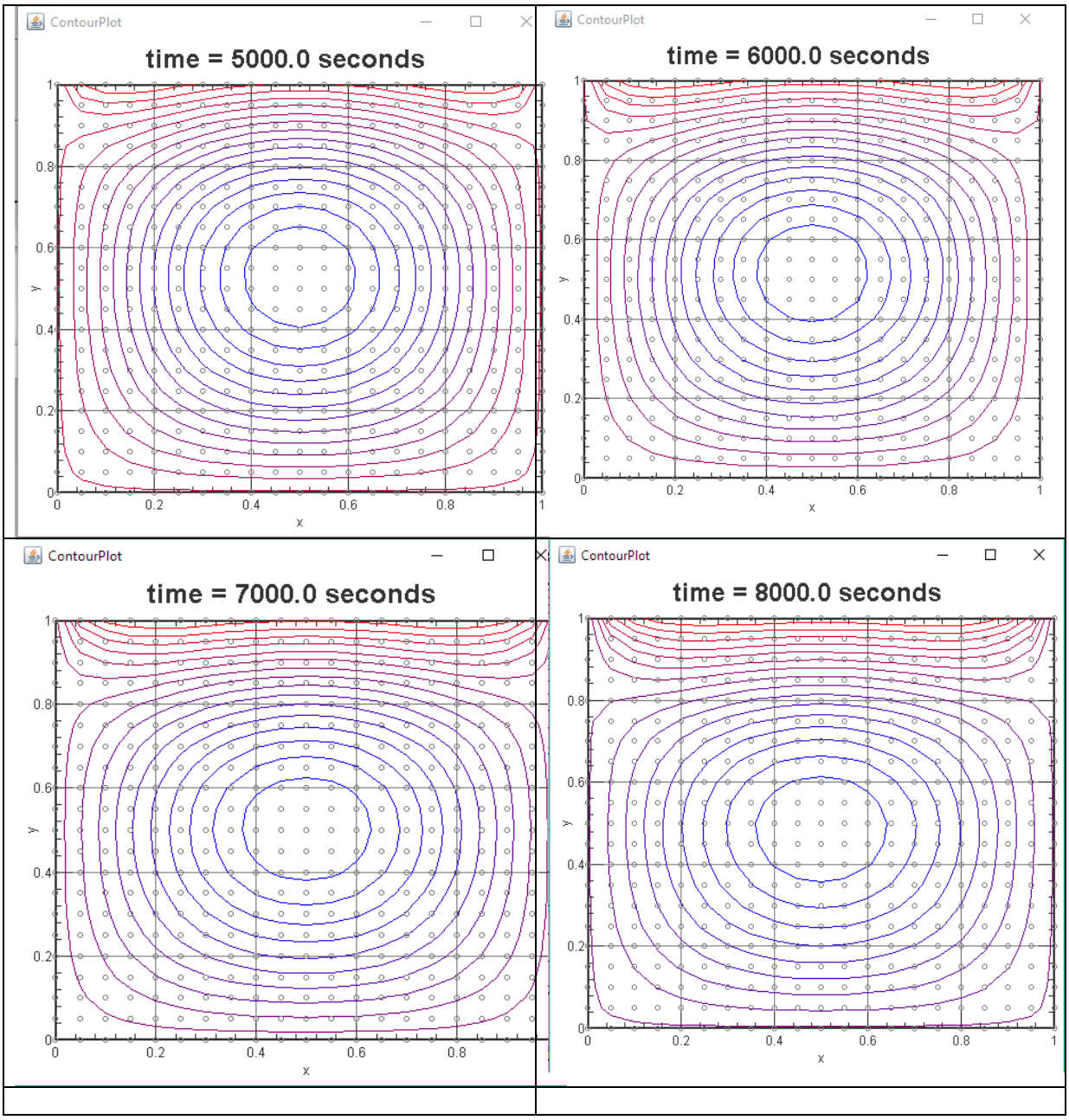
public void surfacePlot(double Tx[][])
{    double b[][]=surfacePlot.dataformat(0,L,0,W,Tx);
     surfacePlot p1;
     p1=new surfacePlot(b);
     p1.plot();
}
public static double[] AXB(double a[][],double b[])
{
//Solution of system of linear equations by LU method
// note that the same calculation can be done by divide method.
//Text.print(a,"K");
//Text.print(b,"f");
int n=a.length;
double c[]=new double[n];
int d[]={1};
int indx[]=new int[n];
double e[][]=new double[n][n];
e=LU(a,indx,d);
c=LUaxb(e,b,indx);
return c;
} //end of AXB

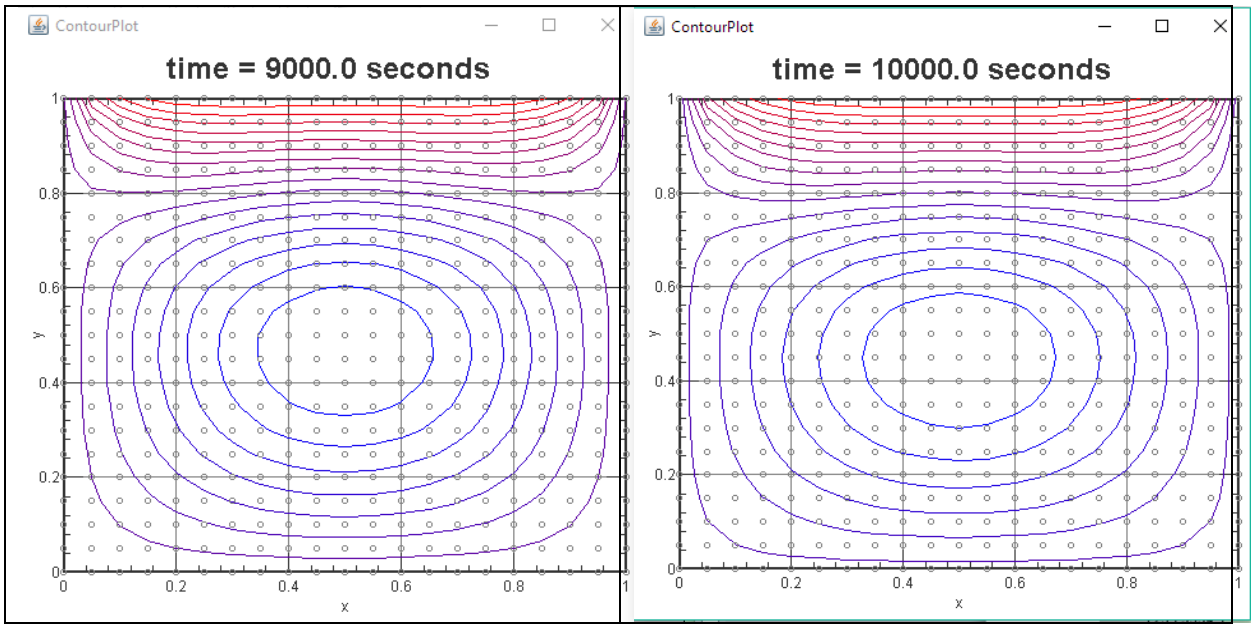
public static void main(String arg[]) //throws RemoteException, VisADException
{double L=1.0;
 double W=1.0;
 int N=20,M=20;
 double hi=50.0;
 double Cp=490.0;
 double ro=7800.0;

 int cond0=0;//x==0
 int cond1=1;//x==L
 int cond2=2;//y==0
 int cond3=3;//y==W
 double Ti0=100.0;
 double Ti1=100.0;
 double Ti2=100.0;
 double Ti3=300.0;
 double TT[][]=new double[N+1][M+1];
 for(int i=0;i<=N;i++)
 {for(int j=0;j<=M;j++)
   TT[i][j]=50.0;
 }
 triangular_grid2 tg=new triangular_grid2(L,W,M,N,TT,10.0,10000.0);
 tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
 tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
 tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
 //tg.setTemperature(cond3,0.0,L,0.0,W,Ti3);
 tg.setConvection(cond3,0.0,L,0.0,W,hi,Ti3);
 double k[]={51,51};
 double fi=0.5;
 tg.solve(1,k,Cp,ro,fi);

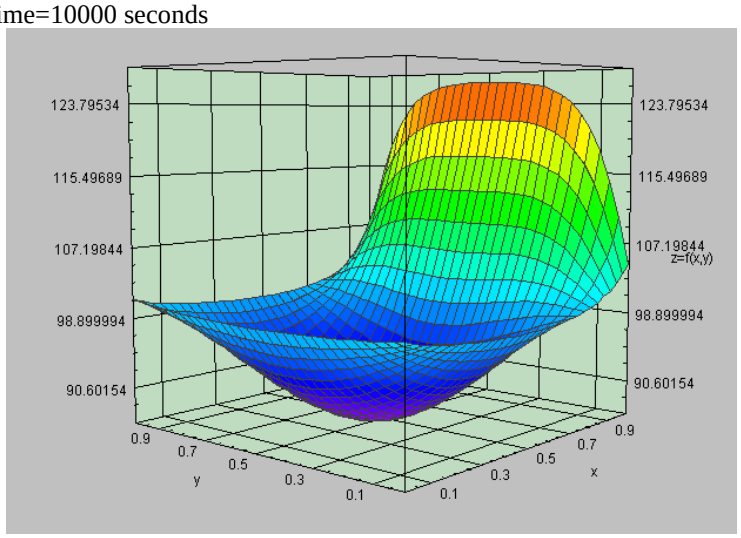
 //Take an output at every 2000 seconds
 for(int ip=0;ip<=1000;ip++)
 {if((ip%100)==0)
   { String s="time = "+ip*tg.dt+" seconds";
                System.out.println(s);
   Text.print(tg.TT[ip],s);
   tg.ContourPlot(tg.TT[ip],s);
   }
 }
//temperature profile t=10 s
 tg.surfacePlot(tg.TT[1000]);
}}
```


ContourPlot
time = 1000.0 seconds
ContourPlot
time = 2000.0 seconds
ContourPlot
time = 3000.0 seconds
ContourPlot
time = 4000.0 seconds
x
y
0
0.2
0.4
0.6
0.8
1


ContourPlot
time = 5000.0 seconds
ContourPlot
time = 6000.0 seconds
ContourPlot
time = 7000.0 seconds
ContourPlot
time = 8000.0 seconds
x
y



time=10000 seconds



time = 2000.0 seconds

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	97.84433...	95.81717...	94.04260...	92.59541...	91.49635...	90.72019...	90.21272...	89.91028...	89.75646...	89.71330...	89.76716...	89.93015...	90.23826...	90.74565...	91.51431...	92.59866...	94.02739...	95.78702...	97.81388...	100.0
100.0	95.81716...	91.89572...	88.46848...	85.67641...	83.55736...	82.06123...	81.08273...	80.49888...	80.20099...	80.11623...	80.21864...	80.53215...	81.12691...	82.10824...	83.59643...	85.69670...	88.46367...	91.86938...	95.78702...	100.0
100.0	94.04255...	88.46840...	83.60347...	79.64424...	76.64158...	74.52248...	73.13650...	72.30885...	71.88553...	71.76356...	71.90661...	72.34928...	73.19193...	74.58506...	76.70021...	79.68702...	83.62218...	88.46364...	94.02737...	100.0
100.0	92.59521...	85.67607...	79.64390...	74.73921...	71.02223...	68.40032...	66.68573...	65.66135...	65.13633...	64.98332...	65.15776...	65.70316...	66.74485...	68.47052...	71.09395...	74.80121...	79.68681...	85.69651...	92.59857...	100.0
100.0	91.49569...	83.55619...	76.64013...	71.02097...	66.76528...	63.76486...	61.80321...	60.63089...	60.02901...	59.85178...	60.04843...	60.66939...	61.85919...	63.83419...	66.84073...	71.09299...	76.69921...	83.59570...	91.51395...	100.0
100.0	90.71828...	82.05774...	74.51787...	68.39529...	63.76077...	60.49468...	58.35989...	57.08386...	56.42781...	56.23282...	56.44375...	57.11594...	58.40770...	60.55599...	63.83071...	68.46646...	74.58157...	82.10580...	90.74443...	100.0
100.0	90.20760...	81.07319...	73.12354...	66.67065...	61.78794...	58.34812...	56.10026...	54.75649...	54.06480...	53.85758...	54.07671...	54.78078...	56.13725...	58.39695...	61.84568...	66.73202...	73.18143...	81.11960...	90.23458...	100.0
100.0	89.89749...	80.47478...	72.27561...	65.62161...	60.58790...	57.04248...	54.72594...	53.34098...	52.62744...	52.41227...	52.63547...	53.35759...	54.75170...	57.07729...	60.63020...	65.66799...	72.32086...	80.51236...	89.92011...	100.0
100.0	89.72674...	80.14451...	71.80679...	65.04080...	59.92284...	56.31841...	53.96342...	52.55530...	51.82929...	51.60925...	51.83404...	52.56521...	53.97901...	56.33984...	59.94940...	65.07051...	71.83637...	80.16958...	89.74212...	100.0
100.0	89.64887...	79.99307...	71.59048...	64.77134...	59.61285...	55.97965...	53.60566...	52.18590...	51.45342...	51.23055...	51.45551...	52.19027...	53.61258...	55.98922...	59.62477...	64.78473...	71.60386...	80.00446...	89.65599...	100.0
100.0	89.63735...	79.96864...	71.55292...	64.72139...	59.55216...	55.91041...	53.53015...	52.10615...	51.37099...	51.14656...	51.37083...	52.10578...	53.52948...	55.90930...	59.55052...	64.71920...	71.55037...	79.96618...	89.63570...	100.0
100.0	89.68621...	80.05980...	71.67693...	64.86828...	59.71332...	56.07944...	53.70290...	52.28022...	51.54509...	51.31993...	51.54275...	52.27522...	53.69478...	56.06779...	59.69820...	64.85050...	71.65831...	80.04330...	89.67572...	100.0
100.0	89.81206...	80.29714...	72.00341...	65.25951...	60.14750...	56.53969...	54.17755...	52.76190...	52.02945...	51.80421...	52.02462...	52.75170...	54.16116...	56.51655...	60.11794...	65.22528...	71.96810...	80.26622...	89.79259...	100.0
100.0	90.05593...	80.75673...	72.63456...	66.01480...	60.98524...	57.42791...	55.09419...	53.69297...	52.96652...	52.74195...	52.95862...	53.67646...	55.06808...	57.39172...	60.93986...	65.96323...	72.58228...	80.71163...	90.02785...	100.0
100.0	90.48828...	81.56675...	73.73919...	67.32876...	62.43644...	58.96265...	56.67605...	55.29906...	54.58302...	54.36014...	54.57135...	55.27497...	56.63868...	58.91202...	62.37457...	67.26027...	73.67149...	81.50962...	90.45331...	100.0
100.0	91.21867...	82.91834...	75.55863...	69.47032...	64.78446...	61.43472...	59.21788...	57.87692...	57.17667...	56.95677...	57.16052...	57.84408...	59.16810...	61.36926...	64.70717...	69.38796...	75.48035...	82.85466...	91.18080...	100.0
100.0	92.41756...	85.08459...	78.41141...	72.77374...	68.36772...	65.18339...	63.05912...	61.76622...	61.08733...	60.87177...	61.06608...	61.72375...	62.99647...	65.10403...	68.27839...	72.68396...	78.33169...	85.02423...	92.38382...	100.0
100.0	94.37446...	88.45859...	82.69546...	77.61578...	73.54365...	70.55404...	68.53899...	67.30339...	66.65033...	66.44023...	66.62343...	67.25061...	68.46353...	70.46281...	73.44759...	77.52808...	82.62769...	88.41637...	94.35578...	100.0
100.0	97.67173...	93.62836...	88.87995...	84.37123...	80.63245...	77.83921...	75.93707...	74.76238...	74.13740...	73.93329...	74.10426...	74.69854...	75.84883...	77.73827...	80.53559...	84.29672...	88.84081...	93.62456...	97.68532...	100.0
100.0	103.7495...	101.4781...	97.44751...	93.33174...	89.84264...	87.21202...	85.41157...	84.29521...	83.69815...	83.49989...	83.65800...	84.21916...	85.30983...	87.10239...	89.74947...	93.28042...	97.45460...	101.5393...	103.8272...	100.0

time = 4000.0 seconds

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	98.90729...	97.84887...	96.86162...	95.97540...	95.21183...	94.58407...	94.09790...	93.75378...	93.54939...	93.48201...	93.55025...	93.75477...	94.09776...	94.58134...	95.20529...	95.96457...	96.84722...	97.83331...	98.89529...	100.0
100.0	97.84816...	95.76674...	93.82647...	92.08540...	90.58563...	89.35279...	88.39810...	87.72241...	87.32115...	87.18902...	87.32344...	87.72579...	88.40045...	89.35166...	90.57894...	92.07225...	93.80789...	95.74633...	97.83269...	100.0
100.0	96.85864...	93.82267...	90.99406...	88.45672...	86.27157...	84.47562...	83.08502...	82.10086...	81.51649...	81.32427...	81.52054...	82.10749...	83.09162...	84.47913...	86.26932...	88.44737...	90.97843...	93.80440...	96.84453...	100.0
100.0	95.96748...	92.07246...	88.44496...	85.19198...	82.39113...	80.08948...	78.30746...	77.04636...	76.29759...	76.05145...	76.30339...	77.05638...	78.31891...	80.09900...	82.39569...	85.18980...	88.43629...	92.06019...	95.95728...	100.0
100.0	95.19474...	90.55514...	86.23554...	82.36282...	79.02892...	76.28956...	74.16875...	72.66791...	71.77677...	71.48387...	71.78394...	72.68072...	74.18444...	76.30474...	79.04043...	82.36849...	86.23496...	90.55009...	95.18926...	100.0
100.0	94.55149...	89.29221...	84.39669...	80.00833...	76.23093...	73.12732...	70.72452...	69.02400...	68.01417...	67.68212...	68.02205...	69.03836...	70.74281...	73.14637...	76.24765...	80.02035...	84.40303...	89.29356...	94.55027...	100.0
100.0	94.04094...	88.28973...	82.93688...	78.13885...	74.00889...	70.61550...	67.98814...	66.12842...	65.02374...	64.66015...	65.03148...	66.14268...	68.00673...	70.63566...	74.02780...	78.15420...	82.94735...	88.29523...	94.04263...	100.0
100.0	93.66062...	87.54267...	81.84842...	76.74404...	72.34989...	68.73884...	65.94239...	63.96246...	62.78588...	62.39804...	62.79256...	63.97486...	65.95876...	68.75694...	72.36740...	76.75898...	81.85947...	87.54939...	93.66345...	100.0
100.0	93.40548...	87.04089...	81.11609...	75.80371...	71.22907...	67.46837...	64.55491...	62.49124...	61.26415...	60.85885...	61.26895...	62.50014...	64.56667...	67.48139...	71.24168...	75.81450...	81.12412...	87.04583...	93.40761...	100.0
100.0	93.27124...	86.77566...	80.72648...	75.29962...	70.62346...	66.77673...	63.79462...	61.68089...	60.42302...	60.00654...	60.42528...	61.68501...	63.79988...	66.78223...	70.62832...	75.30321...	80.72851...	86.77632...	93.27113...	100.0
100.0	93.25771...	86.74609...	80.67705...	75.22667...	70.52475...	66.65226...	63.64679...	61.51427...	60.24381...	59.82201...	60.24314...	61.51281...	63.64437...	66.64870...	70.52005...	75.22107...	80.67114...	86.74081...	93.25434...	100.0
100.0	93.37192...	86.96486...	80.98397...	75.60221...	70.94970...	67.11004...	64.12449...	62.00265...	60.73666...	60.31508...	60.73295...	61.99539...	64.11404...	67.09699...	70.93490...	75.58682...	80.96949...	86.95310...	93.36492...	100.0
100.0	93.63095...	87.46335...	81.68850...	76.47297...	71.94707...	68.19872...	65.27525...	63.19231...	61.94693...	61.53087...	61.94035...	63.17960...	65.25731...	68.17685...	71.92296...	76.44864...	81.66625...	87.44574...	93.62070...	100.0
100.0	94.06526...	88.29678...	82.86233...	77.91955...	73.60091...	70.00262...	67.18226...	65.16504...	63.95546...	63.54996...	63.94645...	65.14777...	67.15816...	69.97368...	73.56958...	77.88854...	82.83452...	88.27514...	94.05286...	100.0
100.0	94.72377...	89.55101...	84.61305...	80.05893...	76.03007...	72.63889...	69.95993...	68.03278...	66.87258...	66.48220...	66.86181...	68.01230...	69.93169...	72.60558...	75.99475...	80.02480...	84.58318...	89.52828...	94.71095...	100.0
100.0	95.68413...	91.35470...	87.09166...	83.04584...	79.38436...	76.25008...	73.74418...	71.92665...	70.82657...	70.45500...	70.81486...	71.90460...	73.71433...	76.21576...	79.34917...	83.01318...	87.06432...	91.33478...	95.67328...	100.0
100.0	97.07723...	93.90332...	90.50453...	87.07279...	83.83663...	80.99175...	78.67796...	76.98140...	75.94775...	75.59727...	75.93597...	76.95958...	78.64926...	80.96017...	83.80624...	87.04696...	90.48526...	93.89108...	97.07136...	100.0
100.0	99.15084...	97.50787...	95.13077...	92.36639...	89.56898...	87.01495...	84.89261...	83.31681...	82.34989...	82.02093...	82.33886...	83.29692...	84.86774...	86.98986...	89.54822...	92.35314...	95.12585...	97.50922...	99.15395...	100.0
100.0	102.4518...	102.6850...	101.3357...	99.17021...	96.74848...	94.44258...	92.48642...	91.01788...	90.11148...	89.80248...	90.10188...	91.00133...	92.46764...	94.42725...	96.74178...	99.17524...	101.3521...	102.7075...	102.4699...	100.0
100.0	108.4010...	110.2747...	109.5360...	107.6921...	105.4862...	103.3401...	101.5026...	100.1169...	99.25969...	98.96749...	99.25191...	100.1045...	101.4914...	103.3368...	105.4970...	107.7205...	109.5808...	110.3283...	108.4461...	100.0

time = 6000.0 seconds

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	99.286699...	98.590806...	97.931804...	97.327133...	96.791841...	96.338320...	95.976241...	95.712685...	95.552388...	95.498026...	95.550436...	95.708717...	95.970175...	96.330139...	96.781708...	97.315519...	97.919629...	98.579621...	99.278999...	100.0
100.0	98.589515...	97.214781...	95.913435...	94.719624...	93.662934...	92.767768...	92.053189...	91.533162...	91.217037...	91.110104...	91.214026...	91.527026...	92.043782...	92.755054...	93.647187...	94.701640...	95.894757...	97.197939...	98.578325...	100.0
100.0	97.926650...	95.906992...	93.995739...	92.242739...	90.691275...	89.377093...	88.328144...	87.564919...	87.101156...	86.944635...	87.097811...	87.558078...	88.317603...	89.362783...	90.673503...	92.222448...	93.974762...	95.888274...	97.914444...	100.0
100.0	97.314309...	94.699097...	92.224796...	89.955664...	87.947566...	86.246686...	84.889174...	83.901561...	83.301658...	83.099570...	83.298493...	83.895050...	84.879066...	86.232863...	87.930300...	89.935891...	92.204363...	94.680949...	97.302583...	100.0
100.0	96.766443...	93.618455...	90.640474...	87.909593...	85.492902...	83.445930...	81.812199...	80.623701...	79.901949...	79.659170...	79.899247...	80.618100...	81.803415...	83.433794...	85.477612...	87.891980...	90.622223...	93.602253...	96.756013...	100.0
100.0	96.294671...	92.687855...	89.275957...	86.147065...	83.377941...	81.032223...	79.159892...	77.797770...	76.970676...	76.692753...	76.968507...	77.793233...	79.152694...	81.022162...	83.365142...	86.132218...	89.260508...	92.674121...	96.285839...	100.0
100.0	95.908512...	91.925943...	88.158314...	84.702645...	81.643656...	79.051782...	76.982514...	75.476874...	74.562607...	74.255577...	74.560864...	75.473208...	76.976654...	79.043534...	81.633104...	84.690357...	88.145506...	91.914555...	95.901199...	100.0
100.0	95.615967...	91.348328...	87.310071...	83.604863...	80.323574...	77.542095...	75.320518...	73.703498...	72.721393...	72.391639...	72.719855...	73.700281...	75.315414...	77.534965...	80.314520...	83.594394...	87.299229...	91.338747...	95.609850...	100.0
100.0	95.424410...	90.969300...	86.751619...	82.879199...	79.447186...	76.535650...	74.208500...	72.513606...	71.483750...	71.137882...	71.482153...	72.510327...	74.203431...	76.528764...	79.438668...	82.869572...	86.741841...	90.960799...	95.419055...	100.0
100.0	95.341729...	90.804005...	86.504257...	82.551770...	79.044106...	76.064410...	73.679883...	71.941452...	70.884348...	70.529106...	70.882449...	71.937645...	73.674191...	76.056960...	79.035225...	82.542067...	86.494688...	90.795890...	95.336723...	100.0
100.0	95.377709...	90.871017...	86.593660...	82.653405...	79.148503...	76.164445...	73.771648...	72.024355...	70.960593...	70.602756...	70.958240...	72.019723...	73.764909...	76.155910...	79.138675...	82.643022...	86.583730...	90.862820...	95.372764...	100.0
100.0	95.545676...	91.195312...	87.053690...	83.223849...	79.803396...	76.880150...	74.528427...	72.806679...	71.756540...	71.402796...	71.753709...	72.801177...	74.520579...	76.870454...	79.792531...	83.212686...	87.043293...	91.186930...	95.540722...	100.0
100.0	95.864660...	91.811919...	87.930686...	84.316001...	81.064475...	78.267559...	76.005359...	74.342362...	73.325214...	72.981968...	73.322039...	74.336251...	75.996775...	78.257163...	81.053089...	84.304579...	87.920291...	91.803710...	95.859888...	100.0
100.0	96.362617...	92.770985...	89.288722...	86.000317...	83.003276...	80.396367...	78.269246...	76.695485...	75.728867...	75.401931...	75.725638...	76.689332...	78.260743...	80.386287...	82.992513...	85.989811...	89.279412...	92.763800...	96.358509...	100.0
100.0	97.082123...	94.145837...	91.216836...	88.369662...	85.709536...	83.350318...	81.397608...	79.938541...	79.036832...	78.730991...	79.033978...	79.933187...	81.390398...	83.342075...	85.701126...	88.361872...	91.210295...	94.141020...	97.079450...	100.0
100.0	98.091853...	96.047312...	93.839845...	91.544881...	89.292357...	87.225462...	85.475290...	84.148210...	83.320933...	83.039397...	83.318983...	84.144694...	85.470884...	87.220971...	89.288512...	91.542151...	93.838316...	96.046698...	98.091686...	100.0
100.0	99.512561...	98.650871...	97.334712...	95.680480...	93.879000...	92.125413...	90.588225...	89.398614...	88.648625...	88.392473...	88.648155...	89.398076...	90.588298...	92.126830...	93.882263...	95.685586...	97.340956...	98.656820...	99.516339...	100.0
100.0	101.58249...	102.24854...	101.95265...	100.96701...	99.608392...	98.152077...	96.813913...	95.752283...	95.074418...	94.842204...	95.075991...	95.755846...	96.820140...	98.161608...	99.621459...	100.98305...	101.96995...	102.26411...	101.59227...	100.0
100.0	104.84127...	107.34235...	108.03723...	107.61979...	106.61377...	105.38989...	104.20853...	103.24951...	102.63054...	102.41833...	102.63465...	103.25816...	104.22241...	105.40957...	106.63927...	107.65000...	108.06944...	107.37172...	104.86047...	100.0
100.0	110.70661...	114.76425...	115.99278...	115.83091...	114.98732...	113.88284...	112.79187...	111.89772...	111.31846...	111.12049...	111.32545...	111.91219...	112.81455...	113.91431...	115.02750...	115.87841...	116.04423...	114.81343...	110.74242...	100.0

time = 8000.0 seconds

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	99.50216...	99.01553...	98.55288...	98.12599...	97.74552...	97.42081...	97.15969...	96.96835...	96.85127...	96.81117...	96.84897...	96.96383...	97.15312...	97.41248...	97.73585...	98.11559...	98.54261...	99.00660...	99.49629...	100.0
100.0	99.01482...	98.05267...	97.13821...	96.29457...	95.54276...	94.90118...	94.38531...	94.00739...	93.77627...	93.69732...	93.77239...	93.99977...	94.37427...	94.88722...	95.52665...	96.27741...	97.12149...	98.03842...	99.00581...	100.0
100.0	98.55017...	97.13490...	95.79011...	94.54961...	93.44420...	92.50092...	91.74253...	91.18704...	90.84750...	90.73180...	90.84266...	91.17756...	91.72880...	92.48362...	93.42432...	94.52857...	95.76982...	97.11787...	98.53963...	100.0
100.0	98.11969...	96.28470...	94.54135...	92.93327...	91.50031...	90.27752...	89.29442...	88.57443...	88.13451...	87.98493...	88.12921...	88.56406...	89.27943...	90.25868...	91.47875...	92.91057...	94.51962...	96.26665...	98.10866...	100.0
100.0	97.73419...	95.52334...	93.42298...	91.48551...	89.75887...	88.28534...	87.10057...	86.23290...	85.70289...	85.52301...	85.69751...	86.22238...	87.08539...	88.26630...	89.73716...	91.46275...	93.40132...	95.50548...	97.72337...	100.0
100.0	97.40390...	94.87091...	92.46432...	90.24400...	88.26484...	86.57541...	85.21677...	84.22168...	83.61393...	83.40797...	83.60873...	84.21152...	85.20213...	86.55709...	88.24401...	90.22224...	92.44371...	94.85400...	97.39372...	100.0
100.0	97.13857...	94.34655...	91.69323...	89.24440...	87.06061...	85.19567...	83.69530...	82.59611...	81.92478...	81.69754...	81.91992...	82.58663...	83.68167...	85.17864...	87.04129...	89.22429...	91.67425...	94.33104...	97.12928...	100.0
100.0	96.94774...	93.96893...	91.13678...	88.52117...	86.18688...	84.19187...	82.58575...	81.40847...	80.68928...	80.44606...	80.68486...	81.39984...	82.57336...	84.17643...	86.16941...	88.50304...	91.11972...	93.95504...	96.93946...	100.0
100.0	96.84117...	93.75705...	90.82229...	88.10877...	85.68394...	83.60884...	81.93630...	80.70921...	79.95921...	79.70570...	79.95527...	80.70152...	81.92528...	83.59514...	85.66847...	88.09277...	90.80728...	93.74488...	96.83394...	100.0
100.0	96.82944...	93.73136...	90.77879...	88.04330...	85.59337...	83.49220...	81.79536...	80.54854...	79.78575...	79.52797...	79.78231...	80.54184...	81.78578...	83.48032...	85.58000...	88.02951...	90.76589...	93.72093...	96.82326...	100.0
100.0	96.92475...	93.91520...	91.03896...	88.36459...	85.96011...	83.89025...	82.21329...	80.97798...	80.22100...	79.96513...	80.21811...	80.97235...	82.20525...	83.88031...	85.94896...	88.35312...	91.02827...	93.90658...	96.91964...	100.0
100.0	97.14217...	94.33697...	91.64161...	89.11873...	86.83472...	84.85592...	83.24404...	82.05175...	81.31916...	81.07135...	81.31688...	82.04732...	83.23774...	84.84816...	86.82604...	89.10984...	91.63334...	94.33030...	97.13823...	100.0
100.0	97.50149...	95.03314...	92.63517...	90.36125...	88.27596...	86.44866...	84.94641...	83.82759...	83.13714...	82.90326...	83.13559...	83.82460...	84.94217...	86.44348...	88.27020...	90.35537...	92.62972...	95.02873...	97.49887...	100.0
100.0	98.03034...	96.05318...	94.08266...	92.16116...	90.35355...	88.73600...	87.38470...	86.36684...	85.73425...	85.51948...	85.73362...	86.36563...	87.38303...	88.73400...	90.35139...	92.15899...	94.08064...	96.05151...	98.02929...	100.0
100.0	98.76996...	97.46777...	96.06918...	94.60619...	93.15098...	91.79445...	90.62842...	89.73336...	89.17082...	88.97914...	89.17133...	89.73438...	90.62996...	91.79645...	93.15332...	94.60867...	96.07148...	97.46953...	98.77088...	100.0
100.0	99.78700...	99.38372...	98.71359...	97.80940...	96.76792...	95.70922...	94.75041...	93.99098...	93.50527...	93.33890...	93.50719...	93.99477...	94.75593...	95.71620...	96.77592...	97.81774...	98.72137...	99.38986...	99.79045...	100.0
100.0	101.1998...	101.9718...	102.1859...	101.9163...	101.3208...	100.5714...	99.82287...	99.19913...	98.78978...	98.64859...	98.79342...	99.20627...	99.83325...	100.5845...	101.3357...	101.9320...	102.2007...	101.9836...	101.2066...	100.0
100.0	103.2446...	105.5197...	106.7309...	107.1090...	106.9383...	106.4714...	105.9103...	105.4065...	105.0645...	104.9457...	105.0702...	105.4176...	105.9265...	106.4918...	106.9617...	107.1338...	106.7546...	105.5390...	103.2559...	100.0
100.0	106.4593...	110.5262...	112.6873...	113.5965...	113.7460...	113.4847...	113.0594...	112.6432...	112.3511...	112.2492...	112.3590...	112.6588...	113.0821...	113.5135...	113.7795...	113.6323...	112.7223...	110.5558...	106.4776...	100.0
100.0	112.2615...	117.8220...	120.4572...	121.5678...	121.8326...	121.6513...	121.2854...	120.9119...	120.6460...	120.5538...	120.6564...	120.9323...	121.3153...	121.6895...	121.8774...	121.6167...	120.5064...	117.8660...	112.2918...	100.0

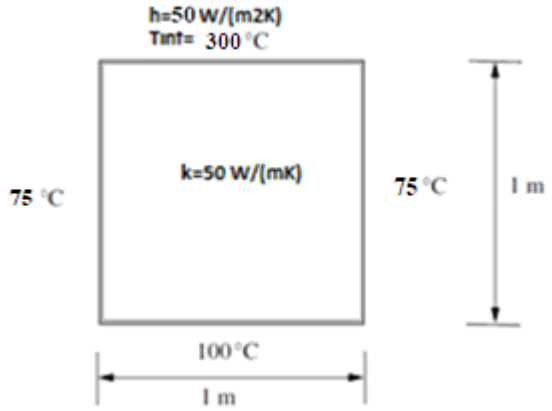
time = 10000.0 seconds

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
100.0	99.65025...	99.30814...	98.98250...	98.68153...	98.41278...	98.18293...	97.99769...	97.86168...	97.77827...	97.74955...	97.77623...	97.85770...	97.99197...	98.17577...	98.40461...	98.67291...	98.97415...	99.30101...	99.64565...	100.0
100.0	99.30855...	98.63280...	97.98977...	97.39555...	96.86496...	96.41121...	96.04559...	95.77718...	95.61268...	95.55619...	95.60916...	95.77031...	96.03574...	96.39894...	96.85103...	97.38098...	97.97584...	98.62117...	99.30133...	100.0
100.0	98.98429...	97.99210...	97.04814...	96.17591...	95.39709...	94.73107...	94.19442...	93.80054...	93.55925...	93.47660...	93.55472...	93.79173...	94.18182...	94.71542...	95.37942...	96.15756...	97.03079...	97.97781...	98.97561...	100.0
100.0	98.68657...	97.40387...	96.18364...	95.05609...	94.04924...	93.18814...	92.49429...	91.98506...	91.67323...	91.56667...	91.66812...	91.97513...	92.48011...	93.17058...	94.02950...	95.03570...	96.16449...	97.38827...	98.67721...	100.0
100.0	98.42433...	96.88571...	95.42195...	94.06919...	92.86097...	91.82744...	90.99451...	90.38317...	90.00892...	89.88128...	90.00357...	90.37278...	90.97969...	91.80914...	92.84046...	94.04809...	95.40226...	96.86977...	98.41486...	100.0
100.0	98.20648...	96.45509...	94.78859...	93.24795...	91.87138...	90.69334...	89.74361...	89.04639...	88.61961...	88.47431...	88.61430...	89.03608...	89.72891...	90.67522...	91.85113...	93.22720...	94.76930...	96.43956...	98.19730...	100.0
100.0	98.04195...	96.12960...	94.30917...	92.62518...	91.11943...	89.82989...	88.78960...	88.02555...	87.55781...	87.39878...	87.55274...	88.01572...	88.77561...	89.81267...	91.10022...	92.60554...	94.29097...	96.11500...	98.03336...	100.0
100.0	97.94000...	95.92732...	94.00986...	92.23418...	90.64447...	89.28133...	88.18042...	87.37117...	86.87557...	86.70725...	86.87092...	87.36215...	88.16759...	89.26555...	90.62690...	92.21624...	93.99327...	95.91404...	97.93221...	100.0
100.0	97.91042...	95.86733...	93.91814...	92.10962...	90.48705...	89.09278...	87.96462...	87.13411...	86.62507...	86.45231...	86.62096...	87.12615...	87.95330...	89.07888...	90.47157...	92.09384...	93.90356...	95.85568...	97.90359...	100.0
100.0	97.96404...	95.97061...	94.06382...	92.28862...	90.68999...	89.31128...	88.19213...	87.36621...	86.85919...	86.68718...	86.85574...	87.35952...	88.18262...	89.29959...	90.67698...	92.27536...	94.05157...	95.96082...	97.95830...	100.0
100.0	98.11344...	96.26119...	94.48057...	92.81229...	91.29973...	89.98685...	88.91524...	88.12102...	87.63214...	87.46625...	87.62944...	88.11579...	88.90780...	89.97770...	91.28954...	92.80188...	94.47094...	96.25348...	98.10891...	100.0
100.0	98.37397...	96.76803...	95.20804...	93.72778...	92.36811...	91.17387...	90.18944...	89.45440...	88.99980...	88.84538...	88.99795...	89.45082...	90.18434...	91.16759...	92.36109...	93.72057...	95.20134...	96.76262...	98.37076...	100.0
100.0	98.76556...	97.52789...	96.29506...	95.09112...	93.95449...	92.93244...	92.07424...	91.42482...	91.01980...	90.88189...	91.01891...	91.42310...	92.07178...	92.92939...	93.95104...	95.08752...	96.29163...	97.52505...	98.76382...	100.0
100.0	99.31579...	98.59013...	97.80448...	96.97110...	96.12828...	95.32971...	94.63351...	94.09314...	93.75095...	93.63388...	93.75116...	94.09354...	94.63404...	95.33032...	96.12888...	96.97158...	97.80476...	98.59019...	99.31570...	100.0
100.0	100.0656...	100.0249...	99.82071...	99.45454...	98.97184...	98.44090...	97.93528...	97.52163...	97.25189...	97.15875...	97.25333...	97.52441...	97.93921...	98.44567...	98.97706...	99.45970...	99.82527...	100.0283...	100.0674...	100.0
100.0	101.0815...	101.9385...	102.4615...	102.6530...	102.5831...	102.3491...	102.0501...	101.7716...	101.5785...	101.5106...	101.5813...	101.7771...	102.0579...	102.3587...	102.5936...	102.6636...	102.4710...	101.9458...	101.0855...	100.0
100.0	102.4813...	104.5007...	105.8957...	106.7106...	107.0766...	107.1435...	107.0479...	106.9001...	106.7803...	106.7365...	106.7847...	106.9086...	107.0600...	107.1585...	107.0932...	106.7275...	105.9112...	104.5128...	102.4881...	100.0
100.0	104.5008...	107.9986...	110.3672...	111.8079...	112.5794...	112.9123...	112.9910...	112.9535...	112.8951...	112.8716...	112.9012...	112.9654...	113.0080...	112.9333...	112.6030...	111.8323...	110.3898...	108.0167...	104.5112...	100.0
100.0	107.6781...	112.9307...	116.2140...	118.1534...	119.2163...	119.7303...	119.9251...	119.9608...	119.9434...	119.9337...	119.9514...	119.9763...	119.9474...	119.7581...	119.2479...	118.1865...	116.2455...	112.9568...	107.6939...	100.0
100.0	113.4310...	120.1281...	123.8388...	125.9365...	127.0763...	127.6377...	127.8659...	127.9251...	127.9224...	127.9182...	127.9323...	127.9445...	127.8938...	127.6729...	127.1168...	125.9797...	123.8812...	120.1653...	113.4560...	100.0

EXAMPLE 6: NON-HOMOGENEOUS, TIME DEPENDENT SOLUTION; CARTESIAN COORDINATES, FINITE ELEMENT SOLUTION

Plate initial temperature is T_i=50 ℃. (t=0). The plate exposed to 100 C temperature at the bottom surface and 75 ℃ at the side surfaces as shown in the figure and convectve heat flow with temperature T_∞=300 ℃ and thermal conductivity h=50 W/(m^{2}K) (t>0). Thermal conductivty of plate is k=51 W/(mK), specific heat Cp=490 J/(kgK) and density ρ=7800 kg/m^3

Calculate temperature profile as a function of time.

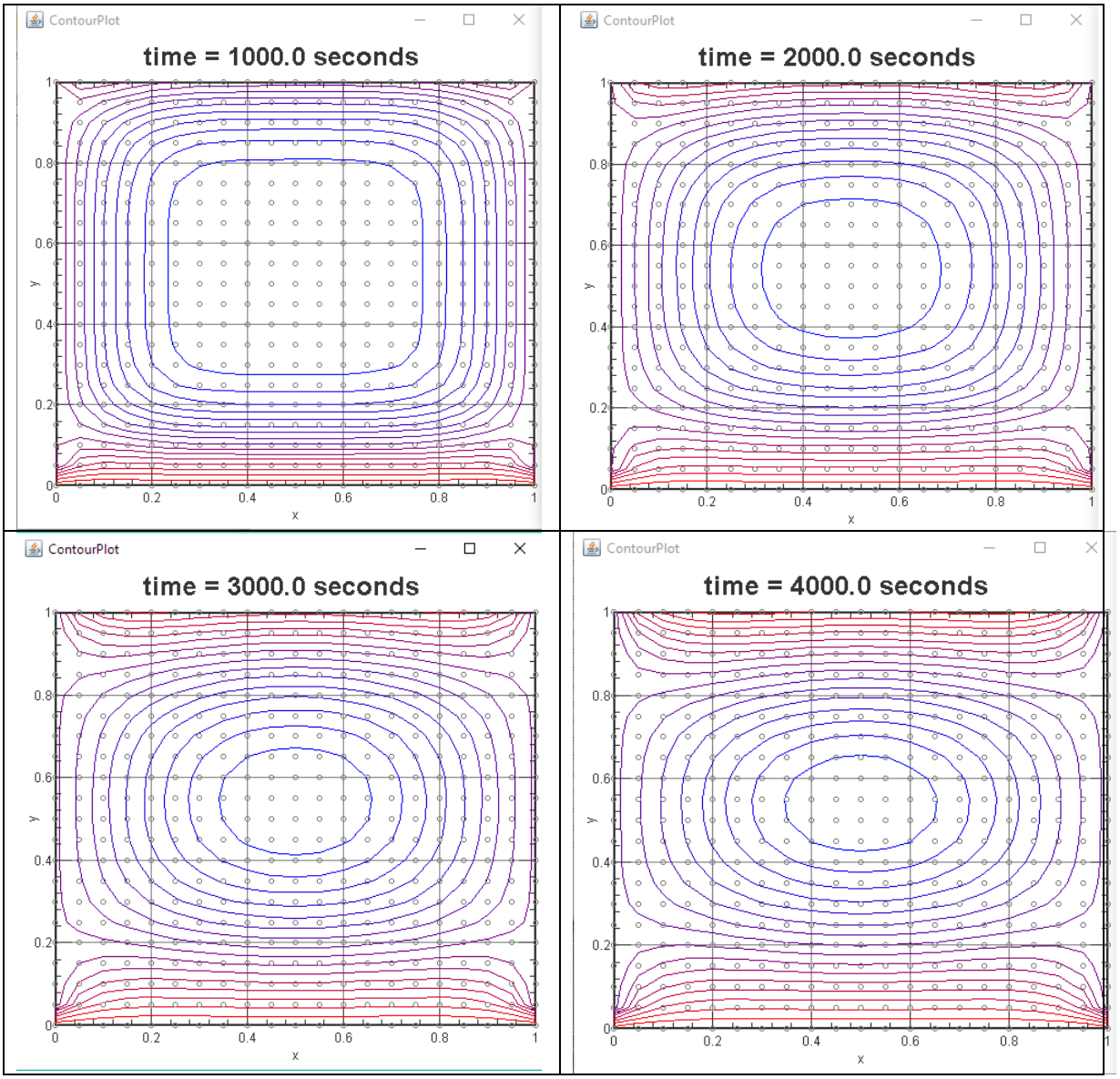


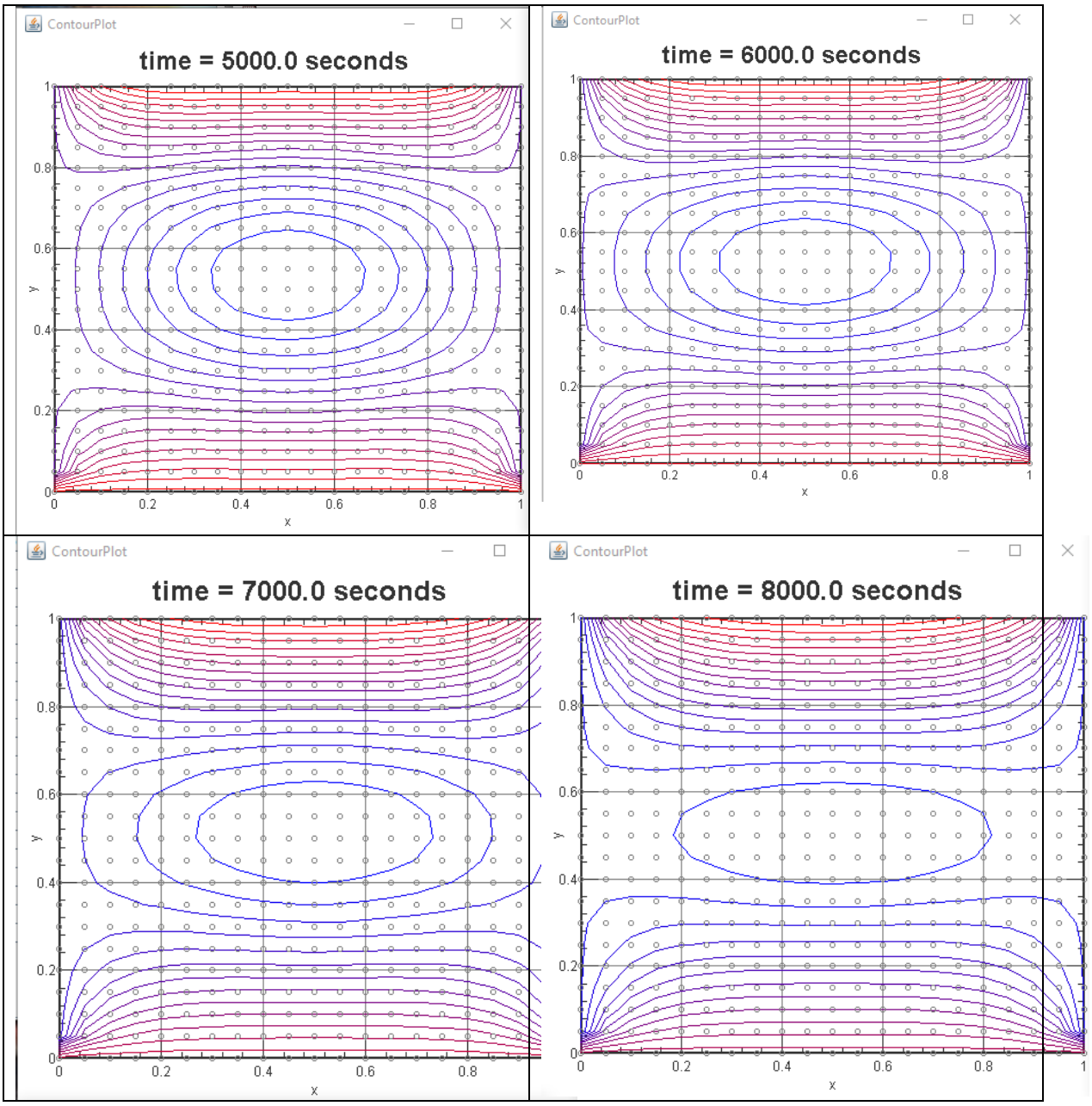

```
public static void main(String arg[]) //throws RemoteException, VisADException
{double L=1.0;
 double W=1.0;
 int N=20,M=20;
 double hi=50.0;
 double Cp=490.0;
 double ro=7800.0;

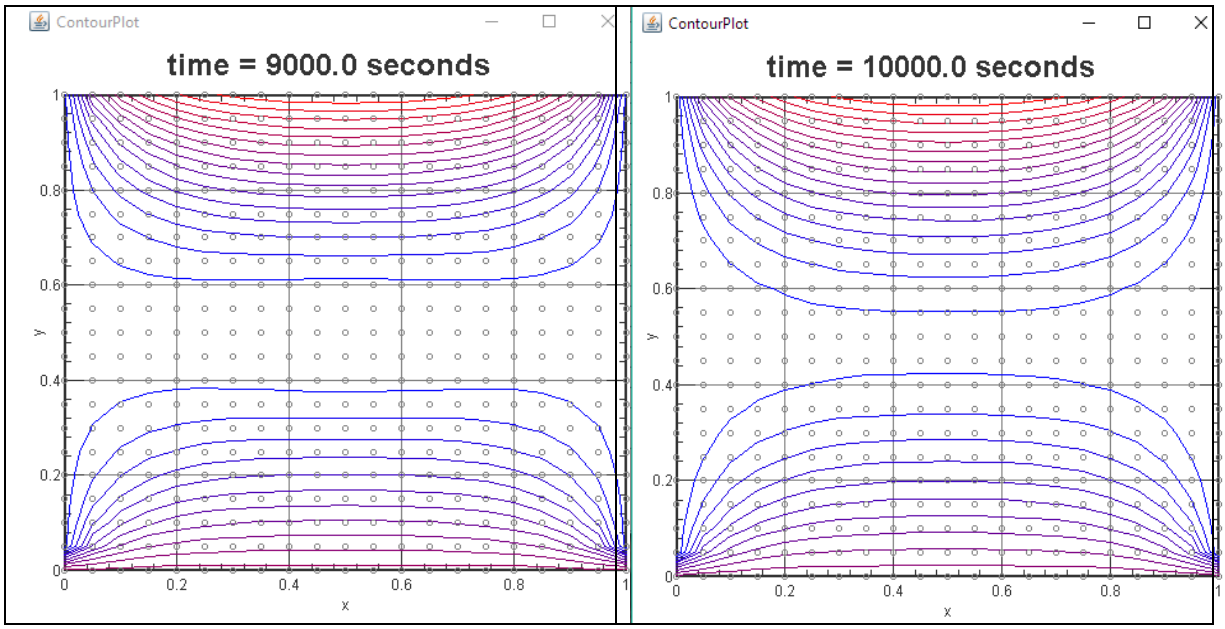
 int cond0=0;//x==0
 int cond1=1;//x==L
 int cond2=2;//y==0
 int cond3=3;//y==W
 double Ti0=75.0;
 double Ti1=75.0;
 double Ti2=100.0;
 double Ti3=300.0;
 double TT[][]=new double[N+1][M+1];
 for(int i=0;i<=N;i++)
 {for(int j=0;j<=M;j++)
   TT[i][j]=50.0;
 }
```

```
triangular_grid2 tg=new triangular_grid2(L,W,M,N,TT,10.0,10000.0);
tg.setTemperature(cond0,0.0,L,0.0,W,Ti0);
tg.setTemperature(cond1,0.0,L,0.0,W,Ti1);
tg.setTemperature(cond2,0.0,L,0.0,W,Ti2);
tg.setConvection(cond3,0.0,L,0.0,W,hi,Ti3);
double k[]={51,51};
double fi=0.5;
tg.solve(1,k,Cp,ro,fi);

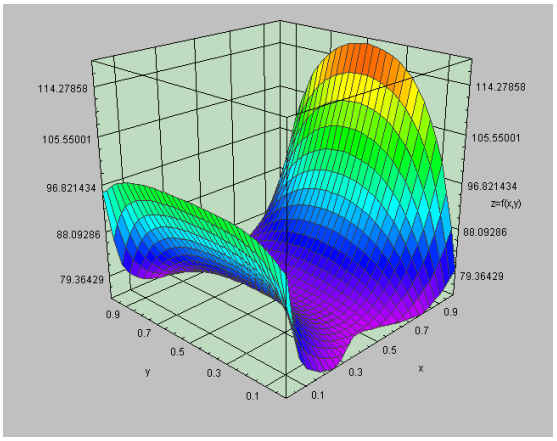
//Take an output at every 2000 seconds
for(int ip=0;ip<=1000;ip++)
{if((ip%100)==0)
  { String s="time = "+ip*tg.dt+" seconds";
            System.out.println(s);
  Text.print(tg.TT[ip],s);
  tg.ContourPlot(tg.TT[ip],s);
  }
 }
 //temperature profile t=10 s
 tg.surfacePlot(tg.TT[1000]);
}
```




ContourPlot
time = 5000.0 seconds
ContourPlot
time = 6000.0 seconds
ContourPlot
time = 7000.0 seconds
ContourPlot
time = 8000.0 seconds
x
y



Time=10000 seconds



time = 10000.0 seconds

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
75.0	87.229309...	91.903096...	93.966679...	94.989316...	95.537405...	95.844329...	96.018494...	96.115020...	96.163070...	96.177116...	96.161579...	96.112110...	96.014311...	95.839098...	95.531436...	94.983018...	93.960580...	91.897889...	87.225955...	75.0
75.0	82.025628...	86.440531...	89.010491...	90.500524...	91.373299...	91.887297...	92.187439...	92.356468...	92.441325...	92.466239...	92.438748...	92.351447...	92.180239...	91.878328...	91.363124...	90.489882...	89.000323...	86.432034...	82.020350...	75.0
75.0	79.456902...	82.872185...	85.207340...	86.724110...	87.682734...	88.275685...	88.632611...	88.837184...	88.940799...	88.971283...	88.937494...	88.830750...	88.623406...	88.264257...	87.669834...	86.710712...	85.194669...	82.861752...	79.450560...	75.0
75.0	77.966402...	80.457632...	82.331470...	83.647255...	84.528255...	85.095278...	85.445269...	85.648647...	85.752226...	85.782615...	85.748494...	85.641390...	85.434907...	85.082450...	84.513833...	83.632361...	82.317492...	80.446240...	77.959570...	75.0
75.0	76.999504...	78.757407...	80.156686...	81.190741...	81.911211...	82.387722...	82.686399...	82.860879...	82.949550...	82.975259...	82.945639...	82.853282...	82.675568...	82.374345...	81.896222...	81.175329...	80.142302...	78.745768...	76.992585...	75.0
75.0	76.333749...	77.534600...	78.522206...	79.274804...	79.811372...	80.170718...	80.396195...	80.526665...	80.591849...	80.610181...	80.587956...	80.519109...	80.385436...	80.157456...	79.796550...	79.259614...	78.508089...	77.523236...	76.327035...	75.0
75.0	75.870725...	76.664043...	77.327440...	77.840183...	78.208006...	78.452874...	78.603306...	78.687174...	78.727039...	78.737404...	78.723318...	78.679954...	78.593037...	78.440234...	78.193907...	77.825770...	77.314086...	76.653332...	75.864422...	75.0
75.0	75.564312...	76.080819...	76.515000...	76.850620...	77.088708...	77.242597...	77.331918...	77.377247...	77.396018...	77.399770...	77.392586...	77.370591...	77.322457...	77.230965...	77.075751...	76.837397...	76.502774...	76.071036...	75.558570...	75.0
75.0	75.393318...	75.753827...	76.056956...	76.289946...	76.452226...	76.552864...	76.606596...	76.629704...	76.636470...	76.636555...	76.633419...	76.623790...	76.598194...	76.542540...	76.440736...	76.278231...	76.046137...	75.745180...	75.388250...	75.0
75.0	75.350164...	75.672652...	75.946166...	76.158827...	76.309018...	76.403737...	76.455470...	76.478582...	76.485978...	76.486518...	76.483388...	76.473561...	76.448338...	76.394974...	76.299266...	76.148885...	75.936982...	75.665312...	75.345861...	75.0
75.0	75.436128...	75.841436...	76.191703...	76.472748...	76.681398...	76.823836...	76.912336...	76.961427...	76.984335...	76.990192...	76.982280...	76.957442...	76.906673...	76.816874...	76.673641...	76.464829...	76.184374...	75.835563...	75.432676...	75.0
75.0	75.659758...	76.276794...	76.817378...	77.261569...	77.604041...	77.851461...	78.018056...	78.120777...	78.175017...	78.191356...	78.173575...	78.117980...	78.014074...	77.846554...	77.598556...	77.255944...	76.812141...	76.272567...	75.657252...	75.0
75.0	76.037342...	77.008689...	77.862856...	78.570745...	79.125152...	79.535613...	79.821418...	80.004867...	80.105779...	80.137588...	80.105039...	80.003426...	79.819358...	79.533055...	79.122259...	78.567732...	77.859995...	77.006323...	76.035899...	75.0
75.0	76.595519...	78.084379...	79.387551...	80.464323...	81.308364...	81.936952...	82.379408...	82.667470...	82.828282...	82.879820...	82.828345...	82.667584...	82.379551...	81.937087...	81.308447...	80.464311...	79.387417...	78.084145...	76.595288...	75.0
75.0	77.377274...	79.576839...	81.478078...	83.029919...	84.235207...	85.128383...	85.756623...	86.166678...	86.396566...	86.470730...	86.397548...	86.168572...	85.759292...	85.131617...	84.238727...	83.033386...	81.481118...	79.579089...	77.378454...	75.0
75.0	78.454908...	81.601057...	84.260826...	86.385687...	88.007512...	89.194594...	90.023302...	90.562265...	90.864179...	90.961810...	90.866204...	90.566187...	90.028861...	89.201392...	88.015012...	86.393216...	84.267605...	81.606257...	78.457771...	75.0
75.0	79.959627...	84.345255...	87.921485...	90.688292...	92.748089...	94.229187...	95.251155...	95.911239...	96.279782...	96.399026...	96.282980...	95.917448...	95.259990...	94.240060...	92.760190...	90.700587...	87.932729...	84.354050...	79.964572...	75.0
75.0	82.155690...	88.130388...	92.731455...	96.137622...	98.595930...	100.32720...	101.50598...	102.26130...	102.68130...	102.81722...	102.68579...	102.27004...	101.51848...	100.34267...	98.613317...	96.155519...	92.748113...	88.143713...	82.163375...	75.0
75.0	85.647960...	93.516339...	99.069745...	102.96608...	105.68916...	107.57027...	108.83641...	109.64237...	110.08909...	110.23381...	110.09499...	109.65387...	108.85292...	107.59086...	105.71254...	102.99054...	99.093054...	93.535643...	85.659646...	75.0
75.0	92.031750...	101.43781...	107.38867...	111.38625...	114.12725...	116.00313...	117.25959...	118.05740...	118.49929...	118.64287...	118.50664...	118.07179...	117.28035...	116.02923...	114.15726...	111.41827...	107.42017...	101.46540...	92.050322...	75.0

15. Partial differential equation solution methods: Integral solutions

INTEGRAL TRANSFORM METHODS

When nonhomogenities in the boundary layers existed, methods like seperation of variables are not always good enough to give us homogeneous solutions, eventhough several methods are existed to convert the problem to the homogeneous one. Integral transform method is derived from seperation of variable methos. Integral transform and inverse integral transform is derived from arbitrary function in

terms of eigenfunctions of the corresponding eigenvalue problem. Consider the following eigenvalue problem:

$\nabla^2\varphi(r) + \lambda^2\varphi(r) = 0$ in region R

With boundary condition:

$k_i \frac{\partial\varphi(r_i)}{\partial n_i} + h_i\varphi(r_i) = 0$ on the boundary S_i

Assuming that eigenfunction $\varphi(\lambda_m, r)$ of this eigenvalue problem satisfy the orthogonality conditions. The general can be applied.

$$\int_R \varphi(\lambda_m, r)\varphi(\lambda_n, r)dv = \begin{cases} 0 & m \neq n \\ N(\lambda_m) & m = n \end{cases}$$

$$N(\lambda_m) = \int_R [\varphi(\lambda_m, r)]^2 dv$$

Now the representation of the function can be given as:

$$T(r,t) = \sum_{m=1}^{\infty} C_m(t)\varphi(\lambda_m, r)$$

By applying the operator $\int_R \varphi(\lambda_m, r)dv$ into the both side of the equation and considering orthogonality condition

$$\int_R \varphi(\lambda_m, r)T(r,t)dv = C_m(t)\int_R [\varphi(\lambda_m, r)]^2 dv = C_m(t)N(\lambda_m)$$

$$C_m(t) = \frac{1}{N(\lambda_m)}\int_R \varphi(\lambda_m, r)T(r,t)dv$$

This expression is introduced into the $T(r,t)$ equation by spliting it into two parts

Inversion formula: $T(r,t) = \sum_{m=1}^{\infty} \frac{\varphi(\lambda_m, r)}{N(\lambda_m)}\bar{T}(\lambda_m, t)$

Integral transform: $\bar{T}(\lambda_m, t) = \int_R \varphi(\lambda_m, r)T(r,t)dv$ where $\bar{T}(\lambda_m, t)$ is called integral transform of the function $T(r,t)$

Now let us look at the heat transfer problem:

$$\nabla^2 T(r,t) + \frac{1}{k}g(r,t) = \frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t}$$ in region R

With the boundary conditions:

$$k_i \frac{\partial T(r_i,t)}{\partial n_i} + h_i T(r_i, t) = f_i(r_i, t)$$ on boundary S_i t>0

$$T(r,t) = F(r)$$ for t=0 in region R

Integrated form of the equation:

$$\int_R \varphi(\lambda_m, r)\nabla^2 T(r,t)dv + \frac{1}{k}\int_R \varphi(\lambda_m, r)g(r,t)dv = \frac{1}{\alpha}\int_R \varphi(\lambda_m, r)\frac{\partial T(r,t)}{\partial t}dv$$

Integral transform of this expression:

$$\int_R \varphi(\lambda_m, r)\nabla^2 T(r,t)dv + \frac{1}{k}\bar{g}(\lambda_m, t) = \frac{1}{\alpha}\frac{\partial \bar{T}(\lambda_m, t)}{\partial t}$$

Where $\bar{g}(\lambda_m, t) = \int_R \varphi(\lambda_m, r)g(r,t)dv$ and $\bar{T}(\lambda_m, t) = \int_R \varphi(\lambda_m, r)T(r,t)dv$

The integral $\int_R \varphi(\lambda_m, r)\nabla^2 T(r,t)dv$ can be evaluated making use of Green's theorem

$$\int_R \varphi(\lambda_m, r)\nabla^2 T(r,t)dv = \int_R T\nabla^2\varphi(\lambda_m, r)\,\mathrm{dv} + \sum_{i=1}^{N}\int_{S_i}\left(\varphi(\lambda_m, r)\frac{\partial T(r,t)}{\partial n_i} - T(r,t)\frac{\partial\varphi(\lambda_m, r)}{\partial n_i}\right)ds_i$$

Where s_i continious boundaries of region R. The integral $\int_R T\nabla^2\varphi(\lambda_m,r)\,\mathrm{dv}$ is evaluated by writing equation $\nabla^2\varphi(r)+\lambda^2\varphi(r)=0$ in region R for the eigenfunction $\varphi(\lambda_m,r)$, multiplying both sides with T(r,t) and integrating in region R and utilizing integral transform.

$\int_R T(r,t)\nabla^2\varphi(\lambda_m,r)dv+\lambda_m^2\int_R T(r,t)\varphi(\lambda_m,r)dv=0$

The surface integral $\int_{S_i}\left(\varphi(\lambda_m,r)\frac{\partial T(r,t)}{\partial n_i}-T(r,t)\frac{\partial\varphi(\lambda_m,r)}{\partial n_i}\right)ds_i$ is evaluated by using boundary condition $k_i\frac{\partial T(r_i,t)}{\partial n_i}+h_iT(r_i,t)=f_i(r_i,t)$ on boundary S_i t>0

$\left(\varphi(\lambda_m,r)\frac{\partial T(r,t)}{\partial n_i}-T(r,t)\frac{\partial\varphi(\lambda_m,r)}{\partial n_i}\right)=\frac{\varphi(\lambda_m,r_i)}{k_i}f_i(r_i,t)$ so equation becomes

$\int_R\varphi(\lambda_m,r)\nabla^2T(r,t)dv=-\lambda_m^2\int_R T(r,t)\varphi(\lambda_m,r)dv+\sum_{i=1}^N\int_{S_i}\left(\frac{\varphi(\lambda_m,r_i)}{k_i}f_i(r_i,t)\right)ds_i$ Substituting this equation into $\int_R\varphi(\lambda_m,r)\nabla^2T(r,t)dv+\frac{1}{k}\bar{g}(\lambda_m,t)=\frac{1}{\alpha}\frac{\partial\bar{T}(\lambda_m,t)}{\partial t}$ equation becomes

$\frac{\partial\bar{T}(\lambda_m,t)}{\partial t}+\alpha\lambda_m^2\bar{T}(\lambda_m,t)=A(\lambda_m,t)$ where $A(\lambda_m,t)=\frac{\alpha}{k}g(\lambda_m,t)+\alpha\sum_{i=1}^N\int_{S_i}\left(\frac{\varphi(\lambda_m,r_i)}{k_i}f_i(r_i,t)\right)ds_i$

Integral transform of the boundary condition $T(r,t)=F(r)$ for t=0 in region R becomes

$\bar{T}(\lambda_m,t)=\int_R\varphi(\lambda_m,r)F(r)dv=\bar{F}(\lambda_m)$ for t=0

Our integral transform equation then becomes:

$$\bar{T}(\lambda_m,t)=\exp(-\alpha\lambda_m^2t)\left[\bar{F}(\lambda_m)+\int_0^t\exp(-\alpha\lambda_m^2t)A(\lambda_m,t)dt\right]$$

Introducing integral transform into the inversion formula gives:

$$T(r,t)=\sum_{m=1}^{\infty}\frac{1}{N(\lambda_m)}\exp(-\alpha\lambda_m^2t)\varphi(\lambda_m,r)\left[\bar{F}(\lambda_m)+\int_0^t\exp(-\alpha\lambda_m^2t)A(\lambda_m,t)dt\right]$$

Where

$$A(\lambda_m,t)=\frac{\alpha}{k}g(\lambda_m,t)+\alpha\sum_{i=1}^N\int_{S_i}\left(\frac{\varphi(\lambda_m,r_i)}{k_i}f_i(r_i,t)\right)ds_i$$

$$\bar{F}(\lambda_m)=\int_R\varphi(\lambda_m,r)F(r)dv$$

$$\bar{g}(\lambda_m,t)=\int_R\varphi(\lambda_m,r)g(r,t)dv$$

$$N(\lambda_m)=\int_R[\varphi(\lambda_m,r)]^2dv$$

Note: in the boundary equation $k_i\frac{\partial T(r_i,t)}{\partial n_i}+h_iT(r_i,t)=f_i(r_i,t)$ on boundary S_i t>0 if $k_i=0$ (which can be happened when a temperature type of boundary is given) the the term $\frac{\varphi(\lambda_m,r_i)}{k_i}$ will be replaced by $-\frac{1}{h_i}\frac{\partial\varphi(\lambda_m,r_i)}{\partial n_i}$

Example problem: Consider the following two dimensional (x,t) heat conduction problem:

$\frac{\partial^2T(x,t)}{\partial x^2}+\frac{1}{k}g(x,t)=\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ in $0\le x\le L$ t>0

$\frac{\partial T(x,t)}{\partial x}=0$ at x=0 t>0

$T(x,t)$=0 at x=l t>0

$T(x,t)=F(x)$ at t=0 $\quad 0\le x\le L$

Solution of the problem then has a form:

$$T(r,t)=\sum_{m=1}^{\infty}\frac{1}{N(\beta_m)}\exp(-\alpha\lambda_m^2 t)X(\beta_m,x)\left[\bar{F}(\beta_m)+\frac{\alpha}{k}\int_0^t \bar{g}(\beta_m,t)dt\right]$$

Where $X(\beta_m,x)=\cos(\beta_m x)$ $\frac{1}{N(\beta_m)}=\frac{2}{L}$ and $\cos(\beta_m L)=0 \rightarrow \beta_m=\frac{(2m-1)\pi}{L}$ m=1..N

$$T(x,t)=\frac{2}{L}\sum_{m=1}^{\infty}\exp(-\alpha\lambda_m^2 t)\cos(\beta_m x)\left[\int_{x=0}^{L}F(x)\cos(\beta_m x)dx+\frac{\alpha}{k}\int_{t=0}^{t}\int_{x=0}^{L}\exp(\alpha\lambda_m^2 t)\cos(\beta_m x)g(x,t)\cos(\beta_m x)dxdt\right]$$

Note: equation involve double integration in order to solve double integration numerical analysis info about numerical information solution is given in below info box

MULTIDIMENSIONAL INTEGRATION BY PARAMETRIC METHODS

A two dimensional integration

$$I(D)=\int\int_D f(x,y)dydx$$

Where D is a region in xy plane. If this region is defined as $a\le x\le b$ and $c(x)\le y\le d(x)$, one way two calculate two dimensional integration is to repeat one dimensional integration twice. In the first step intehration function g(x) is calculated as an integration in y dimension and then function g(x) is solved to get integration result.

$$I=\int_a^b\int_{c(x)}^{d(x)}f(x,y)dydx$$

$$g(x)=\int_{c(x)}^{d(x)}f(x,y)dy,\quad a\le x\le b$$

$$I(D)=\int_a^b g(x)dx$$

Any integration formulation can be used for this purpose. All it has to be done is to evaluate integration Formula twice onece for each independent variable. As our first example Gauss-Legendre formulation will be used to evaluate double integration. Two code is defined. In the first one Gauss-Legendre coefficients are defined as constants, in the second one they are calculated through Legendre polynomials. In the two dimensional integration x dimesion limits are still given as constant values, but y dimension limits c(x) and d(x) should be given as functions for proper evaluation of the integral.

$$I=\int_a^b\int_{c(x)}^{d(x)}f(x,y)dy=\int_a^b\frac{d(x)-c(x)}{2}\sum_{J=1}^{n}c_{n,j}f\left(x,\frac{(d(x)-c(x))r_{n,j}+d(x)+c(x)}{2}\right)dx$$

In this equation $r_{n,j}$ root values and $c_{n,j}$ are equation coefficients. These values can be calculated from Legendre polynomials.

Now we can create numerical solution for the given final formulation.

```
//=====================================================
// Numerical Analysis package in java
// example to show utilisation of integration (integral)
// and differentiation (derivative) functions
// Gauss-Legendre integration with two variables
// constant number of coefficients (60 coefficients)
// and roots
```

```
// Dr. Turhan Coban
// ====================================================
import java.io.*;
abstract class fdi extends Mathd
{
public double a,b;
public fdi(double ai,double bi)
{a=ai;b=bi;}
public fdi()
{a=0;b=1;}
public void setab(double ai,double bi)
{a=ai;b=bi;}
abstract public  double c(double x);
abstract public  double d(double x);
abstract public  double f(double x[]);
}
class f1 extends fdi
{ int m;
  double L,t;
  double beta;
  double alpha;
  if_xj g;
  public f1( if_xj gi,int mi,double alphai,double Li,double ti)
  {super(0.0,Li);g=gi;alpha=alphai;m=mi;L=Li;t=ti;beta=(2.0*m-1.0)*Math.PI/L;}
  //lower integration limit function
  public double c(double x)
  {return 0.0;}
  //upper integration limit function
  public double d(double x)
  {return L;}
  public double f(double x[])
  {
  //function required to integrate

  double ff=exp(alpha*beta*beta*x[1])*g.func(x)*cos(beta*x[0]);
  return ff;
  }
}
class HT_int_tr1
{
public static double[][] gauss_legendre_coefficients(double x1,double x2,int n)
{   //calculates legendre gauss-coefficients as coefficients of the integral
              //for n terms
              double EPS=3.0e-15;
              int m,j,i;
              double z1,z,xm,xl,pp,p3,p2,p1;
    //double x[]=new double[n];
    //double w[]=new double[n];
    double a[][]=new double[2][n];//a[0][i]=x[i]   a[1][i]=w[i]
              m=(n+1)/2;
              xm=0.5*(x2+x1);
              xl=0.5*(x2-x1);
              for (i=1;i<=m;i++)  {
                        z=Math.cos(Math.PI*((i-0.25)/(n+0.5)));
                        do {
                                  p1=1.0;
                                  p2=0.0;
                                  for (j=1;j<=n;j++) {
                                            p3=p2;
                                            p2=p1;
                                            p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
                                  }
                                  pp=n*(z*p1-p2)/(z*z-1.0);
                                  z1=z;
                                  z=z1-p1/pp;
                        } while (Math.abs(z-z1) > EPS);
                        a[0][i-1]=xm-xl*z;
                        a[0][n-i]=xm+xl*z;
                        a[1][i-1]=2.0*xl/((1.0-z*z)*pp*pp);
                        a[1][n-i]=a[1][i-1];
                                  }
return a;
}

public static double I(fdi f)
{
```

```
//integral f(x)dx
//integral of a function by using gauss-legendre quadrature
//coefficients are pre-calculated for 60 terms for [-1,1]
//band then utilises variable transform
 int i,j;
 int n=60;
 double r[],c[];
 r=new double[n];
 c=new double[n];
 r[ 0] = .15532579626752470000E-02;
 r[ 1] = .81659383601264120000E-02;
 r[ 2] = .19989067515846230000E-01;
 r[ 3] = .36899976285362850000E-01;
 r[ 4] = .58719732103973630000E-01;
 r[ 5] = .85217118808615820000E-01;
 r[ 6] = .11611128394758690000E+00;
 r[ 7] = .15107475260334210000E+00;
 r[ 8] = .18973690850537860000E+00;
 r[ 9] = .23168792592899010000E+00;
 r[10] = .27648311523095540000E+00;
 r[11] = .32364763723456090000E+00;
 r[12] = .37268153691605510000E+00;
 r[13] = .42306504319570830000E+00;
 r[14] = .47426407872234120000E+00;
 r[15] = .52573592127765890000E+00;
 r[16] = .57693495680429170000E+00;
 r[17] = .62731846308394490000E+00;
 r[18] = .67635236276543910000E+00;
 r[19] = .72351688476904450000E+00;
 r[20] = .76831207407100990000E+00;
 r[21] = .81026309149462140000E+00;
 r[22] = .84892524739665800000E+00;
 r[23] = .88388716052413100000E+00;
 r[24] = .91478288119138420000E+00;
 r[25] = .94128026789602640000E+00;
 r[26] = .96310002371463720000E+00;
 r[27] = .98001093248415370000E+00;
 r[28] = .99183406163987350000E+00;
 r[29] = .99844674203732480000E+00;
c[ 0] = .39840962480827790000E-02;
c[ 1] = .92332341555455000000E-02;
c[ 2] = .14392353941661670000E-01;
c[ 3] = .19399596284813530000E-01;
c[ 4] = .24201336415292590000E-01;
c[ 5] = .28746578108808720000E-01;
c[ 6] = .32987114941090080000E-01;
c[ 7] = .36877987368852570000E-01;
c[ 8] = .40377947614710090000E-01;
c[ 9] = .43449893600541500000E-01;
c[10] = .46061261118893050000E-01;
c[11] = .48184368587322120000E-01;
c[12] = .49796710293397640000E-01;
c[13] = .50881194874202750000E-01;
c[14] = .51426326446779420000E-01;
c[15] = .51426326446779420000E-01;
c[16] = .50881194874202750000E-01;
c[17] = .49796710293397640000E-01;
c[18] = .48184368587322120000E-01;
c[19] = .46061261118893050000E-01;
c[20] = .43449893600541500000E-01;
c[21] = .40377947614710090000E-01;
c[22] = .36877987368852570000E-01;
c[23] = .32987114941090080000E-01;
c[24] = .28746578108808720000E-01;
c[25] = .24201336415292590000E-01;
c[26] = .19399596284813530000E-01;
c[27] = .14392353941661670000E-01;
c[28] = .92332341555455000000E-02;
c[29] = .39840962480827790000E-02;
for(i=0;i<30;i++) {r[i+30]=-r[i];c[i+30]=c[i];}
int m=n;
double x[],h1,h2,J,a1,b1,c1,d1,k1,k2,JX,Q;
x=new double[2];
//1=========================
a1=f.a;
b1=f.b;
```

```
h1=(b1-a1)/2.0;
h2=(b1+a1)/2.0;
J=0;
//2=========================
for(i=0;i<m;i++)
{
//3=========================
JX=0;
x[0]=h1*r[i]+h2;
d1=f.d(x[0]);
c1=f.c(x[0]);
k1=(d1-c1)/2.0;
k2=(d1+c1)/2.0;
//4=========================
   for(j=0;j<n;j++)
   {
            x[1]=k1*r[j]+k2;
            Q=f.f(x);
            JX+=c[j]*Q;
            }
J=J+c[i]*k1*JX;
}
J=h1*J;
return J;
}
public static void main(String args[]) throws IOException
  { double L=1.0;  //m
    double t=3600;
    double alpha=1.0e-6;
    double T=10;
    double integ1=0;
    double integ2=0;
    double x=0.1;
    double k=50.0; //  W/(mK)
    //x1[0]=x  x1[1]=t
    if_xj g=(double x1[])->100*x1[0];
    if_x F=(x1)->20.0*x1;
    for(int m=1;m<50;m++)
    {f1 ff=new f1(g,m,alpha,L,t);
     double beta=(2.0*m-1.0)*Math.PI/L;
     integ2=I(ff);
     if_x y1=(x1)->F.func(x1)*Math.cos(beta*x1);
     integ1=y1.integral(0.0,t);
     System.out.println("m="+m+"beta="+beta+"integ1="+integ1+"integ2="+integ2);
     T+=Math.exp(-alpha*beta*beta*t)*Math.cos(beta*x)*(y1.integral(0.0,L)+alpha/k*integ2);
    }
    T*=2.0/L;
    System.out.println("x = "+x+"t="+t+"T="+T);
}
 }
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" HT_int_tr1
m=1beta=3.141592653589793integ1=1.6360812510762941E7integ2=-20.26433672879602
m=2beta=9.42477796076938integ1=1.3378589112125183E7integ2=-2.2516818616792733
m=3beta=15.707963267948966integ1=9705177.59864646integ2=-0.8106694773636043
m=4beta=21.991148575128552integ1=8720971.602538412integ2=-0.4136558677234771
m=5beta=28.274333882308138integ1=1.3050998575900134E7integ2=-0.2502757887329145
m=6beta=34.55751918948772integ1=2.2945813251362227E7integ2=-0.16757307063293492
m=7beta=40.840704496667314integ1=3.567774863499815E7integ2=-0.12000678182580861
m=8beta=47.12388980384689integ1=4.630412345322993E7integ2=-0.09016334841184619
m=9beta=53.40707511102649integ1=4.954023133207446E7integ2=-0.07021856133348224
m=10beta=59.690260418206066integ1=4.1972597972257584E7integ2=-0.05623373861661546
m=11beta=65.97344572538566integ1=2.365693112556217E7integ2=-0.04605079541921173
m=12beta=72.25663103256524integ1=-1623589.61221613integ2=-0.03840685994864523
m=13beta=78.53981633974483integ1=-2.7606452680663705E7integ2=-0.03252298469988887
m=14beta=84.82300164692441integ1=-4.782444433133388E7integ2=-0.02789754716134818
m=15beta=91.106186954104integ1=-5.797368951710145E7integ2=-0.024195683589059554
m=16beta=97.38937226128358integ1=-5.738689837913661E7integ2=-0.0211869316094581
m=17beta=103.67255756846318integ1=-4.899992406146612E7integ2=-0.018708473766491245
m=18beta=109.95574287564276integ1=-3.785017090878593E7integ2=-0.016642638294184813
m=19beta=116.23892818282235integ1=-2.878650013768683E7integ2=-0.01490267011736603
m=20beta=122.52211349000193integ1=-2.441563417084229E7integ2=-0.013423471827733192
m=21beta=128.8052987971815integ1=-2.4198906073778387E7integ2=-0.01215542554799932
m=22beta=135.0884841043611integ1=-2.5086898664658315E7integ2=-0.011060179753506549
```

```
m=23beta=141.3716694115407integ1=-2.336915985368206E7integ2=-0.010107695437541886
m=24beta=147.6548547187203integ1=-1.684344505427437E7integ2=-0.00927399499869856
m=25beta=153.93804002589985integ1=-6241877.52025964integ2=-0.00854085875742046
m=26beta=160.22122533307945integ1=4820651.528156524integ2=-0.007901241038355473
m=27beta=166.50441064025904integ1=1.142472281391428E7integ2=-0.0073135524924237295
m=28beta=172.78759594743863integ1=9765027.911342267integ2=-0.006555576851762927
m=29beta=179.0707812546182integ1=-718015.944600611integ2=-0.006291931602300288
m=30beta=185.3539665617978integ1=-1.6548858425989337E7integ2=-0.009976501890803399
m=31beta=191.63715186897738integ1=-3.125090952243381E7integ2=-0.007262575021289047
m=32beta=197.92033717615698integ1=-3.785698714610449E7integ2=0.039051484110085306
m=33beta=204.20352248333654integ1=-3.1842626838082083E7integ2=0.021249021486473132
m=34beta=210.48670779051614integ1=-1.3179771327419568E7integ2=-0.32668676317057493
m=35beta=216.76989309769573integ1=1.3448131605640331E7integ2=-0.23482953724525646
m=36beta=223.05307840487532integ1=4.042899943900705E7integ2=1.5532892857783265
m=37beta=229.3362637120549integ1=6.0261448620811075E7integ2=1.2623235265568737
m=38beta=235.61944901923448integ1=6.852209758517803E7integ2=-4.871138309542406
m=39beta=241.90263432641407integ1=6.5442147005385354E7integ2=-4.359354083492212
m=40beta=248.18581963359367integ1=5.539400103293332E7integ2=9.06071206626194
m=41beta=254.46900494077323integ1=4.454507727295937E7integ2=8.83498426317714
m=42beta=260.75219024795285integ1=3.77897563094229E7integ2=-7.571687802222459
m=43beta=267.0353755551324integ1=3.641735753539598E7integ2=-8.343554685516132
m=44beta=273.318560862312integ1=3.763752315520038 5E7integ2=-2.5427046498639174
m=45beta=279.6017461694916integ1=3.620672003842143E7integ2=-1.5033151341933508
m=46beta=285.88493147667117integ1=2.738455925997177 5E7integ2=6.605926242152686
m=47beta=292.16811678385073integ1=9777865.891168052integ2=6.980448899306837
m=48beta=298.45130209103036integ1=-1.336392960787008E7integ2=3.42632459693947
m=49beta=304.7344873982099integ1=-3.5137308079091646E7integ2=2.6442578297236365
x = 0.1t=3600.0T=12.194514058635923

> Terminated with exit code 0.
```

16. Partial differential equation solution methods: Approximations solutions

APPROXIMATION METHODS IN HEAT CONDUCTION

Approximate methods used an approximation equation such as a polynomial to approximate heat transfer temperature profile. As a first example case let us consider semi-infinite wall with constant surface temperature. Let us solve the problem first as an exact solution by using Laplace transform method.

EXAMPLE 1: SEMI-INFINITE WALL WITH CONSTANT TEMPERATURE APPROXIMATION METHODS

$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ with boundary condition:

$T(x,t) = T_0$ $x=0$ $t > 0$

$T(x,t) = T_i$ $x \to \infty$ $t > 0$

$T(x,t) = T_i$ at $t=0$

Defining $\theta = T(x,t) - T_i$

$\frac{\partial^2 \theta(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta(x,t)}{\partial t}$ with boundary condition:

$\theta(x,t) = \theta_0$ $x=0$ $t > 0$

$\theta(x,t) = 0$ $x \to \infty$ $t > 0$

$\theta(x,t) = 0$ at $t=0$

Laplace transform of the equation is:

$$\frac{\partial^2\theta(x,s)}{\partial x^2}-\frac{s}{\alpha}\theta(x,s)=0$$

$\theta(x,s)=\frac{\theta_0}{s}$ x=0

$\theta(x,s)=0 \quad x\to\infty$

$A^2-\frac{s}{\alpha}=0 \quad A=\sqrt{\frac{s}{\alpha}}$

$$\theta(x,s)=C_1\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)+C_2\exp\left(\sqrt{\frac{s}{\alpha}}x\right)$$

$\theta(x,s)=0 \quad x\to\infty$ ➔ $C_2=0$

$$\theta(x,s)=C_1\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)$$

$\theta(x,s)=\frac{\theta_0}{s}$ x=0 ➔ $C_1=\frac{\theta_0}{s}$

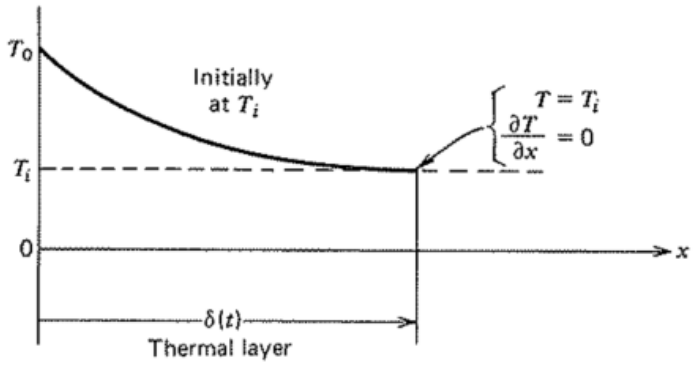
$$\theta(x,s)=\frac{\theta_0}{s}\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)$$

inversion of the Laplace transform yields:

$$\theta(x,t)=\theta_0 erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)=\theta_0\left(1-erf\left(\frac{x}{2\sqrt{\alpha t}}\right)\right)$$

$$\frac{T(x,t)-T_i}{T_0-T_i}=1-erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Now that we have the exact solution let us look at the approximate solution method. It will be assumed that at time t, effect of surface temperature will be penetrate some thermal layer with penetration depth $\delta(t)$. For the remaining profile, temperature will remain constant at initial temperature, T_i.



Considering this condition differential equation can be integrate for the thermal layer only for integration depth $\delta(t)$. Therefore differential equation can be integrate for this integration depth.

$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=\delta(t)}-\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0}=\frac{1}{\alpha}\int_{x=0}^{x=\delta(t)}\frac{\partial T(x,t)}{\partial t}dx$ and considering

$$\frac{\partial}{\partial t}\left[\int_{x=0}^{x=\delta(t)} T(x,t)dx\right] = \int_{x=0}^{x=\delta(t)} \frac{\partial T(x,t)}{\partial t}dx + \left[T\frac{\partial x}{\partial t}\right]_{x=\delta(t)} - \left[T\frac{\partial x}{\partial t}\right]_{x=0}$$

$$\int_{x=0}^{x=\delta(t)} \frac{\partial T(x,t)}{\partial t}dx = \frac{\partial}{\partial t}\left[\int_{x=0}^{x=\delta(t)} T(x,t)dx\right] - T_i\frac{\partial \delta(t)}{\partial t} \quad \text{defining}$$

$\theta(x,t) = \left[\int_{x=0}^{x=\delta(t)} T(x,t)dx\right]$ differential eqaution becomes:

$$-\alpha\frac{\partial T(x,t)}{\partial x}\bigg|_{x=0} = \frac{\partial}{\partial t}\left[\theta(x,t) - T_i\delta(t)\right]$$

In order to solve this differential equation, an approximate temperature profile T(x,t) will be assumed. As a first case, let us take a cubic polynomial

$T(x,t) = a + bx + cx^2 + dx^3$ Coefficients can be found by applying boundary conditions:

$$T(x,t)\big|_{x=0} = T_0 \quad T(x,t)\big|_{x=\delta(t)} = T_i \quad \frac{\partial T(x,t)}{\partial x}\bigg|_{x=\delta(t)} = 0 \quad \frac{\partial^2 T(x,t)}{\partial x^2}\bigg|_{x=0} = 0$$

$$\frac{\partial T(x,t)}{\partial x} = b + 2cx + 3dx^2 \qquad \frac{\partial^2 T(x,t)}{\partial x^2} = 2c + 6dx$$

$\frac{\partial^2 T(x,t)}{\partial x^2}\Big|_{x=0} = 0$ ➔ 2c=0 c=0

$T(x,t)\big|_{x=0} = T_0$ ➔ $T_0 = a$

$\frac{\partial T(x,t)}{\partial x}\Big|_{x=\delta(t)} = 0$ ➔ $b + 3d\delta^2 = 0$ $b = -3d\delta^2$

$T(x,t)\big|_{x=\delta(t)} = T_i$ ➔ $T_i = T_0 + (-3d\delta^2)\delta + d\delta^2$ $d = \frac{T_0 - T_i}{2\delta^3}$ $b = -\frac{3}{2}\frac{T_0 - T_i}{\delta}$

$$T(x,t) - T_i = (T_0 - T_i) - \frac{3}{2}(T_0 - T_i)\frac{x}{\delta} + \frac{1}{2}(T_0 - T_i)\left(\frac{x}{\delta}\right)^3$$

$\frac{T(x,t) - T_i}{T_0 - T_i}_{i} = 1 - \frac{3}{2}\frac{x}{\delta} + \frac{1}{2}\left(\frac{x}{\delta}\right)^3$ This equation can now be substituted into the differential equation

$$\frac{\partial T(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[(T_0 - T_i) - \frac{3}{2}(T_0 - T_i)\frac{x}{\delta} + \frac{1}{2}(T_0 - T_i)\left(\frac{x}{\delta}\right)^3 - T_i\right] = -\frac{3}{2\delta}(T_0 - T_i) + \frac{3}{2}(T_0 - T_i)\frac{x^2}{\delta^3}$$

$$\frac{\partial T(x,t)}{\partial x}\bigg|_{x=0} = -\frac{3}{2\delta}(T_0 - T_i)$$

$$\theta(x,t) = \int_{x=0}^{x=\delta(t)} T(x,t)dx = \int_{x=0}^{x=\delta(t)} \left[(T_0 - T_i) - \frac{3}{2}(T_0 - T_i)\frac{x}{\delta} + \frac{1}{2}(T_0 - T_i)\left(\frac{x}{\delta}\right)^3 + T_i\right]dx$$

$$\theta(x,t) = (T_0 - T_i)\left[x - \frac{3}{4}\frac{x^2}{\delta} + \frac{1}{8}\frac{x^4}{\delta^3} - T_i x\right]_{x=0}^{x=\delta} = (T_0 - T_i)\left[\delta - \frac{3}{4}\delta + \frac{1}{8}\delta\right] - T_i\delta$$

Substituting the results into the differential equation gives

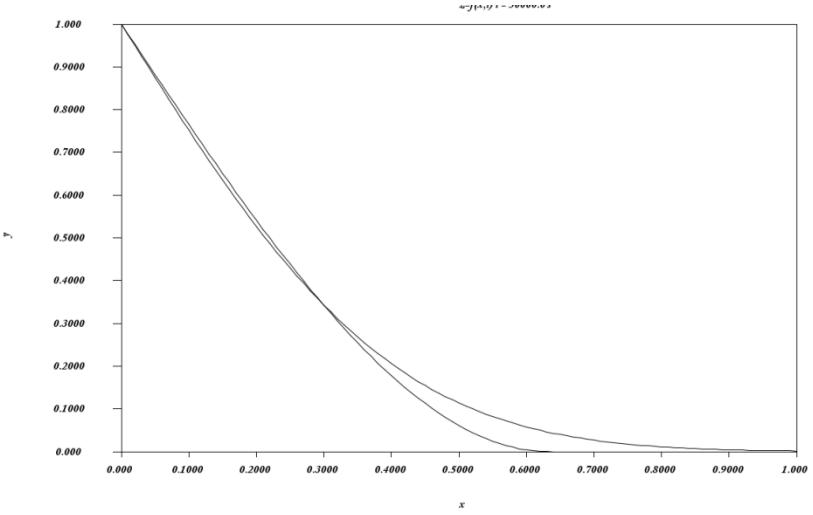
$$\frac{3\alpha}{2\delta}(T_0 - T_i) = \frac{\partial}{\partial t}\left[(T_0 - T_i)\left[\frac{3}{8}\delta\right] + T_i\delta - T_i\delta\right]$$

$\frac{3\alpha}{2\delta} = \frac{\partial}{\partial t}\left[\frac{3}{8}\delta\right]$ $4\alpha\partial t = \delta\partial[\delta]$ $4\alpha t = \frac{\delta^2}{2}$ $\delta(t) = \sqrt{8\alpha t}$ By using this result, together with approximation temperature profile will give us approximation solution. The results can be compare by using a graphic program

Problem 9.1 java solution

```
 class i_wall extends fi_xt
{
public double[] func(double x,double t)
{ double a[]=new double[2];
            double alpha=1.0e-6;
  double delta=Math.sqrt(8.0*alpha*t);
  double eta=x/2/Math.sqrt(alpha*t);
  double xx=x/delta;

  a[0]=erfc(eta);
  if(xx<1) a[1]=1.0-1.5*xx+0.5*xx*xx*xx;
  else a[1]=0.0;
  return a;
}          }
public class timeplot_ntest1
{
public static void main(String args[])
 {
 i_wall ff=new i_wall();
  timeplot_n p=new timeplot_n(ff,0.0,1.0,0.01,0.0,50000.0,0.1,100.0);
  p.start();
  }}
```



For better accuracy, degree of polynomial can be increased. Of course additional boundary conditions to solve thic new equations is required. If a 4th degree polynomial is assumed:

$T(x,t) = a + bx + cx^2 + dx^3 + ex^4$ Coefficients can be found by applying boundary conditions:

$T(x,t)\big|_{x=0} = T_0$ $T(x,t)\big|_{x=\delta(t)} = T_i$ $\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=\delta(t)} = 0$ $\left.\frac{\partial^2 T(x,t)}{\partial x^2}\right|_{x=0} = 0$ $\left.\frac{\partial^2 T(x,t)}{\partial x^2}\right|_{x=\delta(t)} = 0$

$\frac{\partial T(x,t)}{\partial x} = b + 2cx + 3dx^2 + 4ex^3$ $\frac{\partial^2 T(x,t)}{\partial x^2} = 2c + 6dx + 12ex^2$

$\left.\frac{\partial^2 T(x,t)}{\partial x^2}\right|_{x=0} = 0$ ➔ 2c=0 c=0

$T(x,t)\big|_{x=0} = T_0$ ➔ $T_0 = a$

$$\left.\frac{\partial^2 T(x,t)}{\partial x^2}\right|_{x=\delta(t)} = 0 \rightarrow \quad 6d\delta + 12e\delta^2 = 0 \qquad d = -2e\delta$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=\delta(t)} = 0 \rightarrow \quad b + 3d\delta^2 + 4e\delta^3 = b + 3(-2e\delta)\delta^2 + 4e\delta^3 = b - 2e\delta^3 = 0 \qquad b = 2e\delta^3$$

$$T(x,t)\big|_{x=\delta(t)} = T_i \rightarrow \quad T_i = T_0 + (2e\delta^3)\delta + (-2e\delta)\delta^3 + e\delta^4 \quad e = -\frac{T_0 - T_i}{\delta^4} \quad d = \frac{2(T_0 - T_i)}{\delta^3}$$

$$b = -\frac{2(T_0 - T_i)}{\delta}$$

$\frac{T(x,t)-T_i}{T_0-T_i} = 1 - 2\frac{x}{\delta} + 2\left(\frac{x}{\delta}\right)^3 - \left(\frac{x}{\delta}\right)^4$ We can apply this new temperature profile into the differential equation

$$T(x,t) = (T_0 - T_i) - 2(T_0 - T_i)\frac{x}{\delta} + 2(T_0 - T_i)\left(\frac{x}{\delta}\right)^3 - (T_0 - T_i)\left(\frac{x}{\delta}\right)^4 + T_i$$

$$\frac{\partial T(x,t)}{\partial x} = -2(T_0 - T_i)\frac{1}{\delta} + 6(T_0 - T_i)\frac{x^2}{\delta^3} - 4(T_0 - T_i)\frac{x^3}{\delta^4} \qquad \left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = -2(T_0 - T_i)\frac{1}{\delta}$$

$$\theta(x,t) = \int_{x=0}^{x=\delta(t)} T(x,t)dx = \int_{x=0}^{x=\delta(t)} \left[(T_0 - T_i) - 2(T_0 - T_i)\frac{x}{\delta} + 2(T_0 - T_i)\left(\frac{x}{\delta}\right)^3 - (T_0 - T_i)\left(\frac{x}{\delta}\right)^4 + T_i \right] dx$$

$$\theta(x,t) = (T_0 - T_i)\left[x - \frac{x^2}{\delta} + \frac{1}{2}\frac{x^4}{\delta^3} - \frac{1}{5}\frac{x^5}{\delta^4} - T_i x \right]_{x=0}^{x=\delta} = (T_0 - T_i)\left[\delta - \delta + \frac{1}{2}\delta - \frac{1}{5}\delta \right] - T_i\delta = (T_0 - T_i)\left[\frac{3}{10}\delta \right] - T_i\delta$$

Substituting the results into the differential equation gives

$$\frac{2\alpha}{\delta}(T_0 - T_i) = \frac{\partial}{\partial t}\left[(T_0 - T_i)\left[\frac{3}{10}\delta \right] + T_i\delta - T_i\delta \right]$$

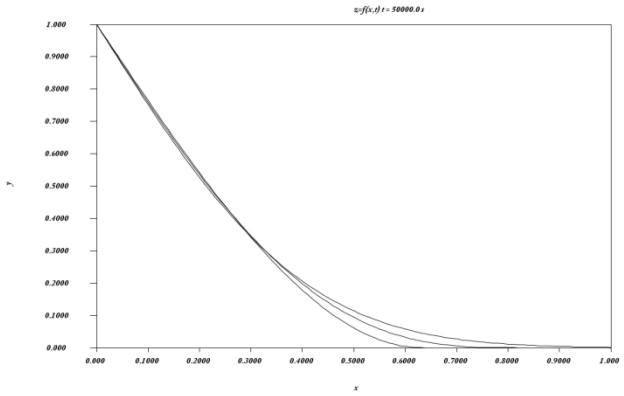
$\frac{20\alpha}{3\delta} = \frac{\partial}{\partial t}[\delta] \qquad \frac{20}{3}\alpha\partial t = \delta\partial[\delta] \qquad \frac{20}{3}\alpha t = \frac{\delta^2}{2} \qquad \delta(t) = \sqrt{\frac{40}{3}\alpha t}$ By using this result, together with approcimation temperature profile will give us approximation solution. The results can be compare by using a graphic program

Problem 9.1 java solution

```
 class i_wall extends fi_xt
{
public double[] func(double x,double t)
{ double a[]=new double[3];
            double alpha=1.0e-6;
  double delta=Math.sqrt(8.0*alpha*t);
  double delta1=Math.sqrt(40.0/3.0*alpha*t);
  double eta=x/2/Math.sqrt(alpha*t);
  double xx=x/delta;
  double xx1=x/delta1;

  a[0]=erfc(eta);
  if(xx<1) a[1]=1.0-1.5*xx+0.5*xx*xx*xx;
  else a[1]=0.0;
  if(xx1<1) a[2]=1.0-2*xx1+2.0*xx1*xx1*xx1-xx1*xx1*xx1*xx1;
  else a[2]=0.0;
  return a;
}          }
public class timeplot_ntest
{
public static void main(String args[])
  {
 i_wall ff=new i_wall();
  timeplot_n p=new timeplot_n(ff,0.0,1.0,0.01,0.0,50000.0,0.1,100.0);
```

```
  p.start();
  }
}
```



Now let us consider 1 dimensional, time dependent (Lumped) solution

EXAMPLE 2: TIME DEPENDENT LUMPED SOLUTION APPROXIMATION METHODS (COLLOCATION METHOD)

$-hA_s(T-T_\infty)=\rho VC_p\frac{dT}{dt}$ where A_s is surface area, V is volume, C_p is specific heat. By changing variables as:

$\theta(t)=\frac{T(t)-T_\infty}{T_i-T_\infty}\quad \tau=\frac{t}{\tau_{\max}}\quad X=\frac{\rho VC_p}{\tau_{\max}hA_s}$ equation becomes:

$$X\frac{d\theta(\tau)}{dt}+\theta(\tau)=0$$

Boundary condition t=0 $\theta=1$ so exact solution is:

$$\theta(\tau)=\exp\left(-\frac{\tau}{X}\right)$$

Now let us create approximate solution, for this the following polynomial is assumed:

Th $\theta(\tau)=1+c_1\tau+c_2\tau^2+c_3\tau^3+c_4\tau^4$ is equation automatically satisfy the boundary condition but if it is applied to differential equation a residue is remained as:

$$\frac{d\theta(\tau)}{d\tau}=c_1+2c_2\tau\ +3c_3\tau^2+4c_4\tau^3$$

$$X\frac{d\theta(\tau)}{dt}+\theta(\tau)=0$$

$$R=X\left(c_1+2c_2\tau\ +3c_3\tau^2+4c_4\tau^3\right)+(1+c_1\tau+c_2\tau^2+c_3\tau^3+c_4\tau^4)$$

$$R=1+(X+\tau)c_1+(2\tau X+\tau^2)c_2+(3\tau^2X+\tau^3)c_3+(4\tau^3+\tau^4)c_4$$

Assuming that equation is valid in the range of $0\le\tau\le1$ the equation will be evaluate at collocation locations such as:

$\tau_1=0.25\ \tau_1=0.5\ \tau_3=0.75\ \tau_4=1.0\ \tau_{\max}=60\ s$. This method is called **Collocation method**

Coefficients can be obtained by solving the system of equations

$$\begin{bmatrix}(X+\tau_1) & (2\tau_1X+\tau_1^2) & (3\tau_1^2X+\tau_1^2) & (4\tau_1^3+\tau_1^4)\\ (X+\tau_2) & (2\tau_2X+\tau_2^2) & (3\tau_2^2X+\tau_2^2) & (4\tau_2^3+\tau_2^4)\\ (X+\tau_3) & (2\tau_3X+\tau_3^2) & (3\tau_2^2X+\tau_2^2) & (4\tau_2^3+\tau_2^4)\\ (X+\tau_4) & (2\tau_4X+\tau_4^2) & (3\tau_4^2X+\tau_4^2) & (4\tau_2^3+\tau_2^4)\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}-1\\-1\\-1\\-1\end{bmatrix}$$

Let us write a small program to do this task:

```
import java.io.*;

class collocation
{
 public static double[] pivotlugauss(double a[][],double b[])
 { //gauss elimination with partial pivoting
 int n=b.length;
 double x[]=new double[n];
 double carpan=0;
 double toplam=0;
 double buyuk;
 double dummy=0;
 //gauss eleme
 int i,j,k,p,ii,jj;
 System.out.println("Original matrix :\n"+Matrix.toString(a));
 for(k=0;k<(n-1);k++)
 { //pivoting
              p=k;
              buyuk=Math.abs(a[k][k]);
              for(ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[ii][k]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              if(p!=k)
              { for(jj=k;jj<n;jj++)
                { dummy=a[p][jj];
                  a[p][jj]=a[k][jj];
                  a[k][jj]=dummy;
                }
                dummy=b[p];
                b[p]=b[k];
                b[k]=dummy;
              }
                System.out.println("pivoted matrix : k="+k+"\n"+Matrix.toString(a));
              //Solving gauss elimination
  for(i=k+1;i<n;i++)
  {  carpan=a[i][k]/a[k][k];
     a[i][k]=0;
     for(j=k+1;j<n;j++)
     {   a[i][j]-=carpan*a[k][j]; }
        b[i]   =b[i]   -carpan*b[k];
              }
 }
 System.out.println("final upper triangular matrix : \n"+Matrix.toString(a));
 //back substitution
 x[n-1]=b[n-1]/a[n-1][n-1];
 for(i=n-2;i>=0;i--)
 {
  toplam=0;
  for(j=i+1;j<n;j++)
  {  toplam+=a[i][j]*x[j];}
  x[i]=(b[i]-toplam)/a[i][i];
 }
 return x;
 }
 public static double[] c(double X, double to[])
 {  double a[][]=new double[4][4];
    a[0][0]=(X+to[0]);a[1][0]=(X+to[1]);a[2][0]=(X+to[2]);a[3][0]=(X+to[3]);
    a[0][1]=(2.0*X*to[0]+to[0]*to[0]);a[1][1]=(2.0*X*to[1]+to[1]*to[1]);
    a[2][1]=(2.0*X*to[2]+to[2]*to[2]);a[3][1]=(2.0*X*to[3]+to[3]*to[3]);
    a[0][2]=(3.0*X*to[0]*to[0]+to[0]*to[0]*to[0]);a[1][2]=(3.0*X*to[1]*to[1]+to[1]*to[1]*to[1]);
              a[2][2]=(3.0*X*to[2]*to[2]+to[2]*to[2]*to[2]);a[3][2]=(3.0*X*to[3]*to[3]+to[3]*to[3]*to[3]);
    a[0][3]=(4.0*X*to[0]*to[0]*to[0]+to[0]*to[0]*to[0]*to[0]);
    a[1][3]=(4.0*X*to[1]*to[1]*to[1]+to[1]*to[1]*to[1]*to[1]);
    a[2][3]=(4.0*X*to[2]*to[2]*to[2]+to[2]*to[2]*to[2]*to[2]);
    a[3][3]=(4.0*X*to[3]*to[3]*to[3]+to[3]*to[3]*to[3]*to[3]);
              double b[]={-1,-1,-1,-1};
              double c[]=pivotlugauss(a,b);
              return c;
 }

 public static void main(String args[]) throws IOException
 {
 double toi[]={0.25,0.5,0.75,1.0};
 double ro=8500; //kg/m^3
 double Cp=400; //J/kgK
```

```
  double h=80; //W/m2K
  double D=0.02;
  double k=20; //W/mK
  double V=Math.PI*D*D*D/6;
  double A=Math.PI*D*D;
  double Bi=h*D/6/k;
  System.out.println("Bi="+Bi);
  double tomax=240;
  double X=ro*V*Cp/(tomax*h*A);
  double c[]=c(X,toi);
  String c1[]={"c1","c2","c3","c4"};
  Text.print(c,c1,"Collocation method for approximation");
  if_x f=to->Math.exp(-to/X);
  if_x f1=to->1+c[0]*to+c[1]*to*to+c[2]*to*to*to+c[3]*to*to*to*to;
  Plot pp=new Plot(f,0,1);
  pp.addFunction(f1,0,1);
  pp.plot();
  }
}
```

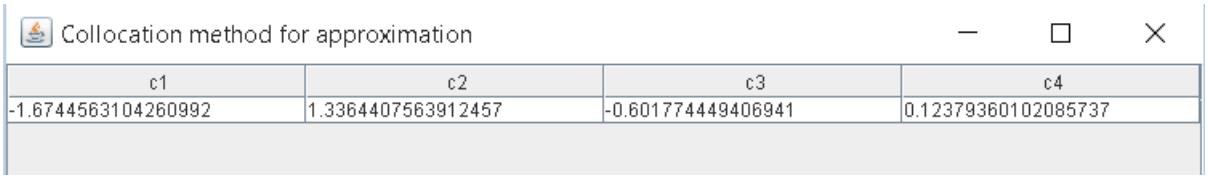
```
---------- Capture Output ----------
> "C:\java\bin\javaw.exe" collocation
Bi=0.013333333333333332
Original matrix :
      0.840277777777778      0.357638888888889      0.126302083333333      0.040798611111111
      1.090277777777778      0.840277777777778      0.567708333333333      0.357638888888889
      1.340277777777778      1.447916666666667      1.417968750000000      1.312500000000000
      1.590277777777778      2.180555555555555      2.770833333333333      3.361111111111111

pivoted matrix : k=0
      1.590277777777778      2.180555555555555      2.770833333333333      3.361111111111111
      1.090277777777778      0.840277777777778      0.567708333333333      0.357638888888889
      1.340277777777778      1.447916666666667      1.417968750000000      1.312500000000000
      0.840277777777778      0.357638888888889      0.126302083333333      0.040798611111111

pivoted matrix : k=1
      1.590277777777778      2.180555555555555      2.770833333333333      3.361111111111111
      0.000000000000000     -0.794532387190684     -1.337762691048035     -1.735159661572052
      0.000000000000000     -0.389844129063561     -0.917275063682678     -1.520226831635128
      0.000000000000000     -0.654688258127122     -1.331945960698690     -1.946703663270257

pivoted matrix : k=2
      1.590277777777778      2.180555555555555      2.770833333333333      3.361111111111111
      0.000000000000000     -0.794532387190684     -1.337762691048035     -1.735159661572052
      0.000000000000000      0.000000000000000     -0.260890327713212     -0.668855864312387
      0.000000000000000      0.000000000000000     -0.229640327713212     -0.516946142090165

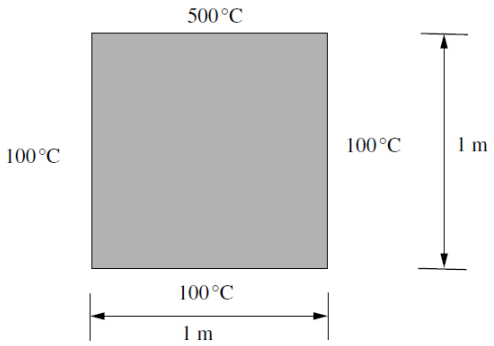
final upper triangular matrix :
      1.590277777777778      2.180555555555555      2.770833333333333      3.361111111111111
      0.000000000000000     -0.794532387190684     -1.337762691048035     -1.735159661572052
      0.000000000000000      0.000000000000000     -0.260890327713212     -0.668855864312387
      0.000000000000000      0.000000000000000      0.000000000000000      0.071792739952420
```



EXAMPLE 3: TWO DIMENSIONAL STEADY-STATE PROBLEM APPROXIMATION WITH COLLOCATION METHOD

Now let us consider the following problem:

A square plate with temperatures of 100 C in three sides and 500 C in the fourth side is given. Plate sides are given as 1 m. Calculate the temperature profile



Differential equation:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial x^2} = 0$$

Polynomial equation

$$T(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{i(n+1)+j} x^i y^j$$

$$\frac{\partial T(x,y)}{\partial x} = \sum_{i=0}^{n} \sum_{j=0}^{m} i * a_{i(n+1)+j} x^{i-1} y^j$$

$$\frac{\partial T(x,y)}{\partial y} = \sum_{i=0}^{n} \sum_{j=0}^{m} j * a_{i(n+1)+j} x^i y^{j-1}$$

$$\frac{\partial^2 T(x,y)}{\partial x^2} = \sum_{i=0}^{n} \sum_{j=0}^{m} i(i-1) a_{i(n+1)+j} x^{i-2} y^j$$

$$\frac{\partial^2 T(x,y)}{\partial y^2} = \sum_{i=0}^{n} \sum_{j=0}^{m} j(j-1) a_{i(n+1)+j} x^i y^{j-2}$$

$$\frac{\partial^2 T(x,y)}{\partial x \partial y} = \sum_{i=0}^{n} \sum_{j=0}^{m} (i-1)(j-1) a_{i(n+1)+j} x^{i-1} y^{j-1}$$

Differential equation becomes:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = \sum_{i=0}^{n} \sum_{j=0}^{m} i(i-1) a_{i(n+1)+j} x^{i-2} y^j + \sum_{i=0}^{n} \sum_{j=0}^{m} j(j-1) a_{i(n+1)+j} x^i y^{j-2} = 0$$

Equation have total K=(n+1)(m+1) coefficient

Equation divided to P=Px*Py pieces (greater than coefficient of polynomial surface)

x=dx*iPx $0 \leq iPx \leq (Px-1)$

y=dy*iPy $0 \leq iPy \leq (Py-1)$

Boundary conditions:

x=0 $0 \leq y \leq (Py-1)$ $T=T_1$

x=L $0 \leq y \leq (Py-1)$ $T=T_2$

y=0 $0 \leq x \leq (Px-1)$ $T=T_3$

y=w $0 \leq x \leq (Px-1)$ $T=T_4$

$$T_1 = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i(n+1)+j} y^j$$

$$T_2 = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i(n+1)+j} L^i y^j$$

$$T_3 = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i(n+1)+j} x^i$$

$$T_4 = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i(n+1)+j} x^i y W^j$$

Not that number of equations are not equal to number of unkowns in the polynomial surface. Therefore Singular value decomposition method is used to solve the equations for finding the polynomial coefficients. A small explanation about SVD numerical method. For further details you can check numerical analysis text books.

Let A be an m x n matrix such that the number of rows m is greater than or equal to the number of columns n. Then there exists:
1) an m x n column orthogonal matrix **U** (a matrix Q is orthogonal if its transpose is equal to its inverse)
2) an n x n diagonal matrix **S,** with positive or zero elements, and

3) an n x n orthogonal matrix V such that:
$\boldsymbol{A=USV^T}$
This is the **Singular Value Decomposition (SVD) of matrix A.** If a matrix can be decomposed this way, its inverse value will be existed as:
$\boldsymbol{A^{-1}=V\ [diag\ (s_1^{-1},s_2^{-1},\ldots,s_n^{-1})]\ (U^T)}$
It should ve note that because of the orhogonal properties of U and V inverses are equal to transpose values. Then a system of equation can be simply solve as $x=A^{-1}b$
Now let us look at these processes in more details with an example

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

The singular values of A are the square roots of the eigenvalues of A^TA

$$A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

Eigenvalues of A^TA is

$\lambda = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$ Root of the eigenvalues constructs the matrix S, by using root of eigenvalues as the diagonal of the matrix. $S = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 6.3245 & 0 \\ 0 & 3.1622 \end{bmatrix}$

And then the inverse can be easily computed by inverting the eigenvalues.

$$S^{-1} = \begin{bmatrix} 1/6.3245 & 0 \\ 0 & 1/3.1622 \end{bmatrix} = \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix}$$

The eigenvectors corresponding to these eigenvalues are:

$$\left[A^T A - \lambda I\right][X^1] = \begin{bmatrix} 25-40 & -15 \\ -15 & 25-40 \end{bmatrix}\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = 0$$

$$\left[A^T A - \lambda I\right][X^2] = \begin{bmatrix} 25-10 & -15 \\ -15 & 25-10 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = 0$$

$$[X] = \begin{bmatrix} 0.707106781186548 & 0.707106781186548 \\ -0.707106781186548 & 0.707106781186548 \end{bmatrix}$$

The eigenvector values will be normalized by using Length factor

$L = \sqrt{(x_1)^2 + (x_2)^2}$ Then the normalized Value will be V vector

$$V = \begin{bmatrix} 0.707106781186548 & 0.707106781186548 \\ -0.707106781186548 & 0.707106781186548 \end{bmatrix}$$

The U vector will be calculated as

$$U = AVS^{-1} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix}$$

An SVD code in java is listed below:

Singular value decomposition linear system of equation solving method by using Jacobi eigenvalue method

```
import java.io.*;
import javax.swing.*;

class SVD
{
double U[][];
double V[][];
double S[][];
double invS[][];

public static double[][] Transpose(double [][] left)
{
//transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
{
for(j=0;j<m;j++)
{
b[j][i]=left[i][j];
}
}
return b;
}

public static double[][] T(double [][] left)
{return Transpose(left);}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
```

```
b[i][i]=1.0;
return b;
}
//Converts vector to diagonal values of matrix
public static double[][] S(double[] left)
{
int n=left.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
b[i][i]=left[i];
return b;
}
//Converts double to diagonal values of matrix
public static double[][] S(double left,int n)
{
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
b[i][i]=left;
return b;
}
//takes invert value of a diagonal matrix
public static double[][] invS(double[][] S)
{
int n=S.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
{  if(S[i][i]==0) b[i][i]=0;
                else b[i][i]=1.0/S[i][i];
}
return b;
}
//square root of a diagonal matrix
public static double[][] sqrtS(double[] left)
{
int n=left.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
b[i][i]=Math.sqrt(left[i]);
return b;
}
//multiply two matrices
public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[m1][n2];
if(n1 != m2)
{
System.out.println("m1="+m1+"n1="+n1+"m2="+m2+"n2="+n2+"inner matrix dimensions must agree");
for(ii=0;ii<n1;ii++)
{
for(jj=0;jj<m2;jj++)
b[ii][jj]=0;
}
return b;
}
for(i=0;i<m1;i++)
{
for(j=0;j<n2;j++)
{
for(k=0;k<n1;k++)
                b[i][j]+=left[i][k]*right[k][j];
}
}
return b;
//end of multiply of two matrices
}
```

```
//multiply a matrix with a double value
public static double[][] multiply(double[][] left,double right)
{
//multiplying a matrix with a constant
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
{
for(j=0;j<m;j++)
b[i][j]=left[i][j]*right;
}
return b;
//end of multiplying a matrix with a constant double
}
//multiply a matrix with a vector
public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
if(n1 != m2)
{
System.out.println("inner matrix dimensions must agree");
for(ii=0;ii<n1;ii++)
{
b[ii]=0;
}
return b;
}
for(i=0;i<m1;i++)
{
b[i]=0;
for(k=0;k<n1;k++)
b[i]+=left[i][k]*right[k];
}
return b;
//end of multiply of a matrix and a vector
}

//substract two matrices
public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
{
for(j=0;j<m1;j++)
{
b[i][j]=b[i][j]+left[i][j];
}
}
for(i=0;i<n2;i++)
{
for(j=0;j<m2;j++)
{
b[i][j]=b[i][j]-right[i][j];
}
}
```

```
return b;
//end of matrix substraction method
}

public SVD(double A[][])
{initilize(A);}

public static double[] solve(double a[][],double b[])
{ SVD q=new SVD(a);
double x[]=multiply(multiply(q.V,q.invS),multiply(q.T(q.U),b));
return x;
}

public  double[] solve(double b[])
{
double x[]=multiply(multiply(V,invS),multiply(T(U),b));
return x;
}

public void initilize(double A[][])
{
double ATA[][]=multiply(Transpose(A),A);
//double aa[][]=Householder(ATA);
double c[][];
c=eigen_jacobi(ATA);//eigenQL(aa);
int n=c[0].length;
int n1=c.length;
double dummy[]=new double[n1];
for(int i=0;i<n;i++)
{for(int k=i+1;k<n;k++)
{if(c[0][i]<c[0][k])
{ for(int j=0;j<n1;j++)
{ dummy[j]=c[j][k];
c[j][k]=c[j][i];
c[j][i]=dummy[j];
}
}
}
}
double x[]=new double[n];
for(int i=0;i<n;i++)
{x[i]=c[0][i];}
S=sqrtS(x);
invS=invS(S);
double xx[][]=new double[n][n];
V=new double[n][n];
U=new double[n][n];
double L[]=new double[n];
for(int i=0;i<n;i++)
{ for(int j=0;j<n;j++)
{xx[i][j]=c[i+1][j];}
}

double bb[]=new double[n];
for(int j=0;j<n;j++)
{ for(int i=0;i<n;i++)
{L[j]+=xx[i][j]*xx[i][j];}
L[j]=Math.sqrt(L[j]);
}
for(int i=0;i<n;i++)
{ for(int j=0;j<n;j++)
{V[i][j]=xx[i][j]/L[i];}
}
U=multiply(A,multiply(V,invS));
}

public static void print(String s)
{JOptionPane.showMessageDialog(null,s,
"Singular value decomposition : ",JOptionPane.PLAIN_MESSAGE);}

public static double[][] eigen_jacobi(double A[][])
{
double eps=1e-15;
double err=1;
int m=100;
int n=A.length;
```

```
        double P[][]=new double[n][n];
        double R[][]=new double[n][n];
        int p=n,q=n;
        double max=-9.99e100;
        double fi;
        P=I(n);
        for(int i=0;i<n;i++)
        {for(int j=i+1;j<n;j++)
        {if(Math.abs(A[i][j]) > max) {max=Math.abs(A[i][j]);p=i;q=j;}}
        }
        while(Math.abs(A[p][q]) > eps )
        {
        fi=0.5*Math.atan(2.0*A[p][q]/(A[p][p]-A[q][q]));
        R=I(n);
        R[p][p]=Math.cos(fi);
        R[q][q]=Math.cos(fi);
        R[p][q]=-Math.sin(fi);
        R[q][p]=Math.sin(fi);
        P=multiply(P,R);
        A=multiply(T(R),multiply(A,R));
        max=-9.99e100;
        for(int i=0;i<n;i++)
        {for(int j=i+1;j<n;j++){if(Math.abs(A[i][j])>max) {max=Math.abs(A[i][j]);p=i;q=j;}}}
        }
        double lambda[]=new double[n];
        for(int i=0;i<n;i++) {lambda[i]=A[i][i];}
        double PP[][]=new double[n+1][n];
        PP[0]=lambda;
        for(int i=0;i<n;i++)
        {for(int j=0;j<n;j++)
        {PP[i+1][j]=P[i][j];}
        }
        return PP;
        }

        public static void main(String args[]) throws IOException
        {
        double a[][]={{3,2,-1},{2,-2,4},{-1,0.5,-1}};
        double b[]={1,-2,0};
        //Dynamic call to SVD
        SVD q=new SVD(a);
        double x[]=q.solve(b);
        System.out.println("S:\n"+Matrix.toString(q.S));
        System.out.println("V:\n"+Matrix.toString(q.V));
        System.out.println("U:\n"+Matrix.toString(q.U));
        //Static call to SVD
        //double x2[]=solve(a,b);
        System.out.println("SINGULAR VALUE DECOMPOSITION : \n"+Matrix.toString(x));
        //System.out.println("STATIC SINGULAR VALUE DECOMPOSITION : \n"+Matrix.toString(x2));
        }
        }
```

A second code is given this program poly2, defines the two dimensional polynomial function and its coefficients and derivatives.

```
 //Polynomial with two independent variable
//polinomial of order k=n*m
public class poly2
{ public int k;
  public double func(double a[],double x,double y,int n,int m)
  {   k=n*m;
     int k1=0;
     double xx=0;
     double total=0.0;
     for(int i=0;i<=n;i++)
     {  for(int j=0;j<=m;j++)
        { k1=i*(m+1)+j;
          xx=pow(x,y,i,j);
          total+=a[k1]*xx;
        }
     }
                return total;
  }

  public double dfuncx(double a[],double x,double y,int n,int m)
```

```
{   k=n*m;
    int k1=0;
    double total=0.0;
    for(int i=0;i<=n;i++)
    {for(int j=0;j<=m;j++)
     {k1=i*(m+1)+j;
      double xx=dpowx(x,y,i,j);
      total+=a[k1]*xx;
      System.out.println("i="+i+"j="+j+"k1="+k1+"xx="+xx+"a="+a[k1]+"total="+total);
     }
    }
                return total;
}

public double dfuncy(double a[],double x,double y,int n,int m)
{   k=n*m;
    int k1=0;
    double total=0.0;
    for(int i=0;i<=n;i++)
    {for(int j=0;j<=m;j++)
     {k1=i*(m+1)+j;
     double xx=dpowy(x,y,i,j);
      total+=a[k1]*xx;
      System.out.println("i="+i+"j="+j+"k1="+k1+"xx="+xx+"a="+a[k1]+"total="+total);
     }
    }
                return total;
}

public double d2funcx2(double a[],double x,double y,int n,int m)
{   k=n*m;
    int k1=0;
    double total=0.0;
    if(a.length<k) System.out.println("error");
    for(int i=0;i<=n;i++)
    {for(int j=0;j<=m;j++)
     {k1=i*(m+1)+j;
      total+=a[k1]*d2powx2(x,y,i,j);
     }
    }
                return total;
}
public double d2funcy2(double a[],double x,double y,int n,int m)
{   k=n*m;
    int k1=0;
    double total=0.0;
    for(int i=0;i<=n;i++)
    {for(int j=0;j<=m;j++)
     {k=i*(m+1)+j;
      total+=a[k]*d2powy2(x,y,i,j);
     }
    }
                return total;
}
public double d2funcxy(double a[],double x,double y,int n,int m)
{   k=n*m;
    int k1=0;
    double total=0.0;
    if(a.length<k) System.out.println("error");
    for(int i=0;i<=n;i++)
    {for(int j=0;j<=m;m++)
     {k=i*(m-1)+j;
      total+=a[k]*d2powxy(x,y,i,j);
     }
    }
                return total;
}
//=====================================
public static double pow(double x,double y,int n,int m)
{ double pow=1.0;
                if(n==0 && m==0) return pow;
 else if(m==0 && n!=0)
 {for(int i=0;i<n;i++)
  {pow*=x;}
 }
 else if(n==0 && m!=0)
```

```
   {for(int j=0;j<m;j++)
    {pow*=y;}
   }
   else
   {for(int i=0;i<n;i++)
    {pow*=x;}
    for(int j=0;j<m;j++)
    {pow*=y;}
   }
   return pow;
  }
  public static double dpowx(double x,double y,int n,int m)
  { double powx=n;
    double powy=1.0;
    for(int i=1;i<n;i++)
    {powx*=x;}
    for(int j=0;j<m;j++)
    {powy*=y;}
    return powx*powy;
  }
  public static double dpowy(double x,double y,int n,int m)
  { double powx=1.0;
    double powy=m;
    for(int i=0;i<n;i++)
    {powx*=x;}
    for(int j=1;j<m;j++)
    {powy*=y;}
    return powx*powy;
  }
  public static double d2powx2(double x,double y,int n,int m)
  { double powx=n*(n-1);
    double powy=1.0;
    for(int i=2;i<n;i++)
    {powx*=x;}
    for(int j=0;j<m;j++)
    {powy*=y;}
    return powx*powy;
  }
  public static double d2powy2(double x,double y,int n,int m)
  { double powx=1.0;
    double powy=m*(m-1);
    for(int i=0;i<n;i++)
    {powx*=x;}
    for(int j=2;j<m;j++)
    {powy*=y;}
    return powx*powy;
   }

  public static double d2powxy(double x,double y,int n,int m)
  { double powx=n;
    double powy=m;
    for(int i=1;i<n;i++)
    {powx*=x;}
    for(int j=1;j<m;j++)
    {powy*=y;}
    return powx*powy;
  }
  public static void main(String arg[])
  { double x=0.1;double y=0.1;
                double a[]={1,2,3,4,5,6,7,8,9};
    poly2 p=new poly2();
    System.out.println(p.func(a,x,y,2,2));
    System.out.println(p.dfuncx(a,x,y,2,2));
    System.out.println(p.dfuncy(a,x,y,2,2));
  }
 }
```

Now we can construct solution of the differential equation by using **collocation** method:

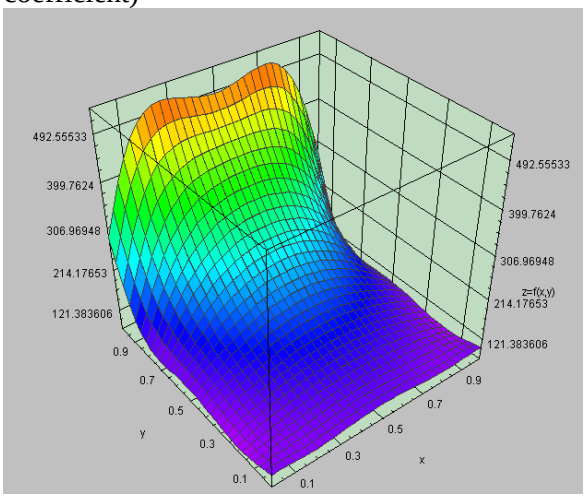
```
 public class HT_poly2_T
{ int P,Px,Py,m,n,k,ip;
double T1,T2,T3,T4;
double dx,dy;
double L,W;
```

```
double A[][];
double B[];
double TT;
double x,y;
 public HT_poly2_T(double T1i,double T2i,double T3i,double T4i,int ni,int mi, int Pxi,int Pyi,double Li,double Wi)
 {Px=Pxi;Py=Pyi;P=Px*Py;n=ni;m=mi;k=(n+1)*(m+1);T1=T1i;T2=T2i;T3=T3i;T4=T4i;W=Wi;L=Li;dx=L/Px;dy=W/Py;TT=0;
 A=new double[P][k];B=new double[P];
 int k1=0;
 for(int ipx=0;ipx<Px;ipx++)
 {for(int ipy=0;ipy<Py;ipy++)
  {   x=dx*ipx;y=dy*ipy;
     ip=ipx*Py+ipy;
                if(ipx==0)
                {TT=T3;
                 for(int i=0;i<=n;i++)
                 {for(int j=0;j<=m;j++)
                  {k1=i*(m+1)+j;
                   A[ip][k1]=poly2.pow(x,y,i,j);
                   B[ip]=TT;
                  }
                 }
                }
                else if(ipx==(Px-1))
                {TT=T4;
                 for(int i=0;i<=n;i++)
                 {for(int j=0;j<=m;j++)
                  {k1=i*(m+1)+j;
                   A[ip][k1]=poly2.pow(x,y,i,j);
                   B[ip]=TT;
                  }
                 }
                }
                else if(ipy==0)
                {TT=T1;
                 for(int i=0;i<=n;i++)
                 {for(int j=0;j<=m;j++)
                  {k1=i*(m+1)+j;
                   A[ip][k1]=poly2.pow(x,y,i,j);
                   B[ip]=TT;
                  }
                 }
                }
                else if(ipy==(Py-1))
                {TT=T2;
                 for(int i=0;i<=n;i++)
                 {for(int j=0;j<=m;j++)
                  {k1=i*(m+1)+j;
                   A[ip][k1]=poly2.pow(x,y,i,j);
                   B[ip]=TT;
                  }
                 }
                }
                else
                {for(int i=0;i<=n;i++)
                 {for(int j=0;j<=m;j++)
                  {k1=i*(m+1)+j;
                   A[ip][k1]=poly2.d2powx2(x,y,i,j)+poly2.d2powy2(x,y,i,j);
                            B[ip]=0.0;
                  }
                 }
                }
 }
}
Text.print(A,"A");
Text.printT(B,"B");
SVD q=new SVD(A);
double a[]=q.solve(B);
Text.printT(a,"a");
double T[][]=new double[Px][Py];
poly2 p=new poly2();
for(int ipx=0;ipx<Px;ipx++)
{for(int ipy=0;ipy<Py;ipy++)
 {   x=dx*ipx;y=dy*ipy;
     ip=ipx*Py+ipy;
     T[ipx][ipy]=p.func(a,x,y,n,m);
  } // end int ipy=0;ipy<Py;ipy++)
```

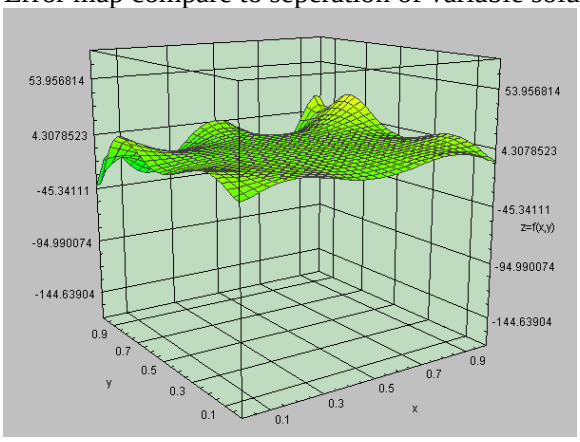
```
 }//end for(int ipx=0;ipx<Px;ipx++)
  Text.print(T,"T");
    double b[][]=surfacePlot.dataformat(0.0,L,0.0,W,T);
    surfacePlot p1;
    p1=new surfacePlot(b);
    p1.plot();
}
public static void main(String arg[])
{HT_poly2_T h=new HT_poly2_T(100,500.0,100.0,100.0,5,5,80,80,1.0,1.0);

}
}
```

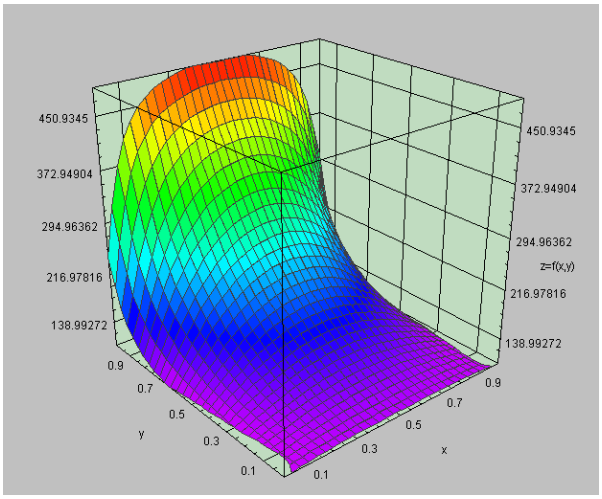
Output of the polynomial function (Temperature profile) n=7,m=7 Px=80,Py=80 (64 polynomial coefficient)



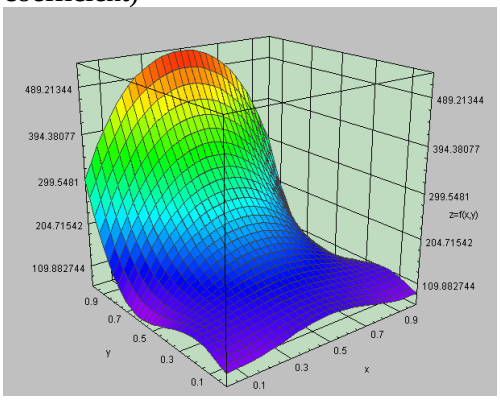
Error map compare to seperation of variable solution



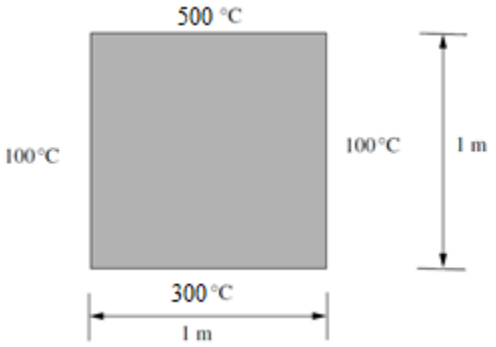
Seperation of variable Solution of this problem



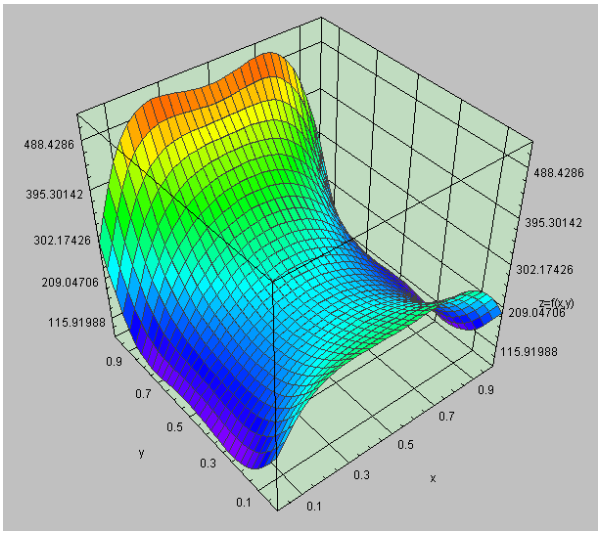
Output of the polynomial function(Temperature profile) n=5,m=5,Px=80,Py=80 (36 polynomial coefficient)



A square plate with temperatures as seen in the figures are given. Plate sides are given as 1 m. Calculate the temperature profile



Output of the polynomial function(Temperature profile) n=7,m=7,Px=80,Py=80



EXAMPLE 4: TIME DEPENDENT LUMPED SOLUTION APPROXIMATION METHODS (LEAST SQUARE METHOD)

As a next approximation method we will consider **least square method**.

$-hA_s(T-T_\infty)=\rho VC_p\dfrac{dT}{dt}$ where A_s is surface area, V is volume, C_p is specific heat. By changing variables as:

$\theta(t)=\dfrac{T(t)-T_\infty}{T_i-T_\infty}\quad \tau=\dfrac{t}{\tau_{\max}}\quad X=\dfrac{\rho VC_p}{\tau_{\max}hA_s}$ equation becomes:

$$X\frac{d\theta(\tau)}{dt}+\theta(\tau)=0$$

Boundary condition t=0 $\theta=1$ so exact solution is:

$$\theta(\tau)=\exp\left(-\frac{\tau}{X}\right)$$

Now let us create approximate solution, for this the following polynomial is assumed:

$\theta(\tau)=1+c_1\tau+c_2\tau^2+c_3\tau^3+c_4\tau^4$ is equation automatically satisfy the boundary condition but if it is applied to differential equation a residue is remained as:

$$\frac{d\theta(\tau)}{d\tau}=c_1+2c_2\tau\ +3c_3\tau^2+4c_4\tau^3$$

$$X\frac{d\theta(\tau)}{dt}+\theta(\tau)=0$$

$$R=X\left(c_1+2c_2\tau\ +3c_3\tau^2+4c_4\tau^3\right)+(1+c_1\tau+c_2\tau^2+c_3\tau^3+c_4\tau^4)$$

$$R=1+(X+\tau)c_1+(2\tau X+\tau^2)c_2+(3\tau^2X+\tau^3)c_3+(4\tau^3+\tau^4)c_4$$

$$R=1+(X+\tau)c_1+(2\tau X+\tau^2)c_2+(3\tau^2X+\tau^3)c_3+(4\tau^3+\tau^4)c_4$$

It is possible to write least square equations as:

$$\frac{1}{2}\frac{\partial}{\partial c_1}\int_{\tau=0}^{\tau=1}R^2(\tau,c_1,c_2,c_3,c_4)d\tau=0$$

$$\frac{1}{2}\frac{\partial}{\partial c_2}\int_{\tau=0}^{\tau=1}R^2(\tau,c_1,c_2,c_3,c_4)d\tau=0$$

$$\frac{1}{2}\frac{\partial}{\partial c_3}\int_{\tau=0}^{\tau=1} R^2(\tau,c_1,c_2,c_3,c_4)d\tau = 0$$

$$\frac{1}{2}\frac{\partial}{\partial c_4}\int_{\tau=0}^{\tau=1} R^2(\tau,c_1,c_2,c_3,c_4)d\tau = 0$$

Considering R as

$$R = 1 + \phi_1 c_1 + \phi_1 c_2 + \phi_1 c_3 + \phi_1 c_4$$

$$\frac{\partial R}{\partial c_1} = \phi_1 \quad \frac{\partial R}{\partial c_2} = \phi_2 \quad \frac{\partial R}{\partial c_3} = \phi_3 \quad \frac{\partial R}{\partial c_4} = \phi_4$$

$$\int_{\tau=0}^{\tau=1} \phi_1\left(1 + \phi_1 c_1 + \phi_1 c_2 + \phi_1 c_3 + \phi_1 c_4\right)d\tau = 0$$

$$\int_{\tau=0}^{\tau=1} \phi_2\left(1 + \phi_1 c_1 + \phi_1 c_2 + \phi_1 c_3 + \phi_1 c_4\right)d\tau = 0$$

$$\int_{\tau=0}^{\tau=1} \phi_3\left(1 + \phi_1 c_1 + \phi_1 c_2 + \phi_1 c_3 + \phi_1 c_4\right)d\tau = 0$$

$$\int_{\tau=0}^{\tau=1} \phi_4\left(1 + \phi_1 c_1 + \phi_1 c_2 + \phi_1 c_3 + \phi_1 c_4\right)d\tau = 0 \text{ so}$$

$$\begin{bmatrix} \int \phi_1\phi_1 d\tau & \int \phi_1\phi_2 d\tau & \int \phi_1\phi_3 d\tau & \int \phi_1\phi_4 d\tau \\ \int \phi_2\phi_1 d\tau & \int \phi_2\phi_2 d\tau & \int \phi_2\phi_3 d\tau & \int \phi_2\phi_4 d\tau \\ \int \phi_3\phi_1 d\tau & \int \phi_3\phi_2 d\tau & \int \phi_3\phi_3 d\tau & \int \phi_3\phi_4 d\tau \\ \int \phi_4\phi_1 d\tau & \int \phi_4\phi_2 d\tau & \int \phi_4\phi_3 d\tau & \int \phi_4\phi_4 d\tau \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -\int \phi_1 d\tau \\ -\int \phi_2 d\tau \\ -\int \phi_3 d\tau \\ -\int \phi_4 d\tau \end{bmatrix}$$

Each integration should be calculated and carried out into the matrix to determine the coefficients.

$$\int_{\tau=0}^{\tau=0} \phi_1\phi_1 d\tau = \int_{\tau=0}^{\tau=1} (X^2 + 2X\tau + \tau^2)d\tau = \left(X^2\tau + X\tau^2 + \frac{\tau^3}{3} \right)\Bigg|_{\tau=0}^{\tau=1} = X^2 + X + \frac{1}{3}$$

$$\int_{\tau=0}^{\tau=0} \phi_1\phi_2 d\tau = \int_{\tau=0}^{\tau=1} (X+\tau)*(2X\tau + \tau^2)d\tau = \int_{\tau=0}^{\tau=1} (2X^2\tau + 3X\tau^2 + \tau^3)d\tau = \left(X^2\tau^2 + X\tau^3 + \frac{\tau^4}{4} \right)\Bigg|_{\tau=0}^{\tau=1} = X^2 + X + \frac{1}{4}$$

$$\int_{\tau=0}^{\tau=0} \phi_1\phi_3 d\tau = \int_{\tau=0}^{\tau=1} (X+\tau)*(3X\tau^2 + \tau^3)d\tau = \int_{\tau=0}^{\tau=1} (3X^2\tau^2 + 4X\tau^3 + \tau^4)d\tau = \left(X^2\tau^3 + X\tau^4 + \frac{\tau^5}{5} \right)\Bigg|_{\tau=0}^{\tau=1} = X^2 + X + \frac{1}{5}$$

$$\int_{\tau=0}^{\tau=0} \phi_1\phi_4 d\tau = \int_{\tau=0}^{\tau=1} (X+\tau)*(3X\tau^3 + \tau^4)d\tau = \int_{\tau=0}^{\tau=1} (4X^2\tau^2 + 5X\tau^4 + \tau^5)d\tau = \left(X^2\tau 4^3 + X\tau^5 + \frac{\tau^6}{6} \right)\Bigg|_{\tau=0}^{\tau=1} = X^2 + X + \frac{1}{6}$$

$$\int_{\tau=0}^{\tau=0} \phi_2\phi_2 d\tau = \int_{\tau=0}^{\tau=1} (2X\tau + \tau^2)*(2X\tau + \tau^2)d\tau = \int_{\tau=0}^{\tau=1} (4X^2\tau^2 + 4X\tau^3 + \tau^4)d\tau = \left(\frac{4}{3}X^2\tau^2 + X\tau^4 + \frac{\tau^5}{5} \right)\Bigg|_{\tau=0}^{\tau=1} = \frac{4}{3}X^2 + X + \frac{1}{5}$$

$$\int_{\tau=0}^{\tau=0} \phi_2\phi_3 d\tau = \int_{\tau=0}^{\tau=1} (2X\tau + \tau^2)*(3X\tau^2 + \tau^3)d\tau = \int_{\tau=0}^{\tau=1} (6X^2\tau^3 + 5X\tau^4 + \tau^5)d\tau = \left(\frac{6}{4}X^2\tau^4 + X\tau^5 + \frac{\tau^6}{6} \right)\Bigg|_{\tau=0}^{\tau=1} = \frac{6}{4}X^2 + X + \frac{1}{6}$$

$$\int_{\tau=0}^{\tau=0} \phi_2\phi_4 d\tau = \int_{\tau=0}^{\tau=1} (2X\tau + \tau^2)*(4X\tau^3 + \tau^4)d\tau = \int_{\tau=0}^{\tau=1} (8X^2\tau^4 + 5X\tau^5 + \tau^6)d\tau = \left(\frac{8}{5}X^2\tau^5 + X\tau^6 + \frac{\tau^7}{7} \right)\Bigg|_{\tau=0}^{\tau=1} = \frac{8}{5}X^2 + X + \frac{1}{7}$$

$$\int_{\tau=0}^{\tau=0} \phi_3\phi_3 d\tau = \int_{\tau=0}^{\tau=1} (3X\tau^2 + \tau^3)*(3X\tau^2 + \tau^3)d\tau = \int_{\tau=0}^{\tau=1} (9X^2\tau^4 + 6X\tau^5 + \tau^6)d\tau = \left(\frac{9}{5}X^2\tau^5 + X\tau^6 + \frac{\tau^7}{7} \right)\Bigg|_{\tau=0}^{\tau=1} = \frac{9}{5}X^2 + X + \frac{1}{7}$$

$$\int_{\tau=0}^{\tau=0} \phi_3\phi_4 d\tau = \int_{\tau=0}^{\tau=1} (3X\tau^2 + \tau^3)*(4X\tau^3 + \tau^4)d\tau = \int_{\tau=0}^{\tau=1} (12X^2\tau^5 + 7X\tau^6 + \tau^7)d\tau = \left(\frac{12}{6}X^2\tau^6 + X\tau^7 + \frac{\tau^8}{8} \right)\Bigg|_{\tau=0}^{\tau=1} = 2X^2 + X + \frac{1}{8}$$

$$\int_{\tau=0}^{\tau=0}\phi_4\phi_4 d\tau=\int_{\tau=0}^{\tau=1}(4X\tau 3+\tau^4)*(4X\tau^3+\tau^4)d\tau=\int_{\tau=0}^{\tau=1}(16X^2\tau^6+8X\tau 7+\tau^8)d\tau=\left(\frac{16}{7}X^2\tau^7+X\tau^8+\frac{\tau 9}{8}\right)\Big|_{\tau=0}^{\tau=1}=\frac{16}{7}X^2+X+\frac{1}{9}$$

$$-\int_{\tau=0}^{\tau=0}\phi_1 d\tau=\int_{\tau=0}^{\tau=1}(X+\tau)d\tau=\left(X\tau+\frac{\tau^2}{2}\right)\Big|_{\tau=0}^{\tau=1}=-\left(X+\frac{1}{2}\right)$$

$$-\int_{\tau=0}^{\tau=0}\phi_2 d\tau=\int_{\tau=0}^{\tau=1}(2X\tau+\tau^2)d\tau=\left(X\tau^2+\frac{\tau^3}{3}\right)\Big|_{\tau=0}^{\tau=1}=-\left(X+\frac{1}{3}\right)$$

$$-\int_{\tau=0}^{\tau=0}\phi_3 d\tau=\int_{\tau=0}^{\tau=1}(3X\tau^2+\tau^3)d\tau=\left(X\tau^3+\frac{\tau^4}{4}\right)\Big|_{\tau=0}^{\tau=1}=-\left(X+\frac{1}{4}\right)$$

$$-\int_{\tau=0}^{\tau=0}\phi_4 d\tau=\int_{\tau=0}^{\tau=1}(3X\tau^4+\tau^4)d\tau=\left(X\tau^4+\frac{\tau^5}{5}\right)\Big|_{\tau=0}^{\tau=1}=-\left(X+\frac{1}{5}\right)$$

Now that all coefficients of the matrix is established, solution can be done by using a small application program

```
import java.io.*;

class residue_least_square
{

 public static double[] pivotlugauss(double a[][],double b[])
 { //gauss elimination with partial pivoting
 int n=b.length;
 double x[]=new double[n];
 double carpan=0;
 double toplam=0;
 double buyuk;
 double dummy=0;
 //gauss eleme
 int i,j,k,p,ii,jj;
 System.out.println("Original matrix :\n"+Matrix.toString(a));
 for(k=0;k<(n-1);k++)
 { //pivoting
            p=k;
            buyuk=Math.abs(a[k][k]);
            for(ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[ii][k]);
              if(dummy > buyuk) {buyuk=dummy;p=ii;}
            }
            if(p!=k)
            { for(jj=k;jj<n;jj++)
              { dummy=a[p][jj];
                a[p][jj]=a[k][jj];
                a[k][jj]=dummy;
              }
              dummy=b[p];
              b[p]=b[k];
              b[k]=dummy;
            }
              System.out.println("pivoted matrix : k="+k+"\n"+Matrix.toString(a));
            //Solving gauss elimination
  for(i=k+1;i<n;i++)
  {  carpan=a[i][k]/a[k][k];
    a[i][k]=0;
    for(j=k+1;j<n;j++)
    {   a[i][j]-=carpan*a[k][j]; }
       b[i]   =b[i]   -carpan*b[k];
            }
 }
 System.out.println("final upper triangular matrix : \n"+Matrix.toString(a));
 //back substitution
 x[n-1]=b[n-1]/a[n-1][n-1];
 for(i=n-2;i>=0;i--)
 {
  toplam=0;
  for(j=i+1;j<n;j++)
  {  toplam+=a[i][j]*x[j];}
  x[i]=(b[i]-toplam)/a[i][i];
 }
```

```
  return x;
  }
  public static double[] c(double X)
  {  double a[][]=new double[4][4];
    a[0][0]=(X*X+X+1.0/3.0);
    a[1][0]=(X*X+X+1.0/4.0);
    a[2][0]=(X*X+X+1.0/5.0);
    a[3][0]=(X*X+X+1.0/6.0);
    a[0][1]=a[1][0];
    a[1][1]=(4.0/3.0*X*X+X+1.0/5.0);
    a[2][1]=(6.0/4.0*X*X+X+1.0/6.0);
    a[3][1]=(8.0/5.0*X*X+X+1.0/7.0);
    a[0][2]=a[2][0];
    a[1][2]=a[2][1];
              a[2][2]=(9.0/5.0*X*X+X+1.0/7.0);
              a[3][2]=(2.0*X*X+X+1.0/8.0);
    a[0][3]=a[3][0];
    a[1][3]=a[3][1];
    a[2][3]=a[3][2];
    a[3][3]=(16.0/7.0*X*X+X+1.0/9.0);;
              double b[]={-(X+1.0/2.0),-(X+1.0/3.0),-(X+1.0/4.0),-(X+1.0/5.0)};
              double c[]=pivotlugauss(a,b);
              return c;
  }

  public static void main(String args[]) throws IOException
  {
  double toi[]={0.25,0.5,0.75,1.0};
  double ro=8500; //kg/m^3
  double Cp=400; //J/kgK
  double h=80; //W/m2K
  double D=0.02;
  double k=20; //W/mK
  double V=Math.PI*D*D*D/6;
  double A=Math.PI*D*D;
  double Bi=h*D/6/k;
  System.out.println("Bi="+Bi);
  double tomax=240;
  double X=ro*V*Cp/(tomax*h*A);
  double c[]=c(X);
  String c1[]={"c1","c2","c3","c4"};
  Text.print(c,c1,"Least square method for approximation");
  if_x f=to->Math.exp(-to/X);
  if_x f1=to->1+c[0]*to+c[1]*to*to+c[2]*to*to*to+c[3]*to*to*to*to;
  Plot pp=new Plot(f,0,1);
  pp.addFunction(f1,0,1);
  pp.plot();
  }
}
```

```
---------- Capture Output ----------
> "C:\java\bin\javaw.exe" residue_least_square
Bi=0.013333333333333332
Original matrix :
      1.272038966049383     1.188705632716049     1.138705632716049     1.105372299382716
      1.188705632716049     1.254848251028807     1.279586226851852     1.290619488536155
      1.138705632716049     1.279586226851852     1.360305059523810     1.412133487654321
      1.105372299382716     1.290619488536155     1.412133487654321     1.497795414462081

pivoted matrix : k=0
      1.272038966049383     1.188705632716049     1.138705632716049     1.105372299382716
      1.188705632716049     1.254848251028807     1.279586226851852     1.290619488536155
      1.138705632716049     1.279586226851852     1.360305059523810     1.412133487654321
      1.105372299382716     1.290619488536155     1.412133487654321     1.497795414462081

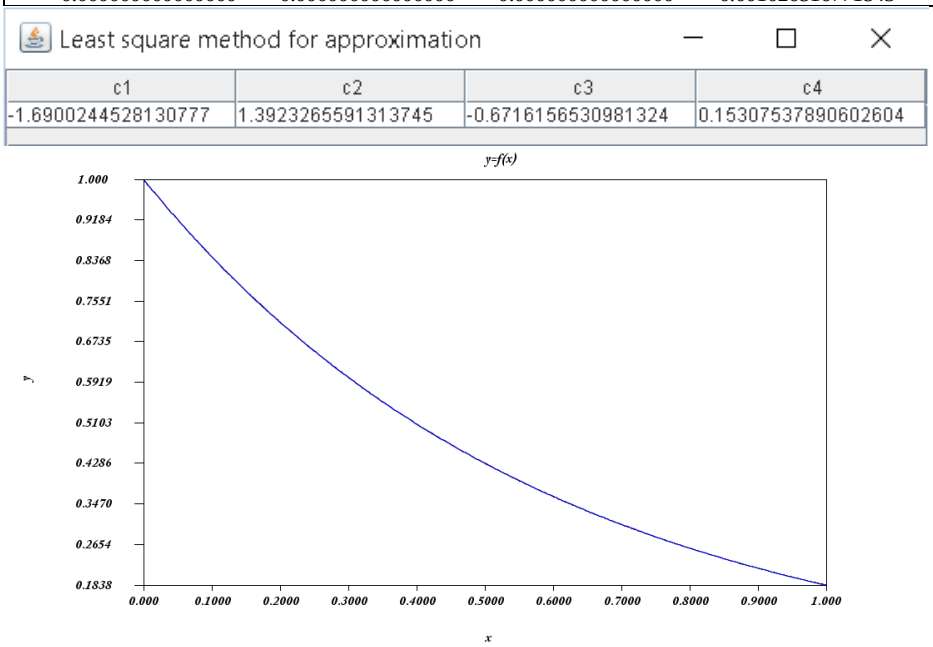
pivoted matrix : k=1
      1.272038966049383     1.188705632716049     1.138705632716049     1.105372299382716
      0.000000000000000     0.257661919158115     0.422624756279086     0.537252575088716
      0.000000000000000     0.215479044806210     0.340956947880412     0.422624756279086
      0.000000000000000     0.144016649981762     0.215479044806210     0.257661919158115

pivoted matrix : k=2
      1.272038966049383     1.188705632716049     1.138705632716049     1.105372299382716
      0.000000000000000     0.257661919158115     0.422624756279086     0.537252575088716
      0.000000000000000     0.000000000000000    -0.020741355156435    -0.042628152076166
```

0.000000000000000	0.000000000000000	-0.012478201139771	-0.026672027964582

final upper triangular matrix :

1.272038966049383	1.188705632716049	1.138705632716049	1.105372299382716
0.000000000000000	0.257661919158115	0.422624756279086	0.537252575088716
0.000000000000000	0.000000000000000	-0.020741355156435	-0.042628152076166
0.000000000000000	0.000000000000000	0.000000000000000	-0.001026516771543



The last method we will look for the approximation is **Galarkin Method**. This Method multiplies residue with a weight function and integrates similar to least square method

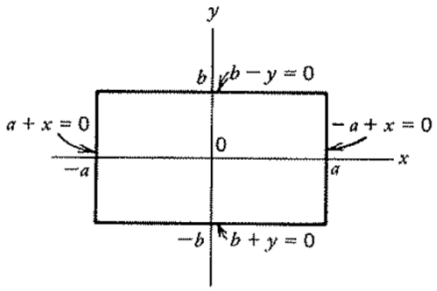
$$\int_{\tau=0}^{\tau=1} w_i R(\tau, c_1, c_2, c_3, c_4) d\tau = 0 \quad \text{i=1..4}$$

w_i is called weight function. One possible way is defining the weight function in here is to utilised derivatives of residue with respect to c_i

$$w_i = \frac{\partial R(\tau, c_1, c_2, c_3, c_4)}{\partial c_i}$$ i=1..4 in this case the results will be same as the previous least square case.

Other considerations of weight functions are also possible. In fact selection of the weight factor is one of the most important criteria in solution. In general, weight functions should be selected so that it should satisfy the boundary conditions. As an example let us consider the following problem with a rectangular region:

EXAMPLE 5: TWO DIMENSIONAL STEADY-STATE PROBLEM APPROXIMATION WITH GALARKIN METHOD



For differential equation:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{1}{k} g = 0$$

Boundary conditions:

$T(x,y)=0 \quad at \;\; x=-a \qquad T(x,y)=0 \quad at \;\; x=-b$

$T(x,y)=0 \quad at \;\; x=a \qquad T(x,y)=0 \quad at \;\; x=b$

Where

$\overline{T}(x,y)=\phi_0(x,y)+\sum_{j=1}^{n} C_j\phi_j(x,y)$ when this function is substituted into the differential equation

$$\frac{\partial^2\overline{T}(x,y)}{\partial x^2}+\frac{\partial^2\overline{T}(x,y)}{\partial y^2}+\frac{1}{k}g=R(C_1,C_2,C_3,..,x,y)$$

Then the Galarkin method to evalute the problem will be

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[\frac{\partial^2\overline{T}(x,y)}{\partial x^2}+\frac{\partial^2\overline{T}(x,y)}{\partial y^2}+\frac{1}{k}g\right]dxdy=\int_{x=-a}^{a}\int_{y=-b}^{b}R(C_1,C_2,C_3,..,x,y)dxdy$$

Before going into the details of approximate solution, let us solve the differential equation with exact solution so that the results can be compared.

This equation can be solved by using Separation of variable methods. In oreder to hmogenise the equation, system is divided up into two equation as:

$T(x,y)=\theta(x,y)+\psi(x)$ and

$$\frac{\partial^2\psi(x)}{\partial x^2}+\frac{g}{k}=0$$

$\psi(x)=0$ at x=-a

$\psi(x)=0$ at x=a

$$\frac{\partial^2\theta(x,y)}{\partial x^2}+\frac{\partial^2\theta(x,y)}{\partial y^2}=0$$

$\theta(x)=0$ at x=-a

$\theta(x)=0$ at x=a

$\theta(x)=-\psi(x)$ at y=-b

$\theta(x)=-\psi(x)$ at y=b

Solution of $\psi(x)$

$\psi(x)=-\frac{g}{k}\frac{x^2}{2}+C_1x+C_2$ applying boundary conditions $C_2=\frac{g}{k}\frac{a^2}{2} \quad C_1=0$

$$\psi(x)=\frac{g}{2k}(a^2-x^2)$$

Solution of $\theta(x,y)$

$\theta(x,y)=X(x)Y(y)$

$$\frac{1}{X(x)}\frac{\partial^2X(x)}{\partial^2x}=-\frac{1}{Y(y)}\frac{\partial^2Y(y)}{\partial^2y}=-\beta^2$$

$$\frac{\partial^2X(x)}{\partial^2x}+\beta^2X(x)=0$$

$$\frac{\partial^2Y(y)}{\partial^2y}-\beta^2Y(y)=0$$

$X(x)=C_1\cos(\beta x)+C_2\sin(\beta x)$

$X(x)=C_3\cosh(-\beta y)+C_4\sinh(\beta y)$

Boundary conditions:

$\theta(x)=0$ at x=-a $\theta(x)=0$ at x=a

$$0=C_1\cos(-\beta a)+C_2\sin(-\beta a)$$

$$0=C_1\cos(\beta a)+C_2\sin(\beta a)$$

Considering that $\cos(-\beta a)=\cos(\beta a)$ and $\sin(-\beta a)=-\sin(\beta a)$

$$0=C_1\cos(\beta a)-C_2\sin(\beta a)$$

$$0=C_1\cos(\beta a)+C_2\sin(\beta a)$$

$$0=C_1\cos(\beta a) \quad \beta a=(2n+1)\frac{\pi}{2} \quad \beta_n=(2n+1)\frac{\pi}{2a}$$

$$0=C_2$$

$$\theta(x,y)=T(x,y)-\psi(x)$$

$$\theta(x,y)=\sum_{n=0}^{\infty}\left[C_3\cosh(\beta_n y)+C_4\sinh h(\beta_n y)\right]\cos(\beta_n x)$$

Boundary conditions:

$$-\psi(x)=\sum_{n=0}^{\infty}\left[C_3\cosh(-\beta_n b)+C_4\sinh h(-\beta_n b)\right]\cos(\beta_n x)$$

$$-\psi(x)=\sum_{n=0}^{\infty}\left[C_3\cosh(\beta_n b)+C_4\sinh h(\beta_n b)\right]\cos(\beta_n x)$$

Considering that $\cosh(-\beta_n b)=\cosh(\beta_n b)$ and $\sin(-\beta_n b)=-\sinh(\beta_n b)$

$$-\psi(x)=\sum_{n=0}^{\infty}\left[C_3\cosh(\beta_n b)-C_4\sinh h(\beta_n b)\right]\cos(\beta_n x)$$

$$-\psi(x)=\sum_{n=0}^{\infty}\left[C_3\cosh(\beta_n b)+C_4\sinh h(\beta_n b)\right]\cos(\beta_n x)$$

So $C_4=0$

$$-\psi(x)=\sum_{n=0}^{\infty}\left[C_n\cosh(\beta_n b)\cos(\beta_n x)\right]$$

$$\frac{-\psi(x)}{\cosh(\beta_n b)}=\sum_{n=0}^{\infty}C_n\cos(\beta_n x)$$

It is known that $\cos(\beta_n x)$ is an othogonal function so orthogonality can be applied.

$$\int_{x=-a}^{x=a}\frac{-\psi(x)\cos(\beta_n x)dx}{\cosh(\beta_n b)}=\int_{x=-a}^{x=a}C_n\cos^2(\beta_n x)dx$$

To solve the integral variable change is aplied as

$$\bar{x}=\frac{x}{a} \text{ and } \psi(\bar{x})=\frac{ga^2}{2k}(1-\bar{x}^2) \quad \bar{\beta}_n=(2n+1)\frac{\pi}{2} \quad \bar{\beta}_n\bar{x}=\beta_n x$$

$$\int_{x=-1}^{x=1}\frac{-\psi(\bar{x})\cos(\bar{\beta}_n\bar{x})d\bar{x}}{\cosh(\bar{\beta}_n\frac{b}{a})}=C_n\int_{x=-1}^{x=1}\cos^2(\bar{\beta}_n\bar{x})d\bar{x}$$

$$\int_{x=-1}^{x=1}\cos^2(\bar{\beta}_n\bar{x})d\bar{x}=\int_{x=-1}^{x=1}\frac{1+\cos(2\bar{\beta}_n\bar{x})}{2}d\bar{x}=\left[\frac{x}{2}+\frac{1}{2\bar{\beta}}\sin(2\bar{\beta}_n)\right]_{-1}^{1}=1$$

$$-\frac{ga^2}{2k\cosh(\bar{\beta}_n\frac{b}{a})}\int_{x=-1}^{x=1}(1-\bar{x}^2)\cos(\bar{\beta}_n\bar{x})d\bar{x}=-\frac{ga^2}{2k\cosh(\bar{\beta}_n\frac{b}{a})}\left[\int_{x=-1}^{x=1}(\cos(\bar{\beta}_n\bar{x})-\bar{x}^2\cos(\bar{\beta}_n\bar{x}))d\bar{x}\right]$$

$$\left[\int_{x=-1}^{x=1}(\cos(\bar{\beta}_n\bar{x})-\bar{x}^2\cos(\bar{\beta}_n\bar{x}))d\bar{x}\right]=\left[\frac{\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n}\Big|_{-1}^{1}-x^2\frac{\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n}\Big|_{-1}^{1}-2x\frac{\cos(\bar{\beta}_n\bar{x})}{\bar{\beta}_n^2}\Big|_{-1}^{1}+2\frac{\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n^3}\Big|_{-1}^{1}\right]$$

$$\left[\int_{x=-1}^{x=1}(\cos(\bar{\beta}_n\bar{x})-\bar{x}^2\cos(\bar{\beta}_n\bar{x}))d\bar{x}\right]=\left[\frac{2\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n}-\frac{2\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n}+\frac{4\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n^3}\right]=\frac{4\sin(\bar{\beta}_n\bar{x})}{\bar{\beta}_n^3}$$

$C_n=-\frac{2ga}{k\bar{\beta}_n^3}\frac{(-1)^n}{\cosh\left(\bar{\beta}_n\frac{b}{a}\right)}$ so the exact solution:

$$T(x,y)=\frac{g}{k}\left[\frac{a^2-x^2}{2}-2a^2\sum_{n=0}^{\infty}\frac{(-1)^n}{\bar{\beta}_n^3}\frac{\cosh\left(\bar{\beta}_n\frac{y}{a}\right)\cos\left(\bar{\beta}_n\frac{x}{a}\right)}{\cosh\left(\bar{\beta}_n\frac{b}{a}\right)}\right]$$

Now that exact solution formula is established, Galarkin approximation

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[\frac{\partial^2\bar{T}(x,y)}{\partial x^2}+\frac{\partial^2\bar{T}(x,y)}{\partial y^2}+\frac{1}{k}g\right]\phi_1 dxdy=0$$

Take

$\phi_1(x,y)=(a^2-x^2)(b^2-y^2)$

$\bar{T}_1(x,y)=C_1\phi_1(x,y)=C_1(a^2-x^2)(b^2-y^2)$

$\frac{\partial^2\bar{T}_1(x,y)}{\partial x^2}=-2C_1(b^2-y^2)$

$\frac{\partial^2\bar{T}_1(x,y)}{\partial y^2}=-2C_1(a^2-x^2)$

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[-2C_1\left((a^2-x^2)+(b^2-y^2)\right)+\frac{g}{k}\right](a^2-x^2)(b^2-y^2)dxdy=0$$

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[-2C_1\left((a^2-x^2)^2(b^2-y^2)+(a^2-x^2)(b^2-y^2)^2\right)+\frac{g}{k}(a^2-x^2)(b^2-y^2)\right]dxdy=0$$

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[-2C_1\left((a^4-2a^2x^2+x^4)(b^2-y^2)+(a^2-x^2)(b^4-2b^2y^2+y^4)\right)+\frac{g}{k}(a^2-x^2)(b^2-y^2)\right]dxdy=0$$

Variable conversion: $\bar{x}=\frac{x}{a}$ and $\bar{y}=\frac{y}{b}$

$$\int_{\bar{x}=-1}^{1}\int_{\bar{y}=-1}^{1}\left[-2C_1\left(a^2(1-2\bar{x}^2+\bar{x}^4)(1-\bar{y}^2)+(1-2\bar{y}^2+\bar{y}^4)(1-\bar{x}^2)\right)+\frac{g}{k}(1-\bar{x}^2)(1-\bar{y}^2)\right]d\bar{x}d\bar{y}=0$$

$$\int_{\bar{x}=-1}^{1}\int_{\bar{y}=-1}^{1}\left[-2C_1\left(a^2(1-2\bar{x}^2+\bar{x}^4)(1-\bar{y}^2)+(1-2\bar{y}^2+\bar{y}^4)(1-\bar{x}^2)\right)+\frac{g}{k}(1-\bar{x}^2)(1-\bar{y}^2)\right]d\bar{x}d\bar{y}=0$$

$$-2C_1\left(a^2(\bar{x}-\frac{2}{3}\bar{x}^3+\frac{1}{5}\bar{x}^5)\Big|_{-1}^{1}(\bar{y}-\frac{1}{3}\bar{y}^3)\Big|_{-1}^{1}\right\}+b^2(\bar{y}-\frac{2}{3}\bar{y}^3+\frac{1}{5}\bar{y}^5)\Big|_{-1}^{1}(\bar{x}-\frac{1}{3}\bar{x}^3)\Big|_{-1}^{1}\Big)+\frac{g}{k}(\bar{x}-\frac{1}{3}\bar{x}^3)\Big|_{-1}^{1}(\bar{y}-\frac{1}{3}\bar{y}^3)\Big|_{-1}^{1}=0$$

$$-2C_1\left(a^2\left[\left(1-\frac{2}{3}+\frac{1}{5}\right)-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]+b^2\left[\left(1-\frac{2}{3}+\frac{1}{5}\right)-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]\right)+$$

$$\frac{g}{k}\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]=0$$

$$-2C_1\left(a^2\frac{16}{15}\frac{4}{3}+b^2\frac{16}{15}\frac{4}{3}\right)+\frac{g}{k}\frac{4}{3}\frac{4}{3}=0$$

$$C_1 = \frac{5}{8}\frac{g/k}{a^2+b^2}$$

$$\overline{T_1}(x,y) = \frac{5}{8}\frac{g/k}{a^2+b^2}(a^2-x^2)(b^2-y^2)$$

Now let us apply a two coefficient approximation

$$\phi_1 = (a^2-x^2)(b^2-y^2)$$

$$\phi_2 = x^2(a^2-x^2)(b^2-y^2)$$

$$T_2(x,y) = C_1\phi_1 + C_2\phi_2$$

$$T_2(x,y) = C_1(a^2-x^2)(b^2-y^2) + C_2(a^2x^2 - x^4)(b^2-y^2)$$

$$\frac{\partial T_2(x,y)}{\partial x} = -2C_1x(b^2-y^2) + C_2(2a^2x - 4x^3)(b^2-y^2)$$

$$\frac{\partial^2 T_2(x,y)}{\partial x^2} = -2C_1(b^2-y^2) + C_2(2a^2 - 12x^2)(b^2-y^2)$$

$$\frac{\partial T_2(x,y)}{\partial y} = -2C_1(a^2 - x^2)y - 2C_2(a^2x^2 - x^4)y$$

$$\frac{\partial^2 T_2(x,y)}{\partial y^2} = -2C_1(a^2 - x^2) - 2C_2{}^2(a^2x^2 - x^4)$$

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial T(x,y)}{\partial y^2} + \frac{1}{k}g\right]\phi_1(x,y)dxdx = 0$$

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\left[\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial T(x,y)}{\partial y^2} + \frac{1}{k}g\right]\phi_2(x,y)dxdx = 0$$

$$\int_{\bar{x}=-a}^{a}\int_{\bar{y}=-a}^{a}\left[\left(-2C_1(a^2-x^2) + C_2(2a^4 - 14a^2x^2 + 12x^4)\right)(b^4 - 2b^2y^2 + y^4) + \left((a^4 - 2a^2x^2 + x^4)C_1 + (a^4x^2 - 2a^2x^4 + x^6)C_2\right)\left((-2)(b^2-y^2)\right) + \frac{g}{k}(a^2-x^2)(b^2-y^2)\right]dxdy = 0$$

$$\int_{\bar{x}=-a}^{a}\int_{\bar{y}=-a}^{a}\left[\left(-2C_1(a^2x^4 - x^2) + C_2(2a^4x^2 - 14a^2x^4 + 12x6)\right)(b^4 - 2b^2y^2 + y^4) + \left((a^4x^2 - 2a^2x^4 + x^6)C_1 + (a^4x^4 - 2a^2x^6 + x^8)C_2\right)\left((-2)(b^2-y^2)\right) + \frac{g}{k}(a^2x^2 - x^4)(b^2-y^2)\right]dxdy = 0$$

$$\left(-2C_1(a^2x - \frac{x^3}{3}) + C_2(2a^4x - \frac{14}{3}a^2x^3 + \frac{12}{5}x^5)\right)\Big|_{-a}^{a}(b^4x - 2b^2\frac{y^3}{3} + \frac{y^5}{5})\Big|_{-b}^{b} +$$

$$\left((a^4 - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5)C_1 + (a^4\frac{x^3}{3} - \frac{2}{5}a^2x^5 + \frac{x^7}{7})C_2\right)\Big|_{-a}^{a}\left((-2)(b^2y - \frac{y^3}{3})\right)\Big|_{-b}^{b} + \frac{g}{k}(a^2x - \frac{x^3}{3})\Big|_{-a}^{a}(b^2y - \frac{y^3}{3})\Big|_{-b}^{b} = 0$$

$$\left(-2C_1(a^2\frac{x^3}{3} - \frac{x^5}{5}) + C_2(2a^4\frac{x^3}{3} - \frac{14}{5}a^2x^5 + \frac{12}{7}x^7)\right)\Big|_{-a}^{a}(b^4x - 2b^2\frac{y^3}{3} + \frac{y^5}{5})\Big|_{-b}^{b} +$$

$$\left((a^4\frac{x^3}{3} - \frac{2}{5}a^2x^5 + \frac{1}{7}x^7)C_1 + (a^4\frac{x^5}{5} - \frac{2}{7}a^2x^7 + \frac{x^9}{9})C_2\right)\Big|_{-a}^{a}\left((-2)(b^2y - \frac{y^3}{3})\right)\Big|_{-b}^{b} + \frac{g}{k}(a^2\frac{x^3}{3} - \frac{x^5}{5})\Big|_{-a}^{a}(b^2y - \frac{y^3}{3})\Big|_{-b}^{b} = 0$$

$$\left(-\frac{8}{3}a^3C_1 - \frac{4}{15}a^5C_2\right)\frac{16}{15}b^5 + \left(\frac{16}{15}a^5C_1 + \frac{16}{105}a^7C_2\right)\frac{(-8)}{3}b^3 + \frac{16}{9}\frac{g}{k}a^3b^3 = 0$$

$$\left(-\frac{8}{15}a^5C_1 - \frac{88}{105}a^7C_2\right)\frac{16}{15}b^5 + \left(\frac{16}{105}a^7C_1 + \frac{158}{315}a^9C_2\right)\frac{(-8)}{3}b^3 + \frac{16}{45}\frac{g}{k}a^5b^3 = 0$$

$$\begin{bmatrix} \frac{8b^2(a^2+b^2)}{5} & \left(\frac{4}{25}a^2b^2 + \frac{8}{35}a^4\right) \\ 8(\frac{a^2}{7} + \frac{b^2}{5}) & \left(\frac{88}{35}a^2b^2 + \frac{79}{21}a^4\right) \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} \frac{g}{k} \\ \frac{g}{k} \end{Bmatrix}$$

```
class fa extends f_xj
{
double a,b,k,g;
public fa(double ai,double bi,double ki,double gi)
{a=ai;b=bi;k=ki;g=gi;}
```

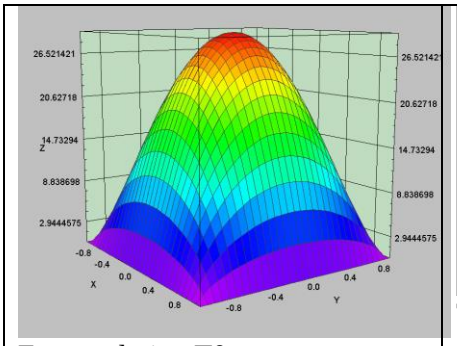
```
public double func(double x[])
{
//çözümü istenen fonksiyon
double ff=0;
double sign=1.0;
double beta=1.0;
double x1,y1,y2,ffx;
for(int n=0;n<200;n++)
{   beta=(2.0*n+1.0)/2.0*PI;
    y1=Math.cosh(beta*x[1]/a);
    x1=Math.cos(beta*x[0]/a);
    y2=Math.cosh(beta*b/a);
                ff+=-2.0*a*a*sign/(beta*beta*beta)*y1*x1/y2;
                sign*=-1.0;
}
ff+=(a*a-x[0]*x[0])/2.0;
ff*=g/k;
double ff1=5.0/8.0*g/k/(a*a+b*b)*(a*a-x[0]*x[0])*(b*b-x[1]*x[1]);

double aa[][]={{8.0/5.0*(b*b+a*a),(4.0/25.0*a*a*b*+8/35*a*a*a*a)},
            {(8.0/7.0*a*a+8.0/5.0*b*b),(88.0/35.0*a*a*b*b+79.0/21.0*a*a*a*a)}};
double bb[]={g/k,g/k};
double C[]=pivotlugauss(aa,bb);
double ff2=C[0]*(a*a-x[0]*x[0])*(b*b-x[1]*x[1])+C[1]*x[0]*x[0]*(a*a-x[0]*x[0])*(b*b-x[1]*x[1]);
return (ff-ff2); //maksimum testi
}

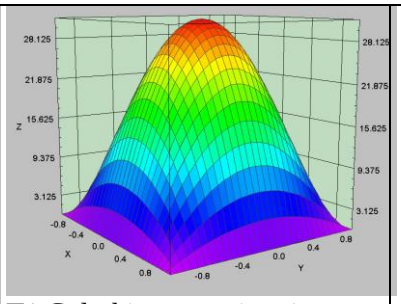
public static double[] pivotlugauss(double a[][],double b[])
  { //gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss eleme
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //Solving gauss elimination
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
          b[i]   =b[i]   -carpan*b[k];
                }
  }
  //back substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
  }
```

```
}

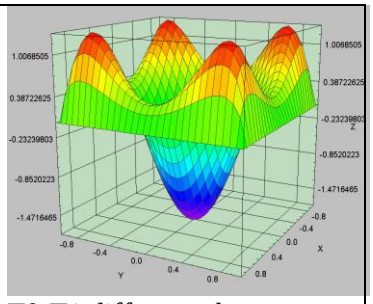
public class surfacePlotE4
{       public static void main(String arg[])
        {  double a=1.0;
           double b=1.0;
           double k=100.0;
           double g=10000.0;
                    fa f=new fa(a,b,k,g);
                    double x[]={0,0};
                    //System.out.println("T(0,0)="+f.func(x));
           surfacePlot p=new surfacePlot(f,-a,a,-b,b);
           p.plot();
        }
}
```



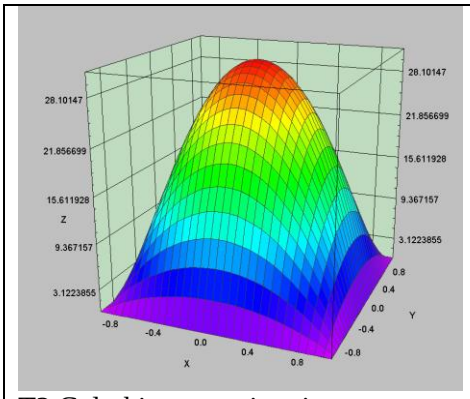
Exact solution T0



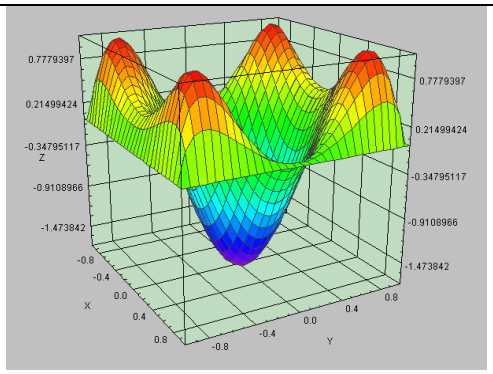
T1 Galarkin approximation



T0-T1 difference between solutions



T2 Galarkin approximation



T0-T2 difference between solutions

17. Eigenvalue applications and singualar value decomposition method

EIGENVALUE AND EIGENVECTOR CALCULATIONS

INTRODUCTION TO EIGENVALUES

Non-zero solution of **[A]{X} = 0** equation is existed, but not necessarily a single solution. A set of {X} vectors can be existed for any given matrix [A]. The equation can also be written as follows for the unit matrix I

[A]{X} = [I]{λ} {X} = 0

[A- λI]{X}= 0

In this equation **λ** is called eigen-value and **{X}** is called eigen-vector. If the equation is written in the open format:

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{11} & a_{22}-\lambda & a_{23} & \ldots & a_{2n} \\ a_{11} & a_{11} & a_{33}-\lambda & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & a_{3n} & \ldots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \ldots \\ X_5 \end{bmatrix} = 0$$

In this chapter, solution methods of eigenvalues and eigenvectors will be investigated. It should be note that eigenvalues could be complex numbers as well.

DOMINANT EIGENVALUE CALCULATION (POWER METHOD)

Assume that A is n x n matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, .., \lambda_n$ with an associated collection of linearly independent eigenvectors $x^1, x^2, x^3, \ldots, x^n$. It is also assumed that A has one eigenvalue λ_1 which is largest in magitude so that

$|\lambda_1| > |\lambda_2| > |\lambda_3| > |\lambda_4| \ldots > |\lambda_{n-1}| > |\lambda_n|$

It shoud be noted that a matrix A does not have to have linealy independent eigenvectors. When it does not, convergence of the power method may fail. Eigen value with the maximum absolute value is called dominant eigen value. If an eigenvalue is dominant it should be either a real value or it should have an eigen vector consist of real values. If x^k is the eigenvector corresponds to eigenvalue λ_k , The following equations can be written.

$A x^k = \lambda_k x^k$, $1 <= k < n$.

Where $\{ x^1, x^2, x^3, x^4, \ldots, x^n \}$ is eigenvector with n components. If A is a symmetric matrix, with n linearly independent eigenvalues. If y^0 is any vector the fact that $\{ x^1, x^2, x^3, x^4, \ldots, x^n \}$ is linearly independent implies that constants c_k exist with

$$y^0 = \sum_{k=1}^{n} c_k x^k$$

If a series of y^k vectors ae constructed

$$y^{k+1} = A y^k \quad k \geq 0$$

$$y^1 = A\left(\sum_{k=1}^{n} c_k x^k \right) = \sum_{k=1}^{n} c_k A x^k = \sum_{k=1}^{n} c_k \lambda_k x^k$$

Generalizing the equation will yield

$$y^i = \sum_{k=1}^{n} c_k \lambda_k^i x^k, \quad i \geq 0$$

remembering that λ_1 is the eigenvalue with the largest magnitude. If λ_1^i is taken out of the paranthesis, the equation becomes

$$y^i = \lambda_1^i \left[c_1 x^1 + \sum_{k=2}^{n} c_k \left(\frac{\lambda_k}{\lambda_1} \right)^i x^k \right], \quad i \geq 0$$

It is known tht $\left|\lambda_k / \lambda_1\right| < 1$. Therefore when $i \to \infty$ $\left|\lambda_k / \lambda_1\right|^i \to 0$.

$$y^i \approx \lambda_1^i c_1 x^1 , \quad i >> 1$$

If vector y is normalised :

$$y^{k+1} = \frac{A y^k}{\left\| A y^k \right\|}$$

Vector y became the eigenvector of the dominant eigenvalue. When the eigenvector is known, eigenvalue can be calculated easily.

$$Ax = \lambda x$$

By multilying both side of te equation with the transpose matrix x^T

$x^T Ax = \lambda x^T x$ and

$$\lambda = \frac{x^T Ax}{x^T x}$$

When k is is large, it is known that y^k is approximating to the dominant eigenvalue . In this case

$$\mu = \frac{(y^k)^T A y^k}{(y^k)^T y^k}, \quad k \geq 0$$

It can be written that with interpolation process, μ approximate to the dominant eigenvalue. Deviation of the μ from the eigen value is

$r^k \cong (\mu_k I - A) y^k$

In this equation when $r^k = 0$ μ_k becomes eigenvalue and y^k becomes eigenvector. In other words if $\|r_k\|$ is small iteration process can be finalize. In the following program computer code from the power method is given

Program Power method to calculate the dominant eigenvalue

```
import java.io.*;
import javax.swing.*;

class NA131
{
  //===========Eigen value calculations ==============
//*************norm methods definition*****************
public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

  public static double norm(double v[],int p)
  {
   //p vector norm
  if(p==0) p=1;
  double total=0;
  double x,y;
  for(int i=0;i<v.length;i++)
    {
   x=Math.abs(v[i]);
   total+=Math.pow(x,p);
    }
  return Math.pow(total,(1.0/p));
  }
  public static double norminf(double v[])
    {
     //infinite vector norm
    double x;
    double max=0;
    for(int i=0;i<v.length;i++)
      {
     x=Math.abs(v[i]);
     if(x>max) max=x;
      }
    return max;
    }
  public static double norminf(double a[][])
       {
         //infinite matrix norm
         double x;
```

```
        double max = 0;
        double total;
      int i,j;

      int n=a.length;
      for(i=0;i<n;i++)
        {
        total=0;
        for(j=0;j<n;j++)
        {total+=Math.abs(a[i][j]);}
        x=total;
        if(x>max) max=x;
        }
      return max;
      }
  public static double norm(double a[][])
     {
     //matrix norm
            double x;
            double max = 0;
            double total;
            int i,j;
            int n=a.length;
            for(j=0;j<n;i++)
              {
              total=0;
              for(i=0;i<n;j++)
              {total+=Math.abs(a[i][j]);}
              x=total;
              if(x>max) max=x;
              }
            return max;
    }
  public static double normE(double a[][])
          {
          //Euclidian matrix norm
          double x;
          double total;
          int i,j;
          total=0;
          int n=a.length;
          for(j=0;j<n;i++)
          {for(i=0;i<n;j++) {total+=a[i][j]*a[i][j];}}
          return Math.sqrt(total);
         }
 public static double[] multiply(double[][] left,double[] right)
 {
 //multiplication of one matrix with one vector
 int ii,jj,i,j,k;
 int m1=left[0].length;
 int n1=left.length;
 int m2=right.length;
 double[] b;
 b=new double[m2];
 if(n1 != m2) {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
  {b[ii]=0;}
  return b;   }
for(i=0;i<m1;i++)
  { b[i]=0;
    for(k=0;k<n1;k++)
              b[i]+=left[i][k]*right[k];}
return b;
//end of multiply of a matrix and a vector
}

 public static double  YT_X(double [] left,double [] right)
 {
 //multiplys a vector transpose with a vector
```

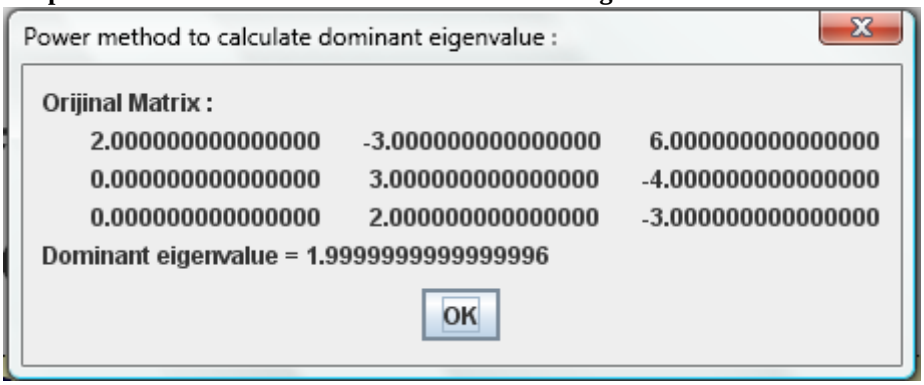
```
  int n=left.length;
  double tot=0;
  for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
  return tot;
  }

  public static double eigen_power(double A[][])
  {
   double eps=1e-15;
   double err=1;
   int m=100;
   int n=A.length;
   double y[]=new double[n];
   double r[]=new double[n];
   double xnorm;
   double mu=1.0;
   for(int i=0;i<n;i++) y[i]=1.0;
   int k=0;
   double mueski;
   double x[]=new double[n];
   do{
                mueski=mu;
                x=multiply(A,y);
                xnorm=norminf(x);
                for(int i=0;i<n;i++) {y[i]=x[i]/xnorm;}
                mu=YT_X(y,x)/YT_X(y,y);
                for(int i=0;i<n;i++) {r[i]=mu*y[i]-x[i];}
                k++;
                err=Math.abs((mu-mueski)/mu);
     }while(err>eps);
   return mu;
  }



  public static void main(String args[]) throws IOException
  {
  //Eigen Value calculations using  method
  //double a[][]={{3.556,-1.778,0},{-1.778,3.556,-1.778},{0,-1.778,3.556}};
  //double a[][]={{-3.9,-0.6,-3.9,0.4},{1,0,0,0},{0,1,0,0},{0,0,1,0}};
  double a[][]={{2,-3,6},{0,3,-4},{0,2,-3}};
  String s="Orijinal Matrix : \n";
  s+=Matrix.toString(a);
  s+="Dominant eigenvalue = "+eigen_power(a);
  JOptionPane.showMessageDialog(null,s,
     "Power method to calculate dominant eigenvalue : ",JOptionPane.PLAIN_MESSAGE);
  }
 }
```

Output 9.2-1 Power method to calculate dominant eigenvalue



Example : Find the biggest eigenvalue of the following matrix by using power method

$$A=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix}$$

Steps of process is shown in spreadheet solution. Starting vector is taken as y={1,1}

1	2	1	3	3
2	5	1	7	7
				7

1	2	0.428571	2.428571	2.428571
2	5	1	5.857143	5.857143
				5.857143

1	2	0.414634	2.414634	2.414634
2	5	1	5.829268	5.829268
				5.829268

1	2	0.414226	2.414226	2.414226
2	5	1	5.828452	5.828452
				5.828452

1	2	0.414214	2.414214	2.414214
2	5	1	5.828428	5.828428
				5.828428

1	2	0.414214	2.414214	2.414214
2	5	1	5.828427	5.828427
				5.828427

λ= 5.828427

Example : Here is a second example:

$$A=\begin{bmatrix}10 & 4 & -6\\ 4 & -6 & 8\\ -6 & 8 & 14\end{bmatrix}$$ Find the Biggest eigenvalue by power method. Starting vector is again taken as y={1,1,1}

3.556	-1.778	0	1	1.778	1.778	
-1.778	3.556	-1.778	1	0	0	
0	-1.778	3.556	1	1.778	1.778	
					1.778	

3.556	-1.778	0	1	3.556	3.556	
-1.778	3.556	-1.778	0	-3.556	3.556	
0	-1.778	3.556	1	3.556	3.556	
				3.556	3.556	1

3.556	-1.778	0	1	5.334	5.334	
-1.778	3.556	-1.778	-1	-7.112	7.112	
0	-1.778	3.556	1	5.334	5.334	
				5.334	7.112	1

3.556	-1.778	0	0.75	4.445	4.445	
-1.778	3.556	-1.778	-1	-6.223	6.223	
0	-1.778	3.556	0.75	4.445	4.445	
				4.445	6.223	-0.125

3.556	-1.778	0	0.714286	4.318	4.318	
-1.778	3.556	-1.778	-1	-6.096	6.096	
0	-1.778	3.556	0.714286	4.318	4.318	
				4.318	6.096	-0.020408163

3.556	-1.778	0	0.708333	4.296833	4.296833	
-1.778	3.556	-1.778	-1	-6.07483	6.074833	
0	-1.778	3.556	0.708333	4.296833	4.296833	
				4.296833	6.074833	-0.003472222
3.556	-1.778	0	0.707317	4.29322	4.29322	
-1.778	3.556	-1.778	-1	-6.07122	6.07122	
0	-1.778	3.556	0.707317	4.29322	4.29322	
				4.29322	6.07122	-0.000594884
3.556	-1.778	0	0.707143	4.2926	4.2926	
-1.778	3.556	-1.778	-1	-6.0706	6.0706	
0	-1.778	3.556	0.707143	4.2926	4.2926	
				4.2926	6.0706	-0.000102041
3.556	-1.778	0	0.707113	4.292494	4.292494	
-1.778	3.556	-1.778	-1	-6.07049	6.070494	
0	-1.778	3.556	0.707113	4.292494	4.292494	
				4.292494	6.070494	-1.75067E-05
3.556	-1.778	0	0.707108	4.292475	4.292475	
-1.778	3.556	-1.778	-1	-6.07048	6.070475	
0	-1.778	3.556	0.707108	4.292475	4.292475	
				4.292475	6.070475	-3.00365E-06
3.556	-1.778	0	0.707107	4.292472	4.292472	
-1.778	3.556	-1.778	-1	-6.07047	6.070472	
0	-1.778	3.556	0.707107	4.292472	4.292472	
				4.292472	6.070472	-5.15345E-07
3.556	-1.778	0	0.707107	4.292472	4.292472	
-1.778	3.556	-1.778	-1	-6.07047	6.070472	
0	-1.778	3.556	0.707107	4.292472	4.292472	
				4.292472	6.070472	-8.84191E-08

$\lambda =$ 6.070472

SMALLEST MAGNITUDE EIGENVALUE(INVERSE POWER METHOD)

If Power method for dominant eigenvalue problem applied in inverse way, smallest magnitude eigenvalue can be obtained.

$$y^k = \frac{A^{-1}y^{k-1}}{\left\|A^{-1}y^{k-1}\right\|}$$

$$\mu_k = \frac{(y^k)^T A^{-1} y^k}{(y^k)^T y^k}$$

When k is large $1/\mu_k$ becomes eigenvalue.

Program Power method to calculate smallest magnitude eigenvalue

```
import java.io.*;
import javax.swing.*;


class NA132
{
  //===========Eigen value calculations ==============
public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}
```

```
public static double norm(double v[],int p)
{
 //p vector norm
if(p==0) p=1;
double total=0;
double x,y;
for(int i=0;i<v.length;i++)
  {
  x=Math.abs(v[i]);
  total+=Math.pow(x,p);
  }
return Math.pow(total,(1.0/p));
}
public static double norminf(double v[])
  {
   //infinite vector norm
  double x;
  double max=0;
  for(int i=0;i<v.length;i++)
    {
    x=Math.abs(v[i]);
    if(x>max) max=x;
    }
  return max;
  }
 public static double norminf(double a[][])
     {
       //infinite matrix norm
       double x;
       double max = 0;
       double total;
     int i,j;

     int n=a.length;
     for(i=0;i<n;i++)
       {
       total=0;
       for(j=0;j<n;j++)
       {total+=Math.abs(a[i][j]);}
       x=total;
       if(x>max) max=x;
       }
     return max;
     }
 public static double norm(double a[][])
    {
    //matrix norm
            double x;
            double max = 0;
            double total;
            int i,j;
            int n=a.length;
            for(j=0;j<n;i++)
              {
              total=0;
              for(i=0;i<n;j++)
              {total+=Math.abs(a[i][j]);}
              x=total;
              if(x>max) max=x;
              }
            return max;
   }
 public static double normE(double a[][])
          {
          //Euclidian matrix norm
          double x;
          double total;
          int i,j;
          total=0;
          int n=a.length;
```

```
            for(j=0;j<n;i++)
            {for(i=0;i<n;j++) {total+=a[i][j]*a[i][j];}}
            return Math.sqrt(total);
           }
 public static double[] multiply(double[][] left,double[] right)
 {
 //multiplication of one matrix with one vector
 int ii,jj,i,j,k;
 int m1=left[0].length;
 int n1=left.length;
 int m2=right.length;
 double[] b;
 b=new double[m2];
 if(n1 != m2) {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
  {b[ii]=0;}
  return b;   }
for(i=0;i<m1;i++)
  { b[i]=0;
    for(k=0;k<n1;k++)
                b[i]+=left[i][k]*right[k];}
return b;
//end of multiply of a matrix and a vector
}

 public static double  YT_X(double [] left,double [] right)
 {
 //multiplys a vector transpose with a vector
 int n=left.length;
 double tot=0;
 for(int i=0;i<n;i++)
 {
 tot+=left[i]*right[i];
 }
 return tot;
 }

 public static double eigen_power(double A[][])
 {
  double eps=1e-15;
  double err=1;
  int m=100;
  int n=A.length;
  double y[]=new double[n];
  double r[]=new double[n];
  double xnorm;
  double mu=1.0;
  for(int i=0;i<n;i++) y[i]=1.0;
  int k=0;
  double mueski;
  double x[]=new double[n];
  do{
                mueski=mu;
                x=multiply(A,y);
                xnorm=norminf(x);
                for(int i=0;i<n;i++) {y[i]=x[i]/xnorm;}
                mu=YT_X(y,x)/YT_X(y,y);
                for(int i=0;i<n;i++) {r[i]=mu*y[i]-x[i];}
                k++;
                err=Math.abs((mu-mueski)/mu);
    }while(err>eps);
  return mu;
 }
   public static void pivot(double a[][],int o[],double s[],int k)
 {// GaussLU için pivot metodu
   int n=a.length;
   int dummy1=0;
   double buyuk;
   double dummy=0;
```

```
  //pivotlama
              int p=k;
              buyuk=Math.abs(a[o[k]][k]);
              for(int ii=k+1;ii<n;ii++)
              {  dummy=Math.abs(a[o[ii]][k]/s[o[ii]]);
                 if(dummy > buyuk) {buyuk=dummy;p=ii;}
              }
              dummy1=o[p];
              o[p]=o[k];
              o[k]=dummy1;
}

public static int[] sirali_indeks(int n)
{   // sirali rakamlari taşiyan bir vektör oluşturur
                int o[]=new int[n];
                for(int i=0;i<n;i++) {o[i]=i;}
                return o;
}

public static double[][] gaussLU(double ai[][],int o[])
{ //kismi pivotlu gauss eleme yöntemi
int n=ai.length;
double a[][]=new double[n][n];
double s[]=new double[n];
double carpan=0;
double toplam=0;
double absaij;

//gauss eleme
int i,j,k,p,ii,jj;
//System.out.println("Orijinal Matris :\n"+Matrix.toString(ai));
for(i=0;i<n;i++)
{for(j=0;j<n;j++){a[i][j]=ai[i][j];}}
for(i=0;i<n;i++)
{ o[i]=i;s[i]=Math.abs(a[i][0]);
  for(j=1;j<n;j++)
    { absaij=Math.abs(a[i][j]);
      if(absaij>s[i]) {s[i]=absaij;}
    }
}
for(k=0;k<(n-1);k++)
{ pivot(a,o,s,k);
              //pivot referansini değiştir
              //System.out.println("pivotlu Matris : k="+k+"\n"+Matrix.toString(a));
              //gauss elemeyi çözme
  for(i=k+1;i<n;i++)
  {  carpan=a[o[i]][k]/a[o[k]][k];
     a[o[i]][k]=carpan;
     for(j=k+1;j<n;j++)
     {   a[o[i]][j]-=carpan*a[o[k]][j]; }
              }
}
return a;
}

public static double[] yerine_koyma(double ai[][],double bi[])
{   int n=ai.length;
                return yerine_koyma(ai,bi,sirali_indeks(n));
}

public static double[][] yerine_koyma(double ai[][],double bi[][])
{   int n=ai.length;
                return yerine_koyma(ai,bi,sirali_indeks(n));
}

public static double[] yerine_koyma(double ai[][],double bi[],int o[])
{ //gauss LU çözümüyle LU indirgenmesi yapilmiş Matris kullanarak linear
  //denklem sistemi kökleri
  int n=ai.length;
  double toplam;
```

```
  double x[]=new double[n];
   double a[][]=new double[n][n];
  double b[]=new double[n];
  int i,j;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};b[i]=bi[i];}
//System.out.println("a=\n "+Matrix.toString(a));
//System.out.println("b=\n "+Matrix.toStringT(b));
for(i=1;i<n;i++)
    {toplam=b[o[i]];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]];};b[o[i]]=toplam;}
 //System.out.println("d=\n "+Matrix.toStringT(b));
x[n-1]=b[o[n-1]]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {toplam=0;
     for(j=i+1;j<n;j++)
      {toplam+=a[o[i]][j]*x[j];};
      x[i]=(b[o[i]]-toplam)/a[o[i]][i];
    }
//System.out.println("d=\n "+Matrix.toStringT(x));
return x;
}

public static double[][] yerine_koyma(double ai[][],double bi[][],int o[])
{
  int n=ai.length;
  int m=bi[0].length;
  double toplam;
  double x[][]=new double[n][m];
  double a[][]=new double[n][n];
  double b[][]=new double[n][m];
  int i,j,k;
for(i=0;i<n;i++)
    {for(j=0;j<n;j++){a[i][j]=ai[i][j];};for(k=0;k<m;k++) {b[i][k]=bi[i][k];}}
//System.out.println("a=\n "+Matrix.toString(a));
//System.out.println("b=\n "+Matrix.toStringT(b));
for(k=0;k<m;k++)
{
for(i=1;i<n;i++)
    {toplam=b[o[i]][k];for(j=0;j<i;j++){toplam-=a[o[i]][j]*b[o[j]][k];};b[o[i]][k]=toplam;}
 //System.out.println("d=\n "+Matrix.toStringT(b));
x[n-1][k]=b[o[n-1]][k]/a[o[n-1]][n-1];
for(i=(n-2);i>=0;i--)
    {toplam=0;
     for(j=i+1;j<n;j++)
      {toplam+=a[o[i]][j]*x[j][k];};
      x[i][k]=(b[o[i]][k]-toplam)/a[o[i]][i];
    }
}//kin sonu
//System.out.println("d=\n "+Matrix.toStringT(x));
return x;
}

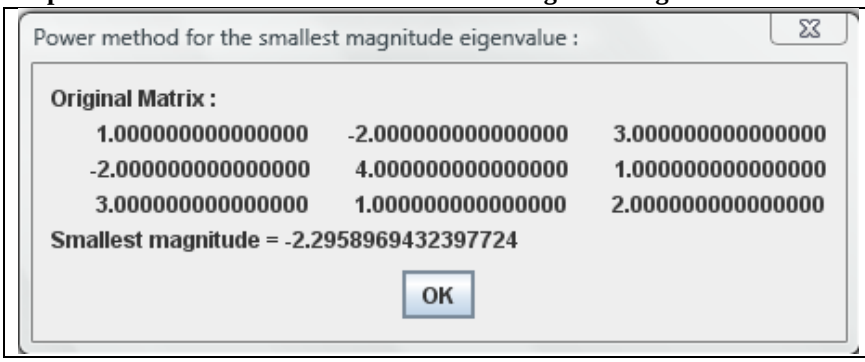
public static double [] AXB(double a[][],double b[])
{
//denklem sistemi çözümü b vektörü için
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[]=yerine_koyma(c,b,q);
return s;
}

public static double [][] AXB(double a[][],double b[][])
{//denklem sistemi çözümü b matrisi için
int n=a.length;
int q[]=new int[n];
double c[][]=gaussLU(a,q);
double s[][]=yerine_koyma(c,b,q);
return s;
}
```

```
public static double eigen_inversepower(double A[][])
{
 double eps=1e-15;
 double err=1;
 int m=100;
 int n=A.length;
 double y[]=new double[n];
 double r[]=new double[n];
 int o[]=new int[n];
 double B[][]=new double[n][n];
 double xnorm;
 double mu=1.0;
 for(int i=0;i<n;i++) y[i]=1.0;
 int k=0;
 double mueski;
 double x[]=new double[n];
 do{
                mueski=mu;
                B=gaussLU(A,o);
                x=yerine_koyma(B,y,o);
                xnorm=norminf(x);
                for(int i=0;i<n;i++) {y[i]=x[i]/xnorm;}
                x=yerine_koyma(B,y,o);
                mu=YT_X(y,x)/YT_X(y,y);
                for(int i=0;i<n;i++) {r[i]=mu*y[i]-x[i];}
                k++;
                err=Math.abs((mu-mueski)/mu);
   }while(err>eps);
 return 1.0/mu;
}

public static void main(String args[]) throws IOException
{
//Eigen Value calculations using  method
//double a[][]={{3.556,-1.778,0},{-1.778,3.556,-1.778},{0,-1.778,3.556}};
//double a[][]={{-3.9,-0.6,-3.9,0.4},{1,0,0,0},{0,1,0,0},{0,0,1,0}};
//double a[][]={{-1,2},{2,2}};
double a[][]={{1,-2,3},{-2,4,1},{3,1,2}};
String s="Original Matrix : \n";
s+=Matrix.toString(a);
s+="Smallest magnitude = "+eigen_inversepower(a);
JOptionPane.showMessageDialog(null,s,
   "Power method for the smallest magnitude eigenvalue : ",JOptionPane.PLAIN_MESSAGE);
}
}
```

Output 9.3-1 Power method for the smallest magnitude eigevalue



Example : Find the smalest eigenvalue of $A=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix}$ by using inverse power method.

Inverse of the A

$$A^{-1}=\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

5	-2	1	3	3
-2	1	1	-1	1
				3

5	-2	1	5.666667	5.666667
-2	1	-0.33333	-2.33333	2.333333
				5.666667

5	-2	1	5.823529	5.823529
-2	1	-0.41176	-2.41176	2.411765
				5.823529

5	-2	1	5.828283	5.828283
-2	1	-0.41414	-2.41414	2.414141
				5.828283

5	-2	1	5.828423	5.828423
-2	1	-0.41421	-2.41421	2.414211
				5.828423

5	-2	1	5.828427	5.828427
-2	1	-0.41421	-2.41421	2.414213
				5.828427

λ=1/5.828427= 0.171573

Example : Find the smalest eigenvalue of $A=\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix}$

by using inverse power method.

Inverse of the matrix A

$$A^{-1}=\begin{bmatrix} 0.42182227 & 0.281215 & 0.14060 \\ 0.28121484 & 0.56243 & 0.28121 \\ 0.14060742 & 0.281215 & 0.42182 \end{bmatrix}$$

0.421822272	0.281215	0.140607	1	0.843645	0.843645	
0.281214848	0.56243	0.281215	1	1.124859	1.124859	
0.140607424	0.281215	0.421822	1	0.843645	0.843645	
					1.124859	
0.421822272	0.281215	0.140607	0.75	0.703037	0.703037	
0.281214848	0.56243	0.281215	1	0.984252	0.984252	
0.140607424	0.281215	0.421822	0.75	0.703037	0.703037	
				0.984252	0.984252	-0.125
0.421822272	0.281215	0.140607	0.714286	0.68295	0.68295	
0.281214848	0.56243	0.281215	1	0.964165	0.964165	
0.140607424	0.281215	0.421822	0.714286	0.68295	0.68295	
				0.964165	0.964165	-0.02041
0.421822272	0.281215	0.140607	0.708333	0.679603	0.679603	
0.281214848	0.56243	0.281215	1	0.960817	0.960817	
0.140607424	0.281215	0.421822	0.708333	0.679603	0.679603	
				0.960817	0.960817	-0.00347
0.421822272	0.281215	0.140607	0.707317	0.679031	0.679031	
0.281214848	0.56243	0.281215	1	0.960246	0.960246	

0.140607424	0.281215	0.421822	0.707317	0.679031	0.679031	
				0.960246	0.960246	-0.00059
0.421822272	0.281215	0.140607	0.707143	0.678933	0.678933	
0.281214848	0.56243	0.281215	1	0.960148	0.960148	
0.140607424	0.281215	0.421822	0.707143	0.678933	0.678933	
				0.960148	0.960148	-0.0001
0.421822272	0.281215	0.140607	0.707113	0.678916	0.678916	
0.281214848	0.56243	0.281215	1	0.960131	0.960131	
0.140607424	0.281215	0.421822	0.707113	0.678916	0.678916	
				0.960131	0.960131	-1.8E-05
0.421822272	0.281215	0.140607	0.707108	0.678913	0.678913	
0.281214848	0.56243	0.281215	1	0.960128	0.960128	
0.140607424	0.281215	0.421822	0.707108	0.678913	0.678913	
				0.960128	0.960128	-3E-06
0.421822272	0.281215	0.140607	0.707107	0.678913	0.678913	
0.281214848	0.56243	0.281215	1	0.960128	0.960128	
0.140607424	0.281215	0.421822	0.707107	0.678913	0.678913	
				0.960128	0.960128	-5.2E-07
0.421822272	0.281215	0.140607	0.707107	0.678913	0.678913	
0.281214848	0.56243	0.281215	1	0.960128	0.960128	
0.140607424	0.281215	0.421822	0.707107	0.678913	0.678913	
				0.960128	0.960128	-8.8E-08
0.421822272	0.281215	0.140607	0.707107	0.678913	0.678913	
0.281214848	0.56243	0.281215	1	0.960128	0.960128	
0.140607424	0.281215	0.421822	0.707107	0.678913	0.678913	
				0.960128	0.960128	-1.5E-08
0.421822272	0.281215	0.140607	0.707107	0.678913	0.678913	
0.281214848	0.56243	0.281215	1	0.960128	0.960128	
0.140607424	0.281215	0.421822	0.707107	0.678913	0.678913	
				0.960128	0.960128	-2.6E-09

λ=1/0.960128= 1.041528

COMPUTATION OF INTERMEDIATE EIGENVALUES BY POWER METHOD

Using power method already given largest and smallest eigenvalue. The remaining eigenvalues can also be found by using power method. The procedure is called deflation. **Hotelling deflation** for symmetric matrices will be described in here. According to this procedure deflated matrix A_i to be used in finding the eigenvector X_i and this eigenvector will be used to define new defleted matrix A_{i+1} acording to relation

$$[A_{i+1}]=[A_i]-\lambda_i\{X^i\}\{X^i\}^T \quad i=1,2,\ldots,n-1$$

Eigenvector $\{X^i\}$ has been normalized as. $\{X^i\}\{X^i\}^T=1$ This process can be defined as $k=\dfrac{1}{|X^i|}$

$$\{X^i\}=k\{X^i\}$$

An example is given by using spreadsheet:

$$A=\begin{bmatrix}12 & -6 & -6\\ 6 & 16 & 2\\ -6 & 2 & 16\end{bmatrix}$$

POWER METHOD

A		
12	6	-6
6	16	2
-6	2	16

y
1
0.5
-0.5

X
18
13
-13

18

X/Xmax
1
0.722222222
-0.722222222

A

12	6	-6
6	16	2
-6	2	16

y

1
0.7222222
-0.722222

X

20.66666667
16.11111111
-16.1111111

X/Xmax

1
0.779569892
-0.779569892

TOTAL 20.66666667 RESİDUE 2.666666667

A

12	6	-6
6	16	2
-6	2	16

y

1
0.7795699
-0.77957

X

21.35483871
16.91397849
-16.9139785

21.35483871

X/Xmax

1
0.79204431
-0.79204431

0.688172043

A

12	6	-6
6	16	2
-6	2	16

y

1
0.7920443
-0.792044

X

21.50453172
17.08862034
-17.0886203

21.50453172

X/Xmax

1
0.794652056
-0.794652056

0.149693012

A

12	6	-6
6	16	2
-6	2	16

y

1
0.7946521
-0.794652

X

21.53582467
17.12512878
-17.1251288

21.53582467

X/Xmax

1
0.795192617
-0.795192617

0.031292948

A

12	6	-6
6	16	2
-6	2	16

y

1
0.7951926
-0.795193

X

21.54231141
17.13269664
-17.1326966

21.54231141

X/Xmax

1
0.795304474
-0.795304474

0.006486736

A

12	6	-6
6	16	2
-6	2	16

y

1
0.7953045
-0.795304

X

21.54365369
17.13426264
-17.1342626

21.54365369

X/Xmax

1
0.795327612
-0.795327612

0.001342282

A

12	6	-6
6	16	2
-6	2	16

1
0.7953276
-0.795328

21.54393134
17.13458656
-17.1345866

21.54393134

1
0.795332397
-0.795332397

0.000277654

λ1=		**21.543654**

1	0.7953324	-0.795	1
			0.7953324
			-0.795332

K^2= 2.265107245 0.4414802

K= 0.664439764

1			0.664439764
0.7953324	*	0.6644398	0.528450471
-0.795332			-0.528450471

X1=

0.6644398	0.6644398	0.5284505	-0.528
0.5284505			
-0.52845			

0.4414802	0.35112351	-0.351123506
0.351123506	0.2792599	-0.2792599
-0.35112351	-0.2792599	0.2792599

0.4414802	0.3511235	-0.351124			9.511096543	7.56448322	-7.564483216
0.3511235	0.2792599	-0.27926	*	21.543654	=	7.564483216	6.01627857	-6.016278572
-0.351124	-0.27926	0.2792599			-7.56448322	-6.01627857	6.016278572

12	6	-6		9.5110965	7.56448322	-7.56448322		2.488903457	-1.564483	1.5644832
6	16	2	-	7.5644832	6.01627857	-6.01627857	=	-1.564483216	9.9837214	8.0162786

-6	2	16	-7.564483	-6.01627857	6.016278572	1.564483216	8.0162786	9.9837214

A

2.4889035	-1.564483	1.5644832	1	2.488903457	0.127215395
-1.564483	9.9837214	8.0162786	1	16.43551678	0.840069048
1.5644832	8.0162786	9.9837214	1	19.56448322	1
				19.56448322	

2.4889035	-1.564483	1.5644832	0.1272154	0.566836127	0.033506943
-1.564483	9.9837214	8.0162786	0.840069	16.20426757	0.957870264
1.5644832	8.0162786	9.9837214	1	16.91697528	1
				16.91697528	

2.4889035	-1.564483	1.5644832	0.0335069	0.149306812	0.008428414
-1.564483	9.9837214	8.0162786	0.9578703	17.5269674	0.989402588
1.5644832	8.0162786	9.9837214	1	17.71469735	1
				17.71469735	

2.4889035	-1.564483	1.5644832	0.0084284	0.037556983	0.002094851
-1.564483	9.9837214	8.0162786	0.9894026	17.88101228	0.997366052
1.5644832	8.0162786	9.9837214	1	17.92823431	1
				17.92823431	

2.4889035	-1.564483	1.5644832	0.0020949	0.009334649	0.000519106
-1.564483	9.9837214	8.0162786	0.9973661	17.97042604	0.999347305
1.5644832	8.0162786	9.9837214	1	17.9821629	1
				17.9821629	

2.4889035	-1.564483	1.5644832	0.0005191	0.002313135	0.000128539
-1.564483	9.9837214	8.0162786	0.9993473	17.99267155	0.999838382
1.5644832	8.0162786	9.9837214	1	17.99557995	1
				17.99557995	

2.4889035	-1.564483	1.5644832	0.0001285	0.000572769	3.18225E-05
-1.564483	9.9837214	8.0162786	0.9998384	17.99818536	0.999959988
1.5644832	8.0162786	9.9837214	1	17.99890552	1
				17.99890552	

2.4889035	-1.564483	1.5644832	3.182E-05	0.000141801	7.87794E-06
-1.564483	9.9837214	8.0162786	0.99996	17.99955075	0.999990095
1.5644832	8.0162786	9.9837214	1	17.99972904	1
				17.99972904	

2.4889035	-1.564483	1.5644832	7.878E-06	3.51041E-05	1.95023E-06
-1.564483	9.9837214	8.0162786	0.9999901	17.99988878	0.999997548
1.5644832	8.0162786	9.9837214	1	17.99993292	1
				17.99993292	

λ2=		**17.999933**

12	6	-6
6	16	2
-6	2	16

1	0	0
0	1	0
0	0	1

C=

		12	6	-6	1	0	0
	0.5	0	13	5	-0.5	1	0
	-0.5	0	5	13	0.5	0	1

		12	6	-6	1	0	0
		0	13	5	-0.5	1	0
	0.38462	0	0	11.076923	0.6923077	-0.38461538	1

inverse A

0.14583	-0.0625	0.090278
-0.0625	0.090278	-0.090278
0.0625	-0.03472	0.090278

			y	X	X/Xmax
0.14583	-0.0625	0.09028	1	0.173611111	1
-0.0625	0.09028	-0.0903	1	-0.0625	-0.36
0.0625	-0.0347	0.09028	1	0.118055556	0.68
				0.173611111	

0.14583	-0.0625	0.09028	1	0.229722222	1
-0.0625	0.09028	-0.0903	-0.36	-0.15638889	-0.680773881
0.0625	-0.0347	0.09028	0.68	0.136388889	0.593712213
				0.229722222	

0.14583	-0.0625	0.09028	1	0.24198072	1
-0.0625	0.09028	-0.0903	-0.680774	-0.17755777	-0.733768262
0.0625	-0.0347	0.09028	0.5937122	0.139737001	0.577471631
				0.24198072	

0.14583	-0.0625	0.09028	1	0.243826705	1
-0.0625	0.09028	-0.0903	-0.733768	-0.18087582	-0.741821219
0.0625	-0.0347	0.09028	0.5774716	0.14011092	0.574633201
				0.243826705	

0.14583	-0.0625	0.09028	1	0.244073768	1
-0.0625	0.09028	-0.0903	-0.741821	-0.18134658	-0.742999058
0.0625	-0.0347	0.09028	0.5746332	0.14013429	0.574147279
				0.244073768	

0.14583	-0.0625	0.09028	1	0.244103515	1
-0.0625	0.09028	-0.0903	-0.742999	-0.18140904	-0.743164409
0.0625	-0.0347	0.09028	0.5741473	0.140131319	0.574065142
				0.244103515	

0.14583	-0.0625	0.09028	1	0.244106434	1
-0.0625	0.09028	-0.0903	-0.743164	-0.18141656	-0.743186296
0.0625	-0.0347	0.09028	0.5740651	0.140129645	0.574051419
				0.244106434	

0.14583	-0.0625	0.09028	1	0.244106563	1
-0.0625	0.09028	-0.0903	-0.743186	-0.18141729	-0.743188922
0.0625	-0.0347	0.09028	0.5740514	0.140129166	0.574049154
				0.244106563	

λ3=		**4.096571**

A program developed for power eigenvalues, and the same problem as above spreadsheet is run

Program : Power method to calculate all eigenvalues

```
import java.io.*;
import javax.swing.*;

class eigen_power
{   double A[][];
              double X[][];
              double lambda[];
              int jj;
              public eigen_power(double Ai[][])
              {jj=0;
               A=Ai;
               int n=A.length;
    X=new double[n][n];
    lambda=new double[n];
              }
  //===========Eigen value calculations ==============
//*************norm methods definition*****************
public static double norm(double v[])
{
// vector norm
double total=0;;
for(int i=0;i<v.length;i++)
  {total+=v[i]*v[i];}
return Math.sqrt(total);
}

  public static double norm(double v[],int p)
  {
   //p vector norm
  if(p==0) p=1;
  double total=0;
  double x,y;
  for(int i=0;i<v.length;i++)
    {
   x=Math.abs(v[i]);
   total+=Math.pow(x,p);
    }
  return Math.pow(total,(1.0/p));
  }
  public static double norminf(double v[])
    {
     //infinite vector norm
    double x;
    double max=0;
    for(int i=0;i<v.length;i++)
      {
      x=Math.abs(v[i]);
      if(x>max) max=x;
      }
    return max;
    }
```

```
  public static double norminf(double a[][])
      {
       //infinite matrix norm
       double x;
       double max = 0;
       double total;
      int i,j;

      int n=a.length;
      for(i=0;i<n;i++)
       {
       total=0;
       for(j=0;j<n;j++)
       {total+=Math.abs(a[i][j]);}
       x=total;
       if(x>max) max=x;
       }
      return max;
      }
  public static double norm(double a[][])
     {
     //matrix norm
            double x;
            double max = 0;
            double total;
            int i,j;
            int n=a.length;
            for(j=0;j<n;i++)
              {
              total=0;
              for(i=0;i<n;j++)
              {total+=Math.abs(a[i][j]);}
              x=total;
              if(x>max) max=x;
              }
            return max;
    }
  public static double normE(double a[][])
          {
          //Euclidian matrix norm
          double x;
          double total;
          int i,j;
          total=0;
          int n=a.length;
          for(j=0;j<n;i++)
          {for(i=0;i<n;j++) {total+=a[i][j]*a[i][j];}}
          return Math.sqrt(total);
         }

  public static double[] multiply(double[][] left,double[] right)
  {
  //multiplication of one matrix with one vector
  int ii,jj,i,j,k;
  int m1=left[0].length;
  int n1=left.length;
  int m2=right.length;
  double[] b;
  b=new double[m2];
  if(n1 != m2) {
  System.out.println("inner matrix dimensions must agree");
  for(ii=0;ii<n1;ii++)
   {b[ii]=0;}
   return b;   }
 for(i=0;i<m1;i++)
   { b[i]=0;
     for(k=0;k<n1;k++)
               b[i]+=left[i][k]*right[k];}
               return b;

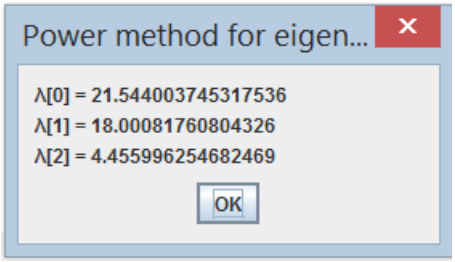
 //end of multiply of a matrix and a vector
 }
 public static double[] multiply(double left,double[] right)
{
//multiplying a vector with a constant
 int i;
```

```
int n=right.length;
double b[];
b=new double[n];
for(i=0;i<n;i++)
  {
  b[i]=left*right[i];
  }
return b;
}
  public static double  YT_X(double [] left,double [] right)
  {
  //multiplys a vector transpose with a vector
  int n=left.length;
  double tot=0;
  for(int i=0;i<n;i++)
  {
  tot+=left[i]*right[i];
  }
  return tot;
  }
public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
                aa[i][j]=left[i]*left[j];
                }
  }
return aa;
}
  public static double[][] multiply(double left,double[][] right)
{
//multiplying a matrix with a constant
int i,j;
int n=right.length;
int m=right[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=right[i][j]*left;
  }
return b;
//end of multiplying a matrix with a constant double
}
public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
  for(j=0;j<m1;j++)
    {
    b[i][j]=b[i][j]+left[i][j];
    }
  }
for(i=0;i<n2;i++)
  {
  for(j=0;j<m2;j++)
    {
```

```
    b[i][j]=b[i][j]-right[i][j];
    }
  }
return b;
}
  public double[] power()
  {double eps=1e-25;
   double err=1;
   int m=300;
   int n=A.length;
   double y[]=new double[n];
   double r[]=new double[n];
   double xnorm;
   double mu=1.0;
   double A1[][]=A;
   for(jj=0;jj<n-1;jj++)
   {
   for(int i=0;i<n;i++) y[i]=1.0;
   int k=0;
   double mueski;
   double x[]=new double[n];
   do{
                    mueski=mu;
                    x=multiply(A,y);
                    xnorm=norminf(x);
                    for(int i=0;i<n;i++) {y[i]=x[i]/xnorm;}

                    mu=YT_X(y,x)/YT_X(y,y);
                    for(int i=0;i<n;i++) {r[i]=mu*y[i]-x[i];}
                    k++;
                    err=Math.abs((mu-mueski)/mu);
     }while(err>eps);
     //for(int i=0;i<n;i++) {r[i]=r[i]/mu;}
     X[jj]=y;
     lambda[jj]=mu;
     double cc=YT_X(X[jj],X[jj]);
     double kk=1.0/norm(X[jj]);
     X[jj+1]=multiply(kk,X[jj]);
               A=substract(A,multiply(lambda[jj], V_VT(X[jj+1])));
    }//end for(jj
    lambda[jj]=NA132.eigen_inversepower(A1);
   return lambda;
  }
  public double[] lambda()
  {return lambda;}
  public String toString()
  {String s="";
   for(int i=0;i<lambda.length;i++)
   {s+="\u03BB'+"["+i+"] = "+lambda[i]+"\n";}
   return s;
  }

  public static void main(String args[]) throws IOException
  {
  double a[][]={{12,6,-6},{6,16,2},{-6,2,16}};
  eigen_power w=new eigen_power(a);
  double c[]=w.power();
  System.out.println(Matrix.toString(c));
  JOptionPane.showMessageDialog(null,w,
  "Power method for eigenvalues : ",JOptionPane.PLAIN_MESSAGE);
  }
}
```



JACOBİ METHOD AND HOUSEHOLDER TRANSFORMATION

Jacobi method is used to calculate eigen value problems for the symmetric matrices. For a symmetric matrix, transpose of the matrix is the same as the matrix itself.

$\{A\}^T=\{A\}$

By using this property of the symmetric matrices a transformation called similarity tranformation can be applied. This transformation converts the matrix into diagonal form. Eigenvalues can be calculated by using this form. Assume matrix P be a nonsingular matrix similarity transformation :

$\{B\}=\{P\}^{-1}\{A\}\{P\}$

After applying similarity transformation eigen values reamain the same, but eigenvectors change. Therefore eigenvalues of matrix B is the same as eigenvalues of the matrix A. Now assume that vector $[y]^k$ is the eigenvector of matrix $\{B\}$ corresponding eigenvalue λ_k . In this case :

$\{B\}[y]^k = \lambda_k [y]^k$

$\{B\}=\{P\}^{-1}\{A\}\{P\}$

If both side of the equation is multiplied with $\{P\}$:

$\{P\}\{B\}=\{P\}\{P\}^{-1}\{A\}\{P\}=\{A\}\{P\}$ and

$\{A\}\{P\}[y]^k = \{P\}\{B\}[y]^k= \lambda_k\{P\}[y]^k$. If $[y]^k$ is eigenvector of $\{B\}$, eigenvector of $\{A\}$ took a velue like $[x]^k$. The relation between $[y]^k$ and $[x]^k$

$[x]^k=\{P\}[y]^k$.

Eigenvector of $\{A\}$ cb be obtained from multiplication of eigenvector $\{B\}$ by $\{P\}$

Symmetric matrices can be converted to the diagonal form by applying orthogonal symmetry .Ortagonal ($\{P\}^{-1}=\{P\}^T$) In order to obtained this convertion angular convertion matrix, $R(\phi)$, can be applied Matrix $R(\phi)$ is a unit matrix except row p and column q.

$R_{pp}=R_{qq} = \cos(\phi)=c$

$R_{pq}= - R_{qp}= - \sin(\phi)=-s$

For example for n=8 p=3 q=5

$$R(\phi)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{pp} & 0 & R_{pq} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{qp} & 0 & R_{qq} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\phi) & 0 & -\sin(\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin(\phi) & 0 & \cos(\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In this equation ϕ is the rotation angle. Rotation matrix R has interesting properties, one of these is multiplication of rotation matrix and transpose value will be given a unit matrix.

$\{R(\phi)\}^T\{R(\phi)\}=I$

Inversion of the rotation matrix is also very simple.

$\{R(\phi)\}^{-1}=\{R(\phi)\}^T$

For the rotation matrix following property is also valid.

$\{R(\phi)\}^T=\{R(-\phi)\}$

If similarity transformation is applied by using R matrix

$\{B\}=\{R(\phi)\}^T\{A\}\{R(\phi)\}$

In this transform angle ϕ can be selected such that $B_{qp}=0$.

$B_{qp}=(A_{qq} - A_{pp})\sin(\phi)\cos(\phi)+A_{pq}[\cos^2(\phi) - \sin^2(\phi)] = 0$

If $\sin(2\phi)=2\sin(\phi)\cos(\phi)$

And remembering that

$\cos(2\phi)=\cos^2(\phi) - \sin^2(\phi)$

$B_{qp}=(A_{qq} - A_{pp})\sin(2\phi)/2+A_{pq}\cos(2\phi) = 0$

ϕ becomes

$$\phi = \frac{1}{2}\arctan\left\{\frac{2A_{pq}}{A_{pp} - A_{qq}}\right\}$$

By using this angle and R matrix similarity transformation is carried out. Aplication of Jacobi method is as follows: P=I is selected. In upper trianglar part of the matrix the value with the biggest magnitude A_{pq} value is found. By using p and q values (indices of the element) angle ϕ is calculated. {P} ={P}{R} tranformation and ve {A} ={R}^{T}{A} {R} tranformation is applied. And biggest magnitude A_{pq} value is found in the upper triangular part of the matrix again and iteration continued until absolute value A_{pq} is close to zero (equal to ε) . Eigenvalues is taken as $\lambda_k=A_{kk}$, and eigen vector becomes P=[x^1,x^2,…,x^n].

One of the problem of Jacobi iteration is the excessive number of iteration steps. If A matrix is pre-transformed before the process above by using Householder transformation, total number of iteration reduces significantly. Householder transformation converts the matrix into upper Hessenberg matrix.

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \dots & B_{1,n-1} & B_{1,n} \\ B_{21} & B_{22} & B_{23} & \dots & B_{2,n-1} & B_{2,n} \\ 0 & B_{32} & B_{33} & \dots & B_{3,n-1} & B_{3,n} \\ 0 & 0 & B_{43} & \dots & B_{4,n-1} & B_{4,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & B_{n,n-1} & B_{n,n} \end{bmatrix}$$

As it is seen from the matrix itself, upper triangular part plus one below the diagonal is existed in this form, rest of the matrix elements are zero. Householder transformations is as follows:

1. Q=I is taken
2. $\alpha = sign(A_{k+1,k})\sqrt{\sum_{j=k+1}^{n} A_{j,k}^2}$ In this equation $sign(A_{k+1,k})$, is the sign of $A_{k+1,k}$. sign(x)= x / |x|
2. $u^k=[0,\dots,0,A_{k+1,k} + \alpha, A_{k+2,k},\dots.., A_{n,k}]^T$.
3. $P = I - \frac{2uu^T}{u^T u}$
4. Q = Q P is applied.
5. $A=PAP^T$.
6. B=A

B is the upper Hessenberg matrix.

An example problem: Upper Hessenberg transformation of Matrix

$$[A] = \begin{bmatrix} 4 & -2 & 6 & 4 \\ -2 & 2 & -1 & 3 \\ 6 & -1 & 22 & 13 \\ 4 & 3 & 13 & 46 \end{bmatrix}$$

And then find the eigenvalues:

Program Jacobi method to calculate eigenvalues of the symmetric matrices.

```
import java.io.*;
import javax.swing.*;

class NA133
{  // Jacobi eigenvalue calculation method
public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
```

```
    b[i][i]=1.0;
return b;
}

public static double[][] multiply(double[][] left,double right)
{
//multiplying a matrix with a constant
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=left[i][j]*right;
  }
return b;
//end of multiplying a matrix with a constant double
}

public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[m1][n2];
 if(n1 != m2)
 {
 System.out.println("inner matrix dimensions must agree");
 for(ii=0;ii<n1;ii++)
   {
   for(jj=0;jj<m2;jj++)
     b[ii][jj]=0;
   }
   return b;
 }
 for(i=0;i<m1;i++)
   {
   for(j=0;j<n2;j++)
     {
     for(k=0;k<n1;k++)
                  b[i][j]+=left[i][k]*right[k][j];
     }
   }
return b;
//end of multiply of two matrices
}

public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
```

```
  for(j=0;j<m1;j++)
    {
    b[i][j]=b[i][j]+left[i][j];
    }
  }
for(i=0;i<n2;i++)
  {
  for(j=0;j<m2;j++)
    {
    b[i][j]=b[i][j]-right[i][j];
    }
  }
return b;
//end of matrix substraction method
}

public static double[][] T(double [][] left)
{
//transpose matrix (if A=a(i,j) T(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
return b;
}

public static double VT_V(double [] left)
{
//multiply a vector transpose with a vector
int n=left.length;
double tot=0;
for(int i=0;i<n;i++)
  {
  tot+=left[i]*left[i];
  }
return tot;
}

public static double[][] V_VT(double [] left)
{
//multiplys a vector transpose with a vector
int n=left.length;
double aa[][]=new double[n][n];
for(int i=0;i<n;i++)
  {
  for(int j=0;j<n;j++)
    {
                aa[i][j]=left[i]*left[j];
                }
  }
return aa;
}

  public static double[] eigen_jacobi(double A[][])
  {
   A=Householder(A);
   double eps=1e-15;
   double err=1;
   int m=100;
   int n=A.length;
   double P[][]=new double[n][n];
   double R[][]=new double[n][n];
```

```
 int p=n,q=n;
 double max=-9.99e100;
 double fi;
 P=I(n);
 for(int i=0;i<n;i++)
 {for(int j=i+1;j<n;j++)
    {if(Math.abs(A[i][j]) > max) {max=Math.abs(A[i][j]);p=i;q=j;}}
 }
 while(Math.abs(A[p][q]) > eps )
 {
  fi=0.5*Math.atan(2.0*A[p][q]/(A[p][p]-A[q][q]));
  R=I(n);
  R[p][p]=Math.cos(fi);
  R[q][q]=Math.cos(fi);
  R[p][q]=-Math.sin(fi);
  R[q][p]=Math.sin(fi);
  P=multiply(P,R);
  A=multiply(T(R),multiply(A,R));
  max=-9.99e100;
  for(int i=0;i<n;i++)
 {for(int j=i+1;j<n;j++){if(Math.abs(A[i][j])>max) {max=Math.abs(A[i][j]);p=i;q=j;}}}
  }
 double lambda[]=new double[n];
 for(int i=0;i<n;i++) {lambda[i]=A[i][i];}
 return lambda;
}

public static double sgn(double x)
{return x/Math.abs(x);}

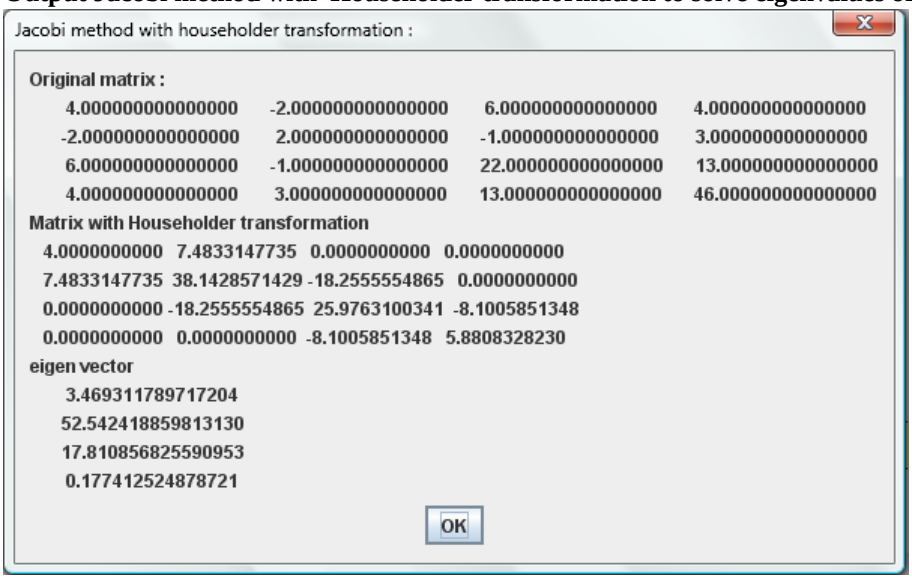
public static double[][] Householder(double A[][])
{ //Householder transformation
 int n=A.length;
 double Q[][]=I(n);
 double alpha;
 double toplam;
 double u[]=new double[n];
 double C[][]=new double[n][n];
 double P[][]=new double[n][n];
 double B[][]=new double[n][n];
 double cc;
 for(int k=0;k<(n-2);k++)
 {
   toplam=0;
   for(int j=k+1;j<n;j++) {toplam+=A[j][k]*A[j][k];}
   toplam=Math.sqrt(toplam);
   alpha=sgn(A[k+1][k])*toplam;
   for(int j=0;j<n;j++)
   {   if(j<=k) u[j]=0.0;
       else if(j==k+1) u[j]=A[j][k]+alpha;
       else u[j]=A[j][k];
   }
   C=V_VT(u);
   cc=2.0/VT_V(u);
   C=multiply(C,cc);
   P=substract(I(n),C);
   Q=multiply(Q,P);
   A=multiply(P,multiply(A,P));
 }
 return A;
 }


public static void main(String args[]) throws IOException
{
//Sinir değer hesaplari
double c[];
c=new double[2];
//double a[][]={{12,6,-6},{6,16,2},{-6,2,16}};
double a[][]={{4,-2,6,4},{-2,2,-1,3},{6,-1,22,13},{4,3,13,46}};
```

```
  String s="Original matrix : \n";
  s+=Matrix.toString(a);

  //c=eigen_jacobi(a);
  s+="Matrix with Householder transformation  \n";
  s+=Matrix.toStringT(Householder(a));
  c=eigen_jacobi(a);
  s+="eigen vector \n";
  s+=Matrix.toStringT(c);
  JOptionPane.showMessageDialog(null,s,
     "Jacobi method with householder transformation : ",JOptionPane.PLAIN_MESSAGE);
 }
}
```

Output Jacobi method with Householder transformation to solve eigenvalues of the symmetric matrices.



Program Jacobi method to calculate eigenvalues of the symmetric matrices, Python version

```
# -*- coding: utf-8 -*-
"""
Created on Sat Aug 25 17:26:02 2018

@author: Mustafa Turhan Çoban
"""
from math import *;

def I(n):
    #unit matrix of dimension n
    B = [[0 for x in range(n)] for y in range(n)];
    for i in range(0,n):
        for j in range(0,n):
            if i==j: B[i][j]=1.0;
    return B;

def AT(A):
    #transpose matrix
    n=len(A);
    m=len(A[0]);
    B = [[0 for x in range(m)] for y in range(n)];
    for i in range(0,n):
        for j in range(0,m):
            B[i][j]=A[j][i];
    return B;
def multiply_m_m(left,right):
    #multiplying two matrisis
    n1=len(left);
    m1=len(left[0]);
    n2=len(right);
    m2=len(right[0]);
    b = [[0 for x in range(n1)] for y in range(m2)];
    if m1 != n2:
```

```
        print("warning : matrix sizes should be same");
        return b;
    for i in range(0,n1):
        for j in range(0,m2):
            for k in range(0,m1):
                b[i][j]+=left[i][k]*right[k][j];
    return b;

def eigen_jacobi(A):
    # eigen value of a symmetric matrix
    # Jacobi method
    eps=1e-15;
    err=1;
    m=100;
    max=-9.99e100;
    n=len(A);
    P= [[0.0 for x in range(n)] for y in range(n)];
    R= [[0.0 for x in range(n)] for y in range(n)];
    p=n;q=n;
    lambda1=[0.0 for x in range(n)];
    P=I(n);
    for i in range(n):
        for j in range(i+1,n):
            if abs(A[i][j]) > max: max=abs(A[i][j]);p=i;q=j;
    while abs(A[p][q]) > eps:
        fi=0.5*atan(2.0*A[p][q]/(A[p][p]-A[q][q]+1e-40));
        R=I(n);
        R[p][p]=cos(fi);
        R[q][q]=cos(fi);
        R[p][q]=-sin(fi);
        R[q][p]=sin(fi);
        P=multiply_m_m(P,R);
        A= multiply_m_m(AT(R), multiply_m_m(A,R));
        max=-9.99e100;
        for i in range(0,n):
            for j in range(i+1,n):
                if abs(A[i][j]) > max: max=abs(A[i][j]);p=i;q=j;
    for i in range(0,n):lambda1[i]=A[i][i];
    PP= [[0.0 for x in range(n)] for y in range(n+1)];
    PP[0]=lambda1;
    for i in range(1,n+1):
        for j in range(0,n):
            PP[i][j]=P[i-1][j];
    return PP;

a=[[4,-2,6,4],[-2,2,-1,3],[6,-1,22,13],[4,3,13,46]];
c=eigen_jacobi(a);
print("eigen values=\n",c[0]);
```

```
runfile('D:/okul/SCO1/eigen_jacobi.py', wdir='D:/okul/SCO1')
eigen values=
 [3.4693117897172057, 0.1774125248787215, 17.810856825590943, 52.54241885981311]
```

CHARACTERISTIC POLYNOMIAL AND LEVERRIER METHOD

Characteristic matrix obtained by substracting of the matrix from eigenvalue matrix(unit matrix and eigenvalue multiplication). Characteristic polynomial obtained when the determinant of this matrix is taken. Roots of characteristic polynomial is given eigenvalues.

$$\Delta(\lambda) = \det\left\{ I\lambda - \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{11} & a_{22} & a_{23} & ... & a_{2n} \\ a_{11} & a_{11} & a_{33} & ... & a_{3n} \\ ... & ... & ... & .... & .... \\ a_{1n} & a_{2n} & a_{3n} & ... & a_{nn} \end{bmatrix} \right\} \qquad \Delta(\lambda) = \det\left\{ \begin{bmatrix} \lambda - a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{11} & \lambda - a_{22} & a_{23} & ... & a_{2n} \\ a_{11} & a_{11} & \lambda - a_{33} & ... & a_{3n} \\ ... & ... & ... & .... & .... \\ a_{1n} & a_{2n} & a_{3n} & ... & \lambda - a_{nn} \end{bmatrix} \right\}$$

Process can be explained by an example problem. If matrix A is given as: $A = \begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix}$, characteristic polynomial of this matrix will be:

$$\Delta(\lambda) = \det(\lambda I - A) = \det\left\{\begin{bmatrix} \lambda-1 & 1 \\ 6 & \lambda-2 \end{bmatrix}\right\} = (\lambda-1)((\lambda-2)-6 = \lambda^2 - 3\lambda - 4$$

Roots of this polynomial can be obtained by using quadratic equation solution:

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = \{4, -1\}$$

So the roots of the characteristic polynomial will give us eigenvalues. Method will be valid for non-symmetric matrices as well. In case of non symmetric matrices complex root finding methods such as Bairstow method should be utilised. In this section Leverrier method to obtain characteristic polynomial will be investigated. Algorithm of the method can be given as:

B_n=A and a_n=-trace(B_n) (trace process add the diagonal elements of the matrix) $trace(B_n) = \sum_{i=0}^{n-1} b_{ii}$

for k=n-2 to 0
{ C_k=(B_{k+1}+a_{k+1}I)
B_k=AC_k
$a_k = -\frac{trace(B_n)}{n-k}$
}

As it is seen from the algorithm, applying of the method is quite simple. In our example program, characteristic polynomial will be obtained by using Leverrier method and then smallest and biggest eigenvalues are obtained by using power method, by applying a search method in this region, region of each eigenvalue is obtained and then solved by using Brent root finding algorithm. Another interesting result of the Leverrier algorithm is that inverse matrix can be obtained from simple relation

$$A^{-1} = \frac{C_0}{a_0}$$

As an example problem let us calculate eigenvalues of :

$$[A] = \begin{bmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{bmatrix}$$

The following first program solves root of the charecteristic polynomial by using brent method.

Program Leverrier method with brent root finding method (real eigen values, symmetric matrix)

```
 public class Leverrier
{
double D[][]; //inverse of matri
public static double trace(double B[][])
{
double sum=0;
int n=B.length;
for(int i=0;i<n;i++)
{sum+=B[i][i];
}
return sum;
}

public static double[][] topla(double A[][],double a)
{
int n=A.length;
int m=A[0].length;
double B[][]=new double[n][m];
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
  if(i!=j) B[i][j]=A[i][j];
  else     B[i][j]=A[i][j]+a;
}
return B;
```

```
}
public static double[][] carp(double left,double[][] right)
{
//multiplying a matrix with a constant
int i,j;
int n=right.length;
int m=right[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=right[i][j]*left;
  }
return b;
//end of multiplying a matrix with a constant double
}
public static double[][] carp(double[][] left,double[][] right)
{
//iki matrisin çarpımı
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[n1][m2];
 if(m1 != n2)
 {
 System.out.println("uyarı: matris boyutları aynı olmalıdır");
 for(ii=0;ii<n1;ii++)
   {
   for(jj=0;jj<m2;jj++)
     b[ii][jj]=0;
   }
   return b;
 }
 for(i=0;i<n1;i++)
   {
   for(j=0;j<m2;j++)
     {
     for(k=0;k<m1;k++){
       //System.out.println(i+"  "+j+"   "+k);
                b[i][j]+=left[i][k]*right[k][j];
                //System.out.println(b[i][j]);
    }
     }
   }
return b;
}
public  double[] polynomial_coefficients(double A[][])
{
int n=A.length;
int m=A[0].length;
double B[][]=new double[n][m];
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
  B[i][j]=A[i][j];
}
System.out.println("i="+(n-1)+"\n"+Matrix.toString(B));
double a[]=new double[n+1];
double C[][]=new double[n][m];
```

```
a[n]=1.0;
a[n-1]=-trace(B);
for(int k=n-2;k>=0;k--)
{
C=topla(B,a[k+1]);
B=carp(A,C);
System.out.println("i="+k+"\n"+Matrix.toString(C));
a[k]=-trace(B)/(double)(n-k);
}
D=carp((1.0/a[0]),C);
return a;
}
//polynom fonksiyonu
public static double func(double e[],double x)
{
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}
public static double[][] findrootlimits(double e[],double x1,double x2,int n,int nbb)
{
		// find root limits
		// n : number of sub regions
		// x1,x2 : limit values
		// nbb = number of roots in the given region
		//
		int nb;
		int i;
		double x,fp,fc,dx;
		double xb[][]=new double[2][nbb];
		nb=0;
		dx=(x2-x1)/n;
		x=x1;
		fp=func(e,x1);
		for (i=1;i<=n;i++)
		{
			x+=dx;
			fc=func(e,x);
			// if the region with a root is found.....
			if (fc*fp < 0.0 || fp==0) {
				xb[0][nb]=x-dx;
				xb[1][nb]=x;
				nb++;
			}
			fp=fc;
			if (nbb == nb) return xb;
		}
if( nb == 0)
		System.out.println("search is completed no roots is found");
else if(nb<nbb)
  {		System.out.println("search is completed only "+nb+" roots are found \n"+
    "you search for"+nbb+" roots");
   double xc[][]=new double[2][nb];
   for (i=0;i<nb;i++) {xc[0][i]=xb[0][i];xc[1][i]=xb[1][i];}
   return xc;
```

```
  }
return xb;
}
public static double brent(double e1[], double x1,double x2)
{
        //referans : Numerical Recipes in C, second edition, William H. Press,
   //Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery
   //cambridge university press
        double ITMAX=200;
   double EPS=3.0e-10;
        int iter;
        double a=x1,b=x2,c,d,e,min1,min2;
        double tol=1.0e-5;
        c=0;
        d=0;
        e=0;
        double fa=func(e1,a),fb=func(e1,b),fc,p,q,r,s,tol1,xm;
        if (fb*fa > 0.0) System.out.println("Root limits is not selected correctly \n results may not be correct");
        fc=fb;
        for (iter=1;iter<=ITMAX;iter++) {
                if (fb*fc > 0.0) {
                        c=a;
                        fc=fa;
                        e=d=b-a;
                }
                if (Math.abs(fc) < Math.abs(fb)) {
                        a=b;
                        b=c;
                        c=a;
                        fa=fb;
                        fb=fc;
                        fc=fa;
                }
                tol1=2.0*EPS*Math.abs(b)+0.5*tol;
                xm=0.5*(c-b);
                if (Math.abs(xm) <= tol1 || fb == 0.0) return b;
                if (Math.abs(e) >= tol1 && Math.abs(fa) > Math.abs(fb)) {
                        s=fb/fa;
                        if (a == c) {
                                p=2.0*xm*s;
                                q=1.0-s;
                        } else {
                                q=fa/fc;
                                r=fb/fc;
                                p=s*(2.0*xm*q*(q-r)-(b-a)*(r-1.0));
                                q=(q-1.0)*(r-1.0)*(s-1.0);
                        }
                        if (p > 0.0)  q = -q;
                        p=Math.abs(p);
                        min1=3.0*xm*q-Math.abs(tol1*q);
                        min2=Math.abs(e*q);
                        if (2.0*p < (min1 < min2 ? min1 : min2)) {
                                e=d;
                                d=p/q;
                        } else {
                                d=xm;
                                e=d;
                        }
                } else {
                        d=xm;
                        e=d;
                }
```

```
                        a=b;
                        fa=fb;
                        if (Math.abs(d) > tol1)
                                b += d;
                        else
                                b += (xm > 0.0 ? Math.abs(tol1) : -Math.abs(tol1));
                        fb=func(e1,b);
                }
                System.out.println("Warning : Maximum number of ,teration is exceeded \n"+
   " result might not be valid");
return b;
}
public  double[] Leverrier(double A[][])
{
int n=A.length;
double a[]=polynomial_coefficients(A);
System.out.println("coefficient of polynomial a[0]..a[n] \n"+Matrix.toStringT(a));
double lambda[]=new double[n];
lambda[n-1]=SCO13C.eigen_power(A);
lambda[0]=SCO13D.eigen_inversepower(A);
double xx[][]=findrootlimits(a,lambda[0]-0.1,lambda[n-1]+0.1,40,n);
System.out.println("root values : \n"+Matrix.toString(xx));
for(int k=1;k<(n-1);k++)
{lambda[k]=brent(a, xx[0][k],xx[1][k]);}
return lambda;
}

public static void main(String arg[])
{
//double A[][]={{1,2},{2,5}};
double A[][]={{12,6,-6},{6,16,2},{-6,2,16}};
Leverrier lv=new Leverrier();
double a[]=lv.Leverrier(A);
System.out.println("A-1 = \n"+Matrix.toString(lv.D));
System.out.println("lambda = \n"+Matrix.toStringT(a));
}
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" Leverrier
i=2
      12.000000000000000       6.000000000000000      -6.000000000000000
       6.000000000000000      16.000000000000000       2.000000000000000
      -6.000000000000000       2.000000000000000      16.000000000000000

i=1
     -32.000000000000000       6.000000000000000      -6.000000000000000
       6.000000000000000     -28.000000000000000       2.000000000000000
      -6.000000000000000       2.000000000000000     -28.000000000000000

i=0
     252.000000000000000    -108.000000000000000     108.000000000000000
    -108.000000000000000     156.000000000000000     -60.000000000000000
     108.000000000000000     -60.000000000000000     156.000000000000000

coefficient of polynomial a[0]..a[n]
   -1728.000000000000000000000000
     564.000000000000000000000000
     -44.000000000000000000000000
       1.000000000000000000000000

mu=24.0k=0
```

0.50000000000000000000000 1.00000000000000000000000 0.50000000000000000000000

mu=19.999999999999996k=1
0.45000000000000000000000 1.00000000000000000000000 0.35000000000000000000000

mu=19.4k=2
0.47938144329896915000000000 1.00000000000000000000000 0.25257731958762886000000000

mu=19.38144329896907k=3
0.52819148936170210000000000 1.00000000000000000000000 0.16329787234042548000000000

mu=19.495744680851065k=4
0.58261486412746920000000000 1.00000000000000000000000 0.07404780093855719000000000

mu=19.643784786641923k=5
0.63873086068579320000000000 1.00000000000000000000000 -0.01582812951398927000000000

mu=19.800728905086782k=6
0.69491073643145510000000000 1.00000000000000000000000 -0.10533123534673466000000000

mu=19.95880194789526k=7
0.75009092671700240000000000 1.00000000000000000000000 -0.19313605066074550000000000

mu=20.114273458980524k=8
0.80340497793886280000000000 1.00000000000000000000000 -0.27794801449205830000000000

mu=20.26453383864906k=9
0.85413007572902460000000000 1.00000000000000000000000 -0.35863633268706885000000000

mu=20.407507789000007k=10
0.90169652733340430000000000 1.00000000000000000000000 -0.43429907605594725000000000

mu=20.541581011888532k=11
0.94569900793388600000000000 1.00000000000000000000000 -0.50429245805862190000000000

mu=20.66560913148607k=12
0.98589607080666040000000000 1.00000000000000000000000 -0.56823262754204770000000000

mu=21.24014861493221k=13
1.00000000000000000000000000 0.97828464133946380000000000 -0.61238264860202060000000000

mu=21.54400373964891k=14
1.00000000000000000000000000 0.94818907437491370000000000 -0.64247821631593220000000000

mu=21.544003744145073k=15
1.00000000000000000000000000 0.92304424904173670000000000 -0.66762304180410150000000000

mu=21.54400374507503k=16
1.00000000000000000000000000 0.90203576507927770000000000 -0.68863152579861760000000000

mu=21.544003745267375k=17
1.00000000000000000000000000 0.88448319147071900000000000 -0.70618409941380700000000000

mu=21.54400374530716k=18
1.00000000000000000000000000 0.86981802854475490000000000 -0.72084926234114240000000000

mu=21.544003745315386k=19
1.00000000000000000000000000 0.85756529521821860000000000 -0.73310199566796250000000000

mu=21.54400374531709k=20
1.00000000000000000000000000 0.84732814512824870000000000 -0.74333914575799100000000000

mu=21.54400374531744k=21
1.000000000000000000000000 0.838775013417067800000000 -0.751892277469184000000000

mu=21.54400374531751k=22
1.000000000000000000000000 0.831628877878783900000000 -0.759038413007470700000000

mu=21.54400374531752k=23
1.000000000000000000000000 0.825658286654899400000000 -0.765009004231355900000000

mu=0.19004597963216294k=0
1.000000000000000000000000 -0.047619047619047620000000 0.809523809523809500000000

mu=0.22192536045782874k=1
1.000000000000000000000000 -0.475953565505804370000000 0.688225538971807700000000

mu=0.22426665037604132k=2
1.000000000000000000000000 -0.591802328644433900000000 0.645751014172066800000000

mu=0.2244077059039705k=3
1.000000000000000000000000 -0.619895378935789700000000 0.633324644965418600000000

mu=0.22441616283344146k=4
1.000000000000000000000000 -0.626576893576217500000000 0.629905193526249400000000

mu=0.22441667278567667k=5
1.000000000000000000000000 -0.628166793713999700000000 0.628990927686914100000000

mu=0.22441670368044042k=6
1.000000000000000000000000 -0.628546737822805100000000 0.628750766608927100000000

mu=0.22441670555850934k=7
1.000000000000000000000000 -0.628637954431483200000000 0.628688463361304100000000

mu=0.2244167056729508k=8
1.000000000000000000000000 -0.628659947150822700000000 0.628672450932751400000000

mu=0.22441670567993627k=9
1.000000000000000000000000 -0.628665269812347700000000 0.628668365191772400000000

mu=0.22441670568036312k=10
1.000000000000000000000000 -0.628666562280800600000000 0.628667328558598500000000

mu=0.22441670568038927k=11
1.000000000000000000000000 -0.628666877029119500000000 0.628667066725289600000000

mu=0.22441670568039082k=12
1.000000000000000000000000 -0.628666953869320900000000 0.628667000829622500000000

mu=0.22441670568039093k=13
1.000000000000000000000000 -0.628666972668676400000000 0.628666984293950300000000

root values :
4.355996254682469 17.754202059924630 21.211803558051650
4.788196441948346 18.186402247190507 21.644003745317526

A-1 =
-0.145833333333333 0.062500000000000 -0.062500000000000
0.062500000000000 -0.090277777777778 0.034722222222222
-0.062500000000000 0.034722222222222 -0.090277777777778

lambda =
4.455996254682469000000000

```
      18.000000089225104000000000
      21.544003745317520000000000


> Terminated with exit code 0.
```

A Second example :

$$[A]=\begin{bmatrix} 4 & -2 & 6 & 4 \\ -2 & 2 & -1 & 3 \\ 6 & -1 & 22 & 13 \\ 4 & 3 & 13 & 46 \end{bmatrix}$$

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" Leverrier
i=3
       4.000000000000000      -2.000000000000000       6.000000000000000       4.000000000000000
      -2.000000000000000       2.000000000000000      -1.000000000000000       3.000000000000000
       6.000000000000000      -1.000000000000000      22.000000000000000      13.000000000000000
       4.000000000000000       3.000000000000000      13.000000000000000      46.000000000000000

i=2
     -70.000000000000000      -2.000000000000000       6.000000000000000       4.000000000000000
      -2.000000000000000     -72.000000000000000      -1.000000000000000       3.000000000000000
       6.000000000000000      -1.000000000000000     -52.000000000000000      13.000000000000000
       4.000000000000000       3.000000000000000      13.000000000000000     -28.000000000000000

i=1
     969.000000000000000     142.000000000000000    -234.000000000000000     -24.000000000000000
     142.000000000000000    1063.000000000000000      77.000000000000000     -99.000000000000000
    -234.000000000000000      77.000000000000000     255.000000000000000     -57.000000000000000
     -24.000000000000000     -99.000000000000000     -57.000000000000000      99.000000000000000

i=0
   -1364.000000000000000   -1492.000000000000000     212.000000000000000     156.000000000000000
   -1492.000000000000000   -1988.000000000000000     196.000000000000000     204.000000000000000
     212.000000000000000     196.000000000000000     -68.000000000000000     -12.000000000000000
     156.000000000000000     204.000000000000000     -12.000000000000000     -36.000000000000000

coefficient of polynomial a[0]..a[n]
     576.000000000000000000000000000
   -3456.000000000000000000000000000
    1193.000000000000000000000000000
     -74.000000000000000000000000000
       1.000000000000000000000000000

root values :
       0.077412524878722       2.705662841625443      17.161039583732414      51.328293701439755
       1.391537683252082       4.019787999998804      18.475164742105775      52.642418859813110

A-1 =
      -2.368055555555555      -2.590277777777778       0.368055555555556       0.270833333333333
      -2.590277777777778      -3.451388888888889       0.340277777777778       0.354166666666667
       0.368055555555556       0.340277777777778      -0.118055555555556      -0.020833333333333
       0.270833333333333       0.354166666666667      -0.020833333333333      -0.062500000000000

lambda =
       0.17741252487872160000000000
       3.46931167646750450000000000
      17.81085541180162200000000000
      52.54241885981315000000000000

> Terminated with exit code 0.
```

The following program will be used Leverier method to find charecteristic matrix then use Horner method to solve it in python code

Program Leverrier method with horner root finding method with sysnthetic division process(complex eigen values) python code

```
 from math import *

def f_poly(c1,x):
     # this function calculates the value of
     # c least square curve fitting function
     n=len(c1);
     if n!=0.0:
       ff=c1[n-1];
       for i in range(n-2,-1,-1):
         ff=ff*x+c1[i];
     else:
       ff=0,0;
     return ff;

def horner(a):
  # roots of a polynomial by using synthetic division process
     n=len(a);
     x1=complex(1.56435,0.1);
     n1=n;
     nmax=150;
     tolerance=1.0e-15;
     n1=n+1;
     ref=3.1;
     y=[complex(0.0,0.0) for i in range(n1-2)];
     n2=n1-2;
     for k in range(0,n2):
           n=len(a);
           x=x1;
           xold=3.4878;
           ref=abs(xold-x);
           b=[complex(0.0,0) for i in range(n)];
           c=[complex(0.0,0.0) for i in range(n-1)];
           b[n-1]=a[n-1];
           ii=0;
           while ref>tolerance and ii<nmax:
             for i in range(n-2,-1,-1):
                 b[i]=a[i]+b[i+1]*x;
             for i in range(n-2,-1,-1):  c[i]=b[i+1];
             Q=f_poly(c,x);
             P=b[0];
             x=x-P/Q;
             ref=abs(xold-x);
             ii=ii+1;
             xold=x;
           a=c;
           n=len(a);
           y[k]=x;
     return y;

def trace(B):
   sum=0;
   n=len(B);
   for i in range(0,n):sum+=B[i][i];
   return sum;

def add(A,a):
   n=len(A);
   m=len(A[0]);
   w, h = 8, 5;
   B = [[0 for x in range(m)] for y in range(n)];
   for i in range(0,n):
     for j in range(0,m):
       if i!=j: B[i][j]=A[i][j];
       else:    B[i][j]=A[i][j]+a;
   return B;

def multiply_c_m(left,right):
   #multiplying a matrix with a constant
    n=len(right);
    m=len(right[0]);
    b = [[0 for x in range(m)] for y in range(n)];
```

```
    for i in range(0,n):
      for j in range(0,m):
        b[i][j]=right[i][j]*left;
   return b;
#end of multiplying a matrix with a constant double

def multiply_m_m(left,right):
  #multiplying two matrisis
  n1=len(left);
  m1=len(left[0]);
  n2=len(right);
  m2=len(right[0]);
  b = [[0 for x in range(n1)] for y in range(m2)];
  if m1 != n2:
    print("uyarı: matris boyutları aynı olmalıdır");
    return b;
  for i in range(0,n1):
     for j in range(0,m2):
         for k in range(0,m1):
            b[i][j]+=left[i][k]*right[k][j];
  return b;

def polynomial_coefficients(A):
  n=len(A);
  m=len(A[0]);
  B = [[0 for x in range(m)] for y in range(n)];
  for i in range(0,n):
     for j in range(0,m):
        B[i][j]=A[i][j];
  a = [0.0 for y in range(n+1)];
  C = [[0.0 for x in range(m)] for y in range(n)];
  a[n]=1.0;
  a[n-1]=-trace(B);
  for k in range(n-2,-1,-1):
    C=add(B,a[k+1]);
    B=multiply_m_m(A,C);
    a[k]=-trace(B)/(n-k);
  return a;

def eigen_horner(A):
  b=polynomial_coefficients(A);
  return horner(b);

A=[[4,1,1],[0,2,2],[-2,0,9]];
print(eigen_horner(A));
```

```
runfile('D:/okul/SCO1/eigen_horner.py', wdir='D:/okul/SCO1')
[(1.7783225824834876+4.930380657631324e-32j), (4.817630421052084-1.1799137694746532e-16j),
(8.40404699646443+5.4124212604319174e-17j)]
```

```
A=[[4,-2,6,4],[-2,2,-1,3],[6,-1,22,13],[4,3,13,46]];
print(eigen_horner(A));
```

```
runfile('D:/okul/SCO1/eigen_horner.py', wdir='D:/okul/SCO1')
[(0.1774125248787216+0j), (3.469311789717206-6.393786517153874e-23j), (17.810856825590946+1.6791691784404488e-23j),
(52.542418859813125-1.6382924242895428e-24j)]
```

Program Leverrier method with LeGuerre root finding method with sysnthetic division process(complex eigen values) python code

```
 # -*- coding: utf-8 -*-
"""
Created on Sun Sep  2 20:11:52 2018

@author: Mustafa Turhan Çoban


"""
from math import *
from cmath import *
def f_poly(c1,x):
     # this function calculates the value of
     # c least square curve fitting function
```

```
    n=len(c1);
    if n!=0.0:
      ff=c1[n-1];
      for i in range(n-2,-1,-1):
        ff=ff*x+c1[i];
    else:
      ff=0,0;
    return ff;

def df_poly(c1,x):
    h=0.001
    return (-f_poly(c1,(x+2.0*h))+8.0*f_poly(c1,(x+h))-8.0*f_poly(c1,(x-h))+f_poly(c1,(x-2.0*h)))/(12.0*h);

def d2f_poly(c1,x):
    dx=0.001
    y1=-1.0/12.0*f_poly(c1,(x-2.0*dx))+4.0/3.0*f_poly(c1,(x-dx));
    y2=-5.0/2.0*f_poly(c1,x)+4.0/3.0*f_poly(c1,(x+dx))-1.0/12.0*f_poly(c1,(x+2.0*dx));
    return (y1+y2)/dx;

def trace(B):
     sum=0;
     n=len(B);
     for i in range(0,n):sum+=B[i][i];
     return sum;

def topla(A,a):
     n=len(A);
     m=len(A[0]);
     B = [[0 for x in range(m)] for y in range(n)];
     for i in range(0,n):
       for j in range(0,m):
         if i!=j: B[i][j]=A[i][j];
         else:    B[i][j]=A[i][j]+a;
     return B;

def carp_m_m(left,right):
     #multiplying two matrisis
     n1=len(left);
     m1=len(left[0]);
     n2=len(right);
     m2=len(right[0]);
     b = [[0 for x in range(n1)] for y in range(m2)];
     if m1 != n2:
       print("warning : matrix sizes should be same");
       return b;
     for i in range(0,n1):
       for j in range(0,m2):
         for k in range(0,m1):
           b[i][j]+=left[i][k]*right[k][j];
     return b;
def eigen_LeGuerre(A):
    #eigen value result is complex use Horner method with synthetic division
    b=polynomial_coefficients(A);
    return LeGuerre_SD(b);

def polynomial_coefficients(A):
     n=len(A);
     m=len(A[0]);
     B = [[0 for x in range(m)] for y in range(n)];
     for i in range(0,n):
       for j in range(0,m):
         B[i][j]=A[i][j];
     a = [0.0 for y in range(n+1)];
     C = [[0.0 for x in range(m)] for y in range(n)];
     a[n]=1.0;
     a[n-1]=-trace(B);
     for k in range(n-2,-1,-1):
       C=topla(B,a[k+1]);
       B=carp_m_m(A,C);
       a[k]=-trace(B)/(n-k);
     return a;
def LeGuerre_SD(a):
  # roots of a polynomial by using synthetic division process
  # complex roots
    n=len(a);
    x1=complex(1.56435,1.76353);
```

```
    x=complex(1.56435,1.87363);
    n1=n;
    nmax=500;
    tolerance=1.0e-15;
    n1=n+1;
    ref=3.1;
    y=[complex(0.0,0.0) for i in range(n1-2)];
    n2=n1-2;
    for k in range(0,n2):
        n=len(a);
        m=n-1;
        x=x1;
        xold=1.4878;
        ref=abs(xold-x);
        b=[complex(0.0,0) for i in range(n)];
        c=[complex(0.0,0.0) for i in range(n-1)];
        b[n-1]=a[n-1];
        ii=0;
        while ref>tolerance and ii<nmax:
          for i in range(n-2,-1,-1):
             b[i]=a[i]+b[i+1]*x;
          for i in range(n-2,-1,-1):  c[i]=b[i+1];
          dfx=df_poly(a,x);
          fx=f_poly(a,x);
          d2fx=d2f_poly(a,x);
          G=dfx/fx;
          H=G*G-d2fx/fx;
          y1=sqrt((m*H-G*G)*(m-1.0));
          z1=G+y1;
          z2=G-y1;
          zbool=z1.real*z1.real+z1.imag*z1.imag> z2.real*z2.real+z2.imag*z2.imag;
          if zbool: z=z1;
          else: z=z2;
          aa=n/z;
          x=x-aa;
          ref=abs(xold-x);
          ii=ii+1;
          xold=x;
        if n==2: y[k]=-a[0]/a[1];
        else:    y[k]=x;
        a=c;
        n=len(a);
    return y;

A=[[4,1,1],[0,2,2],[-2,0,9]];
b= eigen_LeGuerre(A);
print("b=",b);
```

```
runfile('D:/okul/SCO1/LeGuerre.py', wdir='D:/okul/SCO1')
b= [(1.7783225824834867+1.0465662590966393e-240j), (4.817630421052081-1.0052689051413437e-152j), (8.404046996464423-
2.010537810284278e-152j)]
```

In root finding of the charecteristic polynomial, Bairstow method can be a better option comparing brent method due to direct approach of polynamial root finding of Bairstow method. Next version of Leverrier method is given here utilising Bairstow method. It should be note that eigenvalues of non-symmetric matrices can also be found by this method.

Program Leverrier method with several different root finding method s(imaginary eigen values, non-symmetric matrix)

```
 public class Leverrier7
{  //variable root finding version
double D[][]; //inverse of matrix
public static double trace(double B[][])
{
double sum=0;
int n=B.length;
for(int i=0;i<n;i++)
{sum+=B[i][i];
}
return sum;
}

public static double[][] add(double A[][],double a)
```

```
{
int n=A.length;
int m=A[0].length;
double B[][]=new double[n][m];
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
  if(i!=j) B[i][j]=A[i][j];
  else     B[i][j]=A[i][j]+a;
}
return B;
}
public static double[][] multiply(double left,double[][] right)
{
//multiplying a matrix with a constant
int i,j;
int n=right.length;
int m=right[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=right[i][j]*left;
  }
return b;
//end of multiplying a matrix with a constant double
}
public static double[][] multiply(double[][] left,double[][] right)
{
//iki matrisin çarpımı
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[n1][m2];
 if(m1 != n2)
 {
 System.out.println("uyarı: matris boyutları aynı olmalıdır");
 for(ii=0;ii<n1;ii++)
   {
   for(jj=0;jj<m2;jj++)
     b[ii][jj]=0;
   }
   return b;
 }
 for(i=0;i<n1;i++)
   {
   for(j=0;j<m2;j++)
     {
     for(k=0;k<m1;k++){
                    b[i][j]+=left[i][k]*right[k][j];
    }
     }
   }
return b;
}
public  double[] polynomial_coefficients(double A[][])
{
int n=A.length;
int m=A[0].length;
double B[][]=new double[n][m];
for(int i=0;i<n;i++)
{for(int j=0;j<m;j++)
  B[i][j]=A[i][j];
}
double a[]=new double[n+1];
double C[][]=new double[n][m];
a[n]=1.0;
a[n-1]=-trace(B);
for(int k=n-2;k>=0;k--)
{
C=add(B,a[k+1]);
B=multiply(A,C);
a[k]=-trace(B)/(double)(n-k);
```

```
}
D=multiply((1.0/a[0]),C);
return a;
}

public  c[] Leverrier(String name,double A[][])
{
int n=A.length;
double a[]=polynomial_coefficients(A);
c x[];
if(name.equals("Hasan"))
{c x1[]=Hasan.root(a);x=x1;}
else if(name.equals("Halley"))
{c x1[]=Halley.root(a);x=x1;}
else if(name.equals("Lagrange"))
{c x1[]=inverse_quadratic.root(a);x=x1;}
else if(name.equals("LeGuerre"))
{c x1[]=Leguerre.root(a);x=x1;}
else if(name.equals("Muller"))
{c x1[]=muller.root(a);x=x1;}
else if(name.equals("Horner"))
{horner h=new horner();c x1[]=h.root(a);x=x1;}
else // if(name.equals("Bairstow")
{bairstow h=new bairstow();c x1[]=h.root(a);x=x1;}
return x;
}

public static void main(String arg[])
{
double A[][]={{8,-1,-5},{-4,4,-2},{18,-5,-7}};
Leverrier7 lv=new Leverrier7();
double c[]=lv.polynomial_coefficients(A);
System.out.println("Charecteristic polynomial : \n"+Matrix.toString(c));
c a[]=lv.Leverrier("Hasan",A);
System.out.println("lambda = \n"+Matrix.toStringT(a));
}
}
```

An example for the nonsymmetric matrix eigenvalue problem:

$$[A]=\begin{bmatrix} 8 & -1 & -5 \\ -4 & 4 & -2 \\ 18 & -5 & -7 \end{bmatrix}$$

```
public static void main(String arg[])
{
double A[][]={{8,-1,-5},{-4,4,-2},{18,-5,-7}};
Leverrier1 lv=new Leverrier1();
complex a[]=lv.Leverrier("Bairstow",A);
System.out.println("lambda = \n"+Matrix.toStringT(a));
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" Leverrier1
Charecteristic polynomial :
     -20.000000000000000                24.000000000000000                -5.000000000000000                1.000000000000000

lambda =
(   1.0000000000)
(   2.0000000000 -    4.0000000000i )
(   2.0000000000 +    4.0000000000i )
```

```
A=[[8,-1,-5],[-4,4,-2],[18,-5,-7]];
print(eigen_horner(A));
```

```
runfile('D:/okul/SCO1/eigen_horner.py', wdir='D:/okul/SCO1')
[(1+0j), (1.9999999999999998+4j), (2-4j)]
```

Another non-symmetric example

$$[A]=\begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 2 \\ -2 & 0 & 9 \end{bmatrix}$$

```
 public static void main(String arg[])
{
double A[][]={{4,1,1},{0,2,2},{-2,0,9}};
Leverrier7 lv=new Leverrier7();
double c[]=lv.polynomial_coefficients(A);
System.out.println("Charecteristic polynomial : \n"+Matrix.toString(c));
c a[]=lv.Leverrier("Hasan",A);
System.out.println("lambda = \n"+Matrix.toStringT(a));
}
}
```

```
 ---------- Capture Output ----------
> "C:\java\bin\java.exe" Leverrier7
Charecteristic polynomial :
    -72.000000000000000          64.000000000000000          -15.000000000000000          1.000000000000000

lambda =
(  4.8176304211)
(  8.4040469965)
(  1.7783225825)


> Terminated with exit code 0.
```

```
A=[[4,1,1],[0,2,2],[-2,0,9]];
print(eigen_horner(A));
```

```
runfile('D:/okul/SCO1/eigen_horner.py', wdir='D:/okul/SCO1')
[(1.7783225824834876+4.930380657631324e-32j), (4.817630421052084-1.1799137694746532e-16j),
(8.40404699646443+5.4124212604319174e-17j)]
```

```
a=[[8,-1,-5],[-4,4,-2],[18,-5,-7]];
b= eigen_LeGuerre(a);
print("b=",b);
```

```
runfile('D:/okul/SCO1/LeGuerre.py', wdir='D:/okul/SCO1')
b= [(1.0000000000000002-7.974909351313641e-17j), (2+3.9999999999999996j), (2-4j)]
```

QL ALGORITHM

If a symmetric matrix A is considered, This matrix can be devided into two submatrix as Q orthogonal matrix and L lower triangular matrix.

A=Q L . Since Q is an orthogonal matrix $Q^T = Q^{-1}$. If both side of the eqaution is multiplied by Q^T , it becomes

$Q^T A = L$. A new B matrix can be created as:

$B=L\ Q = Q^T A\ Q$. Matrix Bis a similarity transformed form of matrix A, therefore eigenvalues of both matrices are the same.

QL algorithm is based on a series of othogonal transformations. The recurrence relations used is

$\mathbf{A_m=Q_m\ L_m}$ and $\mathbf{A_{m+1}= L_m\ Q_m = Q_m^T\ A_m\ Q_m}$

Because A is a symmetric matrix, eigenvalue will be the real numbers

Program 9.7-1 QL algorithm to calculate eigenvalues of the symmetric matrices.

```
import java.io.*;
import java.io.*;
import javax.swing.*;

class NA134
{
public static double[][] tridiagonal(double b[][], double d[], double e[])
{
//reduces matrix to tridiaonal form by using HOUSEHOLDER TRANSFORMATION
```

```
//this method is used by QL method to calculate eigen values
//and eigen  vectors of a symmetric matrix
        int l,k,j,i;
   int n=b.length;
        double scale,hh,h,g,f;
   double a[][]=new double[n+1][n+1];
   double c[][]=new double[n][n];
   for(i=0;i<n;i++)
     for(j=0;j<n;j++)
       a[i][j]=b[i][j];
        for (i=n;i>=2;i--) {
                l=i-1;
                h=scale=0.0;
                if (l > 1) {
                        for (k=1;k<=l;k++)
                                scale += Math.abs(a[i-1][k-1]);
                        if (scale == 0.0)
                                e[i-1]=a[i-1][l-1];
                        else {
                                for (k=1;k<=l;k++) {
                                        a[i-1][k-1] /= scale;
                                        h += a[i-1][k-1]*a[i-1][k-1];
                                }
                                f=a[i-1][l-1];
                                g=(f >= 0.0 ? -Math.sqrt(h) : Math.sqrt(h));
                                e[i-1]=scale*g;
                                h -= f*g;
                                a[i-1][l-1]=f-g;
                                f=0.0;
                                for (j=1;j<=l;j++) {
                                        a[j-1][i-1]=a[i-1][j-1]/h;
                                        g=0.0;
                                        for (k=1;k<=j;k++)
                                                g += a[j-1][k-1]*a[i-1][k-1];
                                        for (k=j+1;k<=l;k++)
                                                g += a[k-1][j-1]*a[i-1][k-1];
                                        e[j-1]=g/h;
                                        f += e[j-1]*a[i-1][j-1];
                                }
                                hh=f/(h+h);
                                for (j=1;j<=l;j++) {
                                        f=a[i-1][j-1];
                                        e[j-1]=g=e[j-1]-hh*f;
                                        for (k=1;k<=j;k++)
                                                a[j-1][k-1] -= (f*e[k-1]+g*a[i-1][k-1]);
                                }
                        }
                } else
                        e[i-1]=a[i-1][l-1];
                d[i-1]=h;
        }
        d[1-1]=0.0;
        e[1-1]=0.0;
        /* Contents of this loop can be omitted if eigenvectors not
                        wanted except for statement d[i-1]=a[i-1][i-1]; */
        for (i=1;i<=n;i++) {
                l=i-1;
                if (d[i-1] != 0) {
                        for (j=1;j<=l;j++) {
                                g=0.0;
                                for (k=1;k<=l;k++)
                                        g += a[i-1][k-1]*a[k-1][j-1];
                                for (k=1;k<=l;k++)
                                        a[k-1][j-1] -= g*a[k-1][i-1];
                        }
                }
                d[i-1]=a[i-1][i-1];
                a[i-1][i-1]=1.0;
                for (j=1;j<=l;j++) a[j-1][i-1]=a[i-1][j-1]=0.0;
        }
```

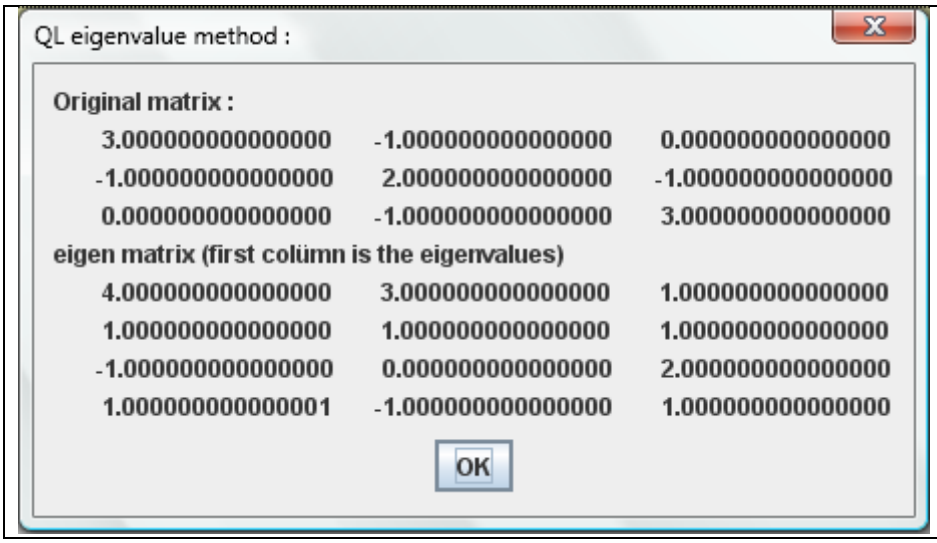
```
return  a;
}

public static double pythag(double a, double b)
{
//this method is used by QL method
            double absa,absb;
            absa=Math.abs(a);
            absb=Math.abs(b);
            if (absa > absb) return absa*Math.sqrt(1.0+(absb/absa)*(absb/absa));
            else return (absb==0.0 ? 0.0 : absb*Math.sqrt(1.0+(absa/absb)*(absa/absb)));
}

public static double[][] QL(double d[], double e[], double a[][])
{
// QL algorithm : eigenvalues of a symmetric matrix reduced to tridiagonal
// form by using method tridiagonal
   int n=d.length;
            int m,l,iter,i,j,k;
            double s,r,p,g,f,dd,c,b;
            for (i=2;i<=n;i++) e[i-2]=e[i-1];
            e[n-1]=0.0;
   double z[][]=new double[n][n];
   for(i=0;i<n;i++)
     for(j=0;j<n;j++)
       z[i][j]=a[i][j];
            for (l=1;l<=n;l++) {
                        iter=0;
                        do {
                                    for (m=l;m<=n-1;m++) {
                                                dd=Math.abs(d[m-1])+Math.abs(d[m]);
                                                if ((double)(Math.abs(e[m-1])+dd) == dd) break;
                                    }
                                    if (m != l) {
          if (iter++ == 30) System.out.println("Too many iterations in QL");
          g=(d[l]-d[l-1])/(2.0*e[l-1]);
          r=Matrix.pythag(g,1.0);
          g=d[m-1]-d[l-1]+e[l-1]/(g+Matrix.SIGN(r,g));
          s=c=1.0;
          p=0.0;
          for (i=m-1;i>=l;i--) {
                                                            f=s*e[i-1];
                                                            b=c*e[i-1];
                                                            e[i]=(r=Matrix.pythag(f,g));
                                                            if (r == 0.0) {
                                                                        d[i] -= p;
                                                                        e[m-1]=0.0;
                                                                        break;
                                                            }
                                                            s=f/r;
                                                            c=g/r;
                                                            g=d[i ]-p;
                                                            r=(d[i-1]-g)*s+2.0*c*b;
                                                            d[i ]=g+(p=s*r);
                                                            g=c*r-b;
                                                            for (k=1;k<=n;k++) {
                                                                        f=z[k-1][i ];
                                                                        z[k-1][i ]=s*z[k-1][i-1]+c*f;
                                                                        z[k-1][i-1]=c*z[k-1][i-1]-s*f;
                                                            }
                                                }
                                                if (r == 0.0 && i >= l) continue;
                                                d[l-1] -= p;
                                                e[l-1]=g;
                                                e[m-1]=0.0;
                                    }
                        } while (m != l);
            }
   return z;
}
```

```
public static double[][] eigenQL(double a[][])
{
 // QL algoritm to solve eigen value problems
 // symmetric matrices only (real eigen values)
 // first column of the matrix returns eigen values
 // second..n+1 column returns eigen vectors.
 // Note : If matrix is not symmetric DO NOT use
int i,j;
 int n=a.length;
 double sum[]=new double[n];;
 double d[]=new double[n];
 double b[][]=new double[n][n];
 double e[]=new double[n];
 double z[][]=new double[n+1][n];
 b=tridiagonal(a,d,e);
 b=QL(d,e,b);
 for(j=0;j<n;j++)
    {
    z[0][j]=d[j];
    for(i=0;i<n;i++)
       {
       z[i+1][j]=b[i][j]/b[0][j];
       if(z[i+1][j]<1e-13) z[i+1][i]=0;
       }
    }
 // result: first row is eigenvalues, the rest is eigenvector
 return z;
}

  public static void main(String args[]) throws IOException
  {
  //Eigen Value calculations
  double c[][];
  c=new double[3][3];
  //double a[][]={{-19,-4.7,6.45},{-4.7,4.6,11.8},  {6.45,11.8,-8.3}};
  double a[][]={{3,-1,0},{-1,2,-1},{0,-1,3}};
  String s="Original matrix : \n";
  s+=Matrix.toString(a);
  c=new double[3][3];
  c=eigenQL(a);
  s+="eigen matrix (first column is the eigenvalues) \n";
  s+=Matrix.toString(c);
  JOptionPane.showMessageDialog(null,s,
     "QL eigenvalue method : ",JOptionPane.PLAIN_MESSAGE);
  }
}
```



QR ALGORİTHM

By using a similarity transformation matrix A can be subdivided into a orthogonal Q and an upper triangular matrix R

$A=Q\,R$

Because Q is an orthogonal matrix $Q^T = Q^{-1}$ can be written. If equation is multiplied with Q^T

$Q^T A = R$, a new matrix B can be created as

$B=R\,Q = Q^T A\,Q$

B and A is realated to each other by similarity transformation, i.e. they have the same eigenvalues. Similarity transforms applied to B several times. When B is converted into an upper triangular form, eigenvalues can be obtained from this upper triangular form. Converting matrix to upper triangular form could be a very long process. In order to shorten the process a householder tranformation can be applied. Matrix does not have to be symmetric in QR algorithm, therfore it can be applied into non-symmetric matrices. Please note that eigen values of the non-symmetric matrices can be complex number, therefore the proces steps should be calculated with complex number algorithms.

Program 9.8-1 QR algorithm

```
 import java.io.*;
import javax.swing.*;


class NA135
{
  //===========Eigen value calculations ==============
public static double[][] balance(double b[][])
{
// balance of a matrix for more accurate eigenvalue
// calculations
double radix=2.0;
double sqrdx=radix*radix;
double c,r,f,s,g;
int m,j,i,last;
int n=b.length;
last=0;
double a[][];
a=new double[n][n];
f=1;
s=1;
for(i=1;i<=n;i++)
  for(j=1;j<=n;j++)
    a[i-1][j-1]=b[i-1][j-1];
while(last==0)
  {
  last=1;
  for(i=1;i<=n;i++)
    {
    c=0;r=0;
    for(j=1;j<=n;j++)
      {
      if(j != i)
              {
              c+=Math.abs(a[j-1][i-1]);
              r+=Math.abs(a[i-1][j-1]);
              } //end of if(j!=..
     } //end of for(j=1...
    if(c != 0 &&  r != 0 )
      {
      g=r/radix;
      f=1.0;
      s=c+r;
      while(c<g)
```

```
            {
            f*=radix;
            c*=sqrdx;
            }
     g=r*radix;
     while(c>g)
            {
            f/=radix;
            c/=sqrdx;
            }
     } //end of if(c != 0 && ....
   if( (c+r)/f < 0.95*s )
     {
     last=0;
     g=1.0/f;
     for(j=1;j<=n;j++)  { a[i-1][j-1]*=g; }
     for(j=1;j<=n;j++)  { a[j-1][i-1]*=f; }
     } //end of if( ((c+r..
   }//end of for(i=1;i<=n....
  } //end of while last==0
return a;
}

public static double[][] Hessenberg(double b[][])
{
// Calculates the hessenberg matrix
// it is used in QR method to calculate eigenvalues
// of a matrix(symmetric or non-symmetric)
int m,j,i;
int n=b.length;
double a[][];
a=new double[n][n];
for(i=0;i<n;i++)
  for(j=0;j<n;j++)
    a[i][j]=b[i][j];
double x,y;
if(n>2)
  {
  for(m=2;m<=(n-1);m++)
     {
     x=0.0;
     i=m;
     for(j=m;j<=n;j++)
       {
       if(Math.abs(a[j-1][m-2]) > Math.abs(x))
              {
              x=a[j-1][m-2];
              i=j;
              } //end of if(Math.abs(..
       }//end of for(j=m,j<=n...
     if(i!=m)
       {
       for(j=(m-1);j<=n;j++)
              {
              y=a[i-1][j-1];
              a[i-1][j-1]=a[m-1][j-1];
              a[m-1][j-1]=y;
              }//end of for(j=(m-1)..
       for(j=1;j<=n;j++)
              {
              y=a[j-1][i-1];
              a[j-1][i-1]=a[j-1][m-1];
```

```
              a[j-1][m-1]=y;
              }//end of for(j=1;j<=n....
      } //end of if(i!=m)
    if(x != 0.0)
      {
      for(i=(m+1);i<=n;i++)
               {
               y=a[i-1][m-2];
               if(y!=0.0)
                 {
                 y=y/x;
                 a[i-1][m-2]=y;
                 for(j=m;j<=n;j++)
                   {
                   a[i-1][j-1]-=y*a[m-1][j-1];
                   }
                 for(j=1;j<=n;j++)
                   {
                   a[j-1][m-1]+=y*a[j-1][i-1];
                   }
                 }//end of if(y!=0..
               }//end of for(i=(m+1)...
      } //end of if(x != 0.0...
    }//end of for(m=2;m<=(n-1)..
  }//end of Hessenberg
for(i=1;i<=n;i++)
  for(j=1;j<=n;j++)
  {
  if(i>(j+1)) a[i-1][j-1]=0;
  }
return a;
}

public static double[][] QR(double b[][])
{
//calculates eigenvalues of a Hessenberg matrix
int n=b.length;
double rm[][]=new double[2][n];
double a[][]=new double[n+1][n+1];
double wr[]=new double[n+1];
double wi[]=new double[n+1];
int nn,m,l,k,j,its,i,mmin;
double z,y,x,w,v,u,t,s,r=0,q=0,p=0,anorm;
for(i=0;i<n;i++)
   for(j=0;j<n;j++)
      a[i+1][j+1]=b[i][j];
anorm=Math.abs(a[1][1]);
for (i=2;i<=n;i++)
   for (j=(i-1);j<=n;j++)
       anorm += Math.abs(a[i][j]);
nn=n;
t=0.0;
while (nn >= 1) {
   its=0;
   do {
      for (l=nn;l>=2;l--) {
      s=Math.abs(a[l-1][l-1])+Math.abs(a[l][l]);
      if (s == 0.0) s=anorm;
      if ((Math.abs(a[l][l-1]) + s) == s) break;
                        }
      x=a[nn][nn];
      if (l == nn) {
```

```
      wr[nn]=x+t;
            wi[nn--]=0.0;
                        }
   else {
       y=a[nn-1][nn-1];
             w=a[nn][nn-1]*a[nn-1][nn];
             if (l == (nn-1)) {
               p=0.5*(y-x);
               q=p*p+w;
               z=Math.sqrt(Math.abs(q));
               x += t;
               if (q >= 0.0) {
                 z=p+Matrix.SIGN(z,p);
                       wr[nn-1]=wr[nn]=x+z;
                       if (z!=0) wr[nn]=x-w/z;
                       wi[nn-1]=wi[nn]=0.0;
                                   }
               else {
                 wr[nn-1]=wr[nn]=x+p;
                       wi[nn-1]= -(wi[nn]=z);
                        }
               nn -= 2;
                                    }
             else {
               if (its == 30) System.out.println("Matrix.java, module QR, Too many iterations in hqr");
               if (its == 10 || its == 20) {
                 t += x;
                       for (i=1;i<=nn;i++) a[i][i] -= x;
                       s=Math.abs(a[nn][nn-1])+Math.abs(a[nn-1][nn-2]);
                       y=x=0.75*s;
                       w = -0.4375*s*s;
                                                  }
               ++its;
               for (m=(nn-2);m>=l;m--) {
               z=a[m][m];
               r=x-z;
               s=y-z;
               p=(r*s-w)/a[m+1][m]+a[m][m+1];
               q=a[m+1][m+1]-z-r-s;
               r=a[m+2][m+1];
               s=Math.abs(p)+Math.abs(q)+Math.abs(r);
               p /= s;
               q /= s;
               r /= s;
               if (m == l) break;
               u=Math.abs(a[m][m-1])*(Math.abs(q)+Math.abs(r));
               v=Math.abs(p)*(Math.abs(a[m-1][m-1])+
         Math.abs(z)+Math.abs(a[m+1][m+1]));
               if ((u+v) == v) break;
                                                 }
               for (i=m+2;i<=nn;i++) {
                 a[i][i-2]=0.0;
                 if (i != (m+2)) a[i][i-3]=0.0;
                                    }
               for (k=m;k<=nn-1;k++) {
               if (k != m) {
                 p=a[k][k-1];
                       q=a[k+1][k-1];
                       r=0.0;
                       if (k != (nn-1)) r=a[k+2][k-1];
                       if ((x=Math.abs(p)+Math.abs(q)+Math.abs(r)) != 0.0) {
                         p /= x;
```

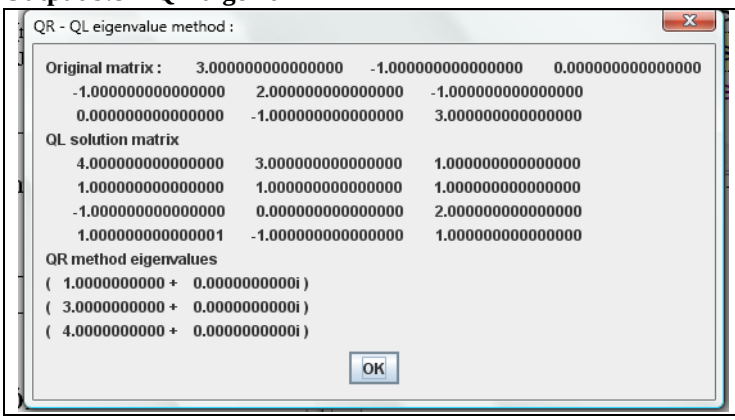
```
                    q /= x;
                    r /= x;
                                                    }                                    }
                  if ((s=Matrix.SIGN(Math.sqrt(p*p+q*q+r*r),p)) != 0.0) {
                    if (k == m) {
                      if (l != m)
                      a[k][k-1] = -a[k][k-1];
                                    }
                    else
                      a[k][k-1] = -s*x;
                    p += s;
                    x=p/s;
                    y=q/s;
                    z=r/s;
                    q /= p;
                    r /= p;
                    for (j=k;j<=nn;j++) {
                    p=a[k][j]+q*a[k+1][j];
                    if (k != (nn-1)) {
                      p += r*a[k+2][j];
                      a[k+2][j] -= p*z;
                              }
                    a[k+1][j] -= p*y;
                    a[k][j] -= p*x;
                                        }
                    mmin = nn<k+3 ? nn : k+3;
                    for (i=l;i<=mmin;i++) {
                      p=x*a[i][k]+y*a[i][k+1];
                      if (k != (nn-1)) {
                        p += z*a[i][k+2];
                              a[i][k+2] -= p*r;
                                    }
                      a[i][k+1] -= p*q;
                      a[i][k] -= p;
                  }
              }
            }
      }
    }
  } while (l < nn-1);
}
for(i=0;i<n;i++)
{
rm[0][i]=wr[i+1];
rm[1][i]=wi[i+1];
}
return rm;
} //end of QR

public static double[][] eigenValue(double b[][])
{
// this routine input a matrix (non symetric or symmetric)
// and calculate eigen values
// method balance can be used prior to this method to balance
// the input matrix
int n=b.length;
double d[][]=new double[2][n];
d=QR(Hessenberg(b));
return d;
}

public static c[] eigenValueC(double b[][])
```

```
{
// this routine input a matrix (non symetric or symmetric)
// and calculate eigen values
// method balance can be used prior to this method to balance
// the input matrix
//output eigenvalues will be in a vector of c form
int n=b.length;
double d[][]=new double[2][n];
d=QR(Hessenberg(b));
c c[]=new c[n];
for(int i=0;i<n;i++)
  {
  c[i]=new c(d[0][i],d[1][i]);
  }
return c;
}
  public static void main(String args[]) throws IOException
  {
  //Eigen Value calculations using  method
  //double a[][]={{3,-1,0},{-1,2,-1},{0,-1,3}};
  //double a[][]={{1,-2,3},{-2,4,1},{3,1,2}};
  double a[][]={{3.556,-1.778,0},{-1.778,3.556,-1.778},{0,-1.778,3.556}};
  String s="Original matrix : ";
  s+=Matrix.toString(a);
  s+="QL solution matrix \n";
  s+=Matrix.toString(SCO13A.eigenQL(a));
  //Eigenvlue calculation
  c c1[];
  int n=a.length;
  c1=new c[n];
  c1=eigenValueC(a);
  s+="QR method eigenvalues \n";
  s+=Matrix.toStringT(c1);
  JOptionPane.showMessageDialog(null,s,
     "QR - QL eigenvalue method : ",JOptionPane.PLAIN_MESSAGE);
  }
}
```

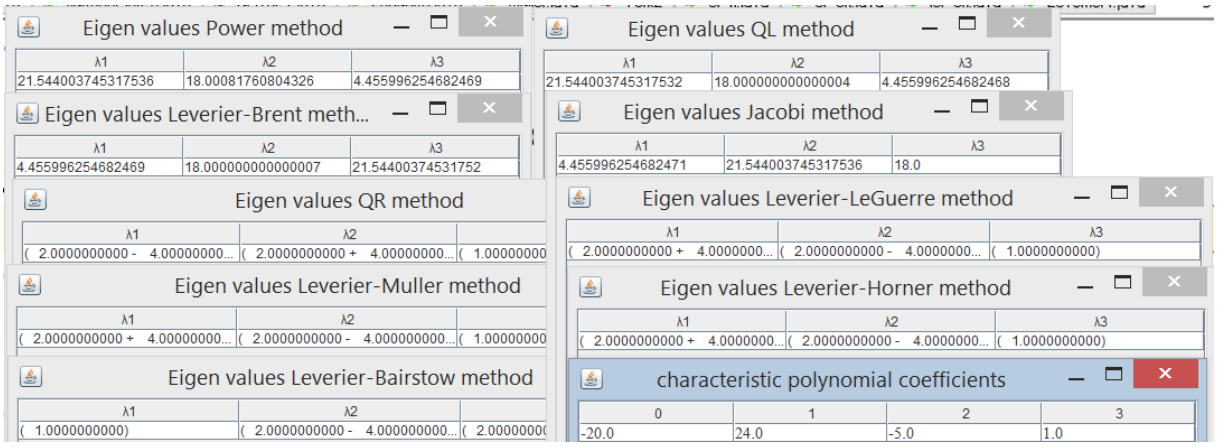
Output 9.8-1 QR algorithm



A general interface program for all different eigenvalues are given for an easy application utilisation. Two eigenvalue problem, one symmetric and one non-symmetric matrix eigenvalue problem is solved and the results are given together as output.

```
 public class eigen
{
public static double[][] Householder(double A[][])
{return NA133.Householder(A);}
```

```
public  static double[] characteristic_polynomial(double A[][])
{ Leverrier2 lv=new Leverrier2();
  return lv.polynomial_coefficients(A);
}
//real eigenvalues by using power method for limit values
//and brent method for the remaining eigenvalues
public  static double[] eigen_LBrent(double A[][])
{Leverrier lv=new Leverrier();
return lv.Leverrier(A);}
//real eigenvalues by using jacobi method
public  static double[] eigen_Jacobi(double A[][])
{return NA133.eigen_jacobi(A);}
//real eigenvalues by using QL method
public  static double[] eigen_QL(double A[][])
{               double w[][]=NA134.eigenQL(A);
                int n=w[0].length;
                double x[]=new double[n];
                for(int i=0;i<n;i++) x[i]=w[0][i];
                return x;
}
public static double[] eigen_power(double A[][])
{eigen_power w=new eigen_power(A);
return w.power();}
//complex eigenvalues using Leverier Bairstow method
public static complex[] eigen_LBairstow(double A[][])
{ Leverrier1 lv=new Leverrier1();
return lv.Leverrier(A);}
public static complex[] eigen_LHorner(double A[][])
{ Leverrier2 lv=new Leverrier2();
return lv.Leverrier(A);}
public static complex[] eigen_LMuller(double A[][])
{ Leverrier3 lv=new Leverrier3();
return lv.Leverrier(A);}
public static complex[] eigen_LLeGuerre(double A[][])
{ Leverrier4 lv=new Leverrier4();
return lv.Leverrier(A);}
//complex eigenvalues using QR method
public static complex[] eigen_QR(double A[][])
{ return NA135.eigenValueC(A);}
public static String[] heading()
{String baslik[]={""+\u03BB'+"1",""+\u03BB'+"2",""+\u03BB'+"3",""+\u03BB'+"4",""+\u03BB'+"5",""+\u03BB'+"6",
""+\u03BB'+"7",""+\u03BB'+"8",""+\u03BB'+"9",""+\u03BB'+"10",""+\u03BB'+"11",""+\u03BB'+"12"};
return baslik;}
public static void print(complex c[],String s)
{Text.print(c,heading(),s);}
public static void print(double c[],String s)
{Text.print(c,heading(),s);}
public static void main(String arg[])
{
double A[][]={{8,-1,-5},{-4,4,-2},{18,-5,-7}};
double b[]=characteristic_polynomial(A);
Text.print(b,"characteristic polynomial coefficients");
complex c[]=eigen_LBairstow(A);
print(c,"Eigen values Leverier-Bairstow method");
complex c1[]=eigen_LHorner(A);
print(c1,"Eigen values Leverier-Horner method");
complex c2[]=eigen_LMuller(A);
print(c2,"Eigen values Leverier-Muller method");
complex c3[]=eigen_LLeGuerre(A);
print(c3,"Eigen values Leverier-LeGuerre method");
complex c4[]=eigen_QR(A);
print(c4,"Eigen values QR method");
double A1[][]={{12,6,-6},{6,16,2},{-6,2,16}};
double d[]=eigen_LBrent(A1);
print(d,"Eigen values Leverier-Brent method");
double d1[]=eigen_Jacobi(A1);
print(d1,"Eigen values Jacobi method");
double d2[]=eigen_QL(A1);
print(d2,"Eigen values QL method");
double d3[]=eigen_power(A1);
print(d3,"Eigen values Power method");
}}
```



ROOTS OF THE POLYNOMIALS AND COMPANION MATRIX

Consider a nth degree polynomial

$f(x)= a_n x^n + \ldots + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

If the coefficients of the polynomial are divided by a_n , it becomes

$f(x)= x^n + b_{n-1} x^4 + \ldots + b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0$

The roots of this polynomial is the same as the eigenvalues of the Matrix

$$A = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \ldots & -b_1 & -b_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & \ldots & 1 & 0 \end{bmatrix}$$

This matrix is called a companion matrix. Since the matrix is not symmetric, eigen values or root of the polynomials can be complex values. QR algorithm can be used to calculate this complex eigenvalues.

Program 9.9-1 QR method to solve companion matrix (root of the nth degree polynomials)

```
 import java.util.*;
import java.awt.*;
import javax.swing.*;
import java.awt.event.*;


class NA136 extends JFrame implements ActionListener
{ private static final long serialVersionUID = 875678563L;
public static double[][] balancedEigenValue(double b[][])
{
// this routine input a matrix (non symetric or symmetric)
// and calculates eigen values
// method balance is used to balance the matrix previous to
// actual calculations
int n=b.length;
double d[][]=new double[2][n];
d=NA135.QR(NA135.Hessenberg(NA135.balance(b)));
return d;
}
public static c[] poly_rootsC(double c[])
{
// roots of a degree n polynomial
// P(x)=c[n]*x^n+c[n-1]*x^(n-1)+....+c[1]*x+c[0]=0;
// roots are returned as c variables
int n=c.length-1;
double a[][]=new double[n][n];
a=characteristic_matrix(c);
double d[][]=new double[2][n];
d=balancedEigenValue(a);
```

```
c e[]=new c[n];
for(int i=0;i<n;i++)
  e[i]=new c(d[0][i],d[1][i]);
return e;
}

public static double SIGN(double a,double b)
{
//returns the value of double a with sign of double b;
//if a=-2, b= 3 SIGN(a,b) returns  2
//if a=-2, b=-3 SIGN(a,b) returns -2
//if a= 2, b=-3 SIGN(a,b) returns -2
//if a= 2, b= 3 SIGN(a,b) returns  2
if(b!=0)
  return Math.abs(a)*b/Math.abs(b);
else
  return Math.abs(a);
}
public static double[][] characteristic_matrix(double c[])
{
//this routine converts polynomial coefficients to a matrix
//with the same eigenvalues (roots)
int n=c.length-1;
int i;
double a[][]=new double[n][n];
for(i=0;i<n;i++)
{
  a[0][i]=-c[i+1]/c[0];
}
for(i=0;i<n-1;i++)
{
  a[i+1][i]=1;
}
return a;
}

public static double[][] tridiagonal(double b[][], double d[], double e[])
{
//reduces matrix to tridiaonal form by using HOUSEHOLDER TRANSFORMATION
//this method is used by QL method to calculate eigen values
//and eigen  vectors of a symmetric matrix
        int l,k,j,i;
    int n=b.length;
        double scale,hh,h,g,f;
    double a[][]=new double[n+1][n+1];
    double c[][]=new double[n][n];
    for(i=0;i<n;i++)
      for(j=0;j<n;j++)
        a[i][j]=b[i][j];
        for (i=n;i>=2;i--) {
                l=i-1;
                h=scale=0.0;
                if (l > 1) {
                        for (k=1;k<=l;k++)
                                scale += Math.abs(a[i-1][k-1]);
                        if (scale == 0.0)
                                e[i-1]=a[i-1][l-1];
                        else {
                                for (k=1;k<=l;k++) {
                                        a[i-1][k-1] /= scale;
                                        h += a[i-1][k-1]*a[i-1][k-1];
                                }
                                f=a[i-1][l-1];
                                g=(f >= 0.0 ? -Math.sqrt(h) : Math.sqrt(h));
                                e[i-1]=scale*g;
                                h -= f*g;
                                a[i-1][l-1]=f-g;
                                f=0.0;
                                for (j=1;j<=l;j++) {
                                        a[j-1][i-1]=a[i-1][j-1]/h;
```

```
                                        g=0.0;
                                        for (k=1;k<=j;k++)
                                                g += a[j-1][k-1]*a[i-1][k-1];
                                        for (k=j+1;k<=l;k++)
                                                g += a[k-1][j-1]*a[i-1][k-1];
                                        e[j-1]=g/h;
                                        f += e[j-1]*a[i-1][j-1];
                                }
                                hh=f/(h+h);
                                for (j=1;j<=l;j++) {
                                        f=a[i-1][j-1];
                                        e[j-1]=g=e[j-1]-hh*f;
                                        for (k=1;k<=j;k++)
                                                a[j-1][k-1] -= (f*e[k-1]+g*a[i-1][k-1]);
                                }
                        }
                } else
                        e[i-1]=a[i-1][l-1];
                d[i-1]=h;
        }
        d[1-1]=0.0;
        e[1-1]=0.0;
        /* Contents of this loop can be omitted if eigenvectors not
                        wanted except for statement d[i-1]=a[i-1][i-1]; */
        for (i=1;i<=n;i++) {
                l=i-1;
                if (d[i-1] != 0) {
                        for (j=1;j<=l;j++) {
                                g=0.0;
                                for (k=1;k<=l;k++)
                                        g += a[i-1][k-1]*a[k-1][j-1];
                                for (k=1;k<=l;k++)
                                        a[k-1][j-1] -= g*a[k-1][i-1];
                        }
                }
                d[i-1]=a[i-1][i-1];
                a[i-1][i-1]=1.0;
                for (j=1;j<=l;j++) a[j-1][i-1]=a[i-1][j-1]=0.0;
        }
return  a;
}

public static double pythag(double a, double b)
{
//this method is used by QL method
        double absa,absb;
        absa=Math.abs(a);
        absb=Math.abs(b);
        if (absa > absb) return absa*Math.sqrt(1.0+(absb/absa)*(absb/absa));
        else return (absb==0.0 ? 0.0 : absb*Math.sqrt(1.0+(absa/absb)*(absa/absb)));
}

public static double[][] QL(double d[], double e[], double a[][])
{
// QL algorithm : eigenvalues of a symmetric matrix reduced to tridiagonal
// form by using method tridiagonal
   int n=d.length;
        int m,l,iter,i,j,k;
        double s,r,p,g,f,dd,c,b;
        for (i=2;i<=n;i++) e[i-2]=e[i-1];
        e[n-1]=0.0;
   double z[][]=new double[n][n];
   for(i=0;i<n;i++)
     for(j=0;j<n;j++)
       z[i][j]=a[i][j];
        for (l=1;l<=n;l++) {
                iter=0;
                do {
                        for (m=l;m<=n-1;m++) {
                                dd=Math.abs(d[m-1])+Math.abs(d[m]);
```

```
                                        if ((Math.abs(e[m-1])+dd) == dd) break;
                                }
                                if (m != l) {
        if (iter++ == 30) System.out.println("Too many iterations in QL");
        g=(d[l]-d[l-1])/(2.0*e[l-1]);
        r=pythag(g,1.0);
        g=d[m-1]-d[l-1]+e[l-1]/(g+SIGN(r,g));
        s=c=1.0;
        p=0.0;
        for (i=m-1;i>=l;i--) {
                                                f=s*e[i-1];
                                                b=c*e[i-1];
                                                e[i]=(r=pythag(f,g));
                                                if (r == 0.0) {
                                                        d[i] -= p;
                                                        e[m-1]=0.0;
                                                        break;
                                                }
                                                s=f/r;
                                                c=g/r;
                                                g=d[i ]-p;
                                                r=(d[i-1]-g)*s+2.0*c*b;
                                                d[i ]=g+(p=s*r);
                                                g=c*r-b;
                                                for (k=1;k<=n;k++) {
                                                        f=z[k-1][i ];
                                                        z[k-1][i ]=s*z[k-1][i-1]+c*f;
                                                        z[k-1][i-1]=c*z[k-1][i-1]-s*f;
                                                }
                                        }
                                        if (r == 0.0 && i >= l) continue;
                                        d[l-1] -= p;
                                        e[l-1]=g;
                                        e[m-1]=0.0;
                                }
                        } while (m != l);
                }
   return z;
}

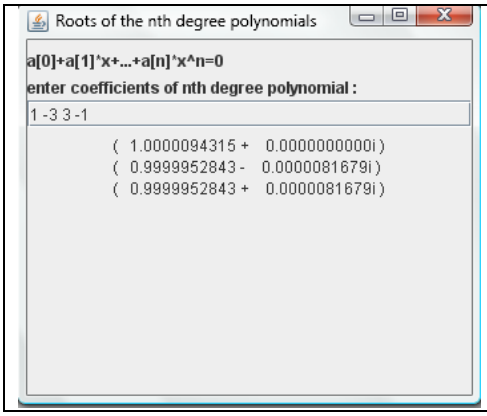
public static double[][] eigenQL(double a[][])
{
 // QL algoritm to solve eigen value problems
 // symmetric matrices only (real eigen values)
 // first column of the matrix returns eigen values
 // second..n+1 column returns eigen vectors.
 // Note : If matrix is not symmetric DO NOT use
 // this method use eigenValue method (a QR algorithm)
 int i,j;
 int n=a.length;
 double sum[]=new double[n];;
 double d[]=new double[n];
 double b[][]=new double[n][n];
 double e[]=new double[n];
 double z[][]=new double[n+1][n];
 b=tridiagonal(a,d,e);
 b=QL(d,e,b);
 for(j=0;j<n;j++)
   {
   z[0][j]=d[j];
   for(i=0;i<n;i++)
     {
     z[i+1][j]=b[i][j]/b[0][j];
     if(z[i+1][j]<1e-13) z[i+1][i]=0;
     }
   }
 // result: first row is eigenvalues, the rest is eigenvector
 return z;
}
```

```
    private JLabel prompt1,prompt2;
    private JTextField input;
    JTextArea t;
    JPanel YaziPaneli;
    int n;
    String s;
    Container c;
    public NA136()
    {
       super("Roots of the nth degree polynomials");
       c=getContentPane();
       c.setLayout(new FlowLayout());
       YaziPaneli=new JPanel();
       YaziPaneli.setFont(new Font("Serif",Font.BOLD,12));
       YaziPaneli.setLayout( new GridLayout(3,1) );
       t=new JTextArea();
       t.setBackground(c.getBackground());
       prompt1= new JLabel("a[0]+a[1]*x+...+a[n]*x^n=0");
       prompt2= new JLabel("enter coefficients of nth degree polynomial : ");
       input = new JTextField(30);
       input.setBackground(c.getBackground());
       YaziPaneli.add(prompt1);
       YaziPaneli.add(prompt2);
       YaziPaneli.add(input);
       c.add(YaziPaneli);
       c.add(t);
       input.addActionListener(this);
    }

    public void  actionPerformed(ActionEvent e)
    {
          s=input.getText();
          StringTokenizer token=new StringTokenizer(s);
          t.setText("");
          n=token.countTokens()-1;
          int m=n+1;
          double a[]=new double[m];
          c z[]=new c[n];
          for(int i=0;i<n;i++)
          {
          z[i]=new c();
          }
          int j=0;
              while(token.hasMoreTokens())
              {
              Double ax=new Double(token.nextToken());
              a[j++]=ax.doubleValue();
              }
              z=poly_rootsC(a);
              int i=0;
              t.setText(Matrix.toStringT(z));
              input.setText("");
    }

     public static void main(String[] args)
     {
     NA136 pencere= new NA136();
     pencere.addWindowListener(new BasicWindowMonitor());
     pencere.setSize(350,300);
     pencere.setVisible(true);
     }
}
```

Output 9.7-1 Roots of the nth degree polynomial by using QR algorithm



SINGULAR VALUE DECOMPOSITION FOR SOLVING SYSTEM OF EQUATIONS

Let A be an m x n matrix such that the number of rows m is greater than or equal to the number of columns n. Then there exists:
1) an m x n column orthogonal matrix **U** (a matrix Q is orthogonal if its transpose is equal to its inverse)
2) an n x n diagonal matrix **S,** with positive or zero elements, and

3) an n x n orthogonal matrix V such that:

$$\boldsymbol{A=USV^T}$$

This is the **Singular Value Decomposition (SVD) of matrix A.** If a matrix can be decomposed this way, its inverse value will be existed as:

$$\boldsymbol{A^{-1}=V\ [diag\ (s_1^{-1},s_2^{-1},\ldots,s_n^{-1})]\ (U^T)}$$

It should ve note that because of the orhogonal properties of U and V inverses are equal to transpose values. Then a system of equation can be simply solve as $x=A^{-1}b$
Now let us look at these processes in more details with an example

$$A=\begin{bmatrix}4 & 0\\ 3 & -5\end{bmatrix}$$

The singular values of A are the square roots of the eigenvalues of A^TA

$$A^T=\begin{bmatrix}4 & 3\\ 0 & -5\end{bmatrix}$$

$$A^TA=\begin{bmatrix}4 & 3\\ 0 & -5\end{bmatrix}\begin{bmatrix}4 & 0\\ 3 & -5\end{bmatrix}=\begin{bmatrix}25 & -15\\ -15 & 25\end{bmatrix}$$

Eigenvalues of A^TA is

$$\lambda=\begin{bmatrix}40\\ 10\end{bmatrix}$$ Root of the eigenvalues constructs the matrix S, by using eigenvalues as the diagonal of the matrix. And then the inverse can be easily computed by inverting the eigenvalues.

$$S=\begin{bmatrix}\sqrt{40} & 0\\ 0 & \sqrt{10}\end{bmatrix}=\begin{bmatrix}6.3245 & 0\\ 0 & 3.1622\end{bmatrix}$$

$$S^{-1}=\begin{bmatrix}1/6.3245 & 0\\ 0 & 1/3.1622\end{bmatrix}=\begin{bmatrix}0.1581 & 0\\ 0 & 0.3162\end{bmatrix}$$

The eigenvectors corresponding to these eigenvalues are:

$$\left[A^TA-\lambda I\right]\left[X^1\right]=\begin{bmatrix}25-40 & -15\\ -15 & 25-40\end{bmatrix}\begin{bmatrix}x_1^1\\ x_2^1\end{bmatrix}=0$$

$$\left[A^TA-\lambda I\right]\left[X^2\right]=\begin{bmatrix}25-10 & -15\\ -15 & 25-10\end{bmatrix}\begin{bmatrix}x_1^2\\ x_2^2\end{bmatrix}=0$$

$$[X] = \begin{bmatrix} 0.707106781186548 & 0.707106781186548 \\ -0.707106781186548 & 0.707106781186548 \end{bmatrix}$$

The eigenvector values will be normalized by using Length factor

$L = \sqrt{(x_1)^2 + (x_2)^2}$ Then the normalized Value will be V vector

$$V = \begin{bmatrix} 0.707106781186548 & 0.707106781186548 \\ -0.707106781186548 & 0.707106781186548 \end{bmatrix}$$

The U vector will be calculated as

$$U = AVS^{-1} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix}$$

A program is developed to calculate singular value decomposition by using jacobi method. The example problem solved in this program as follows:

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & 0.5 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Program Singular value decomposition linear system of equation solving method by using Jacobi eigenvalue method

```
import java.io.*;
import javax.swing.*;

class SVD
{
double U[][];
double V[][];
double S[][];
double invS[][];

public static double[][] Transpose(double [][] left)
{
//transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
{
for(j=0;j<m;j++)
{
b[j][i]=left[i][j];
}
}
return b;
}

public static double[][] T(double [][] left)
{return Transpose(left);}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
b[i][i]=1.0;
return b;
}
//Converts vector to diagonal values of matrix
public static double[][] S(double[] left)
{
int n=left.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
```

```
b[i][i]=left[i];
return b;
}
//Converts double to diagonal values of matrix
public static double[][] S(double left,int n)
{
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
b[i][i]=left;
return b;
}
//takes invert value of a diagonal matrix
public static double[][] invS(double[][] S)
{
int n=S.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
{  if(S[i][i]==0) b[i][i]=0;
                else b[i][i]=1.0/S[i][i];
}
return b;
}
//square root of a diagonal matrix
public static double[][] sqrtS(double[] left)
{
int n=left.length;
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
b[i][i]=Math.sqrt(left[i]);
return b;
}
//multiply two matrices
public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[m1][n2];
if(n1 != m2)
{
System.out.println("m1="+m1+"n1="+n1+"m2="+m2+"n2="+n2+"inner matrix dimensions must agree");
for(ii=0;ii<n1;ii++)
{
for(jj=0;jj<m2;jj++)
b[ii][jj]=0;
}
return b;
}
for(i=0;i<m1;i++)
{
for(j=0;j<n2;j++)
{
for(k=0;k<n1;k++)
                b[i][j]+=left[i][k]*right[k][j];
}
}
return b;
//end of multiply of two matrices
}
//multiply a matrix with a double value
public static double[][] multiply(double[][] left,double right)
{
//multiplying a matrix with a constant
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
```

```
{
for(j=0;j<m;j++)
b[i][j]=left[i][j]*right;
}
return b;
//end of multiplying a matrix with a constant double
}
//multiply a matrix with a vector
public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right.length;
double[] b;
b=new double[m2];
if(n1 != m2)
{
System.out.println("inner matrix dimensions must agree");
for(ii=0;ii<n1;ii++)
{
b[ii]=0;
}
return b;
}
for(i=0;i<m1;i++)
{
b[i]=0;
for(k=0;k<n1;k++)
b[i]+=left[i][k]*right[k];
}
return b;
//end of multiply of a matrix and a vector
}

//substract two matrices
public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
{
for(j=0;j<m1;j++)
{
b[i][j]=b[i][j]+left[i][j];
}
}
for(i=0;i<n2;i++)
{
for(j=0;j<m2;j++)
{
b[i][j]=b[i][j]-right[i][j];
}
}
return b;
//end of matrix substraction method
}

public SVD(double A[][])
{initilize(A);}

public static double[] solve(double a[][],double b[])
{ SVD q=new SVD(a);
double x[]=multiply(multiply(q.V,q.invS),multiply(q.T(q.U),b));
```

```
return x;
}

public  double[] solve(double b[])
{
double x[]=multiply(multiply(V,invS),multiply(T(U),b));
return x;
}

public void initilize(double A[][])
{
double ATA[][]=multiply(Transpose(A),A);
//double aa[][]=Householder(ATA);
double c[][];
c=eigen_jacobi(ATA);//eigenQL(aa);
int n=c[0].length;
int n1=c.length;
double dummy[]=new double[n1];
for(int i=0;i<n;i++)
{for(int k=i+1;k<n;k++)
{if(c[0][i]<c[0][k])
{ for(int j=0;j<n1;j++)
{ dummy[j]=c[j][k];
c[j][k]=c[j][i];
c[j][i]=dummy[j];
}
}
}
}
double x[]=new double[n];
for(int i=0;i<n;i++)
{x[i]=c[0][i];}
S=sqrtS(x);
invS=invS(S);
double xx[][]=new double[n][n];
V=new double[n][n];
U=new double[n][n];
double L[]=new double[n];
for(int i=0;i<n;i++)
{ for(int j=0;j<n;j++)
{xx[i][j]=c[i+1][j];}
}

double bb[]=new double[n];
for(int j=0;j<n;j++)
{ for(int i=0;i<n;i++)
{L[j]+=xx[i][j]*xx[i][j];}
L[j]=Math.sqrt(L[j]);
}
for(int i=0;i<n;i++)
{ for(int j=0;j<n;j++)
{V[i][j]=xx[i][j]/L[i];}
}
U=multiply(A,multiply(V,invS));
}

public static void print(String s)
{JOptionPane.showMessageDialog(null,s,
"Singular value decomposition : ",JOptionPane.PLAIN_MESSAGE);}

public static double[][] eigen_jacobi(double A[][])
{
double eps=1e-15;
double err=1;
int m=100;
int n=A.length;
double P[][]=new double[n][n];
double R[][]=new double[n][n];
int p=n,q=n;
double max=-9.99e100;
double fi;
P=I(n);
for(int i=0;i<n;i++)
{for(int j=i+1;j<n;j++)
{if(Math.abs(A[i][j]) > max) {max=Math.abs(A[i][j]);p=i;q=j;}}
}
```

```
while(Math.abs(A[p][q]) > eps )
{
fi=0.5*Math.atan(2.0*A[p][q]/(A[p][p]-A[q][q]));
R=I(n);
R[p][p]=Math.cos(fi);
R[q][q]=Math.cos(fi);
R[p][q]=-Math.sin(fi);
R[q][p]=Math.sin(fi);
P=multiply(P,R);
A=multiply(T(R),multiply(A,R));
max=-9.99e100;
for(int i=0;i<n;i++)
{for(int j=i+1;j<n;j++){if(Math.abs(A[i][j])>max) {max=Math.abs(A[i][j]);p=i;q=j;}}}
}
double lambda[]=new double[n];
for(int i=0;i<n;i++) {lambda[i]=A[i][i];}
double PP[][]=new double[n+1][n];
PP[0]=lambda;
for(int i=0;i<n;i++)
{for(int j=0;j<n;j++)
{PP[i+1][j]=P[i][j];}
}
return PP;
}

public static void main(String args[]) throws IOException
{
double a[][]={{3,2,-1},{2,-2,4},{-1,0.5,-1}};
double b[]={1,-2,0};
//Dynamic call to SVD
SVD q=new SVD(a);
double x[]=q.solve(b);
System.out.println("S:\n"+Matrix.toString(q.S));
System.out.println("V:\n"+Matrix.toString(q.V));
System.out.println("U:\n"+Matrix.toString(q.U));
//Static call to SVD
//double x2[]=solve(a,b);
System.out.println("SINGULAR VALUE DECOMPOSITION : \n"+Matrix.toString(x));
//System.out.println("STATIC SINGULAR VALUE DECOMPOSITION : \n"+Matrix.toString(x2));
}
}
```

Output 9.8-1 Singular value decomposition linear system of equation solving method by using Jacobi eigenvalue method results

```
---------- Capture Output ----------
                    "D:\java\bin\javaw.exe" SVD
S:
5.126196943738645       0.000000000000000       0.000000000000000
0.000000000000000       3.734641762863597       0.000000000000000
0.000000000000000       0.000000000000000       0.156702894292504

V:
0.347022679391807       0.881301927095452      -0.320752510957550
-0.450285808796899      0.456574244972559       0.767321737750118
0.822689461694906      -0.121847741569467       0.555279368868110

U:
-0.133079756513558      0.985074940622975       0.109165652853519
0.953021289812956       0.096947557746556       0.286969671237626
-0.272103288421038      -0.142227045260808      0.951699147854431

SINGULAR VALUE DECOMPOSITION :
1.000000000000170000000000
-2.000000000000420000000000
-2.000000000000300000000000

> Terminated with exit code 0.
```

Program Singular value decomposition linear system of equation solving method by using Jacobi eigenvalue method (Python code)

```
# -*- coding: utf-8 -*-
"""
Created on Sun Aug 26 07:28:01 2018
```

```
@author: Mustafa Turhan Çoban
"""
from math import *;
from eigen_jacobi import *;
def SVD(A):
    #singular value decomposition
    B=AT(A);
    ATA=multiply_m_m(B,A);
    c=eigen_jacobi(ATA);
    n=len(c[0]);
    n1=len(c);
    dummy=[0.0 for x in range(n1)];
    V = [[0.0 for x in range(n)] for y in range(n)];
    U = [[0.0 for x in range(n)] for y in range(n)];
    xx = [[0.0 for x in range(n)] for y in range(n)];
    x=[0.0 for x in range(n)];
    for i in range(n):
        for k in range(i+1,n):
            if c[0][i]<c[0][k]:
                for j in range(n1):
                    dummy[j]=c[j][k];
                    c[j][k]=c[j][i]
                    c[j][i]=dummy[j];
    S = [[0.0 for x in range(n)] for y in range(n)];
    SI = [[0.0 for x in range(n)] for y in range(n)];
    for i in range(0,n):
        x[i]=c[0][i];
        S[i][i]=sqrt(x[i]);
        SI[i][i]=1.0/S[i][i];
    L=[0.0 for x in range(n)];
    for i in range(n):
        for j in range(n):
            xx[i][j]=c[i+1][j];
    for j in range(0,n):
        for i in range(0,n):
            L[j]+=xx[i][j]*xx[i][j];

    for i in range(0,n):
        for j in range(0,n):
            V[i][j]=xx[i][j]/L[j];
    VT=AT(V);
    U=multiply_m_m(A,V);
    U=multiply_m_m(U,SI);
    UT=AT(U);
    #AA=multiply_m_m(U,multiply_m_m(S,VT));
    #BB=multiply_m_m(V,multiply_m_m(SI,UT));
    #CC=multiply_m_m(AA,BB);
    x=[S,V,U];
    return x;

def invB(B):
    #inverse matrix (n!=m)
    x=SVD(B);
    S=x[0];
    V=x[1];
    U=x[2];
    SI = invS(S);
    UT=AT(U);
    return multiply_m_m(V,multiply_m_m(SI,UT));

def multiply_m_v(left,right):
    #multiplication of one matrix with one vector
    m1=len(left[0]);
    n1=len(left);
    m2=len(right);
    b = [0.0 for i in range(m2)];
    if n1 != m2:
        print("inner matrix dimensions must agree");
        for ii in range(n1):b[ii]=0;
        return b;
    for i in range(m1):
        b[i]=0;
        for k in range(n1):
            b[i]+=left[i][k]*right[k];

def multiply_v_m(left,right):
```

```
    #multiplication of one vector with one matrix
    m2=len(right[0]);
    n2=len(right);
    m1=len(left);
    b = [0.0 for i in range(m1)];
    if n2 != m1:print("inner matrix dimensions must agree");
    for ii in range(n2): b[ii]=0;
    for i in range(m2):
        b[ii]=0;
        for k in range(m1):
            b[i]+=right[i][k]*left[k];
    return b;

def invS(S):
    n=len(S);
    SI = [[0.0 for x in range(n)] for y in range(n)];
    for i in range(0,n):
        SI[i][i]=1.0/S[i][i];
    return SI;

def solve(A,B):
    #solution of the system of equation by using SVD
    AI=invB(A);
    X=multiply_v_m(B,AI);
    return X;

A=[[3,2,-1],[2,-2,4],[-1,0.5,-1]];
C=SVD(A);
S=C[0];
V=C[1];
U=C[2];
print("S=\n",S);
print("V=\n",V);
print("U=\n",U);
B=[1,-2,0];
X=solve(A,B);
print("X=",X);
```

```
runfile('D:/okul/SCO1/SVD.py', wdir='D:/okul/SCO1')
Reloaded modules: eigen_jacobi
S=
 [[5.126196943738645, 0.0, 0.0], [0.0, 3.734641762863597, 0.0], [0.0, 0.0, 0.1567028942925037]]
V=
 [[0.3470226793918069, 0.8813019270954525, -0.3207525109575496], [-0.4502858087968988, 0.45657424497255894,
0.7673217377501177], [0.8226894616949058, -0.12184774156946693, 0.5552793688681098]]
U=
 [[-0.13307975651355772, 0.9850749406229755, 0.10916565285351665], [0.9530212898129558, 0.0969475577465563,
0.28696967123762784], [-0.2721032884210384, -0.1422270452608084, 0.9516991478544303]]

X= [1.000000000000028, -2.000000000000068, -2.0000000000000497]
```

SINGULAR VALUE DECOMPOSITION FOR SOLVING POLYNOMIAL LEAST SQUARE CURVE FITTING PROBLEM

The singular value decomposition can be applied in many different areas. One of which is solving the polynomial least square problem. Let A be an $m \times n$ matrix, with $m > n$, and $\mathbf{b}$ is a vector. Note that degree of polynomial is (n-1). For a given data set $[x_i,y_i]$ i=0..(m-1) A and b can be represented as:

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \dots & \dots & .. & \dots & \dots \\ 1 & x_{m-2} & x_{m-2}^2 & \dots & x_{m-2}^{n-1} \\ 1 & x_{m-1} & x_{m-1}^2 & \dots & x_{m-1}^{n-1} \end{bmatrix} \quad b = \begin{bmatrix} y_0 \\ y_1 \\ \dots \\ y_{m-2} \\ y_{m-1} \end{bmatrix}$$

The least squares objective is to find a vector $\mathbf{x}$ that will minimize $\|A\mathbf{x} - \mathbf{b}\|_2$. Suppose that the singular value decomposition of A is known, $A = U S V^T$, Then

$$\left\|Ax-b\right\|_2 = \left\|USV^T x - UU^T b\right\|_2 = \left\|SV^T x - U^T b\right\|_2$$

Can be written. If $z = V^T x$ and $c = U^T b$ is defined, Equation becomes:

$$\left\|Ax-b\right\|_2 = \left\|Sz-c\right\|_2 = \left\{\sum_{i=1}^{k}(s_i z_i - c_i)^2 + \sum_{i=k+1}^{m}(c_i)^2\right\}$$

The norm is minimized when the derivatives set equal to zero

$$2(s_i z_i - c_i)\frac{dz_i}{dx_i} = 2(s_i z_i - c_i)V^T = 0 \quad i = 1..k$$

Therefore z_i can be chosen as

$$z_i = \left\{\begin{array}{ll}\left(\frac{c_i}{s_i}\right) & i <= k \\ arbitrarynumber & i > k\end{array}\right\}$$

From these least square coefficients can be calculated as x=Vz. A program developed to solve SVD version of polynomial least square approach:

Program Polynomial least square solution by using singular value decomposition

```
import java.io.*;
import java.util.*;
import javax.swing.*;

class SVDLeastsquare
{

public static double[][] Transpose(double [][] left)
{
//transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
return b;
}

public static double[][] T(double [][] left)
{return Transpose(left);}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
    b[i][i]=1.0;
return b;
}
//Converts vector to diagonal values of matrix
public static double[][] S(double[] left,int m)
{
int n=left.length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
    b[i][i]=left[i];
return b;
}

//Converts double to diagonal values of matrix
```

```
public static double[][] S(double left,int n)
{
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=left;
return b;
}

//takes invert value of a diagonal matrix
public static double[][] invS(double[][] S)
{
int m=S.length;
int n=S[0].length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
   {  if(S[i][i]==0) b[i][i]=0;
                else b[i][i]=1.0/S[i][i];
              }
return b;
}

//square root of a diagonal matrix
public static double[][] sqrtS(double[] left,int m)
{
int n=left.length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
   b[i][i]=Math.sqrt(left[i]);
return b;
}
//multiply two matrices
public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[n1][m2];
 if(m1 !=n2 )
 {
 System.out.println("inner matrix dimensions must agree"+"m1="+m1+"n2="+n2);
 return b;
 }
 for(i=0;i<n1;i++)
   {
   for(j=0;j<m2;j++)
     {
     for(k=0;k<m1;k++)
                 {b[i][j]+=left[i][k]*right[k][j];
                 }
     }
   }
return b;
//end of multiply of two matrices
}

//multiply a matrix with a double value
public static double[][] multiply(double[][] left,double right)
{
//multiplying a matrix with a constant
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=left[i][j]*right;
  }
```

```
return b;
//end of multiplying a matrix with a constant double
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int n1=left.length;
int m1=left[0].length;

int m2=right.length;
double[] b;
b=new double[m2];
 for(i=0;i<n1;i++)
   {
     b[i]=0;
    for(k=0;k<m1;k++)
                {b[i]+=left[i][k]*right[k];
                }
   }
return b;
//end of multiply of a matrix and a vector
}

//substract two matrices
public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
  for(j=0;j<m1;j++)
    {
    b[i][j]=b[i][j]+left[i][j];
    }
  }
for(i=0;i<n2;i++)
  {
  for(j=0;j<m2;j++)
    {
    b[i][j]=b[i][j]-right[i][j];
    }
  }
return b;
//end of matrix substraction method
}

public static double[] SVD_LSQ(double xi[],double yi[],int n_leastsquare)
{ // Singular value decomposition least square approximation
  double U[][];
  double V[][];
  double S[][];
  double invS[][];
  double A[][]=A(xi,n_leastsquare);
  int m=A.length;
  int m1=A[0].length;
  System.out.println("A="+Matrix.toString(A));
  double ATA[][]=multiply(Transpose(A),A);
  System.out.println("ATA="+Matrix.toString(ATA));
  //double aa[][]=Householder(ATA);
  double cc[][];
  cc=eigen_jacobi(ATA);//eigenQL(aa);
   System.out.println("c="+Matrix.toString(cc));
  int n=cc[0].length;
```

```
  int n1=cc.length;
  double dummy[]=new double[n1];
  for(int i=0;i<n;i++)
  {for(int k=i+1;k<n;k++)
    {if(cc[0][i]<cc[0][k])
      { for(int j=0;j<n1;j++)
        { dummy[j]=cc[j][k];
          cc[j][k]=cc[j][i];
          cc[j][i]=dummy[j];
        }
      }
    }
  }
  double x[]=new double[n];
  for(int i=0;i<n;i++)
  {x[i]=cc[0][i];}
  S=sqrtS(x,m1);
  System.out.println("S="+Matrix.toString(S));
  invS=invS(S);
  System.out.println("invS="+Matrix.toString(invS));
  double xx[][]=new double[n][n];
  V=new double[n][n];
  U=new double[n][n];
  double L[]=new double[n];

  for(int i=0;i<n;i++)
  { for(int j=0;j<n;j++)
    {xx[i][j]=cc[i+1][j];}
  }

  double bb[]=new double[n];
  for(int j=0;j<n;j++)
  { for(int i=0;i<n;i++)
    {L[j]+=xx[i][j]*xx[i][j];}
    L[j]=Math.sqrt(L[j]);
  }
  for(int i=0;i<n;i++)
  { for(int j=0;j<n;j++)
    {V[i][j]=xx[i][j]/L[i];}
  }
  System.out.println("VT="+Matrix.toString(T(V)));
  U=multiply(A,multiply(V,invS));
   System.out.println("U="+Matrix.toString(U));
  double c[]=multiply(T(U),yi);
   System.out.println("c="+Matrix.toString(c));
   double z[]=new double[n];
   for(int i=0;i<n;i++) z[i]=c[i]/S[i][i];
   System.out.println("z="+Matrix.toString(z));
   double a[]=multiply(V,z);
   System.out.println("a="+Matrix.toString(a));
   return a;
}
public static double func(double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=e[n-1];
  for(int i=n-2;i>=0;i--)
   { ff=ff*x+e[i]; }
  }
else
  ff=0;
return ff;
}

public static double error(double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
```

```
 total+=(y[k]-func(e,x[k]))*(y[k]-func(e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcSVD_LSQ(double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
               //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
               // ara değer sayısı 2 dir
                int n=xi.length;
               int nn=(n-1)*(aradegersayisi+1)+1;
               double z[][]=new double[2][nn];
   double E[]=SVD_LSQ(xi,yi,polinomkatsayisi);
   System.out.println("coefficients :\n"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=func(E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=func(E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=func(E,z[0][k]);
   return z;
 }
 public static void print(String s)
 {JOptionPane.showMessageDialog(null,s,
    "Singular value decomposition : ",JOptionPane.PLAIN_MESSAGE);}

 public static double[][] eigen_jacobi(double A[][])
 {               //Eigen value calculation
  double eps=1e-15;
  double err=1;
  int m=100;
  int n=A.length;
  double P[][]=new double[n][n];
  double R[][]=new double[n][n];
  int p=n,q=n;
  double max=-9.99e100;
  double fi;
  P=I(n);
  for(int i=0;i<n;i++)
  {for(int j=i+1;j<n;j++)
      {if(Math.abs(A[i][j]) > max) {max=Math.abs(A[i][j]);p=i;q=j;}}
  }
  while(Math.abs(A[p][q]) > eps )
  {
   fi=0.5*Math.atan2(2.0*A[p][q],(A[p][p]-A[q][q]));
   R=I(n);
   R[p][p]=Math.cos(fi);
   R[q][q]=Math.cos(fi);
   R[p][q]=-Math.sin(fi);
   R[q][p]=Math.sin(fi);
   P=multiply(P,R);
   A=multiply(T(R),multiply(A,R));
   max=-9.99e100;
   for(int i=0;i<n;i++)
  {for(int j=i+1;j<n;j++){if(Math.abs(A[i][j])>max) {max=Math.abs(A[i][j]);p=i;q=j;}}}
   }
  double lambda[]=new double[n];
  for(int i=0;i<n;i++) {lambda[i]=A[i][i];}
  double PP[][]=new double[n+1][n];
  PP[0]=lambda;
  for(int i=0;i<n;i++)
  {for(int j=0;j<n;j++)
    {PP[i+1][j]=P[i][j];}
  }
  return PP;
 }

 public static double[] gausswithpartialpivot(double a[][],double b[])
 { // Gauss elimination with partial pivoting
 int n=b.length;
 double x[]=new double[n];
```

```
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
           b[i]   =b[i]   -carpan*b[k];
                }
  }
  // backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[][] A(double xi[],int n_leastsquare)
{
int n=xi.length;
int m=n_leastsquare+1;
int l=xi.length;
int i,j,k;
double A[][];
A=new double[n][m];
double B[];
B=new double[n];
double X[];
X=new double[n];
double cc=1.0;
for(i=0;i<n;i++)
  {
     cc=1.0;
                for(j=0;j<m;j++)
     {
                A[i][j]=cc;
                cc*=xi[i];
     }
  }
return A;
}

  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1=JOptionPane.showInputDialog("Data file name (a.txt)");
  // JFileChooser fc=new JFileChooser();
```

```
  // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file = fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  int n=Integer.parseInt(JOptionPane.showInputDialog("polinom derecesi (n)"));
  double z[][]=funcSVD_LSQ(x,y,n,4);
  System.out.println("Polynomial Least Square Curve fitting\n"+Matrix.toStringT(z));
  Plot pp=new Plot(a[0],a[1]);
  pp.setPlabel("Polynomial Singular Value Decomposition Least Square Curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y=f(x)");
  pp.setPlotType(0,22);
  pp.addData(z[0],z[1]);
  pp.setPlotType(1,0);
  pp.setGrid(1,1);
  pp.plot();
  }
}
```

Girdi verisi:

```
0 1
.25 1.2840
0.5 1.6487
0.75 2.1170
1 2.7183
```

Output 9.11-1 Polynomial least square application

```
---------- Capture Output ----------
> "C:\turhan\java\bin\javaw.exe" SVDLeastsquare
A=          1.000000000000000       0.000000000000000       0.000000000000000
            1.000000000000000       0.250000000000000       0.062500000000000
            1.000000000000000       0.500000000000000       0.250000000000000
            1.000000000000000       0.750000000000000       0.562500000000000
            1.000000000000000       1.000000000000000       1.000000000000000

ATA=        5.000000000000000       2.500000000000000       1.875000000000000
            2.500000000000000       1.875000000000000       1.562500000000000
            1.875000000000000       1.562500000000000       1.382812500000000

c=          7.353236163159007       0.878108943148838       0.026467393692153
            0.798724502630908      -0.592863568473215       0.102722724234778
            0.471216204284530       0.510173089573159      -0.719498928070819
            0.374158332416491       0.623086035674515       0.686854667619525

S=          2.711685115045441       0.000000000000000       0.000000000000000
            0.000000000000000       0.937074673197840       0.000000000000000
            0.000000000000000       0.000000000000000       0.162688025656939

invS=       0.368774381085631       0.000000000000000       0.000000000000000
            0.000000000000000       1.067150813699214       0.000000000000000
            0.000000000000000       0.000000000000000       6.146733885066024

VT=         0.798724502630908       0.471216204284530       0.374158332416491
           -0.592863568473215       0.510173089573159       0.623086035674515
            0.102722724234778      -0.719498928070819       0.686854667619525

U=          0.294549134115641      -0.632674839508811       0.631409249820203
            0.346616000605336      -0.455009909469034      -0.210363306816900
            0.415930368028146      -0.194227333182423      -0.524396756007415
            0.606301605673112       0.576683558131304       0.430753667951314

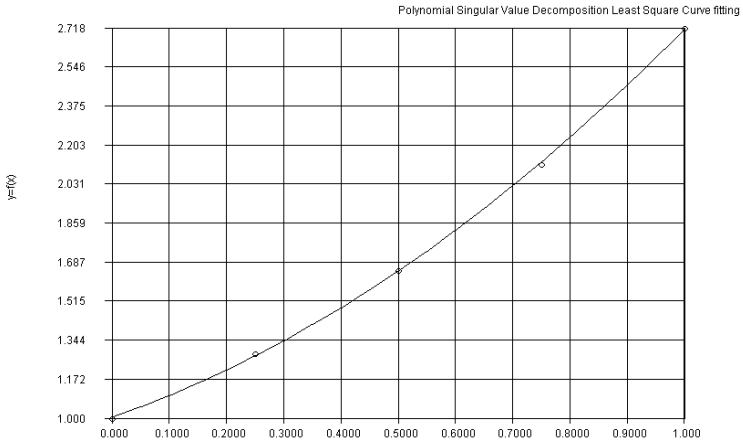
c=      4.13723419578719600000000000        0.34732169503952750000000000        0.00991447389033983700000000
0.00000000000000000000000000        0.00000000000000000000000000

z=      1.52570597995773040000000000        0.37064462947682210000000000        0.06094163261435424400000000

a=      1.00513714285714000000000000        0.86418285714286660000000000        0.84365714285713820000000000

coefficients :
      1.00513714285714000000000000
      0.86418285714286660000000000
      0.84365714285713820000000000
```

```
Polynomial Least Square Curve fitting
  0.0000000000   1.0051371429
  0.0500000000   1.0504554286
  0.1000000000   1.0999920000
  0.1500000000   1.1537468571
  0.2000000000   1.2117200000
  0.2500000000   1.2739114286
  0.3000000000   1.3403211429
  0.3500000000   1.4109491429
  0.4000000000   1.4857954286
  0.4500000000   1.5648600000
  0.5000000000   1.6481428571
  0.5500000000   1.7356440000
  0.6000000000   1.8273634286
  0.6500000000   1.9233011429
  0.7000000000   2.0234571429
  0.7500000000   2.1278314286
  0.8000000000   2.2364240000
  0.8500000000   2.3492348571
  0.9000000000   2.4662640000
  0.9500000000   2.5875114286
  1.0000000000   2.7129771429
```



The same least square concept can be apply to a more general linear least square formula as well. If a general least square equation is defined as

$$f(x) = \sum_{i=0}^{n} a_i \phi_i(x)$$

Then A matrix becomes:

$$A = \begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \phi_2(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_n(x_1) \\ \dots & \dots & .. & \dots & \dots \\ \phi_0(x_{m-1}) & \phi_1(x_{m-1}) & \phi_2(x_{m-1}) & \dots & \phi_n(x_{m-1}) \\ \phi_0(x_m) & \phi_1(x_m) & \phi_2(x_m) & \dots & \phi_n(x_m) \end{bmatrix}$$

The remaining of the problem will be similar to the previous case of polynomial least square, since this definition is more general, it will also cover the polynomial least square as well ($\phi_0(x) = 1,\ \phi_1(x) = x,\ \phi_2(x) = x^2, \dots$). A modified version of the program is given for this case. As an example the same polynomial least square problem is solved.

Program General linear least square solution by using singular value decomposition

```
import java.io.*;
import java.util.*;
import javax.swing.*;

abstract class f_xr
{  abstract double func(double x,int equation_ref);}
```

```
class fa extends f_xr
{
            double func(double x,int i)
            {
    double xx=1.0;
     if(i==0) xx=1.0;
     else if(i==1) xx=x;
     else if(i==2) xx=x*x;
     else if(i==3) xx=x*x*x;
     else if(i==4) xx=x*x*x*x;
     else if(i==5) xx=x*x*x*x*x;
     else if(i==6) xx=x*x*x*x*x*x;
     else if(i==7) xx=x*x*x*x*x*x*x;
     else if(i==8) xx=x*x*x*x*x*x*x*x;
     else if(i==9) xx=x*x*x*x*x*x*x*x*x;

  return xx;
            }
}

class SVDGeneralleastsquare
{

public static double[][] Transpose(double [][] left)
{
//transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
return b;
}

public static double[][] T(double [][] left)
{return Transpose(left);}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
    b[i][i]=1.0;
return b;
}
//Converts vector to diagonal values of matrix
public static double[][] S(double[] left,int m)
{
int n=left.length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
    b[i][i]=left[i];
return b;
}

//Converts double to diagonal values of matrix
public static double[][] S(double left,int n)
{
```

```
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
   b[i][i]=left;
return b;
}

//takes invert value of a diagonal matrix
public static double[][] invS(double[][] S)
{
int m=S.length;
int n=S[0].length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
   {  if(S[i][i]==0) b[i][i]=0;
               else b[i][i]=1.0/S[i][i];
             }
return b;
}

//square root of a diagonal matrix
public static double[][] sqrtS(double[] left,int m)
{
int n=left.length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
   b[i][i]=Math.sqrt(left[i]);
return b;
}
//multiply two matrices
public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[n1][m2];
 if(m1 !=n2 )
 {
 System.out.println("inner matrix dimensions must agree"+"m1="+m1+"n2="+n2);
 return b;
 }
 for(i=0;i<n1;i++)
   {
   for(j=0;j<m2;j++)
     {
     for(k=0;k<m1;k++)
                {b[i][j]+=left[i][k]*right[k][j];
                }
     }
   }
return b;
//end of multiply of two matrices
}

//multiply a matrix with a double value
public static double[][] multiply(double[][] left,double right)
{
//multiplying a matrix with a constant
int i,j;
int n=left.length;
int m=left[0].length;
```

```
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=left[i][j]*right;
  }
return b;
//end of multiplying a matrix with a constant double
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int n1=left.length;
int m1=left[0].length;

int m2=right.length;
double[] b;
b=new double[m2];
 for(i=0;i<n1;i++)
   {
    b[i]=0;
    for(k=0;k<m1;k++)
            {b[i]+=left[i][k]*right[k];
            }
   }
return b;
//end of multiply of a matrix and a vector
}

//substract two matrices
public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
  for(j=0;j<m1;j++)
    {
    b[i][j]=b[i][j]+left[i][j];
    }
  }
for(i=0;i<n2;i++)
  {
  for(j=0;j<m2;j++)
    {
    b[i][j]=b[i][j]-right[i][j];
    }
  }
return b;
//end of matrix substraction method
}

public static double[] SVD_LSQ(f_xr f,double xi[],double yi[],int n_leastsquare)
```

```
{ // Singular value decomposition least square approximation
  double U[][];
  double V[][];
  double S[][];
  double invS[][];
  double A[][]=A(f,xi,n_leastsquare);
  int m=A.length;
  int m1=A[0].length;
  System.out.println("A="+Matrix.toString(A));
  double ATA[][]=multiply(Transpose(A),A);
  System.out.println("ATA="+Matrix.toString(ATA));
  //double aa[][]=Householder(ATA);
  double cc[][];
  cc=eigen_jacobi(ATA);//eigenQL(aa);
   System.out.println("c="+Matrix.toString(cc));
  int n=cc[0].length;
  int n1=cc.length;
  double dummy[]=new double[n1];
  for(int i=0;i<n;i++)
  {for(int k=i+1;k<n;k++)
    {if(cc[0][i]<cc[0][k])
      { for(int j=0;j<n1;j++)
        { dummy[j]=cc[j][k];
          cc[j][k]=cc[j][i];
          cc[j][i]=dummy[j];
        }
      }
    }
  }
  double x[]=new double[n];
  for(int i=0;i<n;i++)
  {x[i]=cc[0][i];}
  S=sqrtS(x,m1);
  System.out.println("S="+Matrix.toString(S));
  invS=invS(S);
  System.out.println("invS="+Matrix.toString(invS));
  double xx[][]=new double[n][n];
  V=new double[n][n];
  U=new double[n][n];
  double L[]=new double[n];

  for(int i=0;i<n;i++)
  { for(int j=0;j<n;j++)
    {xx[i][j]=cc[i+1][j];}
  }

  double bb[]=new double[n];
  for(int j=0;j<n;j++)
  { for(int i=0;i<n;i++)
    {L[j]+=xx[i][j]*xx[i][j];}
    L[j]=Math.sqrt(L[j]);
  }
  for(int i=0;i<n;i++)
  { for(int j=0;j<n;j++)
    {V[i][j]=xx[i][j]/L[i];}
  }
  System.out.println("VT="+Matrix.toString(T(V)));
  U=multiply(A,multiply(V,invS));
   System.out.println("U="+Matrix.toString(U));
  double c[]=multiply(T(U),yi);
   System.out.println("c="+Matrix.toString(c));
   double z[]=new double[n];
   for(int i=0;i<n;i++) z[i]=c[i]/S[i][i];
   System.out.println("z="+Matrix.toString(z));
   double a[]=multiply(V,z);
   System.out.println("a="+Matrix.toString(a));
   return a;
```

```
}
public static double func(f_xr f,double e[],double x)
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=0;
  for(int i=0;i<e.length;i++)
   { ff+=e[i]*f.func(x,i); }
  }
else
  ff=0;
return ff;
}

public static double error(f_xr f,double x[],double y[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(y[k]-func(f,e,x[k]))*(y[k]-func(f,e,x[k]));
 }
total=Math.sqrt(total);
return total;
}

public static double[][]  funcSVD_LSQ(f_xr f,double xi[],double yi[],int polinomkatsayisi,int aradegersayisi)
 {
              //aradegersayisi:  x--o--o--x--o--o--x zincirinde x deneysel noktalar ise
              // ara değer sayısı 2 dir
               int n=xi.length;
              int nn=(n-1)*(aradegersayisi+1)+1;
              double z[][]=new double[2][nn];
   double E[]=SVD_LSQ(f,xi,yi,polinomkatsayisi);
   System.out.println("coefficients :\n"+Matrix.toStringT(E));
   double dx=0;
   int k=0;
   int i;
   for(i=0;i<(n-1);i++)
      {z[0][k]=xi[i];z[1][k]=func(f,E,z[0][k]);k++;
      for(int j=0;j<aradegersayisi;j++)
      {dx=(xi[i+1]-xi[i])/((double)aradegersayisi+1.0);
      z[0][k]=z[0][k-1]+dx;z[1][k]=func(f,E,z[0][k]);k++;}
      }
      z[0][k]=xi[i];z[1][k]=func(f,E,z[0][k]);
   return z;
 }

  public static void print(String s)
  {JOptionPane.showMessageDialog(null,s,
    "Singular value decomposition : ",JOptionPane.PLAIN_MESSAGE);}

  public static double[][] eigen_jacobi(double A[][])
  {             //Eigen value calculation
   double eps=1e-15;
   double err=1;
   int m=100;
   int n=A.length;
   double P[][]=new double[n][n];
   double R[][]=new double[n][n];
   int p=n,q=n;
   double max=-9.99e100;
```

```
 double fi;
 P=I(n);
 for(int i=0;i<n;i++)
 {for(int j=i+1;j<n;j++)
    {if(Math.abs(A[i][j]) > max) {max=Math.abs(A[i][j]);p=i;q=j;}}
 }
 while(Math.abs(A[p][q]) > eps )
 {
  fi=0.5*Math.atan2(2.0*A[p][q],(A[p][p]-A[q][q]));
  R=I(n);
  R[p][p]=Math.cos(fi);
  R[q][q]=Math.cos(fi);
  R[p][q]=-Math.sin(fi);
  R[q][p]=Math.sin(fi);
  P=multiply(P,R);
  A=multiply(T(R),multiply(A,R));
  max=-9.99e100;
  for(int i=0;i<n;i++)
 {for(int j=i+1;j<n;j++){if(Math.abs(A[i][j])>max) {max=Math.abs(A[i][j]);p=i;q=j;}}}
  }
 double lambda[]=new double[n];
 for(int i=0;i<n;i++) {lambda[i]=A[i][i];}
 double PP[][]=new double[n+1][n];
 PP[0]=lambda;
 for(int i=0;i<n;i++)
 {for(int j=0;j<n;j++)
   {PP[i+1][j]=P[i][j];}
 }
 return PP;
}

public static double[] gausswithpartialpivot(double a[][],double b[])
{ // Gauss elimination with partial pivoting
int n=b.length;
double x[]=new double[n];
double carpan=0;
double toplam=0;
double buyuk;
double dummy=0;
//gauss elimination
int i,j,k,p,ii,jj;
for(k=0;k<(n-1);k++)
{ //partial pivoting
            p=k;
            buyuk=Math.abs(a[k][k]);
            for(ii=k+1;ii<n;ii++)
            {  dummy=Math.abs(a[ii][k]);
               if(dummy > buyuk) {buyuk=dummy;p=ii;}
            }
            if(p!=k)
            { for(jj=k;jj<n;jj++)
              { dummy=a[p][jj];
                a[p][jj]=a[k][jj];
                a[k][jj]=dummy;
              }
              dummy=b[p];
              b[p]=b[k];
              b[k]=dummy;
            }
            //
  for(i=k+1;i<n;i++)
  {  carpan=a[i][k]/a[k][k];
     a[i][k]=0;
     for(j=k+1;j<n;j++)
     {   a[i][j]-=carpan*a[k][j]; }
         b[i]   =b[i]   -carpan*b[k];
            }
```

```
  }
  // backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[][] A(f_xr f,double xi[],int n_leastsquare)
{
int n=xi.length;
int m=n_leastsquare+1;
int l=xi.length;
int i,j,k;
double A[][];
A=new double[n][m];
double B[];
B=new double[n];
double X[];
X=new double[n];
double cc=1.0;
for(i=0;i<n;i++)
  {  for(j=0;j<m;j++) { A[i][j]=f.func(xi[i],j);}}
return A;
}

  public static void main(String args[]) throws IOException
  {
  double x[];
  double y[];
  String s1=JOptionPane.showInputDialog("Data file name (a.txt)");
  // JFileChooser fc=new JFileChooser();
  // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file =
fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  x=a[0];
  y=a[1];
  int n=Integer.parseInt(JOptionPane.showInputDialog("polinom derecesi (n)"));
  fa f1=new fa();
  double z[][]=funcSVD_LSQ(f1,x,y,n,4);
  System.out.println("Polynomial Least Square Curve fitting\n"+Matrix.toStringT(z));
  Plot pp=new Plot(a[0],a[1]);
  pp.setPlabel("Polynomial Singular Value Decomposition Least Square Curve fitting");
  pp.setXlabel("x");
  pp.setYlabel("y=f(x)");
  pp.setPlotType(0,22);
  pp.addData(z[0],z[1]);
  pp.setPlotType(1,0);
  pp.setGrid(1,1);
  pp.plot();
  }
}
```

```
---------- Capture Output ----------
> "D:\java\bin\javaw.exe" SVDGeneralleastsquare
A=      1.000000000000000       0.000000000000000       0.000000000000000
        1.000000000000000       0.250000000000000       0.062500000000000
        1.000000000000000       0.500000000000000       0.250000000000000
        1.000000000000000       0.750000000000000       0.562500000000000
        1.000000000000000       1.000000000000000       1.000000000000000

ATA=        5.000000000000000       2.500000000000000       1.875000000000000
        2.500000000000000       1.875000000000000       1.562500000000000
```

```
      1.875000000000000       1.562500000000000       1.382812500000000

c=       7.353236163159007       0.878108943148838       0.026467393692153
       0.798724502630908      -0.592863568473215       0.102722724234778
       0.471216204284530       0.510173089573159      -0.719498928070819
       0.374158332416491       0.623086035674515       0.686854667619525

S=       2.711685115045441       0.000000000000000       0.000000000000000
       0.000000000000000       0.937074673197840       0.000000000000000
       0.000000000000000       0.000000000000000       0.162688025656939

invS=       0.368774381085631       0.000000000000000       0.000000000000000
       0.000000000000000       1.067150813699214       0.000000000000000
       0.000000000000000       0.000000000000000       6.146733885066024

VT=       0.798724502630908       0.471216204284530       0.374158332416491
      -0.592863568473215       0.510173089573159       0.623086035674515
       0.102722724234778      -0.719498928070819       0.686854667619525

U=       0.294549134115641      -0.632674839508811       0.631409249820203
       0.346616000605336      -0.455009009469034      -0.210363306816900
       0.415930368028146      -0.194227333182423      -0.524396756007415
       0.502492236384071       0.149670189351023      -0.310691097751344
       0.606301605673112       0.576683558131304       0.430753667951314

c=       4.137234195787196000000000           0.347321695039527500000000000           0.009914473890339837000000
0.000000000000000000000000000       0.000000000000000000000000000

z=       1.525705979957730400000000           0.370644629476822100000000000           0.060941632614354244000000

a=       1.005137142857140000000000           0.864182857142866600000000000           0.843657142857138200000000

coefficients :
       1.005137142857140000000000
       0.864182857142866600000000000
       0.843657142857138200000000

Polynomial Least Square Curve fitting
   0.0000000000   1.0051371429
   0.0500000000   1.0504554286
   0.1000000000   1.0999920000
   0.1500000000   1.1537468571
   0.2000000000   1.2117200000
   0.2500000000   1.2739114286
   0.3000000000   1.3403211429
   0.3500000000   1.4109491429
   0.4000000000   1.4857954286
   0.4500000000   1.5648600000
   0.5000000000   1.6481428571
   0.5500000000   1.7356440000
   0.6000000000   1.8273634286
   0.6500000000   1.9233011429
   0.7000000000   2.0234571429
   0.7500000000   2.1278314286
   0.8000000000   2.2364240000
   0.8500000000   2.3492348571
   0.9000000000   2.4662640000
   0.9500000000   2.5875114286
   1.0000000000   2.7129771429
```

As the last example let us look at SVD polynomial curve fitting for two dimensional case (can be applied to multidimensions) Curve fitting data is created by using the function

$f(x, y) = 1 + x + y + xy + x^2 + y^2$

0	0	1	0	0.3	1.39	0	0.6	1.96	0	1	3
0.1	0	1.11	0.1	0.3	1.53	0.1	0.6	2.13	0.1	1	3.21
0.2	0	1.24	0.2	0.3	1.69	0.2	0.6	2.32	0.2	1	3.44
0.3	0	1.39	0.3	0.3	1.87	0.3	0.6	2.53	0.3	1	3.69
0.4	0	1.56	0.4	0.3	2.07	0.4	0.6	2.76	0.4	1	3.96
0.5	0	1.75	0.5	0.3	2.29	0.5	0.6	3.01	0.5	1	4.25
0.6	0	1.96	0.6	0.3	2.53	0.6	0.6	3.28	0.6	1	4.56
0.7	0	2.19	0.7	0.3	2.79	0.7	0.6	3.57	0.7	1	4.89
0.8	0	2.44	0.8	0.3	3.07	0.8	0.6	3.88	0.8	1	5.24
0.9	0	2.71	0.9	0.3	3.37	0.9	0.6	4.21	0.9	1	5.61
1	0	3	1	0.3	3.69	1	0.6	4.56	1	1	6
0	0.1	1.11	0	0.4	1.56	0	0.7	2.19			

0.1	0.1	1.23	0.1	0.4	1.71	0.1	0.8	2.63	
0.2	0.1	1.37	0.2	0.4	1.88	0.2	0.8	2.84	
0.3	0.1	1.53	0.3	0.4	2.07	0.3	0.8	3.07	
0.4	0.1	1.71	0.4	0.4	2.28	0.4	0.8	3.32	
0.5	0.1	1.91	0.5	0.4	2.51	0.5	0.8	3.59	
0.6	0.1	2.13	0.6	0.4	2.76	0.6	0.8	3.88	
0.7	0.1	2.37	0.7	0.4	3.03	0.7	0.8	4.19	
0.8	0.1	2.63	0.8	0.4	3.32	0.8	0.8	4.52	
0.9	0.1	2.91	0.9	0.4	3.63	0.9	0.8	4.87	
1	0.1	3.21	1	0.4	3.96	.0	0.9	2.71	
0	0.2	1.24	0	0.5	1.75	0.1	0.9	2.91	
0.1	0.2	1.37	0.1	0.5	1.91	0.2	0.9	3.13	
0.2	0.2	1.52	0.2	0.5	2.09	0.3	0.9	3.37	
0.3	0.2	1.69	0.3	0.5	2.29	0.4	0.9	3.63	
0.4	0.2	1.88	0.4	0.5	2.51	0.5	0.9	3.91	
0.5	0.2	2.09	0.5	0.5	2.75	0.6	0.9	4.21	
0.6	0.2	2.32	0.6	0.5	3.01	0.7	0.9	4.53	
0.7	0.2	2.57	0.7	0.5	3.29	0.8	0.9	4.87	
0.8	0.2	2.84	0.8	0.5	3.59	0.9	0.9	5.23	
0.9	0.2	3.13	0.9	0.5	3.91	1	0.9	5.61	
1	0.2	3.44	1	0.5	4.25		0.8	5.24	

Program General multidimensional linear least square solution by using singular value decomposition

```
import java.io.*;
import java.util.*;
import javax.swing.*;

// surface curve fitting
//f(x,y)=a0+a1*x+a2*y+a3*x^2+a4*y^2+a5*xy+a6*x^3+a7*y^3+a8*x^2y
//      +a9*xy^2+a10*x^4+a11*y^4+a12*x^2y^2+a13x^3y+a14*xy^3
abstract class f_xir
{
  abstract double func(double x[],int equation_ref);
}

class fa extends f_xir
{
              double func(double x[],int i)
              { // a general surface curve fitting model
   double xx=0.0;
   //density function
   if(i==0)      {xx=1.0;          }
   else if (i==1) {xx=x[0];         }
   else if (i==2) {xx=x[1];         }
   else if (i==3) {xx=x[0]*x[0];    }
   else if (i==4) {xx=x[1]*x[1];    }
   else if (i==5) {xx=x[0]*x[1];    }
   else if (i==6) {xx=x[0]*x[0]*x[0];}
   else if (i==7) {xx=x[1]*x[1]*x[1];}
   else if (i==8) {xx=x[0]*x[0]*x[1];}
   else if (i==9) {xx=x[0]*x[1]*x[1];}
   else if (i==10){xx=x[0]*x[0]*x[0]*x[0];}
   else if (i==11){xx=x[1]*x[1]*x[1]*x[1];}
   else if (i==12){xx=x[0]*x[1]*x[1]*x[1];}
   else if (i==13){xx=x[0]*x[0]*x[1]*x[1];}
   else if (i==14){xx=x[0]*x[0]*x[0]*x[1];}
   else if (i==15){xx=x[0]*x[0]*x[0]*x[1];}
   return xx;
              }
}


class SVDGeneralleastsquare1
{

public static double[][] Transpose(double [][] left)
{
//transpose matrix (if A=a(i,j) Transpose(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
for(i=0;i<n;i++)
  {
```

```
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
return b;
}

public static double[][] T(double [][] left)
{return Transpose(left);}

public static double[][] I(int n)
{
//unit matrix
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
    b[i][i]=1.0;
return b;
}
//Converts vector to diagonal values of matrix
public static double[][] S(double[] left,int m)
{
int n=left.length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
    b[i][i]=left[i];
return b;
}

//Converts double to diagonal values of matrix
public static double[][] S(double left,int n)
{
double b[][];
b=new double[n][n];
for(int i=0;i<n;i++)
    b[i][i]=left;
return b;
}

//takes invert value of a diagonal matrix
public static double[][] invS(double[][] S)
{
int m=S.length;
int n=S[0].length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
    {  if(S[i][i]==0) b[i][i]=0;
                else b[i][i]=1.0/S[i][i];
              }
return b;
}

//square root of a diagonal matrix
public static double[][] sqrtS(double[] left,int m)
{
int n=left.length;
double b[][];
b=new double[m][n];
for(int i=0;i<n;i++)
    b[i][i]=Math.sqrt(left[i]);
return b;
}
//multiply two matrices
public static double[][] multiply(double[][] left,double[][] right)
{
//multiplication of two matrices
int ii,jj,i,j,k;
int m1=left[0].length;
int n1=left.length;
int m2=right[0].length;
int n2=right.length;
double[][] b;
b=new double[n1][m2];
```

```
 if(m1 !=n2 )
 {
 System.out.println("inner matrix dimensions must agree"+"m1="+m1+"n2="+n2);
 return b;
 }
 for(i=0;i<n1;i++)
   {
   for(j=0;j<m2;j++)
     {
     for(k=0;k<m1;k++)
                 {b[i][j]+=left[i][k]*right[k][j];
                 }
     }
   }
return b;
//end of multiply of two matrices
}

//multiply a matrix with a double value
public static double[][] multiply(double[][] left,double right)
{
//multiplying a matrix with a constant
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[n][m];
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    b[i][j]=left[i][j]*right;
  }
return b;
//end of multiplying a matrix with a constant double
}

public static double[] multiply(double[][] left,double[] right)
{
//multiplication of one matrix with one vector
int ii,jj,i,j,k;
int n1=left.length;
int m1=left[0].length;

int m2=right.length;
double[] b;
b=new double[m2];
 for(i=0;i<n1;i++)
   {
     b[i]=0;
     for(k=0;k<m1;k++)
                 {b[i]+=left[i][k]*right[k];
                 }
   }
return b;
//end of multiply of a matrix and a vector
}

//substract two matrices
public static double[][] substract(double[][] left,double[][] right)
{
//addition of two matrices
int n1=left.length;
int m1=left[0].length;
int n2=right.length;
int m2=right[0].length;
int nMax,mMax;
int i,j;
if(m1>=m2) mMax=m1;
else       mMax=m2;
if(n1>=n2) nMax=n1;
else       nMax=n2;
double b[][];
b=new double[nMax][mMax];
for(i=0;i<n1;i++)
  {
  for(j=0;j<m1;j++)
```

```
    {
    b[i][j]=b[i][j]+left[i][j];
    }
  }
for(i=0;i<n2;i++)
  {
  for(j=0;j<m2;j++)
    {
    b[i][j]=b[i][j]-right[i][j];
    }
  }
return b;
//end of matrix substraction method
}
public static double[] SVD_LSQ(f_xir f,double xi[][],double yi[],int n_leastsquare)
{ // Singular value decomposition least square approximation
  double U[][];
  double V[][];
  double S[][];
  double invS[][];
  double A[][]=A(f,xi,n_leastsquare);
  int m=A.length;
  int m1=A[0].length;
  System.out.println("A="+Matrix.toString(A));
  double ATA[][]=multiply(Transpose(A),A);
  System.out.println("ATA="+Matrix.toString(ATA));
  //double aa[][]=Householder(ATA);
  double cc[][];
  cc=eigen_jacobi(ATA);//eigenQL(aa);
   System.out.println("c="+Matrix.toString(cc));
  int n=cc[0].length;
  int n1=cc.length;
  double dummy[]=new double[n1];
  for(int i=0;i<n;i++)
  {for(int k=i+1;k<n;k++)
    {if(cc[0][i]<cc[0][k])
      { for(int j=0;j<n1;j++)
        { dummy[j]=cc[j][k];
          cc[j][k]=cc[j][i];
          cc[j][i]=dummy[j];
        }
      }
    }
  }
  double x[]=new double[n];
  for(int i=0;i<n;i++)
  {x[i]=cc[0][i];}
  S=sqrtS(x,m1);
  System.out.println("S="+Matrix.toString(S));
  invS=invS(S);
  System.out.println("invS="+Matrix.toString(invS));
  double xx[][]=new double[n][n];
  V=new double[n][n];
  U=new double[n][n];
  double L[]=new double[n];

  for(int i=0;i<n;i++)
  { for(int j=0;j<n;j++)
    {xx[i][j]=cc[i+1][j];}
  }

  double bb[]=new double[n];
  for(int j=0;j<n;j++)
  { for(int i=0;i<n;i++)
    {L[j]+=xx[i][j]*xx[i][j];}
    L[j]=Math.sqrt(L[j]);
  }
  for(int i=0;i<n;i++)
  { for(int j=0;j<n;j++)
    {V[i][j]=xx[i][j]/L[i];}
  }
  System.out.println("VT="+Matrix.toString(T(V)));
  U=multiply(A,multiply(V,invS));
   System.out.println("U="+Matrix.toString(U));
  double c[]=multiply(T(U),yi);
   System.out.println("c="+Matrix.toString(c));
```

```
   double z[]=new double[n];
   for(int i=0;i<n;i++) z[i]=c[i]/S[i][i];
   System.out.println("z="+Matrix.toString(z));
   double a[]=multiply(V,z);
   System.out.println("a="+Matrix.toString(a));
   return a;
}
public static double func(f_xir f,double e[],double x[])
{
// this function calculates the value of
// least square curve fitting function
int n=e.length;
double ff;
if(n!=0.0)
  { ff=0;
  for(int i=0;i<e.length;i++)
   { ff+=e[i]*f.func(x,i); }
  }
else
  ff=0;
return ff;
}

public static double error(f_xir f,double x[][],double z[],double e[])
{
//calculates absolute square root error of a least square approach
double n=x.length;
 int k;
 double xt[][]=T(x);
 double total=0;
 for(k=0;k<n;k++)
 {
 total+=(z[k]-func(f,e,xt[k]))*(z[k]-func(f,e,xt[k]));
 }
total=Math.sqrt(total);
return total;
}
  public static void print(String s)
  {JOptionPane.showMessageDialog(null,s,
     "Singular value decomposition : ",JOptionPane.PLAIN_MESSAGE);}
  public static double[][] eigen_jacobi(double A[][])
  {               //Eigen value calculation
   double eps=1e-15;
   double err=1;
   int m=100;
   int n=A.length;
   double P[][]=new double[n][n];
   double R[][]=new double[n][n];
   int p=n,q=n;
   double max=-9.99e100;
   double fi;
   P=I(n);
   for(int i=0;i<n;i++)
   {for(int j=i+1;j<n;j++)
      {if(Math.abs(A[i][j]) > max) {max=Math.abs(A[i][j]);p=i;q=j;}}
   }
   while(Math.abs(A[p][q]) > eps )
   {
    fi=0.5*Math.atan2(2.0*A[p][q],(A[p][p]-A[q][q]));
    R=I(n);
    R[p][p]=Math.cos(fi);
    R[q][q]=Math.cos(fi);
    R[p][q]=-Math.sin(fi);
    R[q][p]=Math.sin(fi);
    P=multiply(P,R);
    A=multiply(T(R),multiply(A,R));
    max=-9.99e100;
    for(int i=0;i<n;i++)
   {for(int j=i+1;j<n;j++){if(Math.abs(A[i][j])>max) {max=Math.abs(A[i][j]);p=i;q=j;}}}
    }
   double lambda[]=new double[n];
   for(int i=0;i<n;i++) {lambda[i]=A[i][i];}
   double PP[][]=new double[n+1][n];
   PP[0]=lambda;
   for(int i=0;i<n;i++)
   {for(int j=0;j<n;j++)
```

```
    {PP[i+1][j]=P[i][j];}
   }
   return PP;
  }

  public static double[] gausswithpartialpivot(double a[][],double b[])
  { // Gauss elimination with partial pivoting
  int n=b.length;
  double x[]=new double[n];
  double carpan=0;
  double toplam=0;
  double buyuk;
  double dummy=0;
  //gauss elimination
  int i,j,k,p,ii,jj;
  for(k=0;k<(n-1);k++)
  { //partial pivoting
                p=k;
                buyuk=Math.abs(a[k][k]);
                for(ii=k+1;ii<n;ii++)
                {  dummy=Math.abs(a[ii][k]);
                   if(dummy > buyuk) {buyuk=dummy;p=ii;}
                }
                if(p!=k)
                { for(jj=k;jj<n;jj++)
                  { dummy=a[p][jj];
                    a[p][jj]=a[k][jj];
                    a[k][jj]=dummy;
                  }
                  dummy=b[p];
                  b[p]=b[k];
                  b[k]=dummy;
                }
                //
    for(i=k+1;i<n;i++)
    {  carpan=a[i][k]/a[k][k];
       a[i][k]=0;
       for(j=k+1;j<n;j++)
       {   a[i][j]-=carpan*a[k][j]; }
           b[i]   =b[i]   -carpan*b[k];
                }
  }
  // backward substitution
  x[n-1]=b[n-1]/a[n-1][n-1];
  for(i=n-2;i>=0;i--)
  {
    toplam=0;
    for(j=i+1;j<n;j++)
    {  toplam+=a[i][j]*x[j];}
    x[i]=(b[i]-toplam)/a[i][i];
  }
  return x;
}

public static double[][] A(f_xir f,double xi[][],int n_leastsquare)
{
int n=xi[0].length;
double txi[][]=T(xi);
int m=n_leastsquare+1;
int l=xi.length;
int i,j,k;
double A[][];
A=new double[n][m];
double cc=1.0;
for(i=0;i<n;i++)
  {  for(j=0;j<m;j++) { A[i][j]=f.func(txi[i],j);}}
return A;
}
public static double[][] MT(double [][] left)
{
//transpose matrix (if A=a(i,j) T(A)=a(j,i)
int i,j;
int n=left.length;
int m=left[0].length;
double b[][];
b=new double[m][n];
```

```
for(i=0;i<n;i++)
  {
  for(j=0;j<m;j++)
    {
    b[j][i]=left[i][j];
    }
  }
return b;
}

  public static void main(String args[]) throws IOException
  {
  double z[];
  String s1=JOptionPane.showInputDialog("Data file name (a.txt)");
  // JFileChooser fc=new JFileChooser();
  // if (fc.showOpenDialog(null) == JFileChooser.APPROVE_OPTION) {File file = fc.getSelectedFile();s1=file.getName(); }
  double a[][]=Text.readDoubleT(s1);
  double x[][]={a[0],a[1]};
  z=a[2];
  int n=Integer.parseInt(JOptionPane.showInputDialog("polinom derecesi (n)"));
  fa f1=new fa();
  double A[][]=A(f1,x,n);
  double E[]=SVD_LSQ(f1,x,z,n);
  double x1[]={0.5,0.5};
  double z1=func(f1,E,x1);
  System.out.println("z="+z1);
  }
}
```

```
---------- Capture Output ----------
> "E:\co\java\bin\java.exe" SVDGeneralleastsquare1
A=      1.000000000000000       0.000000000000000       0.000000000000000       0.000000000000000       0.000000000000000
0.000000000000000       0.000000000000000
       1.000000000000000       0.100000000000000       0.000000000000000       0.010000000000000       0.000000000000000
0.000000000000000       0.001000000000000
       1.000000000000000       0.200000000000000       0.000000000000000       0.040000000000000       0.000000000000000
0.000000000000000       0.008000000000000
       1.000000000000000       0.300000000000000       0.000000000000000       0.090000000000000       0.000000000000000
0.000000000000000       0.027000000000000
       1.000000000000000       0.400000000000000       0.000000000000000       0.160000000000000       0.000000000000000
0.000000000000000       0.064000000000000
       1.000000000000000       0.500000000000000       0.000000000000000       0.250000000000000       0.000000000000000
0.000000000000000       0.125000000000000
       1.000000000000000       0.600000000000000       0.000000000000000       0.360000000000000       0.000000000000000
0.000000000000000       0.216000000000000
       1.000000000000000       0.700000000000000       0.000000000000000       0.490000000000000       0.000000000000000
0.000000000000000       0.343000000000000
       1.000000000000000       0.800000000000000       0.000000000000000       0.640000000000000       0.000000000000000
0.000000000000000       0.512000000000000
       1.000000000000000       0.900000000000000       0.000000000000000       0.810000000000000       0.000000000000000
0.000000000000000       0.729000000000000
       1.000000000000000       1.000000000000000       0.000000000000000       1.000000000000000       0.000000000000000
0.000000000000000       1.000000000000000
       1.000000000000000       0.000000000000000       0.100000000000000       0.000000000000000       0.010000000000000
0.000000000000000       0.000000000000000
       1.000000000000000       0.100000000000000       0.100000000000000       0.010000000000000       0.010000000000000
0.010000000000000       0.001000000000000
       1.000000000000000       0.200000000000000       0.100000000000000       0.040000000000000       0.010000000000000
0.020000000000000       0.008000000000000
       1.000000000000000       0.300000000000000       0.100000000000000       0.090000000000000       0.010000000000000
0.030000000000000       0.027000000000000
       1.000000000000000       0.400000000000000       0.100000000000000       0.160000000000000       0.010000000000000
0.040000000000000       0.064000000000000
       1.000000000000000       0.500000000000000       0.100000000000000       0.250000000000000       0.010000000000000
0.050000000000000       0.125000000000000
       1.000000000000000       0.600000000000000       0.100000000000000       0.360000000000000       0.010000000000000
0.060000000000000       0.216000000000000
       1.000000000000000       0.700000000000000       0.100000000000000       0.490000000000000       0.010000000000000
0.070000000000000       0.343000000000000
       1.000000000000000       0.800000000000000       0.100000000000000       0.640000000000000       0.010000000000000
0.080000000000000       0.512000000000000
       1.000000000000000       0.900000000000000       0.100000000000000       0.810000000000000       0.010000000000000
0.090000000000000       0.729000000000000
       1.000000000000000       1.000000000000000       0.100000000000000       1.000000000000000       0.010000000000000
0.100000000000000       1.000000000000000
```

1.000000000000000 0.000000000000000 0.200000000000000 0.000000000000000 0.040000000000000 0.000000000000000 0.000000000000000
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55.000000000000000 38.499999999999990 26.400000000000000 30.250000000000004 18.480000000000004
18.479999999999997 25.332999999999990
52.700000000000010 26.400000000000000 36.809999999999990 18.479999999999997 29.333000000000000
18.480000000000004 14.520000000000000
38.499999999999990 30.250000000000004 18.479999999999997 25.332999999999990 12.936000000000000
14.520000000000000 22.082500000000000
36.809999999999990 18.480000000000004 29.333000000000000 12.936000000000000 25.055700000000005
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26.400000000000000 18.479999999999997 18.480000000000004 14.520000000000000 14.751000000000001
12.936000000000000 12.159840000000003
30.250000000000004 25.332999999999990 14.520000000000000 22.082500000000000 10.164000000000001
12.159840000000003 19.784050000000000

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0.000000000000000      0.000000000000000          0.000000000000000
```
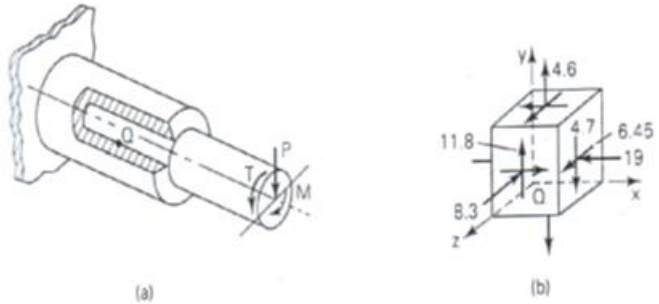

PROBLEM 2

A steel shaft is placed inside of a iron base as shown in the figure . T torsion and M beding moment is applied to the shaft. Under this forces stress matrix at point Q is as follows (MPa)



(a)

(b)

-19	***-4.7***	***6.45***
-4.7	**4.6**	**11.8**
6.45	**11.8**	**-8.3**

Calculate the principle stresses at point Q

Note : Principle stresses are the eigenvalues of the matrix

PROBLEM 3

$A=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$ Calculate the eigenvalues of the matrix

PROBLEM 4

$A=\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$ Calculate the eigenvalues of the matrix

PROBLEM 5

$A=\begin{bmatrix}3 & -2 & 0\\ 2 & 1 & 4\\ 1 & -1 & 2\end{bmatrix}$ Calculate the eigenvalues of the matrix

PROBLEM 6

$A=\begin{bmatrix}4 & -1 & 3 & 9 & -5\\ -1 & 2 & 7 & 12 & 7\\ 3 & 7 & 0 & -11 & 8\\ 2 & -4 & 14 & 1 & -5\\ 11 & -8 & 6 & -2 & 13\end{bmatrix}$ Calculate the eigenvalues of the matrix

PROBLEM 7

$A=\begin{bmatrix}8 & -1 & -5\\ -4 & 4 & -2\\ 18 & -5 & -7\end{bmatrix}$ calculate eigenvalues and eigenvectors of he mtrix

PROBLEM 8

$A=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}=\begin{bmatrix}10 & 4 & -6\\ 4 & -6 & 8\\ -6 & 8 & 14\end{bmatrix}MPa$ Normal and shear stresses are given. Find the principle stresses

PROBLEM 9

Solve the following system of equation by using Singular value decomposition method.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

PROBLEM 10 Solve the following system of equation by using SVD and Gauss elimination methods, compare the results

$$\begin{bmatrix} 8 & -1 & -5 \\ -4 & 4 & -2 \\ 18 & -5 & -7 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

PROBLEM 11 Solve the roots of polynomial $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$ by using companion matrix and eigenvalue.

PROBLEM 12 Solve the roots of polynomial $f(x) = x^3 - 3x^2 + 3x - 1$ by using companion matrix and eigenvalue.

PROBLEM 13 The following data is given. Fit the polynomial curve by using SVD

X	y
0	-2.20
0.1	-1.90
0.2	0
0.3	-0.1
0.4	-0.01
0.5	-0.3
0.6	-0.2
0.7	-0.09
0.8	0.4
0.9	-0.1
1.0	-1.1
1.1	-1.3
1.2	-2
1.3	-1.8
1.4	-0.02
1.5	1

PROBLEM 14 Fit the linear function **$f(x)=a_0*\exp(x)+a_1x+a_2/x$** the data given in problem 12 by using SVD

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