also proposed. This differs from Maxwell's theory in that the latter merely postulates changes of specific resistance and specific inductive capacity from point to point of the dielectric, while our theory is distinctly chemical. We consider that our results on mixed films are best explained by the theory we propose, though the difficulty of disproving Maxwell's theory is almost equal to the difficulty of establishing it, and we do not wish to imply that some sort of explanation on this theory may not be constructed to fit in with our observations. This is a point, however, on which we are still engaged. The matter may, perhaps, be best summed up in the statement that the evidence we have against Maxwell's theory is nearly worthless; but that we do not consider this theory necessary if our theory of conduction be accepted.

II. "On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion." By OSBORNE REYNOLDS, F.R.S., &c. Received April 25, 1894.

(Abstract.)

The equations of motion of viscous fluid (obtained by grafting on certain terms to the abstract equations of the Eulerian form so as to adapt these equations to the case of fluids subject to stresses depending in some hypothetical manner on the rates of distortion, which equations Navier* seems to have first introduced in 1822, and which were much studied by Cauchy† and Poisson‡) were finally shown by St. Venant§ and Sir Gabriel Stokes,|| in 1845, to involve no other assumption than that the stresses, other than that of pressure uniform in all directions, are linear functions of the rates of distortion with a co-efficient depending on the physical state of the fluid.

By obtaining a singular solution of these equations as applied to the case of pendulums in steady periodic motion Sir G. Stokes¶ was able to compare the theoretical results with the numerous experiments that had been recorded, with the result that the theoretical calculations agreed so closely with the experimental determinations as seemingly to prove the truth of the assumption involved. This was also the result of comparing the flow of water through uniform tubes with the flow calculated from a singular solution of the equations so long as the tubes were small and the velocities slow. On the other

* 'Mém. de l'Académie,' t. vi, p. 389.
† 'Mém. des Savants Etrangers,' t. 1, p. 40.
‡ 'Mém. de l'Académie,' t. x, p. 345.
§ 'B.A. Report,' 1846.
|| 'Cambridge Trans.,' 1845.
¶ 'Cambridge Trans.,' vol. ix, 1857.

hand, these results, both theoretical and practical, were directly at variance with common experience as to the resistance encountered by larger bodies moving with higher velocities through water, or by water moving with greater velocities through larger tubes. This discrepancy Sir G. Stokes considered as probably resulting from eddies which rendered the actual motion other than that to which the singular solution referred and not as disproving the assumption.

In 1850, after Joule's discovery of the Mechanical Equivalent of Heat, Stokes showed, by transforming the equations of motion—with arbitrary stresses—so as to obtain the equation of ("Vis-viva") energy, that this equation contained a definite function, which represented the difference between the work done on the fluid by the stresses and the rate of increase of the energy per unit of volume, which function, he concluded, must, according to Joule, represent the Vis-viva converted into heat.

This conclusion was obtained from the equations irrespective of any particular relation between the stresses and the rates of distortion. Sir G. Stokes, however, translated the function into an expression in terms of the rates of distortion, which expression has since been named by Lord Rayleigh the *Dissipation Function.*

In 1883 the author succeeded in proving, by means of experiments with colour bands—the results of which were communicated to the Society*—that when water is caused by pressure to flow through a uniform smooth pipe, the motion of the water is *direct*, *i.e.*, parallel to the sides of the pipe, or *sinuous*, *i.e.*, crossing and recrossing the pipe, according as U_m, the mean velocity of the water, as measured by dividing Q, the discharge by Δ, the area of the section of the pipe, is below or above a certain value given by $K\mu/D\rho$, where D is the diameter of the pipe, ρ the density of the water, and K a numerical constant, the value of which according to the author's experiments and, as he was able to show, to all the experiments by Poiseuille and Darcy, is for pipes of circular section between

1,900 and 2,000,

or, in other words, steady direct motion in round tubes is stable or unstable according as

$$\rho\frac{DU_m}{\mu} < 1900 \quad \text{or} \quad > 2000$$

the number K being thus a criterion of the possible maintenance of sinuous or eddying motion.

The experiments also showed that K was equally a criterion of the law of the resistance to be overcome—which changes from a

* 'Phil. Trans.,' 1883, Part III, p. 935.

resistance proportional to the velocity and in exact accordance with the theoretical results obtained from the singular solution of the equation, when direct motion changes to sinuous, *i.e.*, when

$$\rho \frac{DU_m}{\mu} = K.$$

In the same paper it was pointed out that the existence of this sudden change in the law of motion of fluids between solid surfaces when

$$DU_m = \frac{\mu}{\rho} K,$$

proved the dependence of the manner of motion of the fluid on a relation between the product of the dimensions of the pipe multiplied by the velocity of the fluid and the product of the molecular dimensions multiplied by the molecular velocities which determine the value of μ for the fluid, also that the equations of motion for viscous fluid contained evidence of this relation.

These experimental results completely removed the discrepancy previously noticed, showing that, whatever may be the cause, in those cases in which the experimental results do not accord with those obtained by the singular solution of the equations, the actual motions of the water are different. But in this there is only a partial explanation, for there remains the mechanical or physical significance of the existence of the criterion to be explained.

In the present paper the author applies the dynamical theory to the motion of incompressible viscous fluids to show—

(*a.*) That the adoption of the conclusion arrived at by Sir Gabriel Stokes, that the dissipation function represents the rate at which heat is produced, adds a definition to the meaning of u, v, w—the components of mean or fluid velocity—which was previously wanting;

(*b.*) That as the result of this definition the equations are true, and are only true, as applied to fluid in which the mean-motions of the matter, excluding the heat motions, are steady;

(*c.*) That the evidence of the possible existence of such steady mean-motions, while at the same time the conversion of the energy of these mean-motions into heat is going on, proves the existence of some *discriminative cause* by which the *periods* in space and time of the mean-motion are prevented from approximating in magnitude to the corresponding *periods* of the heat motions; and also proves the existence of some general action by which the energy of mean-motion is continually *transformed* into the energy of heat-motion without passing through any intermediate stage;

(*d.*) That as applied to fluid in unsteady mean-motion (excluding the heat-motions), however steady the mean integral flow may be, the equations are approximately true in a degree which increases with the ratios of the magnitudes of the *periods*, in time and space, of the mean-motion to the magnitude of the corresponding periods of the heat-motions;

(*e.*) That if the *discriminative cause* and the *action of transformation* are the result of general properties of matter, and not of properties which affect only the ultimate motions, there must exist evidence of similar actions as between mean-mean-motion, in directions of mean flow, and the periodic mean-motions taken relative to the mean-mean-motion but excluding heat-motions. And that such evidence must be of a general and important kind, such as the unexplained laws of the resistance of fluid motions, the law of the universal dissipation of energy and the second law of thermodynamics;

(*f.*) That the *generality* of the effects of the properties on which the *action of transformation* depends is proved by the evidence that resistance, other than proportional to the velocity, is caused by the relative (eddying) mean-motion.

(*g.*) That the existence of the *discriminative cause* is directly proved by the existence of the *criterion*, the dependence of which on circumstances which limit the magnitudes of the periods of relative-mean-motion, as compared with the heat motion, also proves the *generality* of the effects of the properties on which it depends.

(*h.*) That the proof of the generality of the effects of the properties on which the discriminative cause and the action of transformation depend, shows that—if in the equations of motion the mean-mean-motion is distinguished from the relative-mean-motion in the same way as the mean-motion is distinguished from the heat-motions—(1) the equations must contain expressions for the *transformation* of the energy of mean-mean-motion to energy of relative-mean-motion; and (2) that the equation, when integrated over a complete system, must show that the possibility of relative-mean-motion depends on the ratio of the possible magnitudes of the periods of relative-mean-motion, as compared with the corresponding magnitude of the periods of the heat-motions.

(*i.*) That when the equations are transformed so as to distinguish between the mean-mean-motions of infinite periods and the relative-mean-motion of finite periods, there result two distinct systems of equations, one system for mean-mean-motion, as affected by relative-mean-motions and heat-motion, the other system for relative-mean-motion as affected by mean-mean-motion and heat-motions.

(*j.*) That the equation of energy of mean-mean-motion, as obtained from the first system, shows that the rate of increase of energy is diminished by conversion into heat, and by transformation of energy of mean-mean-motion in consequence of the relative-mean-motion, which transformation is expressed by a function identical in form with that which expresses the conversion into heat; and that the equation of energy of relative-mean-motion, obtained from the second system, shows that this energy is increased only by transformation of energy from mean-mean-motion expressed by the same function, and diminished only by the conversion of energy of relative-mean-motion into heat.

(*k*.) That the difference of the two rates (1) transformation of energy of mean-mean-motion into energy of relative-mean-motion as expressed by the transformation function, (2) the conversion of energy of relative-mean-motion into heat, as expressed by the function expressing dissipation of the energy of relative-mean-motion, affords a discriminating equation as to the conditions under which relative-mean-motion can be maintained.

(*l*.) That this discriminating equation is independent of the energy of relative-mean-motion, and expresses a relation between variations of mean-mean-motion of the first order, the space periods of relative-mean-motion and μ/ρ such that any circumstances which determine the maximum periods of the relative-mean-motion determine the conditions of mean-mean-motion under which relative mean-motion will be maintained—determine *the criterion :*

(*m*.) That as applied to water in steady mean flow between parallel plane surfaces, the boundary conditions and the equation of continuity impose limits to the maximum space periods of relative-mean-motion such that the discriminating equation affords definite proof that when an indefinitely small sinuous or relative disturbance exists it must fade away if

$$\rho\frac{DU_m}{\mu}$$

is less than a certain number, which depends on the shape of the section of the boundaries and is constant as long as there is geometrical similarity. While for greater values of this function, in so far as the discriminating equation shows, the energy of sinuous motion may increase until it reaches to a definite limit, and rules the resistance.

(*n*.) That besides thus affording a mechanical explanation of the existence of the criterion K, the discriminating equation shows the purely geometrical circumstances on which the value of K depends, and although these circumstances must satisfy geometrical conditions required for steady mean-motion other than those imposed by the conservations of mean energy and momentum, the theory admits of the determination of an inferior limit to the value of K under any definite boundary conditions, which, as determined for the particular case, is

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This is below the experimental value for round pipes, and is about half what might be expected to be the experimental value for a flat pipe, which leaves a margin to meet the other kinematical conditions for steady mean-mean-motion.

(*o*.) That the discriminating equation also affords a definite expression for the resistance, which proves that, with smooth fixed boundaries, the conditions of dynamical similarity under any geometrical similar circumstances depend only on the value of

$$\frac{\rho}{\mu^2}\frac{dp}{dx}b^3,$$

where b is one of the lateral dimensions of the pipe; and that the expression for this resistance is complex, but shows that above the critical velocity the relative-mean-motion is limited, and that the resistances increase as a power of the velocity higher than the first.

III. "On certain Functions connected with Tesseral Harmonics, with Applications." By A. H. LEAHY, M.A., late Fellow of Pembroke College, Cambridge, Professor of Mathematics at Firth College, Sheffield. Communicated by Professor W. M. HICKS, F.R.S. Received March 24, 1894.

(Abstract.)

The transformation of a zonal harmonic referred to a pole on a sphere to another pole on the same sphere, and its expression in a series containing the $2n+1$ harmonics of the same order referred to this new pole, is an operation frequently employed in physical research. The purpose of this paper is the investigation of certain functions of the angular distance between the poles which occur when a general tesseral harmonic is transformed from one pole and plane to another pole and another plane of reference. If the coordinates of any point on the sphere when referred to the first pole are β' and γ'; β' denoting the colatitude, and γ' the longitude; and if the coordinates of the same point when referred to the second pole are δ' and q; δ' denoting the colatitude and q the longitude referred to a plane through the two poles, it is shown that

$$\mathrm{P}(n,m,\mu')\cos m\gamma' = \cos m\gamma\left\{u_{m,0}\,.\,\mathrm{P}_n(\nu) + 2\Sigma\frac{n-r\,!}{n+r\,!}\,u_{mr}\,.\,\mathrm{P}\,(n,r,\nu)\cos rq\right\}$$

$$+\sin m\gamma\,.\,2\Sigma\frac{n-r\,!}{n+r\,!}\,v_{mr}\,.\,\mathrm{P}\,(n,r,\nu)\sin rq,$$